Kolmogorov Derandomization of Probabilistic Games

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Abstract

Using Kolmogorov Game Derandomization, upper bounds of the Kolmogorov complexity of deterministic winning players against deterministic environments can be proved. This paper extends this result, generalizing to probabilistic games. This applies to computable or uncomputable environments. We apply this result to the classic even-odds game. In addition, we start with an illustrative example of game derandomization, involving minotaurs and labyrinths.

1 The Minatour and the Labyrinth

A hero is trapped in a labyrinth, which consists of long corridors connecting to small rooms. The intent of the the hero is to reach the goal room, which has a ladder in its center reaching the outside. The downside is the hero is blindfolded. The upside is there is a minotaur present to guide the hero.

At every room, the minotaur tells the hero the number of corridors n leading out (including the one which the hero just came from). The hero states a number between 1 and n and the minotaur takes the hero to corresponding door. However the hero faces another obstacle, in that the minotaur is trying to trick him. This means the mapping the minotaur uses is a function of all the hero's past actions. Thus if a hero returns to the same room, he may be facing a different mapping than before. This process continues for a very large number of turns. The question is how much information is needed by the hero to find the exit? Using **Kolmogorov Game Derandomization**, we get the following surprising good news for the hero. Let c be the number of corridors and d be the number of doors in the goal room.

The hero can find the exit using $\log(c/d) + \epsilon$ *bits.*

The error term ϵ is logarithmic and also is dependent on the information the halting sequence has about the entire construct, which is negligible except

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in exotic cases. Assuming the **Independence Postulate** [Lev84, Lev13], one cannot find such exotic constructs in the physical world.

The reasoning for this is as follows. Take a random hero who chooses a corridor with uniform probability. Then the hero is performing a random walk on the graph (of the labyrinth). Assuming the number of turns is greater than the graph's mixing time, the probability the hero is at exit at the end is not less than d/bc, for some fixed constant b. Then the following theorem can be applied. $\mathbf{K}(x)$ is the prefix Kolmogorov complexity of x. $\mathbf{I}(x;\mathcal{H}) = \mathbf{K}(x) - \mathbf{K}(x|\mathcal{H})$ is the amount of information the halting sequence $\mathcal{H}$ has about x.

Theorem 1 ([Eps23a]) *If probabilistic agent* $\mathbf{p}$ *wins against environment* $\mathbf{q}$ *with at least probability* p*, then there is a deterministic agent of complexity* $K(\mathbf{p}) - \log p + \mathbf{I}(\langle p, \mathbf{p}, \mathbf{q}\rangle;\mathcal{H})$ *that wins against* $\mathbf{q}$.

2 Setup

The main result of this paper is to extend Theorem 6 to probabilistic environments. Before we do so, we introduce some key tools necessary to prove this fact. We use $x <^+ y$, $x >^+ y$ and $x =^+ y$ to denote $x < y + O(1)$, $x + O(1) > y$ and $x = y \pm O(1)$, respectively. In addition, $x <^{\log} y$ and $x >^{\log} y$ denote $x < y + O(\log y)$ and $x + O(\log x) > y$, respectively. We say $[A] = 1$ if mathematical statement A is true, and $[A] = 0$, otherwise. The function $\mathbf{m}(x)$ is a universal lower-computable semi-measure. Mutual information between strings is $\mathbf{I}(x : y) = \mathbf{K}(x) + \mathbf{K}(y) - \mathbf{K}(x, y)$.

A probability P over $\mathbb{N}$ is elementary if it has finite support and its range is a subset of $\mathbb{Q}$. Elementary probabilities can be encoded into finite strings or natural numbers. The randomness deficiency of $x \in \mathbb{N}$ with respect to elementary probability P and $y \in \mathbb{N}$ is $\mathbf{d}(x|P, y) = \lceil -\log P(x)\rceil - \mathbf{K}(x|P, y)$.

Definition 1 (Stochasticity) *The stochasticity of* $x \in \mathbb{N}$ *with respect to* $y \in \mathbb{N}$ *is* $\mathbf{Ks}(x|y) = \min\{\mathbf{K}(P|y) + 3\log\max\{\mathbf{d}(x|P, y), 1\} : P \textit{ is elementary}\}$.

Lemma 1 ([Eps23b, Lev16]) $\mathbf{Ks}(x|y) <^{\log} \mathbf{I}(x;\mathcal{H}|y)$.

Lemma 2 ([Eps22]) *For partial computable function* f*,*
$\mathbf{I}(f(x);\mathcal{H}) <^+ \mathbf{I}(x;\mathcal{H}) + \mathbf{K}(f)$.

A Win/No-Halt game is a series of interactions between an agent $\mathbf{p}$ and an environment $\mathbf{q}$. Each round starts with $\mathbf{p}$ initiating a move, which is chosen out of $\mathbb{N}$ and then $\mathbf{q}$ responds with a number or $\mathbf{q}$ can halt. Agent $\mathbf{p}$ wins if $\mathbf{q}$ halts the game, otherwise the game can continue potentially forever. Thus $\mathbf{p}$ is a function $(\mathbb{N}\times\mathbb{N})^* \mapsto \mathbb{N}$ and $\mathbf{q}$ is a function $\mathbb{N}\times(\mathbb{N}\times\mathbb{N})^* \mapsto \mathbb{N}\cup\{\emptyset\}$. Both $\mathbf{p}$ and $\mathbf{q}$ are are assumed to be computable, however, uncomputable environments are studied in Section 6. Both the agent and environment can be probabilistic in their choice actions. Thus the probabilities of each action are uniformly computable to any degree of accuracy.

3 Probabilistic Games

In this section we prove Kolmogorov Game Derandomization over probabilistic environments. This is an extension to Theorem 6, enabling the characterization of all environments that are probabilistically computable.

The main proof uses the notion of a *game fragment*. A game fragment $\mathcal{F}$ is a finite tree, where each edge has a number $n \in \mathbb{N}$ representing an action. On the odd levels, the edges are coupled with rational weights in $[0, 1]$ and the summation of weights on each level is less than 1. Such fragments $\mathcal{F}$ can be coupled with an probabilistic agent $\mathbf{p}$, who fills in the weights of each even level edges with its probabilistic action. In such a coupling, the weight of each path is the product of the probabilities along each edge of the path. $\text{Weight}(\mathbf{p}, \mathcal{F})$ is the sum of the weights of each path. If each path represents a winning interaction with an environment $\mathbf{q}$ and the weights correspond to $\mathbf{q}$'s probabilities of those particular action then $\text{Weight}(\mathbf{p}, \mathcal{F})$ is not more than the probability that $\mathbf{p}$ wins against $\mathbf{q}$.

Theorem 2 *Let* $\mathbf{p}$ *be a probabilistic agent and* $\mathbf{q}$ *be a probabilistic environment. If* $\mathbf{p}$ *Wins in the Win/No-Halt game against* $\mathbf{q}$ *with probability* $> 2^{-s}$*,* $s \in \mathbb{N}$*, then there is a deterministic agent of complexity* $<^{\log} \mathbf{K}(\mathbf{p}) + 2s + \mathbf{I}(\langle \mathbf{p}, s, \mathbf{q}\rangle; \mathcal{H})$ *that wins with probability* $> 2^{-s-1}$.

Proof. We relativize the universal Turing machine to $\langle \mathbf{p}, s\rangle$. Thus this information is on an auxiliary tape and implicitly in the conditional of all complexity terms. Let $\mathcal{F}$ be a game fragment corresponding the environment $\mathbf{q}$ such that each path is a winning interaction and $\text{Weight}(\mathbf{p}, \mathcal{F}) > 2^{-s}$ and also $\mathbf{K}(\mathcal{F}|\mathbf{q}) = O(1)$. Note that the actions of $\mathcal{F}$ are rationals which lower bound $\mathbf{q}$'s computable action probabilities. Let Q be an elementary probability measure that realizes $\mathbf{Ks}(\mathcal{F})$ and $d = \max\{\mathbf{d}(\mathcal{F}|Q), 1\}$. Without loss of generality, one can limit the support of Q to encodings of game fragments $\mathcal{G}$ such that $\text{Weight}(\mathcal{G}, \mathbf{p}) > 2^{-s}$. This can be done by defining a new probability Q' that is Q conditioned on the above property, which is straightforward but tedious. Let m be the longest path and ℓ be the largest action number of any game fragment in the support of Q. We define a probability P over deterministic agents $\mathbf{g}$ defined up to m steps and up to ℓ actions. Each action of the deterministic agent is determined by the corresponding probability of actions in that turn by $\mathbf{p}$. Using backwards induction, for each math fragment $\mathcal{G}$ in the support of Q,

$$\mathbf{E}_{\mathbf{g}\sim P}[\text{Weight}(\mathbf{g}, \mathcal{G})] > 2^{-s}.$$

Let N be a number to be specified later. Assume we randomly define N determinstic agents $\{\mathbf{g}_i\}_{i=1}^N$, each drawn i.i.d. from P. For math fragment $\mathcal{G}$ in the support of Q, $X_{\mathcal{G}} = \frac{1}{N}\sum \text{Weight}(\mathbf{g_i}, \mathcal{G})$. Each such $X_{\mathcal{G}}$ is a random variable. By the Hoeffding's inequality,

$$\Pr(X_{\mathcal{G}} \leq 2^{-s-1}) < 2\exp(-N2^{-2s-2}).$$

Let $N = d2^{2s+3}$. Then it is possible to find a set of N deterministic agents such that

$$Q(\{\mathcal{G} : X_{\mathcal{G}} \leq 2^{-s-1}\}) < e^{-d}.$$

In the above formula, each $X_{\mathcal{G}}$ is a fixed value and no longer a random variable. It must be that $X_{\mathcal{F}} > 2^{-s-1}$. Otherwise using Q-test $t(\mathcal{G}) = [X_{\mathcal{G}} \leq 2^{-s}]e^d$,

$$1.44d < \log t(\mathcal{F}) <^+ \mathbf{d}(\mathcal{F}) =^+ d.$$

This is a contradiction for large enough d which we can assume without loss of generality. Thus since $X_{\mathfrak{F}} > 2^{-s-1}$ there exists deterministic agent $\mathbf{g}_i$ such that $\text{Weight}(\mathbf{g}_i, \mathcal{F}) > 2^{-s-1}$. Thus $\mathbf{g}_i$ wins against $\mathbf{q}$ with probability more than 2^{-s-1}. So,

$$\begin{aligned}
\mathbf{K}(\mathbf{g}_i|s,\mathbf{p}) &<^+ \log N + \mathbf{K}(N|s,\mathbf{p}) \\
\mathbf{K}(\mathbf{g}_i) &<^+ \mathbf{K}(s,\mathbf{p}) + <^+ 2s + \log d + \mathbf{K}(d,Q|s,\mathbf{p}) \\
&<^{\log} \mathbf{K}(\mathbf{p}) + 2s + 3\log d + \mathbf{K}(Q|s,\mathbf{p}) \\
&<^{\log} \mathbf{K}(\mathbf{p}) + 2s + \mathbf{Ks}(\mathfrak{F})|s,\mathbf{p}) && (1) \\
&<^{\log} \mathbf{K}(\mathbf{p}) + 2s + \mathbf{Ks}(\mathfrak{F}) + O(\log \mathbf{K}(s,\mathbf{p})) && (2) \\
&<^{\log} \mathbf{K}(\mathbf{p}) + 2s + \mathbf{I}(\mathfrak{F};\mathcal{H}) && (3) \\
&<^{\log} \mathbf{K}(\mathbf{p}) + 2s + \mathbf{I}(\langle \mathbf{p}, s, \mathbf{q}\rangle;\mathcal{H}). && (4)
\end{aligned}$$

Equations 1 and 2 follow from the definition of stochasticity, **Ks**. Equation 3 follows from Lemma 1. Equation 4 follows from Lemma 4. □

Corollary 1 *Let computable $\epsilon \in (0,1)$. Let $\mathbf{p}$ be a probabilistic agent and $\mathbf{q}$ be a probabilistic environment. If $\mathbf{p}$ Wins in the Win/No-Halt game against $\mathbf{q}$ with probability $> 2^{-s}$, $s \in \mathbb{N}$, then there is a deterministic agent of complexity $<^{\log} \mathbf{K}(\mathbf{p}) + 2s + \mathbf{I}(\langle \mathbf{p}, s, \mathbf{q}\rangle;\mathcal{H}) + O_\epsilon(1)$ that wins with probability $> \epsilon 2^{-s}$.*

Corollary 2 *Let computable $\epsilon \in (0,1)$ and computable $p \in (0,1)$. Let $\mathbf{p}$ be a probabilistic agent and $\mathbf{q}$ be a probabilistic environment. If $\mathbf{p}$ Wins in the Win/No-Halt game against $\mathbf{q}$ with probability $> p$, then there is a deterministic agent of complexity $<^{\log} \mathbf{K}(\mathbf{p}) + \mathbf{I}(\langle \mathbf{p}, \mathbf{q}\rangle;\mathcal{H}) + O_{p,\epsilon}(1)$ that wins with probability $> \epsilon p$.*

4 Even-Odds

We define the following game, entitled EVEN-ODDS. There are N rounds. The player starts out with a score of 0. At the start of each round, the environment $\mathbf{q}$ secretly records a bit $e_i \in \{0,1\}$. The player sends $\mathbf{q}$ a bit b_i and the environment responds with e_i. The agent gets a point if $e_i \oplus b_i = 1$. Otherwise the agent loses a point. The environment $\mathbf{q}$ can be any probabilistic algorithm. There are N rounds.

Theorem 3 *For large enough number of rounds, N, given any probabilistic environment $\mathbf{q}$ there is a deterministic agent $\mathbf{p}$ of complexity $\mathbf{K}(\mathbf{p}) <^{\log} \mathbf{I}(\mathbf{q};\mathcal{H})$ that can achieve a score of $\sqrt{N}$ with probability $> 1/21$.*

Proof. We describe a probabilistic agent $\mathbf{p}'$. At round i, $\mathbf{p}'$ submits 0 with probability 1/2. Otherwise it submits 1. By the central limit theorem, for large enough N, the score of the probabilistic agent divided by $\sqrt{N}$ is $S \sim \mathcal{N}(0,1)$. Let $\Phi(x) = \Pr[S > x]$. A common bound for $\Phi(x)$ is

$$\Phi(x) > \frac{1}{2\pi}\frac{x}{x^2+1}e^{-x^2/2}$$
$$\Phi(1) > \frac{1}{4\pi}e^{-1/2} = p.$$

Describe a Win/No-Halt game where the player $\mathbf{p}'$ wins if it has a score of at least $\sqrt{N}$. Thus $\mathbf{p}'$ wins with probability greater than p. Thus by Corollary 2, with $\epsilon = 1/(21p)$, there exists a deterministic agent $\mathbf{p}$ that can beat $\mathbf{q}$ with complexity

$$\mathbf{K}(\mathbf{p}) <^{\log} \mathbf{K}(\mathbf{p}') + \mathbf{I}(\langle \mathbf{p}', \mathbf{q}\rangle;\mathcal{H}) <^{\log} \mathbf{I}(\mathbf{q};\mathcal{H}).$$

Furthermore $\mathbf{p}$ wins with probability $> 1/21$. □

5 The Minatour Revisited

Suppose the minatour has gotten fed up with the hero, who can find the exit using a very small amount of information. The minatour decides to use chance to his advantage. At every room the minatour computes a probability over all possible mappings of numbers to doors and selects a mapping at random. This probability is a functions of all the hero's previous actions. However due to derandomization of probabilistic games, the hero can achieve the following results. Let c be the number of corridors and d be the number of doors in the goal room.

Theorem 4 *There is a positive constant b, where given any labyrinth and probabilistic minatour (L, M), there is a deterministic hero $\mathbf{p}$ of complexity $\mathbf{K}(\mathbf{p}) <^{\log} 2\log(c/d) + \mathbf{I}(\langle L, M\rangle;\mathcal{H})$ that can find the goal room with probability not less than bd/c.*

We leave the proof to the reader.

6 Uncomputable Environments

In this section, we derive the results of Theorems 6 and 2 with respect to uncomputable environments. We will use the following mutual information term between infinite sequences.

Definition 2 ([Lev74]) *For* $\alpha, \beta \in \{0,1\}^\infty$,
$\mathbf{I}(\alpha : \beta) = \log \sum_{x,y \in \{0,1\}^*} \mathbf{m}(x|\alpha)\mathbf{m}(y|\beta)2^{\mathbf{I}(x:y)}$.

Proposition 1 $\mathbf{I}(x;\mathcal{H}) <^+ \mathbf{I}(\alpha : \mathcal{H}) + \mathbf{K}(x|\alpha)$.

Now a probabilistic environment $\mathbf{q}$ is of the form $\mathbb{N} \times (\mathbb{N} \times \mathbb{N})^* \to [0,1]$. We fix a computable function ℓ such that for every environment $\mathbf{q}$ there is an infinite sequence α such that $\ell(\alpha, \cdot)$ computes $\mathbf{q}$. Let $\ell[\mathbf{q}]$ be the set of all such infinite sequences α.

Definition 3 *For probabilistic environment* $\mathbf{q}$, $\mathbf{I}(\mathbf{q} : \mathcal{H}) = \inf_{\alpha \in \ell[\mathbf{q}]} \mathbf{I}(\alpha : \mathcal{H})$.

Theorem 5 *Let* $\mathbf{p}$ *be a probabilistic agent and* $\mathbf{q}$ *be a (potentially uncomputable) probabilistic environment. If* $\mathbf{p}$ *Wins in the Win/No-Halt game against* $\mathbf{q}$ *with probability* $> 2^{-s}$, $s \in \mathbb{N}$, *then there is a deterministic agent of complexity* $<^{\log} \mathbf{K}(\mathbf{p}) + 2s + \mathbf{I}(\langle \mathbf{p}, s, \mathbf{q} \rangle : \mathcal{H})$ *that wins with probability* $> 2^{-s-1}$.

Proof. Using $\mathbf{p}$, s, any encoding $\alpha \in \ell[\mathbf{q}]$, and ℓ, one can construct the math fragment $\mathcal{F}$ described in the proof of Theorem 2. Let $\alpha \in \ell[\mathbf{q}]$ and $\mathbf{I}(\alpha : \mathcal{H}) < \mathbf{I}(\mathbf{q} : \mathcal{H}) + 1$. Thus $\mathbf{K}(\mathcal{F}|\mathbf{p}, s, \alpha) = O(1)$. Using Proposition 1, the definition of $\mathbf{I}(\mathbf{q} : \mathcal{H})$, and the reasoning of the proof of Theorem 2, this theorem follows. □

The follow Theorem extends Theorem 6 to uncomputable environments.

Theorem 6 *If probabilistic agent* $\mathbf{p}$ *wins against deterministic, and potentially uncomputable, environment* $\mathbf{q}$ *with at least probability* p, *then there is a deterministic agent of complexity* $\mathbf{K}(\mathbf{p}) - \log p + \mathbf{I}(\langle p, \mathbf{p}, \mathbf{q} \rangle : \mathcal{H})$ *that wins against* $\mathbf{q}$.

Proof. This follows from using the same reasoning as the proof for Theorem 5 and the proof of Theorem 6 found in [Eps23a]. □

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