On the Formal Theory of Game Encodings

Sam Epstein

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Abstract

This paper initiates a new formal theory of the encodings of two player games. Given a simple probabilistic player that wins with non negligible probability, one can prove the existence of a simple winning deterministic player. All players who play zero sum repeated games for n rounds will admit a trivially simple adversary who can achieve a score of $c\sqrt{n}$, for some constant c. If there are a large number of complicated probabilistic players who wins against an adversary with non-negligible probability, then there is a simple deterministic player who wins against that adversary.

1 Introduction

For the first time, the encoding length of players in sequential games is researched. The only previous work connecting game theory and description complexity is work that uses games in the proofs on theorems in Algorithmic Information Theory. The games studied in this paper are sequential interactions between players and adversaries. They exchange numbers over turns and at some point, the adversary declares that the player has won or lost. The players have complete information on their opponents' turns. Thus the type of games studied is very general. Notions such as a "simple" player and a"complicated" player are formally defined. Deep connections between the encoding lengths of probabilistic players and deterministic players are studied. Surprisingly, adversaries who lose to a large number of players will lose to a simple player. All deterministic players of zero-sum repeated games, whether computable or uncomputable, will have a weak spot, in that there exists a trivially simple adversary who can achieve a score of $c\sqrt{n}$ in n rounds, for some constant c dependent on the game. We start with an illustrative game, taking place on the island of Crete.

1.1 The Minotaur and the Labyrinth

A hero is trapped in a labyrinth, which consists of long corridors connecting to small rooms. The intent of the the hero is to reach the goal room, which has a ladder in its center reaching the outside. The downside is the hero is blindfolded. The upside is there is a minotaur present to guide the hero.

At every room, the minotaur tells the hero the number of corridors n leading out (including the one which the hero just came from). The hero states a number between 1 and n and the minotaur takes the hero to corresponding door. However the hero faces another obstacle, in that the minotaur is trying to trick him. This means the mapping the minotaur uses is a function of all the hero's past actions. Thus if a hero returns to the same room, he may be facing a different mapping than before. This process continues for a very large number of turns. The question is how much information is needed by the hero to find the exit? Using **Game Derandomization**, we get the following surprising good news for the hero. Let c be the number of corridors and d be the number of doors in the goal room.

The hero can find the exit using $\log(c/d) + \epsilon$ *bits.*

The error term ϵ is logarithmic and also is dependent on the information the halting sequence has about the entire construct, which is negligible except in exotic cases. Assuming the **Independence Postulate** [Lev84, Lev13], **IP** one cannot find such exotic constructs in the physical world.

2 Setup

We use $x <^+ y$, $x >^+ y$ and $x =^+ y$ to denote $x < y + O(1)$, $x + O(1) > y$ and $x = y \pm O(1)$, respectively. Furthermore, $\overset{*}{<} f$, $\overset{*}{>} f$ denotes $< O(1)f$ and $> f/O(1)$. In addition, $x <^{\log} y$ and $x >^{\log} y$ denote $x < y + O(\log y)$ and $x + O(\log x) > y$, respectively. The prefix Kolmogorov complexity of a string is $\mathbf{K}(x)$. Mutual information between strings is $\mathbf{I}(x : y) = \mathbf{K}(x) + \mathbf{K}(y) - \mathbf{K}(x, y)$. For strings, x, y, $\mathbf{I}(x; y) = \mathbf{K}(x) - \mathbf{K}(x|y)$. The halting sequence is $\mathcal{H} \in \{0, 1\}^\infty$.

We study Win/Lose game which are a series of interactions between an agent $\mathbf{p}$ and an environment $\mathbf{q}$. Each round starts with $\mathbf{p}$ initiating a move, which is chosen out of $\mathbb{N}$ and then $\mathbf{q}$ responds with a number out of $\mathbb{N}$. Agent $\mathbf{p}$ wins if at some point $\mathbf{q}$ declares that $\mathbf{p}$ wins. Agent $\mathbf{p}$ loses if at some point $\mathbf{q}$ declares that $\mathbf{p}$ loses. If $\mathbf{q}$ does not make a declaration, then the game is undefined. Thus $\mathbf{p}$ is a function $(\mathbb{N}\times\mathbb{N})^* \mapsto \mathbb{N}$ and $\mathbf{q}$ is a function $\mathbb{N}\times(\mathbb{N}\times\mathbb{N})^* \mapsto \mathbb{N}\cup\{W, L\}$. Both $\mathbf{p}$ and $\mathbf{q}$ are are assumed to be computable, however, uncomputable environments are also. Both the agent and environment can be probabilistic in their choice actions. Thus the probabilities of each action are uniformly computable to any degree of accuracy.

3 Game Derandomization

All work in this paper is from [Eps24]. Remarkably, deterministic agents can be constructed from probabilistic agents. If there is an probabilistic agent that wins with non-negligible probability against an environment, then there exists a simple deterministic agent that will win. This can be achieved with a novel reworking of the code of the probabilistic player.

Theorem 1. *If probabilistic agent* $\mathbf{p}$ *wins against deterministic environment* $\mathbf{q}$ *with at least probability* p*, then there is a deterministic agent of Kolmogorov complexity* $<^{\log} \mathbf{K}(\mathbf{p}) - \log p + \mathbf{I}(\langle \mathbf{p}, \mathbf{q}\rangle; \mathcal{H})$ *that wins against* $\mathbf{q}$.

Theorem 2. *Let* $\mathbf{q}$ *be a deterministic environment and* $\mathbf{p}$ *be a probabilistic player that wins with probability* p*. For* $s = \lceil -\log p \rceil + 1$*,* $r < s$*,* $r \in \mathbb{N}$*, there is a probabilistic agent of complexity* $<^{\log} s - r + \mathbf{I}(\langle \mathbf{p}, s, \mathbf{q}\rangle; \mathcal{H}) + \mathbf{K}(r)$ *that wins with probability* $> 2^{-r}$.

In fact, this result can be extended to probabilistic environments. If a probabilistic agent wins against a probabilistic environment with probability greater 2^{-s} then there exists a deterministic agent that can win against the environment with probability greater than 2^{-s-1}.

Theorem 3. *Let* $\mathbf{p}$ *be a probabilistic agent and* $\mathbf{q}$ *be a probabilistic environment. If* $\mathbf{p}$ *wins in the Win/Lose game against* $\mathbf{q}$ *with probability* $> 2^{-s}$*,* $s \in \mathbb{N}$*, then there is a deterministic agent of complexity* $<^{\log} \mathbf{K}(\mathbf{p}) + 2s + \mathbf{I}(\langle \mathbf{p}, \mathbf{q}\rangle; \mathcal{H})$ *that wins with probability* $> 2^{-s-1}$.

4 Computability of Environments

In this section, we derive the results of Theorems 1 and 3 with respect to uncomputable environments. We will use the following mutual information term between infinite sequences.

Definition 1 ([Lev74])**.** *For* $\alpha, \beta \in \{0,1\}^\infty$,
$\mathbf{I}(\alpha : \beta) = \log \sum_{x,y \in \{0,1\}^*} 2^{\mathbf{I}(x:y) - \mathbf{K}(x|\alpha) - \mathbf{K}(y|\beta)}$.

We fix a computable function ℓ such that for every environment $\mathbf{q}$ there is an infinite sequence α such that $\ell(\alpha, \cdot)$ computes $\mathbf{q}$. Let $\ell[\mathbf{q}]$ be the set of all such infinite sequences α.

Definition 2. *For probabilistic environment* $\mathbf{q}$*,* $\mathbf{I}(\mathbf{q} : \mathcal{H}) = \inf_{\alpha \in \ell[\mathbf{q}]} \mathbf{I}(\alpha : \mathcal{H})$.

Due to **IP**, [Lev84, Lev13], there is no physical way to find or produce environements $\mathbf{q}$ with high $\mathbf{I}(\mathbf{q} : \mathcal{H})$. They are exotic and simply don't exist in the physical world.

Theorem 4. *If probabilistic agent* $\mathbf{p}$ *wins against deterministic and uncomputable environment* $\mathbf{q}$ *with at least probability* p*, then there is a deterministic agent of complexity* $<^{\log} \mathbf{K}(\mathbf{p}) - \log p + \mathbf{I}(\langle \mathbf{p}, \mathbf{q}\rangle : \mathcal{H})$ *that wins against* $\mathbf{q}$.

Theorem 5. *Let* $\mathbf{p}$ *be a probabilistic agent and* $\mathbf{q}$ *be a uncomputable probabilistic environment. If* $\mathbf{p}$ *Wins in the Win/Lose game against* $\mathbf{q}$ *with probability* $> 2^{-s}$*,* $s \in \mathbb{N}$*, then there is a deterministic agent of complexity* $<^{\log} \mathbf{K}(\mathbf{p}) + 2s + \mathbf{I}(\langle \mathbf{p}, \mathbf{q}\rangle : \mathcal{H})$ *that wins with probability* $> 2^{-s-1}$.

5 Even-Odds

We define the following game, entitled *Even-Odds*. There are N rounds. At each round i, the player and the environment simultaneously output a bit, with p_i qnd e_i. The agent gets a point if $p_i \oplus e_i = 1$, otherwise the agent loses a point. The environment $\mathbf{q}$ can be any probabilistic algorithm.

Theorem 6. *For large enough number of rounds, N, given any deterministic environment $\mathbf{q}$ there is a deterministic agent $\mathbf{p}$ of complexity $\mathbf{K}(\mathbf{p}) <^{\log} \mathbf{I}(\mathbf{q}; \mathcal{H})$ that can achieve a score of $\sqrt{N}$.*

Theorem 7. *For large enough number of rounds, N, given any probabilistic environment $\mathbf{q}$ there is a deterministic agent $\mathbf{p}$ of complexity $\mathbf{K}(\mathbf{p}) <^{\log} \mathbf{I}(\mathbf{q}; \mathcal{H})$ that can achieve a score of $\sqrt{N}$ with probability $> 1/4$.*

6 Zero-Sum Repeated Games

In this section we generalize from the Even-Odds game of Section 5 to all zero-sum repeated games. A simultaneous game between two players A and B are defined as follows. At each turn, both players simultaneously play an action $a, b \in \mathbb{N}$. Each action is a function of the previous turns. Thus both A and B are of the form $(\mathbb{N} \times \mathbb{N})^* \to \mathbb{N}$. This process continues of N turns. The determination of the outcome after N turns is dependent on each such game.

A *Zero-Sum Repeated Game* is when the simultaneous game is a series of identical zero-sum stage games $\mathcal{G}$. The payoffs of the stage game $\mathcal{G}$ are assumed to be rationals. Each player starts with a score of 0. A zero sum stage game is when the actions of A and B are chosen from $\{1, \ldots, n\}$. After the actions occur, each player is given a penalty or a prize. The total prizes and penalties for each player sum to 0.

Theorem 8. *For repetition of a zero-sum stage game that has n actions, there is a constant $c \in \mathbb{R}_{>0}$, where over large enough turns N, for all computable deterministic players B, there is a computable deterministic player A that can achieve a score greater than $c\sqrt{N}$ with complexity $\mathbf{K}(A) <^{\log} \mathbf{K}(n) + \mathbf{I}(\langle B, N, \mathcal{G}\rangle; \mathcal{H})$.*

Theorem 9. *For repetition of a zero-sum stage game that has n actions, there is a constant $c \in \mathbb{R}_{>0}$ where over large enough turns N, for all computable probabilistic players B, there is a computable deterministic player A with complexity $\mathbf{K}(A) <^{\log} \mathbf{K}(n) + \mathbf{I}(\langle B, N, \mathcal{G}\rangle; \mathcal{H})$ that can achieve a score greater than $c\sqrt{N}$ with probability $> 1/4$.*

Theorem 10. *For repetition of a zero-sum stage game that has n actions, there is a constant $c \in \mathbb{R}_{>0}$ over large enough turns N, for all uncomputable deterministic players B, there is a computable deterministic player A that can achieve a score greater than $c\sqrt{N}$ with complexity $\mathbf{K}(A) <^{\log} \mathbf{K}(n) + \mathbf{I}(\langle B, N, \mathcal{G}\rangle : \mathcal{H})$.*

7 Many Winning Players

If there is a large number of players who win against an environment, then there exists a simple deterministic player who can win against the environment. This is true if the players are deterministic or probabilistic.

Theorem 11. *If 2^m deterministic players $\mathbf{r}$ of Kolmogorov complexity $\mathbf{K}(\mathbf{r}) \leq n$ win against an environment $\mathbf{q}$, then there exists a deterministic player $\mathbf{p}$ with $\mathbf{K}(\mathbf{p}) <^{\log} n - m + \mathbf{I}(\mathbf{q};\mathcal{H})$ that wins against $\mathbf{q}$.*

Theorem 12. *If 2^m probabilistic players $\mathbf{r}$ of Kolmogorov complexity $\mathbf{K}(\mathbf{r}) \leq n$ win against an environment $\mathbf{q}$ with probability at least p, then there exists a deterministic player $\mathbf{p}$ with $\mathbf{K}(\mathbf{p}) <^{\log} n - m - \log p + \mathbf{I}(\mathbf{q};\mathcal{H})$ that wins against $\mathbf{q}$.*

References

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