AIT Blog

On The Curious Lack of Algorithmic Information In Quantum States

Samuel Epstein[*]

December 24, 2022

For classical algorithmic information theory, random strings have a high amount of self information, with $\mathbf{K}(x) =^+ \mathbf{I}(x:x)$. We can generalize from strings to arbitrary signals, formalized by probability measure over strings.

Definition 1 (Information, Signals)
For semi-measures p and q over $\{0,1\}^*$, $\mathbf{I}(p:q) = \log \sum_{x,y\in\{0,1\}^*} 2^{\mathbf{I}(x:y)} p(x)q(y)$.

As shown in http://www.jptheorygroup.org/doc/InfoProb.pdf, this measure observes conservation inequalities over deteministic or randomnized processing. Thus processing cannot increase information between two signals. In addition information of probabilities can be extended to infinite sequences or general spaces. If the the probability measure is concentrated at a single point, then it contains self-information equal to the complexity of that point. If the probability measure is spread out, then it is white noise, and contains no self-information. Some examples are as follows.

Example 1

- *In general, a probability p, will have low $\mathbf{I}(p:p)$ if it has large measure on simple strings, or low measure on a large number of complex strings, or some combination of the two.*
- *If probability p is concentrated on a single string x, then $\mathbf{I}(p:p) = \mathbf{K}(x)$.*
- *The uniform distribution U_n over strings of length n has self information equal to (up to an additive constant) $\mathbf{K}(n)$.*
- *There are semi-measures that have infinite self information. Let α_n be the n bit prefix of a Martin Löf random sequence α and $n \in [2,\infty)$. Semi-measure $p(x) = [x = \alpha_n]n^{-2}$ has $\mathbf{I}(p:p) = \infty$.*
- *The universal semi-measure $\mathbf{m}$ has no self information.*

This blog explains a curious fact: most quantum states have negligible algorithmic self-information and given a measurement, the overwhelming majority of pure quantum states will produce random noise. For algorithmic information $\mathbf{I}_Q$ between quantum states we refer the reader to the definition $\mathbf{I}$ in [Eps19]. The following theorem shows that self information of states is negligible.

[*]JP Theory Group. samepst@jptheorygroup.org

Given a quantum state $|\psi\rangle$, a measurement, or POVM, E produces a probability measure $E|\psi\rangle$ over strings. This probability represents the classical information, or *signal* produced from the measurement. We refer readers to https://en.wikipedia.org/wiki/POVM for an introduction to quantum measurements. Given a measurement E, for an overwhelming majority of quantum states $|\psi\rangle$, the signal (probability) produced will be white noise, i.e. have no meaningful information, i.e. $\mathbf{I}(E|\psi\rangle : E|\psi\rangle)$ is negligible.

Theorem 1 ([Eps19]) *Let Λ be the uniform distribution on the n qubit space.*

- $\int 2^{\mathbf{I}_Q(|\psi\rangle : |\psi\rangle)} d\Lambda = O(1)$.
- *Relativized to POVM E,* $\int 2^{\mathbf{I}(E|\psi\rangle : E|\psi\rangle)} d\Lambda = O(1)$.

This is result is in constrast to the fact that most pure quantum states will have a very large *algorithmic quantum entropy*, using any definition from [G01́, Vit00, BvL01]. Thus most quantum pure states contain high quantum algorithmic entropy, low self algorithmic information, and will most likely produce random noise (or the empty signal) given a quantum measurement. One interesting note is that the measurement theorem is derived from the strangest proof I've ever written, leveraging *upper computable* tests!

References

[BvL01] A. Berthiaume, W. van Dam, and S. Laplante. Quantum Kolmogorov Complexity. *Journal of Computer and System Sciences*, 63(2), 2001.

[Eps19] S. Epstein. Algorithmic No-Cloning Theorem. *IEEE Transactions on Information Theory*, 65(9), 2019.

[G01́] P. Gács. Quantum Algorithmic Entropy. *Journal of Physics A Mathematical General*, 34(35), 2001.

[Vit00] P Vitányi. Three Approaches to the Quantitative Definition of Information in an Individual Pure Quantum State. In *Proceedings of the 15th Annual IEEE Conference on Computational Complexity*, COCO '00, page 263. IEEE Computer Society, 2000.