

Outliers are in the Physical World

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Abstract

The Independence Postulate (IP) is a finitary Church Turing Thesis, postulating that certain infinite and finite sequences cannot be found in nature, a.k.a. have high “physical addresses”. In this paper we show that IP implies that outliers are found in the physical world.

1 Introduction

$\mathbf{K}(x|y)$ is the conditional prefix complexity. For probability p over \mathbb{N} , the deficiency of randomness is $\mathbf{d}(a|p) = \lfloor -\log p(a) \rfloor - \mathbf{K}(a|P)$. $\mathbf{I}(a; \mathcal{H}) = \mathbf{K}(a) - \mathbf{K}(a|\mathcal{H})$, where $\mathcal{H} \in \{0, 1\}^\infty$ is the halting sequence. The indicator function of a mathematical statement A is denoted by $[A]$, where if A is true then $[A] = 1$, otherwise $[A] = 0$. An elementary probability measure \mathbb{N} has finite support and range in \mathbb{Q} . The stochasticity of $a \in \mathbb{N}$, conditional to $b \in \mathbb{N}$, is measured by

$$\begin{aligned} \Lambda(a|b) &= \min\{\mathbf{K}(Q|b) + 3 \log \max\{\mathbf{d}(a|Q, b), 1\} \\ &\quad : Q \text{ is an elementary probability measure}\}. \\ \Lambda(a|b) &< \Lambda(a) + O(\log \mathbf{K}(b)). \end{aligned}$$

The information between two sequences is

Definition 1 For $\alpha, \beta \in \{0, 1\}^{*\infty}$, $\mathbf{I}(\alpha : \beta) = \log \sum_{x, y \in \{0, 1\}^*} 2^{\mathbf{K}(x) + \mathbf{K}(y) - \mathbf{K}(x, y) - \mathbf{K}(x|\alpha) - \mathbf{K}(y|\beta)}$.

The Independence Postulate [Lev84, Lev13] is as follows:

IP: Let α be a sequence defined with an n -bit mathematical statement, and a sequence β can be located in the physical world with a k -bit instruction set. Then $\mathbf{I}(\alpha : \beta) < k + n + c$ for some small absolute constant c .

Lemma 1 ([EL11]) For $x \in \mathbb{N}$, $\Lambda(x) <^{\log} \mathbf{I}(x; \mathcal{H})$.

Theorem 1 For computable probability p over \mathbb{N} and for $D \subset \{0, 1\}^*$, $|D| = 2^s$, $s < \max_{a \in D} \mathbf{d}(a|p) + \Lambda(D) + \mathbf{K}(s) + \mathbf{K}(p) + O(\log \mathbf{K}(s)\mathbf{K}(p))$.

Proof. We relativize the universal Turing machine U to p and s . Let Q be an elementary probability distribution that realizes $\Lambda(D)$. Let $d = \mathbf{d}(D|Q)$ be the deficiency of randomness of D with respect to Q . Let V be the combined elements of encoded sets of size 2^s in the support of Q . Suppose each element of V is selected independently with probability $cd2^{-s}$, where c is a constant to be chosen later. The selected set is F , and $\mathbf{E}[p(F)] \leq cd2^{-s}$. Furthermore

$$\begin{aligned} &\mathbf{E}[Q(\{G : |G| = 2^s, G \cap F = \emptyset\})] \\ &\leq \sum_G Q(G)(1 - cd2^{-s})^{2^s} < e^{-cd}. \end{aligned}$$

Thus a finite set F can be chosen such that $p(F) \leq 2cd2^{-s}$ and $Q(\{G : |G| = 2^s, G \cap F = \emptyset\}) \leq e^{1-cd}$. Now it must be that $D \cap F \neq \emptyset$. Otherwise, using the Q -test, $t(G) = [|G| = 2^s, G \cap F = \emptyset]e^{cd-1}$, we have

$$\begin{aligned} \mathbf{K}(D|Q, d, c) &<^+ -\log Q(D) - (\log e)cd \\ (\log e)cd &<^+ -\log Q(D) - \mathbf{K}(D|Q) + \mathbf{K}(d, c) \\ (\log e)cd &<^+ d + \mathbf{K}(d, c), \end{aligned}$$

which is a contradiction for large enough c solely dependent on the universal Turing machine U . Thus there is an $a \in D \cap F$, where

$$\begin{aligned} \mathbf{K}(a) &<^+ -\log p(a) + \log d - s + \mathbf{K}(d) + \mathbf{K}(Q) \\ s &<^+ \mathbf{d}(a|p) + \Lambda(D). \end{aligned}$$

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Removing the relativization of p and s

$$\begin{aligned} s &< -\log p(a) - \mathbf{K}(a|p, s) + \Lambda(D|p, s) \\ s &< \max_{a \in D} \mathbf{d}(a|p) + \Lambda(D) + \mathbf{K}(s) + \mathbf{K}(p) \\ &+ O(\log \mathbf{K}(s)\mathbf{K}(p)). \end{aligned}$$

□

For $\tau \in \mathbb{N}^{\mathbb{N}}$, $\langle \tau \rangle = \langle \tau[1] \rangle \langle \tau[2] \rangle \langle \tau[3] \rangle \dots$. Let $\tau(n)$ be the first 2^n unique numbers found in τ . The sequence τ is assumed to have an infinite amount of unique numbers, and represents a series of observations.

Theorem 2 *For probability p over \mathbb{N} and $\tau \in \mathbb{N}^{\mathbb{N}}$, let s_τ be the smallest number such that $\max_{a \in \tau(n)} \mathbf{d}(a|p) > n - 4\mathbf{K}(n) - s_\tau$. Then $s_\tau <^{\log} \mathbf{I}(\langle \tau \rangle : \mathcal{H}) + \mathbf{K}(p)$.*

Proof. By Theorem 1, Lemma 1, and the fact that $\mathbf{I}(x; \mathcal{H}) <^+ \mathbf{I}(\alpha : \mathcal{H}) + \mathbf{K}(x|\alpha)$,

$$\begin{aligned} n &< \max_{a \in \tau(n)} \mathbf{d}(a|p) + \mathbf{I}(\tau(n); \mathcal{H}) + \mathbf{K}(p) + 2\mathbf{K}(n) \\ &+ O(\log(\mathbf{I}(\tau(n); \mathcal{H}) + \mathbf{K}(p))), \\ n &< \max_{a \in \tau(n)} \mathbf{d}(a|p) + 4\mathbf{K}(n) + \mathbf{I}(\langle \tau \rangle : \mathcal{H}) + \mathbf{K}(p) \\ &+ O(\log(\mathbf{I}(\langle \tau \rangle : \mathcal{H}) + \mathbf{K}(p))), \\ n - 4\mathbf{K}(n) - (\mathbf{I}(\langle \tau \rangle : \mathcal{H}) + \mathbf{K}(p)) \\ &+ O(\log(\mathbf{I}(\langle \tau \rangle : \mathcal{H}) + \mathbf{K}(p))) < \max_{a \in \tau(n)} \mathbf{d}(a|p). \end{aligned}$$

□

Let k be the physical address of an infinite sequence of numbers τ . The halting sequence \mathcal{H} can be described in a mathematical statement of size equal to some small constant c_{HM} . Then by Theorem 2, for some sequence $\tau \in \mathbb{N}^{\mathbb{N}}$ and probability p over \mathbb{N} , \mathbf{IP} states

$$s_\tau <^{\log} \mathbf{I}(\langle \tau \rangle : \mathcal{H}) + \mathbf{K}(p) <^{\log} k + c_{\text{HM}} + c + \mathbf{K}(p).$$

Thus sequences τ with large s_τ , as defined in Theorem 2, will have large physical address. Thus it is hard to find physical sequences which do not have large outliers, and completely impossible to find sequences with no outliers. As the complexity of the probability p in the randomness deficiency term increases, the bounds loosen.

References

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