AIT Blog

A New Proof to the Outliers Theorem

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In this blog entry, I present a new proof to the Outliers Theorem in [Eps21]. The proof to this theorem is similar to the proof of Proposition 1 in [Epsb] which is analogously similar to the proof in [Lev16]. It is different from the proof in [Lev16] in that it doesn't use left-total machines. It implies that large sets of strings with low deficiency of randomness will be non-stochastic, which implies that they have high mutual information with the halting sequence. In fact this works for any computable pairing function, not just probability measures. There are more direct proofs that show that algorithms must produce sets which contain elements that have high randomness deficiencies. These proofs can be found in [Eps22b] and achieve better bounds than if you used stochasticity theorems to produce the algorithmic no-go theorems. However the theorem in this blog and the theorems in [Eps21] are beneficial because one can use conservation properties of the mutual information function to prove new and interesting facts. This can be seen in [Epsa] which contains a result that thermodynamic entropy must oscillate in the presence of dynamics. The proof to this theorem uses properties of the mutual information with the halting sequence.

The hope was that the new proof in [Epsb] could lead to a Resource Bounded EL Theorem. However it turns out that such a theorem basically already existed in the literature, which can be found in [AF09]. The results in this paper could be quickly reworked into a Resource EL Theorem, with applications in Resource Bounded Derandomization [Eps22a]. However it could be that the proof in this blog entry and [Epsb] will have other interesting applications.

New Proof

A probability is *elementary*, if it has finite support and rational values. The deficiency of randomness of x relative to a elementary probability measure Q is $\mathbf{d}(x|Q) = -\log Q(x) - \mathbf{K}(x|Q)$.

Definition 1 (Stochasticisty) *A string x is (α,β)-stochastic if there exists an elementary probability measure Q such that*

$$\mathbf{K}(Q) \leq \alpha \text{ and } \mathbf{d}(x|Q) \leq \beta.$$

Theorem 1 *Let W be a positive computable function on strings and be c be a large constant. If D is (α,β)-stochastic relative to s and $\sum_{x\in D}\mathbf{m}(x)/W(x) \geq 2^s$ then there exists $x \in D$ with*

$$\log\frac{\mathbf{m}(x)}{W(x)} > s - \alpha - \log\beta - \mathbf{K}(\log\beta, s) - c.$$

This implies a slightly weaker form of Theorem 1 in [Epsb], which is when $W(a) = 1$.

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Proof. Let Q be a probability measure that realizes the stochasticity of D. Assume β is large. Let $\gamma = 2^{s-1}$ and $f(x) = \mathbf{m}(x)/W(x)$. Let f_* be a function on strings and $f_1, f_2, \dots$ be a finite or infinite series of function on strings such that

- In the infinite case, we have $f_* = 0$.
- The sequence $f_1, f_2, \dots$ can be uniformly computed given a program for W.
- All functions are nonnegative, and each function f_1, f_2 is non-zero for finitely many strings.
- $\sum_x f_i(x) = \gamma/\beta$ for all integers i and $\sum_x f_*(x) \leq \gamma/\beta$.
- $f = f_* + \sum_i f_i$.

We consider the infinite case. The finite case is similar.

Construction of a lower-semicomputable probability bounded test g on sets X of strings. Let $g_0 = 1$. We construct g_1, g_2, together with a sequence $z_1, z_2, \dots$ Assume that for some $i \geq 0$, we already defined g_i and $z_1, \dots, z_i$. Let z_{i+1} be chosen such that the function

$$g_{i+1}(X) = \begin{cases} g_i(X) & \text{if } g_i(X) \geq \exp(\beta) \\ g_i(X)\exp(\frac{\beta}{\gamma} f_i(X)) & \text{if } g_i(X) < \exp(\beta) \text{ and } X \text{ is disjoint from } \{z_1, \dots, z_i\} \\ 0 & \text{otherwise.} \end{cases}$$

satisfies

$$\frac{\gamma}{\beta} W(z_i) + \mathbb{E}_{X\sim Q} g_i(X) <^+ \sum_x f_i(x) W(x) + \mathbb{E}_{X\sim Q} g_{i-1}(X). \tag{1}$$

Let $g(X) = \exp(\beta)$ if some $g_i(X) \geq \exp(\beta)$ and 0 otherwise. *End of construction.*

We first prove that there always exists a z_i that satisfies the condition. Assume we select z_i randomly with probability $\frac{\beta}{\gamma} f_i(x)$. This is possible because these probabilities sum up to one by the definition of f_i. We show that the expected value of the left hand side in 1 equals the right hand side. Indeed $\mathbb{E}W(x) = \frac{\beta}{\gamma}\sum_x f_i(x)W(x)$. We also have

$$\mathbb{E}_{z_i\sim(\beta/\gamma)f_i} g_i(X) \leq (1 - (\beta/\gamma) f_i(X)) \cdot g_{i-1}(X)\exp((\beta/\gamma) f_i(X)) \leq g_{i-1}(X)$$
$$\mathbb{E}_{z_i\sim(\beta/\gamma)f_i, X\sim Q} g_i(X) \leq \mathbb{E}_{X\sim Q} g_{i-1}(X).$$

Hence the required z_i exists. We sum Equation 1 for all i, and obtain

$$\frac{\gamma}{\beta}\sum_i W(z_i) + \lim_i \mathbb{E}_{X\sim Q} g_i(X) \leq \sum_x f(x)W(x) + \mathbb{E}_{X\sim Q} g_0(X) \leq 1 + 1.$$

This implies that $\sum_i W(z_i) \leq 2\frac{\beta}{\gamma}$ and $\mathbb{E}g \leq 2$. Hence $g/2$ is an expection bounded test and $\mathbf{d}(X|\beta,\gamma) >^+ \log g(X)$. Note that if there is some set X with $\sum_{x\in X} f(x) \geq 2^{-s} = 2\gamma$ and X does not contain any element z_i, then $g_i(X) = \exp\sum_{j=1}^i \frac{\beta}{\gamma} f_j(X)$ and thus there is some i where $g_i(X) > \exp\beta$. Thus $g(X) = \exp\beta$ and hence X is not (α,β)-stochastic relative to s. So

$$1.44\beta \leq \log g(X) \leq \mathbf{d}(X|\beta,\gamma) \leq \mathbf{d}(X|s) + 2\log\beta.$$

Thus any D that satisfies the conditions of the theorem (i.e. has large β) must contain some element z_i, otherwise $1.44\beta \leq \mathbf{d}(D|s) + 2\log\beta <^{\log} \beta$, causing a contradiction. Since $\sum_i \frac{\gamma}{\beta} W(z_i) \leq 1$, and the list $z_1, z_2, \dots$ can be enumerated, we have that $\frac{\gamma}{\beta} W(z_i) \overset{*}{<} \mathbf{m}(z_i|\beta, s)$, and thus

$$\log\frac{\mathbf{m}(z_i)}{W(z_i)} >^+ \log\frac{\gamma}{\beta} - \mathbf{K}(\beta, s),$$

which proves the theorem. $\square$

References

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