Euler Method

1 Introduction

Up to this point, practically every differential equation that we've been presented with could be solved with pen and paper. The problem is that these are the exceptions rather than the rule. The vast majority of first-order differential equations can't be solved analytically. In these cases, we resort to numerical methods that will allow us to approximate solutions to differential equations.
Many different methods can be used to approximate the solutions to a differential equation. The Euler Method, introduced by the Swiss mathematician Leonhard Euler in the 18th century, is one of the simplest and most widely used numerical methods. It is a first-order method to solve the differential equations numerically.

2 Euler's method

The goal of the Euler Method is to approximate the curve of the solution by finding out the tangent lines at different points along the curve that are a certain step size apart. The Euler method works on the approximation that the approximated value of the curve at a certain point is quite close to the true value at that point computed via making substitutions in the true solution. To illustrate the Euler Method, consider a general first-order differential equation of the form,

$$\frac{dy}{dt} = f(t, y) \tag{1}$$

and let's assume that the true solution of the equation is g(t) and the initial value condition is

$$y(t_0) = y_0 \tag{2}$$

Euler method asserts that at a point t_{i+1},

$$g(t_{i+1}) \approx y_{i+1}$$

To approximate the solution to (1) near t $= t_0$ we start with the two pieces of information that we do know about the solution. First, we know the value of the solution at t $= t_0$ from the initial condition. Second, we also know the value of the derivative at t $= t_0$.

$$\frac{dy}{dt}|(t = t_0) = f(t_0, y_0) \tag{3}$$

From this, the equation of tangent line can be written as follows:

$$y = y_0 + f(t_0, y_0)(t - t_0) \tag{4}$$

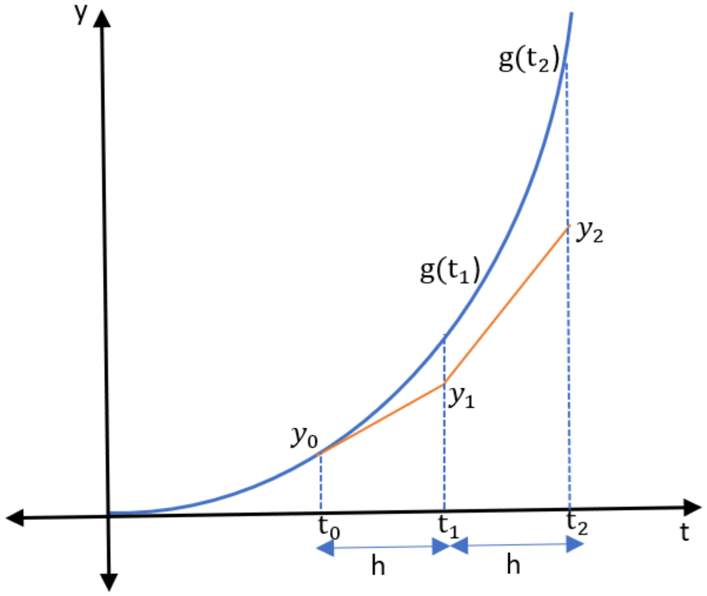

Figure 1: Tangent Line Approximation via Euler's Method

If t_1 is close enough to t_0 then the point y_1 on the tangent line should be fairly close to the actual value $g(t_1)$ of the solution at t_1

$$y_1 = y_0 + f(t_0, y_0)(t_1 - t_0) \tag{5}$$

Let's consider that y_1 is a good approximation to the solution and construct a tangent line through the point (t_1,y_1) that has slope f(t_1,y_1). Now, to get an approximation to the solution $g(t_2)$ at t=t_2, we get,

$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1) \tag{6}$$

So, we will continue in this fashion to compute the next approximation via a loop. In general, the expression becomes

$$y_{n+1} = y_n + hf_n(t_n, y_n) \tag{7}$$

where

$$t_{n+1} = t_n + h \tag{8}$$

where h is the step size
The smaller the step size, the better the approximation, but there is a limit to the reduction in the step size. The next section covers more on this.

3 Error Analysis

To see how the error terms and finally the truncation errors are approximated for the Euler method, we have taken a simpler case of a first-order differential equation, which can be extended to higher orders in the same manner but will be complex.
The Governing Equation is:

$$\frac{df}{dt} = \frac{f(t + \Delta t) - f(t)}{\Delta t} \tag{9}$$

If we write the Taylor expansion of the RHS, we get

$$\frac{df}{dt} = (\frac{f(t)}{\Delta t} + \frac{df(t)}{dt} + \frac{\Delta t}{2!}\frac{d^2 f(t)}{dt^2} + \frac{(\Delta t)^2}{3!}\frac{d^3 f(t)}{dt^3} + ...) - \frac{f(t)}{\Delta t} \tag{10}$$

Hence, we get

$$\frac{df}{dt} = \frac{df}{dt} + \frac{\Delta t}{2!}\frac{d^2 f(t)}{dt^2} + \frac{(\Delta t)^2}{3!}\frac{d^3 f(t)}{dt^3} + ... \tag{11}$$

Only first term of the RHS matches with the first term of RHS, and the rest are error terms. Finally, we can write

$$\frac{df}{dt} = \frac{df}{dt} + error(O(\Delta t)) \tag{12}$$

From eq.(12), we can say that the local error is of order Δt.
In the same manner, local error can be approximated for the Euler Method using Taylor expansion:

$$y(t+h) = y(t) + hy'(t) + O(h^2) \tag{13}$$

3.1 Local Truncation Error

The local truncation error for the Euler Method is of order $O(h^2)$ (here, h=Δt), where h is the step size. This means that the error introduced in a single step is proportional to the second power of the step size.

3.2 Global Truncation Error

The global truncation error, which accumulates over all of the steps, is of order $O(h)$. This is because the local errors accumulate over approximately $1/h$ steps.

This suggests that reducing the step size `h` improves the accuracy, but there is a trade-off in computational cost.

4 Implementing Euler Method in C++

```
// Implementing Euler method

for(double t=0 ;t<100 ;t+=h) // h is the step size
{
    a(t+h)= a(t) + h*b(t);
    /* b(t) represents the derivative of a, and h is the step size*/
}
```

This loop will stop executing as soon as the value of t becomes greater than 100, you can change this value if you wish to observe the behavior of the curve for a longer time.
A nice exercise is as follows: *CHANGE THE STEP SIZE h IN THE PROGRAM; YOU CAN SUBSTITUTE DIFFERENT VALUES OF h SUCH AS 1,0.1,0.5,0.01, ETC. TO OBSERVE HOW THE CURVE CHANGES AND AT WHAT VALUE OF h THE EULER METHOD APPROXIMATION DIVERGES.*

5 Conclusion

Euler method, the simplest numerical method of first-order is used to approximate the solutions of ordinary differential equations. However, the simplicity of the Euler Method comes with limitations. The method's accuracy is dependent on the step size `h`; smaller step sizes yield better approximations but at the cost of increased computational effort.
The method is also prone to accumulating significant errors over many iterations, particularly when the solution curve is rapidly changing or when a coarse step size is used. This makes the Euler Method less suitable for problems requiring high precision or long time intervals.

Despite these limitations, the Euler Method serves as an important introduction to more advanced numerical techniques, such as higher-order Runge-Kutta methods, which offer improved accuracy and stability. Understanding the Euler Method is essential for anyone studying numerical analysis or applied mathematics, as it lays the groundwork for more complex methods used in various scientific and engineering applications.