Outliers are in the Physical World

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July 13, 2022

Abstract

The Independence Postulate (IP) is a finitary Church Turing Thesis, postulating that certain infinite and finite sequences cannot be found in nature, i.e. have high "physical addresses". In this paper we show that IP implies that outliers are found in the physical world.

Outliers

$\mathbf{K}(x|y)$ is the conditional prefix Kolmogorov complexity. For probability p over $\mathbb{N}$, the deficiency of randomness is $\mathbf{d}(a|p) = \lfloor -\log p(a) \rfloor - \mathbf{K}(a)$. $\mathbf{I}(a;\mathcal{H}) = \mathbf{K}(a) - \mathbf{K}(a|\mathcal{H})$, where $\mathcal{H}$ is the halting sequence. An elementary probability measure $\mathbb{N}$ has finite support and range in $\mathbb{Q}$. Stochasticity is

$$\Lambda(a|b) = \min\{\mathbf{K}(Q|b) + 3\log\max\{\mathbf{d}(a|Q,b),1\}$$
$$: Q \text{ is an elementary probability measure}\}.$$
$$\Lambda(a|b) < \Lambda(a) + O(\log \mathbf{K}(b)).$$

Definition 1 (Information) $\mathbf{I}(\alpha:\beta) = \log \sum_{x,y\in\{0,1\}^*} 2^{\mathbf{K}(x)+\mathbf{K}(y)-\mathbf{K}(x,y)-\mathbf{K}(x|\alpha)-\mathbf{K}(y|\beta)}$.

The Independence Postulate [Lev84, Lev13] statement is:

IP: *Let α be a sequence defined with an n-bit mathematical statement, and a sequence β can be located in the physical world with a k-bit instruction set. Then $\mathbf{I}(\alpha:\beta) < k+n+c$ for some small absolute constant c.*

It is well known in the literature that stochastic numbers have high mutual information with the halting sequence. One detailed proof is in [Eps21].

Lemma 1 $\Lambda(x) <^{\log} \mathbf{I}(x;\mathcal{H})$.

Theorem 1 *For computable probability p over $\mathbb{N}$ and for $D\subset\mathbb{N}$, $|D| = 2^s$, $s < \max_{a\in D}\mathbf{d}(a|p) + \Lambda(D) + \mathbf{K}(s) + \mathbf{K}(p) + O(\log \mathbf{K}(s)\mathbf{K}(p))$.*

Proof. We relativize the universal Turing machine to p and s. Let Q be an elementary probability measure that realizes $\Lambda(D)$. Let $d = \mathbf{d}(D|Q)$. Let $F \subseteq \mathbb{N}$ be a random set where each element of F is selected independently with probability $cd2^{-s}$, where $c\in\mathbb{N}$ is chosen later. $\mathbf{E}[p(F)] \le cd2^{-s}$. Furthermore

$$\mathbf{E}[Q(\{G : |G| = 2^s, G\cap F = \emptyset\})]$$
$$\le \sum_G Q(G)(1-cd2^{-s})^{2^s} < e^{-cd}.$$

Thus finite $W \subset \mathbb{N}$ can be chosen such that $p(W) \le 2cd2^{-s}$ and $Q(\{G : |G| = 2^s, G\cap W = \emptyset\}) \le e^{1-cd}$. $D\cap W \ne \emptyset$, otherwise, using the Q-test, $t(G) = e^{cd-1}$ if $(|G| = 2^s, G\cap W = \emptyset)$ and $t(G) = 0$ otherwise, we have

$$\mathbf{K}(D|Q,d,c) <^+ -\log Q(D) - (\log e)cd$$
$$(\log e)cd <^+ -\log Q(D) - \mathbf{K}(D|Q) + \mathbf{K}(d,c)$$
$$(\log e)cd <^+ d + \mathbf{K}(d,c),$$

which is a contradiction for large c. Thus there is an $a \in D\cap W$, where

$$\mathbf{K}(a) <^+ -\log p(a) + \log d - s + \mathbf{K}(d) + \mathbf{K}(Q)$$
$$s <^+ \mathbf{d}(a|p) + \Lambda(D).$$

Removing the relativization of p and s

$$s < -\log p(a) - \mathbf{K}(a|p,s) + \Lambda(D|p,s)$$
$$s < \max_{a\in D}\mathbf{d}(a|p) + \Lambda(D) + \mathbf{K}(s) + \mathbf{K}(p)$$
$$+ O(\log \mathbf{K}(s)\mathbf{K}(p)).$$

□

For $\tau \in \mathbb{N}^{\mathbb{N}}$, $\langle\tau\rangle = \langle\tau[1]\rangle\langle\tau[2]\rangle\langle\tau[3]\rangle\ldots$ Let $\tau(n)$ be the first 2^n unique numbers found in τ.

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The sequence τ is assumed to have an infinite amount of unique numbers, and represents a series of observations.

Theorem 2 *For probability p over $\mathbb{N}$ and $\tau \in \mathbb{N}^{\mathbb{N}}$, let s_τ be the smallest number such that $\max_{a\in\tau(n)} \mathbf{d}(a|p) > n - 4\mathbf{K}(n) - s_\tau$. Then $s_\tau <^{\log} \mathbf{I}(\langle\tau\rangle : \mathcal{H}) + \mathbf{K}(p)$.*

Proof. By Theorem 1, Lemma 1, and the fact that $\mathbf{I}(x;\mathcal{H}) <^{+} \mathbf{I}(\alpha:\mathcal{H}) + \mathbf{K}(x|\alpha)$,

$$\begin{aligned}
n &< \max_{a\in\tau(n)} \mathbf{d}(a|p) + \mathbf{I}(\tau(n);\mathcal{H}) + \mathbf{K}(p) + 2\mathbf{K}(n)\\
&\quad + O(\log(\mathbf{I}(\tau(n);\mathcal{H}) + \mathbf{K}(p))),\\
n &< \max_{a\in\tau(n)} \mathbf{d}(a|p) + 4\mathbf{K}(n) + \mathbf{I}(\langle\tau\rangle:\mathcal{H}) + \mathbf{K}(p)\\
&\quad + O(\log(\mathbf{I}(\langle\tau\rangle:\mathcal{H}) + \mathbf{K}(p))),\\
&n - 4\mathbf{K}(n) - (\mathbf{I}(\langle\tau\rangle:\mathcal{H}) + \mathbf{K}(p)\\
&\quad + O(\log(\mathbf{I}(\langle\tau\rangle:\mathcal{H}) + \mathbf{K}(p)))) < \max_{a\in\tau(n)} \mathbf{d}(a|p).
\end{aligned}$$

□

Let k be the physical address of an infinite sequence of numbers τ. The halting sequence $\mathcal{H}$ can be described by a small mathematical statement. By Theorem 2, for some sequence $\tau \in \mathbb{N}^{\mathbb{N}}$ and probability p over $\mathbb{N}$, **IP** states

$$s_\tau <^{\log} \mathbf{I}(\langle\tau\rangle:\mathcal{H}) + \mathbf{K}(p) <^{\log} k + c + \mathbf{K}(p).$$

Thus sequences τ with large s_τ, as defined in Theorem 2, will have large physical addresses. Thus it is hard to find physical sequences which do not have large outliers, and completely impossible to find sequences with no outliers.

References

[Eps21] Samuel Epstein. All sampling methods produce outliers. *IEEE Transactions on Information Theory*, 67(11):7568–7578, 2021.

[Lev84] L. A. Levin. Randomness conservation inequalities; information and independence in mathematical theories. *Information and Control*, 61(1):15–37, 1984.

[Lev13] L. A. Levin. Forbidden information. *J. ACM*, 60(2), 2013.