The No Synchronization Law

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Abstract

    This paper looks at the intersection of algorithmic information theory and physics, namely quantum mechanics, thermodynamics, and black holes. We discuss theorems which characterize the barrier between the quantum world and the classical realm. The No Synchronization Law is detailed, which says separate and isolated physical systems evolving over time cannot have thermodynamic algorithmic entropies that are in synch. We look at future work involving the Kolmogorov complexity of black holes.

1 Introduction

The Turing machine is an abstract computational device, and its universality formulizes the study of algorithms and complexity agnostic to the machine running the programs. This definition leads to the (modern) Church-Turing thesis, which says that any computational model can be simulated in polynomial time with a probabilistic Turing machine.

However it has become apparent one cannot simply separate computation from the physical model used to perform it. This was made clear with the introduction of the quantum Turing machine (QTM) [DP85], on which the famous Shor's Factoring Algorithm [Sho94] can be run efficiently. Using a universal QTM, one can define BQP, which is the class of decision problems solvable in polynomial time by a QTM, with at most $1/3$ probability of error [BV93]. There is a large group of researchers working on what useful quantum algorithms are in BQP.

However, an overlooked area of research is the study of the algorithmic entropy level of individual (pure or mixed) quantum states. Simple states such as $|0^n\rangle$ should have $O(1)$ entropy, whereas a typical state $|\psi\rangle$ should have $n+O(1)$ entropy. As shown in [Eps24] there is a wide set of interesting implications and consequences to the study of this area.

One of the main implications of study into this area is the further characterization of the barrier between the quantum realm and the the classical world. It is easy to see (folklore) there exist no quantum algorithm that can transform a (pure) quantum state into its classical description, up to any accuracy. It has also become apparent that a majority quantum states, when measured by a fixed POVM, will produce either the empty signal or pure white noise (or a combination of the two). Furthermore, in the context of quantum decoherence, most non-pointer states will decohere into a classical probability that is pure white noise. This is a consequence of the fact that quantum states have virtually no self-information, which, as we will see, is shown with two independent information measures.

This begs the question, why is objective reality so ordered? I suspect it is due to the following reasoning. Due to conservation inequalities (Theorem 2), a quantum operation cannot increase the self-information of a (pure or mixed) quantum state. However a wave function collapse, i.e.

measurement, can cause an uptake in algorithmic information of the quantum state. This collapse enables nonnegligible signals from further measurements and also enables partial information cloning. However it is still an open problem on how to best formally say this.

Another application of algorithmic randomness to physics is to determine an objective way to measure the algorithmic content of classical information. As shown with QTMs, algorithms depend on the physical model used to run them. The question is, does the physical model used change the amount information measured in strings. Müller's Theorem [Mue07, Mul09] answers this question in the negative; quantum mechanics does not offer any benefit to compressing classical information. The quantitative amount of information in individual strings is not dependent on the physical model used to measure it.

1.1 The No Synchronization Law

Classical thermodynamics is the study of substances and changes to their properties such as volume, temperature, and pressure. Substances, such as a gas or a liquid, is modeled as a point in a phase space Ω. There is a volume measure over Ω, denoted by L. Boltzmann introduced an entropy measure of points ω in Ω using partitions $\{\Gamma_i\}$ of the phase space. The Boltzmann entropy of $\omega \in \Gamma_i$ is $\log L(\Gamma_i)$.

There are several issues with this definition. The Boltzmann entropy depends on the choice of the partition. Another digit of precision will decrease it by around $\log 10$. This issue was addressed in [Gac94], which introduced a new entropy measure $\mathbf{G}$, which is the negative logarithm of a universal lower computable L-test over Ω. This definition requires a lot of mathematical framework (see Chapter 13 of [Eps24]), as Ω can be any computable metric space and L can be any computable (not-necessarily probability) measure over Ω.

However there are many benefits to using $\mathbf{G}$ to model the algorithmic thermodynamic entropy of a state. One property of $\mathbf{G}$ is that it oscillates either under discrete ergodic dynamics or under a tranformation group. The proof of this uses the same mathematical machinery to prove that outliers must be present in all large enough set of observations.

This paper will discuss a surprising result in [Eps24] that given two separated and isolated physical systems Ω_1 and Ω_2, under mild conditions, their corresponding algorithmic entropies must be out of sync with respect to ergodic dynamics. Let L_1 and L_2 be the measures over Ω_1 and Ω_2. Let $\mathbf{G}_1$ and $\mathbf{G}_2$ be the algorithmic entropies corresponding to Ω_1 and Ω_2, respectively. Let R_1 and R_2 be discrete ergodic transformations over $\Omega_1 \ni \alpha$ and $\Omega_2 \ni \beta$, respectively.

No Synchronization Law: *Under mild conditions,* $\sup_{t \in \mathbb{N}} \left| \mathbf{G}_1(R_1^t(\alpha)) - \mathbf{G}_2(R_2^t(\beta)) \right| = \infty$.

2 The Algorithmic Content of Quantum States

In Algorithmic Information Theory there are two main definitions of the information content of individual strings $x \in \{0,1\}^*$, plain complexity $\mathbf{C}(x)$ and prefx-free complexity $\mathbf{K}(x)$ and they are logarithmically equivalent. In in my manuscript, Algorithmic Physics, [Eps24], 4 definitions are studied. By examining the common traits of these measures, one can distill algorithmic properties of quantum states.

Let $\mathcal{Q}_n$ be the Hilbert space containing all n qubit quantum states. A matrix, or quantum state, is elementary if all its coefficients are roots of polynomial equations with rational coefficients. One can perform the standard linear algebra algorithms over elementary numbers. Let $D(\sigma, \rho)$ be the trace distance. The four main definitions are as follows:

2.1 Vitányi Complexity

Vitányi complexity, [Vit01], of a pure state $|\psi\rangle$ is equal to the minimum size of a program to a universal Turing machine that outputs an approximation that is an elementary pure state $|\theta\rangle$ of the target state plus a score of their closeness.

$$\mathbf{Hv}(|\psi\rangle) = \min_{\text{Elementary } |\theta\rangle \in \mathcal{Q}_n} \mathbf{K}(|\theta\rangle |n) - \log |\langle\psi|\theta\rangle|^2.$$

2.2 Gács Complexity

Gács complexity, [Gó1], takes a different approach than Vitányi complexity. The Kolmogorov complexity of a string x is equal to, up to an additive factor, $-\log \mathbf{m}(x)$, where $\mathbf{m}$ is a universal lower-computable semi-measure. Similarly Gács complexity is defined using the following universal lower computable semi-density matrix, with

$$\boldsymbol{\mu} = \sum_{\text{Elementary } |\phi\rangle \in \mathcal{Q}_n} \mathbf{m}(|\phi\rangle |n) |\phi\rangle\langle\phi|.$$

The parameter n represents number of qubits used. The Gács entropy of a mixed state σ is defined by

$$\mathbf{Hg}(\sigma) = -\log \operatorname{Tr}\boldsymbol{\mu}\sigma.$$

2.3 Quantum Unitary Complexity

Suppose that Alice wants to send a (possibly mixed) n qubit quantum state σ to Bob. Alice has access to two channels, a quantum channel and a classical channel. Alice can choose to send $m \leq n$ qubits ρ on the quantum channel and P regular bits p on the classical channel, representing an encoding of unitary operation V, and m, where $U(p) = (V, m)$. Bob, upon receiving ρ and p, constructs the unitary operation V, and then applies it to ρ (tensored with $|0^{n-m}\rangle$) to produce $\sigma' = V(|0^{n-m}\rangle \rho \langle 0^{n-m}|)V^*$. Bob is required to produce σ exactly, however there does exist a variant where only an approximation is necessary.

A quantum unitary pair (V, m) consists of two parts, (1) an elementary unitary transform over $\mathcal{Q}_n$, and (2) the number of qbits $m \leq n$ that are connected to the quantum input. $\mathcal{C}_{n,m}$ be the set of all quantum pairs over n qbits with an input size of m. For density matrix σ, its quantum unitary complexity [Eps24] is

$$\mathbf{Huc}(\sigma) = \min\{\mathbf{K}(V, m) + m \colon (V, m) \in \mathcal{C}_{n,m}, \xi \text{ is an } m \text{ qubit mixed state}, \sigma = V(|0^{n-m}\rangle \xi \langle 0^{n-m}|)V^*\}.$$

2.4 BvL Complexity

BvL complexity is the size of the smallest program to a universal QTM $\mathfrak{U}$ that approximates the target (pure or mixed) quantum state. This definition was formulated by [BvL01] and the universal QTM $\mathfrak{U}$ used is from [Mul08, Mul09]. The universal QTM has the following property: for every QTM M and mixed state σ for which $M(\sigma)$ is defined, there is mixed state σ' such that $D(\mathfrak{U}(\sigma'), M(\sigma)) < \delta$, for every $\delta \in \mathbb{Q}_{>0}$ where $\|\sigma'\| <^+ \|\sigma\| + \mathbf{K}(M, \delta)$. The BvL complexity of quantum state σ is

$$\mathbf{Hbvl}(\rho) = \min_{\sigma} \left\{ \|\sigma\| : \forall_k, D(\mathfrak{U}(\sigma, k), \rho) < \frac{1}{k} \right\}.$$

BvL enjoys a special status over the other complexity measures. Whereas $\mathbf{Hv}$, $\mathbf{Hg}$, $\mathbf{Huc}$ can be considered to be algorithmic entropy scores, $\mathbf{Hbvl}$ can be considered to be the minimal amount of quantum mechanical resources needed to construct a quantum state.

3 Algorithmic Properties of Quantum States

Each of the measures above have slightly different properties, but by examining them collectively, one gets a general picture on algorithmic properties of quantum state. The properties examined for the quantum measures are as follows. We use $\mathbf{H}$ to denote a generic measure.

- Subadditivity. $\mathbf{H}(\sigma \otimes \rho) <^+ \mathbf{H}(\sigma) + \mathbf{H}(\rho)$.

- Monotonicity. $\mathbf{H}(\sigma) <^+ \mathbf{H}(\sigma \otimes \rho)$.

- Unitary Transform. $\mathbf{H}(U\sigma U^*) =^+ \mathbf{H}(\sigma) \pm \mathbf{K}(U)$.

- Addition Inequality. For elementary σ, $\mathbf{H}(\sigma) + \mathbf{H}(\rho|\langle\sigma\rangle, \lceil\mathbf{H}(\sigma)\rceil) <^+ \mathbf{H}(\sigma \otimes \rho)$.

- No Cloning Theorem. There exists states $|\psi\rangle^m$ such that $\mathbf{H}(|\psi\rangle^m)$ which have a much higher complexity than $\mathbf{H}(|\psi\rangle)$. More formally, $\log \binom{m+2^n-1}{m} <^+ \max_{|\psi\rangle} \mathbf{H}(|\psi\rangle^m) <^+ \log \binom{m+2^n-1}{m} + \mathbf{K}(n, m)$.

- Müller's Theorem. For $x \in \{0, 1\}^n$, $\mathbf{H}(|x\rangle)$ is equal to $\mathbf{C}(x)$, $\mathbf{K}(x)$, or $\mathbf{K}(x|n)$.

- Quantum EL Theorem. A projection P of large rank and large $\min_{|\psi\rangle \in \text{Image}(P)} \mathbf{H}(|\psi\rangle)$ will have large mutual information with the halting sequence.

- Quantum Levin/Schnorr Theorem. An infinite quantum Martin Löf sequence [NS19] will have $\mathbf{H}$ incompressible prefixes.

	Hv	Hg	Huc	Hbvl
Subadditivity	✓	✓	✓	
Monotonicity		✓	✓	
Addition Inequality		✓		
Unitary Transform	✓	✓	✓	✓
No Cloning Theorem	✓	✓	✓	✓
Müller's Theorem	✓	✓	✓	✓
Quantum EL Theorem	✓	✓	✓	
Quantum Levin/Schnorr Theorem			✓	

Figure 1: Properties of four measures of the algorithmic content of quantum states.

Figure 1 describes the different properties of the quantum complexity measures. Though they have separate properties, there is an overall picture on what it means to quantify the algorithmic content of quantum states. Notice that all measures have a version of Müller's theorem. Thus they can be seen as generalizations to the classical measures $\mathbf{C}$ and $\mathbf{K}$. The term $\mathbf{Hbvl}$ takes a special importance because it can be seen as a measure of the amount of quantum mechanical resources needed to approximate or construct a state. This fact, in my opinion, implies Müller's Theorem is the greatest accomplishment of the application of algorithmic information theory to physics.

4 Mutual Information Between Quantum States

In this section we look at two definitions on the algorithmic mutual information between quantum states and some of their implications. One approach to define information between two mixed as the deficiency of randomness over $\boldsymbol{\mu} \otimes \boldsymbol{\mu}$, with the mutual information of σ and ρ being $\mathbf{d}(\sigma \otimes \rho | \boldsymbol{\mu} \otimes \boldsymbol{\mu})$. This is a counter argument for the hypothesis that the states are independently chosen according to the universal semi-density matrix $\boldsymbol{\mu}$. This parallels the classical algorithmic case, where $\mathbf{I}(x : y) =^+$ $\mathbf{d}((x, y) | \mathbf{m} \otimes \mathbf{m}) =^+ \mathbf{K}(x) + \mathbf{K}(y) - \mathbf{K}(x, y)$. Let $\mathcal{C}_{C \otimes D}$ be the set of all lower computable matrices $A \otimes B$, such that $\mathrm{Tr}(A \otimes B)(C \otimes D) \leq 1$. lower algorithmic probability of a lower computable matrix σ is $\underline{\mathbf{m}}(\sigma | x) = \sum \{\mathbf{m}(q | x) : q \text{ lower computes } \sigma\}$. Let $\mathfrak{C}_{C \otimes D} = \sum_{A \otimes B \in \mathcal{C}_{C \otimes D}} \underline{\mathbf{m}}(A \otimes B | n) A \otimes B$.

Definition 1 *The mutual information between two quantum states σ, ρ is defined to be $\mathbf{I}_{\mathrm{d}}(\sigma : \rho) = \log \mathrm{Tr} \mathfrak{C}_{\boldsymbol{\mu} \otimes \boldsymbol{\mu}}(\sigma \otimes \rho)$.*

The second approach using classical information as a model, where $\mathbf{I}(x : y) = \mathbf{K}(x) + \mathbf{K}(y) - \mathbf{K}(x, y)$.

Definition 2 $\mathbf{I}_{\mathrm{g}}(\sigma : \rho) = \mathbf{Hg}(\sigma) + \mathbf{Hg}(\rho) - \mathbf{Hg}(\sigma \otimes \rho)$.

Though the information terms are very different, they share the following key property, in that an overwhelming majority of pure quantum states have no self-information.

Theorem 1 ([Eps24]) *Let Λ be the uniform distribution over all n qubit pure states.*

- $\int 2^{\mathbf{I}_{\mathrm{d}}(|\psi\rangle : |\psi\rangle)} d\Lambda = O(1)$.

- $\int 2^{\mathbf{I}_{\mathrm{g}}(|\psi\rangle : |\psi\rangle)} d\Lambda = O(1)$.

The information terms enjoy conservation inequalities.

Theorem 2 ([Eps24])

- *Relativized to elementary quantum operation ε, $\mathbf{I}_{\mathrm{d}}(\varepsilon(\rho) : \sigma) <^+ \mathbf{I}_{\mathrm{d}}(\rho : \sigma)$.*

- *Relativized to elementary unitary transform A, $\mathbf{I}_{\mathrm{g}}(A\rho A^* : \sigma) <^+ \mathbf{I}_{\mathrm{g}}(\rho : \sigma)$.*

5 Information Between Classical Probabilities

In order to better understand the barrier between the quantum and classical realms, we introduce a new information term over probabilities. This is due to the fact that quantum states decohere into probability measures and a POVM will also produce a probability from the measured quantum states. For probabilities p, q over strings, their mutual information is defined as follows.

$$\mathbf{I}_{\mathrm{prob}}(p : q) = \log \sum_{x, y \in \{0, 1\}^*} 2^{\mathbf{I}(x : y)} p(x) q(y).$$

The information term is $\mathbf{I}(x : y) = \mathbf{K}(x) + \mathbf{K}(y) - \mathbf{K}(x, y)$. This definition obeys conservation inequalities over randomized processing $f : \{0, 1\}^* \times \{0, 1\}^* \to \mathbb{R}_{\geq 0}$, where $fp(x) = \sum_z f(x | z) p(z)$.

Theorem 3 ([Eps24]) *Relativized to random processing f,*

$$\mathbf{I}_{\mathrm{prob}}(fp : q) <^+ \mathbf{I}_{\mathrm{prob}}(p : q).$$

If $\mathbf{I}_{\text{prob}}(p : p)$ is small then either p represents an empty signal, or white noise, or some combination of the two. Furthermore no amount of deterministic or randomized processing can increase p's "signal".

We recall that a POVM E is a finite or infinite set of positive semi-definite matrices E_k such that $\sum_k E_k = \mathbf{1}$. Given a quantum state σ, a POVM induces a probability measure $E\sigma(k) = \text{Tr}\sigma E_k$, where $E\sigma(i)$ can be interpreted as the probability of measuring i when σ is applied to E. Remarkably, algorithmic self-information of a quantum state upper bounds the signal produced from a POVM.

Theorem 4 ([Eps24]) *Relativized to POVM E, $\mathbf{I}_{\text{prob}}(E\sigma : E\sigma) <^{\log} \mathbf{I}_{\text{d}}(\sigma : \sigma)$.*

This theorem, combined with Theorem 1, shows that the measurements of most quantum states produce the empty signal, white noise, or some combination of the two. In fact one can show this directly, with the following theorem.

Theorem 5 ([Eps24]) *Let Λ be the uniform distribution over the unit sphere of Q_n. Relativized to POVM E,*

$$\int 2^{\mathbf{I}_{\text{prob}}(E|\psi\rangle : E|\psi\rangle)} d\Lambda = O(1).$$

6 No Synchronization Law

In this section, we detail the No-Synchronization Law. In algorithmic thermodynamics studied in [Eps24], computable measure spaces where used. This involves a computable metric space S and a computable (not-necessarily probability) measure over S. The metric space has a dense set of ideal points $x, y \in S$ such that their metric distance $d(x, y)$ is computable. A measure μ is computable if it is a constructive point in the space of all Borel measures over S. It can be shown [HR09], that one can define a universal lower computable μ test $\mathbf{t}_\mu$ such that $\int \mathbf{t}_\mu d\mu \leq 1$. The algorithmic entropy [Gac94] of a point $\alpha \in S$ is defined to be $\mathbf{G}_\mu(\alpha) = -\log \mathbf{t}_\mu(\alpha)$. This entropy score has a max value of $\log \mu(S)$ and can take arbitrary negative values, even $-\infty$.

In my opinion, the definition $\mathbf{G}_\mu$ answers how to apply algorithmic randomness to thermodynamics. The term is very easy to work with, as there are many properties that can be proved with $\mathbf{t}_\mu$ which then can be directly translated into properties about $\mathbf{G}_\mu$. We detail one such property, that two separate, isolated physical systems that are typical must be out of sync.

Theorem 6 ([Eps24]) *Let $(\mathcal{X} \times \mathcal{Y}, \mu \times \nu)$ be a non-atomic computable product measure space. Let R_X^t and R_Y^t be ergodic transformation groups over $\mathcal{X}$ and $\mathcal{Y}$ respectively. Let $(\alpha, \beta) \in X \times Y$. If $\mathbf{G}_{\mu \times \nu}(\alpha, \beta) > -\infty$ and (α, β) has finite mutual information with the halting sequence then $\sup_{t \in \mathbb{N}} |\mathbf{G}_\mu(R_X^t(\alpha)) - \mathbf{G}_\nu(R_Y^t(\beta))| = \infty$.*

The notion of "finite mutual information with the halting sequence" merits some further explanation. For $(\alpha, \beta) \in \mathcal{X} \times \mathcal{Y}$, let $\Pi \subset \{0, 1\}^\infty$ be the set of all encoded infinite sequences consisting of ideal points that converge quickly (exponentially) to (α, β). The point (α, β) has finite mutual information with the halting sequence $\mathcal{H}$ if there is a sequence $\gamma \in \Pi$ such that $\mathbf{I}(\gamma : \mathcal{H}) < \infty$, where $\mathbf{I}$ is any suitable information term between infinite sequences. Thus only exotic points have infinite mutual information with the halting sequence. Note, that one can replace "finite mutual information with the halting sequence" with the weaker condition of "computable".

Is there a classical law of thermodynamics that says the same thing? Or is this a property only seen in algorithmic thermodynamic entropy $\mathbf{G}$? What further properties about thermodynamics can be discerned using $\mathbf{G}$?

7 Looking Forward: Black Holes

An overwhelming majority of research in Algorithmic Physics has been in quantum mechanics or thermodynamics. However this represents just the starting point for the investigation of the intersection of physics with algorithmic information theory. For example, Brown and Susskind introduced [BS18] the Kolmogorov complexity of black holes. They did this by associating black holes with the $SU(n)$ space, then partitioning this space into cells, assigning each cells a number c, then defining the Kolmogorov complexity of all unitary transforms in that cell with $\mathbf{K}(c)$. Each cell can be seen as a node of a graph G and two vertices v_1, v_2 share an edge if there are to unitary transforms $U_1 \in v_1$ and $U_2 \in v_2$ such that $U_2 = gU_1$, where g is a simple gate. The evolution of the complexity of a black hole can be seen as the evolution of the Kolmogorov complexity of the nodes of a random walk over G.

Brown and Susskind [BS18] conjectured that the complexity of a black hole will increase linearly for an exponential amount of time before reaching a maximum and fluctuating continuously after that. This would match the properties of the Einstein-Rosen bridge between two entangled black holes, and is called the Complexity/Volume Correspondence. This conjecture was proven to be true for quantum circuit complexity [HFK$^+$22], but it remains open for Kolmogorov complexity. In [Eps24], it was shown if G is an expander graph, then linear growth of Kolmogorov complexity of black holes must occur (from [HFK$^+$22], one can assume proving expander properties for G may land someone a Nature publication!).

References

[BS18]   A. Brown and L. Susskind. Second law of quantum complexity. *Phys. Rev. D*, 97, 2018.

[BV93]   E. Bernstein and U. Vazirani. Quantum complexity theory. In *Proceedings of the Twenty-Fifth Annual ACM Symposium on Theory of Computing*, page 11–20, New York, NY, USA, 1993. Association for Computing Machinery.

[BvL01]   A. Berthiaume, W. van Dam, and S. Laplante. Quantum Kolmogorov Complexity. *Journal of Computer and System Sciences*, 63(2), 2001.

[DP85]   D. Deutsch and R. Penrose. Quantum theory, the church–turing principle and the universal quantum computer. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 400(1818):97–117, 1985.

[Eps24]   S. Epstein. Algorithmic Physics, 2024. http://www.jptheorygroup.com/doc/APhysics.pdf.

[Gó1]   P. Gács. Quantum Algorithmic Entropy. *Journal of Physics A Mathematical General*, 34(35), 2001.

[Gac94]   P. Gacs. The boltzmann entropy and randomness tests. In *Proceedings Workshop on Physics and Computation. PhysComp '94*, pages 209–216, 1994.

[HFK$^+$22] J. Haferkamp, P. Faist, N. Kothakonda, J. Eisert, and N. Halpern. Linear growth of quantum circuit complexity. *Nature Physics*, 18(5):528–532, 2022.

[HR09]   M. Hoyrup and C. Rojas. Computability of probability measures and martin-löf randomness over metric spaces. *Information and Computation*, 207(7):830–847, 2009.

[Mue07]   M. Mueller. Quantum kolmogorov complexity and the quantum turing machine. *CoRR*, abs/0712.4377, 2007.

[Mul08]   M. Muller. Strongly Universal Quantum Turing Machines and Invariance of Kolmogorov Complexity. *IEEE Transactions on Information Theory*, 54(2), 2008.

[Mul09]   M. Muller. On the quantum kolmogorov complexity og classical strings. *International Journal of Quantum Information*, 07(04):701–711, 2009.

[NS19]   A. Nies and V. Scholz. Martin-Löf Random Quantum States. *Journal of Mathematical Physics*, 60(9), 2019.

[Sho94]   P. Shor. Algorithms for quantum computation: discrete logarithms and factoring. In *Proceedings 35th Annual Symposium on Foundations of Computer Science*, pages 124–134, 1994.

[Vit01]   P.M.B. Vitanyi. Quantum kolmogorov complexity based on classical descriptions. *IEEE Transactions on Information Theory*, 47(6):2464–2479, 2001.