Conservation Inequalities Over Infinite Quantum Spin Systems

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Abstract

In algorithmic information theory, information and randomness deficiency terms enjoy a special property, in that they cannot be increased by a computable (even possibly randomized) transformation. Conservation of randomness and information has been proven for computable transformations with respect to computable probabilities over numbers and infinite sequences. In addition conservation inequalities has been proven for quantum operations over finite quantum systems. In this paper, we extend conservation inequalities of randomness and information to infinite quantum spin states. Conservation of NS randomness is proven over a broad class of computable dynamics with respect to infinite quantum spin chains. We introduce new terms of randomness deficiency and information over infinite quantum states and prove conservation over changes in a finite number of qubits.

1 Introduction

In this paper, we prove conservation of randomness and information with respect to infinite quantum spin chains. We first prove conservation over *NS Randomness*, [NS19], which is analogous to ML randomness of infinite sequences. One main difference between the two definitions is that quantum states are NS random at a particular order $\delta \in [0,1]$, where the smaller the δ the more "random" the infinite quantum sequence is. In the classical case, when an ML random sequence is transformed with a computable function, there is no guarantee that the output will remain ML random. However the situation is very different in the quantum case. In fact, conservation of NS randomness is proven over a broad class of computable "admissible" dynamics. In this paper, we use the Heisenberg picture for dynamics. Thus dynamics τ_t are strongly continuous one parameter group of $^*-$automorphisms over the self-adjoint (observables) elements of the quantum C^* algebra. The first result of this paper is as follows.

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Theorem. *For infinite quantum spin state Z, computable $t > 0$, admissible computable dynamics τ_t, if Z is NS random at order δ, then $Z \circ \tau_t$ is NS random at order ϵ, for all $\epsilon > \delta$. Furthermore, if Z is not NS random at order δ then $Z \circ \tau_t$ is not NS random at order ϵ for all $\epsilon < \delta$.*

For the second part of the paper, we introduce two definitions, the *randomness deficiency* of one state Z with respect to another state Y, $\mathbf{d}(Z|Y)$ and the amount of *algorithmic information* between two stats, $\mathbf{I}(Z : Y)$. We show conservation of randomness with respect an elementary unitary transform U, with

$$\mathbf{d}(Z \circ U | Y \circ U^*) <^+ \mathbf{d}(Z|Y).$$

We also prove information non-growth with respect an elementary unitary transform, with,

$$\mathbf{I}(Z \circ U : Y) <^+ \mathbf{I}(Z : Y).$$

These results mirror established conservation inequalities over systems of finite qubits with respect to elementary quantum operations. In this finite case, these conservation inequalities led to many interesting characterizations of finite dimensional Hilbert spaces in which the quantum states reside. The hope is that this same program of defining and leveraging conservation inequalities can be applied to infinite quantum systems.

2 Algebras

Before we introduce NS random sequences, we revisit the notion of C^* algebras and functional states. A C^* algebra, $\mathcal{M}$, is a Banach algebra and a function $* : \mathcal{M} \to \mathcal{M}$ such that

1. For every $x \in \mathcal{M}$, $x^{**} = x$.
2. For every $x, y \in \mathcal{M}$, $(x + y)^* = x^* + y^*$ and $(xy)^* = y^* x^*$.
3. For every $\lambda \in \mathbb{C}$ and $x \in \mathcal{M}$, $(\lambda x)^* = \overline{\lambda} x^*$.
4. For all $x \in \mathcal{M}$, $\|x^* x\| = \|x\| \|x^*\|$.

A $^*-$algebra does not necessarily have property (4). A C^* algebra $\mathcal{M}$ is unital if it admits a multiplicative identity $\mathbf{1}$. Set of positive elements of $\mathcal{M}$ represent the quantum observables of the system. A state over unital $\mathcal{M}$ is a positive linear functional $Z : \mathcal{M} \to \mathbb{C}$ such that $Z(\mathbf{1}) = 1$. Since states are positive, they are continuous. States are used to define NS random sequences. The set of states of $\mathcal{M}$ is denoted by $S(\mathcal{M})$. A state is tracial if $Z(x^* x) = Z(x x^*)$, for all $x \in \mathcal{M}$. An element A of a C^* algebra is unitary if $A^* A = AA^* = \mathbf{1}$. A $^*-$homomorphism between two C^*-algebras $\mathcal{M}$ and $\mathcal{N}$ is a mapping $\pi : \mathcal{M} \to \mathcal{N}$ such that

- $\pi(\alpha A + \beta B) = \alpha \pi(A) + \beta \pi(B)$,

- $\pi(AB) = \pi(A)\pi(B)$,
- $\pi(A^*) = \pi(A)^*$.

By definition, $*-$homomorphisms π are contractive, with $\|A\| \geq \|\pi(A)\|$. an $^*-$automorphism is a $^*-$homomorphism from a C^* algebra to itself. An inner $^*-$automorphism τ over a C^* algebra M is of the form $\tau(A) = UAU^*$ for some unitary element $U \in M$.

3 Direct Limits

A *direct system* of C^*-algebras is a family (A_i, π_i^j) of C^*-algebras and also $^*-$homomorphisms $\pi_i^j : A_i \to A_j$ for all $j \geq i$ indexed by a directed set (partially ordered and every finite subset has a majorant), with

- π_i^i is the identity map
- $\pi_j^k \circ \pi_i^j = \pi_i^k$ for all $i \leq j \leq k$.

Consider the product $^*-$algebra $\prod A_i$ and let A' be the $^*-$subalgebra of all elements $a = \{a_i\}$ such that there is an index i_0 with $\pi_i^j(a_i) = a_j$ for all $i_0 \leq i \leq j$. Since each π_i^j is norm-decreasing, the net $\{\|a_i\|\}$ is convergent and we define $\|a\| = \lim \|a_i\|$. If N is the kernel of $\|\cdot\|$ then the Banach space completion of the quotient A'/N is a C^*-algebra. This C^* algebra is called the *direct limit* of (A_i, π_i^j).

Suppose A is the direct limit of direct system (A_i, π_i^j). For each element $a \in A_i$, we associate the sequence $a' = \{a'_j\}$ in A' where $a'_j = 0$ for $j < i$ and $a'_j = \pi_i^j(a)$ for $j \geq i$. The map $\pi_i : A_i \to A$ is defined by taking a to the image a', which is a $^*-$homomorphism from A_i to A.

4 CAR Algebras

In this chapter, we restrict our attention to the special case where A_n are $2^n \times 2^n$ complex matrices and the directed set of indices is equal to $\mathbb{N}$. Furthermore, π_n^{n+1} is the map

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

This direct limit is known as a CAR algebra. We denote A_k by $\mathcal{M}_k$ and A by $\mathcal{M}_\infty$. The $^*-$homomorphisms from $\mathcal{M}_k$ to $\mathcal{M}_\infty$ is denoted by $\pi_k : \mathcal{M}_k \to \mathcal{M}_\infty$. Each state $\rho \in S(\mathcal{M}_k)$, can be identified with a density matrix S such that $\rho(X) = \mathrm{Tr}SX$, for all $X \in \mathcal{M}_k$. States that cannot be represented as the convex combination of other states are called pure states. Otherwise they are called mixed states. The tracial state $\tau_n \in S(\mathcal{M}_n)$ corresponds to the matrix $2^{-n}I_{2^n}$. A state $Z \in S(\mathcal{M}_\infty)$ over $\mathcal{M}_\infty$ can be identified with a sequence of density matrices $\{\rho_n\}$ that are coherent under partial traces, with $\mathrm{Tr}_{\mathcal{M}_{n+1}}\rho_{n+1} = \rho_n$.

We use $Z\lceil n$ to denote the restriction of state Z to the algebra $\mathcal{M}_n$. There is a unique tracial state $\tau \in S(\mathcal{M}_\infty)$, where $\tau\lceil n = \tau_n$. A projection $p \in \mathcal{M}_\infty$ is a self adjoint positive element such that $p = p^2$. A special projection $p \in \mathcal{M}_n$ is a projection represented by an elementary matrix. An elementary matrix has entries that are algebraic, that is roots of polynomials with rational coefficients. Elementary matrices can be associated with numbers and all the standard algorithms of linear algebra can be performed on such matrices.

5 NS Randomness

An NS Σ_1^0 set is a computable sequence of special projections $\{p_i\}$ in $\mathcal{M}_\infty$ with $p_i \le p_{i+1}$ over all i. Since $0 \le p_i \le p_{i+1} \le \mathbf{1}$, $\lim_{i\to\infty} p_i$ exists and is a member of $\mathcal{M}_\infty$. And since states are continuous, for state ρ and NS Σ_1^0 set G, $\rho(G) = \rho(\lim_i p_i) = \sup_i \rho(p_i)$.

We define NS tests. But initially, we will provide the definition for the classical Martin Löf random sequence, to provide a point of reference. A classical Martin Löf test, is a sequence $\{U_n\}$ of uniformly Σ_1^0 sets of infinite sequences $U_n \subseteq \{0,1\}^\infty$ such that $\mu(U_n) \le 2^{-n}$. An infinite sequence $\alpha \in \{0,1\}^\infty$ is Martin-Löf random if there is no Martin Löf test $\{U_n\}$ such that $\alpha \in \bigcap_n U_n$. There is a universal Martin Löf test $\{V_n\}$ such that if $\alpha \notin \bigcap_n V_n$, then α is random.

Mirroring the classical case, a NS test is an effective sequence of NS Σ_1^0 sets $\langle G^r \rangle$ such that $\tau(G^r) \le 2^{-r}$. Unlike a classical test, which can either pass or fail a sequence, a NS test can pass a quantum state up to a particular order. For $\delta \in [0,1]$, state $Z \in S(\mathcal{M}_\infty)$ fails test $\langle G^r \rangle$ at order δ if $Z(G^r) > \delta$ for all r. Otherwise Z passes the test at order δ. We says Z passes a NS test if it passes it at all orders $\delta \in [0,1]$. A state is NS random at order δ if it passes every NS test at order δ. A state is NS random if it NS random at every order $\delta > 0$.

Theorem 1 ([NS19])**.** *There exists a universal NS test $\langle R^n \rangle$, where for each NS test $\langle G^k \rangle$ and each state Z and for each n there exists a k such that $Z(R^n) \ge Z(G^k)$.*

6 Interactions

An interaction Φ is a function from $\mathbb{N}$ to Hermitian elements of $\mathcal{M}_n$ such that $\Phi(n) \in \mathcal{M}_n$. Each $\Phi(n)$ represents the energy of interaction of the set of the first n particles. In the spin system the particles are considered to be fixed at the lattice sites hence the total energy of interaction in a subset consists of the interaction energy of all subsystems. This total energy is defined to be the Hamiltonian $H_\Phi(n)$, with

$$H_\Phi(n) = \sum_{m \le n} \Phi(m).$$

Note that $\Phi(n) \in \mathcal{M}_n$ for each n, so using the homomorphism $\pi_n^m : \mathcal{M}_n \to \mathcal{M}_m$, the above term is technically $\sum_{m \leq n} \pi_m^n(\Phi(m))$. An interaction is *admissible* if $\sum_{n \geq 1} \Phi(n) \in \mathcal{M}_\infty$ and we define $\overline{\Phi} \in \mathcal{M}_\infty$ to be the element equal to this sum. An interaction Φ is *computable* if each $\Phi(n)$ is a Hermitian matrix uniformly computable in n. By the Heisenberg picture, the dynamical evolution of any system can be defined in terms of the evolution of the associated observables. For a finite spin system confined to $\mathcal{M}_n$, with interaction Φ and Hamiltonian H_Φ, this evolution is given by the Heisenberg relations $\tau_t^n : \mathcal{M}_n \to \mathcal{M}_n$, with the one parameter $^*-$automorphism group

$$\tau_t^n(A) = e^{itH_\Phi(n)} A e^{-itH_\Phi(n)}.$$

Note that since $H_\Phi(n)$ is Hermitian, $e^{itH_\Phi(n)}$ is unitary.

Proposition 1. *If* $\lim_{n\to\infty} \|A - A_n\| = 0$ *then* $\lim_{n\to\infty} \|e^A - e^{A_n}\| = 0$.

Proof. First we show that $\lim_{n\to\infty} \left\|e^A - e^{A_n}\right\| < \infty$. This term is not more than $\left\|e^A\right\| + \lim_{n\to\infty} \left\|e^{A_n}\right\| \leq \left\|e^A\right\| + e^{\lim_{n\to\infty} \|A_n\|} < \infty$. So,

$$\lim_{n\to\infty} \left\|e^A - e^{A_n}\right\| \leq \lim_{n\to\infty} \sum_{k\geq 0} \frac{\left\|A^k - A_n^k\right\|}{k!} = \sum_{k\geq 0} \lim_{n\to\infty} \frac{\left\|A^k - A_n^k\right\|}{k!} = 0.$$

□

The following theorem is a reworking of Theorem 6.2.6 in [BR79] to the special case of CAR algebras.

Theorem 2. *If Φ is an admissible interaction, then there is a strongly continuous one-parameter group of inner $^*-$automorphisms τ of $\mathcal{M}_\infty$ with*

$$\lim_{n\to\infty} \|\tau_t(A) - \tau_t^n(A)\| = 0,$$

for all $A \in \mathcal{M}_\infty$ and $t \in \mathbb{R}$, where $\tau_t(A) = e^{it\overline{\Phi}} A e^{-it\overline{\Phi}}$ and $\tau_t^n(A) = e^{itH_\Phi(n)} A e^{-itH_\Phi(n)}$.

Proof. We have that $\lim_{n\to\infty} \|\overline{\Phi} - \pi_n(H_\Phi(n))\| = 0$. So due to Proposition 1, for all $t > 0$,

$$\lim_{n\to\infty} \left\|e^{it\overline{\Phi}} - e^{it\pi_n(H_\Phi(n))}\right\| = 0.$$

This implies that for all $t > 0$ and $A \in \mathcal{M}_\infty$,

$$\lim_{n\to\infty} \left\|e^{it\overline{\Phi}} A e^{-it\overline{\Phi}} - \tau_t^n(A)\right\| = 0.$$

So, it is readily apparent that $\tau_t : A \mapsto e^{it\overline{\Phi}} A e^{-itH\overline{\Phi}}$ is a one-parameter group of inner $^*-$automorphisms satisfying the theorem. □

7 Conservation of NS Randomness

Proposition 2. *If Z is a state and τ is a $^*-$automorphism, then $Z \circ \tau$ is also a state.*

Proof. $Z \circ \tau$ is linear because $Z(\tau(\alpha A + \beta B)) = Z(\alpha\tau(A) + \beta\tau(B)) = \alpha Z(\tau(A)) + \beta Z(\tau(B))$. Further $Z \circ \tau$ sends positive elements to positive elements because $Z(\tau(AA^*)) = Z(\tau(A)\tau(A^*)) = Z(\tau(A)\tau(A)^*) \geq 0$. Furthermore $Z(\tau(1)) = Z(1) = 1$. □

Lemma 1. *For computable interaction Φ, one can compute, uniformly in computable $t \in \mathbb{R}$, a projection q, and $m \in \mathbb{N}$, a sequence of elementary unitary transformations $\{U_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} U_k = e^{itH_\Phi(m)}$ and for all density matrices ρ, $\mathrm{Tr}\rho e^{itH_\Phi(m)} U_k q U_k^* e^{-itH_\Phi(m)} > \mathrm{Tr}\rho q - 2^{-k}$.*

Proof. We use the fact that for matrix X, $e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}$. So if X is computable, then so is e^X. The set of elementary unitary matrices is dense in $SU(2^n)$ so given a computable Φ, one can compute a series of elementary matrices $\{U_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} U_k = e^{itH_\phi(m)}$. So for any given elementary density matrix σ and $\epsilon > 0$, one can compute an number $k(\sigma, \epsilon)$ such that $\mathrm{Tr}\sigma e^{itH_\Phi(m)} U_{k(\sigma,\epsilon)} q U^*_{k(\sigma,\epsilon)} e^{-itH_\Phi(m)} > \mathrm{Tr}\sigma q - \epsilon$. Given $\epsilon > 0$, one can construct a finite set of elementary density matrices D_ϵ such that for all density matrices ρ, there is a density matrix $\sigma \in D_\epsilon$, where $\mathrm{Tr}|\rho - \sigma| < \epsilon$. Let $\{V_j\}$ be uniformly computable enumerable set of elementary density matrices, where $V_j = U_{\max\{k(\sigma, 2^{-j-2}) : \sigma \in D_{2^{-j-2}}\}}$. So for all density matrices ρ, there is a elementary matrix $\rho_0 \in D_{2^{-j-2}}$ so

$$\begin{aligned}
&\mathrm{Tr}\rho e^{itH_\Phi(m)} V_k q V_k^* e^{-itH_\Phi(m)}\\
>&\mathrm{Tr}\sigma e^{itH_\Phi(m)} V_k q V_k^* e^{-itH_\Phi(m)} - 2^{-k-2}\\
>&\mathrm{Tr}\sigma q - 2^{-k-1}\\
>&\mathrm{Tr}\rho q - 2^{-k}.
\end{aligned}$$

□

Theorem 3. *Given admissible computable interaction Φ, with corresponding dynamics τ_t, and computable $t \in \mathbb{R}^+$, if state Z is not NS random order δ, then for all $0 < \delta' < \delta$, $Z \circ \tau_t$ is not NS random at order δ'.*

Proof. Let $\langle G^r \rangle$, $G^r = \langle p_k^r \rangle$ be the NS test that Z fails with at order δ. So $Z(G^r) > \delta$, for all r. So for all r there is a large enough k such that $\mathrm{Tr}Z\lceil n_k^r p_k^r > \delta$. Fix any computable ϵ, such that $0 < \epsilon < \delta$. We show that for each $n \in \mathbb{N}$, $Z \circ \tau_t^n$ fails at order δ'. For each k, r, and n using Lemma 1, we compute elementary unitary matrix $U_{n,r,k}$ such that for all density matrices ρ, $\mathrm{Tr}\rho e^{itH_\Phi(n)} U_{n,r,k} p_k^r U^*_{n,r,k} e^{-itH_\Phi(n)} > \mathrm{Tr}\rho p_k^r - \epsilon$. Define the test $\langle G^r(n) \rangle$ with

$G^r(n) = \langle p(n)^r_k \rangle$, where $p(n)^r_k = U_{n,r,k} p^r_k U^*_{n,r,k}$. So for all r and large enough k,

$$\begin{aligned}
&Z(\tau^n(G^r(n)))\\
\geq &Z(\tau^n(p(n)^r_k))\\
\geq &\mathrm{Tr} Z_{n_k} e^{itH_\Phi(n)} U_{n,r,k} p^r_k U^*_{n,r,k} e^{-itH_\Phi(n)}\\
\geq &\mathrm{Tr} Z_{n_k} p^r_k - \epsilon\\
> &\delta - \epsilon,
\end{aligned}$$

So for each $n \in \mathbb{N}$, $Z \circ \tau^n_t$ fails an NS test at order δ', for all $\delta' < \delta$. Denote $\langle W^r \rangle$, $W^r = \langle q^r_k \rangle$ to be the NS test constructed in the following fashion, with $q^r_k = \bigvee_{j \leq k} p(j)^{r+j}_k$. So $Z(\tau^n_t(W^r)) > \epsilon$ for all $n, r \in \mathbb{N}$. Due to Theorem 2, for all $r \in \mathbb{N}$, there exists a one parameter $^*-$automorphism group τ_t, where $\lim_{n\to\infty} \|\tau_t(\Psi^r) - \tau^n_t(\Psi^r)\| = 0$. Since Z is continuous, for all $r \in \mathbb{N}$ $Z(\tau_t(W^r)) = \lim_{n\to\infty} Z(\tau^n_t(\Psi^r)) > \epsilon$. So $Z \circ \tau_t$ fails NS test $\langle W^r \rangle$ at any order $\delta' < \delta$. □

Lemma 2. *For computable interaction Φ, one can compute, uniformly in computable $t \in \mathbb{R}$, a projection q, and $m \in \mathbb{N}$, a sequence of elementary unitary transformations $\{U_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} U_n = e^{itH_\Phi(m)}$ and for all density matrices ρ, $\mathrm{Tr}\rho U_n q U^*_n > \mathrm{Tr}\rho e^{itH_\Phi(m)} q e^{-itH_\Phi(m)} - 2^{-n}$.*

Proof. The proof is the same as the one for Lemma 1. □

Theorem 4. *Given admissible computable interaction Φ, with corresponding dynamics τ_t, and computable $t \in \mathbb{R}^+$, if state Z is NS random at order δ, then for all $\delta < \delta' < 1$, $Z \circ \tau_t$ is NS random at order δ'.*

Proof. Fix computable $\delta' > \delta$. It must be for all $n \in \mathbb{N}$, $Z \circ \tau^n_t$ passes all NS tests at order δ'. Assume not. Then there exists an NS test $\langle G^r \rangle$, $G^r = \langle p^r_k \rangle$ and a n such that for all r, $Z(\tau^n_t(G^r)) > \delta'$. For each k and r, using Lemma 2, we compute elementary unitary matrix $U_{k,r}$ such that for all density matrices ρ, $\mathrm{Tr}\rho U_{k,r} p^r_k U_{k,r} > \mathrm{Tr}\rho e^{itH(n)} p^r_k e^{-itH(n)} - \epsilon$. The computable term ϵ is chosen later. Define test $\langle H^r \rangle$, $H^r = \langle q^r_k \rangle$, where $q^r_k = U_{k,r} p^r_k U^*_{k,r}$. So for all r and large enough k,

$$Z(H^r) \geq \mathrm{Tr} Z_{n_k} U_{k,r} p^r_k U^*_{k,r} > \mathrm{Tr} Z_{n_k} e^{itH_\Phi(n)} p^r_k e^{-itH_\Phi(n)} - \epsilon > \delta' - \epsilon > \delta,$$

for proper choice of ϵ, causing a contradiction. So for the universal NS test $\langle W^r \rangle$, for all $n \in \mathbb{N}$, each state $Z \circ \tau^n_t$ passes $\langle W^r \rangle$ at order δ'. Due to Theorem 2, for all $r \in \mathbb{N}$, $\lim_{n\to\infty} \|\tau_t(W^r) - \tau^n_t(W^r)\| = 0$. Since Z is continuous, for all $r \in \mathbb{N}$, $Z(\tau_t(W^r)) = \lim_{n\to\infty} Z(\tau^n_t(W^r)) < \delta'$. So $Z \circ \tau_t$ passes every NS test at order $\delta' > \delta$ and thus it is NS random at each order δ'. □

Corollary 1. *Given admissible computable interaction Φ and computable $t \in \mathbb{R}^+$, computable state Z is NS random at order δ iff $Z \circ \tau_t$ is NS random at order δ.*

Corollary 2. *Let $\{U_n(t)\}_{n=1}^{\infty}$ be a sequence of strongly continuous unitary groups, where $U_n(t) \in \mathcal{M}_n$ is uniformly computable in $n \in \mathbb{N}$ and $t \in \mathbb{R}$. Assume V is a strongly continuous unitary group where for all t, $V(t) \in \mathcal{M}_\infty$ and $\lim_{n\to\infty} U_n(t) = V(t)$. Then for one parameter $^*-$automorphism group $\chi_t(A) = V(t)AV(t)^*$, $A \in \mathcal{M}_\infty$, computable $t \in \mathbb{R}$, we have*

1. *If state Z is NS random at order δ then $Z \circ \chi_t$ is NS random at order δ', for all $\delta' < \delta$.*
2. *If state Z is not NS random order δ then $Z \circ \chi_t$ is not NS random at order δ', for all $\delta' > \delta$.*

8 Conservation Inequalities

In this section, we show conservation inequalities related to infinite quantum spin states. We define the notion of the *randomness deficiency* of one state with respect to another state. This definition is invariant to changes to a finite number of qubits. Also in this section, we define *information* between two quantum infinite states. We prove that changing a finite number of bits will not affect the information measure. It is still an open question whether there is information non-growth laws respect to dynamics derived from arbitrary admissible computable interactions, but I suspect it is not true.

A semi-density matrix is a positive matrix of trace less than or equal to 1. A semi-state Y is a sequence of semi-density matrices $\{Y_n\}_{n=[Y]}^{\infty}$, $[Y] \in \mathbb{N}$, where $Y_n \geq \mathrm{Tr}_{n+1} Y_{n+1}$. Given a unitary matrix $U \in \mathcal{M}_m$, we define $Y \circ U = \{(U \otimes I_{2^n-2^m})Y_n(U \otimes I_{2^n-2^m})^*\}_{n=[Y\circ U]}^{\infty}$, where $[Y \circ U] = \max\{m, [Y]\}$. For semi-state Y, a Y-test $s = \{s_n\}_{n=[s]}^{\infty}$, $[s] \in \mathbb{N}$, is a increasing sequence of elementary positive elements, where $s_n \in \mathcal{M}_n$, $s_n \leq s_{n+1}$, and $\mathrm{Tr} Y_n s_n \leq 1$ for all $n \geq \max\{[Y], [s]\}$. We say $Y(s) = \sup_{n\geq \max\{[Y],[s]\}} \mathrm{Tr} Y_n s_n$. So if $Y(s) \leq 1$, s is a Y test, and $s \in \mathcal{S}_Y$. For unitary transform $U \in \mathcal{M}_m$, $s \circ U = \{(U \otimes I_{2^n-2^m})s(U \otimes I_{2^n-2^m})^*\}_{n=[s\circ U]}^{\infty}$, where $[s \circ U] = \max\{[s], m\}$. The algorithmic probability of a test s is equal to $\mathbf{m}(s) = \sum\{2^{-\|p\|} : M(p,n) = \langle s_n\rangle, n \in \mathbb{N}\}$, where M be a universal prefix-free Turing machine.

Definition 1 (Randomness Deficiency)**.** *The randomness deficiency of state Z with respect to semi-state Y is $\mathbf{d}(Z|Y) = \log \sum_{s\in\mathcal{S}_Y} \mathbf{m}(s)Z(s)$.*

Theorem 5 (Conservation of Randomness)**.** *Relativized to elementary unitary matrix $U \in \mathcal{M}_m$, for state Z, semi-state Y, $\mathbf{d}(Z \circ U^*|Y \circ U) <^+ \mathbf{d}(Z|Y)$.*

Proof. If $s \in \mathcal{S}_{Y\circ U}$ then $s \circ U^* \in S_Y$. This is because

$$\begin{aligned}
1 \geq& Y \circ U(s) \\
\geq& \sup_{n\geq\max\{[Y],m,[s]\}} \mathrm{Tr}(U \otimes I_{2^n-2^m})Y(U \otimes I_{2^n-2^m})^* s_n \\
=& \sup_{n\geq\max\{[Y],m,[s]\}} \mathrm{Tr} Y(U \otimes I_{2^n-2^m})^* s_n (U \otimes I_{2^n-2^m}) \\
=& Y(s \circ U^*).
\end{aligned}$$

So

$$\begin{aligned}
&\mathbf{d}(Z \circ U^*|Y \circ U)\\
=&\log \sup_{s \in \mathcal{S}_{Y \circ U}} \mathbf{m}(s) Z(s \circ U^*)\\
<^+&\log \sup_{s \in \mathcal{S}_{Y \circ U}} \mathbf{m}(s \circ U^*) Z(s \circ U^*)\\
<^+&\log \sup_{s \in \mathcal{S}_{Y}} \mathbf{m}(s) Z(s)\\
=^+&\mathbf{d}(Z|Y).
\end{aligned}$$

□

A semi-state $Y = \{Y_n\}_{n=[Y]}^{\infty}$ is lower computable if each $Y_n \in \mathcal{M}_n$ is lower computable, uniformly in n. The algorithmic probability of a lower computable test Y is $\mathbf{m}(Y) = \sum\{2^{-\|p\|} : M(p,n) \text{ lower computes } Y_n\}$. Using standard techniques in algorithmic information theory, one can prove there exists a universal lower computable semi-state $\mathbf{Y}$ such that for every lower computable semi-state Y, $\mathbf{Y} \overset{*}{>} \mathbf{m}(Y)Y$.

Given two semi-states X and Y, a (X,Y) test is a pair (s,r) where s is an X-test and r is a Y test and $[s]=[r]$. We say $(s,r) \in \mathcal{S}_{(X,Y)}$. The algorithmic probability of such a pair is $\mathbf{m}((s,r)) = \sum\{2^{-\|p\|} : M(p,0,n) = \langle s_n\rangle, M(p,1,n) = \langle r_n\rangle\}$.

Definition 2 (Information)**.**
For states Z, X, $\mathbf{I}(Z:X) = \log \sum_{(s,r)\in\mathcal{S}_{(\mathbf{Y},\mathbf{Y})}} \mathbf{m}((s,r))(Z(s)+X(r))$.

Theorem 6 (Conservation of Information)**.** *Relativized to elementary unitary matrix $U \in \mathcal{M}_n$, states Z and X, $\mathbf{I}(Z \circ U : X) <^+ \mathbf{I}(Z:X)$.*

Proof. For semi states A, B, C, and D, let $\mathbf{d}(A,B|C,D) = \log \sum_{(s,r)\in\mathcal{S}_{(C,D)}} A(s) + B(t)$. Using the same reasoning as in Theorem 5, one can show

$$\mathbf{d}(A \circ U, B|C \circ U^*, D) <^+ \mathbf{d}(A,B|C,D).$$

Thus

$$\mathbf{d}(Z \circ U, X|\mathbf{Y} \circ U^*, \mathbf{Y}) <^+ \mathbf{d}(Z,X|\mathbf{Y},\mathbf{Y}).$$

Due to the definition of $\mathbf{Y}$, we have that $\mathbf{Y} \overset{*}{>} \mathbf{Y} \circ U^*$ which implies $\mathbf{d}(Z \circ U, X|\mathbf{Y},\mathbf{Y}) <^+ \mathbf{d}(Z \circ U, X|\mathbf{Y} \circ U^*, \mathbf{Y})$. So

$$\begin{aligned}
&\mathbf{I}(Z \circ U : X)\\
=&\mathbf{d}(Z \circ U, X|\mathbf{Y},\mathbf{Y})\\
<^+&\mathbf{d}(Z \circ U, X|\mathbf{Y} \circ U^*, \mathbf{Y})\\
<^+&\mathbf{d}(Z,X|\mathbf{Y},\mathbf{Y})\\
=^+&\mathbf{I}(Z:X).
\end{aligned}$$

□

References

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