Outliers are in the Physical World

Samuel Epstein*

July 13, 2022

Abstract

The Independence Postulate (IP) is a finitary Church Turing Thesis, postulating that certain infinite and finite sequences cannot be found in nature, a.k.a. have high "physical addresses". In this paper we show that IP implies that outliers are found in the physical world.

1 Introduction

$\mathbf{K}(x|y)$ is the conditional prefix complexity. For probability p over $\mathbb{N}$, the deficiency of randomness is $\mathbf{d}(a|p) = \lfloor -\log p(a)\rfloor - \mathbf{K}(a|P)$. $\mathbf{I}(a;\mathcal{H}) = \mathbf{K}(a) - \mathbf{K}(a|\mathcal{H})$, where $\mathcal{H} \in \{0,1\}^\infty$ is the halting sequence. Let $[A] = 1$, where if A is true otherwise $[A] = 0$. An elementary probability measure $\mathbb{N}$ has finite support and range in $\mathbb{Q}$. The stochastiity of $a \in \mathbb{N}$, conditional to $b \in \mathbb{N}$, is measured by

$$\Lambda(a|b) = \min\{\mathbf{K}(Q|b) + 3\log\max\{\mathbf{d}(a|Q,b),1\}$$
$$: Q \text{ is an elementary probability measure}\}.$$
$$\Lambda(a|b) < \Lambda(a) + O(\log \mathbf{K}(b)).$$

The information between two sequences is

Definition 1 *For* $\alpha,\beta\in\{0,1\}^{*\infty}$, $\mathbf{I}(\alpha:\beta) = \log\sum_{x,y\in\{0,1\}^*} 2^{\mathbf{K}(x)+\mathbf{K}(y)-\mathbf{K}(x,y)-\mathbf{K}(x|\alpha)-\mathbf{K}(y|\beta)}$.

The Independence Postulate [Lev84, Lev13] is as follows:

IP: *Let α be a sequence defined with an n-bit mathematical statement, and a sequence β can be located in the physical world with a k-bit instruction set. Then $\mathbf{I}(\alpha:\beta) < k+n+c$ for some small absolute constant c.*

Lemma 1 ([EL11]) *For $x \in \mathbb{N}$, $\Lambda(x) <^{\log} \mathbf{I}(x;\mathcal{H})$.*

Theorem 1 *For computable probability p over $\mathbb{N}$ and for $D \subset \{0,1\}^*$, $|D| = 2^s$, $s < \max_{a\in D}\mathbf{d}(a|p) + \Lambda(D) + \mathbf{K}(s) + \mathbf{K}(p) + O(\log\mathbf{K}(s)\mathbf{K}(p))$.*

Proof. We relativize the universal Turing machine U to p and s. Let Q be an elementary probability distribution that realizes $\Lambda(D)$. Let $d = \mathbf{d}(D|Q)$. Let V be the combined elements of encoded sets of size 2^s in the support of Q. Suppose each element of V is selected independently with probability $cd2^{-s}$, where $c \in \mathbb{N}$ is chosen later. The selected set is F, and $\mathbf{E}[p(F)] \le cd2^{-s}$. Furthermore

$$\mathbf{E}[Q(\{G : |G| = 2^s, G\cap F = \emptyset\})]$$
$$\le \sum_G Q(G)(1-cd2^{-s})^{2^s} < e^{-cd}.$$

Thus finite $F \subset \mathbb{N}$ can be chosen such that $p(F) \le 2cd2^{-s}$ and $Q(\{G : |G| = 2^s, G\cap F = \emptyset\}) \le e^{1-cd}$. Now it must be that $D \cap F \neq \emptyset$. Otherwise, using the Q-test, $t(G) = [|G| = 2^s, G\cap F = \emptyset]e^{cd-1}$, we have

$$\mathbf{K}(D|Q,d,c) <^+ -\log Q(D) - (\log e)cd$$
$$(\log e)cd <^+ -\log Q(D) - \mathbf{K}(D|Q) + \mathbf{K}(d,c)$$
$$(\log e)cd <^+ d + \mathbf{K}(d,c),$$

which is a contradiction for large c. Thus there is an $a \in D\cap F$, where

$$\mathbf{K}(a) <^+ -\log p(a) + \log d - s + \mathbf{K}(d) + \mathbf{K}(Q)$$
$$s <^+ \mathbf{d}(a|p) + \Lambda(D).$$

Removing the relativization of p and s

$$s < -\log p(a) - \mathbf{K}(a|p,s) + \Lambda(D|p,s)$$
$$s < \max_{a\in D}\mathbf{d}(a|p) + \Lambda(D) + \mathbf{K}(s) + \mathbf{K}(p)$$
$$+ O(\log\mathbf{K}(s)\mathbf{K}(p)).$$

*JP Theory Group. samepst@jptheorygroup.org

$\square$

For $\tau \in \mathbb{N}^{\mathbb{N}}$, $\langle\tau\rangle = \langle\tau[1]\rangle\langle\tau[2]\rangle\langle\tau[3]\rangle\dots$ Let $\tau(n)$ be the first 2^n unique numbers found in τ. The sequence τ is assumed to have an infinite amount of unique numbers, and represents a series of observations.

Theorem 2 *For probability p over $\mathbb{N}$ and $\tau \in \mathbb{N}^{\mathbb{N}}$, let s_τ be the smallest number such that $\max_{a\in\tau(n)} \mathbf{d}(a|p) > n - 4\mathbf{K}(n) - s_\tau$. Then $s_\tau <^{\log} \mathbf{I}(\langle\tau\rangle : \mathcal{H}) + \mathbf{K}(p)$.*

Proof. By Theorem 1, Lemma 1, and the fact that $\mathbf{I}(x;\mathcal{H}) <^+ \mathbf{I}(\alpha : \mathcal{H}) + \mathbf{K}(x|\alpha)$,

$$\begin{aligned}
n &< \max_{a\in\tau(n)} \mathbf{d}(a|p) + \mathbf{I}(\tau(n);\mathcal{H}) + \mathbf{K}(p) + 2\mathbf{K}(n)\\
&\quad + O(\log(\mathbf{I}(\tau(n);\mathcal{H}) + \mathbf{K}(p))),\\
n &< \max_{a\in\tau(n)} \mathbf{d}(a|p) + 4\mathbf{K}(n) + \mathbf{I}(\langle\tau\rangle : \mathcal{H}) + \mathbf{K}(p)\\
&\quad + O(\log(\mathbf{I}(\langle\tau\rangle : \mathcal{H}) + \mathbf{K}(p))),\\
&n - 4\mathbf{K}(n) - (\mathbf{I}(\langle\tau\rangle : \mathcal{H}) + \mathbf{K}(p)\\
&\quad + O(\log(\mathbf{I}(\langle\tau\rangle : \mathcal{H}) + \mathbf{K}(p)))) < \max_{a\in\tau(n)} \mathbf{d}(a|p).
\end{aligned}$$

$\square$

Let k be the physical address of an infinite sequence of numbers τ. The halting sequence $\mathcal{H}$ can be described in a mathematical statement of size equal to some small constant c_{HM}. Then by Theorem 2, for some sequence $\tau \in \mathbb{N}^{\mathbb{N}}$ and probability p over $\mathbb{N}$, **IP** states

$$s_\tau <^{\log} \mathbf{I}(\langle\tau\rangle : \mathcal{H}) + \mathbf{K}(p) <^{\log} k + c_{\mathrm{HM}} + c + \mathbf{K}(p).$$

Thus sequences τ with large s_τ, as defined in Theorem 2, will have large physical address. Thus it is hard to find physical sequences which do not have large outliers, and completely impossible to find sequences with no outliers. As the complexity of the probability p in the randomness deficiency term increases, the bounds loosen.

References

[EL11] Samuel Epstein and Leonid Levin. On sets of high complexity strings. *CoRR*, abs/1107.1458, 2011.

[Lev84] L. A. Levin. Randomness conservation inequalities; information and independence in mathematical theories. *Information and Control*, 61(1):15–37, 1984.

[Lev13] L. A. Levin. Forbidden information. *J. ACM*, 60(2), 2013.