

# A Short Proof on the Existence of Anomalies

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## Abstract

The Independence Postulate (IP) is a finitary Church-Turing Thesis, saying mathematical sequences are independent from physical ones. IP implies the existence of anomalies.

## Anomalies

$\mathbf{K}(x|y)$  is the conditional prefix Kolmogorov complexity. For probability  $p$  over  $\mathbb{N}$ , randomness deficiency is  $\mathbf{d}(a|p, b) = \lfloor -\log p(a) \rfloor - \mathbf{K}(a|\langle p \rangle, b)$  and measures the extent of the refutation against the hypothesis  $p$  given the result  $a$  [G21].  $\mathbf{d}(a|p) = \mathbf{d}(a|p, \emptyset)$ .  $\mathbf{I}(a; \mathcal{H}) = \mathbf{K}(a) - \mathbf{K}(a|\mathcal{H})$ , where  $\mathcal{H}$  is the halting sequence.  $<^+ f$  is  $< f + O(1)$  and  $<^{\log} f$  is  $< f + O(\log(f+1))$ . Stochasticity is  $\Lambda(a|b) = \min\{\mathbf{K}(Q|b) + 3 \log \max\{\mathbf{d}(a|Q, b), 1\} : Q \text{ has finite support and a range in } \mathbb{Q}\}$ .  $\Lambda(a|b) < \Lambda(a) + O(\log \mathbf{K}(b))$ . The following definition is from [Lev74].

**Definition 1 (Information)**  $\mathbf{I}(\alpha : \beta) = \log \sum_{x,y} 2^{\mathbf{K}(x) + \mathbf{K}(y) - \mathbf{K}(x,y) - \mathbf{K}(x|\alpha) - \mathbf{K}(y|\beta)}$ .

The Independence Postulate states [Lev13]:

**IP:** Let  $\alpha$  be a sequence defined with an  $n$ -bit mathematical statement, and a sequence  $\beta$  can be located in the physical world with a  $k$ -bit instruction set. Then  $\mathbf{I}(\alpha : \beta) < k + n + c$  for some small absolute constant  $c$ .

There are many proofs in the literature that non-stochastic numbers have high mutual information with the halting sequence. One such detailed proof is in [Eps21].

**Lemma 1**  $\Lambda(x) <^{\log} \mathbf{I}(x; \mathcal{H})$ .

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**Lemma 2** For probability  $p$  over  $\mathbb{N}$ ,  $D \subset \mathbb{N}$ ,  $|D| = 2^s$ ,  $s < \max_{a \in D} \mathbf{d}(a|p) + \Lambda(D) + \mathbf{K}(s) + O(\log \mathbf{K}(s, p))$ .

**Proof.** We relativize the universal Turing machine to  $\langle s, p \rangle$ . Let  $Q$  be a probability measure that realizes  $\Lambda(D)$ , with  $d = \max\{\mathbf{d}(D|Q), 1\}$ . Let  $F \subseteq \mathbb{N}$  be a random set where each element  $a \in \mathbb{N}$  is selected independently with probability  $cd2^{-s}$ , where  $c \in \mathbb{N}$  is chosen later.  $\mathbf{E}[p(F)] \leq cd2^{-s}$ . Furthermore

$$\begin{aligned} & \mathbf{E}[Q(\{G : |G| = 2^s, G \cap F = \emptyset\})] \\ & \leq \sum_G Q(G)(1 - cd2^{-s})^{2^s} < e^{-cd}. \end{aligned}$$

Thus finite  $W \subset \mathbb{N}$  can be chosen such that  $p(W) \leq 2cd2^{-s}$  and  $Q(\{G : |G| = 2^s, G \cap W = \emptyset\}) \leq e^{1-cd}$ .  $D \cap W \neq \emptyset$ , otherwise, using the  $Q$ -test,  $t(G) = e^{cd-1}$  if  $(|G| = 2^s, G \cap W = \emptyset)$  and  $t(G) = 0$  otherwise, we have

$$\begin{aligned} \mathbf{K}(D|Q, d, c) & <^+ -\log Q(D) - (\log e)cd \\ (\log e)cd & <^+ -\log Q(D) - \mathbf{K}(D|Q) + \mathbf{K}(d, c) \\ (\log e)cd & <^+ d + \mathbf{K}(d, c), \end{aligned}$$

which is a contradiction for large enough  $c$ . Thus there is an  $a \in D \cap W$ , where

$$\begin{aligned} \mathbf{K}(a) & <^+ -\log p(a) + \log d - s + \mathbf{K}(d) + \mathbf{K}(Q) \\ s & <^+ \mathbf{d}(a|p) + \Lambda(D). \end{aligned}$$

Making the relativization of  $\langle s, p \rangle$  explicit,

$$\begin{aligned} s & < -\log p(a) - \mathbf{K}(a|s, p) + \Lambda(D|s, p) \\ s & < \max_{a \in D} \mathbf{d}(a|p) + \Lambda(D) + \mathbf{K}(s) \\ & + O(\log \mathbf{K}(s, p)). \quad \square \end{aligned}$$

Let  $\tau \in \mathbb{N}^{\mathbb{N}}$  represent a series of observations. Assuming  $\tau$  has an infinite amount of unique numbers,  $\tau(n)$  is the first  $2^n$  unique numbers of  $\tau$ .

**Theorem 1** For probability  $p$  over  $\mathbb{N}$ ,  $\tau \in \mathbb{N}^{\mathbb{N}}$ , let  $s_{\tau,p} = \sup_n (n - 3\mathbf{K}(n) - \max_{a \in \tau(n)} \mathbf{d}(a|p))$ . Then  $s_{\tau,p} <^{\log} \mathbf{I}(\langle \tau \rangle : \mathcal{H}) + O(\log \mathbf{K}(p))$ .

**Proof.** By Lemmas 1 and 2, and the fact that  $\mathbf{I}(x; \mathcal{H}) <^+ \mathbf{I}(\alpha : \mathcal{H}) + \mathbf{K}(x|\alpha)$ ,

$$\begin{aligned} n &< \max_{a \in \tau(n)} \mathbf{d}(a|p) + \mathbf{I}(\tau(n); \mathcal{H}) + \mathbf{K}(n) \\ &\quad + O(\log \mathbf{I}(\tau(n); \mathcal{H}) \mathbf{K}(p) \mathbf{K}(n)), \\ n &< \max_{a \in \tau(n)} \mathbf{d}(a|p) + 2\mathbf{K}(n) + \mathbf{I}(\langle \tau \rangle : \mathcal{H}) \\ &\quad + O(\log \mathbf{I}(\langle \tau \rangle : \mathcal{H}) \mathbf{K}(p) \mathbf{K}(n)), \\ n - 3\mathbf{K}(n) - \max_{a \in \tau(n)} \mathbf{d}(a|p) &<^{\log} \mathbf{I}(\langle \tau \rangle : \mathcal{H}) + O(\log \mathbf{K}(p)). \square \end{aligned}$$

Let  $k$  be a physical address of  $\tau$ .  $\mathcal{H}$  can be described by a small mathematical statement. By Theorem 1 and IP,

$$\begin{aligned} s_{\tau,p} &<^{\log} \mathbf{I}(\langle \tau \rangle : \mathcal{H}) + O(\log \mathbf{K}(p)) \\ &<^{\log} k + c + O(\log \mathbf{K}(p)). \end{aligned}$$

It's hard to find observations with small anomalies and impossible to find observations with no anomalies.

## References

- [Eps21] Samuel Epstein. All sampling methods produce outliers. *IEEE Transactions on Information Theory*, 67(11):7568–7578, 2021.
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