AIT Blog

A Quantum EL Theorem

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In this blog entry, we introduce a Quantum EL theorem: non exotic projections of large rank must have simple quantum pure states in their images. Simplicity is measured according to the classical information content of a pure state. It is similar to the definition in [Vit00] except a classical Turing machine is used instead of a quantum Turing machine.

Definition 1 (Complexity of a Quantum Pure State)
For n qubit state $|\phi\rangle$, $\mathbf{K}(|\phi\rangle|n) = \min\{\mathbf{K}(|\psi\rangle|n) - \log|\langle\phi|\psi\rangle|^2 : |\psi\rangle$ is an elementary pure state$\}$.

Definition 2 (Computable Operators) *For computable operator A, $\mathbf{I}(A;\mathcal{H}|y) = \min\{\mathbf{K}(p|y) - \mathbf{K}(p|y,\mathcal{H}) : p$ is a program that computes $A\}$. $\mathcal{H} \in \{0,1\}^\infty$ is the halting sequence.*

Theorem 1 (Quantum EL Theorem) *Fix an n qubit Hilbert space. Let P be a computable projection of rank $> 2^m$. Then, $\min_{|\phi\rangle\in\mathrm{Image}(P)} \mathbf{K}(|\phi\rangle|n) <^{\log} 3(n-m) + \mathbf{I}(P;\mathcal{H}|n)$.*

Corollary 1 *Fix an n qubit Hilbert space. Let ρ be a density matrix of rank $> 2^m$. Then, $\min_{|\phi\rangle\in\mathrm{Image}(\rho)} \mathbf{K}(|\phi\rangle|n) <^{\log} 3(n-m) + \mathbf{I}(\rho;\mathcal{H}|n)$.*

The corollary is due to conservation of information. More specifically, if operator P is the projection onto the image of density matrix ρ, then $\mathbf{K}(P|\rho) = O(1)$ and also $\mathbf{I}(P;\mathcal{H}) <^+ \mathbf{I}(\rho;\mathcal{H})$. Thus the theorem applies to any quantum operator. This also applies to algorithmic quantum entropy [GÓ1] since it is less than $\mathbf{K}(|\phi\rangle)$. Another application is that if a quantum measurement can (even approximately) detect quantum algorithmic complexity, then it is exotic, in that it has high mutual information with the halting sequence. Put another way, if you receive a measurement, you cannot use it to infer the algorithmic complexity of the collapsed state. One note is that the theorem can be generalized to arbitrary (i.e. uncomputable) operators A. The error term is $\inf_{\langle A\rangle} \mathbf{I}(\langle A\rangle : \mathcal{H})$, where $\langle A\rangle$ is any appropriate infinite encoding of the operator and $\mathbf{I}$ is the mutual information term between infinite sequences.

This blog post is another example of a result in the intersection of AIT and physics. Future work in algorithmic physics involves proving that algorithmic thermodynamic entropy must oscillate in the presence of dynamics. Another avenue of future work is conservation of algorithmic randomness and information with respect to the most general transformation of a qubit, a quantum operation.

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References

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