

Randomness Conservation Over Infinite Quantum Spin Systems

Sam Epstein*

November 8, 2024

Abstract

Conservation of randomness has been proven for computable transformations with respect to computable probabilities over numbers, infinite sequences, and a broad set of computable topologies. In addition conservation of randomness has been proven for quantum operations over finite quantum systems. In this paper, conservation of NS randomness is proven over a broad class of computable dynamics with respect to infinite quantum spin chains.

1 Introduction

For a computable probability p over \mathbb{N} , deficiency of randomness is defined by $\mathbf{d}(a|p) = \lceil -\log p(a) \rceil - \mathbf{K}(a)$, where \mathbf{K} is the prefix free Kolmogorov complexity. It is a score for how atypical a is with respect to the model p . For a computable transformation $f : \mathbb{N} \rightarrow \mathbb{N}$, fp denotes the probability $fp(a) = \sum_{f(b)=a} p(b)$. One of the foundational developments of algorithmic information theory is *conservation of randomness*, where [G21]

$$\mathbf{d}(fa|fp) < \mathbf{d}(a|p) + O(1).$$

This simple inequality has many applications, including defining information non-growth laws. Similarly, for computable probability P over $\{0, 1\}^\infty$, one can define deficiency randomness of $\alpha \in \{0, 1\}^\infty$ with respect to P as $\mathbf{D}(\alpha|P) = \sup_n \lceil -\log P(\alpha[0..n]) \rceil - \mathbf{K}(\alpha[0..n])$. For a given computable transformation $F : \{0, 1\}^\infty \rightarrow \{0, 1\}^\infty$ between infinite sequences one gets the following conservation law, with [G21]

$$\mathbf{D}(F\alpha|FP) < \mathbf{D}(\alpha|P) + O(1).$$

In addition, one can define the deficiency of randomness of one mixed quantum state σ with respect to another quantum mixed state ρ , with $\mathbf{d}(\sigma|\rho)$. In [Eps24],

*samepst@jpththeorygroup.com

conservation of randomness between mixed states is proven with respect to elementary quantum operations ϵ , with

$$\mathbf{d}(\epsilon(\sigma)|\epsilon(\rho)) < \mathbf{d}(\sigma|\rho) + O(1).$$

In this paper, we prove conservation of randomness with respect to infinite quantum spin chains Z . The notion of randomness used is *NS Randomness*, [NS19], which is analogous to ML randomness of infinite sequences. One main difference between the two definitions is that quantum states are NS random at a particular order $\delta \in [0, 1]$, where the smaller the δ the more “random” the infinite quantum sequence is. In the classical case, when an ML random sequence is transformed with a computable function, there is no guarantee that the output will remain ML random. However the situation is very different in the quantum case. In fact, conservation of NS randomness is proven over a broad class of computable “admissible” dynamics. In this paper, we use the Heisenberg picture for dynamics. Thus dynamics τ_t are strongly continuous one parameter group of $*$ -automorphisms over the self-adjoint (observables) elements of the quantum C^* algebra. The result of this paper is as follows.

Theorem. *For infinite quantum spin state Z , computable $t > 0$, admissible computable dynamics τ_t , if Z is NS random at order δ , then $Z \circ \tau_t$ is NS random at order ϵ , for all $\epsilon > \delta$. Furthermore, if Z is not NS random at order δ then $Z \circ \tau_t$ is not NS random at order ϵ for all $\epsilon < \delta$.*

2 Algebras

Before we introduce NS random sequences, we revisit the notion of C^* algebras and functional states. A C^* algebra, \mathcal{M} , is a Banach algebra and a function $*$: $\mathcal{M} \rightarrow \mathcal{M}$ such that

1. For every $x \in \mathcal{M}$, $x^{**} = x$.
2. For every $x, y \in \mathcal{M}$, $(x + y)^* = x^* + y^*$ and $(xy)^* = y^*x^*$.
3. For every $\lambda \in \mathbb{C}$ and $x \in \mathcal{M}$, $(\lambda x)^* = \bar{\lambda}x^*$.
4. For all $x \in \mathcal{M}$, $\|x^*x\| = \|x\|\|x^*\|$.

A $*$ -algebra does not necessarily have property (4). A C^* algebra \mathcal{M} is unital if it admits a multiplicative identity $\mathbf{1}$. Set of positive elements of \mathcal{M} represent the quantum observables of the system. A state over unital \mathcal{M} is a positive linear functional $Z : \mathcal{M} \rightarrow \mathbb{C}$ such that $Z(\mathbf{1}) = 1$. Since states are positive, they are continuous. States are used to define NS random sequences. The set of states of \mathcal{M} is denoted by $S(\mathcal{M})$. A state is tracial if $Z(x^*x) = Z(xx^*)$, for all $x \in \mathcal{M}$. An element A of a C^* algebra is unitary if $A^*A = AA^* = \mathbf{1}$. A $*$ -homomorphism between two C^* -algebras \mathcal{M} and \mathcal{N} is a mapping $\pi : \mathcal{M} \rightarrow \mathcal{N}$ such that

- $\pi(\alpha A + \beta B) = \alpha\pi(A) + \beta\pi(B)$,
- $\pi(AB) = \pi(A)\pi(B)$,
- $\pi(A^*) = \pi(A)^*$.

By definition, $*$ -homomorphisms π are contractive, with $\|A\| \geq \|\pi(A)\|$. an $*$ -automorphism is a $*$ -homomorphism from a C^* algebra to itself. An inner $*$ -automorphism τ over a C^* algebra M is of the form $\tau(A) = UAU^*$ for some unitary element $U \in M$.

3 Direct Limits

A *direct system* of C^* -algebras is a family (A_i, π_i^j) of C^* -algebras and also $*$ -homomorphisms $\pi_i^j : A_i \rightarrow A_j$ for all $j \geq i$ indexed by a directed set (partially ordered and every finite subset has a majorant), with

- π_i^i is the identity map
- $\pi_j^k \circ \pi_i^j = \pi_i^k$ for all $i \leq j \leq k$.

Consider the product $*$ -algebra $\prod A_i$ and let A' be the $*$ -subalgebra of all elements $a = \{a_i\}$ such that there is an index i_0 with $\pi_i^j(a_i) = a_j$ for all $i_0 \leq i \leq j$. Since each π_i^j is norm-decreasing, the net $\{\|a_i\|\}$ is convergent and we define $\|a\| = \lim \|a_i\|$. If N is the kernel of $\|\cdot\|$ then the Banach space completion of the quotient A'/N is a C^* -algebra. This C^* algebra is called the *direct limit* of (A_i, π_i^j) .

Suppose A is the direct limit of direct system (A_i, π_i^j) . For each element $a \in A_i$, we associate the sequence $a' = \{a'_j\}$ in A' where $a'_j = 0$ for $j < i$ and $a'_j = \pi_i^j(a)$ for $j \geq i$. The map $\pi_i : A_i \rightarrow A$ is defined by taking a to the image a' , which is a $*$ -homomorphism from A_i to A .

4 CAR Algebras

In this chapter, we restrict our attention to the special case where A_n are $2^n \times 2^n$ complex matrices and the directed set of indices is equal to \mathbb{N} . Furthermore, π_n^{n+1} is the map

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

This direct limit is known as a CAR algebra. We denote A_k by \mathcal{M}_k and A by \mathcal{M}_∞ . The $*$ -homomorphisms from \mathcal{M}_k to \mathcal{M}_∞ is denoted by $\pi_k : \mathcal{M}_k \rightarrow \mathcal{M}_\infty$. Each state $\rho \in S(\mathcal{M}_k)$, can be identified with a density matrix S such that $\rho(X) = \text{Tr}SX$, for all $X \in \mathcal{M}_k$. States that cannot be represented as the convex combination of other states are called pure states. Otherwise they are called mixed states. The tracial state $\tau_n \in S(\mathcal{M}_n)$ corresponds to the matrix $2^{-n}I_{2^n}$. A state $Z \in S(\mathcal{M}_\infty)$ over \mathcal{M}_∞ can be identified with a sequence of density

matrices $\{\rho_n\}$ that are coherent under partial traces, with $\text{Tr}_{\mathcal{M}_{n+1}}\rho_{n+1} = \rho_n$. We use $Z|_n$ to denote the restriction of state Z to the algebra \mathcal{M}_n . There is a unique tracial state $\tau \in S(\mathcal{M}_\infty)$, where $\tau|_n = \tau_n$. A projection $p \in \mathcal{M}_\infty$ is a self adjoint positive element such that $p = p^2$. A special projection $p \in \mathcal{M}_n$ is a projection represented by an elementary matrix. An elementary matrix has entries that are algebraic, that is roots of polynomials with rational coefficients. Elementary matrices can be associated with numbers and all the standard algorithms of linear algebra can be performed on such matrices.

5 NS Randomness

An NS Σ_1^0 set is a computable sequence of special projections $\{p_i\}$ in \mathcal{M}_∞ with $p_i \leq p_{i+1}$ over all i . Since $0 \leq p_i \leq p_{i+1} \leq \mathbf{1}$, $\lim_{i \rightarrow \infty} p_i$ exists and is a member of \mathcal{M}_∞ . And since states are continuous, for state ρ and NS Σ_1^0 set G , $\rho(G) = \rho(\lim_i p_i) = \sup_i \rho(p_i)$.

We define NS tests. But initially, we will provide the definition for the classical Martin L f random sequence, to provide a point of reference. A classical Martin L f test, is a sequence $\{U_n\}$ of uniformly Σ_1^0 sets of infinite sequences $U_n \subseteq \{0,1\}^\infty$ such that $\mu(U_n) \leq 2^{-n}$. An infinite sequence $\alpha \in \{0,1\}^\infty$ is Martin-L f random if there is no Martin L f test $\{U_n\}$ such that $\alpha \in \bigcap_n U_n$. There is a universal Martin L f test $\{V_n\}$ such that if $\alpha \notin \bigcap_n V_n$, then α is random.

Mirroring the classical case, a NS test is an effective sequence of NS Σ_1^0 sets $\langle G^r \rangle$ such that $\tau(G^r) \leq 2^{-r}$. Unlike a classical test, which can either pass or fail a sequence, a NS test can pass a quantum state up to a particular order. For $\delta \in [0, 1]$, state $Z \in S(\mathcal{M}_\infty)$ fails test $\langle G^r \rangle$ at order δ if $Z(G^r) > \delta$ for all r . Otherwise Z passes the test at order δ . We say Z passes a NS test if it passes it at all orders $\delta \in [0, 1]$. A state is NS random at order δ if it passes every NS test at order δ . A state is NS random if it is NS random at every order $\delta > 0$.

Theorem 1 ([NS19]). *There exists a universal NS test $\langle R^n \rangle$, where for each NS test $\langle G^k \rangle$ and each state Z and for each n there exists a k such that $Z(R^n) \geq Z(G^k)$.*

6 Interactions

An interaction Φ is a function from \mathbb{N} to Hermitian elements of \mathcal{M}_n such that $\Phi(n) \in \mathcal{M}_n$. Each $\Phi(n)$ represents the energy of interaction of the set of the first n particles. In the spin system the particles are considered to be fixed at the lattice sites hence the total energy of interaction in a subset consists of the interaction energy of all subsystems. This total energy is defined to be the Hamiltonian $H_\Phi(n)$, with

$$H_\Phi(n) = \sum_{m \leq n} \Phi(m).$$

Note that $\Phi(n) \in \mathcal{M}_n$ for each n , so using the homomorphism $\pi_n^m : \mathcal{M}_n \rightarrow \mathcal{M}_m$, the above term is technically $\sum_{m \leq n} \pi_m^n(\Phi(m))$. An interaction is *admissible* if $\sum_{n \geq 1} \Phi(n) \in \mathcal{M}_\infty$ and we define $\bar{\Phi} \in \mathcal{M}_\infty$ to be the element equal to this sum. An interaction Φ is *computable* if each $\Phi(n)$ is a Hermitian matrix uniformly computable in n . By the Heisenberg picture, the dynamical evolution of any system can be defined in terms of the evolution of the associated observables. For a finite spin system confined to \mathcal{M}_n , with interaction Φ and Hamiltonian H_Φ , this evolution is given by the Heisenberg relations $\tau_t^n : \mathcal{M}_n \rightarrow \mathcal{M}_n$, with the one parameter $*$ -automorphism group

$$\tau_t^n(A) = e^{itH_\Phi(n)} A e^{-itH_\Phi(n)}.$$

Note that since $H_\Phi(n)$ is Hermitian, $e^{itH_\Phi(n)}$ is unitary.

Proposition 1. *If $\lim_{n \rightarrow \infty} \|A - A_n\| = 0$ then $\lim_{n \rightarrow \infty} \|e^A - e^{A_n}\| = 0$.*

Proof. First we show that $\lim_{n \rightarrow \infty} \|e^A - e^{A_n}\| < \infty$. This term is not more than $\|e^A\| + \lim_{n \rightarrow \infty} \|e^{A_n}\| \leq \|e^A\| + e^{\lim_{n \rightarrow \infty} \|A_n\|} < \infty$. So,

$$\lim_{n \rightarrow \infty} \|e^A - e^{A_n}\| \leq \lim_{n \rightarrow \infty} \sum_{k \geq 0} \frac{\|A^k - A_n^k\|}{k!} = \sum_{k \geq 0} \lim_{n \rightarrow \infty} \frac{\|A^k - A_n^k\|}{k!} = 0.$$

□

The following theorem is a reworking of Theorem 6.2.6 in [BR79] to the special case of CAR algebras.

Theorem 2. *If Φ is an admissible interaction, then there is a strongly continuous one-parameter group of inner $*$ -automorphisms τ of \mathcal{M}_∞ with*

$$\lim_{n \rightarrow \infty} \|\tau_t(A) - \tau_t^n(A)\| = 0,$$

for all $A \in \mathcal{M}_\infty$ and $t \in \mathbb{R}$, where $\tau_t(A) = e^{it\bar{\Phi}} A e^{-it\bar{\Phi}}$ and $\tau_t^n(A) = e^{itH_\Phi(n)} A e^{-itH_\Phi(n)}$.

Proof. We have that $\lim_{n \rightarrow \infty} \|\bar{\Phi} - \pi_n(H_\Phi(n))\| = 0$. So due to Proposition 1, for all $t > 0$,

$$\lim_{n \rightarrow \infty} \left\| e^{it\bar{\Phi}} - e^{it\pi_n(H_\Phi(n))} \right\| = 0.$$

This implies that for all $t > 0$ and $A \in \mathcal{M}_\infty$,

$$\lim_{n \rightarrow \infty} \left\| e^{it\bar{\Phi}} A e^{-it\bar{\Phi}} - \tau_t^n(A) \right\| = 0.$$

So, it is readily apparent that $\tau_t : A \mapsto e^{it\bar{\Phi}} A e^{-it\bar{\Phi}}$ is a one-parameter group of inner $*$ -automorphisms satisfying the theorem. □

7 Results

Proposition 2. *If Z is a state and τ is a $*$ -automorphism, then $Z \circ \tau$ is also a state.*

Proof. $Z \circ \tau$ is linear because $Z(\tau(\alpha A + \beta B)) = Z(\alpha \tau(A) + \beta \tau(B)) = \alpha Z(\tau(A)) + \beta Z(\tau(B))$. Further $Z \circ \tau$ sends positive elements to positive elements because $Z(\tau(AA^*)) = Z(\tau(A)\tau(A^*)) = Z(\tau(A)\tau(A)^*) \geq 0$. Furthermore $Z(\tau(1)) = Z(1) = 1$. \square

Lemma 1. *For computable interaction Φ , one can compute, uniformly in computable $t \in \mathbb{R}$, a projection q , and $m \in \mathbb{N}$, a sequence of elementary unitary transformations $\{U_k\}_{k=1}^\infty$ such that $\lim_{k \rightarrow \infty} U_k = e^{itH_\Phi(m)}$ and for all density matrices ρ , $\text{Tr} \rho e^{itH_\Phi(m)} U_k q U_k^* e^{-itH_\Phi(m)} > \text{Tr} \rho q - 2^{-k}$.*

Proof. We use the fact that for matrix X , $e^X = \sum_{k=0}^\infty \frac{X^k}{k!}$. So if X is computable, then so is e^X . The set of elementary unitary matrices is dense in $SU(2^n)$ so given a computable Φ , one can compute a series of elementary matrices $\{U_k\}_{k=1}^\infty$ such that $\lim_{k \rightarrow \infty} U_k = e^{itH_\Phi(m)}$. So for any given elementary density matrix σ and $\epsilon > 0$, one can compute an number $k(\sigma, \epsilon)$ such that $\text{Tr} \sigma e^{itH_\Phi(m)} U_{k(\sigma, \epsilon)} q U_{k(\sigma, \epsilon)}^* e^{-itH_\Phi(m)} > \text{Tr} \sigma q - \epsilon$. Given $\epsilon > 0$, one can construct a finite set of elementary density matrices D_ϵ such that for all density matrices ρ , there is a density matrix $\sigma \in D_\epsilon$, where $\text{Tr} |\rho - \sigma| < \epsilon$. Let $\{V_j\}$ be uniformly computable enumerable set of elementary density matrices, where $V_j = U_{\max\{k(\sigma, 2^{-j-2}): \sigma \in D_{2^{-j-2}}\}}$. So for all density matrices ρ , there is a elementary matrix $\rho_0 \in D_{2^{-j-2}}$ so

$$\begin{aligned} & \text{Tr} \rho e^{itH_\Phi(m)} V_k q V_k^* e^{-itH_\Phi(m)} \\ & > \text{Tr} \sigma e^{itH_\Phi(m)} V_k q V_k^* e^{-itH_\Phi(m)} - 2^{-k-2} \\ & > \text{Tr} \sigma q - 2^{-k-1} \\ & > \text{Tr} \rho q - 2^{-k}. \end{aligned}$$

\square

Theorem 3. *Given admissible computable interaction Φ and computable $t \in \mathbb{R}^+$, if state Z is not NS random order δ , then for all $0 < \delta' < \delta$, $Z \circ \tau_t$ is not NS random at order δ' .*

Proof. Let $\langle G^r \rangle$, $G^r = \langle p_k^r \rangle$ be the NS test that Z fails with at order δ . So $Z(G^r) > \delta$, for all r . So for all r there is a large enough k such that $\text{Tr} Z |n_k^r p_k^r| > \delta$. Fix any computable ϵ , such that $0 < \epsilon < \delta$. We show that for each $n \in \mathbb{N}$, $Z \circ \tau_t^n$ fails at order δ' . For each k, r , and n using Lemma 1, we compute elementary unitary matrix $U_{n,r,k}$ such that for all density matrices ρ , $\text{Tr} \rho e^{itH_\Phi(n)} U_{n,r,k} p_k^r U_{n,r,k}^* e^{-itH_\Phi(n)} > \text{Tr} \rho p_k^r - \epsilon$. Define the test $\langle G^r(n) \rangle$ with

$G^r(n) = \langle p(n)_k^r \rangle$, where $p(n)_k^r = U_{n,r,k} p_k^r U_{n,r,k}^*$. So for all r and large enough k ,

$$\begin{aligned}
& Z(\tau^n(G^r(n))) \\
& \geq Z(\tau^n(p(n)_k^r)) \\
& \geq \text{Tr} Z_{n_k} e^{itH_\Phi(n)} U_{n,r,k} p_k^r U_{n,r,k}^* e^{-itH_\Phi(n)} \\
& \geq \text{Tr} Z_{n_k} p_k^r - \epsilon \\
& > \delta - \epsilon,
\end{aligned}$$

So for each $n \in \mathbb{N}$, $Z \circ \tau_t^n$ fails an NS test at order δ' , for all $\delta' < \delta$. Denote $\langle W^r \rangle$, $W^r = \langle q_k^r \rangle$ to be the NS test constructed in the following fashion, with $q_k^r = \bigvee_{j \leq k} p(j)_k^{r+j}$. So $Z(\tau_t^n(W^r)) > \epsilon$ for all $n, r \in \mathbb{N}$. Due to Theorem 2, for all $r \in \mathbb{N}$, there exists a one parameter *-automorphism group τ_t , where $\lim_{n \rightarrow \infty} \|\tau_t(\Psi^r) - \tau_t^n(\Psi^r)\| = 0$. Since Z is continuous, for all $r \in \mathbb{N}$ $Z(\tau_t(W^r)) = \lim_{n \rightarrow \infty} Z(\tau_t^n(W^r)) > \epsilon$. So $Z \circ \tau_t$ fails NS test $\langle W^r \rangle$ at any order $\delta' < \delta$. \square

Lemma 2. *For computable interaction Φ , one can compute, uniformly in computable $t \in \mathbb{R}$, a projection q , and $m \in \mathbb{N}$, a sequence of elementary unitary transformations $\{U_n\}_{n=1}^\infty$ such that $\lim_{n \rightarrow \infty} U_n = e^{itH_\Phi(m)}$ and for all density matrices ρ , $\text{Tr} \rho U_n q U_n^* > \text{Tr} \rho e^{itH_\Phi(m)} q e^{-itH_\Phi(m)} - 2^{-n}$.*

Proof. The proof is the same as the one for Lemma 1. \square

Theorem 4. *Given admissible computable interaction Φ and computable $t \in \mathbb{R}^+$, if state Z is NS random at order δ , then for all $\delta < \delta' < 1$, $Z \circ \tau_t$ is NS random at order δ' .*

Proof. Fix computable $\delta' > \delta$. It must be for all $n \in \mathbb{N}$, $Z \circ \tau_t^n$ passes all NS tests at order δ' . Assume not. Then there exists an NS test $\langle G^r \rangle$, $G^r = \langle p_k^r \rangle$ and a n such that for all r , $Z(\tau_t^n(G^r)) > \delta'$. For each k and r , using Lemma 2, we compute elementary unitary matrix $U_{k,r}$ such that for all density matrices ρ , $\text{Tr} \rho U_{k,r} p_k^r U_{k,r}^* > \text{Tr} \rho e^{itH(n)} p_k^r e^{-itH(n)} - \epsilon$. The computable term ϵ is chosen later. Define test $\langle H^r \rangle$, $H^r = \langle q_k^r \rangle$, where $q_k^r = U_{k,r} p_k^r U_{k,r}^*$. So for all r and large enough k ,

$$Z(H^r) \geq \text{Tr} Z_{n_k} U_{k,r} p_k^r U_{k,r}^* > \text{Tr} Z_{n_k} e^{itH_\Phi(n)} p_k^r e^{-itH_\Phi(n)} - \epsilon > \delta' - \epsilon > \delta,$$

for proper choice of ϵ , causing a contradiction. So for the universal NS test $\langle W^r \rangle$, for all $n \in \mathbb{N}$, each state $Z \circ \tau_t^n$ passes $\langle W^r \rangle$ at order δ' . Due to Theorem 2, for all $r \in \mathbb{N}$, $\lim_{n \rightarrow \infty} \|\tau_t(W^r) - \tau_t^n(W^r)\| = 0$. Since Z is continuous, for all $r \in \mathbb{N}$, $Z(\tau_t(W^r)) = \lim_{n \rightarrow \infty} Z(\tau_t^n(W^r)) < \delta'$. So $Z \circ \tau_t$ passes every NS test at order $\delta' > \delta$ and thus it is NS random at each order δ' . \square

Corollary 1. *Given admissible computable interaction Φ and computable $t \in \mathbb{R}^+$, computable state Z is NS random at order δ iff $Z \circ \tau_t$ is NS random at order δ .*

Corollary 2. *Let $\{U_n(t)\}_{n=1}^\infty$ be a sequence of strongly continuous unitary groups, where $U_n(t) \in \mathcal{M}_n$ is uniformly computable in $n \in \mathbb{N}$ and $t \in \mathbb{R}$. Assume V is a strongly continuous unitary group where for all t , $V(t) \in \mathcal{M}_\infty$ and $\lim_{n \rightarrow \infty} U_n(t) = V(t)$. Then for one parameter $*$ -automorphism group $\chi_t(A) = V(t)AV(t)^*$, $A \in \mathcal{M}_\infty$, computable $t \in \mathbb{R}$, we have*

1. *If state Z is NS random at order δ then $Z \circ \chi_t$ is NS random at order δ' , for all $\delta' < \delta$.*
2. *If state Z is not NS random order δ then $Z \circ \chi_t$ is not NS random at order δ' , for all $\delta' > \delta$.*

References

- [BR79] O. Bratteli and D. Robinson. *Operator Algebras and Quantum Statistical Mechanics*. Number v. 2 in Operator Algebras and Quantum Statistical Mechanics. Springer, 1979.
- [Eps24] Samuel Epstein. Algorithmic physics, 2024.
- [G21] Peter Gács. Lecture notes on descriptional complexity and randomness. *CoRR*, abs/2105.04704, 2021.
- [NS19] A. Nies and V. Scholz. Martin-Löf Random Quantum States. *Journal of Mathematical Physics*, 60(9), 2019.