AIT Blog

Two Modest Lemmas

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This blog post contains two small lemmas that might be of independent interest. In general, the blog posting will slow down as I intend to write a survey over the material covered. I still intend to post blogs of papers of interest, but with a slower rate. As of today, the survey will contain the following contents.

1. Outliers in strings, sequences, and general spaces
2. Machine Learning and AIT
3. Clusters
4. Sets Have Simple Members Theorem
5. Resource Bounded EL Theorems
6. Derandomization (resourse free and resourse bounded) in particular its connection with Classical Information Theory and also parameterized instances
7. Quantum Information Theory, Many Worlds Theory

Computable Probability

In my October 11th blog post, I demonstrated the utility of so-called left-total machines. In this section, we show how to make an semi-computable semi-measure, $\mathbf{m}$, computable by using left-total machines. This enables a greater range of flexibility in proving results when $\mathbf{m}$ is computable, as shown in my September 28th blog post. Let $\mathbf{K}$ be the prefix-free Kolmogorov complexity. Let $\mathbf{I}(a;\mathcal{H}) = \mathbf{K}(a) - \mathbf{K}(a|\mathcal{H})$, where $\mathcal{H}$ is the halting sequence.

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Definition 1 *For* $D \subseteq \{0,1\}^*$, $\overline{\mathbf{m}}(D) = \min\{\mathbf{m}(P)P(D) : \textit{probability } P \textit{ is total computable}\}$.

Lemma 1 $-\log \overline{\mathbf{m}}(D) <^{\log} -\log \mathbf{m}(D) + \mathbf{I}(D;\mathcal{H})$.

Proof. Let $(p0)^- = (p1)^- = p$. We define the following computable semi-measure, with $\mathbf{m}_b(x) = \sum\{2^{-\|p\|} : U(p) = x, p \lhd b \text{ or } p \sqsupseteq b\}$. If b and b^- are total then $\mathbf{m}_b(x) \leq \mathbf{m}_{b^-}(x)$. Let $s = \lceil -\log \mathbf{m}(D)\rceil + 1$. Let b be the shortest total string such that $\mathbf{m}_b(D) \geq 2^{-s}$. Thus b^- is not total. Thus $-\log \overline{\mathbf{m}}(D) <^+ -\log \mathbf{m}(\mathbf{m}_b)\mathbf{m}_b(D) <^+ s + \mathbf{K}(b)$. We show that $\mathbf{K}(b) <^{\log} \mathbf{I}(D;\mathcal{H}) + \mathbf{K}(s)$. From Lemma 2 in [Eps22], we have that $\mathbf{I}(f(a);\mathcal{H}) <^+ \mathbf{I}(a;\mathcal{H}) + \mathbf{K}(f)$ and so $\mathbf{I}(b;\mathcal{H}) <^+ \mathbf{I}(D;\mathcal{H}) + \mathbf{K}(b|D)$. Now since b is total and b^- is not, b^- is a prefix of border, the binary expansion of Chaitin's Omega, and thus b is random. Furthermore b is simple relative to the halting sequence, with $\mathbf{K}(b|\mathcal{H}) <^+ \mathbf{K}(\|b\|)$. Thus $\mathbf{K}(b) <^{\log} \mathbf{I}(b;\mathcal{H})$. Now we prove that $\mathbf{K}(b|D) <^+ \mathbf{K}(\|b\|) + \mathbf{K}(s)$. There is an algorithm that can enumerate total strings of length $\|b\|$ and return the first string c such that $\mathbf{m}_c(D) \geq 2^{-s-1}$. This string is indeed b, as shown in Figure 1. □

Mutual Information with the Halting Sequence

The following lemma presents a non intuitive inequality about the mutual information with the halting sequence.

Lemma 2 $\mathbf{I}(x;\mathcal{H}/y) <^{\log} \mathbf{I}(\langle x,y\rangle;\mathcal{H})$.

Proof.

$$\begin{aligned}\mathbf{I}(x;\mathcal{H}/y) &= \mathbf{K}(x/y) - \mathbf{K}(x/y,\mathcal{H})\\ &<^+ \mathbf{K}(x,y) - \mathbf{K}(y) + \mathbf{K}(\mathbf{K}(y)/y) - \mathbf{K}(x/y,\mathcal{H}).\end{aligned}$$

Due to Theorem 3.3.1 in [G2́1], $\mathbf{K}(\mathbf{K}(y)/y) <^{\log} \mathbf{I}(y;\mathcal{H})$, so

$$\begin{aligned}\mathbf{I}(x;\mathcal{H}/y) &< \mathbf{K}(x.y) - \mathbf{K}(y) + \mathbf{I}(y;\mathcal{H}) - \mathbf{K}(x/y,\mathcal{H}) + O(\log \mathbf{I}(y;\mathcal{H}))\\ &< \mathbf{K}(x,y) - \mathbf{K}(y/\mathcal{H}) - \mathbf{K}(x/y,\mathcal{H}) + O(\log \mathbf{I}(y;\mathcal{H}))\\ &< \mathbf{K}(x,y) - \mathbf{K}(x,y/\mathcal{H}) + O(\log \mathbf{I}(y;\mathcal{H}))\\ &<^{\log} \mathbf{I}(\langle x,y\rangle;\mathcal{H}).\end{aligned}$$

The last inequality is due Lemma 2 in [Eps22], which states that $\mathbf{I}(y;\mathcal{H}) <^+ \mathbf{I}(\langle x,y\rangle;\mathcal{H})$.

References

[Eps22] S. Epstein. The outlier theorem revisited. *CoRR*, abs/2203.08733, 2022.

[G2́1] Peter Gács. Lecture notes on descriptional complexity and randomness. *CoRR*, abs/2105.04704, 2021.

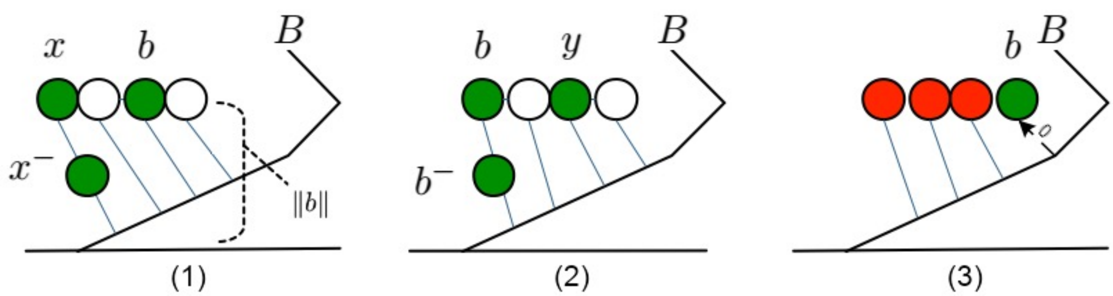

Figure 1: The above diagram represents the domain of the universal left-total Turing machine U with 0s branching to the left and 1s branching to the right. It shows all the total strings of length $\|b\|$, including b. The large diagonal line is the border sequence, B. A string c is marked green if $\mathbf{m}_c(D) \geq 2^{-s-1}$. By definition, b is a shortest green string. If x is green and total, and $x \lhd y$, and y is total, then y is green, since $\mathbf{bb}(x) \leq \mathbf{bb}(y)$. Furthermore, if x is green and total and x^- is total, then x^- is green, as $\mathbf{bb}(x) \leq \mathbf{bb}(x^-)$. It cannot be that there is a green $x \lhd b$ with $\|x\| = \|b\|$. Otherwise, x^- is total, and thus, it is green, causing a contradiction because it is shorter than b. This is shown in part (1). Furthermore, there cannot be a green y, with $b \lhd y$ and $\|y\| = \|b\|$. Otherwise, b^- is total and thus green, contradicting the definition of b. This is shown in part (2). Thus, b is unique, and since b^- is not total, b^- is a prefix of the border, as shown in part (3). Thus, an algorithm returning a green string of length $\|b\|$ will return b.