Algorithms and Data Structures Coursework

4. Prove or disprove each of the following statements. We will assume that $x > 0$, and all functions are asymptotically positive. That is, for some constant k, $f(x) > 0$ for all $x \geqslant k$

   (a) $2x^4$ is $O(x^3 + 3x + 2)$ [2]

   My Solution:
   Assume that there are constants k and C such that
   $$2x^4 \leqslant C \cdot (x^3 + 3x + 2)$$
   when $x \geqslant k$
   $$\frac{2}{C} \leqslant \frac{1}{x} + \frac{3}{x^3} + \frac{2}{x^4}$$
   For values of x greater than 1, as the value of x increases $\frac{1}{x} + \frac{3}{x^3} + \frac{2}{x^4}$ tends to 0 and so this does not hold

   (b) $4x^3 + 2x^2 \cdot \log x + 1$ is $O(x^3)$ [2]

   My Solution:
   As $x > \log x$ for all $x > 0$
   $$f(x) = 4x^3 + 2x^2 \log x + 1 \leqslant 4x^3 + 2x^3 + 1$$
   As $x^3 \geqslant 1$ for all $x \geqslant 1$
   $$f(x) = 4x^3 + 2x^2 \log x + 1 \leqslant 4x^3 + 2x^3 + 1 \leqslant 7x^3$$
   For $x \geqslant 1$. Because the above inequality holds for every positive $x \geqslant 1$, using $k = 1$ and $C = 16$ as witnesses, we get
   $$|f(x)| \leqslant C \cdot |x^3|$$
   For every $x \geqslant k$

(c) $3x^2 + 7x + 1$ is $\omega(x \cdot \log x)$ [2]

My Solution:

$$f = \omega(g) \Leftrightarrow g = o(f)$$

So by proving that

$$x \log x = o(3x^2 + 7x + 1)$$

It is true that

$$3x^2 + 7x + 1 = \omega(x \cdot \log x)$$

That would be true if:

$$\lim_{x\to\infty} \frac{x \log x}{3x^2 + 7x + 1} = 0$$

As $x > \log x \quad \forall \quad x > 0$ if $\lim_{x\to\infty} \dfrac{x^2}{3x^2+7x+1} = 0$ then $\lim_{x\to\infty} \dfrac{x \log x}{3x^2+7x+1} = 0$

As $x^2 < 3x^2 + 7x + 1$ then $x > \log x \quad \forall \quad x > 0$ if $\lim_{x\to\infty} \dfrac{x^2}{3x^2+7x+1} = 0$ Therefore

$$3x^2 + 7x + 1 = \omega(x \cdot \log x)$$

With the witness $k = 1$ and $C = 1$

(d) $x^2 + 4x$ is $\Omega(x \cdot \log x)$ [2]

My Solution:
$f(x) = x^2 + 4x$
$g(x) = x \cdot \log x$

$$|f(x)| \geqslant C \cdot |g(x)|$$

As for $x > 0$ $x > \log x$

$$x^2 + 4x \geqslant x^2 \geqslant x \cdot \log x$$

So it is true with the witnesses $c = 1$ and $k = 1$

(e) $f(x) + g(x)$ is $\Theta(f(x) + g(x))$ [2]

My Solution:
As you don't know anything about the complexity of $f(x)$ and $g(x)$ it is impossible to say what the complexity of their sum is

Total for Question 4: 10