

A Short Proof of the Existence of Anomalies

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Abstract

The Independence Postulate (IP) is a finitary Church-Turing Thesis, postulating that mathematical sequences are independent from physical ones. IP implies that anomalies are found in the physical world.

Anomalies

$\mathbf{K}(x|y)$ is the conditional prefix Kolmogorov complexity. For probability p over \mathbb{N} , the deficiency of randomness is $\mathbf{d}(a|p) = \lfloor -\log p(a) \rfloor - \mathbf{K}(a)$. $\mathbf{I}(a; \mathcal{H}) = \mathbf{K}(a) - \mathbf{K}(a|\mathcal{H})$, where \mathcal{H} is the halting sequence. An elementary probability measure over \mathbb{N} has finite support and a range in \mathbb{Q} . Stochasticity is

$$\begin{aligned} \Lambda(a|b) &= \min\{\mathbf{K}(Q|b) + 3 \log \max\{\mathbf{d}(a|Q, b), 1\} \\ &\quad : Q \text{ is an elementary probability measure}\}. \\ \Lambda(a|b) &< \Lambda(a) + O(\log \mathbf{K}(b)). \end{aligned}$$

The following definition is from [Lev74].

Definition 1 (Information) $\mathbf{I}(\alpha : \beta) = \log \sum_{x,y} 2^{\mathbf{K}(x) + \mathbf{K}(y) - \mathbf{K}(x,y) - \mathbf{K}(x|\alpha) - \mathbf{K}(y|\beta)}$.

The Independence Postulate [Lev84, Lev13] statement is:

IP: *Let α be a sequence defined with an n -bit mathematical statement, and a sequence β can be located in the physical world with a k -bit instruction set. Then $\mathbf{I}(\alpha : \beta) < k + n + c$ for some small absolute constant c .*

There are many proofs in the literature that stochastic numbers have high mutual information with the halting sequence. One detailed proof is in [Eps21].

Lemma 1 $\Lambda(x) <^{\log} \mathbf{I}(x; \mathcal{H})$.

Lemma 2 *For computable probability p over \mathbb{N} and for $D \subset \mathbb{N}$, $|D| = 2^s$, $s < \max_{a \in D} \mathbf{d}(a|p) + \Lambda(D) + \mathbf{K}(s) + \mathbf{K}(p) + O(\log \mathbf{K}(s)\mathbf{K}(p))$.*

Proof. We relativize the universal Turing machine to p and s . Let Q be an elementary probability measure that realizes $\Lambda(D)$. Let $d = \mathbf{d}(D|Q)$. Let $F \subseteq \mathbb{N}$ be a random set where each element $a \in \mathbb{N}$ is selected independently with probability $cd2^{-s}$, where $c \in \mathbb{N}$ is chosen later. $\mathbf{E}[p(F)] \leq cd2^{-s}$. Furthermore

$$\begin{aligned} &\mathbf{E}[Q(\{G : |G| = 2^s, G \cap F = \emptyset\})] \\ &\leq \sum_G Q(G)(1 - cd2^{-s})^{2^s} < e^{-cd}. \end{aligned}$$

Thus finite $W \subset \mathbb{N}$ can be chosen such that $p(W) \leq 2cd2^{-s}$ and $Q(\{G : |G| = 2^s, G \cap W = \emptyset\}) \leq e^{1-cd}$. $D \cap W \neq \emptyset$, otherwise, using the Q -test, $t(G) = e^{cd-1}$ if $(|G| = 2^s, G \cap W = \emptyset)$ and $t(G) = 0$ otherwise, we have

$$\begin{aligned} \mathbf{K}(D|Q, d, c) &<^+ -\log Q(D) - (\log e)cd \\ (\log e)cd &<^+ -\log Q(D) - \mathbf{K}(D|Q) + \mathbf{K}(d, c) \\ (\log e)cd &<^+ d + \mathbf{K}(d, c), \end{aligned}$$

which is a contradiction for large c . Thus there is an $a \in D \cap W$, where

$$\begin{aligned} \mathbf{K}(a) &<^+ -\log p(a) + \log d - s + \mathbf{K}(d) + \mathbf{K}(Q) \\ s &<^+ \mathbf{d}(a|p) + \Lambda(D). \end{aligned}$$

Removing the relativization of p and s

$$\begin{aligned} s &< -\log p(a) - \mathbf{K}(a|p, s) + \Lambda(D|p, s) \\ s &< \max_{a \in D} \mathbf{d}(a|p) + \Lambda(D) + \mathbf{K}(s) + \mathbf{K}(p) \\ &\quad + O(\log \mathbf{K}(s)\mathbf{K}(p)). \end{aligned}$$

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□

For $\tau \in \mathbb{N}^{\mathbb{N}}$, $\langle \tau \rangle = \langle \tau[1] \rangle \langle \tau[2] \rangle \langle \tau[3] \rangle \dots$. Let $\tau(n)$ be the first 2^n unique numbers found in τ . The sequence τ is assumed to have an infinite amount of unique numbers, and represents a series of observations.

Theorem 1 *For probability p over \mathbb{N} and $\tau \in \mathbb{N}^{\mathbb{N}}$, let $s_{\tau,p}$ be the smallest number such that $\max_{a \in \tau(n)} \mathbf{d}(a|p) > n - 4\mathbf{K}(n) - s_{\tau,p}$. Then $s_{\tau,p} <^{\log} \mathbf{I}(\langle \tau \rangle : \mathcal{H}) + \mathbf{K}(p)$.*

Proof. By Lemmas 1 and 2, and the fact that $\mathbf{I}(x; \mathcal{H}) <^+ \mathbf{I}(\alpha : \mathcal{H}) + \mathbf{K}(x|\alpha)$,

$$\begin{aligned} n &< \max_{a \in \tau(n)} \mathbf{d}(a|p) + \mathbf{I}(\tau(n); \mathcal{H}) + \mathbf{K}(p) + 2\mathbf{K}(n) \\ &\quad + O(\log(\mathbf{I}(\tau(n); \mathcal{H}) + \mathbf{K}(p))), \\ n &< \max_{a \in \tau(n)} \mathbf{d}(a|p) + 4\mathbf{K}(n) + \mathbf{I}(\langle \tau \rangle : \mathcal{H}) + \mathbf{K}(p) \\ &\quad + O(\log(\mathbf{I}(\langle \tau \rangle : \mathcal{H}) + \mathbf{K}(p))), \\ n - 4\mathbf{K}(n) - (\mathbf{I}(\langle \tau \rangle : \mathcal{H}) + \mathbf{K}(p)) \\ &\quad + O(\log(\mathbf{I}(\langle \tau \rangle : \mathcal{H}) + \mathbf{K}(p))) < \max_{a \in \tau(n)} \mathbf{d}(a|p). \end{aligned}$$

□

Let k be the physical address of an infinite sequence of numbers $\tau \in \mathbb{N}^{\mathbb{N}}$. The halting sequence can be described by a small mathematical statement. By Theorem 1 and IP,

$$s_{\tau,p} <^{\log} \mathbf{I}(\langle \tau \rangle : \mathcal{H}) + \mathbf{K}(p) <^{\log} k + c + \mathbf{K}(p).$$

Thus sequences τ with large $s_{\tau,p}$ will have large physical addresses. So, assuming IP, it's hard to find observations which do not have large anomalies, and impossible to find observations with no anomalies.

References

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