KEY - A, Slot

① Evaluate $\int_C z e^{\pi z}\left[\frac{1}{z^4-16}+1\right] dz$, where C is the ellipse $9x^2+y^2=9$, using Cauchy's Residue theorem.

Sol:

$$\int_C z e^{\pi z}\left(\frac{1}{z^4-16}+1\right) dz = \underbrace{\int_C \frac{z e^{\pi z}}{z^4-16}\, dz}_{\text{I}} + \underbrace{\int_C z e^{\pi z}\, dz}_{\text{II}}$$

I: $z^4-16=0 \Rightarrow z=\pm 2i$ and $z=\pm 2$

Clearly $z=\pm 2i$ lies inside C and $z=\pm 2$ lies outside C

$\because e^{2\pi i}=1$

$$\text{Res}\left[\frac{z e^{\pi z}}{z^4-16}\right]_{z=2i} = \left[\frac{z e^{\pi z}}{z^3}\right]_{z=2i} = \frac{1}{-16}$$

Illy

$$\text{Res}\left[\frac{z e^{\pi z}}{z^4-16}\right]_{z=-2i} = -\frac{1}{16}$$

II. By Laurent's series expansion, we get

$$z e^{\pi z} = z + z^2\pi + \frac{z^3\pi^2}{2!} + \cdots$$

But it has no negative powers, $\text{Res}\left[z e^{\pi z}\right]=0$

$$\therefore \int_C z e^{\pi z}\left(\frac{1}{z^4-16}+1\right) dz = 2\pi i\left(-\frac{1}{16}-\frac{1}{16}+0\right)$$

$$= 2\pi i\left(-\frac{2}{16}\right)$$

$$= -\frac{\pi i}{4}$$

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(2) Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta$

sol

put $z = e^{i\theta}$

$$\int_0^{2\pi} \frac{\cos\theta}{5-4\cos\theta}\, d\theta = Re \int_C \frac{e^{i3\theta}}{5-4\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)}\, d\theta$$

$$= Re \int_C \frac{(e^{i\theta})^3}{5-4\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)}\, d\theta$$

$$= Re \int_C \frac{z^3}{5-4\left(\frac{z+z^{-1}}{2}\right)} \frac{dz}{iz}$$

$$= Re \int_C -\frac{1}{2i} \left(\left(\frac{z^3}{z-2}\right) \Big/ z-\tfrac{1}{2}\right) dz$$

clearly $z = 2$ lies outside $|z| = 1$ and

$z = \frac{1}{2}$ lies inside $|z| = 1$

$\therefore Res\left[f(z)\right]_{z=1/2} = -\frac{1}{12}$

$$\therefore \quad Re \int_C -\frac{1}{2i} \frac{\left(z^3/z-2\right)}{z-\frac{1}{2}}\, dz = Re\,\pi i\left(\text{sum of residues}\right)$$

$$= Re.\ 2\pi i * -\tfrac{1}{2i} * -\tfrac{1}{12}$$

$$= \frac{\pi}{12}$$

$$\therefore \int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta = \pi/12$$

(3) Gauss - Jordan Elimination :

$$[A|b] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 2 & -1 & 1 & 1 & 0 \\ 3 & 2 & -1 & -1 & 1 \\ 1 & 1 & 3 & -3 & -8 \end{array}\right] \simeq \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 0 & -3 & -1 & 3 & 4 \\ 0 & -1 & -4 & 2 & 7 \\ 0 & 0 & 2 & -2 & -6 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1. \end{array}$$

$$\simeq \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 4 & -2 & -7 \\ 0 & -3 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & -6 \end{array}\right] R_2 \leftrightarrow R_3 \ \& \ R_2 \to \frac{R_2}{-1}$$

$$\simeq \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 4 & -2 & -7 \\ 0 & 0 & 11 & -3 & -17 \\ 0 & 0 & 1 & -1 & -3 \end{array}\right] \begin{array}{l} R_3 \to R_3 + 3R_2 \\ R_4 \to \frac{R_4}{2} \end{array} \simeq \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 4 & -2 & -7 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 11 & -3 & -17 \end{array}\right] R_3 \leftrightarrow R_4$$

$$\simeq \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 4 & -2 & -7 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 8 & 16 \end{array}\right] R_4 \to R_4 - 11R_3 \quad \simeq \left[\begin{array}{cccc|c} 1 & 0 & -3 & 1 & 5 \\ 0 & 1 & 4 & -2 & -7 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 \to R_1 - R_2 \\ \\ \\ R_4 \to \frac{R_4}{8} \end{array}$$

$$\simeq \left[\begin{array}{cccc|c} 1 & 0 & -3 & 1 & 5 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] R_2 \to R_2 - 4R_3 \quad \simeq \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -4 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] R_1 \to R_1 + 3R_3$$

$$\cong \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 \to R_1 + 2R_4 \\ R_2 \to R_2 - 2R_4 \\ R_3 \to R_3 + R_4 \\ \\ \end{array}$$

$$\therefore \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix}$$

4. Cayley - Hamilton Theorem: Given $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

ch. equation : $\lambda^3 + \lambda^2 - 5\lambda - 5 = 0$

Cayley - Hamilton theorem holds $\therefore$ $A^3 + A^2 - 5A - 5I = 0$

$$A^3 = \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix}; \quad A^2 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{5}\left[A^2 + A - 5I\right] = \frac{1}{5}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

Eigen values: $A \to -1, \pm\sqrt{5}$

$A^{-1} \to -1, \pm \frac{1}{\sqrt{5}}$

$A^{T} \to -1, \pm\sqrt{5}$

5. a. (i) $W = \{(x,y,z) \in \mathbb{R}^3 \mid xyz = 0\}$

Ans: Not a subspace.

$\because$ $(1,0,1) \in W$, $(0,1,1) \in W$

But $(1,0,1) + (0,1,1) = (1,1,2) \notin W$

Addition fails.

(ii) $W = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2-z^2=0\}$

Ans: Not a subspace.

$\because$ $(1,0,1)$, $(0,1,1) \in W$

But $(1,0,1)+(0,1,1) = (1,1,2) \notin W$

Addition fails.

b) (i) $\{x,\ 2x-x^2,\ 3x+2x^2\} \subseteq P_2(\mathbb{R})$

$$\begin{vmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{vmatrix} = 0$$

$\therefore$ Given vectors are linearly dependent.

i.e., Not Independent.

ii) $\left\{ \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -1 & 7 \end{bmatrix} \right\} \subseteq M_{2\times2}(\mathbb{R})$

$$a\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} + b\begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix} + d\begin{bmatrix} -1 & 0 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$-a+3b-d=0$ — ①

$a+2c=0$ — ② $\Rightarrow a=-2c. \Rightarrow c=-\frac{a}{2}$

$-2a+b-3c-d=0$ — ③

$2a+b+c+7d=0$ — ④

From ①, $2c+3b-d=0$

From ③, $c+b-d=0$

$$c+2b=0$$

$\Rightarrow c=-2b$ or $b=-\frac{c}{2}$

From ①, $4c+3\cdot\left(-\frac{c}{2}\right)-d=0$

$8c-3c-2d=0$

$5c-2d=0$

$\Rightarrow d=\frac{5c}{2} \Rightarrow c=\frac{2}{5}d$

$\therefore c=-\frac{a}{2}$

$c=-2b$

$c=\frac{2}{5}d$

Hence the vector equation has no trivial solution.

$\therefore$ The Given vectors are Linearly Dependent.