AIT Blog

Resolving Four Open Problems in Quantum Information Theory

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In this blog entry, I talk about the resolution of 4 open problems in Quantum Information Theory. The problems were posed as questions in [Ǵ01]. Quantum information theory studies ultimate capabilities of noisy physical systems, governed by the laws of quantum mechanics, to preserve information and correlations. Quantum Information Theory encompasses subjects as diverse as quantum computation, quantum algorithms, quantum complexity theory, quantum communication complexity, entanglement theory, quantum key distribution, quantum error correction, and even the experimental implementation of quantum protocols.

Quantum description complexity, started in [BvL01, Ǵ01], measures the algorithmic complexity of quantum states. In [BvL01], complexity was defined as the shortest program to a universal quantum Turing machine that produces the target state, which we denote as **QC**. The measure **QC** has been refered to as Quantum Kolmogorov complexity. In [Ǵ01], the complexity of a state was measured using a universal lower semicomputable mixed state, which we will call **H**. As stated in [Ǵ01], one of the possible applications of **H** is to gain new insights into von Neumann entropy. In [Ǵ01], four open problems were posed as questions.

1. Does smallness of **H** imply smallness of **QC**?
2. What is the proper quantum generalization of classical information non-growth laws?
3. What addition theorems apply to **H**?
4. Does **H** obey strong superadditivity?

In the recent literature [Eps19, Eps20] problems (1-3) are resolved, which I will briefly review. In this blog entry, I resolve the 4th open question in the negative: **H** is neither strongly superadditive or strongly subadditive. A function **L** from quantum mixed states to whole numbers is strongly subadditive if there exists a constant $c \in \mathbb{N}$ such that for all mixed states ρ_{123}, $\mathbf{L}(\rho_{123}) + \mathbf{L}(\rho_2) < \mathbf{L}(\rho_{12}) + \mathbf{L}(\rho_{23}) + c$. Similarly **L** is strongly superadditive if there exists a constant $c \in \mathbb{N}$ such that for all mixed states ρ_{123}, $\mathbf{L}(\rho_{12}) + \mathbf{L}(\rho_{23}) < \mathbf{L}(\rho_{123}) + \mathbf{L}(\rho_2) + c$. I'll also pose some new questions about the intersection of AIT and physics.

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Conventions

We use $\mathcal{H}_n$ to denote a Hilbert space with n dimensions, spanned by bases $|\beta_1\rangle, \ldots, |\beta_n\rangle$. A qubit is a unit vector in the Hilbert space $\mathcal{G} = \mathcal{H}_2$, spanned by vectors $|0\rangle$, $|1\rangle$. To model n qubits, we use a unit vector in $\mathcal{H}_{2^n}$, spanned by basis vectors $|i\rangle$, where $i \in [1..2^n]$. A pure quantum state $|\psi\rangle$ of length n is a unit vector in $\mathcal{H}_{2^n}$. Its corresponding element in the dual space is denoted by $\langle\psi|$. The conjugate transpose of a matrix A is A^*. The tensor product of two matrices A and B is $A \otimes B$. Tr is used to denote the trace of a matrix.

Let $H_A \otimes H_B$ be a Hilbert space and O be an operator acting on this composite space. Then $O = \sum_{i,j} c_{ij} M_i \otimes N_j$, where M_i and N_j are operators acting on H_A and H_B respectively. The partial trace over H_A is $\mathrm{Tr}_{H_A}(O) = \sum_{i,j} c_{ij} \mathrm{Tr}(M_i) N_j$. Furthermore $\mathrm{Tr}(I \otimes M)\rho_{12} = \mathrm{Tr} M \rho_2$. Density matrices are used to represent mixed states, and are self-adjoint, positive definite matrices with trace equal to 1. Semi-density matrices are density matrices except they may have a trace in [0,1]. The maximally mixed state is $2^{-n} I$.

Pure quantum states are elementary if their values are complex numbers with rational coefficients, and thus they can be represented with finite strings. Thus elementary quantum states $|\phi\rangle$ can be encoded as strings and assigned Kolmogorov complexities $\mathbf{K}(|\phi\rangle)$, and algorithmic probabilities $\mathbf{m}(|\phi\rangle)$. They are equal to the complexity (and algorithmic probability) of the strings that encodes the states.

In [GÓ1], a universal lower computable semi-density matrix, $\boldsymbol{\mu}$ was introduced. It can be defined (up to a multiplicative constant) by $\boldsymbol{\mu}^n = \sum_{\text{elementary } n \text{ qubit } |\phi\rangle} \mathbf{m}(|\phi\rangle / n) |\phi\rangle \langle\phi|$, where the summation is over all n qubit elementary pure quantum states, and $\mathbf{m}$ is the algorithmic probability. The quantum algorithmic entropy of an n qubit mixed state ρ is $\mathbf{H}(\rho) = \lceil -\log \mathrm{Tr} \boldsymbol{\mu}^n \rho \rceil$. This definition of algorithmic entropy generalizes the definition of $\underline{H}$ in [GÓ1] to mixed states.

Results

Problem 1. Quantum Kolmogorov complexity, defined in [BvL01], uses a universal quantum Turing machine to define the complexity of a pure quantum state. The input and output tape of this machine consists of symbols of the type 0, 1, and #. The input is an ensemble $\{p_i\}$ of pure states $|\psi_i\rangle$ of the same length n, where $p_i \geq 0$ and $\sum_i p_i = 1$. This ensemble can be represented as a mixed state of n qubits. If, during the operation of the quantum Turing machine, all computational branches halt at a time t, then the contents on the output tape are considered the output of the quantum Turing machine. The quantum Kolmogorov complexity of a pure state, $\mathbf{QC}[\epsilon](|\psi\rangle)$ is the size of the smallest (possibly mixed state) input to a universal quantum Turing machine such that fidelity between the output and $|\psi\rangle$ is at least ϵ. The fidelity between a mixed state output σ and $|\psi\rangle$ is $\langle\psi| \sigma |\psi\rangle$.

Theorem. [Eps20] $\mathbf{QC}[\langle\psi| \boldsymbol{\mu} |\psi\rangle \mathbf{H}^{-O(1)}(|\psi\rangle)](|\psi\rangle) <^{\log} \mathbf{H}(|\psi\rangle)$.

Problem 2. In AIT, Information has been shown to be conserved with respect to randomized transformations for a large number of information terms over strings or infinite sequences. This property can be shown to be true for quantum states as well. In [Eps19], an information term $\mathbf{I}$ between quantum mixed states was defined. This term is the summation of so called quantum tests, each weighted by their complexity. Using this definition, conservation was proven over unitary transform. In fact, this result can be strengthed to any computable quantum operation.

Theorem. [Eps19] *For density matrices σ and ρ, relativized to elementary unitary transform A,* $\mathbf{I}(A\sigma A^* : \rho) =^+ \mathbf{I}(\sigma : \rho)$.

Problem 3. The addition theorem for classical entropy asserts that the joint entropy for a pair of random variables is equal to the entropy of one plus the conditional entropy of the other, with $\mathcal{H}(\mathcal{X}) + \mathcal{H}(\mathcal{Y}/\mathcal{X}) = \mathcal{H}(\mathcal{X}, \mathcal{Y})$. For algorithmic entropy, the chain rule is slightly more nuanced, with $\mathbf{K}(x) + \mathbf{K}(y/x, \mathbf{K}(x)) =^+ \mathbf{K}(x, y)$. An analogous relationship cannot be true for $\mathbf{H}$ since as shown in Theorem 15 of [GÓ1], there exists elementary $|\phi\rangle$ where $\mathbf{H}(|\phi\rangle\,|\phi\rangle) - \mathbf{H}(|\phi\rangle)$ can be arbitrarily large, and $\mathbf{H}(|\phi\rangle\,/\,|\phi\rangle) =^+ 0$. However, the following theorem shows that a chain rule inequality does hold for $\mathbf{H}$.

Theorem. [Eps19] *For semi-density matrices σ, ρ, elementary ρ,* $\mathbf{H}(\rho) + \mathbf{H}(\sigma/\langle\rho, \mathbf{H}(\rho)\rangle) <^+ \mathbf{H}(\sigma \otimes \rho)$.

Problem 4. The following two theorems address the last open problem in [GÓ1]. It is still an open question as to whether $\mathbf{H}$ is subadditive, that is $\mathbf{H}(\rho_{12}) <^+ \mathbf{H}(\rho_1) + \mathbf{H}(\rho_2)$. This property holds when $\rho_{12} = \rho_1\rho_2$ but it is unknown whether this property holds for arbitrary mixed states.

Theorem. $\mathbf{H}$ *is not strongly subadditive.*

Proof. We fix the number of qubits n, and for $i \in [1..2^n]$, $|i\rangle$ is the ith basis state of the n qubit space. Let $|\psi\rangle = \sum_{i=1}^{2^n} 2^{-n/2}\,|i\rangle\,|i\rangle$. The pure state $|\psi\rangle$ is elementary, with $\mathbf{K}(|\psi\rangle\,/2n) =^+ 0$. We define the the $3n$ qubit mixed state $\rho_{123} = .5\,|\psi\rangle\langle\psi| \otimes |1\rangle\langle 1| + .5\,|1\rangle\langle 1| \otimes |\psi\rangle\langle\psi|$. $\rho_{12} = .5\,|\psi\rangle\langle\psi| + .5\,|1\rangle\langle 1| \otimes 2^{-n}I$. $\rho_{23} = .5 * 2^{-n}I \otimes |1\rangle\langle 1| + .5\,|\psi\rangle\langle\psi|$. $\rho_2 = 2^{-n}I$. $\mathbf{H}(\rho_{12}) =^+ -\log \mathrm{Tr}\boldsymbol{\mu}^{2n}\rho_{12} <^+ -\log \mathrm{Tr}\boldsymbol{\mu}^{2n}\,|\psi\rangle\langle\psi| <^+ -\log \mathbf{m}(|\psi\rangle\,/2n)|\,\langle\psi|\psi\rangle\,|^2 <^+ 0$. Similarly, $\mathbf{H}(\rho_{23}) =^+ 0$. $\mathbf{H}(\rho_2) =^+ n$. So $\mathbf{H}(\rho_{123}) + \mathbf{H}(\rho_2) >^+ n$ and $\mathbf{H}(\rho_{12}) + \mathbf{H}(\rho_{23}) =^+ 0$, proving that $\mathbf{H}$ is not strongly subadditive.

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New Questions on AIT and Physics

Currently there is relatively little overlap in the literature of Algorithmic Information Theory and physics. This combination represents new opportunities for applying properties and relationships found in AIT to the foundations of our physical laws. Here are some new questions.

1. In [NS19], the notion of Quantum Martin Löf Random States was introduced for infinite quantum spin chains. One question is, how do such infinite quantum states transform? Is Quantum Martin Löf Randomness preserved over such transformations?

2. In [Eps19], the notion of a typical quantum state was defined. Is there such a thing as a typical particle? i.e. a typical wave function? Certaintly one can define typical measurements of a wave function.

3. In [Gac94], two notions of algorithmic thermodynamic entropy were introduced. Can the Theorem 7 of my September 30th blog post be applied to prove fluctuations of thermodynamic entropy? In [Gac94], a reformulation of Maxwell's demon was achieved. Can recent results of fluctuations of randomness be used to revisit Maxwell's demon?

References

[BvL01] A. Berthiaume, W. van Dam, and S. Laplante. Quantum Kolmogorov Complexity. *Journal of Computer and System Sciences*, 63(2), 2001.

[Eps19] S. Epstein. Algorithmic no-cloning theorem. *IEEE Transactions on Information Theory*, 65(9):5925–5930, 2019.

[Eps20] Samuel Epstein. An extended coding theorem with application to quantum complexities. *Information and Computation*, 275, 2020.

[Ǵ01] P. Gács. Quantum Algorithmic Entropy. *Journal of Physics A Mathematical General*, 34(35), 2001.

[Gac94] P. Gacs. The boltzmann entropy and randomness tests. In *Proceedings Workshop on Physics and Computation. PhysComp '94*, pages 209–216, 1994.

[NS19] A. Nies and V. Scholz. Martin-Löf Random Quantum States. *Journal of Mathematical Physics*, 60(9), 2019.