AIT Blog

Uniform Tests and Algorithmic Thermodynamic Entropy

Samuel Epstein*

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For definitions in this post, we use [HR09]. A computable metric space X is a metric space with a dense set of ideal points on which the distance function is computable. A computable probability is defined by a computable sequence of converging points in the corresponding space of Borel probability measures, $\mathcal{M}(X)$, over X. A uniform test takes in a description of a probability measure μ and produces a lower computable μ test. There exists a universal uniform test, from [G2́1, HR09]. We extend the result from [Eps22] to computable metric spaces.

Theorem 1 *Given computable non-atomic probability measures μ and λ over a computable metric space X and universal uniform test $\mathbf{t}(\cdot,\cdot)$. For all n, $\lambda(\{\alpha : \mathbf{t}(\mu,\alpha) > 2^n\}) \overset{*}{>} 2^{-n-\mathbf{K}(n)}$.*

Reworking the above theorem, one can get a result in algorithmic physics. To define algorithmic fine-grain entropy, we use a slightly modified version of the definition in [Gac94], and I refer to that paper for the motivation of the definition. First, note that all the results [HR09] can be easily extended to arbitrary nonnegative measures. This can be achieved by defining the product space of $\mathcal{M}(X)$ and $\mathbb{R}_{\geq 0}$, where the second metric space defines the size of the measure. Given a measure $\mu \in \mathcal{M}(X) \times \mathbb{R}_{\geq 0}$, the algorithmic fine grained entropy of a point $\alpha \in X$ is as follows.

Definition 1 (Algorithmic Fine-Grained Entropy) $H(\alpha) = -\log \mathbf{t}(\mu, \alpha)$.

One can then prove that this term will oscillate in the presence of dynamics. Dynamics can be defined using group theory.

Definition 2 (Transformation Group) *Let M denote a computable metric space and G a topological group each element of which is a homeomorphism of M onto itself:*

$$f(g;x) = g(x) = x' \in M; g \in G, x \in M.$$

The pair (G, M) will be called a topological transformation group if for every pair of elements $g1, g2$ of G, and every $x \in M$, $g1(g2(x)) = (g1g2)(x)$ and if

$$x' = g(x) = f(g;x)$$

is continuous simultaneously in $x \in M$ and $g \in G$.

Theorem 2 (Oscillation of Thermodynamic Entropy) *Let L be the Lebesgue measure over $\mathbb{R}$. For one dimensional topological transformation group (G^t, X) acting on computable metric space X, for all $\alpha \in X$, $L\{t \in [0,1] : H(G^t\alpha) < \max_\beta H(\beta) - n\} \overset{*}{>} 2^{-n-\mathbf{K}(n)}$.*

*JP Theory Group. samepst@jptheorygroup.org

The Stability Theorem 5 in [Gac94] can be updated with the results in [HR09]. Let $\Pi(\cdot)$ be a set of disjoint uniformly enumerable open sets in the metric space X.

Definition 3 (Algorithmic Coarse Grained Entropy) $H(\Pi_i) = \mathbf{K}(i|\mu) + \log \mu(\Pi_i)$.

Proposition 1 *If $\mu(\Pi_i)$ is uniformly computable and $\alpha \in \Pi_i$ then $H(\alpha) <^+ H(\Pi_i) + \mathbf{K}(\Pi)$.*

Lemma 1 (Stability) $\mu\{\alpha \in \Pi_i : H(\alpha) < H(\Pi_i) - \mathbf{K}(\Pi) - m\} \overset{*}{<} 2^{-m}\mu(\Pi_i)$.

Its interesting to note that the proof for this theorem follows from first using a combinatorial argument about finite strings and then applying this result to prove a property of randomness deficiencies and then transfering this result to universal uniform tests and then finally algorithmic thermodynamic entropy. An open question is whether other such transfers can be proven, resulting in further characterization of H.

I have one more result in physics, which is conservation of algorithmic quantum randomness deficiency and information with respect to quantum operations. These two results, combined with the Quantum EL Theorem, will be combined into a paper. The goal is make headway into the intersection of AIT and physics, dubbed *algorithmic physics*. Another area to look into would be the connection of AIT with special and general relativity, and black hole entropy.

References

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