How to Compress the Solution

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Abstract

Using derandomization, we provide an upper bound on the compression size of solutions to the graph coloring problem. In general, if solutions to a combinatorial problem exist with high probability and the probability is simple, then there exists a simple solution to the problem. Otherwise the problem instance has high mutual information with the halting problem.

1 Introduction

In mathematics, the probabilistic method is a constructive method of proving the existence of a certain type of mathematical object. This method, pioneered by Paul Erdos, involves choosing objects from a certain class randomly, and showing objects of a certain type occur with non-zero probability. Thus objects of a certain type are guaranteed to exist. For more information about the probabilistic method, we refer readers to [AS04]. Recent results have shown that there is a strong connection between probabilistic method and the compression sizes of mathematical objects, i.e. their Kolmogorov complexity, $\mathbf{K}$.

> *If the probabilistic method can be used to prove the existence of an object, then bounds on its Kolmogorov complexity can be proven as well.*

If there is a simple probability such that objects of a certain mathematical type occur with large probability, then there exists an object of that type that is simple, i.e. has low Kolmogorov complexity. More formally, if object x has P-probability of at least p of randomally occuring, then

$$\mathbf{K}(x) <^{\log} \mathbf{K}(P) - \log p + \epsilon.$$

The ϵ term is the amount of information that the mathematical construct has with the halting sequence, which can obviously considered to be a negligible amount, except for exotic cases.

This inequality occurs the application of the EL Theorem [Lev16, Eps19]. Producing bounds of the Kolmogorov complexity of an object through probabilistic means is called derandomization. In [Eps22a], derandomization was applied to 22 examples, including a number of games. In [Eps22b], derandomization was used to show the tradeoff in the capacity of classical channels and codebook complexity. In addition, time-resource bounded derandomization was introduced.

I'd recommend derandomization as an area of research for masters students or researchers who are interested in moving into algorithmic information theory. This is because the majority of the technical effort resides in the domain to which derandomization is applied.

In this paper, we show a canonical derandomization example, that of graph vertex coloring. The proof requires an invocation of the EL theorem, an invocation of a conservation theorem, and some straightforward probabilistic arguments.

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2 Results

The function $\mathbf{K}(x|y)$ is the conditional prefix Kolmogorov complexity. Algorithmic probability is $\mathbf{m}(x) = \{2^{-\|p\|} : U(p) = x\}$, where U is the universal Turing machine. The function $\mathbf{m}$ is universal, in that for any semi-measure P, $O(1)\mathbf{m}(x) > 2^{-\mathbf{K}(P)}P(x)$. Thus for set $D \subseteq \{0,1\}^*$, computable probability P, $O(1)\mathbf{m}(D) > 2^{-\mathbf{K}(P)}P(D)$. $\mathbf{I}(a;\mathcal{H}) = \mathbf{K}(a) - \mathbf{K}(a|\mathcal{H})$, where $\mathcal{H} \in \{0,1\}^\infty$ is the halting sequence. $<^+ f$ is $<f+O(1)$ and $<^{\log} f$ is $<f+O(\log(f+1))$. The following lemma is an information non-growth law. There are many such laws, this one is over asymmetric information with the halting sequence.

Lemma 1 ([Eps22a])
For partial computable $f : \mathbb{N} \to \mathbb{N}$, for all $a \in \mathbb{N}$, $\mathbf{I}(f(a);\mathcal{H}) <^+ \mathbf{I}(a;\mathcal{H}) + \mathbf{K}(f)$.

The following result is the EL Theorem [Lev16, Eps19]. It was originally formulated as a statement about learning. However since that time, there has been several unexpected applications. In this paper, the EL Theorem is used for derandomization

Theorem 1 (EL Theorem) *For finite $D \subset \{0,1\}^*$, $-\log \max_{x \in D} \mathbf{m}(x) <^{\log} -\log \sum_{x \in D} \mathbf{m}(x) + \mathbf{I}(D;\mathcal{H})$.*

For graph $G = (V, E)$, with undirected edges, a k-coloring is a function $f : V \to \{1, \ldots, k\}$ such that if $(v, u) \in E$, then $f(v) \neq f(u)$. One example of graph coloring is cell phone towers which each need to operate at a certain frequency. Cell phone towers which are two close together cannot have the same frequency due to interference. Thus towers can be represented by vertices, and edges represent interference, with a graph coloring representing an assignment of frequencies to towers.

Theorem 2 *For graph $G = (V, E)$, $|V| = n$ with max degree d, there is a k coloring f with $2d \leq k$, and $\mathbf{K}(f) <^{\log} \mathbf{K}(n, k) + 2nd/k + \mathbf{I}((G, k);\mathcal{H})$.*

Proof. Assume that each vertex is randomly given a color in $\{1, \ldots, k\}$. The probability that each vertex does not have a conflict with the other vertices that are is $\frac{k-d}{k}$. Thus, the probability that the uniform random color assignments is a proper coloring is $\left(\frac{k-d}{k}\right)^n$. Let the finite set $D \subset \{0,1\}^*$ represent all encodings of proper coloring. Thus there is a simple total computable function f that on input G and k can output D, with $\mathbf{K}(f) = O(1)$. Let P the uniform probability over all color assignments to the vertices of G (even ones that are not a proper coloring). Thus, bearing in mind that $2d \leq k$,

$$-\log P(D) < -n \log \left(1 - \frac{d}{k}\right) \leq \frac{2nd}{k}.$$

Thus, due to the definition of the universal semi-measure $\mathbf{m}$,

$$-\log \mathbf{m}(D) <^+ \mathbf{K}(P) + P(D)$$

Thus by Theorem 1 and Lemma 1, there is a coloring $f \in D$ with

$$\begin{aligned}\mathbf{K}(f) &<^{\log} -\log \mathbf{m}(D) + \mathbf{I}(D;\mathcal{H})\\ &<^{\log} \mathbf{K}(P) - \log P(D) + \mathbf{I}(D;\mathcal{H})\\ &<^{\log} \mathbf{K}(n, k) + 2nd/k + \mathbf{I}((G, k);\mathcal{H}).\end{aligned}$$

□

3 Discussion

Future work involves finding instances of the probabilistic method, and applying derandomization to them. In particular, the Lovász Local Lemma, [EL], has been particularly computatible with derandomization. We present the first proved consequence of LLL and show how it is compatible with derandomization. We leave the proof to the reader.

A *hypergraph* is a pair $J = (V, E)$ of vertices V and edges $E \subseteq \mathcal{P}(V)$. Thus each edge can connect ≥ 2 vertices. A hypergraph is *k-regular* of the size $|e| = k$ for all edges $e \in E$. A 2-regular hypergraph is just a simple graph. A valid *C-coloring* of a hypergraph (V, E) is a mapping $f : V \to \{1, \ldots, C\}$ where every edge $e \in E$ is not *monochromatic* $|\{f(v) : v \in e\}| > 1$. The following classic result

Theorem. [Probabilistic Method] *Let $G = (V, E)$ be a k-regular hypergraph. If for each edge f, there are at most $2^{k-1}/e - 1$ edges $h \in E$ such that $h \cap f \neq \emptyset$, then there exists a valid 2-coloring of G.*

We can now use derandomization, to produce bounds on the Kolmogorov complexity of the simpliest such 2-coloring of G.

Theorem. [Derandomization] *Let $G = (V, E)$ be a k-regular hypergraph with $|E| = m$. If, for each edge f, there are at most $2^{k-1}/e - 1$ edges $h \in E$ such that $h \cap f \neq \emptyset$, then there exists a valid 2-coloring x of G with*

$$\mathbf{K}(x) <^{\log} \mathbf{K}(n) + 4me/2^k + \mathbf{I}(G; \mathcal{H}).$$

The conjecture is that one can produce a suite of derandomization theorems, each one mapping to Kolmogorov complexity with different time and space constraints, and access to a certain number of random bits. So far, resource bounded derandomization does not lend itself to games. This is because the environment must be polynomial time computable which means the agent can efficiently simulate it, making the results trivial.

References

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