

# Game Derandomization

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## Abstract

Using Kolmogorov Game Derandomization, upper bounds of the Kolmogorov complexity of deterministic winning players against deterministic environments can be proved. This paper gives improved upper bounds of the Kolmogorov complexity of such players. This paper also generalizes this result to probabilistic games. This applies to computable, lower computable, and uncomputable environments. We characterize the classic even-odds game in the context of algorithmic information theory. In addition, we start with an illustrative example of game derandomization, taking place on the island of Crete.

## 1 The Minotaur and the Labyrinth

A hero is trapped in a labyrinth, which consists of long corridors connecting to small rooms. The intent of the hero is to reach the goal room, which has a ladder in its center reaching the outside. The downside is the hero is blindfolded. The upside is there is a minotaur present to guide the hero.

At every room, the minotaur tells the hero the number of corridors  $n$  leading out (including the one which the hero just came from). The hero states a number between 1 and  $n$  and the minotaur takes the hero to corresponding door. However the hero faces another obstacle, in that the minotaur is trying to trick him. This means the mapping the minotaur uses is a function of all the hero's past actions. Thus if a hero returns to the same room, he may be facing a different mapping than before. This process continues for a very large number of turns. The question is how much information is needed by the hero to find the exit? Using **Kolmogorov Game Derandomization**, we get the following surprising good news for the hero. Let  $c$  be the number of corridors and  $d$  be the number of doors in the goal room.

*The hero can find the exit using  $\log(c/d) + \epsilon$  bits.*

The error term  $\epsilon$  is logarithmic and also is dependent on the information the halting sequence has about the entire construct, which is negligible except

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in exotic cases. Assuming the **Independence Postulate** [Lev84, Lev13], one cannot find such exotic constructs in the physical world.

The reasoning for this is as follows. Take a random hero who chooses a corridor with uniform probability. Then the hero is performing a random walk on the graph (of the labyrinth). Assuming the number of turns is greater than the graph's mixing time, the probability the hero is at exit at the end is not less than  $d/bc$ , for some fixed constant  $b$ . Then the following theorem can be applied.  $\mathbf{K}(x)$  is the prefix Kolmogorov complexity of  $x$ .  $\mathbf{I}(x; \mathcal{H}) = \mathbf{K}(x) - \mathbf{K}(x|\mathcal{H})$  is the amount of information the halting sequence  $\mathcal{H}$  has about  $x$ . Note that in Section 8, the bounds of the following theorem are improved.

**Theorem 1** ([Eps23b]). *If probabilistic agent  $\mathbf{p}$  wins against environment  $\mathbf{q}$  with at least probability  $p$ , then there is a deterministic agent of Kolmogorov complexity  $<^{\log} \mathbf{K}(\mathbf{p}) - \log p + \mathbf{I}(\langle \mathbf{p}, \mathbf{q} \rangle; \mathcal{H})$  that wins against  $\mathbf{q}$ .*

## 2 Setup

We introduce some key tools necessary for this paper.  $O_{a_1, \dots, a_n}(1)$  is a constant dependent on parameters  $a_1, \dots, a_n$ . For  $n \in \mathbb{N}$ ,  $\langle n \rangle = 01^n 0$ . We use  $x <^+ y$ ,  $x >^+ y$  and  $x =^+ y$  to denote  $x < y + O(1)$ ,  $x + O(1) > y$  and  $x = y \pm O(1)$ , respectively. Furthermore,  $^*f, ^*f$  denotes  $< O(1)f$  and  $> f/O(1)$ . In addition,  $x <^{\log} y$  and  $x >^{\log} y$  denote  $x < y + O(\log y)$  and  $x + O(\log x) > y$ , respectively.  $(x0)^- = (x1)^- = x$ . We say  $x \sqsubseteq y$  if  $xz = y$  for some  $z \in \{0, 1\}^*$ . We say  $[A] = 1$  if mathematical statement  $A$  is true, and  $[A] = 0$ , otherwise. The function  $\mathbf{m}(x)$  is a universal lower-computable semi-measure. A continuous semi-measure is a function  $\omega : \{0, 1\}^* \rightarrow [0, 1]$  such that  $\omega(\emptyset) = 1$  and  $\omega(x) \geq \omega(x0) + \omega(x1)$ . For prefix free set  $G$ ,  $\omega(G) = \sum_{x \in G} \omega(x)$ . Note that it could be that  $\omega(x) > \omega(\{x0, x1\})$ .  $\mathbf{M}$  is a universal lower-computable continuous semi-measure. Thus for prefix free set  $G \subset \{0, 1\}^*$ , lower computable continuous semi-measure  $\omega$ ,  $\mathbf{M}(G) \stackrel{*}{>} \mathbf{m}(\omega)\omega(G)$ . Monotone Kolmogorov complexity is  $\mathbf{Km}(x) = \min\{\|p\| : U(p) \sqsupseteq x\}$ .

This is a slightly different convention then the literature, in that the universal Turing machine  $U$  must halt.

Mutual information between strings is  $\mathbf{I}(x : y) = \mathbf{K}(x) + \mathbf{K}(y) - \mathbf{K}(x, y)$ . For strings,  $x, y$ ,  $\mathbf{I}(x; y) = \mathbf{K}(x) - \mathbf{K}(x|y)$ .

A probability  $P$  over  $\mathbb{N}$  is elementary if it has finite support and its range is a subset of  $\mathbb{Q}$ . Elementary probabilities can be encoded into finite strings or natural numbers. The randomness deficiency of  $x \in \mathbb{N}$  with respect to elementary probability  $P$  and  $y \in \mathbb{N}$  is  $\mathbf{d}(x|P, y) = \lceil -\log P(x) \rceil - \mathbf{K}(x|P, y)$ .

**Definition 1** (Stochasticity). *The stochasticity of  $x \in \mathbb{N}$  with respect to  $y \in \mathbb{N}$  is  $\mathbf{Ks}(x|y) = \min\{\mathbf{K}(P|y) + 3 \log \max\{\mathbf{d}(x|P, y), 1\} : P \text{ is elementary}\}$ .*

**Lemma 1** ([Eps23c, Lev16]).  $\mathbf{Ks}(x|y) <^{\log} \mathbf{I}(x; \mathcal{H}|y)$ .

**Lemma 2** ([Eps22]). *For partial computable function  $f$ ,  $\mathbf{I}(f(x); \mathcal{H}) <^+ \mathbf{I}(x; \mathcal{H}) + \mathbf{K}(f)$ .*

A Win/No-Halt game is a series of interactions between an agent  $\mathbf{p}$  and an environment  $\mathbf{q}$ . Each round starts with  $\mathbf{p}$  initiating a move, which is chosen out of  $\mathbb{N}$  and then  $\mathbf{q}$  responds with a number or  $\mathbf{q}$  can halt. Agent  $\mathbf{p}$  wins if  $\mathbf{q}$  halts the game, otherwise the game can continue potentially forever. Thus  $\mathbf{p}$  is a function  $(\mathbb{N} \times \mathbb{N})^* \mapsto \mathbb{N}$  and  $\mathbf{q}$  is a function  $\mathbb{N} \times (\mathbb{N} \times \mathbb{N})^* \mapsto \mathbb{N} \cup \{\emptyset\}$ . Both  $\mathbf{p}$  and  $\mathbf{q}$  are assumed to be computable, however, lower computable and uncomputable environments are studied in Section 7. Both the agent and environment can be probabilistic in their choice actions. Thus the probabilities of each action are uniformly computable to any degree of accuracy.

### 3 Alternative Proof

In this section we provide an alternative proof to Theorem 1 than the one presented in [Eps23b]. This proof relies on the Monotone EL Theorem, whereas the one in [Eps23b] relies on stochastic processes and is much more extensive.

**Theorem 2** (Monotone EL Theorem, [Eps24]). *For prefix free set  $G$ ,  $\min_{x \in G} \mathbf{Km}(x) <^{\log} -\log \mathbf{M}(G) + \mathbf{I}(G; \mathcal{H})$ .*

**Theorem 3.** *If probabilistic agent  $\mathbf{p}$  wins against environment  $\mathbf{q}$  with at least probability  $p$ , then there is a deterministic agent of Kolmogorov complexity  $<^{\log} \mathbf{K}(\mathbf{p}) - \log p + \mathbf{I}(\langle \mathbf{p}, \mathbf{q} \rangle; \mathcal{H})$  that wins against  $\mathbf{q}$ .*

*Proof.* One can associate  $(\mathbb{N} \times \mathbb{N})^*$  to  $\mathbb{N}$  in the natural way. Thus  $\mathbf{p}$  can be seen as a stochastic process over a set of random variables  $\{X_i\}_{i=1}^\infty$ . So for each  $X_i$ , the index  $i$  encodes a history of interactions with  $\mathbf{q}$ . So an interaction with  $\mathbf{q}$  is a set  $\{(a_i, b_i)\}_{i=1}^n$ , where each  $a_i \in \mathbb{N}$  encodes the history of the interaction and  $b_i$  is the agent's action. By encoding each number  $n \in \mathbb{N}$  into a string  $\langle n \rangle$ ,  $\mathbf{p}$  can be seen as a computable measure over  $\{0, 1\}^\infty$ . With this encoding, an interaction is an open set in the Cantor space. Let  $s = \lceil -\log p \rceil + 1$ . Let  $G \subset \{0, 1\}^\infty$  be a clopen set, computable from  $\mathbf{q}$ ,  $\mathbf{p}$  and  $s$  where each of its elements encode a winning interaction and  $\mathbf{p}(G) > 2^{-s}$ . Thus  $\mathbf{K}(G|\mathbf{p}, \mathbf{q}, s) = O(1)$ . Using the Monotone EL Theorem 2, There exists  $y$  such that  $y \sqsupseteq x \in G$  and

$$\begin{aligned} \mathbf{K}(y) &<^{\log} -\log \mathbf{M}(G) + \mathbf{I}(G; \mathcal{H}) \\ &<^{\log} \mathbf{K}(\mathbf{p}) + s + \mathbf{I}(\langle \mathbf{p}, s, \mathbf{q} \rangle; \mathcal{H}) \\ &<^{\log} \mathbf{K}(\mathbf{p}) - \log p + \mathbf{I}(\langle \mathbf{p}, \mathbf{q} \rangle; \mathcal{H}) \end{aligned} \tag{1}$$

Equation 1 is due to Lemma 2. This  $y \in \{0, 1\}^*$  encodes a finite series of numbers, which in turn encodes a series of actions  $a \in \mathbb{N}$  at a number of histories  $h \in (\mathbb{N} \times \mathbb{N})^*$ . These actions, when paired with  $\mathbf{q}$  will result in a winning interaction for the agent.  $\square$

### 4 Probabilistic Games

In this section we prove Kolmogorov Game Derandomization over probabilistic environments. This is an extension to Theorem 1, enabling the characterization

of all computable and probabilistic environments.

The main proof uses the notion of a *game fragment*. A game fragment  $\mathcal{F}$  is a finite tree, where each edge has a number  $n \in \mathbb{N}$  representing an action. On the odd levels, the edges are coupled with rational weights in  $[0, 1]$  and the summation of weights on edges with the same parent node is less than 1. Such fragments  $\mathcal{F}$  can be coupled with a probabilistic agent  $\mathbf{p}$ , who fills in the weights of each even level edges with its probabilistic action. In such a coupling, the weight of each path is the product of the probabilities along each edge of the path.  $\text{Weight}(\mathbf{p}, \mathcal{F})$  is the sum of the weights of each path.

**Claim 1.** *If there is a fragment  $\mathcal{F}$  where each path from the root to a leaf represents a winning interaction with an environment  $\mathbf{q}$  and the weights of  $\mathcal{F}$  are not more than  $\mathbf{q}$ 's probabilities of those particular actions, then  $\text{Weight}(\mathbf{p}, \mathcal{F})$  is not more than the probability that  $\mathbf{p}$  wins against  $\mathbf{q}$ .*

**Theorem 4.** *Let  $\mathbf{p}$  be a probabilistic agent and  $\mathbf{q}$  be a probabilistic environment. If  $\mathbf{p}$  wins in the Win/No-Halt game against  $\mathbf{q}$  with probability  $> 2^{-s}$ ,  $s \in \mathbb{N}$ , then there is a deterministic agent of complexity  $<^{\log} \mathbf{K}(\mathbf{p}) + 2s + \mathbf{I}(\langle \mathbf{p}, \mathbf{q} \rangle; \mathcal{H})$  that wins with probability  $> 2^{-s-1}$ .*

*Proof.* We relativize the universal Turing machine to  $\langle \mathbf{p}, s \rangle$ . Thus this information is on an auxiliary tape and implicitly in the conditional of all complexity terms. Let  $\mathcal{F}$  be a game fragment corresponding the environment  $\mathbf{q}$  such that each path is a winning interaction and  $\text{Weight}(\mathbf{p}, \mathcal{F}) > 2^{-s}$  and also  $\mathbf{K}(\mathcal{F}|\mathbf{q}) = O(1)$ . Note that the actions of  $\mathcal{F}$  are rationals which lower bound  $\mathbf{q}$ 's computable action probabilities. Let  $Q$  be an elementary probability measure that realizes  $\mathbf{Ks}(\mathcal{F})$  and  $d = \max\{\mathbf{d}(\mathcal{F}|Q), 1\}$ . Without loss of generality, one can limit the support of  $Q$  to encodings of game fragments  $\mathcal{G}$  such that  $\text{Weight}(\mathcal{G}, \mathbf{p}) > 2^{-s}$ . This can be done by defining a new probability  $Q'$  that is  $Q$  conditioned on the above property, which is straightforward but tedious. Let  $m$  be the longest path and  $\ell$  be the largest action number of any game fragment in the support of  $Q$ . We define a probability  $P$  over deterministic agents  $\mathbf{g}$  defined up to  $m$  steps and up to  $\ell$  actions. Each action of the deterministic agent is determined by the corresponding probability of actions in that turn by  $\mathbf{p}$ . Using backwards induction, for each math fragment  $\mathcal{G}$  in the support of  $Q$ ,

$$\mathbf{E}_{\mathbf{g} \sim P}[\text{Weight}(\mathbf{g}, \mathcal{G})] > 2^{-s}.$$

Let  $N$  be a number to be specified later. Assume we randomly define  $N$  deterministic agents  $\{\mathbf{g}_i\}_{i=1}^N$ , each drawn i.i.d. from  $P$ . For math fragment  $\mathcal{G}$  in the support of  $Q$ ,  $X_{\mathcal{G}} = \frac{1}{N} \sum_{i=1}^N \text{Weight}(\mathbf{g}_i, \mathcal{G})$ . Each such  $X_{\mathcal{G}}$  is a random variable. By the Hoeffding's inequality,

$$\Pr(X_{\mathcal{G}} \leq 2^{-s-1}) < 2\exp(-N2^{-2s-2}).$$

Let  $N = d2^{2s+3}$ . Then it is possible to find a set of  $N$  deterministic agents such that

$$Q(\{\mathcal{G} : X_{\mathcal{G}} \leq 2^{-s-1}\}) < e^{-d}.$$

In the above formula, each  $X_{\mathcal{G}}$  is a fixed value and no longer a random variable. It must be that  $X_{\mathcal{F}} > 2^{-s-1}$ . Otherwise using  $Q$ -test  $t(\mathcal{G}) = [X_{\mathcal{G}} \leq 2^{-s-1}]e^d$ ,

$$1.44d < \log t(\mathcal{F}) <^+ \mathbf{d}(\mathcal{F}) =^+ d.$$

This is a contradiction for large enough  $d$  which we can assume without loss of generality. Thus since  $X_{\mathcal{F}} > 2^{-s-1}$  there exists deterministic agent  $\mathbf{g}_i$  such that  $\text{Weight}(\mathbf{g}_i, \mathcal{F}) > 2^{-s-1}$ . Thus, by Claim 1,  $\mathbf{g}_i$  wins against  $\mathbf{q}$  with probability more than  $2^{-s-1}$ . So,

$$\begin{aligned} \mathbf{K}(\mathbf{g}_i|s, \mathbf{p}) &<^+ \log N + \mathbf{K}(\{\mathbf{g}_i\}|s, \mathbf{p}) \\ \mathbf{K}(\mathbf{g}_i) &<^+ \mathbf{K}(s, \mathbf{p}) + 2s + \log d + \mathbf{K}(d, Q|s, \mathbf{p}) \\ &<^{\log} \mathbf{K}(\mathbf{p}) + 2s + 3 \log d + \mathbf{K}(Q|s, \mathbf{p}) \\ &<^{\log} \mathbf{K}(\mathbf{p}) + 2s + \mathbf{Ks}(\mathcal{F}|s, \mathbf{p}) \tag{2} \\ &<^{\log} \mathbf{K}(\mathbf{p}) + 2s + \mathbf{Ks}(\mathcal{F}) + O(\log \mathbf{K}(s, \mathbf{p})) \tag{3} \\ &<^{\log} \mathbf{K}(\mathbf{p}) + 2s + \mathbf{I}(\mathcal{F}; \mathcal{H}) \tag{4} \\ &<^{\log} \mathbf{K}(\mathbf{p}) + 2s + \mathbf{I}(\langle \mathbf{p}, s, \mathbf{q} \rangle; \mathcal{H}) \tag{5} \\ &<^{\log} \mathbf{K}(\mathbf{p}) + 2s + \mathbf{I}(\langle \mathbf{p}, \mathbf{q} \rangle; \mathcal{H}). \tag{6} \end{aligned}$$

Equations 2 and 3 follow from the definition of stochasticity,  $\mathbf{Ks}$ . Equation 4 follows from Lemma 1. Equation 5 follows from Lemma 5 and the fact that  $\mathcal{F}$  is computable from  $\mathbf{p}$ ,  $\mathbf{q}$ , and  $s$ . Equation 6 is due to the logarithmic precision of the inequality and  $\mathbf{K}(s) = O(\log s)$   $\square$

**Corollary 1.** *Let  $\epsilon \in (0, 1)$  be computable. Let  $\mathbf{p}$  be a probabilistic agent and  $\mathbf{q}$  be a probabilistic environment. If  $\mathbf{p}$  wins in the Win/No-Halt game against  $\mathbf{q}$  with probability  $> 2^{-s}$ ,  $s \in \mathbb{N}$ , then there is a deterministic agent of complexity  $<^{\log} \mathbf{K}(\mathbf{p}) + 2s + \mathbf{I}(\langle \mathbf{p}, \mathbf{q} \rangle; \mathcal{H}) + O_{\epsilon}(1)$  that wins with probability  $> \epsilon 2^{-s}$ .*

**Corollary 2.** *Let  $\epsilon \in (0, 1)$  and  $p \in (0, 1)$  both be computable. Let  $\mathbf{p}$  be a probabilistic agent and  $\mathbf{q}$  be a probabilistic environment. If  $\mathbf{p}$  wins in the Win/No-Halt game against  $\mathbf{q}$  with probability  $> p$ , then there is a deterministic agent of complexity  $<^{\log} \mathbf{K}(\mathbf{p}) + \mathbf{I}(\langle \mathbf{p}, \mathbf{q} \rangle; \mathcal{H}) + O_{p, \epsilon}(1)$  that wins with probability  $> \epsilon p$ .*

## 5 Even-Odds

We define the following game, entitled EVEN-ODDS. There are  $N$  rounds. The player starts out with a score of 0. At the start of each round, the environment  $\mathbf{q}$  secretly records a bit  $e_i \in \{0, 1\}$ . The player sends  $\mathbf{q}$  a bit  $b_i$  and the environment responds with  $e_i$ . The agent gets a point if  $e_i \oplus b_i = 1$ . Otherwise the agent loses a point. The environment  $\mathbf{q}$  can be any probabilistic algorithm. There are  $N$  rounds.

**Theorem 5.** *For large enough number of rounds,  $N$ , given any probabilistic environment  $\mathbf{q}$  there is a deterministic agent  $\mathbf{p}$  of complexity  $\mathbf{K}(\mathbf{p}) <^{\log} \mathbf{I}(\mathbf{q}; \mathcal{H})$  that can achieve a score of  $\sqrt{N}$  with probability  $> 1/21$ .*

*Proof.* We describe a probabilistic agent  $\mathbf{p}'$ . At round  $i$ ,  $\mathbf{p}'$  submits 0 with probability  $1/2$ . Otherwise it submits 1. By the central limit theorem, for large enough  $N$ , the score of the probabilistic agent divided by  $\sqrt{N}$  is  $S \sim \mathcal{N}(0, 1)$ . Let  $\Phi(x) = \Pr[S > x]$ . A common bound for  $\Phi(x)$  is

$$\begin{aligned}\Phi(x) &> \frac{1}{2\pi} \frac{x}{x^2 + 1} e^{-x^2/2} \\ \Phi(1) &> \frac{1}{4\pi} e^{-1/2} = p.\end{aligned}$$

One can construct a Win/No-Halt game where the player  $\mathbf{p}'$  wins if it has a score of at least  $\sqrt{N}$ . Thus  $\mathbf{p}'$  wins with probability greater than  $p$ . Thus by Corollary 2, with  $\epsilon = 1/(21p)$ , there exists a deterministic agent  $\mathbf{p}$  that can beat  $\mathbf{q}$  with complexity

$$\mathbf{K}(\mathbf{p}) <^{\log} \mathbf{K}(\mathbf{p}') + \mathbf{I}(\langle \mathbf{p}', \mathbf{q} \rangle; \mathcal{H}) <^{\log} \mathbf{I}(\mathbf{q}; \mathcal{H}).$$

Furthermore  $\mathbf{p}$  wins with probability  $> 1/21$ .  $\square$

## 6 Revisiting Crete

Suppose the minatour has gotten fed up with the hero, who can find the exit using a very small amount of information. The minatour decides to use chance to his advantage. At every room the minatour computes a probability over all possible mappings of numbers to doors and selects a mapping at random. This probability is a function of all the hero's previous actions. However due to derandomization of probabilistic games, the hero can achieve the following results. Let  $c$  be the number of corridors and  $d$  be the number of doors in the goal room.

**Theorem 6.** *Given any labyrinth and probabilistic minatour  $(L, M)$ , there is a deterministic hero  $\mathbf{p}$  of complexity  $\mathbf{K}(\mathbf{p}) <^{\log} 2 \log(c/d) + \mathbf{I}(\langle L, M \rangle; \mathcal{H})$  that can find the goal room with probability greater than  $d/4.03c$ .*

*Proof.* We assume the number of turns  $N$  is long enough such that given a random walk of  $N$  steps, the probability of being at the goal room is  $> d/2.01c$ . Let  $s \in \mathbb{N}$  be the largest number such that  $2^{-s} < d/2.01c$ . Applying Corollary 1, where  $\epsilon = 4.02/4.03$  and where the probabilistic agent  $\mathbf{p}$  chooses each door with uniform probability, one gets a deterministic agent of complexity  $<^{\log} \mathbf{K}(\mathbf{p}) + 2s + \mathbf{I}(\langle \mathbf{p}, s, (L, M) \rangle; \mathcal{H}) + O_\epsilon(1) <^{\log} 2 \log(c/d) + \mathbf{I}(\langle L, M \rangle; \mathcal{H})$ . This agent wins with probability  $> \epsilon 2^{-s} \geq \epsilon d/4.02c \geq d/4.03c$ .  $\square$

## 7 Computability of Environments

**Claim 2.** *Theorem 4 and Corollaries 1 and 2 also apply to probabilistic environments  $\mathbf{q}$  with lower computable probabilities since  $\mathbf{K}(\mathcal{F}|\mathbf{p}, s, \mathbf{q}) = O(1)$ ,*

where  $\langle \mathbf{p}, s, \mathbf{q} \rangle$  consists of a program to compute  $\mathbf{p}$ , the number  $s$ , and a program to lower compute  $\mathbf{q}$ . This is because one lower enumerates the probabilities of  $\mathbf{q}$  until one can find the corresponding game fragment  $\mathcal{F}$  such that  $\text{Weight}(\mathbf{p}, \mathcal{F}) > 2^{-s}$ .

In this section, we derive the results of Theorems 1 and 4 with respect to uncomputable environments. We will use the following mutual information term between infinite sequences.

**Definition 2** ([Lev74]). For  $\alpha, \beta \in \{0, 1\}^\infty$ ,  $\mathbf{I}(\alpha : \beta) = \log \sum_{x, y \in \{0, 1\}^*} \mathbf{m}(x|\alpha) \mathbf{m}(y|\beta) 2^{\mathbf{I}(x:y)}$ .

**Proposition 1.**  $\mathbf{I}(x; \mathcal{H}) <^+ \mathbf{I}(\alpha : \mathcal{H}) + \mathbf{K}(x|\alpha)$ .

Now a probabilistic environment  $\mathbf{q}$  is of the form  $\mathbb{N} \times (\mathbb{N} \times \mathbb{N})^* \rightarrow [0, 1]$ . We fix a computable function  $\ell$  such that for every environment  $\mathbf{q}$  there is an infinite sequence  $\alpha$  such that  $\ell(\alpha, \cdot)$  computes  $\mathbf{q}$ . Let  $\ell[\mathbf{q}]$  be the set of all such infinite sequences  $\alpha$ .

**Definition 3.** For probabilistic environment  $\mathbf{q}$ ,  $\mathbf{I}(\mathbf{q} : \mathcal{H}) = \inf_{\alpha \in \ell[\mathbf{q}]} \mathbf{I}(\alpha : \mathcal{H})$ .

**Theorem 7.** Let  $\mathbf{p}$  be a probabilistic agent and  $\mathbf{q}$  be a (potentially uncomputable) probabilistic environment. If  $\mathbf{p}$  Wins in the Win/No-Halt game against  $\mathbf{q}$  with probability  $> 2^{-s}$ ,  $s \in \mathbb{N}$ , then there is a deterministic agent of complexity  $<^{\log} \mathbf{K}(\mathbf{p}) + 2s + \mathbf{I}(\langle \mathbf{p}, \mathbf{q} \rangle : \mathcal{H})$  that wins with probability  $> 2^{-s-1}$ .

*Proof.* Using  $\mathbf{p}$ ,  $s$ , any encoding  $\alpha \in \ell[\mathbf{q}]$ , and  $\ell$ , one can construct the math fragment  $\mathcal{F}$  described in the proof of Theorem 4. Let  $\alpha \in \ell[\mathbf{q}]$  and  $\mathbf{I}(\alpha : \mathcal{H}) < \mathbf{I}(\mathbf{q} : \mathcal{H}) + 1$ . Thus  $\mathbf{K}(\mathcal{F}|\mathbf{p}, s, \alpha) = O(1)$ . Using Proposition 1, the definition of  $\mathbf{I}(\mathbf{q} : \mathcal{H})$ , and the reasoning of the proof of Theorem 4, this theorem follows.  $\square$

The follow Theorem extends Theorem 1 to uncomputable environments.

**Theorem 8.** If probabilistic agent  $\mathbf{p}$  wins against deterministic, and potentially uncomputable, environment  $\mathbf{q}$  with at least probability  $p$ , then there is a deterministic agent of complexity  $<^{\log} \mathbf{K}(\mathbf{p}) - \log p + \mathbf{I}(\langle \mathbf{p}, \mathbf{q} \rangle : \mathcal{H})$  that wins against  $\mathbf{q}$ .

*Proof.* This follows from using the same reasoning as the proof for Theorem 7 and the proof of Theorem 1 found in [Eps23b].  $\square$

## 8 New Bounds

In fact, as shown in this section, the bounds of Theorems 1 and 4 can be improved. These new bounds use a summation of all probabilistic agent's winning probabilities, weighted by their algorithmic probabilities. A semi-agent is a probabilistic player whose action probabilities don't necessarily sum to 1 at each turn. The leftover probability represents the chance that the semi-agent freezes and does not win. A semi-agent  $\mathbf{p}$  is modelled by the function

$(\mathbb{N} \times \mathbb{N})^* \times \mathbb{N} \rightarrow [0, 1]$ . A semi-agent  $\mathbf{p}$  is lower computable if the probability of  $\mathbf{p}$  performing a finite set of actions given the environment's action is lower computable. More formally, for every action  $a \in \mathbb{N}$  of the player and response  $e \in \mathbb{N}$  of the environment and history  $h \in (\mathbb{N} \times \mathbb{N})^*$ ,  $\mathbf{p}(a, h) \in [0, 1]$ ,  $p(a, h)$  is lower computable, and  $\sum_{b \in \mathbb{N}} \mathbf{p}(b, (h, (a, e))) \leq \mathbf{p}(a, h)$ . Also note that, in general, a lower computable semi-agent will not have (even lower) computable probabilities of an action given its history. Let  $\text{Win}(\mathbf{p}, \mathbf{q})$  is the winning probability of semi-agent  $\mathbf{p}$  against environment  $\mathbf{q}$ . If  $\mathbf{p}$  is lower computable, then  $\text{Win}(\mathbf{p}, \mathbf{q})$  is lower computable. The algorithmic probability of a semi-agent is  $\mathbf{m}(\mathbf{p}) = \sum \{2^{-\|\ell\|} : U_\ell(\cdot) \text{ lower computes } \mathbf{p}\}$ . Thus if  $\mathbf{p}$  is not lower computable, then  $\mathbf{m}(\mathbf{p}) = 0$ . Let  $\mathcal{A}$  be the set of all lower computable semi-agents. The set  $\mathcal{A}$  can be enumerated in the standard way of algorithmic information theory.

**Definition 4.** For (deterministic or probabilistic) environment  $\mathbf{q}$ , let  $\xi(\mathbf{q}) = \lceil -\log \sum_{\mathbf{p} \in \mathcal{A}} \mathbf{m}(\mathbf{p}) \text{Win}(\mathbf{p}, \mathbf{q}) \rceil$  is a score of how hard it is for each semi-agent to win against  $\mathbf{q}$ , weighted by its algorithmic probability.

**Remark 1.** Let  $\Omega = \sum \{2^{-\|p\|} : U(p) \text{ halts}\}$  be Chaitin's Omega and  $\Omega^t = \sum \{2^{-\|p\|} : U(p) \text{ halts in time } t\}$ . For a string  $x$ , let  $BB(x) = \min\{t : \Omega^t > 0.x + 2^{-\|x\|}\}$ . Note that  $BB(x)$  is undefined if  $0.x + 2^{-\|x\|} > \Omega$ . For  $n \in \mathbb{N}$ , let  $\mathbf{bb}(n) = \max\{BB(x) : \|x\| \leq n\}$ .  $\mathbf{bb}^{-1}(m) = \arg \min_n \{\mathbf{bb}(n-1) < m \leq \mathbf{bb}(n)\}$ . Let  $bb(n) = \arg \max_x \{BB(x) : \|x\| \leq n\}$ .

**Lemma 3.** For  $n = \mathbf{bb}^{-1}(m)$ ,  $\mathbf{K}(bb(n)|m, n) = O(1)$ .

*Proof.* Enumerate strings of length  $n$ , starting with  $0^n$ , and return the first string  $y$  such that  $BB(y) \geq m$ . This string  $y$  is equal to  $bb(n)$ , otherwise  $BB(y^-)$  is defined and  $BB(y^-) \geq BB(y) \geq m$ . Thus  $\mathbf{bb}(n-1) \geq m$ , causing a contradiction.  $\square$

**Proposition 2.**

1.  $\mathbf{K}(bb(n)) >^+ n$ .
2.  $\mathbf{K}(bb(n)|\mathcal{H}) <^+ \mathbf{K}(n)$ .

**Theorem 9.** Let  $\mathbf{q}$  be a computable deterministic environment. There is a deterministic agent of complexity  $<^{\log} \xi(\mathbf{q}) + \mathbf{I}(\mathbf{q}; \mathcal{H})$  that wins against  $\mathbf{q}$ .

*Proof.* Let  $\mathbf{p}$  be the universal lower-computable semi-agent, where

$$\mathbf{p} = \sum_{\mathbf{p}' \in \mathcal{A}} \mathbf{m}(\mathbf{p}') \mathbf{p}'.$$

Let  $s = \lceil -\log \text{Win}(\mathbf{p}, \mathbf{q}) \rceil + 1 = \xi(\mathbf{q}) + 1$ . Let  $\mathbf{p}^c$  be  $\mathbf{p}$  after enumerating  $c$  steps. Let  $m$  be the smallest number such that  $\text{Win}(\mathbf{p}^m, \mathbf{q}) > 2^{-s-1}$ . Let  $n = \mathbf{bb}^{-1}(m)$ ,  $k = \mathbf{bb}(n)$ . Let  $\mathbf{p}^k$  be  $\mathbf{p}$  after enumerating  $k$  steps, and completed



to an agent (from a semi-agent). Thus  $\text{Win}(\mathbf{p}^k, \mathbf{q}) > 2^{-s-1}$ . Let  $b = bb(n)$ . Theorem 1, conditioned on  $b$  gives a deterministic agent  $\mathbf{r}$  such that

$$\begin{aligned} \mathbf{K}(\mathbf{r}|b) &<^{\log} \mathbf{K}(\mathbf{p}^k|b) + s + \mathbf{I}(\langle \mathbf{p}^k, \mathbf{q} \rangle; \mathcal{H}|b) \\ &<^{\log} s + \mathbf{I}(\mathbf{q}; \mathcal{H}|b) \\ \mathbf{K}(\mathbf{r}) &<^{\log} s + \mathbf{K}(b) + \mathbf{K}(\mathbf{q}|b) - \mathbf{K}(\mathbf{q}|b, \mathcal{H}) \\ &<^{\log} s + \mathbf{K}(b, \mathbf{q}) + \mathbf{K}(\mathbf{K}(b)) - \mathbf{K}(\mathbf{q}|b, \mathcal{H}) \end{aligned} \quad (7)$$

$$\begin{aligned} &<^{\log} s + \mathbf{K}(\mathbf{q}) + \mathbf{K}(b|\mathbf{q}, s) - \mathbf{K}(\mathbf{q}|\mathcal{H}) \\ &<^{\log} s + \mathbf{K}(\mathbf{q}) + \mathbf{K}(b|\mathbf{q}, s, n) + \mathbf{K}(n) - \mathbf{K}(\mathbf{q}|\mathcal{H}) \\ &<^{\log} s + \mathbf{K}(\mathbf{q}) - \mathbf{K}(\mathbf{q}|\mathcal{H}). \end{aligned} \quad (8)$$

The equations follow from Proposition 2. Equation 7 follows from the chain rule. Equation 8 follows from Lemma 3.  $\square$

The following corollary is an update to Theorem 4 in [EB11] which is derived from Theorem 6 in [VV05]. It also follows from [Lev16, Eps23a]. In fact, theorems of this general form are ubiquitous in algorithmic information theory. It says that if an environment has a lot of deterministic players of bounded complexity who win against it, then there exists a simple deterministic player that wins.

**Corollary 3.** *If  $2^m$  deterministic players  $\mathbf{r}$  of Kolmogorov complexity  $\mathbf{K}(\mathbf{r}) \leq n$  win against an environment  $\mathbf{q}$ , then there exists a deterministic player  $\mathbf{p}$  with  $\mathbf{K}(\mathbf{p}) <^{\log} n - m + \mathbf{I}(\mathbf{q}; \mathcal{H})$  that wins against  $\mathbf{q}$ .*

**Corollary 4.** *If  $2^m$  probabilistic players  $\mathbf{r}$  of Kolmogorov complexity  $\mathbf{K}(\mathbf{r}) \leq n$  win against an environment  $\mathbf{q}$  with probability  $p$ , then there exists a deterministic player  $\mathbf{p}$  with  $\mathbf{K}(\mathbf{p}) <^{\log} n - m - \log p + \mathbf{I}(\mathbf{q}; \mathcal{H})$  that wins against  $\mathbf{q}$ .*

The following theorem provides bounds using information with the environment. It is a followup to Theorem 2 in [EB11] which is a rearranging of Theorem 2 in [VV04].

**Theorem 10.** *For deterministic environment  $\mathbf{q}$ , there exists a deterministic agent  $\mathbf{p}$  with  $\mathbf{K}(\mathbf{p}) <^{\log} \min_{\mathbf{r} \in \mathcal{A}} \mathbf{I}(\mathbf{r}; \mathbf{q}) - \log \text{Win}(\mathbf{r}, \mathbf{q}) + \mathbf{I}(\mathbf{q}; \mathcal{H})$  that wins against  $\mathbf{q}$ .*

*Proof.* For  $n \in \mathbb{N}$ , let  $\mathcal{A}(\mathbf{q}, n) = \{\mathbf{r} : \mathbf{r} \in \mathcal{A}, \text{Win}(\mathbf{r}, \mathbf{q}) > 2^{-n}\}$ . Let  $s = \lfloor -\log \sum_{\mathbf{r} \in \mathcal{A}(\mathbf{q}, n)} \mathbf{m}(\mathbf{r}) \rfloor$ . So we have that

$$\xi(\mathbf{q}) <^+ s + n. \quad (9)$$

Since  $\mathcal{A}(\mathbf{q}, n)$  is lower computable given  $s, n$ , and  $\mathbf{q}$ , we have that  $m(\mathbf{r}) = \lceil \mathbf{r} \in$

$\mathcal{A}(\mathbf{q}, n)]2^s \mathbf{m}(\mathbf{r})$  is a lower computable semimeasure. So if  $\mathbf{r} \in \mathcal{A}(\mathbf{q}, n)$ , then

$$\begin{aligned} \mathbf{m}(\mathbf{r}|\mathbf{q}, n, s) &\stackrel{*}{>} m(\mathbf{r}) \stackrel{*}{=} 2^s \mathbf{m}(\mathbf{r}) \\ s &<^+ \mathbf{K}(\mathbf{r}) - \mathbf{K}(\mathbf{r}|\mathbf{q}, n, s) \\ s &<^{\log} \mathbf{K}(\mathbf{r}) - \mathbf{K}(\mathbf{r}|\mathbf{q}) + \mathbf{K}(n) \\ s &<^{\log} \min_{\mathbf{r} \in \mathcal{A}(\mathbf{q}, n)} \mathbf{K}(\mathbf{r}) - \mathbf{K}(\mathbf{r}|\mathbf{q}) + \mathbf{K}(n). \end{aligned} \quad (10)$$

Combining Equations 9 and 10 and Theorem 9, there is a deterministic agent  $\mathbf{p}$  that wins against  $\mathbf{q}$ , where for all  $n$ ,

$$\begin{aligned} \mathbf{K}(\mathbf{p}) &<^{\log} \min_{\mathbf{r} \in \mathcal{A}(\mathbf{q}, n)} \mathbf{I}(\mathbf{r}; \mathbf{q}) + n + \mathbf{I}(\mathbf{q}; \mathcal{H}) \\ &<^{\log} \min_{\mathbf{r} \in \mathcal{A}} \mathbf{I}(\mathbf{r}; \mathbf{q}) - \log \text{Win}(\mathbf{r}, \mathbf{q}) + \mathbf{I}(\mathbf{q}; \mathcal{H}). \end{aligned}$$

□

**Theorem 11.** *Let  $\mathbf{q}$  be a computable probabilistic environment. Let  $2^{-s} < \xi(\mathbf{q})$ ,  $s \in \mathbb{N}$ . There is a deterministic agent of complexity  $<^{\log} 2s + \mathbf{I}(\mathbf{q}; \mathcal{H})$  that wins with probability  $> 2^{-s-1}$ .*

*Proof.* The proof analogously to that of Theorem 9. □

How does one interpret these new bounds? Let  $\mathcal{B}$  be the set of all total computable probabilistic agents and  $\mathbf{q}$  be a deterministic environment. Using some reasoning, one can prove the following inequality

$$\min_{\mathbf{p} \in \mathcal{B}} \mathbf{K}(\mathbf{p}) - \log \text{Win}(\mathbf{p}, \mathbf{q}) <^{\log} \xi(\mathbf{q}) + \mathbf{I}(\mathbf{q}; \mathcal{H}).$$

Thus only exotic environments will have an averaged win rate that is better than the weighted max win rate. However, the bounds proved in this section are more amenable to manipulation, as shown with Corollaries 3 and 4 and Theorem 10. This also turned out to be the case with Gács entropy and Vitányi entropy, where their difference is only over exotic quantum states but Gács entropy has many applications in algorithmic physics.

## 9 Partial Derandomization

With game derandomization, a probabilistic player that wins with probability  $p$  can be transformed into a deterministic one, with a complexity term of  $-\log p$  in the inequality. In this section we show a general tradeoff of bits versus winning probabilities. A probabilistic agent can be transformed into a more successful probabilistic agent, at the cost of its complexity. The theorem of this section puts bounds on this tradeoff. The proof requires a slight modification to the proof of the Monotone EL Theorem [Eps24].

**Theorem 12** (Modified Monotone EL Theorem). *For prefix free set  $G$ ,  $m = \lceil -\log \mathbf{M}(G) \rceil$ ,  $n < m$ ,  $n \in \mathbb{N}$ , there exists a set  $F$  of size  $2^n$  where there exists  $y \in F$  and  $x \in G$  such that  $y \supseteq x$  and*

$$\mathbf{K}(F) <^{\log} m - n + \mathbf{I}(G; \mathcal{H}) + \mathbf{K}(n).$$

**Theorem 13.** *Let  $\mathbf{q}$  be a deterministic environment and  $\mathbf{p}$  be a probabilistic player that wins with probability  $p$ . For  $s = \lceil -\log p \rceil + 1$ ,  $r < s$ ,  $r \in \mathbb{N}$ , there is a probabilistic agent of complexity  $<^{\log} s - r + \mathbf{I}(\langle \mathbf{p}, s, \mathbf{q} \rangle; \mathcal{H}) + \mathbf{K}(r)$  that wins with probability  $> 2^{-r}$ .*

*Proof.* Let  $s = \lceil -\log p \rceil + 1$ . Using the same reasoning as the proof of Theorem 3, there exists a clopen set  $G \subset \{0, 1\}^\infty$  where each element encodes a winning interaction with the environment  $\mathbf{q}$ ,  $\mathbf{p}$  can be interpreted as a measure over the Cantor space,  $\mathbf{p}(G) > 2^{-s}$ , and  $\mathbf{K}(G|\mathbf{p}, s, \mathbf{q}) = O(1)$ . Using Theorem 12, one gets a prefix free set  $F$ , such that there exists  $y \in F$  and  $x \in G$  such that  $y \supseteq x$ ,  $m = \lceil -\log \mathbf{M}(F) \rceil$ , where

$$\begin{aligned} \mathbf{K}(F) &<^{\log} m - r + \mathbf{I}(F; \mathcal{H}) + \mathbf{K}(r) \\ \mathbf{K}(F) &<^{\log} \mathbf{K}(\mathbf{p}) + s - r + \mathbf{I}(F; \mathcal{H}) + \mathbf{K}(r) \\ \mathbf{K}(F) &<^{\log} \mathbf{K}(\mathbf{p}) + s - r + \mathbf{I}(\langle \mathbf{p}, s, \mathbf{q} \rangle; \mathcal{H}) + \mathbf{K}(r). \end{aligned} \quad (11)$$

Equation 11 is due to Lemma 2. Each such string in  $F$  represents a deterministic player. At least one such deterministic player has a winning interaction with  $\mathbf{q}$ . It is not hard to construct a probabilistic player  $\mathbf{p}'$  that wins with probability  $|F|^{-1}$ . Indeed, at every given history  $h \in (\mathbb{N} \times \mathbb{N})^*$  the player  $\mathbf{p}'$  chooses action  $a \in \mathbb{N}$  with probability  $n_a/m_a$ , where  $m_a$  is the number of deterministic players that are consistent with  $h$ , and  $n_a$  are the number of such agents that choose  $n_a$  as their action. If  $m_a$  is 0, then  $\mathbf{p}'$  can choose any action.  $\square$

**Corollary 5.** *Let  $\mathbf{q}$  be a deterministic environment. For  $r < \xi(\mathbf{q})$ ,  $r \in \mathbb{N}$ , there is a probabilistic agent  $\mathbf{p}$ , with  $\mathbf{K}(\mathbf{p}) <^{\log} \xi(\mathbf{q}) - r + \mathbf{I}(\langle \xi(\mathbf{q}), \mathbf{q} \rangle; \mathcal{H}) + \mathbf{K}(r)$  that wins with probability  $> 2^{-r}$ .*

*Proof.* This follows from Theorem 13 and the reasoning of proof of Theorem 9.  $\square$

## 10 Lose/No-Halt Games

A Lose/No-Halt game is a series of interactions where the player and environment exchange natural numbers. At any given time, the environment can choose to declare that the player has lost. Otherwise, if the game continues forever, the player has won. However Lose/No-Halt games present several barriers in their characterization which the Win/No-Halt games do not have. One such obstacle to derandomizing Lose/No-Halt games is as follows.

**Theorem 14.** *There exists a Lose/No-Halt game consisting of a computable environment  $\mathbf{q}$ , a computable probabilistic player  $\mathbf{p}$  that wins with positive probability, and no deterministic player that can beat  $\mathbf{q}$ .*

*Proof.* The environment  $\mathbf{q}$  gives a single action: 0. In addition, the player loses if she plays a number other than 0 or 1. We inductively define a computable environment  $\mathbf{q}$ . At each of the environment turn at  $n$ , the interaction consists of a bit string  $x$  of length  $2n - 1$ . Let  $y$  be the odd bits of  $x$ , i.e. the player's actions. Let  $\mathbf{K}_t(c) = \min\{\|p\| : U(p) = c \text{ in } \leq t \text{ steps}\}$ . If  $\|z\| - \mathbf{K}_n(z) > d$  for any  $z \sqsubseteq x$ , then environment  $\mathbf{q}$  outputs a loss for the action and the game ends. Otherwise the game continues. This means  $\mathbf{q}$  is computable. Thus a player who randomly chooses a bit will output with non zero probability a winning sequence that has less than  $d$  randomness deficiency. In addition there is no deterministic player who can output an infinite sequence with bounded randomness deficiency.  $\square$

## 11 Conclusion

With this paper, characterizations of the Kolmogorov complexity of deterministic players against deterministic or probabilistic environments that are computable, lower computable, and uncomputable are given. As shown in Section 7, the computability of the environment does not play much of a factor in bounds proved. Instead, the key factor is the amount of information the halting sequence has about the environment. There is a general obstacle in derandomizing Lose/No-Halt games, but there might be further assumptions about such games that could be made to circumvent this. Another open avenue of research is resource bounded derandomization of games. This would characterize time bounded Kolmogorov complexity of winning players to environments that calculate their turns in a polynomial amount of time. This would require an infinite sequence of such games and corresponding players. This research line would build upon the research line of [Eps23b], which introduced Kolmogorov research bounded derandomization. Another open problem is whether there exists a more straightforward proof that does not use stochasticity,  $\mathbf{K}$ s. Such an advancement could be applied to the proofs of other theorems which have  $\mathbf{I}(\cdot; \mathcal{H})$  in their inequalities.

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