AIT Blog

Uniform Tests and Algorithmic Thermodynamic Entropy

Samuel Epstein*

December 22, 2022

For definitions in this post, we use [HR09]. A computable metric space X is a metric space with a dense set of ideal points on which the distance function is computable. A computable probability is defined by a computable sequence of converging points in the corresponding space of Borel probability measures, $\mathcal{M}(X)$, over X. A uniform test takes in a description of a probability measure μ and produces a lower computable μ test. There exists a universal uniform test, from [G2́1, HR09]. We extend the result from [Eps22] to computable metric spaces.

Theorem 1 *Given computable non-atomic probability measures μ and λ over a computable metric space X and universal uniform test $\mathbf{t}(\cdot,\cdot)$. For all n, $\lambda(\{\alpha : \mathbf{t}(\mu,\alpha) > 2^n\}) \overset{*}{>} 2^{-n-\mathbf{K}(n)}$.*

Reworking the above theorem, one can get a result in algorithmic physics. To define algorithmic fine-grain entropy, we use a slightly modified version of the definition in [Gac94], and I refer to that paper for the motivation of the definition. First, note that all the results [HR09] can be easily extended to arbitrary nonnegative measures which are used to represent volume in the space. This can be achieved by defining the product space of $\mathcal{M}(X)$ and $\mathbb{R}_{\geq 0}$, where the second metric space defines the size of the measure. Given a measure $\mu \in \mathcal{M}(X) \times \mathbb{R}_{\geq 0}$, the algorithmic fine grained entropy of a point $\alpha \in X$ is as follows.

Definition 1 (Algorithmic Fine-Grained Entropy) $\mathbf{H}_\mu(\alpha) = -\log \mathbf{t}(\mu,\alpha)$.

One can then prove that this term will oscillate in the presence of dynamics. Dynamics can be defined using group theory.

Definition 2 (Transformation Group) *Let M denote a computable metric space and G a topological group each element of which is a homeomorphism of M onto itself:*

$$f(g;x) = g(x) = x' \in M; g \in G, x \in M.$$

The pair (G, M) will be called a topological transformation group if for every pair of elements $g1, g2$ of G, and every $x \in M$, $g1(g2(x)) = (g1g2)(x)$ and if

$$x' = g(x) = f(g;x)$$

is continuous simultaneously in $x \in M$ and $g \in G$.

Theorem 2 (Oscillation of Thermodynamic Entropy) *Let L be the Lebesgue measure over $\mathbb{R}$. For one dimensional topological transformation group (G^t, X) with time being $\mathbb{R}$, acting on computable metric space X, for all $\alpha \in X$, $L\{t \in [0,1] : \mathbf{H}_\mu(G^t\alpha) < \log \mu(X) - n\} \overset{*}{>} 2^{-n-\mathbf{K}(n)}$.*

*JP Theory Group. samepst@jptheorygroup.org

Obviously, the above theorem does not hold in the case of static dynamics. I also will include some other secondary results that will hopefully round out the forthcoming paper. The Stability Theorem 5 in [Gac94] can be updated with the integration results in [HR09]. Let $\Pi(\cdot)$ be a set of disjoint uniformly enumerable open sets in the metric space X.

Definition 3 (Algorithmic Coarse Grained Entropy) $\mathbf{H}_\mu(\Pi_i) = \mathbf{K}(i|\mu) + \log \mu(\Pi_i)$.

Coarse grained entropy is an excellent approximation of fine grained entropy, as shown by the following two results.

Proposition 1 *If $\mu(\Pi_i)$ is uniformly computable and $\alpha \in \Pi_i$ then* $\mathbf{H}_\mu(\alpha) <^+ \mathbf{H}_\mu(\Pi_i) + \mathbf{K}(\Pi)$.

Lemma 1 (Stability) $\mu\{\alpha \in \Pi_i : \mathbf{H}_\mu(\alpha) < \mathbf{H}_\mu(\Pi_i) - \mathbf{K}(\Pi) - m\} \overset{*}{<} 2^{-m}\mu(\Pi_i)$.

We revisit an rather interesting result in [Gac94] that translates directly with the slightly new definitions of this blog entry. It states that if dynamics are used to increase or decrease algorithmic thermodynamic entropy by a non trivial amount, then the encoded dynamics shares algorithmic information with the ending or starting state, respectively. Put another way,

> *if you want to increase the entropy of a state, you need information about its ending state and if you want to decrease the entropy of a state, you need information about its starting state.*

The following definition introduces information between a point $\alpha \in X$ of the metric space and a sequence t. The term $\mathbf{H}_\mu(\alpha|t)$ is the fine grained algorithmic entropy of α when the universal Turing machine is relativized to the sequence t.

Definition 4 (Information) *For $\alpha \in X_1$, $\beta \in X_2$ and $t \in \{0,1\}^* \cup \{0,1\}^\infty$,*

- $\mathbf{I}(\alpha; t) = \mathbf{H}_\mu(\alpha) - \mathbf{H}_\mu(\alpha|t)$.
- $\mathbf{I}(\alpha : \beta) = \mathbf{H}_{\mu_1}(\alpha) + \mathbf{H}_{\mu_2}(\beta) - \mathbf{H}_{\mu_1 \times \mu_2}((\alpha, \beta))$.

Proposition 2 ([Gac94]) *If transformation group G is measure-preserving, then* $-\mathbf{I}(\alpha; \langle t \rangle) <^+ \mathbf{H}_\mu(G^t\alpha) - \mathbf{H}_\mu(\alpha) <^+ \mathbf{I}(G^t\alpha; \langle t \rangle)$.

Proposition 3 (Conservation of Information) $\mathbf{I}(G^t\alpha : \beta) <^+ \mathbf{I}(\alpha : \beta) + 2\mathbf{K}(t)$.

We revisit Maxwell's demon, providing yet another interpretation. This is done by reworking the Entropy Balance Theorem 9 in [Gac94]to the specific case of finite sequences. Let X be a computable metric space and μ its corresponding computable measure. We use $\{0,1\}^n$, the finite space of n bit sequences, and we use the counting measure with it. Thus if $x \in \{0,1\}^n$, $\mathbf{H}_\#(x) =^+ \mathbf{K}(x|n)$. For a starting point $\alpha \in X$, we couple it with an empty sequence 0^n, with dynamics G^t over discrete time, with a set time $t = 1$, producing $(x, \alpha') = G^1(0^n, \alpha)$.

Proposition 4 (Maxwell's Demon) $\mathbf{H}_\mu(\alpha) - \mathbf{H}_\mu(\alpha') <^+ \mathbf{K}(x|n)$.

Thus after α decreases in thermodynamic entropy, the contents of the register fills up. The original theorem is more general, and is over general spaces rather than sequences, and is over arbitrary time. Thus, putting Propositions 2 and 4 together, if one wants to lower the thermodynamic entropy of a state, the information of the state must be encoded into the dynamics or an independent environment can be coupled with the system which will absorb the entropy.

Its interesting to note that the proof for the above theorem follows from first using a combinatorial argument about finite strings and then applying this result to prove a property of randomness deficiencies of sequences and then transfering this result to universal uniform tests and then finally algorithmic thermodynamic entropy. An open question is whether other such transfers can be proven, resulting in further characterization of $\mathbf{H}_\mu$.

I have one more result in physics, which is conservation of algorithmic quantum randomness deficiency and information with respect to quantum operations. All these results will go into two papers: one containing the Quantum EL Theorem and the conservation inequalities, and another containing all the results of AIT and thermodynamics. The goal is make headway into the intersection of AIT and physics, dubbed *algorithmic physics*. Another area to look into would be the connection of AIT with special and general relativity, and black hole entropy.

References

[Eps22] S. Epstein. The outlier theorem revisited. *CoRR*, abs/2203.08733, 2022.

[Ǵ21] Peter Gács. Lecture notes on descriptional complexity and randomness. *CoRR*, abs/2105.04704, 2021.

[Gac94] P. Gacs. The boltzmann entropy and randomness tests. In *Proceedings Workshop on Physics and Computation. PhysComp '94*, pages 209–216, 1994.

[HR09] M. Hoyrup and C. Rojas. Computability of probability measures and martin-löf randomness over metric spaces. *Information and Computation*, 207(7):830–847, 2009.