On Kolmogorov Structure Functions

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Abstract

All strings with low mutual information with the halting sequence will have flat Kolmogorov Structure Functions, in the context of Algorithmic Statistics. Assuming the Independence Postulate, strings with non-negligible information with the halting sequence are purely mathematical constructions, and cannot be found in nature. Thus Algorithmic Statistics does not study strings in the physical world. We also discuss issues with set-restricted Kolmogorov Structure Functions.

1 Introduction

In statistics, one tries to determine a model (such as a parameter for a distribution) from data which is assumed to have noise. In the Minimum Description Principle [Gru07], the model that describes information with the shortest code is assumed to be the best model. The data is described as a two part code, where the first part is the model and the second part is the noise. In one of his last works, Kolmogorov suggested a two part code for individual strings $x \in \{0,1\}^*$ based off Kolmogorov Complexity. The first part (the model) is a set D containing x, the second part (the noise) is the code of x given D, of size $\lceil \log |D| \rceil$. Other works examined probabilities and also total computable functions as models [Vit02]. Kolmogorov suggested the following *structure function* at the Tallinn conference in Estonia, 1973.

$$\mathbf{H}_k(x) = \min\{\log |S| : x \in S, \mathbf{K}(S) \leq k\}.$$

The function $\mathbf{K}$ is the prefix Kolmogorov complexity. This definition is used for the following function, which is a central definition of *Algorithmic Statistics* [VS15, VS17, VV04a],

$$k \mapsto k + \mathbf{H}_k(x) - \mathbf{K}(x).$$

This function's equivalence to several other definitions is the main theorem of Algorithmic Statistics [SSV24]. Furthermore, Theorem 1 of [VS17] showed that any shape of the structure function is possible.

The structure function is flat for all strings with low mutual information with the halting sequence. Assuming the *Independence Postulate*, [Lev84, Lev13], strings with non-negligible mutual information with the halting sequence are exotic, in that they cannot be found in nature. Such strings are purely mathematical constructions.

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2 Bounds

We review the results of [GTV01], in particular Theorem of III.24, which I don't think is widely known. $\mathbf{m}(x)$ is the algorithmic probability. The amount of information that the halting sequence $\mathcal{H} \in \{0,1\}^\infty$ has about $x \in \{0,1\}^*$ is $\mathbf{I}(x;\mathcal{H}) = \mathbf{K}(x) - \mathbf{K}(x|\mathcal{H})$. We use $x <^+ y$, $x >^+ y$ and $x =^+ y$ to denote $x < y + O(1)$, $x + O(1) > y$ and $x = y \pm O(1)$, respectively. In addition, $x <^{\log} y$ and $x >^{\log} y$ denote $x < y + O(\log y)$ and $x + O(\log x) > y$, respectively. For $x, y \in \{0,1\}^*$, $x \sqsubseteq y$ if $y = xz$ for some $z \in \{0,1\}^*$. $[A] = 1$ if mathematical statement A is true, and $[A] = 0$ otherwise.

Let $S_k = \{x : \mathbf{K}(x) \leq k\}$. Let $N_k = |S_k|$ where $\log N_k =^+ k - \mathbf{K}(k)$, due to [GTV01]. Let I_k^x be the index of x in an enumeration of S_k. For $\mathbf{K}(x) = k$, let m_x be the longest joint prefix of I_k^x and N_k. So $m_x 0 \sqsubseteq I_k^x$ and $m_x 1 \sqsubseteq N_k$. Let $S_x = \{y : m_x 0 \sqsubseteq I_k^y\}$. So

$$\begin{aligned} \log|S_x| &=^+ k - \mathbf{K}(k) - \|m_x\| \\ \mathbf{K}(S_x) &<^+ \mathbf{K}(k) + \mathbf{K}(m_x) <^+ \mathbf{K}(k) + \|m_x\| + \mathbf{K}(\|m_x\|). \end{aligned}$$

Theorem 1 ([GTV01])**.**

$$\|m_x\| < \mathbf{K}(\mathbf{K}(x)) + \mathbf{I}(x;\mathcal{H}) + O(\log \mathbf{I}(x;\mathcal{H})).$$

Proof. Let $\nu(y) = c[\mathbf{K}(y) \leq k]\mathbf{m}(y) 2^{\|m_y\|}/(\|m_y\|^2)$. For proper choice of c, ν is a semimeasure and computable relative to $\mathcal{H}$ and k. So $\mathbf{K}(x|\mathcal{H}, k) <^+ -\log \nu(x) =^+ \mathbf{K}(x) - \|m_x\| + 2\log\|m_x\|$. □

Note that with some additional effort, the $\mathbf{K}(\mathbf{K}(x))$ term can be eliminated.

Corollary 1. *For $x \in \{0,1\}^*$, $n = \mathbf{K}(x)$, for all $m \leq n$, $m \in \mathbb{W}$, there is a set $S \ni x$ such that $|S| = 2^m$ and $\mathbf{K}(S) + m <^{\log} n + \mathbf{I}(x;\mathcal{H})$.*

Claim 1. *Thus there exists a set $S \ni x$ such that $\mathbf{K}(S) <^{\log} 2\mathbf{K}(\mathbf{K}(x)) + \mathbf{I}(x;\mathcal{H})$ and $\mathbf{K}(S) + \log|S| <^+ \mathbf{K}(x) + \mathbf{K}(\mathbf{K}(x)) + O(\log(\mathbf{I}(x;\mathcal{H}) + \mathbf{K}(\mathbf{K}(x))))$. This fact combined with the following proposition characterizes the Kolmogorov Structure Function.*

Proposition 1. *Let $S \ni x$. For all $s < \log|S|$ there exists a set $S' \ni x$ such that $|S'| \leq |S|2^{-s}$ and $\mathbf{K}(S') <^+ \mathbf{K}(S) + s + \mathbf{K}(s)$.*

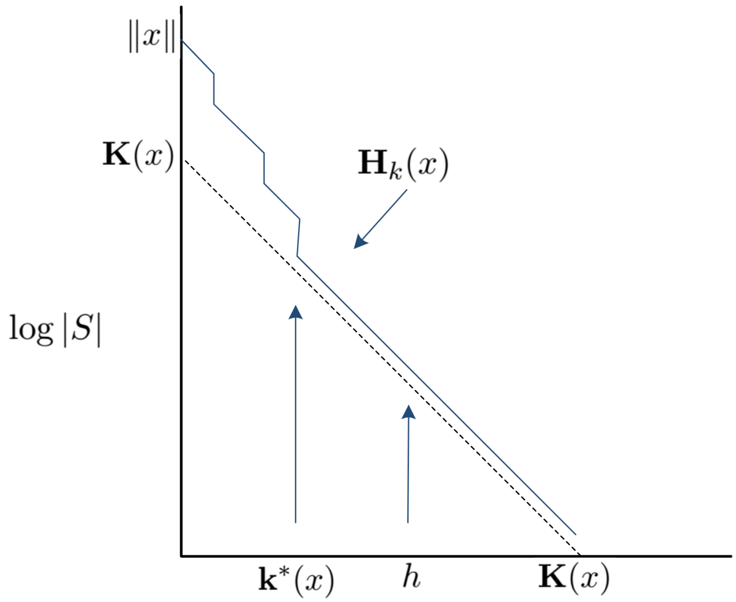

Figure 1: A visual representation of the Kolmogorov structure function $\mathbf{H}_k(x)$. The amount of information that the halting sequence has about x is $h = \mathbf{I}(x; \mathcal{H})$. Since h is negligible for almost all x, the structure function is almost always flat.

The minimal sufficient statistic for $x \in \{0,1\}^*$ is

$$\mathbf{k}^*(x) = \min\{k : \mathbf{H}_k(x) + k = \mathbf{K}(x)\}.$$

This is the location in which the Kolmogorov Structure Function reaches the boundary point and becomes flat. Due to Theorem 1, $\mathbf{k}^*(x) <^{\log} \mathbf{K}(\mathbf{K}(x)) + \mathbf{I}(x; \mathcal{H})$ (but note that the $\mathbf{K}(\mathbf{K}(x))$ term can be eliminated). A visualization of the Kolmogorov Structure Function can be seen in Figure 1.

3 Set-Restricted Structure Functions

One potential method to create strings with non-simple Kolmogorov Structure Functions is to restrict the sets under consideration. Thus for a set of sets $\mathcal{S}$,

$$\mathbf{H}_k^{\mathcal{S}}(x) = \min\{\log|S| : x \in S \in \mathcal{S}, \mathbf{K}(S) \leq k\}.$$

This would banish the pesky set S_x defined in the last section. This was studied in Section 6 of [VS15]. However there is an inherent obstacle to proving such functions can have any shape. Proofs to statements (such as Theorem 10 in [VS15]) of such effect use a shape function R to (non-recursively) construct a string x whose structure function has that shape R (up to a degree of precision depending on $\mathcal{S}$). Thus the proof can be thought of as a program to produce x given R and $\mathcal{H}$, with $\mathbf{K}(x|\mathcal{H}) <^+ \mathbf{K}(R)$. Thus proofs saying that for every shape R there is a set x such that $\mathbf{H}^{\mathcal{S}}_k(x)$ has shape R (up to a certain precision) also implies that $\mathbf{I}(x;\mathcal{H}) >^+ \mathbf{K}(x) - \mathbf{K}(R)$. In general, the Independence Postulate states if a string can be described by a small mathematical statement but has high Kolmogorov complexity then it cannot be found in the physical world. This presents an obstacle for constructive proofs in Algorithmic Information Theory.

4 Curious Properties of the Halting Sequence

The question is what can be expected on h in Figure 1? It is true that $|\{x : \mathbf{I}(x;\mathcal{H}) < k\}| < \infty$ for all $k \in \mathbb{N}$. In this section we provide an explicit lower bound on $\mathbf{I}(x;\mathcal{H})$, though one that grows at an uncountably slow rate.

Let $\Omega = \sum\{2^{-\|p\|} : U(p) \text{ halts}\}$ be Chaitin's Omega and $\Omega^t = \sum\{2^{-\|p\|} : U(p) \text{ halts in time } t\}$. For a string x, let $BB(x) = \min\{t : \Omega^t > 0.x + 2^{-\|x\|}\}$. Note that $BB(x)$ is undefined if $0.x + 2^{-\|x\|} > \Omega$. For $n \in \mathbb{N}$, let $\mathbf{bb}(n) = \max\{BB(x) : \|x\| \leq n\}$. $\mathbf{bb}^{-1}(m) = \arg\min_n\{\mathbf{bb}(n-1) < m \leq \mathbf{bb}(n)\}$. Let $bb(n) = \arg\max_x\{BB(x) : \|x\| \leq n\}$.

Lemma 1 ([Eps22])**.** *For partial computable function f, $\mathbf{I}(f(x);\mathcal{H}) <^+ \mathbf{I}(x;\mathcal{H}) + \mathbf{K}(f)$.*

Lemma 2. *For $n = \mathbf{bb}^{-1}(m)$, $\mathbf{K}(bb(n)|m,n) = O(1)$.*

Proof. Enumerate strings of length n, starting with 0^n, and return the first string y such that $BB(y) \geq m$. This string y is equal to $bb(n)$, otherwise $BB(y^-)$ is defined and $BB(y^-) \geq BB(y) \geq m$. Thus $\mathbf{bb}(n-1) \geq m$, causing a contradiction. □

Proposition 2.

1. $\mathbf{K}(bb(n)) >^+ n$.
2. $\mathbf{K}(bb(n)|\mathcal{H}) <^+ \mathbf{K}(n)$.

Theorem 2. $\mathbf{I}(x;\mathcal{H}) >^{\log} \mathbf{bb}^{-1}(\|x\|)$.

Proof. Let $n = \mathbf{bb}^{-1}(\|x\|)$ and $b = bb(n)$. Thus by Lemma 2, $\mathbf{K}(b|x,n) = O(1)$. So

$$\begin{aligned}
&\mathbf{K}(x) - \mathbf{K}(x|\mathcal{H}) \\
>^+\ &\mathbf{K}(x,b) - \mathbf{K}(x,b|\mathcal{H}) - \mathbf{K}(n) \qquad (1)\\
>^+\ &\mathbf{K}(b) - \mathbf{K}(b|\mathcal{H}) - \mathbf{K}(n) \qquad (2)\\
>^{\log}\ &n.
\end{aligned}$$

Equation 1 is due to Lemma 2 and Equation 2 is due to Lemma 1. □

5 Discussion

The Independence Postulate [Lev84, Lev13] states:

> **IP**: *Let α be a sequence defined with an n-bit mathematical statement (e.g., in PA or set theory), and a sequence β can be located in the physical world with a k-bit instruction set (e.g., ip-address). Then* $\mathbf{I}(\alpha : \beta) < k + n + c$ *for some small absolute constant c.*

When I first learned of **IP**, I didn't realize how much of impact it could have on different fields of study. For example, **IP** and the Many Worlds Theory [Eve57] are in conflict because measuring the spin of a million electrons results in the creation of a world where a large prefix of Chaitin's Omega, Ω, is found at a small address. Furthermore, **IP** causes issues in Constructor Theory [Deu13], which characterizes tasks in physics as either possible or impossible. This raises the question: "Is it possible or impossible to find large prefixes of Ω?". The answer causes trouble for either Constructor Theory or **IP**.

This note reiterates that **IP** implies Algorithmic Statistics does not study strings in the physical world. The intention is not to denigrate the theory; a majority of my work (including [Eps24a, Eps23c, Eps23b, Eps24b, Eps23a]) is descendent from Algorithmic Statistics, particularly [VV04b]. My interpretation of the Kolmogorov Structure Function is that it (and its equivalent definitions) provide a means to know that a string x has high $\mathbf{I}(x; \mathcal{H})$.

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