

A Short Proof on the Existence of Anomalies

Samuel Epstein*

July 25, 2022

Abstract

The Independence Postulate (IP) is a finitary Church-Turing Thesis, saying mathematical sequences are independent from physical ones. IP implies the existence of anomalies.

Anomalies

$\mathbf{K}(x|y)$ is the conditional prefix Kolmogorov complexity. For probability p over \mathbb{N} , randomness deficiency is $\mathbf{d}(a|p, b) = \lfloor -\log p(a) \rfloor - \mathbf{K}(a|\langle p \rangle, b)$ and measures the extent of the refutation against the hypothesis p given the result a [G21]. $\mathbf{d}(a|p) = \mathbf{d}(a|p, \emptyset)$. $\mathbf{I}(a; \mathcal{H}) = \mathbf{K}(a) - \mathbf{K}(a|\mathcal{H})$, where \mathcal{H} is the halting sequence. $<^+ f$ is $< f + O(1)$ and $<^{\log} f$ is $< f + O(\log(f+1))$. Stochasticity is $\Lambda(a|b) = \min\{\mathbf{K}(Q|b) + 3 \log \max\{\mathbf{d}(a|Q, b), 1\} : Q \text{ has finite support and a range in } \mathbb{Q}\}$. $\Lambda(a|b) < \Lambda(a) + O(\log \mathbf{K}(b))$. The following definition is from [Lev74].

Definition 1 (Information) $\mathbf{I}(\alpha : \beta) = \log \sum_{x,y} 2^{\mathbf{K}(x) + \mathbf{K}(y) - \mathbf{K}(x,y) - \mathbf{K}(x|\alpha) - \mathbf{K}(y|\beta)}$.

The Independence Postulate states [Lev13]:

IP: Let α be a sequence defined with an n -bit mathematical statement, and a sequence β can be located in the physical world with a k -bit instruction set. Then $\mathbf{I}(\alpha : \beta) < k + n + c$ for some small absolute constant c .

There are many proofs in the literature that stochastic numbers have high mutual information with the halting sequence. One such detailed proof is in [Eps21].

Lemma 1 $\Lambda(x) <^{\log} \mathbf{I}(x; \mathcal{H})$.

*JP Theory Group. samepst@jpththeorygroup.org

Lemma 2 For probability p over \mathbb{N} , $D \subset \mathbb{N}$, $|D| = 2^s$, $s < \max_{a \in D} \mathbf{d}(a|p) + \Lambda(D) + \mathbf{K}(s) + O(\log \mathbf{K}(s, p))$.

Proof. We relativize the universal Turing machine to $\langle s, p \rangle$. Let Q be a probability measure that realizes $\Lambda(D)$, with $d = \max\{\mathbf{d}(D|Q), 1\}$. Let $F \subseteq \mathbb{N}$ be a random set where each element $a \in \mathbb{N}$ is selected independently with probability $cd2^{-s}$, where $c \in \mathbb{N}$ is chosen later. $\mathbf{E}[p(F)] \leq cd2^{-s}$. Furthermore

$$\begin{aligned} & \mathbf{E}[Q(\{G : |G| = 2^s, G \cap F = \emptyset\})] \\ & \leq \sum_G Q(G)(1 - cd2^{-s})^{2^s} < e^{-cd}. \end{aligned}$$

Thus finite $W \subset \mathbb{N}$ can be chosen such that $p(W) \leq 2cd2^{-s}$ and $Q(\{G : |G| = 2^s, G \cap W = \emptyset\}) \leq e^{1-cd}$. $D \cap W \neq \emptyset$, otherwise, using the Q -test, $t(G) = e^{cd-1}$ if $(|G| = 2^s, G \cap W = \emptyset)$ and $t(G) = 0$ otherwise, we have

$$\begin{aligned} \mathbf{K}(D|Q, d, c) & <^+ -\log Q(D) - (\log e)cd \\ (\log e)cd & <^+ -\log Q(D) - \mathbf{K}(D|Q) + \mathbf{K}(d, c) \\ (\log e)cd & <^+ d + \mathbf{K}(d, c), \end{aligned}$$

which is a contradiction for large enough c . Thus there is an $a \in D \cap W$, where

$$\begin{aligned} \mathbf{K}(a) & <^+ -\log p(a) + \log d - s + \mathbf{K}(d) + \mathbf{K}(Q) \\ s & <^+ \mathbf{d}(a|p) + \Lambda(D). \end{aligned}$$

Making the relativization of $\langle s, p \rangle$ explicit,

$$\begin{aligned} s & < -\log p(a) - \mathbf{K}(a|s, p) + \Lambda(D|s, p) \\ s & < \max_{a \in D} \mathbf{d}(a|p) + \Lambda(D) + \mathbf{K}(s) \\ & + O(\log \mathbf{K}(s, p)). \quad \square \end{aligned}$$

Let $\tau \in \mathbb{N}^{\mathbb{N}}$ represent a series of observations. Assuming τ has an infinite amount of unique numbers, $\tau(n)$ is the first 2^n unique numbers of τ .

Theorem 1 For probability p over \mathbb{N} , $\tau \in \mathbb{N}^{\mathbb{N}}$, let $s_{\tau,p} = \sup_n (n - 3\mathbf{K}(n) - \max_{a \in \tau(n)} \mathbf{d}(a|p))$. Then $s_{\tau,p} <^{\log} \mathbf{I}(\langle \tau \rangle : \mathcal{H}) + O(\log \mathbf{K}(p))$.

Proof. By Lemmas 1 and 2, and the fact that $\mathbf{I}(x; \mathcal{H}) <^+ \mathbf{I}(\alpha : \mathcal{H}) + \mathbf{K}(x|\alpha)$,

$$\begin{aligned} n &< \max_{a \in \tau(n)} \mathbf{d}(a|p) + \mathbf{I}(\tau(n); \mathcal{H}) + \mathbf{K}(p) + \mathbf{K}(n) \\ &\quad + O(\log \mathbf{I}(\tau(n); \mathcal{H}) \mathbf{K}(p) \mathbf{K}(n)), \\ n &< \max_{a \in \tau(n)} \mathbf{d}(a|p) + 2\mathbf{K}(n) + \mathbf{I}(\langle \tau \rangle : \mathcal{H}) + \mathbf{K}(p) \\ &\quad + O(\log \mathbf{I}(\langle \tau \rangle : \mathcal{H}) \mathbf{K}(p) \mathbf{K}(n)), \\ n - 3\mathbf{K}(n) - \max_{a \in \tau(n)} \mathbf{d}(a|p) &<^{\log} \mathbf{I}(\langle \tau \rangle : \mathcal{H}) + \mathbf{K}(p). \square \end{aligned}$$

Let k be a physical address of τ . \mathcal{H} can be described by a small mathematical statement. By Theorem 1 and IP,

$$\begin{aligned} s_{\tau,p} &<^{\log} \mathbf{I}(\langle \tau \rangle : \mathcal{H}) + O(\log \mathbf{K}(p)) \\ &<^{\log} k + c + O(\log \mathbf{K}(p)). \end{aligned}$$

It's hard to find observations with small anomalies and impossible to find observations with no anomalies.

References

- [Eps21] Samuel Epstein. All sampling methods produce outliers. *IEEE Transactions on Information Theory*, 67(11):7568–7578, 2021.
- [G21] Peter Gács. Lecture notes on descriptive complexity and randomness. *CoRR*, abs/2105.04704, 2021.
- [Lev74] L. A. Levin. Laws of Information Conservation (Non-growth) and Aspects of the Foundations of Probability Theory. *Problemy Peredachi Informatsii*, 10(3):206–210, 1974.
- [Lev13] L. A. Levin. Forbidden information. *J. ACM*, 60(2), 2013.