

A Chain Rule for the Randomness Deficiency Function

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Abstract

This paper is an exposition of the addition equality theorem for algorithmic entropy in [G01], applied to the Cantor space. This application implies that randomness deficiency of infinite sequences obeys the chain rule, analogous to the finite Kolmogorov complexity case. This is a generalization of van Lambalgen's Theorem. It is unclear whether this result is folklore, but in any case, this paper presents a dedicated proof of the equality. In addition, a dual integration trick shortens the proof.

1 Introduction

Prefix free Kolmogorov complexity, \mathbf{K} , obeys the chain rule, with for $x, y \in \{0, 1\}^*$,

$$\mathbf{K}(x, y) =^+ \mathbf{K}(x) + \mathbf{K}(y|x, \mathbf{K}(x)).$$

In this paper, we apply the addition equality theorem for algorithmic entropy in [G01] to the specific case of infinite sequences. We also shorten the proof using an integration trick. The consequence to this is a result about randomness deficiency \mathbf{D} , where for computable probability μ , for infinite sequences, $\mathbf{D}(\alpha|\mu, x) = \sup_n -\log \mu(\alpha[0..n]) - \mathbf{K}(\alpha[0..n]|x)$. The randomness deficiency over the space $\{0, 1\}^\infty \times \{0, 1\}^\infty$, is $\mathbf{D}(\alpha, \beta|\mu, \nu) = \sup_n -\log \mu(\alpha[0..n]) - \log \nu(\beta[0..n]) - \mathbf{K}(\alpha[0..n]|\beta[0..n])$. The discrete case for $\mathbf{d}(x|p) = -\log p(x) - \mathbf{K}(x)$ is trivial. The result detailed in this paper is as follows.

Theorem. ([G01]) *Relativized to probabilities μ and ν over $\{0, 1\}^\infty$,*

$$\mathbf{D}(\alpha, \beta|\mu, \nu) =^+ \mathbf{D}(\alpha|\mu) + \mathbf{D}(\beta|\nu, (\alpha, \lceil \mathbf{D}(\alpha|\mu) \rceil)).$$

This is a generalization of van Lambalgen's Theorem, which states (α, β) is ML random iff α is ML random and β is ML random with respect to α . If one were to take the complexities of the probabilities μ and ν into account (that is, they are no longer $O(1)$) then the theorem statement and proof become more nuanced. This generalization can be seen in [G01]. An open question is whether \mathbf{D} follows the linear inequalities that parallel Shannon entropy, as Kolmogorov complexity was shown to do [HRSV00].

2 Results

For the nonnegative real function f , we use $<^+ f$, $>^+ f$, and $=^+ f$ to denote $< f + O(1)$, $> f - O(1)$, and $= f \pm O(1)$. As shown in [G01], $2^{\mathbf{D}(\alpha|\mu)} =^* \mathbf{t}_\mu(\alpha)$ where \mathbf{t}_μ is a universal lower computable μ -test.

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Furthermore, similar arguments can be used to show that $2^{\mathbf{D}(\alpha, \beta | \mu, \nu)} \stackrel{*}{=} \mathbf{t}_{\mu, \nu}(\alpha, \beta)$, where $\mathbf{t}_{\mu, \nu}$ is a universal lower computable test over $\{0, 1\}^\infty \times \{0, 1\}^\infty$. This can be seen in Proposition 1. For measure μ and lower continuous function f over $\{0, 1\}^\infty$, we use the notation $\mu^x f(x) = \int_{x \in \{0, 1\}^\infty} f(x) d\mu(x)$. Throughout this section, the universal Turing machine is assumed to be relativized to probabilities μ and ν over $\{0, 1\}^\infty$. This means that there is an $O(1)$ sized program that can compute $\mu(x\{0, 1\}^\infty)$ uniformly in $x \in \{0, 1\}^*$, and similarly for ν . The following proposition generalizes Theorem 2.3.4 to the $\{0, 1\}^\infty \times \{0, 1\}^\infty$ space.

Proposition 1 $2^{\mathbf{d}(\alpha, \beta | \mu, \nu)} \stackrel{*}{=} \mathbf{t}_{\mu, \nu}(\alpha, \beta)$.

Proof. For the $<^*$ inequality, note that

$$\begin{aligned} f_{\mu, \nu}(\alpha, \beta) &= \sum_{n=1}^{\infty} \sum_{x, y \in \{0, 1\}^n} 1_{x\{0, 1\}^\infty, y\{0, 1\}^\infty}(\alpha, \beta) \frac{\mathbf{m}(x, y)}{\mu(x)\nu(y)} \\ &= \sum_{n=1}^{\infty} \sum_{x, y \in \{0, 1\}^n} \frac{\mathbf{m}(\alpha[0..n]\beta[0..n])}{\mu(x)\nu(y)} \\ &\geq \sup_n \frac{\mu(\alpha[0..n]\beta[0..n])}{\mu(\alpha[0..n])\nu(\beta[0..n])}. \end{aligned}$$

So

$$\log \mathbf{t}_{\mu, \nu}(\alpha, \beta) >^+ \log f_{\mu, \nu}(\alpha, \beta) >^+ \mathbf{D}(\alpha, \beta | \mu, \nu).$$

The $>^+$ inequality is as follows. Note that $1_{x, y}(\alpha, \beta) = 1_{x\{0, 1\}^\infty, y\{0, 1\}^\infty}(\alpha, \beta)$. Since $d(\alpha, \beta) = \lceil \log \mathbf{t}_{\mu, \nu}(\alpha, \beta) \rceil$ takes only integer values and is lower computable,

$$2^{d(\alpha, \beta)} = \sup_i 2^{k_i} 1_{x_i, y_i}(\alpha, \beta) \geq \frac{1}{2} \sum_i 2^{k_i} 1_{x_i, y_i}(\alpha, \beta),$$

with the property that for all i , $\|x_i\| = \|y_i\|$, and if $1_{x_i, x_j}(\alpha, \beta) = 1_{x_j, x_j}(\alpha, \beta) = 1$ and $i < j$ then $k_i < k_j$. Let $\gamma(x, y) = \sum_{y_i=y, x_i=x} 2^{k_i}$.

$$\sum_i 2^{k_i} 1_{x_i, y_i}(\alpha, \beta) = \sum_{n=1}^{\infty} \sum_{x, y \in \{0, 1\}^n} 1_{x, y}(\alpha, \beta) \gamma(x, y).$$

Since $\mu^x \nu^y 2^{d(\alpha, \beta)} \leq 2$, we have that $\sum_{x, y \in \{0, 1\}^*} \mu(x)\nu(y)\gamma(x, y) \leq 2$, so $\mu(x)\nu(y)\gamma(x, y) <^* \mathbf{m}(x, y)$ and $\gamma(x, y) = 0$ if $\|x\| \neq \|y\|$. So

$$2^{d(\alpha, \beta)} <^* \sup_{n \geq 1, x, y \in \{0, 1\}^n} 1_{x, y}(\alpha, \beta) \frac{\mathbf{m}(x, y)}{\mu(x)\nu(y)} \stackrel{*}{=} \sup_n \frac{\mathbf{m}(\alpha[0..n]\beta[0..n])}{\mu(\alpha[0..n])\nu(\beta[0..n])}.$$

□

Proposition 2 $\log \nu^y 2^{\mathbf{D}(x, y | \mu, \nu)} <^+ \mathbf{D}(x | \mu)$.

Proof. Let $f(x, \mu, \nu) = \log \nu^y 2^{\mathbf{D}(x, y | \mu, \nu)}$. The function f is lower computable and has $\mu^x 2^{f(x, \mu, \nu)} \leq 1$. The proposition follows from the universal properties of \mathbf{t}_μ , where $2^f <^* \mathbf{t}_\mu$. □

Proposition 3 If $i < j$, then

$$i + \mathbf{D}(x | \mu, j) <^+ j + \mathbf{D}(x | \mu, i).$$

Proof. By the properties of \mathbf{D} , we have

$$\mathbf{D}(x|\mu, j) <^+ \mathbf{D}(x|\mu, i) + \mathbf{K}(j - i) <^+ \mathbf{D}(x|\mu, i) + j - i.$$

□

Definition 1 Let $F : \{0, 1\}^\infty \rightarrow \mathbb{Z} \cup \{-\infty, \infty\}$ be a lower semicomputable function. An (μ, F) -test is a function $t : \{0, 1\}^\infty \times \{0, 1\}^\infty \rightarrow \mathbb{Z} \cup \{-\infty, \infty\}$ that is lower semicomputable and $\mu^x t(x, y) \leq 2^{F(y)}$. There exists a maximal (μ, F) test, $\mathbf{t}_{(\mu, F)}$, such that $t <^* \mathbf{t}_{(\mu, F)}$.

Proposition 4 Let $F : \{0, 1\}^\infty \rightarrow \mathbb{Z} \cup \{-\infty, \infty\}$ be an upper semicomputable function,. For all x and with $\mathbf{t}_{(\nu, F)}(y) > -\infty$,

$$\mathbf{t}_{(\nu, F)}(x, y) \stackrel{*}{=} 2^{F(y)} \mathbf{t}_\nu(x|y, -F(y)).$$

Proof. To prove the inequality $>^*$, let $g(x, y, m) = \max_{i \geq m} 2^{-i} \mathbf{t}_\nu(x|y, i)$. This function is lower computable, and decreasing in m . The function $g(x, y) = g_\nu(x, y, -F(y))$ is lower semicomputable since $-F$ is upper semi-computable. The multiplicative form of Proposition 3 implies

$$\begin{aligned} g(x, y, m) &\stackrel{*}{=} 2^{-m} \mathbf{t}_\nu(x|y, m) \\ g(x, y) &\stackrel{*}{=} 2^{F(y)} \mathbf{t}_\nu(x|y, -F(y)). \end{aligned}$$

Since \mathbf{t}_ν is a test:

$$\begin{aligned} \nu^x 2^{-m} \mathbf{t}_\nu(x|y, m) &\leq 2^{-m} \\ \nu^x g(x, y) &\stackrel{*}{<} 2^{F(y)}, \end{aligned}$$

which implies $g(x, y) <^* \mathbf{t}_{(\nu, F)}(x, y)$ by the optimality of $\mathbf{t}_{(\nu, F)}$. We now consider the upper bound. Let $\mathbf{t}'_{(\nu, F)}(x, y, m)$ be the modification of $\mathbf{t}_{(\nu, F)}$, which is a lower computable function such that $\nu^x \mathbf{t}'_{(\nu, F)}(x, y, m) \leq 2^{-m+1}$ and if $\nu^x \mathbf{t}_{(\nu, F)}(x, y) \leq 2^{-m}$ then $\mathbf{t}'_{(\nu, F)}(x, y, m) = \mathbf{t}_{(\nu, F)}(x, y)$. The function $2^{m-1} \mathbf{t}'_{(\nu, F)}(x, y, m)$ is a test conditioned on y, m so it has $<^* \mathbf{t}_\nu(x|y, m)$. Substituting $-F(y)$ for m , we have that $\nu^x \mathbf{t}_{(\nu, F)} \leq 2^{-m}$ and so

$$\mathbf{t}_{(\nu, F)}(x, y) = \mathbf{t}'_{(\nu, F)}(x, y, F_\nu(y)) \stackrel{*}{<} 2^{F(y)+1} \mathbf{t}_\nu(x|y, -F(y)).$$

□

The following Theorem is a specific case of Theorem 4.5.2 in [G01], to the Cantor space and with $O(1)$ complexities for the probabilities. The proof is shortened.

Theorem 1 Relativized to computable probabilities μ and ν over $\{0, 1\}^\infty$,

$$\mathbf{D}(x, y|\mu, \nu) =^+ \mathbf{D}(x|\mu) + \mathbf{D}(y|\nu, (x, \lceil \mathbf{D}(x|\mu) \rceil)).$$

Proof. Let $f(x, y) = 2^{\mathbf{D}(x, y|\mu, \nu)}$. Proposition 2 implies there exists $c \in \mathbb{N}$ with $\nu^y f(x, y) \leq 2^{\mathbf{D}(x|\mu)+c}$. Let $F(x, \mu) = \lceil \mathbf{D}(x|\mu) \rceil$. Note that if h is a lower computable function such that $\nu^y h(x, y) <^* 2^{\mathbf{D}(x|\mu)}$, then $\mu^x \nu^y h(x, y) <^* \mu^x \mathbf{t}_\mu(x) <^* 1$, so $h <^* f$, so f is a universal F -test. Proposition 4 (swapping x and y) gives

$$\begin{aligned} \mathbf{D}(x, y|\mu, \nu) &= \log f(x, y) =^+ F(x) + \mathbf{D}(y|\nu, (x, -F(x))) \\ \mathbf{D}(x, y|\mu, \nu) &=^+ \mathbf{D}(x|\mu) + \mathbf{D}(y|\nu, (x, \lceil \mathbf{D}(x|\mu) \rceil)). \end{aligned}$$

□

References

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- [HRSV00] D. Hammer, A. Romashchenko, A. Shen, and N. Vereshchagin. Inequalities for shannon entropy and kolmogorov complexity. *Journal of Computer and System Sciences*, 60(2):442–464, 2000.