

A Chain Rule for the Randomness Deficiency Function

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Abstract

This paper is an exposition of the addition equality theorem for algorithmic entropy in [G01], applied to the Cantor space. This application implies that randomness deficiency of infinite sequences obeys the chain rule, analogous to the finite Kolmogorov complexity case. This is a generalization of van Lambalgen's Theorem. It is unclear whether this result is folklore, but in any case, this paper presents a dedicated proof of the equality. In addition, a dual integration trick shortens the proof.

1 Introduction

Prefix free Kolmogorov complexity, \mathbf{K} , obeys the chain rule, with for $x, y \in \{0, 1\}^*$,

$$\mathbf{K}(x, y) =^+ \mathbf{K}(x) + \mathbf{K}(y|x, \mathbf{K}(x)).$$

In this paper, we apply the addition equality theorem for algorithmic entropy in [G01] to the specific case of infinite sequences. We also shorten the proof using an integration trick. The consequence to this is a result about randomness deficiency \mathbf{D} , where for computable probability μ , for infinite sequences, $\mathbf{D}(\alpha|\mu, x) = \sup_n -\log \mu(\alpha[0..n]) - \mathbf{K}(\alpha[0..n]|x)$. The randomness deficiency over the space $\{0, 1\}^\infty \times \{0, 1\}^\infty$, is $\mathbf{D}(\alpha, \beta|\mu, \nu) = \sup_n -\log \mu(\alpha[0..n]) - \log \nu(\beta[0..n]) - \mathbf{K}(\alpha[0..n]|\beta[0..n])$. The discrete case for $\mathbf{d}(x|p) = -\log p(x) - \mathbf{K}(x)$ is trivial. The result detailed in this paper is as follows.

Theorem. ([G01]) *Relativized to probabilities μ and ν over $\{0, 1\}^\infty$,*

$$\mathbf{D}(\alpha, \beta|\mu, \nu) =^+ \mathbf{D}(\alpha|\mu) + \mathbf{D}(\beta|\nu, (\alpha, \lceil \mathbf{D}(\alpha|\mu) \rceil)).$$

This is a generalization of van Lambalgen's Theorem, which states (α, β) is ML random iff α is ML random and β is ML random with respect to α . If one were to take the complexities of the probabilities μ and ν into account (that is, they are no longer $O(1)$) then the theorem statement and proof become more nuanced. This generalization can be seen in [G01]. An open question is whether \mathbf{D} follows the linear inequalities that parallel Shannon entropy \mathcal{H} , as Kolmogorov complexity was shown to do [HRSV00]:

Conjecture. *Given $\{\alpha_1 \dots \alpha_n\} \in \{0, 1\}^{\infty n}$, and random variables $\{\beta_1, \dots, \beta_n\}$ is it the case that over all $W \subseteq \{1, \dots, n\}$, and $\lambda_W \in \mathbb{R}$,*

$$\sum_W \lambda_W \mathbf{D}(\alpha^W) \leq 0 \Leftrightarrow \sum_W \lambda_W \mathcal{H}(\beta^W) \geq 0?$$

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2 Results

For the nonnegative real function f , we use $<^+ f$, $>^+ f$, and $=^+ f$ to denote $< f + O(1)$, $> f - O(1)$, and $= f \pm O(1)$. The Kolmogorov complexity of a lower computable function f is $\mathbf{K}(f)$, the size of the shortest program that enumerates it. As shown in [G01], $2^{\mathbf{D}(\alpha|\mu)} \stackrel{*}{=} \mathbf{t}_\mu(\alpha)$ where \mathbf{t}_μ is a universal lower computable μ -test. Furthermore, a modification of the proof Theorem 2.3.4 in [G01] to the $\{0, 1\}^\infty \times \{0, 1\}^\infty$ space can be used to show that $2^{\mathbf{D}(\alpha, \beta|\mu, \nu)} \stackrel{*}{=} \mathbf{t}_{\mu, \nu}(\alpha, \beta)$, where $\mathbf{t}_{\mu, \nu}$ is a universal lower computable test over $\{0, 1\}^\infty \times \{0, 1\}^\infty$. For measure μ and lower continuous function f over $\{0, 1\}^\infty$, we use the notation $\mu^x f(x) = \int_{x \in \{0, 1\}^\infty} f(x) d\mu(x)$. Throughout this section, the universal Turing machine is assumed to be relativized to probabilities μ and ν over $\{0, 1\}^\infty$. This means that there is an $O(1)$ sized program that can compute $\mu(x\{0, 1\}^\infty)$ uniformly in $x \in \{0, 1\}^*$, and similarly for ν .

Proposition 1 $\log \nu^y 2^{\mathbf{D}(x, y|\mu, \nu)} <^+ \mathbf{D}(x|\mu)$.

Proof. Let $f(x, \mu, \nu) = \log \nu^y 2^{\mathbf{D}(x, y|\mu, \nu)}$. The function f is lower computable and has $\mu^x 2^{f(x, \mu, \nu)} \leq 1$. The proposition follows from the universal properties of \mathbf{t}_μ , where $2^f \stackrel{*}{<} \mathbf{t}_\mu$. \square

Proposition 2 If $i < j$, then

$$i + \mathbf{D}(x|\mu, j) <^+ j + \mathbf{D}(x|\mu, i).$$

Proof. By the properties of \mathbf{D} , we have

$$\mathbf{D}(x|\mu, j) <^+ \mathbf{D}(x|\mu, i) + \mathbf{K}(j - i) <^+ \mathbf{D}(x|\mu, i) + j - i.$$

\square

Definition 1 Let $F : \{0, 1\}^\infty \rightarrow \mathbb{Z} \cup \{-\infty, \infty\}$ be a lower semicomputable function. An (μ, F) -test is a function $t : \{0, 1\}^\infty \times \{0, 1\}^\infty \rightarrow \mathbb{Z} \cup \{-\infty, \infty\}$ that is lower semicomputable and $\mu^x t(x, y) \leq 2^{F(y)}$. There exists a maximal (μ, F) test, $\mathbf{t}_{(\mu, F)}$, such that $t \stackrel{*}{<} \mathbf{t}_{(\mu, F)}$.

Proposition 3 Let $F : \{0, 1\}^\infty \rightarrow \mathbb{Z} \cup \{-\infty, \infty\}$ be an upper semicomputable function of Kolmogorov complexity $O(1)$. For all x and with $\mathbf{t}_{(\nu, F)}(y) > -\infty$,

$$\mathbf{t}_{(\nu, F)}(x, y) \stackrel{*}{=} 2^{F(y)} \mathbf{t}_\nu(x|y, -F(y)).$$

Proof. To prove the inequality $\stackrel{*}{>}$, let $g(x, y, m) = \max_{i \geq m} 2^{-i} \mathbf{t}_\nu(x|y, i)$. This function is lower computable, and decreasing in m . The function $g(x, y) = g_\nu(x, y, -F(y))$ is lower semicomputable since $-F$ is upper semi-computable. The multiplicative form of Proposition 2 implies

$$\begin{aligned} g(x, y, m) &\stackrel{*}{=} 2^{-m} \mathbf{t}_\nu(x|y, m) \\ g(x, y) &\stackrel{*}{=} 2^{F(y)} \mathbf{t}_\nu(x|y, -F(y)). \end{aligned}$$

Since \mathbf{t}_ν is a test:

$$\begin{aligned} \nu^x 2^{-m} \mathbf{t}_\nu(x|y, m) &\leq 2^{-m} \\ \nu^x g(x, y) &\stackrel{*}{<} 2^{F(y)}, \end{aligned}$$

which implies $g(x, y) \stackrel{*}{<} \mathbf{t}_{(\nu, F)}(x, y)$ by the optimality of $\mathbf{t}_{(\nu, F)}$. We now consider the upper bound. Since, given fixed y , $2^{-F(y)}\mathbf{t}_{(\nu, F)}(x, y)$ is an x -test conditional on y and $-F(y)$, we have

$$2^{-F(y)}\mathbf{t}_{(\nu, F)}(x, y) \stackrel{*}{<} \mathbf{t}(x|y, -F(y))/\mathbf{m}(\mathbf{t}_{(\nu, F)}) \stackrel{*}{<} \mathbf{t}(x|y, -F(y))2^{\mathbf{K}(F)} \stackrel{*}{<} \mathbf{t}(x|y, -F(y)).$$

□

The following Theorem is a specific case of Theorem 4.5.2 in [G01], to the Cantor space and with $O(1)$ complexities for the probabilities. The proof is shortened by noting that f is a universal F -test.

Theorem 1 *Relativized to probabilities μ and ν over $\{0, 1\}^\infty$,*

$$\mathbf{D}(x, y|\mu, \nu) =^+ \mathbf{D}(x|\mu) + \mathbf{D}(y|\nu, (x, \lceil \mathbf{D}(x|\mu) \rceil)).$$

Proof. Let $f(x, y) = 2^{\mathbf{D}(x, y|\mu, \nu)}$. Proposition 1 implies there exists $c \in \mathbb{N}$ with $\nu^y f(x, y) \leq 2^{\mathbf{D}(x|\mu)+c}$. Let $F(x, \mu) = \lceil \mathbf{D}(x|\mu) \rceil$. Note that if h is a lower computable function such that $\nu^y h(x, y) \stackrel{*}{<} 2^{\mathbf{D}(x|\mu)}$, then $\mu^x \nu^y h(x, y) \stackrel{*}{<} \mu^x \mathbf{t}_\mu(x) \stackrel{*}{<} 1$, so $h \stackrel{*}{<} f$, so f is a universal F -test. Proposition 3 (swapping x and y) gives

$$\begin{aligned} \mathbf{D}(x, y|\mu, \nu) &= \log f(x, y) =^+ F(x) + \mathbf{D}(y|\nu, (x, -F(x))) \\ \mathbf{D}(x, y|\mu, \nu) &=^+ \mathbf{D}(x|\mu) + \mathbf{D}(y|\nu, (x, \lceil \mathbf{D}(x|\mu) \rceil)). \end{aligned}$$

□

References

- [G01] P. Gács. Quantum Algorithmic Entropy. *Journal of Physics A Mathematical General*, 34(35), 2001.
- [HRSV00] D. Hammer, A. Romashchenko, A. Shen, and N. Vereshchagin. Inequalities for shannon entropy and kolmogorov complexity. *Journal of Computer and System Sciences*, 60(2):442–464, 2000.