Three Eggs from the Chicken

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In this note, we show an example of taking a theorem produced from a non-constructive probabilistic proof and produce three derandomization theorems, one that involves Kolmogorov complexity, one that involves resource boundedd Kolmogorov complexity, and one involving games.

Our note deals with hypergraphs. A *hypergraph* is a pair $J = (V, E)$ of vertices V and edges $E \subseteq \mathcal{P}(V)$. Thus each edge can connect ≥ 2 vertices. A hypergraph is *k-regular* of the size $|e| = k$ for all edges $e \in E$. A 2-regular hypergraph is just a simple graph. A valid *C-coloring* of a hypergraph (V, E) is a mapping $f : V \to \{1, \ldots, C\}$ where every edge $e \in E$ is not *monochromatic* $|\{f(v) : v \in e\}| > 1$. The following classic result [EL] is the first proved consequence of Lovász Local Lemma.

Theorem (Probabilistic Method) *Let $J = (V, E)$ be a k-regular hypergraph. If for each edge f, there are at most $2^{k-1}/e - 1$ edges $h \in E$ such that $h \cap f \neq \emptyset$, then there exists a valid 2-coloring of J.*

We can now use derandomization, from [Eps22a], to produce bounds on the Kolmogorov complexity of the simpliest such 2-coloring of G.

Theorem 1 (Derandomization) *Let $J = (V, E)$ be a k-regular hypergraph with $|E| = m$. If, for each edge f, there are at most $2^{k-1}/e - 1$ edges $h \in E$ such that $h \cap f \neq \emptyset$, then there exists a valid 2-coloring x of J with*

$$\mathbf{K}(x) <^{\log} \mathbf{K}(n) + 4me/2^k + \mathbf{I}(J; \mathcal{H}).$$

The function $\mathbf{K}$ is the prefix free Kolmogorov complexity. $\mathbf{I}(J; \mathcal{H}) = \mathbf{K}(J) - \mathbf{K}(J|\mathcal{H})$ is the amount of asymmetric information the halting sequence $\mathcal{H} \in \{0, 1\}^\infty$ has about the graph J. For nonnegative function f, $<^{\log} f$ is defined to be $< f + O(\log(f + 1))$. We can now use resource derandomization, from [Eps22b] , to achieve bounds for the smallest time-bounded Kolmogorov complexity $\mathbf{K}^t(x) = \min\{p : U(p) = x \text{ in } t(\|x\|) \text{ steps}\}$ of a 2-coloring of J.

Assumption 1 ***Crypto*** *is the assumption that there exists a language in $\mathbf{DTIME}(2^{O(n)})$ that does not have size $2^{o(n)}$ circuits with Σ_2^p gates.*

Theorem 2 (Resource Bounded Derandomization) *Assume **Crypto**. Let $J_n = (V, E)$ be a $k(n)$-regular hypergraph where $|V| = n$ and $|E| = m(n)$, uniformly polynomial time computable in n. Furthermore, for each edge f in J_n there are at most $2^{k(n)-1}/e - 1$ edges $h \in E$ such that $h \cap f \neq \emptyset$. Then there is a polynomial p, and a valid 2-coloring x of J_n with*

$$\mathbf{K}^p(x) < 4m(n)e/2^{k(n)} + O(\log n).$$

We define the following game involving hypergraphs that is from [Eps23]. The player has access to a list of vertices and the goal of the player is to produce a valid 2-coloring of the hypergraph. We assume that for each edge f of the graph, there are at most $2^{k-1}/e - 1$ edges h such that $f \cap h \neq \emptyset$.

The game proceeds as follows. For the first round, environment gives the number of vertices to the player. The player has n vertices, each with starting color 1. At each subsequent turn, the environment sends to the player the edges which are monochromatic. The player can change the color of up to k vertices and sends these changes to the environment. The game ends when the player has a valid 2-coloring of the graph.

Theorem 3 (Game Derandomization) *For $k \geq 6$, there exists a player $\mathbf{p}$ that can beat the environment $\mathbf{q}$ in $(1+\epsilon)n/k$ turns, with Kolmogorov complextiy $\mathbf{K}(\mathbf{p}) <^{\log} \mathbf{I}(\mathbf{q};\mathcal{H}) - \log\epsilon$, where $\epsilon \in (0,1)$.*

Comparing Theorems 1 and 3 is interesting because it shows if your program is involved in an interaction with an environment where the contents of the hypergraph are disclosed, then tighter bounds on its Kolmogorov complexity can be proved.

References

[EL] P. Erdos and L. Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. *Infinite and finite sets*, 10:609–627.

[Eps22a] S. Epstein. 22 examples of solution compression via derandomization. *CoRR*, abs/2208.11562, 2022.

[Eps22b] S. Epstein. Derandomization under different resource constraints. *CoRR*, abs/2211.14640, 2022.

[Eps23] Samuel Epstein. Game derandomization, 2023. https://www.jptheorygroup.org/doc/Game.pdf.