The Randomness Deficiency Function and the Shift Operator

Samuel Epstein*

August 3, 2023

Abstract

Almost surely, the difference between the randomness deficiencies of two infinite sequences will be unbounded with respect to repeated iterations of the shift operator.

1 Introduction

In [Eps23], a result was proven about thermodynamics and product spaces. It was shown that all typical states of product spaces cannot have their marginal algorithmic thermodynamic entropies in synch during the course computable ergodic dynamics. This result was over all computable metric spaces, using the foundation of [HR09]. This paper shows the special case of the Cantor space and the shift operator, which could be of independent interest from algorithmic physics. It is proved using the uniform measure, but with a little bit of work, it can be generalized to two different computable probability measures.

The result is as follows. Let $\mathbf{K}$ be the prefix free Kolmogorov complexity. Let $\mathbf{m}$ be the algorithmic probability and $\mathbf{I}(x:y) = \mathbf{K}(x)+\mathbf{K}(y)-\mathbf{K}(x,y)$ be the mutual information term. The mutual information between two infinite sequences [Lev74] is $\mathbf{I}(\alpha:\beta) = \log\sum_{x,y\in\{0,1\}^*}\mathbf{m}(x|\alpha)\mathbf{m}(y|\beta)2^{\mathbf{I}(x:y)}$. The halting sequence is $\mathcal{H}\in\{0,1\}^\infty$. The shift operator is σ, where $\sigma(\alpha_1\alpha_2\alpha_3\dots) = \alpha_2\alpha_3\dots$. The uniform measure over $\{0,1\}^\infty$ is λ. The randomness deficiency of $\alpha\in\{0,1\}^\infty$ is $\mathbf{D}(\alpha) = \sup_n(n-\mathbf{K}(\alpha[0..n]))$. For infinite sequences $\alpha,\beta\in\{0,1\}^\infty$, (α,β) encodes them with alternating bits.

Theorem.

(a) If (α,β) is ML Random and $\mathbf{I}((\alpha,\beta):\mathcal{H})<\infty$ then $\sup_n\left|\mathbf{D}\left(\sigma^{(n)}\alpha\right)-\mathbf{D}\left(\sigma^{(n)}\beta\right)\right|=\infty$.

(b) For $\lambda\times\lambda$ almost surely, $\sup_n\left|\mathbf{D}\left(\sigma^{(n)}\alpha\right)-\mathbf{D}\left(\sigma^{(n)}\beta\right)\right|=\infty$.

2 Conventions

For positive real functions f, by $<^+f$, $>^+f$, $=^+f$, and $<^{\log}f$, $>^{\log}f$, $\sim f$ we denote $\leq f+O(1)$, $\geq f-O(1)$, $=f\pm O(1)$ and $\leq f+O(\log(f+1))$, $\geq f-O(\log(f+1))$, $=f\pm O(\log(f+1))$. Furthermore, $\overset{*}{<}f$, $\overset{*}{>}f$ denotes $<O(1)f$ and $>f/O(1)$. The term and $\overset{*}{=}f$ is used to denote $\overset{*}{>}f$ and $\overset{*}{<}f$.

The amount of mutual information that a string has with the halting sequence is $\mathbf{I}(x;\mathcal{H}) = \mathbf{K}(x)-\mathbf{K}(x|\mathcal{H})$. One can see that $\mathbf{I}(x;\mathcal{H}) <^+ \mathbf{I}(\alpha:\mathcal{H})+\mathbf{K}(x|\alpha)$. For a set $D\subseteq\{0,1\}^*$, $\mathbf{m}(D) = \sum_{x\in D}\mathbf{m}(x)$. Let $\Omega=\sum\{2^{-\|p\|}:U(p)\text{ halts}\}$ be Chaitin's Omega, $\Omega_n\in\mathbb{Q}_{\geq 0}$ be be the rational formed from the first n bits of Ω, and $\Omega^t = \sum\{2^{-\|p\|}:U(p)\text{ halts in time }t\}$. For $n\in\mathbb{N}$, let $\mathbf{bb}(n) = \min\{t:\Omega_n<\Omega^t\}$. $\mathbf{bb}^{-1}(m) = \arg\min_n\{\mathbf{bb}(n-1)<m\leq\mathbf{bb}(n)\}$. Let $\Omega[n]\in\{0,1\}^*$ be

*JP Theory Group. samepst@jptheorygroup.org

the first n bits of Ω. The function $\mathbf{t}$ is a universal lower computable $\lambda\times\lambda$ test, where if t is a lower computable $\lambda\times\lambda$ test, then $t \overset{*}{<} \mathbf{t}$. If $\mathbf{t}(\alpha,\beta)=\infty$, then (α,β) is not ML random.

Lemma 1 ([Eps23]) *For $n=\mathbf{bb}^{-1}(m)$, $\mathbf{K}(\Omega[n]|m,n)=O(1)$.*

Theorem 1 ([GHR10]) *Let $C\subseteq\{0,1\}^\infty$ be a clopen set and $A_n=(\mathbf{1}_C+\mathbf{1}_C\circ\sigma+\cdots+\mathbf{1}_C\circ\sigma^{(n-1)})/n-\lambda(C)$. There is a computable function $n(\delta,\epsilon)$, such that*

$$\lambda\{\alpha : \sup_{n>n(\delta,\epsilon)} |A_n(\alpha)|>\delta\}<\epsilon.$$

Theorem 2 (EL Theorem [Lev16, Eps19]) *For finite set $D\subset\{0,1\}^*$, $\min_{x\in D}\mathbf{K}(x) <^{\log} -\log\mathbf{m}(D)+\mathbf{I}(\langle D\rangle;\mathcal{H})$.*

Theorem 3 ([Ver21, Lev74, Gei12]) $\Pr_\mu(\mathbf{I}(\alpha:\mathcal{H})>n) \overset{*}{<} 2^{-n+\mathbf{K}(\mu)}$.

3 Results

Theorem 4

(a) If (α,β) is ML Random and $\mathbf{I}((\alpha,\beta):\mathcal{H})<\infty$ then $\sup_n|\mathbf{D}(\sigma^{(n)}\alpha)-\mathbf{D}(\sigma^{(n)}\beta)|=\infty$.

(b) For $\lambda\times\lambda$ a. s., $\sup_n\left|\mathbf{D}\left(\sigma^{(n)}\alpha\right)-\mathbf{D}\left(\sigma^{(n)}\beta\right)\right|=\infty$.

Proof. (a). Assume not. Then there exists $c\in\mathbb{N}$, $c>\max_t|\mathbf{D}(\alpha)-\mathbf{D}(\beta)|$. Fix $n\in\mathbb{N}$. Let $U_n=\{\alpha:\mathbf{D}(\alpha)>n\}$. It is easy to see that $\lambda(U_n)>2^{-n-2\log n-d}$, for some constant d. Given n, one can compute a clopen set $V_n\subset U_n$ with $p_n=-\log\lambda(V_n)$ and $n+2\log n+d<p_n<n+2\log n+d+1$. Let $A^n_m=(\mathbf{1}_{V_n}+\mathbf{1}_{V_n}\circ\sigma+\cdots+\mathbf{1}_{V_n}\circ\sigma^{(m-1)})/m-2^{-p_n}$ and $B^n_m=A^n_m+2^{-p_n}$, which are both computable. Noting that σ is a computable ergodic transform, by Theorem 1, given $\delta,\epsilon>0$, there is a computable $m(\delta,\epsilon,n)$ such that $\lambda\{\gamma,:\sup_{m>m(\delta,\epsilon,n)}|A^n_m(\gamma)|>\delta\}<\epsilon$. Let $m_n=m(2^{-p_n}-2^{-1.5p_n},2^{-n},n)$. Let $W_n=\{\gamma:\sup_{m>m_n}|A^n_m(\gamma)|>2^{-p_n}-2^{-1.5p_n}\}$. Either (1) there is an infinite number of n where $\alpha\in W_n$, or (2) there is an infinite number of n where $\alpha\notin W_n$.

Case (1). Each W_n is an effectively open set, computable uniformly in n. Furthermore, $\mu(W_n)<2^{-n}$. Thus $t(\gamma,\lambda)=\sup_n[\gamma\in W_n]\mathbf{m}(n)2^n$ is a $\lambda\times\lambda$ test. So $\infty=t(\alpha,\beta)\overset{*}{<}\mathbf{t}(\alpha,\beta)$, which (α,β) is not ML random, causing a contradiction.

Case (2). Fix one such $n\in\mathbb{N}$, where $\alpha\notin W_n$. Thus $\sup_{m>m_n}|2^{-p_n}-B^n_m(\alpha)|\le 2^{-p_n}-2^{-1.5p_n}$ implies $\sup_{m>m_n}B^n_m(\alpha)\ge 2^{-1.5p_n}$. Each $\sigma^{(-\ell)}V_n$ is an effectively open set, uniformly in k and ℓ. So for all $m>m_n$, there are at least $2^{-1.5p_n}m$ indices ℓ, where $\alpha\in\sigma^{(-\ell)}V_n$. Let $b_n=\mathbf{bb}^{-1}(m_n+1)$ and N be the smallest power of 2 not less than $\mathbf{bb}(b_n)$. Thus, due to Lemma 1, $\mathbf{K}(N|(\alpha,\beta))<^+\mathbf{K}(n,b_n)$. Thus there are at least $2^{-1.5(n+2\log n+d+1)}N$ indices $\ell\in[1,\ldots,N]$ where $\alpha\in\sigma^{(-\ell)}V_n$. Let $D\subseteq\{0,1\}^N$, where if $x\in D$ then $\alpha\in\sigma^{(-\mathrm{Num}(x))}V_n$ and $|D|\ge 2^{-1.5(n+2\log n+d+1)}N$. The function $\mathrm{Num}:\{0,1\}^N\to\{1,2,\ldots,N\}$ converts strings to numbers in the natural way. Thus $\mathbf{K}(D|(\alpha,\beta))<^+\mathbf{K}(n,b_n)$. Let $\mathrm{Uniform}(N)$ be the uniform measure over $\{0,1\}^N$. By the EL Theorem (Theorem 2) applied to Uniform(N) $\overset{*}{<}\mathbf{m}/\mathbf{m}(N)$, and the definition of $\mathbf{I}$, there exists

$x_n \in D$, with

$$\begin{aligned}
\mathbf{K}(x_n) &<^{\log} \mathbf{K}(\mathrm{Uniform}(N)) - \log|D| + \mathbf{I}(D;\mathcal{H}) \\
&<^{\log} \mathbf{K}(N) + 1.5n + 3\log n + \mathbf{I}((\alpha,\beta):\mathcal{H}) + \mathbf{K}(n,b_n) \\
&<^{\log} \mathbf{K}(\Omega[b_n]) + 1.5n + \mathbf{I}((\alpha,\beta):\mathcal{H}) + \mathbf{K}(b_n). \qquad (1)
\end{aligned}$$

Due to Lemma 1, $\mathbf{K}(\Omega[b_n])|(\alpha,\beta),n,b_n) = O(1)$. Furtheremore, it is well known that for bits of Chaitin's Omega, $\mathbf{K}(\Omega[b_n]|\mathcal{H}) <^+ \mathbf{K}(b_n)$ and that $b_n <^+ \mathbf{K}(\Omega[b_n])$. So

$$b_n <^+ \mathbf{K}(\Omega[b_n]) <^{\log} \mathbf{I}(\Omega[b_n];\mathcal{H}) <^{\log} \mathbf{I}((\alpha,\beta):\mathcal{H}) + \mathbf{K}(b_n,n) <^{\log} \mathbf{I}((\alpha,\beta):\mathcal{H}) + \mathbf{K}(n) \qquad (2)$$

Combining Equations 1 and 2 together, we get

$$\mathbf{K}(x_n) <^{\log} 1.5n + 2\mathbf{I}((\alpha,\beta):\mathcal{H}).$$

We define the test

$$\begin{aligned}
&t_{n,x_n}(\gamma,\lambda) = \left[(\gamma\times\lambda) \in \left(\sigma^{(-\mathrm{Num}(x_n))}V_n\right) \times \left(\sigma^{(-\mathrm{Num}(x_n))}\{\xi : \mathbf{D}(\xi) > n-c\}\right)\right] 2^{2n-c}, \\
&\mathbf{t}(\gamma,\lambda) \overset{*}{>} \sum_n \mathbf{m}(t_{n,x_n}) t_{n,x_n}(\gamma,\lambda) \\
&\overset{*}{>} \sum_n \left[\gamma\times\lambda) \in \left(\sigma^{(-\mathrm{Num}(x_n))}V_n\right) \times \left(\sigma^{(-\mathrm{Num}(x_n))}\{\xi : \mathbf{D}(\xi) > n-c\}\right)\right] \frac{2^{.5n-2\mathbf{I}((\alpha,\beta):\mathcal{H})}}{(n+\mathbf{I}((\alpha,\beta):\mathcal{H}))^{O(1)}}.
\end{aligned}$$

In recap, since $\alpha \notin W_n$, $|A^n_N(\alpha)| > 2^{-p_n} - 2^{-1.5p_n}$, so $B^n_N > 2^{-1.5p_n} > 2^{-1.5(n+2\log n+d)}$. Thus one can create a large enough set $D \subset \{0,1\}^N$, and find a simple enough $x_n \in D$ such that $\alpha \in \sigma^{(-\mathrm{Num}(x_n))}V_n$. By assumption of the theorem

$$(\alpha,\beta) \in \left(\sigma^{(-\mathrm{Num}(x_n))}V_n\right) \times \left(\sigma^{(-\mathrm{Num}(x_n))}\{\xi : \mathbf{D}(\xi) > n-c\}\right).$$

Thus $\mathbf{m}(t_n,x_n)t_{n,x_n}(\alpha,\beta) = \frac{2^{.5n-2\mathbf{I}((\alpha,\beta):\mathcal{H}}}{(n+\mathbf{I}((\alpha,\beta):\mathcal{H}))^{O(1)}}$. Furthermore, since $\mathbf{I}(\alpha,\beta) < \infty$ and there is an infinite number of n where $(\alpha,\beta) \notin W_n$, $\mathbf{t}(\alpha,\beta) = \infty$, so (α,β) is not ML-random, causing a contradiction.

(b) By the results of (a), if $\sup_n |\mathbf{D}\left(\sigma^{(n)}\alpha\right) - \mathbf{D}\left(\sigma^n(\beta)\right)| < \infty$, then (α,β) is not ML random or $\mathbf{I}((\alpha,\beta):\mathcal{H}) = \infty$. By Theorem 3, $\lambda\{\gamma : \mathbf{I}(\gamma:\mathcal{H}) = \infty\} = 0$. Thus (α,β) is in a λ null set, and thus also a $\lambda\times\lambda$ null set. □

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