

On Kolmogorov Structure Functions

Sam Epstein*

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Abstract

All strings with low mutual information with the halting sequence will have flat Kolmogorov Structure Functions, in the context of Algorithmic Statistics. Assuming the Independence Postulate, strings with non-negligible information with the halting sequence are purely mathematical constructions, and cannot be found in the physical world. We also discuss issues with set-restricted Kolmogorov Structure Functions.

1 Introduction

In statistics, one tries to determine a model (such as a parameter for a distribution) from data which is assumed to have noise. In the Minimum Description Principle [Gru07], the model that describes information with the shortest code is assumed to be the best model. The data is described as a two part code, where the first part is the model and the second part is the noise. In one of his last works, Kolmogorov suggested a two part code for individual strings $x \in \{0, 1\}^*$ based off Kolmogorov Complexity. The first part (the model) is a set D containing x , the second part (the noise) is the code of x given D , of size $\lceil \log |D| \rceil$. Other works examined probabilities and also total computable functions as models [Vit02]. Kolmogorov suggested the following *structure function* at the Tallinn conference in Estonia, 1973.

$$\mathbf{H}_k(x) = \min\{\log |S| : x \in S, \mathbf{K}(S) \leq k\}.$$

The function \mathbf{K} is the prefix Kolmogorov complexity. This definition is used for the following function, which is a central definition of *Algorithmic Statistics* [VS15, VS17, VV04],

$$k \mapsto k + \mathbf{H}_k(x) - \mathbf{K}(x).$$

This function's equivalence to several other definitions is the main theorem of Algorithmic Statistics [SSV24]. Furthermore, Theorem 1 of [VS17] showed that any shape of the structure function is possible.

The structure function is flat for all strings with low mutual information with the halting sequence. Assuming the *Independence Postulate*, [Lev84, Lev13], strings with non-negligible mutual information with the halting sequence are exotic, in that they cannot be found in nature. Such strings are purely mathematical constructions.

*samepst@jpththeorygroup.org

2 Bounds

We review the results of [GTV01], in particular Theorem of III.24, which I don't think is widely known. $\mathbf{m}(x)$ is the algorithmic probability. The amount of information that the halting sequence $\mathcal{H} \in \{0,1\}^\infty$ has about $x \in \{0,1\}^*$ is $\mathbf{I}(x; \mathcal{H}) = \mathbf{K}(x) - \mathbf{K}(x|\mathcal{H})$. We use $x <^+ y$, $x >^+ y$ and $x =^+ y$ to denote $x < y + O(1)$, $x + O(1) > y$ and $x = y \pm O(1)$, respectively. In addition, $x <^{\log} y$ and $x >^{\log} y$ denote $x < y + O(\log y)$ and $x + O(\log x) > y$, respectively. For $x, y \in \{0,1\}^*$, $x \sqsubseteq y$ if $y = xz$ for some $z \in \{0,1\}^*$. $[A] = 1$ if mathematical statement A is true, and $[A] = 0$ otherwise.

Let $S_k = \{x : \mathbf{K}(x) \leq k\}$. Let $N_k = |S_k|$ where $\log N_k =^+ k - \mathbf{K}(k)$, due to [GTV01]. Let I_k^x be the index of x in an enumeration of S_k . For $\mathbf{K}(x) = k$, let m_x be the longest joint prefix of I_k^x and N_k . So $m_x 0 \sqsubseteq I_k^x$ and $m_x 1 \sqsubseteq N_k$. Let $S_x = \{y : m_x 0 \sqsubseteq I_k^y\}$. So

$$\begin{aligned} \log |S_x| &=^+ k - \mathbf{K}(k) - \|m_x\| \\ \mathbf{K}(S_x) &<^+ \mathbf{K}(k) + \mathbf{K}(m_x) <^+ \mathbf{K}(k) + \|m_x\| + \mathbf{K}(\|m_x\|). \end{aligned}$$

Theorem 1 ([GTV01]).

$$\|m_x\| < \mathbf{K}(\mathbf{K}(x)) + \mathbf{I}(x; \mathcal{H}) + O(\log \mathbf{I}(x; \mathcal{H})).$$

Proof. Let $\nu(y) = c[\mathbf{K}(y) \leq k] \mathbf{m}(y) 2^{\|m_y\|} / (\|m_y\|^2)$. For proper choice of c , ν is a semimeasure and computable relative to \mathcal{H} and k . So $\mathbf{K}(x|\mathcal{H}, k) <^+ -\log \nu(x) =^+ \mathbf{K}(x) - \|m_x\| + 2 \log \|m_x\|$. \square

Note that with some additional effort, the $\mathbf{K}(\mathbf{K}(x))$ term can be eliminated.

Corollary 1. For $x \in \{0,1\}^*$, $n = \mathbf{K}(x)$, for all $m \leq n$, $m \in \mathbb{W}$, there is a set $S \ni x$ such that $|S| = 2^m$ and $\mathbf{K}(S) + m <^{\log} n + \mathbf{I}(x; \mathcal{H})$.

Claim 1. Thus there exists a set $S \ni x$ such that $\mathbf{K}(S) <^{\log} 2\mathbf{K}(\mathbf{K}(x)) + \mathbf{I}(x; \mathcal{H})$ and $\mathbf{K}(S) + \log |S| <^+ \mathbf{K}(x) + \mathbf{K}(\mathbf{K}(x)) + O(\log(\mathbf{I}(x; \mathcal{H}) + \mathbf{K}(\mathbf{K}(x))))$. This fact combined with the following proposition characterizes the Kolmogorov Structure Function.

Proposition 1. Let $S \ni x$. For all $s < \log |S|$ there exists a set $S' \ni x$ such that $|S'| \leq |S| 2^{-s}$ and $\mathbf{K}(S') <^+ \mathbf{K}(S) + s + \mathbf{K}(s)$.

3 Set-Restricted Structure Functions

One potential method to create strings with non-simple Kolmogorov Structure Functions is to restrict the sets under consideration. Thus for a set of sets \mathcal{S} ,

$$\mathbf{H}_k^{\mathcal{S}}(x) = \min\{\log |S| : x \in S \in \mathcal{S}, \mathbf{K}(S) \leq k\}.$$

This would banish the pesky set S_x defined in the last section. This was studied in Section 6 of [VS15]. However there is an inherent obstacle to proving such functions can have any shape. Proofs to statements (such as Theorem 10 in [VS15]) of such effect use a shape function R to (non-recursively) construct a string x whose structure function has that shape R (up to a degree of precision depending on \mathcal{S}). Thus the proof can be thought of as a program to produce x given R and \mathcal{H} , with $\mathbf{K}(x|\mathcal{H}) <^+ \mathbf{K}(R)$. Thus proofs saying that for every shape R there is a set x such that $\mathbf{H}_k^{\mathcal{S}}(x)$ has shape R (up to a certain precision) also implies that $\mathbf{I}(x; \mathcal{H}) >^+ \mathbf{K}(x) - \mathbf{K}(R)$. In general, the Independence Postulate states if a string can be described by a small mathematical statement but has high Kolmogorov complexity then it cannot be found in the physical world. This presents an obstacle for constructive proofs in Algorithmic Information Theory.

4 Discussion

The Independence Postulate [Lev84, Lev13] states:

IP: *Let α be a sequence defined with an n -bit mathematical statement (e.g., in PA or set theory), and a sequence β can be located in the physical world with a k -bit instruction set (e.g., ip-address). Then $\mathbf{I}(\alpha : \beta) < k + n + c$ for some small absolute constant c .*

When I first learned of **IP**, I didn't realize how much of impact it could have on different fields of study. For example, **IP** and the Many Worlds Theory are in conflict because measuring the spin of a million electrons results in the creation of a world where a large prefix of Chaitin's Omega, Ω is found at a small address. Furthermore, **IP** causes issues in Constructor Theory, which characterizes tasks in physics as either possible or impossible. This raises the question: "Is it possible or impossible to find large prefixes of Ω ?" The answer causes trouble for either Constructor Theory or **IP**.

This note reiterates the fact that **IP** implies that Algorithmic Statistics does not study strings in the physical world. The intention is not to denigrate the theory; a majority of my work is descendent from Algorithmic Statistics. My interpretation of the Kolmogorov Structure Function is that it provides a means to know that a string x has high $\mathbf{I}(x; \mathcal{H})$.

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