AIT Blog

Conservation Inequalities over Quantum Operations

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December 27, 2022

In [Eps19], algorithmic notions of randomness and information between two quantum mixed states were introduced. These notions were shown to satisfy conservation inequalities with respect to unitary transforms and partial traces. This blog entry generalizes these results by proving conservation of randomness and information with respect to quantum operations. Quantum operations model not only reversible unitary transforms of isolated systems, but also transient interactions with the environment and the effects of measurements. Thus, quantum operations are the most general physically realizable transform that can be applied to a quantum state. We show a computable quantum operation cannot increase the deficiency of randomness of one state with respect to another. Similarly, a quantum operation cannot increase the algorithmic mutual information shared between two states.

1 Conventions

We use $\mathcal{H}_n$ to denote a Hilbert space with n dimensions, spanned by bases $|\beta_1\rangle, \ldots, |\beta_n\rangle$. A qubit is a unit vector in the Hilbert space $\mathcal{G} = \mathcal{H}_2$, spanned by vectors $|0\rangle$, $|1\rangle$. To model n qubits, we use a unit vector in $\mathcal{H}_{2^n}$, spanned by basis vectors $|x\rangle$, where x is a string of size n.

A pure quantum state $|\psi\rangle$ of length n is a unit vector in $\mathcal{H}_{2^n}$. Its corresponding element in the dual space is denoted by $\langle\phi|$. The conjugate transpose of a a matrix A is A^*. The tensor product of two matrices A and B is $A \otimes B$. Tr is used to denote the trace of a matrix, and for Hilbert space $\mathcal{H}_X \otimes \mathcal{Y}$, the partial trace with respect to Y is Tr_Y.

For positive semi-definite matrices A and B, we say $B \preceq A$, iff $A - B$ is positive semi-definite. For functions f whose range are Hermitian matrices, we use $\stackrel{*}{<} f$ and $\stackrel{*}{>} f$ to denote $\preceq f/O(1)$ and $\succeq f/O(1)$. We use $\stackrel{*}{=} f$ to denote $\stackrel{*}{<} f$ and $\stackrel{*}{>} f$.

Density matrices are used to represent mixed states, and are self-adjoint, positive definite matrices with trace equal to 1. Semi-density matrices are used in this paper, and they are density matrices except they may have a trace in [0,1].

Pure and mixed quantum states are elementary if their values are complex numbers with rational coefficients, and thus they can be represented with finite strings. Thus elementary quantum states $|\phi\rangle$ and ρ can be enncoded as strings, $\langle|\phi\rangle\rangle$ and $\langle\rho\rangle$, and assigned Kolmogorov complexities $\mathbf{K}(|\phi\rangle)$, $\mathbf{K}(\rho)$ and algorithmic probabilities $\mathbf{m}(|\phi\rangle)$ and $\mathbf{m}(\rho)$. They are equal to the complexity (and algorithmic probability) of the strings that encodes the states.

More generally, a complex matrix A is elementary if its entries are complex numbers with rational coefficients and can be encoded as $\langle A\rangle$, and has a Kolmogorov complexity $\mathbf{K}(A)$ and algorithmic probability $\mathbf{m}(A)$.

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In [GÓ1], a universal lower computable semi-density matrix, $\boldsymbol{\mu}$ was introduced. It is the quantum analogy to $\mathbf{m}$. It can be defined (up to a multiplicative constant) by

$$\boldsymbol{\mu}_{/x} = \sum_{\text{elementary } |\phi\rangle} \mathbf{m}(|\phi\rangle\, |x, n)\, |\phi\rangle \langle\phi|\,,$$

where the summation is over all n qubit elementary pure quantum states. We use $\boldsymbol{\mu}$ to denote $\boldsymbol{\mu}_{/\emptyset}$.

A matrix is computable if its entries can be computed to any degree of precision. We say a semi-density matrix ρ is lower computable if there a program $p \in \{0,1\}^*$ such that when given to the universal Turing machine U, outputs, with or without halting, a finite or infinite sequence of elementary matrices ρ_i such that $\rho_i \preceq \rho_{i+1}$ and $\lim_{i\to\infty} \rho_i = \rho$. If U reads $\leq \|p\|$ bits on the input tape, then we say p lower computes ρ. From [GÓ1] Theorem 2, if q lower computes ρ, when $\mathbf{m}(q|n)\rho \overset{*}{<} \boldsymbol{\mu}$.

2 Quantum Operations

A map transforming a quantum state σ to $\varepsilon(\sigma)$ is a quantum operation if it satisfies the following three requirements

1. The map of ε is positive and trace preserving, with $\mathrm{Tr}(\sigma) = \mathrm{Tr}(\varepsilon(\sigma))$.
2. The map is linear with $\varepsilon(\sum_i p_i\sigma_i) = \sum_i p_i\varepsilon(\sigma_i)$.
3. The map is completely positive, were any map of the form $\varepsilon \otimes \mathbf{1}$ acting on the extended Hilbert space is also positive.

The operator $\mathbf{1}$ is the identity matrix. Another means to describe quantum operations is through a series of operators. A quantum operation ε on mixed state σ_A can be seen as the appending of an ancillia state σ_b, applying a unitary transform U, then tracing out the ancillia system with

$$\varepsilon(\sigma_A) = \mathrm{Tr}_B \left(U(\sigma_A \otimes \sigma_B)U^*\right). \tag{1}$$

A third way to characterize a quantum operation is through Kraus operators, which can be derived using an algebraic reworking of Equation 1. Map ε is a quantum operation iff it can be represented (not necessarily uniquely) using a set of matrices $\{M_i\}$ where $\varepsilon(\sigma) = \sum_i M_i \varepsilon M_i^*$ and $\sum_i M_i^* M_i \leq \mathbf{1}$.

A quantum operation ε is computable if it admits a represented of the form in Equation 1 where B, U, and σ_B are each computable, in that they each can be computable to arbtirary precision with a program. Each computable quantum operation admits an computable Kraus operator representation $\{M_i\}$, in that each M_i is an computable matrix.

3 Conservation of Randomness and Information

In [GÓ1], the deficiency of randomness of a mixed state σ with respect to computable mixed state ρ was introduced. A positive semi-definite matrix ν is a ρ-test if it is lower computable and $\mathrm{Tr}\rho\nu \leq 1$. Since ρ is computable, the set of ρ-tests, $\{\nu_i\}$, is enumerable. Thus the deficiency of randomness of σ with respect to ρ was defined to be $\mathrm{Tr}\sigma \sum_i \mathbf{m}(i)\nu_i$. Like the classical variant, this measured the level of typicality of σ with respect to ρ.

In [Eps19], the deficiency of a randomness of a mixed state σ with respect to an arbitrary (not necessarily computable) matrix ρ was introduced. Like [Ǵ01], ν is a ρ-test, $\nu \in \mathcal{T}_\rho$, if it is positive semi-definite and lower computable and $\mathrm{Tr}\rho\nu \leq 1$. The lower probability of a lower computable mixed state was defined, with $\underline{\mathbf{m}}(\nu|x) = \sum\{\mathbf{m}(q|x) : q \text{ lower computes } \nu\}$. The deficiency of randomness of σ with respect to ρ is defined as follows.

Definition 1 ([Eps19])
For n qubit semi-density matrices σ and ρ, $\mathbf{d}(\sigma|\rho) = \log \mathrm{Tr}\sigma \sum_{\nu \in \mathcal{T}_\rho} \underline{\mathbf{m}}(\nu|n)\nu$.

In [Eps19], the algorithmic information of two mixed states σ and ρ was introduced, using notions of quantum tests seen in the deficiency of randomness definition. Let $\mathcal{C}_{C\otimes D}$ be the set of all lower computable matrices of the form $A \otimes B$, where $\mathrm{Tr}(A \otimes B)(C \otimes D) \leq 1$. Let $\mathfrak{C}_{C\otimes D} = \sum_{A\otimes B \in \mathcal{C}_{C\otimes D}} \underline{\mathbf{m}}(A \otimes B|n) A \otimes B$ be an aggregation of $C \otimes D$ tests of the form $A \otimes B$, weighted by their lower probability. Using $\mathfrak{C}$, we get the following definition of information.

Definition 2 ([Eps19])
For semi-density matrices σ and ρ, $\mathbf{I}(\sigma : \rho) = \log \mathrm{Tr}\mathfrak{C}_{\boldsymbol{\mu}\otimes\boldsymbol{\mu}}(\sigma \otimes \rho)$.

The following theorem shows conservation of randomness with respect to elementary quantum operations. It generalizes Theorems 2 and 3 from [Eps19].

Theorem 1 (Randomness Conservation) *Relativized to computable quantum operation ε, for semi-density matrices ρ, σ, $\mathbf{d}(\varepsilon(\rho)|\varepsilon(\sigma)) <^+ \mathbf{d}(\rho|\sigma)$.*

The following theorem shows information nongrowth with respect to elementary quantum operations. It generalizes Theorems 5 and 10 from [Eps19].

Theorem 2 (Information Conservation) *Relativized to computable quantum operation ε, for semi-density matrices ρ, σ, $\mathbf{I}(\varepsilon(\rho) : \sigma) <^+ \mathbf{I}(\rho : \sigma)$.*

References

[Eps19] S. Epstein. Algorithmic no-cloning theorem. *IEEE Transactions on Information Theory*, 65(9), 2019.

[Ǵ01] P. Gács. Quantum Algorithmic Entropy. *Journal of Physics A Mathematical General*, 34(35), 2001.