AIT Blog

On Creating Pairs of Derandomization Theorems

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I've uploaded a new paper to my site, with the intention of eventually uploading it to arXiv and submission for publication. The main contribution is a resource bounded EL Theorem and a general formula for resource bounded randomization, in the sense of [Eps22]. In this blog, I show an example of taking a theorem produced from a non-constructive probabilistic proof and producing a pair of derandomization theorems, one that is resource free and one that is resource bounded.

A *hypergraph* is a pair $J = (V, E)$ of vertices V and edges $E \subseteq \mathcal{P}(V)$. Thus each edge can connect ≥ 2 vertices. A hypergraph is *k-uniform* of the size $|e| = k$ for all edges $e \in E$. A 2-uniform hypergraph is just a simple graph. A valid *C-coloring* of a hypergraph (V, E) is a mapping $f : V \to \{1, \ldots, C\}$ where every edge $e \in E$ is not *monochromatic* $|\{f(v) : v \in e\}| > 1$. The following classic result uses Lovasz Local Lemma.

Theorem. **[Probabilistic Method]** *Let $G = (V, E)$ be a k-regular hypergraph. If for each edge f, there are at most $2^{k-1}/e-1$ edges $h \in E$ such that $h \cap f \neq \emptyset$, then there exists a valid 2-coloring of G.*

We can now use derandomization, in the sense of [Eps22], to produce bounds on the Kolmogorov complexity of the simpliest such 2-coloring of G.

Theorem. **[Derandomization]** *Let $G = (V, E)$ be a k-regular hypergraph. If, for each edge f, there are at most $2^{k-1}/e - 1$ edges $h \in E$ such that $h \cap f \neq \emptyset$, then there exists a valid 2-coloring x of G with*

$$\mathbf{K}(x) <^{\log} \mathbf{K}(n, k) + 2ne/2^{k-1} + \mathbf{I}(G; \mathcal{H}).$$

The function $\mathbf{K}$ is the prefix free Kolmogorov complexity. $\mathbf{I}(G; \mathcal{H}) = \mathbf{K}(G) - \mathbf{K}(G|\mathcal{H})$ is the amount of asymmetric information the halting sequence $\mathcal{H} \in \{0,1\}^\infty$ has about the graph G. We can now use resource derandomization, introduced in `http://www.jptheorygroup.org/doc/Resource.pdf`, to achieve bounds for the smallest time-bounded Kolmogorov complexity $\mathbf{K}^t(x) = \min\{p : U(p) = x \text{ in } t(\|x\|) \text{ steps}\}$ of a 2-coloring of G. **Crypto** is the assumption that there exists a language in $\mathbf{DTIME}(2^{O(n)})$ that does not have size $2^{o(n)}$ circuits with Σ_2^p gates.

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Theorem. **[Resource Bounded Derandomization]** *Assume* ***Crypto****. Let $G_n = (V, E)$ be a k-regular hypergraph where $\|V\| = n$, uniformly polynomial time computable in n. Furthermore, for each edge f in G_n there are at most $2^{k-1}/e - 1$ edges $h \in E$ such that $h \cap f \neq \emptyset$. Then there is a polynomial p, and a valid 2-coloring x of G_n with*

$$\mathbf{K}^p(x) < 2ne/2^{k-1} + O(\log n).$$

The conjecture is that one can produce a suite of derandomization theorems, each one mapping to Kolmogorov complexity with different time and space constraints, and access to a certain number of random bits. In my uploaded paper, I used derandomization to show the tradeoff between codebook compression rate and channel capacity, so I believe there are a lot of applications of derandomization. However the codebook is of exponential size, so it is not suitable for resource-bounded derandomization.

References

[Eps22] S. Epstein. 22 examples of solution compression via derandomization. *CoRR*, abs/2208.11562, 2022.