A Small Theorem for Small $\mathbf{m}$

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Abstract

If a semi measure is greater than the universal semi-measure $\mathbf{m}$ up to a multiplicative constant, then it is exotic in that it has infinite mutual information with the halting sequence.

1 Introduction

In this note, we show that semi measures that majorize the algorithmic probability have infinite mutual information with the halting sequence. For a probability p over $\{0,1\}^*$, $[p] \subset \{0,1\}^\infty$ is the set of infinite sequences $\beta \in [p]$ such that $U_x(\beta)$ outputs the bit representation of $p(x)$. The algorithm U is a standard universal Turing machine. $\mathbf{K}(x|y)$ is the prefix Kolmogorov complexity. $\mathbf{m}$ is the algorithmic probability. $\mathbf{I}(x:y) = \mathbf{K}(x)+\mathbf{K}(y)-\mathbf{K}(x,y)$ is the mutual information between two strings. For infinite sequences $\alpha,\beta \in \{0,1\}^\infty$, $\mathbf{I}(\alpha:\beta) = \log \sum_{x,y\in\{0,1\}^*} \mathbf{m}(x|\alpha)\mathbf{m}(y|\beta)2^{\mathbf{I}(x:y)}$ [Lev74]. The halting sequence is $\mathcal{H}$. The amount of mutual information between a probability p and $\mathcal{H}$ is $\mathbf{I}(p:\mathcal{H}) = \inf_{\beta\in[p]} \mathbf{I}(\beta:\mathcal{H})$.

Theorem. *If $\mathbf{w}$ is a semimeasure on $\{0,1\}^*$ and $\mathbf{m} < O(1)\mathbf{w}$ then $\mathbf{I}(\mathbf{w}:\mathcal{H}) = \infty$.*

The amount of information that $\mathcal{H}$ has about $x \in \{0,1\}^*$ is $\mathbf{I}(x;\mathcal{H}) = \mathbf{K}(x) - \mathbf{K}(x|\mathcal{H})$. For positive real functions f, by $<^+ f$, $>^+ f$, $=^+ f$, and $<^{\log} f$, $>^{\log} f$, $\sim f$ we denote $\leq f+O(1)$, $\geq f-O(1)$, $= f\pm O(1)$ and $\leq f+O(\log(f+1))$, $\geq f-O(\log(f+1))$, $= f\pm O(\log(f+1))$. Furthermore, $\overset{*}{<} f$, $\overset{*}{>} f$ denotes $< O(1)f$ and $> f/O(1)$. The term and $\overset{*}{=} f$ is used to denote $\overset{*}{>} f$ and $\overset{*}{<} f$. The chain rule states $\mathbf{K}(x) + \mathbf{K}(y|x,\mathbf{K}(x)) =^+ \mathbf{K}(x,y)$.

2 Kolmogorov Complexity is Exotic

We cover material on busy beaver functions. Let $\Omega = \sum\{2^{-\|p\|} : U(p) \text{ halts}\}$ be Chaitin's Omega, $\Omega_n \in \mathbb{Q}_{\geq 0}$ be be the rational formed from the first n bits of Ω, and $\Omega^t = \sum\{2^{-\|p\|} : U(p) \text{ halts in time } t\}$. For $n\in\mathbb{N}$, let $\mathbf{bb}(n) = \min\{t : \Omega_n < \Omega^t\}$. $\mathbf{bb}^{-1}(m) = \arg\min_n\{\mathbf{bb}(n-1) < m \leq \mathbf{bb}(n)\}$. Let $\Omega[n] \in \{0,1\}^*$ be the first n bits of Ω. For $t \in \mathbb{N}$ define the function $\mathbf{m}^t(x) = \sum\{2^{-\|p\|} : U(p) = x \text{ in } t \text{ steps}\}$ and for $n \in \mathbb{N}$, we have $\mathbf{m}_n(x) = \sum\{2^{-\|p\|} : U(p) = x \text{ in } \mathbf{bb}(n) \text{ steps}\}$.

Lemma 1 *For $n = \mathbf{bb}^{-1}(m)$, $\mathbf{K}(\Omega[n]|m,n) = O(1)$.*

Proof. For a string x, let $BB(x) = \inf\{t : \Omega^t > 0.x\}$. Enumerate strings of length n, starting with 0^n, and return the first string x such that $BB(x) \geq m$. This string x is equal to $\Omega[n]$, otherwise let y be the largest common prefix of x and $\Omega[n]$. Thus $BB(y) = \mathbf{bb}(\|y\|) \geq BB(x) \geq m$, which means $\mathbf{bb}^{-1}(m) \leq \|y\| < n$, causing a contradiction. □

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Lemma 2 $\mathbf{I}(f(a);\mathcal{H}) <^+ \mathbf{I}(a;\mathcal{H}) + \mathbf{K}(f)$.

Proof.

$$\mathbf{I}(a;\mathcal{H}) = \mathbf{K}(a) - \mathbf{K}(a|\mathcal{H}) >^+ \mathbf{K}(a, f(a)) - \mathbf{K}(a, f(a)|\mathcal{H}) - \mathbf{K}(f).$$

The chain rule applied twice results in

$$\begin{aligned}
\mathbf{I}(a;\mathcal{H}) + \mathbf{K}(f) &>^+ \mathbf{K}(f(a)) + \mathbf{K}(a|f(a), \mathbf{K}(f(a))) - (\mathbf{K}(f(a)|\mathcal{H}) + \mathbf{K}(a|f(a), \mathbf{K}(f(a)|\mathcal{H}), \mathcal{H}) \\
&=^+ \mathbf{I}(f(a);\mathcal{H}) + \mathbf{K}(a|f(a), \mathbf{K}(f(a))) - \mathbf{K}(a|f(a), \mathbf{K}(f(a)|\mathcal{H}), \mathcal{H}) \\
&=^+ \mathbf{I}(f(a);\mathcal{H}) + \mathbf{K}(a|f(a), \mathbf{K}(f(a))) - \mathbf{K}(a|f(a), \mathbf{K}(f(a)), \mathbf{K}(f(a)|\mathcal{H}), \mathcal{H}) \\
&>^+ \mathbf{I}(f(a);\mathcal{H}).
\end{aligned}$$

□

Lemma 3 *A relation* $X = \{(x_i, c_i)\}_{i=1}^{2^n} \subset \{0,1\}^* \times \mathbb{N}$, $|\mathbf{K}(x_i) - c_i| \leq s$, *has* $n <^{\log} 2s + 2\mathbf{I}(X;\mathcal{H})$.

Proof. We relativize the universal Turing machine to (n, s), which can be done due to the precision of the theorem. Let $T = \min\{t : \lceil -\log \mathbf{m}_t(x_i)\rceil - c_i < s + 1\}$. Let $N = \mathbf{bb}^{-1}(T)$ and $M = \mathbf{bb}(N)$. So for all x_i, $-\log \mathbf{m}_M(x_i) - \mathbf{K}(x_i) <^+ 2s$. Let Q be an elementary probability measure that realizes $\mathbf{Ks}(X)$ and $d = \max\{\mathbf{d}(X|Q), 1\}$. Without loss of generality, the support of Q is restricted to binary relations $B \subset \{0,1\}^* \times \mathbb{N}$ of size 2^n. Let $B_1 = \bigcup\{y : (y, c) \in B\}$. Let $S = \bigcup\{B_1 : B \in \text{Support}(Q)\}$. We randomly select each string in S to be in a set R independently with probability $d2^{-n}$. Thus $\mathbf{E}[\mathbf{m}_M(R)] \leq d2^{-n}$. For $B \in \text{Support}(Q)$,

$$\mathbf{E}_R \mathbf{E}_{B\sim Q}[[R \cap B_1 = \emptyset]] = \mathbf{E}_{B\sim Q} \Pr(R \cap B_1 = \emptyset) = (1 - d2^{-n})^{2^n} < e^{-d}.$$

Thus there exists a set $R \subseteq S$ such that $\mathbf{m}_M(R) \leq 2{\cdot}2^{-n}$ and $\mathbf{E}_{B\sim Q}[[R \cap B_1 = \emptyset]] < 2e^{-d}$. Let $t(B) = .5[R \cap B_1 = \emptyset]2^d$. t is a Q-test, with $\mathbf{E}_{B\sim Q}[t(B)] \leq 1$. It must be that $t(X) \neq 0$, otherwise,

$$1.44d - 1 < \log t(X) <^+ \mathbf{d}(X|Q) + \mathbf{K}(t|Q) <^+ d + \mathbf{K}(d),$$

which is a contradiction for large enough d, which one can assume without loss of generality. Thus $t(X) \neq 0$ and $R \cap X_1 \neq \emptyset$. Furthermore, if $y \in R$, $\mathbf{K}(y) <^+ -\log \mathbf{m}_M(x) - n + \log d + \mathbf{K}(d, M, R)$. So for $x \in R \cap X_1$, .

$$\begin{aligned}
\mathbf{K}(x) &<^+ -\log \mathbf{m}_M(x) - n + \log d + \mathbf{K}(d, M, R) \\
\mathbf{K}(x) &<^+ \mathbf{K}(x) + 2s - n + \log d + \mathbf{K}(M) + \mathbf{K}(R, d) \\
n &<^+ 2s + \mathbf{K}(M) + \log d + \mathbf{K}(Q, d) \\
n &<^+ 2s + \mathbf{K}(\Omega[N]) + \mathbf{Ks}(X) \\
n &<^+ 2s + \mathbf{K}(\Omega[N]) + \mathbf{I}(X;\mathcal{H}) \qquad (1)
\end{aligned}$$

From Lemma 1, $\mathbf{K}(\Omega[N]|T, N) =^+ \mathbf{K}(\Omega[N]|X, N) = O(1)$. Furthermore it is well known for the bits of Chaitin's Omega, $N <^+ \mathbf{K}(\Omega[N])$ and $\mathbf{K}(\Omega[N]|\mathcal{H}) <^+ \mathbf{K}(N)$. So, using Lemma 2,

$$N <^+ \mathbf{K}(\Omega[N]) <^{\log} \mathbf{I}(\Omega[N];\mathcal{H}) <^{\log} \mathbf{I}(X;\mathcal{H}) + \mathbf{K}(N) <^{\log} \mathbf{K}(X;\mathcal{H}). \qquad (2)$$

So combining Equations 1 and 2, one gets

$$n <^{\log} 2s + 2\mathbf{I}(X;\mathcal{H}).$$

□

3 Results

Theorem 1 *If* $\mathbf{w}$ *is a semimeasure on* $\{0,1\}^*$ *and* $\mathbf{m} \overset{*}{<} \mathbf{w}$ *then* $\mathbf{I}(\mathbf{w}:\mathcal{H}) = \infty$.

Proof. Note that $\mathbf{w}$ has full support since $\mathbf{m}$ does. One can also assume that for each $x \in \{0,1\}^*$, $-\log \mathbf{w}(x) \in \mathbb{N}$. Let $N \subseteq \mathbb{N}$ be a set of numbers n such that $\mathbf{w}(\{0,1\}^n) < 1/n$. Obviously $|N| = \infty$. Fix $n \in N$. We have $X_n = \{x : \mathbf{w}(x) < 2^{-n-\log n+O(1)}\}$. Some simple math shows that $|X_n| \overset{*}{>} 2^n$. So for each $x \in X_n$, $\mathbf{K}(x) >^+ -\log \mathbf{w}(x) >^+ n + \log n$. We also have that for each $x \in \{0,1\}^n$, $\mathbf{K}(x) <^+ n+\mathbf{K}(n)$. Let $Y_n = \{(x, n+\log n) : x \in X_n\}$. So for each $(x,c) \in Y_n$, $|\mathbf{K}(x)-c| <^+ \log n$. So applying Lemma 3 to Y_n, we get $n <^{\log} \mathbf{I}(Y_n;\mathcal{H}) <^{\log} \mathbf{I}(\mathbf{w}:\mathcal{H}) + \mathbf{K}(n) <^{\log} \mathbf{I}(\mathbf{w}:\mathcal{H})$. Since $|N| = \infty$, $\mathbf{I}(\mathbf{w}:\mathcal{H}) = \infty$. □

References

[Lev74] L. A. Levin. Laws of Information Conservation (Non-growth) and Aspects of the Foundations of Probability Theory. *Problemy Peredachi Informatsii*, 10(3):206–210, 1974.