

# AIT Blog

## Uniform Tests and Algorithmic Thermodynamic Entropy

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For definitions in this post, we use [HR09]. A computable metric space  $X$  is a metric space with a dense set of ideal points on which the distance function is computable. A computable probability is defined by a computable sequence of converging points in the corresponding space of probability measures,  $\mathcal{M}(X)$ , over  $X$ . A uniform test takes in a description of a probability measure  $\mu$  and produces a lower computable  $\mu$  test. There exists a universal uniform test, from [G21, HR09]. We extend the result from [Eps22] to computable metric spaces.

**Theorem 1** *Given computable non-atomic probability measures  $\mu$  and  $\lambda$  over a computable metric space  $X$  and universal uniform test  $\mathbf{t}(\cdot, \cdot)$ . For all  $n$ ,  $\lambda(\{\alpha : \mathbf{t}(\mu, \alpha) > 2^n\}) > 2^{-n-\mathbf{K}(n)-O(1)}$ .*

Reworking the above theorem, one can get a result in algorithmic physics. To define algorithmic fine-grain entropy, we use a slightly modified version of the definition in [Gac94]. First, note that all the results [HR09] can be easily extended to arbitrary nonnegative measures. This can be achieved by defining the product space of  $\mathcal{M}(X)$  and  $\mathbb{R}_{\geq 0}$ , where the second metric space defines the size of the measure. Given a measure  $\mu \in \mathcal{M}(X) \times \mathbb{R}_{\geq 0}$ , the algorithmic fine grained entropy of a point  $\alpha \in X$  is as follows.

**Definition 1 (Algorithmic Fine-Grained Entropy)**  $H(\alpha) = -\log \mathbf{t}(\mu, \alpha)$ .

One can then prove that this term will oscillate in the presence of dynamics. Dynamics can be defined using group theory.

**Definition 2 (Transformation Group)** *Let  $M$  denote a computable metric space and  $G$  a topological group each element of which is a homeomorphism of  $M$  onto itself:*

$$f(g; x) = g(x) = x' \in M; g \in G, x \in M.$$

*The pair  $(G, M)$  will be called a topological transformation group if for every pair of elements  $g_1, g_2$  of  $G$ , and every  $x \in M$ ,  $g_1(g_2(x)) = (g_1g_2)(x)$  and if*

$$x' = g(x) = f(g; x)$$

*is continuous simultaneously in  $x \in M$  and  $g \in G$ .*

**Theorem 2 (Oscillation of Thermodynamic Entropy)** *Let  $L$  be the Lebesgue measure over  $\mathbb{R}$ . For one dimensional topological transformation group  $(G^t, X)$  acting on computable metric space  $X$ , for all  $\alpha \in X$ ,  $L\{t \in [0, 1] : H(G^t\alpha) < \max_{\beta} H(\beta) - n\}^* > 2^{-n-\mathbf{K}(n)-O(1)}$ .*

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Its interesting to note that the proof for this theorem follows from first using a combinatorial argument about finite strings and then applying this result to prove a property of randomness deficiencies and then transferring this result to universal tests and then finally algorithmic thermodynamic entropy. An open question is whether other such transfers can occur to further characterize  $H$ .

## References

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