Runge-Kutta Method

1 Introduction

The Runge-Kutta Method is a numerical method used to approximate the solution to ordinary differential equations. The essence of the Runge-Kutta method lies in its ability to estimate the solution at a given point by considering not just the slope at the initial point (as in Euler's method) but also by incorporating information about the slope at several intermediate points within each step. This approach allows the method to approximate the true solution curve better, resulting in more accurate and stable results.
Among the Runge-Kutta methods of different orders, the one discussed in this documentation is the Runge-Kutta method of fourth order (often referred to simply as "RK4"). RK4 evaluates the slope four times within each step and uses a weighted average of these slopes to compute the next value of the solution. This method ensures the accuracy and efficiency of the calculation.

2 Runge-Kutta Method

Let us have the following equation and eventually, we want to find y(t),

$$\frac{d^2y}{dt^2} + A\frac{dy}{dt} + By = 0 \tag{1}$$

In **Runge-Kutta method**, numerical integration of ordinary differential equations is achieved by using trial steps that correspond to intervals designed to cancel out lower-order error terms.
Starting with the first-order approximation,

$$\frac{dy}{dt} = \frac{y(t+\Delta t) - y(t)}{\Delta t} = f(y,t) \tag{2}$$

$$y(t+\Delta t) = \Delta t f(y,t) + y(t) \tag{3}$$

$$y(t+1) = y(t) + \Delta t f(y,t) \tag{4}$$

2.1 Geometrical Interpretation

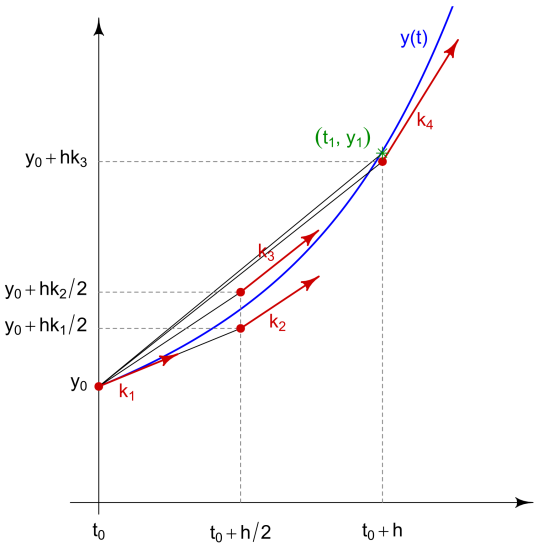

Figure 1: Approximation via Runge-Kutta Method

The time interval is of size h and it is further divided into sub-intervals to ensure that the numerically computed curve is as close as possible to the actual curve.
The RK4 method evaluates the slope at four points within each step:

- k_1: Initial slope at the beginning of the interval
- k_2 and k_3: Two estimates of the slope at the midpoint
- k_4: Slope at the end of the interval

The final slope is a weighted average of these four slopes, with more weight given to the midpoint estimates.

2.2 Formulation of the Method

For a step size h,

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{5}$$

$$k_1 = hf(t_n, y_n) \tag{6}$$
$$k_2 = hf(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}) \tag{7}$$
$$k_3 = hf(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}) \tag{8}$$
$$k_4 = hf(t_n + h, y_n + k_3) \tag{9}$$
$$t_{n+1} = t_n + h \tag{10}$$

3 Error Analysis

3.1 Local Truncation Error

The local truncation error for RK4 is of order $O(h^5)$, where h is the step size. This means that the error introduced in a single step decreases with the fifth power of the step size.

3.2 Global Truncation Error

The global truncation error, which accumulates over all steps, is of order $O(h^4)$. This is because the local errors accumulate over approximately $1/h$ steps.

3.3 Error Term Derivation

To understand how these error terms arise, consider the Taylor series expansion of the solution:

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2!}y''(t) + \frac{h^3}{3!}y'''(t) + \frac{h^4}{4!}y^{(4)}(t) + O(h^5) \tag{11}$$

The RK4 method matches this expansion up to the h^4 term, resulting in a local error of $O(h^5)$.

4 Implementing Runge-Kutta Method in C++

The following is how you can implement the Runge-Kutta method as a numerical technique in a C++ program:
Let's define a **for loop** that will perform the same function as discussed above in Section 2.2:

```
// Part of the code for implementing the Runge-Kutta method

for(double t=0 ;t<500 ;t+=h) // h is the step size
{ /* f() is a function that will perform the necessary operation where t,y, and v are all
     initial values of the parameters involved */
    double k1 = h * f(t,y,v);
    double k2 = h * f(t + 0.5 * h, y + 0.5 * k1, v + 0.5 * k1);
    double k3 = h * f(t + 0.5 * h, y + 0.5 * k2, v + 0.5 * k2);
    double k4 = h * f(t + h, y + k3, v + k3);
    y += (1.0 / 6.0 * k1 + 1.0 / 3.0 * k2 + 1.0 / 3.0 * k3 + 1.0 / 6.0 * k4);
}
```

5 Comparison with Other Methods

5.1 Euler Method

The Euler method is the simplest numerical method for solving ODEs, but it's less accurate than RK4. It has a global truncation error of $O(h)$.

5.2 Ralston Method

The Ralson method, which is an improvement to the second-order Runge-Kutta method, has a global truncation error of $O(h^2)$. It's more accurate than Euler but less accurate than RK4.

5.3 RK4 Advantages

RK4 offers a good balance between accuracy and computational cost. It's significantly more accurate than lower-order methods without requiring evaluation of higher derivatives.

6 Applications

The RK4 method is widely used in various fields, including:

- Physics: Simulating particle motion, analyzing oscillators
- Engineering: Control systems, circuit analysis
- Biology: Population dynamics, enzyme kinetics
- Finance: Option pricing, interest rate models

7 Conclusion

The fourth-order Runge-Kutta method is a powerful tool for numerically solving ODEs. Its balance of accuracy and efficiency makes it suitable for a wide range of applications. However, for very high-precision requirements or stiff equations, more advanced methods might be necessary.

References

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