AIT Blog

A Quantum EL Theorem

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In this blog entry, we introduce a Quantum EL theorem. Non exotic projections of large rank must have simple quantum pure states in their images. Simplicity is measured according to the classical information content of a pure state. It is similar to the definition in [Vit00] except a classical Turing machine is used instead of a quantum Turing machine.

Definition 1 (Complexity of a Quantum Pure State)
For n qubit state $|\phi\rangle$, $\mathbf{K}(|\phi\rangle) = \min\{\mathbf{K}(|\psi\rangle|n) - \log|\langle\phi|\psi\rangle|^2 : |\psi\rangle$ is an elementary pure state$\}$.

Definition 2 (Computable Operators) *For computable operator A, $\mathbf{I}(A;\mathcal{H}|y) = \min\{\mathbf{K}(p|y) - \mathbf{K}(p|y,\mathcal{H}) : p$ is a program that computes $A\}$. $\mathcal{H} \in \{0,1\}^\infty$ is the halting sequence.*

Theorem 1 (Quantum EL Theorem) *Fix an n qubit Hilbert space and relativize to n. Let P be a computable projection of rank $> 2^m$. Then, $\min_{|\phi\rangle\in\mathrm{Image}(P)} \mathbf{K}(|\phi\rangle) <^{\log} 3(n-m) + \mathbf{I}(P;\mathcal{H}|n)$.*

Corollary 1 *Fix an n qubit Hilbert space and relativize to n. Let ρ be a density matrix of rank $> 2^m$. Then, $\min_{|\phi\rangle\in\mathrm{Image}(\rho)} \mathbf{K}(|\phi\rangle) <^{\log} 3(n-m) + \mathbf{I}(\rho;\mathcal{H}|n)$.*

The corollary is due to conservation of information. More specifically, if operator P is the projection onto the image of density matrix ρ, then $\mathbf{K}(P|\rho) = O(1)$ and also $\mathbf{I}(P;\mathcal{H}) <^{+} \mathbf{I}(\rho;\mathcal{H})$. Thus the theorem applies to any quantum operator. This also applies to algorithmic quantum entropy [Ǵ01] since it is less than $\mathbf{K}(|\phi\rangle)$. Another application is that there are no non-exotic quantum measurements that can detect quantum algorithmic complexity. This blog post is another example of a result in the intersection of AIT and physics. Future work in algorithmic physics involves proving that algorithmic thermodynamic entropy must oscillate in the presence of dynamics. Another avenue of future work is conservation of algorithmic randomness and information with respect to the most general transformation, quantum operations.

References

[Ǵ01] P. Gács. Quantum Algorithmic Entropy. *Journal of Physics A Mathematical General*, 34(35), 2001.

[Vit00] P Vitányi. Three Approaches to the Quantitative Definition of Information in an Individual Pure Quantum State. In *Proceedings of the 15th Annual IEEE Conference on Computational Complexity*, COCO '00, page 263. IEEE Computer Society, 2000.

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