A Short Proof on the Existence of Anomalies

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Abstract

The Independence Postulate (IP) is a finitary Church-Turing Thesis, postulating that mathematical sequences are independent from physical ones. IP implies that anomalies are found in the physical world.

Anomalies

$\mathbf{K}(x|y)$ is the conditional prefix Kolmogorov complexity. For probability p over $\mathbb{N}$, the deficiency of randomness is $\mathbf{d}(a|p) = \lfloor -\log p(a) \rfloor - \mathbf{K}(a)$. $\mathbf{I}(a;\mathcal{H}) = \mathbf{K}(a) - \mathbf{K}(a|\mathcal{H})$, where $\mathcal{H}$ is the halting sequence. An elementary probability measure over $\mathbb{N}$ has finite support and a range in $\mathbb{Q}$. $<^+ f$ is $< f + O(1)$ and $<^{\log} f$ is $< f + O(\log(f+1))$. Stochasticity is

$$\Lambda(a|b) = \min\{\mathbf{K}(Q|b) + 3\log\max\{\mathbf{d}(a|Q,b),1\}$$
$$: Q \text{ is an elementary probability measure}\}.$$
$$\Lambda(a|b) < \Lambda(a) + O(\log \mathbf{K}(b)).$$

The following definition is from [Lev74] .

Definition 1 (Information) $\mathbf{I}(\alpha:\beta) = \log \sum_{x,y} 2^{\mathbf{K}(x) + \mathbf{K}(y) - \mathbf{K}(x,y) - \mathbf{K}(x|\alpha) - \mathbf{K}(y|\beta)}$.

The Independence Postulate [Lev84, Lev13] statement is:

IP: *Let α be a sequence defined with an n-bit mathematical statement, and a sequence β can be located in the physical world with a k-bit instruction set. Then $\mathbf{I}(\alpha:\beta) < k + n + c$ for some small absolute constant c.*

There are many proofs in the literature that stochastic numbers have high mutual information with the halting sequence. One detailed proof is in [Eps21].

Lemma 1 $\Lambda(x) <^{\log} \mathbf{I}(x;\mathcal{H})$.

Lemma 2 *For computable probability p over $\mathbb{N}$ and for $D \subset \mathbb{N}$, $|D| = 2^s$, $s < \max_{a \in D} \mathbf{d}(a|p) + \Lambda(D) + \mathbf{K}(s) + \mathbf{K}(p) + O(\log \mathbf{K}(s)\mathbf{K}(p))$.*

Proof. We relativize the universal Turing machine to p and s. Let Q be an elementary probability measure that realizes $\Lambda(D)$. Let $d = \max\{\mathbf{d}(D|Q),1\}$. Let $F \subseteq \mathbb{N}$ be a random set where each element $a \in \mathbb{N}$ is selected independently with probability $cd2^{-s}$, where $c \in \mathbb{N}$ is chosen later. $\mathbf{E}[p(F)] \leq cd2^{-s}$. Furthermore

$$\mathbf{E}[Q(\{G : |G| = 2^s, G \cap F = \emptyset\})]$$
$$\leq \sum_G Q(G)(1 - cd2^{-s})^{2^s} < e^{-cd}.$$

Thus finite $W \subset \mathbb{N}$ can be chosen such that $p(W) \leq 2cd2^{-s}$ and $Q(\{G : |G| = 2^s, G \cap W = \emptyset\}) \leq e^{1-cd}$. $D \cap W \neq \emptyset$, otherwise, using the Q-test, $t(G) = e^{cd-1}$ if $(|G| = 2^s, G \cap W = \emptyset)$ and $t(G) = 0$ otherwise, we have

$$\mathbf{K}(D|Q,d,c) <^+ -\log Q(D) - (\log e)cd$$
$$(\log e)cd <^+ -\log Q(D) - \mathbf{K}(D|Q) + \mathbf{K}(d,c)$$
$$(\log e)cd <^+ d + \mathbf{K}(d,c),$$

which is a contradiction for large c. Thus there is an $a \in D \cap W$, where

$$\mathbf{K}(a) <^+ -\log p(a) + \log d - s + \mathbf{K}(d) + \mathbf{K}(Q)$$
$$s <^+ \mathbf{d}(a|p) + \Lambda(D).$$

Making the relativization of p and s explicit,

$$s < -\log p(a) - \mathbf{K}(a|p,s) + \Lambda(D|p,s)$$
$$s < \max_{a \in D} \mathbf{d}(a|p) + \Lambda(D) + \mathbf{K}(s) + \mathbf{K}(p)$$
$$+ O(\log \mathbf{K}(s)\mathbf{K}(p)).$$

$\square$

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For $\tau \in \mathbb{N}^{\mathbb{N}}$, $\langle\tau\rangle = \langle\tau[1]\rangle\langle\tau[2]\rangle\langle\tau[3]\rangle\dots$ Let $\tau(n)$ be the first 2^n unique numbers found in τ. The sequence τ is assumed to have an infinite amount of unique numbers, and represents a series of observations.

Theorem 1 *For probability p over $\mathbb{N}$ and $\tau \in \mathbb{N}^{\mathbb{N}}$, let $s_{\tau,p}$ be the smallest number such that $\max_{a\in\tau(n)}\mathbf{d}(a|p) > n - 4\mathbf{K}(n) - s_{\tau,p}$. Then $s_{\tau,p} <^{\log} \mathbf{I}(\langle\tau\rangle : \mathcal{H}) + \mathbf{K}(p)$.*

Proof. By Lemmas 1 and 2, and the fact that $\mathbf{I}(x;\mathcal{H}) <^{+} \mathbf{I}(\alpha : \mathcal{H}) + \mathbf{K}(x|\alpha)$,

$$n < \max_{a\in\tau(n)}\mathbf{d}(a|p) + \mathbf{I}(\tau(n);\mathcal{H}) + \mathbf{K}(p) + 2\mathbf{K}(n)$$
$$+ O(\log(\mathbf{I}(\tau(n);\mathcal{H}) + \mathbf{K}(p))),$$
$$n < \max_{a\in\tau(n)}\mathbf{d}(a|p) + 4\mathbf{K}(n) + \mathbf{I}(\langle\tau\rangle : \mathcal{H}) + \mathbf{K}(p)$$
$$+ O(\log(\mathbf{I}(\langle\tau\rangle : \mathcal{H}) + \mathbf{K}(p))),$$
$$n - 4\mathbf{K}(n) - (\mathbf{I}(\langle\tau\rangle : \mathcal{H}) + \mathbf{K}(p)$$
$$+ O(\log(\mathbf{I}(\langle\tau\rangle : \mathcal{H}) + \mathbf{K}(p)))) < \max_{a\in\tau(n)}\mathbf{d}(a|p).$$

$\square$

Let k be the physical address of an infinite sequence of numbers $\tau \in \mathbb{N}^{\mathbb{N}}$. The halting sequence can be described by a small mathematical statement. By Theorem 1 and IP,

$$s_{\tau,p} <^{\log} \mathbf{I}(\langle\tau\rangle : \mathcal{H}) + \mathbf{K}(p) <^{\log} k + c + \mathbf{K}(p).$$

Thus sequences τ with large $s_{\tau,p}$ will have large physical addresses. So, assuming IP, it's hard to find observations which do not have large anomalies, and impossible to find observations with no anomalies. This inequality also applies to a sequence of reals.

References

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