Three Eggs from the Chicken

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In this note, we show an example of taking a theorem produced from a non-constructive probabilistic proof and produce a three derandomization theorems, one that involves Kolmogorov complexity, one that involves resource boundedd Kolmogorov complexity, and one involving games.

Our note deals with hypergraphs. A *hypergraph* is a pair $J = (V, E)$ of vertices V and edges $E \subseteq \mathcal{P}(V)$. Thus each edge can connect ≥ 2 vertices. A hypergraph is *k-regular* of the size $|e| = k$ for all edges $e \in E$. A 2-regular hypergraph is just a simple graph. A valid *C-coloring* of a hypergraph (V, E) is a mapping $f : V \to \{1, \ldots, C\}$ where every edge $e \in E$ is not *monochromatic* $|\{f(v) : v \in e\}| > 1$. The following classic result [EL] is the first proved consequence of Lovász Local Lemma.

Theorem (Probabilistic Method) *Let $J = (V, E)$ be a k-regular hypergraph. If for each edge f, there are at most $2^{k-1}/e-1$ edges $h \in E$ such that $h \cap f \neq \emptyset$, then there exists a valid 2-coloring of J.*

We can now use derandomization, from [Eps22a], to produce bounds on the Kolmogorov complexity of the simpliest such 2-coloring of G.

Theorem 1 (Derandomization) *Let $J = (V, E)$ be a k-regular hypergraph with $|E| = m$. If, for each edge f, there are at most $2^{k-1}/e - 1$ edges $h \in E$ such that $h \cap f \neq \emptyset$, then there exists a valid 2-coloring x of J with*

$$\mathbf{K}(x) <^{\log} \mathbf{K}(n) + 4me/2^k + \mathbf{I}(J; \mathcal{H}).$$

The function $\mathbf{K}$ is the prefix free Kolmogorov complexity. $\mathbf{I}(J; \mathcal{H}) = \mathbf{K}(J) - \mathbf{K}(J|\mathcal{H})$ is the amount of asymmetric information the halting sequence $\mathcal{H} \in \{0, 1\}^\infty$ has about the graph J. For nonnegative function f, $<^{\log} f$ is defined to be $< f + O(\log(f + 1))$. We can now use resource derandomization, from [Eps22b] , to achieve bounds for the smallest time-bounded Kolmogorov complexity $\mathbf{K}^t(x) = \min\{p : U(p) = x \text{ in } t(\|x\|) \text{ steps}\}$ of a 2-coloring of J.

Assumption 1 ***Crypto*** *is the assumption that there exists a language in $\mathbf{DTIME}(2^{O(n)})$ that does not have size $2^{o(n)}$ circuits with Σ_2^p gates.*

Theorem 2 (Resource Bounded Derandomization) *Assume **Crypto**. Let $J_n = (V, E)$ be a $k(n)$-regular hypergraph where $|V| = n$ and $|E| = m(n)$, uniformly polynomial time computable in n. Furthermore, for each edge f in J_n there are at most $2^{k(n)-1}/e - 1$ edges $h \in E$ such that $h \cap f \neq \emptyset$. Then there is a polynomial p, and a valid 2-coloring x of J_n with*

$$\mathbf{K}^p(x) < 4m(n)e/2^{k(n)} + O(\log n).$$

We define the following game involving hypergraphs that is from [Eps23]. The player has access to a list of vertices and the goal of the player is to produce a valid 2-coloring of the hypergraph. We assume that for each edge f of the graph, there are at most $2^{k-1}/e-1$ edges h such that $f \cap h \neq \emptyset$.

The game proceeds as follows. For the first round, environment gives the number of vertices to the player. The player has n vertices, each with starting color 1. At each subsequent turn, the environment sends to the player the edges which are monochromatic. The player can change the color of up to k vertices and sends these changes to the environment. The game ends when the player has a valid 2-coloring of the graph.

Theorem 3 (Game Derandomization) *For $k \geq 6$, there exists a player $\mathbf{p}$ that can beat the environment $\mathbf{q}$ in $(1+\epsilon)n/k$ turns, with Kolmogorov complextiy $\mathbf{K}(\mathbf{p}) <^{\log} \mathbf{I}(\mathbf{q};\mathcal{H}) - \log \epsilon$, where $\epsilon \in (0,1)$.*

References

[EL] P. Erdos and L. Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. *Infinite and finite sets*, 10:609–627.

[Eps22a] S. Epstein. 22 examples of solution compression via derandomization. *CoRR*, abs/2208.11562, 2022.

[Eps22b] S. Epstein. Derandomization under different resource constraints. *CoRR*, abs/2211.14640, 2022.

[Eps23] Samuel Epstein. Game derandomization, 2023. `https://www.jptheorygroup.org/doc/Game.pdf`.