A Short Proof of the Existence of Anomalies

Samuel Epstein*

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Abstract

The Independence Postulate (IP) is a finitary Church-Turing Thesis, postulating that mathematical sequences are independent from physical ones. IP implies that anomalies are found in the physical world.

Anomalies

$\mathbf{K}(x|y)$ is the conditional prefix Kolmogorov complexity. For probability p over $\mathbb{N}$, the deficiency of randomness is $\mathbf{d}(a|p) = \lfloor -\log p(a)\rfloor - \mathbf{K}(a)$. $\mathbf{I}(a;\mathcal{H}) = \mathbf{K}(a) - \mathbf{K}(a|\mathcal{H})$, where $\mathcal{H}$ is the halting sequence. An elementary probability measure over $\mathbb{N}$ has finite support and a range in $\mathbb{Q}$. Stochasticity is

$$\begin{aligned}&\Lambda(a|b) = \min\{\mathbf{K}(Q|b) + 3\log\max\{\mathbf{d}(a|Q,b),1\}\\ &: Q \text{ is an elementary probability measure}\}.\\ &\Lambda(a|b) < \Lambda(a) + O(\log \mathbf{K}(b)).\end{aligned}$$

The following definition is from [Lev74] .

Definition 1 (Information) $\mathbf{I}(\alpha:\beta)=\log\sum_{x,y}2^{\mathbf{K}(x)+\mathbf{K}(y)-\mathbf{K}(x,y)-\mathbf{K}(x|\alpha)-\mathbf{K}(y|\beta)}$.

The Independence Postulate [Lev84, Lev13] statement is:

IP: *Let α be a sequence defined with an n-bit mathematical statement, and a sequence β can be located in the physical world with a k-bit instruction set. Then $\mathbf{I}(\alpha:\beta) < k+n+c$ for some small absolute constant c.*

There are many proofs in the literature that stochastic numbers have high mutual information with the halting sequence. One detailed proof is in [Eps21].

Lemma 1 $\Lambda(x) <^{\log} \mathbf{I}(x;\mathcal{H})$.

Lemma 2 *For computable probability p over $\mathbb{N}$ and for $D\subset\mathbb{N}$, $|D| = 2^s$, $s < \max_{a\in D}\mathbf{d}(a|p) + \Lambda(D) + \mathbf{K}(s) + \mathbf{K}(p) + O(\log\mathbf{K}(s)\mathbf{K}(p))$.*

Proof. We relativize the universal Turing machine to p and s. Let Q be an elementary probability measure that realizes $\Lambda(D)$. Let $d = \mathbf{d}(D|Q)$. Let $F\subseteq\mathbb{N}$ be a random set where each element $a\in\mathbb{N}$ is selected independently with probability $cd2^{-s}$, where $c\in\mathbb{N}$ is chosen later. $\mathbf{E}[p(F)] \le cd2^{-s}$. Furthermore

$$\begin{aligned}&\mathbf{E}[Q(\{G:|G|=2^s, G\cap F=\emptyset\})]\\ &\le \sum_G Q(G)(1-cd2^{-s})^{2^s} < e^{-cd}.\end{aligned}$$

Thus finite $W\subset\mathbb{N}$ can be chosen such that $p(W)\le 2cd2^{-s}$ and $Q(\{G:|G|=2^s, G\cap W=\emptyset\})\le e^{1-cd}$. $D\cap W\ne\emptyset$, otherwise, using the Q-test, $t(G) = e^{cd-1}$ if $(|G|=2^s, G\cap W=\emptyset)$ and $t(G)=0$ otherwise, we have

$$\begin{aligned}&\mathbf{K}(D|Q,d,c) <^+ -\log Q(D) - (\log e)cd\\ &\quad(\log e)cd <^+ -\log Q(D) - \mathbf{K}(D|Q) + \mathbf{K}(d,c)\\ &\quad(\log e)cd <^+ d + \mathbf{K}(d,c),\end{aligned}$$

which is a contradiction for large c. Thus there is an $a\in D\cap W$, where

$$\begin{aligned}&\mathbf{K}(a) <^+ -\log p(a) + \log d - s + \mathbf{K}(d) + \mathbf{K}(Q)\\ &\quad s <^+ \mathbf{d}(a|p) + \Lambda(D).\end{aligned}$$

Removing the relativization of p and s

$$\begin{aligned}&s < -\log p(a) - \mathbf{K}(a|p,s) + \Lambda(D|p,s)\\ &s < \max_{a\in D}\mathbf{d}(a|p) + \Lambda(D) + \mathbf{K}(s) + \mathbf{K}(p)\\ &\quad + O(\log\mathbf{K}(s)\mathbf{K}(p)).\end{aligned}$$

□

*JP Theory Group. samepst@jptheorygroup.org

For $\tau \in \mathbb{N}^{\mathbb{N}}$, $\langle\tau\rangle = \langle\tau[1]\rangle\langle\tau[2]\rangle\langle\tau[3]\rangle\ldots$ Let $\tau(n)$ be the first 2^n unique numbers found in τ. The sequence τ is assumed to have an infinite amount of unique numbers, and represents a series of observations.

Theorem 1 *For probability p over $\mathbb{N}$ and $\tau \in \mathbb{N}^{\mathbb{N}}$, let $s_{\tau,p}$ be the smallest number such that $\max_{a\in\tau(n)} \mathbf{d}(a|p) > n - 4\mathbf{K}(n) - s_{\tau,p}$. Then $s_{\tau,p} <^{\log} \mathbf{I}(\langle\tau\rangle : \mathcal{H}) + \mathbf{K}(p)$.*

Proof. By Lemmas 1 and 2, and the fact that $\mathbf{I}(x;\mathcal{H}) <^{+} \mathbf{I}(\alpha : \mathcal{H}) + \mathbf{K}(x|\alpha)$,

$$\begin{aligned}
n &< \max_{a\in\tau(n)} \mathbf{d}(a|p) + \mathbf{I}(\tau(n);\mathcal{H}) + \mathbf{K}(p) + 2\mathbf{K}(n)\\
&\quad + O(\log(\mathbf{I}(\tau(n);\mathcal{H}) + \mathbf{K}(p))),\\
n &< \max_{a\in\tau(n)} \mathbf{d}(a|p) + 4\mathbf{K}(n) + \mathbf{I}(\langle\tau\rangle : \mathcal{H}) + \mathbf{K}(p)\\
&\quad + O(\log(\mathbf{I}(\langle\tau\rangle : \mathcal{H}) + \mathbf{K}(p))),\\
n &- 4\mathbf{K}(n) - (\mathbf{I}(\langle\tau\rangle : \mathcal{H}) + \mathbf{K}(p)\\
&\quad + O(\log(\mathbf{I}(\langle\tau\rangle : \mathcal{H}) + \mathbf{K}(p)))) < \max_{a\in\tau(n)} \mathbf{d}(a|p).
\end{aligned}$$

□

Let k be the physical address of an infinite sequence of numbers $\tau \in \mathbb{N}^{\mathbb{N}}$. The halting sequence can be described by a small mathematical statement. By Theorem 1 and **IP**,

$$s_{\tau,p} <^{\log} \mathbf{I}(\langle\tau\rangle : \mathcal{H}) + \mathbf{K}(p) <^{\log} k + c + \mathbf{K}(p).$$

Thus sequences τ with large $s_{\tau,p}$ will have large physical addresses. So, assuming **IP**, it's hard to find observations which do not have large anomalies, and impossible to find observations with no anomalies.

References

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