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Uniform Tests and Algorithmic Thermodynamic Entropy

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For definitions in this post, we use [HR09]. A computable metric space X is a metric space with a dense set of ideal points on which the distance function is computable. A computable probability is defined by a computable sequence of converging points in the corresponding space of probability measures, $\mathcal{M}(X)$, over X. A uniform test takes in a description of a probability measure μ and produces a lower computable μ test. There exists a universal uniform test, from [G2́1, HR09]. We extend the result from [Eps22] to computable metric spaces.

Theorem 1 *Given computable non-atomic probability measures μ and λ over a computable metric space X and universal uniform test $\mathbf{t}(\cdot,\cdot)$. For all n, $\lambda(\{\alpha : \mathbf{t}(\mu,\alpha) > 2^n\}) > 2^{-n-\mathbf{K}(n)-O(1)}$.*

Reworking the above theorem, one can get a result in algorithmic physics. To define algorithmic fine-grain entropy, we use a slightly modified version of the definition in [Gac94]. First, note that all the results [HR09] can be easily extended to arbitrary nonnegative measures. This can be achieved by defining the product space of $\mathcal{M}(X)$ and $\mathbb{R}_{\geq 0}$, where the second metric space defines the size of the measure. Given a measure $\mu \in \mathcal{M}(X) \times \mathbb{R}_{\geq 0}$, the algorithmic fine grained entropy of a point $\alpha \in X$ is as follows.

Definition 1 (Algorithmic Fine-Grained Entropy) $H(\alpha) = -\log \mathbf{t}(\mu,\alpha)$.

One can then prove that this term will oscillate in the presence of dynamics. Dynamics can be defined using group theory.

Definition 2 (Transformation Group) *Let M denote a computable metric space and G a topological group each element of which is a homeomorphism of M onto itself:*

$$f(g;x) = g(x) = x' \in M; g \in G, x \in M.$$

The pair (G,M) will be called a topological transformation group if for every pair of elements $g1, g2$ of G, and every $x \in M$, $g1(g2(x)) = (g1g2)(x)$ and if

$$x' = g(x) = f(g;x)$$

is continuous simultaneously in $x \in M$ and $g \in G$.

Theorem 2 (Oscillation of Thermodynamic Entropy) *Let L be the Lebesgue measure over $\mathbb{R}$. For one dimensional topological transformation group (G^t, X) acting on computable metric space X, for all $\alpha \in X$, $L\{t \in [0,1] : H(G^t\alpha) < \max_\beta H(\beta) - n\} \overset{*}{>} 2^{-n-\mathbf{K}(n)-O(1)}$.*

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References

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