AIT Blog

On Creating Pairs of Derandomization Theorems

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I've uploaded a new paper to my site, titled *Derandomization under Different Resource Constraints* with the intention of eventually uploading it to arXiv and submission for publication.

http://www.jptheorygroup.org/doc/Resource.pdf

The main contribution is a resource bounded EL Theorem and a general formula for resource bounded derandomization, in the sense of [Eps22]. In this blog, I show an example of taking a theorem produced from a non-constructive probabilistic proof and produce a pair of derandomization theorems, one that is resource free and one that is resource bounded. This methodology supports the following claim.

Claim. *If the existence of an object can be proven with the probabilistic method, then bounds on its Kolmogorov complexity can be proven as well.*

We show an example using hypergraphs. A *hypergraph* is a pair $J = (V, E)$ of vertices V and edges $E \subseteq \mathcal{P}(V)$. Thus each edge can connect ≥ 2 vertices. A hypergraph is *k-regular* of the size $|e| = k$ for all edges $e \in E$. A 2-regular hypergraph is just a simple graph. A valid *C-coloring* of a hypergraph (V, E) is a mapping $f : V \to \{1, \ldots, C\}$ where every edge $e \in E$ is not *monochromatic* $|\{f(v) : v \in e\}| > 1$. The following classic result uses Lovasz Local Lemma.

Theorem. **[Probabilistic Method]** *Let $G = (V, E)$ be a k-regular hypergraph. If for each edge f, there are at most $2^{k-1}/e-1$ edges $h \in E$ such that $h \cap f \neq \emptyset$, then there exists a valid 2-coloring of G.*

We can now use derandomization, in the sense of [Eps22], to produce bounds on the Kolmogorov complexity of the simpliest such 2-coloring of G.

Theorem. **[Derandomization]** *Let $G = (V, E)$ be a k-regular hypergraph with $|E| = m$. If, for each edge f, there are at most $2^{k-1}/e - 1$ edges $h \in E$ such that $h \cap f \neq \emptyset$, then there exists a valid 2-coloring x of G with*

$$\mathbf{K}(x) <^{\log} \mathbf{K}(n) + 4me/2^k + \mathbf{I}(G; \mathcal{H}).$$

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The function $\mathbf{K}$ is the prefix free Kolmogorov complexity. $\mathbf{I}(G;\mathcal{H}) = \mathbf{K}(G) - \mathbf{K}(G|\mathcal{H})$ is the amount of asymmetric information the halting sequence $\mathcal{H} \in \{0,1\}^\infty$ has about the graph G. We can now use resource derandomization, introduced in http://www.jptheorygroup.org/doc/Resource.pdf, to achieve bounds for the smallest time-bounded Kolmogorov complexity $\mathbf{K}^t(x) = \min\{p : U(p) = x \text{ in } t(\|x\|) \text{ steps}\}$ of a 2-coloring of G. **Crypto** is the assumption that there exists a language in $\mathbf{DTIME}(2^{O(n)})$ that does not have size $2^{o(n)}$ circuits with Σ_2^p gates.

Theorem. **[Resource Bounded Derandomization]** *Assume **Crypto**. Let $G_n = (V,E)$ be a $k(n)$-regular hypergraph where $|V| = n$ and $|E| = m(n)$, uniformly polynomial time computable in n. Furthermore, for each edge f in G_n there are at most $2^{k(n)-1}/e - 1$ edges $h \in E$ such that $h \cap f \neq \emptyset$. Then there is a polynomial p, and a valid 2-coloring x of G_n with*

$$\mathbf{K}^p(x) < 4m(n)e/2^{k(n)} + O(\log n).$$

The conjecture is that one can produce a suite of derandomization theorems, each one mapping to Kolmogorov complexity with different time and space constraints, and access to a certain number of random bits. In my uploaded paper, I used derandomization to show the tradeoff between codebook compression rate and channel capacity, so I believe there are a lot of applications of derandomization. However the codebook is of exponential size, so it is not suitable for resource-bounded derandomization. In [Eps22], derandomization was used on games, where probabilistic players can be turned into winning deterministic ones. So far, resource bounded derandomization does not lend itself to games. This is because the environment must be polynomial time computable which means the agent can efficiently simulate it, making the results trivial.

References

[Eps22] S. Epstein. 22 examples of solution compression via derandomization. *CoRR*, abs/2208.11562, 2022.