

A Chain Rule for Randomness Deficiency

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Abstract

This paper is an exposition of the addition equality theorem for algorithmic entropy in [G01], applied to the specific case of infinite sequences. This application implies that randomness deficiency of infinite sequences obeys the chain rule, analogous to the finite Kolmogorov complexity case. This is a generalization of van Lambalgen's Theorem. It is unclear whether this result is folklore, but in any case this paper presents a dedicated proof of the equality.

1 Introduction

Prefix free Kolmogorov complexity, \mathbf{K} , obeys the chain rule, with for $x, y \in \{0, 1\}^*$,

$$\mathbf{K}(x, y) =^+ \mathbf{K}(x) + \mathbf{K}(y|x, \mathbf{K}(x)).$$

In this paper, we apply the addition equality theorem for algorithmic entropy in [G01] to the specific case of infinite sequences. The consequence to this is a result about randomness deficiency \mathbf{D} , where for computable probability μ , for infinite sequences, $\mathbf{D}(\alpha|\mu, x) = \sup_n -\log \mu(\alpha[0..n]) - \mathbf{K}(\alpha[0..n]|x)$. The randomness deficiency over the space $\{0, 1\}^\infty \times \{0, 1\}^\infty$, is $\mathbf{D}(\alpha, \beta|\mu, \nu) = \sup_n -\log \mu(\alpha[0..n]) - \log \nu(\beta[0..n]) - \mathbf{K}(\alpha[0..n]\beta[0..n])$. The discrete case for $\mathbf{d}(x|p) = -\log p(x) - \mathbf{K}(x)$ is trivial. The result detailed in this paper is as follows.

Theorem. ([G01]) *Relativized to probabilities μ and ν over $\{0, 1\}^\infty$,*

$$\mathbf{D}(\alpha, \beta|\mu, \nu) =^+ \mathbf{D}(\alpha|\mu) + \mathbf{D}(\beta|\nu, (\alpha, \lceil \mathbf{D}(\alpha|\mu) \rceil)).$$

This is a generalization of van Lambalgen's Theorem, which states (α, β) is ML random iff α is ML random and β is ML random with respect to α . If one were to take the complexities of the probabilities μ and ν into account (that is, they are no longer $O(1)$) then the theorem statement and proofs become more nuanced. This generalization can be seen in [G01].

2 Results

As shown in [G01], $2^{\mathbf{D}(\alpha|\mu)} \stackrel{*}{=} \mathbf{t}_\mu(\alpha)$ where \mathbf{t}_μ is a universal lower computable μ -test. Furthermore, similar arguments can be used to show that $2^{\mathbf{D}(\alpha, \beta|\mu, \nu)} \stackrel{*}{=} \mathbf{t}_{\mu, \nu}(\alpha, \beta)$, where $\mathbf{t}_{\mu, \nu}$ is a universal lower computable test over $\{0, 1\}^\infty \times \{0, 1\}^\infty$. This can be seen in Proposition ???. For measure μ and lower continuous function f over $\{0, 1\}^\infty$, we use the notation $\mu^x f(x) = \int_{x \in \{0, 1\}^\infty} f(x) d\mu(x)$. Throughout this section, the universal Turing machine is assumed to be relativized to probabilities μ and ν over $\{0, 1\}^\infty$. This means that there is an $O(1)$ sized program that can compute $\mu(x\{0, 1\}^\infty)$ uniformly in $x \in \{0, 1\}^*$, and similarly for ν .

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Proposition 1 $2^{d(\alpha, \beta | \mu, \nu)} \stackrel{*}{=} \mathbf{t}_{\mu, \nu}(\alpha, \beta)$.

Proof. For the $<^*$ inequality, note that

$$\begin{aligned} f_{\mu, \nu}(\alpha, \beta) &= \sum_{n=1}^{\infty} \sum_{x, y \in \{0, 1\}^n} 1_{x\{0, 1\}^\infty, y\{0, 1\}^\infty}(\alpha, \beta) \frac{\mathbf{m}(x, y)}{\mu(x)\nu(y)} \\ &= \sum_{n=1}^{\infty} \sum_{x, y \in \{0, 1\}^n} \frac{\mathbf{m}(\alpha[0..n]\beta[0..n])}{\mu(x)\nu(y)} \\ &\geq \sup_n \frac{\mu(\alpha[0..n]\beta[0..n])}{\mu(\alpha[0..n])\nu(\beta[0..n])}. \end{aligned}$$

So

$$\log \mathbf{t}_{\mu, \nu}(\alpha, \beta) >^+ \log f_{\mu, \nu}(\alpha, \beta) >^+ \mathbf{D}(\alpha, \beta | \mu, \nu).$$

The $>^+$ inequality is as follows. Note that $1_{x, y}(\alpha, \beta) = 1_{x\{0, 1\}^\infty, y\{0, 1\}^\infty}(\alpha, \beta)$. Since $d(\alpha, \beta) = \lceil \log \mathbf{t}_{\mu, \nu}(\alpha, \beta) \rceil$ takes only integer values and is lower computable,

$$2^{d(\alpha, \beta)} = \sup_i 2^{k_i} 1_{x_i, y_i}(\alpha, \beta) \geq \frac{1}{2} \sum_i 2^{k_i} 1_{x_i, y_i}(\alpha, \beta),$$

with the property that for all i , $\|x_i\| = \|y_i\|$, and if $1_{x_i, x_j}(\alpha, \beta) = 1_{x_j, x_j}(\alpha, \beta) = 1$ and $i < j$ then $k_i < k_j$. Let $\gamma(x, y) = \sum_{y_i=y, x_i=x} 2^{k_i}$.

$$\sum_i 2^{k_i} 1_{x_i, y_i}(\alpha, \beta) = \sum_{n=1}^{\infty} \sum_{x, y \in \{0, 1\}^n} 1_{x, y}(\alpha, \beta) \gamma(x, y).$$

Since $\mu^x \nu^y 2^{d(\alpha, \beta)} \leq 2$, we have that $\sum_{x, y \in \{0, 1\}^*} \mu(x)\nu(y)\gamma(x, y) \leq 2$, so $\mu(x)\nu(y)\gamma(x, y) \stackrel{*}{<} \mathbf{m}(x, y)$ and $\gamma(x, y) = 0$ if $\|x\| \neq \|y\|$. So

$$2^{d(\alpha, \beta)} \stackrel{*}{<} \sup_{n \geq 1, x, y \in \{0, 1\}^n} 1_{x, y}(\alpha, \beta) \frac{\mathbf{m}(x, y)}{\mu(x)\nu(y)} \stackrel{*}{=} \sup_n \frac{\mathbf{m}(\alpha[0..n]\beta[0..n])}{\mu(\alpha[0..n])\nu(\beta[0..n])}.$$

□

Proposition 2 $-\mathbf{D}(x | \mu) <^+ -\log \nu^y 2^{\mathbf{D}(x, y | \mu, \nu)}$.

Proof. Let $f(x, \mu, \nu) = -\log \nu^y 2^{\mathbf{D}(x, y | \mu, \nu)}$. The function f is upper computable and has $\mu^x 2^{-f(x, \mu, \nu)} \leq 1$. The proposition follows from the universal properties of \mathbf{t}_μ , where $2^{-f} \stackrel{*}{<} \mathbf{t}_\mu$. □

Proposition 3 For a computable function $f : N^2 \rightarrow \mathbb{N}$,

$$-\mathbf{D}(x | \mu, y) <^+ \mathbf{K}(z) - \mathbf{D}(x | \mu, f(y, z)).$$

Proof. The function

$$g_\mu(x, y) = \sum_z 2^{\mathbf{D}(x | \mu, f(y, z)) - \mathbf{K}(z)},$$

is lower computable and $\mu^x g_\mu(x, y) \leq \sum_z 2^{-\mathbf{K}(z)} \leq 1$. So $g_\mu(x, y) \stackrel{*}{<} 2^{\mathbf{D}(x | \mu, y)}$. The left hand side is a summation, so the inequality holds for each element of the sum, proving the proposition. □

Proposition 4 If $i < j$, then

$$i - \mathbf{D}(x | \mu, i) <^+ j - \mathbf{D}(x | \mu, j).$$

Proof. Using Proposition 3, with $f(i, n) = i + n$, we have

$$-\mathbf{D}(x|\mu, i) + \mathbf{D}(x|\mu, j) <^+ \mathbf{K}(j - i) <^+ j - i.$$

□

Definition 1 Let $F : \{0, 1\}^\infty \rightarrow \mathbb{Z} \cup \{-\infty, \infty\}$ be an upper semicomputable function. An (μ, F) -test is a function $t : \{0, 1\}^\infty \times \{0, 1\}^\infty \rightarrow \mathbb{Z} \cup \{-\infty, \infty\}$ that is lower semicomputable and $\mu^x t(x, y) \leq 2^{-F(y)}$. There exists a maximal (μ, F) test, $\mathbf{t}_{(\mu, F)}$, such that $t <^* \mathbf{t}_{(\mu, F)}$.

Proposition 5 Let $F : \{0, 1\}^\infty \rightarrow \mathbb{Z} \cup \{-\infty, \infty\}$ be an upper semicomputable function,. For all x and with $\mathbf{t}_{(\nu, F)}(y) > -\infty$,

$$\mathbf{t}_{(\nu, F)}(x, y) \stackrel{*}{=} 2^{-F(y)} \mathbf{t}_\nu(x|y, F(y)).$$

Proof. To prove the inequality $\stackrel{*}{>}$, let $g(x, y, m) = \max_{i \geq m} 2^{-i} \mathbf{t}_\nu(x|y, i)$. This function is lower computable, and decreasing in m . Let $g(x, y) = g_\nu(x, y, F(y))$ is lower semicomputable since F is upper semi-computable. The multiplicative form of Proposition 4 implies

$$\begin{aligned} g(x, y, m) &\stackrel{*}{=} 2^{-m} \mathbf{t}_\nu(x|y, m) \\ g(x, y) &\stackrel{*}{=} 2^{-F(y)} \mathbf{t}_\nu(x|y, F(y)). \end{aligned}$$

Since \mathbf{t}_ν is a test:

$$\begin{aligned} \nu^x 2^{-m} \mathbf{t}_\nu(x|y, m) &\leq 2^{-m} \\ \nu^x g(x, y) &\stackrel{*}{<} 2^{-F(y)}, \end{aligned}$$

which implies $g(x, y) \stackrel{*}{<} \mathbf{t}_{(\nu, F)}(x, y)$ by the optimality of $\mathbf{t}_{(\nu, F)}$. We now consider the upper bound. Let $\mathbf{t}'_{(\nu, F)}(x, y, m)$ be the modification of $\mathbf{t}_{(\nu, F)}$, which is a lower computable function such that $\nu^x \mathbf{t}'_{(\nu, F)}(x, y, m) \leq 2^{-m+1}$ and if $\nu^x \mathbf{t}_{(\nu, F)}(x, y) \leq 2^{-m}$ then $\mathbf{t}'_{(\nu, F)}(x, y, m) = \mathbf{t}_{(\nu, F)}(x, y)$. The function $2^{m-1} \mathbf{t}'_{(\nu, F)}(x, y, m)$ is a test conditioned on y, m so it has $\stackrel{*}{<} \mathbf{t}_\nu(x|y, m)$. Substituting $F(y)$ for m , we have that $\nu^x \mathbf{t}_{(\nu, F)} \leq 2^{-m}$ and so

$$\mathbf{t}_{(\nu, F)}(x, y) = \mathbf{t}'_{(\nu, F)}(x, y, F(y)) \stackrel{*}{<} 2^{-F(y)+1} \mathbf{t}_\nu(x|y, F(y)).$$

□

Theorem 1 ([G01]) Relativized to computable probabilities μ and ν over $\{0, 1\}^\infty$,

$$\mathbf{D}(x, y|\mu, \nu) =^+ \mathbf{D}(x|\mu) + \mathbf{D}(y|\nu, (x, \lceil \mathbf{D}(x|\mu) \rceil)).$$

Proof. We first prove the $<^+$ inequality. Let $G(x, y, m) = \min_{i \geq m} i - \mathbf{D}((y|\nu, (x, i)))$, which is upper computable and increasing in m . So the function

$$G(x, y) = G(x, y, \lceil -\mathbf{D}(x|\mu) \rceil).$$

which is also upper computable because m is replaced with an upper computable function $\lceil -\mathbf{D}(x|\mu) \rceil$. Proposition 3 implies

$$\begin{aligned} G(x, y, m) &=^+ m - \mathbf{D}(y|\nu, (x, m)), \\ G(x, y) &=^+ -\mathbf{D}(x|\mu) - \mathbf{D}(y|\nu, (x, \lceil -\mathbf{D}(x|\mu) \rceil)). \end{aligned}$$

So

$$\begin{aligned}\nu^y 2^{-m+\mathbf{H}(y|\nu, (x, m))} &\leq 2^{-m} \\ \nu^y 2^{-G(x, y)} &\stackrel{*}{<} 2^{\mathbf{D}(x|\mu)}.\end{aligned}$$

Integrating over x gives $\mu^x \nu^y 2^{-G(x, y)} \stackrel{*}{<} 1$, implying $-\mathbf{D}(x, y|\mu, \nu) <^+ G(x, y)$.

To prove the $>^+$ inequality, let $f(x, y) = 2^{\mathbf{D}(x, y|\mu, \nu)}$. Proposition 2 implies there exists $c \in \mathbb{N}$ with $\nu^y f(x, y) \leq 2^{\mathbf{D}(x|\mu)+c}$. Let $F(x, \mu) = \lceil -\mathbf{D}(x|\mu) \rceil$. Note that if h is a lower computable function such that $\nu^y h(x, y) \stackrel{*}{<} 2^{\mathbf{D}(x|\mu)}$, then $\mu^x \nu^y h(x, y) \stackrel{*}{<} \mu^x \mathbf{t}_\mu(x) \stackrel{*}{<} 1$, so $h \stackrel{*}{<} f$, so f is a universal F -test. Proposition 5 (substituting y for x and (x, μ) for y) gives

$$-\mathbf{D}(x, y|\mu, \nu) = -\log f(x, y) >^+ F(x) - \mathbf{D}(y|\nu, (x, F(x))).$$

□

References

- [G01] P. Gács. Quantum Algorithmic Entropy. *Journal of Physics A Mathematical General*, 34(35), 2001.