White Noise in Quantum Mechanics

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Abstract

Every typical quantum state is effectively white noise. Any application of a POVM will produce the empty signal, white noise, or some combindation of the two. No amount of proccessing by quantum operations can increase its information content. The only way to increase its signal is the random projection of the quantum state to a subspace with a measurement. This is the case with all but an exponentially small fraction of quantum states. Thus, serious doubts about the relevance of quantum information theory are raised.

1 Introduction

Quantum information theory refers to the use of properties of quantum mechanics to perform information processing and transmission. It has numerous subfields, including quantum computing, quantum algorithms, quantum key distribution, quantum complexity theory, quantum teleportation, and quantum error correction. It leverages quantum superpositions and entanglements as resources to define methods and algorithms that classical mechanics cannot achieve.

However, this paper shows that all but an exponentially small amount of quantum states are effectively white noise. Thus, characterizing them as "garbage" would not be inaccurate. From these states, no information can be obtained with measurements, and, due to conservation inequalities, no amount of processing through a quantum channel can increase its "signal content".

The facts detailed in this paper need to be reconciled with modern quantum information theory. How does one deal with the fact that virtually all quantum states are effectively white noise, with no value with respect to information processing or transmission? The only quantum states with high signal content are classical basis states such as $|x\rangle$, for $x \in \{0,1\}^*$, and states that are close to them in the Hilbert space. This begs the question:

> *Are quantum methods only effective on classical states or quantum states that are very close to them?*

If this is correct, then serious concerns about the validity of many areas of quantum information theory must be raised.

One can ask, why is objective reality so ordered? Due to conservation inequalities (Theorem 3), a quantum operation cannot increase the self-information of a (pure or mixed) quantum state. However, a wave function collapse, i.e. measurement, can cause an uptake in algorithmic information of the quantum state, as shown in Theorem 7. This collapse enables non-negligible signals from further measurements and also enables partial information cloning.

2 Results

The main reference for the work in this paper is [Eps]. In Algorithmic Information Theory, the information content of individual strings $x \in \{0,1\}^*$ is denoted by prefx-free Kolmogorov complexity $\mathbf{K}(x)$. The information between two strings $x, y \in \{0,1\}^*$ is $\mathbf{I}(x:y) = \mathbf{K}(x) + \mathbf{K}(y) - \mathbf{K}(x,y)$. The universal lower computable semi-measure is $\mathbf{m}(x)$. A matrix or quantum state is elementary if all its coefficients are roots of polynomial equations with rational coefficients. Elementary constructs can be encoded into numbers, and thus their Kolmogorov complexities are well defined. One can perform the standard linear algebra algorithms over elementary numbers. For positive real function f, $<^+ f$ means $< f + O(1)$. When we say that a term is "relativized", we mean that it is encoded in an auxillary tape of the universal Turing machine.

In order to better understand the barrier between the quantum and classical realms, we introduce a information term over probabilities. This is because quantum states decohere into probability measures and a POVM will also produce a probability from the measured quantum states. For probabilities p, q over strings, their mutual information is defined as follows.

Definition 1 (Information, Probabilites) *For probabilities p and q over $\mathbb{N}$,*

$$\mathbf{I}_{\mathrm{prob}}(p:q) = \log \sum_{x,y \in \{0,1\}^*} 2^{\mathbf{I}(x:y)} p(x) q(y).$$

This definition obeys conservation inequalities over randomized processing $f : \{0,1\}^* \times \{0,1\}^* \to \mathbb{R}_{\geq 0}$, where $fp(x) = \sum_z f(x|z)p(z)$. The following result is from Theorem 3 from [Eps].

Theorem 1 *Relativized to random processing f,*

$$\mathbf{I}_{\mathrm{prob}}(fp:q) <^+ \mathbf{I}_{\mathrm{prob}}(p:q).$$

If $\mathbf{I}_{\mathrm{prob}}(p:p)$ is small then either p represents an empty signal, or white noise, or some combination of the two. Furthermore, no amount of deterministic or randomized processing can increase p's "signal".

A positive-semidefinite matrix A is lower computable if there is a program q that outputs a series of elementary positive semidefinite matrices $\{A_i\}$ such that $A_i \leq A_{i+1}$ and $\lim_{i\to\infty} A_i = A$. Furthermore, we say that q lower computes A. Let $\mathcal{C}_{C\otimes D}$ be the set of all lower computable matrices $A \otimes B$, such that $\mathrm{Tr}(A\otimes B)(C \otimes D) \leq 1$. The lower algorithmic probability of a lower computable matrix σ is $\underline{\mathbf{m}}(\sigma|x) = \sum\{\mathbf{m}(q|x) : q \text{ lower computes } \sigma\}$. Let $\mathfrak{C}_{C\otimes D} = \sum_{A\otimes B \in \mathcal{C}_{C\otimes D}} \underline{\mathbf{m}}(A\otimes B|n) A \otimes B$. The universal lower computable semi-denisty matrix is $\boldsymbol{\mu} = \sum_{\text{Elementary } n \text{ qubit } |\phi\rangle} \mathbf{m}(|\phi\rangle\, |n)\, |\phi\rangle\langle\phi|$. For more information about $\boldsymbol{\mu}$, see [Eps].

Definition 2 (Information, Quantum States) *The mutual information between two quantum states σ, ρ is defined to be $\mathbf{I}(\sigma:\rho) = \log \mathrm{Tr}\mathfrak{C}_{\boldsymbol{\mu}\otimes\boldsymbol{\mu}}(\sigma \otimes \rho)$.*

For motivation for this definition, we refer readers to Definition 17 in [Eps]. Quantum information $\mathbf{I}(\sigma:\rho)$ is radically different from classical information $\mathbf{I}(x:y)$ in that overwhelming majority of pure quantum states have no self-information. The following result is from Theorem 56 in [Eps].

Theorem 2 *Let Λ be the uniform distribution over all n qubit pure states.*

$$\int 2^{\mathbf{I}(|\psi\rangle:|\psi\rangle)} d\Lambda = O(1).$$

Like classical Shannon information and Kolmogorov information, quantum information is subject to conservation inequalities. This means there is no means to increase the self-information of a quantum state, with profound consequences to quantum information theory.

Theorem 3 *Relativized to elementary quantum operation* ε, $\mathbf{I}(\varepsilon(\rho):\sigma)$.

Measurements are modeled with a POVM E, which is a is a finite set of positive semi-definite matrices E_k such that $\sum_k E_k = \mathbf{1}$. Given a quantum state σ, a POVM induces a probability measure $E\sigma(k) = \mathrm{Tr}\sigma E_k$, where $E\sigma(i)$ can be interpreted as the probability of measuring i when σ is applied to E. The size of POVM E is $|E|$. Remarkably, algorithmic self-information of a quantum state upper bounds the signal produced from a POVM. The following result is from Theorem 65 in [Eps].

Theorem 4 *Relativized to elementary POVM* E, $\mathbf{I}_{\mathrm{prob}}(E\sigma : E\sigma) <^+ \mathbf{I}(\sigma:\sigma) + \log\log|E|$.

This theorem, combined with Theorem 2, shows that the measurements of most quantum states produce the empty signal, white noise, or some combination of the two. In fact one can show this directly with the following result, which can be found in Theorem 67 in [Eps].

Theorem 5 *Let* Λ *be the uniform distribution over all* n *qubit pure states. Relativized to elementary POVM* E,

$$\int 2^{\mathbf{I}_{\mathrm{prob}}(E|\psi\rangle : E|\psi\rangle)} d\Lambda = O(1).$$

In addition, an overwhelming amount of quantum states cannot clone information. The self-information of a quantum state upper bounds the amount of cloneable information it has.

Theorem 6 *Relativized to elementary POVMs* E, F, *and elementary quantum operation* ϵ, *if* $\epsilon(\nu), = \sigma\otimes\rho$, *then*

$$\begin{aligned} \mathbf{I}(\sigma:\rho) &<^+ \mathbf{I}(\nu:\nu), \\ \mathbf{I}_{\mathrm{prob}}(E\sigma : F\rho) &<^+ \mathbf{I}(\nu:\nu) + \log\log\max\{|E|,|F|\}. \end{aligned}$$

So, a quantum state's algorithmic self-information upper bounds the "signal" produced from measurements as well as the amount of information that can be cloned with a quantum operation. In addition, an overwhelming majority of quantum states (pure and mixed) will have negligible self-information. The question is how does the ordered classical world emerge from quantum states with no self-information?

The answer is that the random projection of a state from a measurement can provides an uptake of its algorithmic self-information. To show this, we rely on PVMs and Quantum Bayesianism [SBC01]. PVMs is short for projection value measure. A PVM $P = \{\Pi_i\}$ is a collection of projectors Π_i with $\sum_i \Pi_i = \mathbf{1}$, and $\mathrm{Tr}\Pi_i\Pi_j = 0$ when $i \neq j$. When a measurement occurs, with probability $\langle\psi|\Pi_i|\psi\rangle$, the value i is measured, and the state collapses to

$$|\psi'\rangle = \Pi_i|\psi\rangle / \sqrt{\langle\psi|\Pi_i|\psi\rangle}.$$

Further measurements of $|\psi'\rangle$ by P will always result in the i measurement, so $P|\psi'\rangle(i) = 1$. Quantum Bayesianism [SBC01] deals with distributions over quantum states. When a PVM is applied to (apriori) distributions over states, a new distribution is created from the probabilistic laws of PVMs. In previous sections, the uniform distribution over pure states Λ is used. The purely classical uniform distribution over the basis states has

$$\frac{1}{2^n}\sum_{x\in\{0,1\}^n} \mathbf{I}(|x\rangle : |x\rangle) >^+ n.$$

Let F be a PVM of 2^{n-c} projectors, of an n qubit space and let Λ_F be the distribution of pure states when F is measured over the uniform distribution Λ. Thus, the support of Λ_F represents the subspace consisting of the F-collapsed states from the purely quantum uniform distribution Λ. Note that if F has too few projectors, it lacks discretionary power to produce a meaningful signal when states are in distribution Λ_F. The following result is from Theorem 69 in [Eps].

Theorem 7 *Relativized to elementary PVM F, $n - 2c - \log n <^+ \log \int 2^{\mathbf{I}(|\psi\rangle:|\psi\rangle)} d\Lambda_F$.*

References

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[SBC01] R. Schack, T.. Brun, and C. Caves. Quantum bayes rule. *Phys. Rev. A*, 64:014305, 2001.