Polar Coordinates - Exam Questions

1 Example 1 - Finding areas

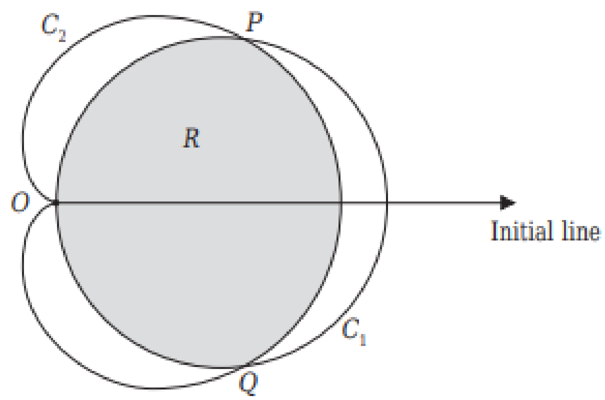

Figure 1

The curve C_1 with equation

$$r = 7\cos\theta, \quad -\frac{\pi}{2} < \theta \leqslant \frac{\pi}{2}$$

and the curve C_2 with equation

$$r = 3(1 + \cos\theta), \quad -\pi < \theta \leqslant \pi$$

are shown on Figure 1.

The curves C_1 and C_2 both pass through the pole and intersect at the point P and the point Q.

(a) Find the polar coordinates of P and the polar coordinates of Q.

(3)

The regions enclosed by the curve C_1 and the curve C_2 overlap, and the common region R is shaded in Figure 1.

(b) Find the area of R.

(7)

Set the two curves equal to each other and simplify

$$7\cos\theta = 3(1 + \cos\theta)$$

$$4\cos\theta = 3$$

$$\cos\theta = \frac{3}{4}$$

Substitute the value of $\cos\theta$ to find the radius

$$r = 7\cos\theta = 7 \times \frac{3}{4} = \frac{21}{4}$$

Write in polar form

$$P : \left(\frac{21}{4}, 0.7727\right) \qquad Q : \left(\frac{21}{4}, -0.7227\right)$$

Figure out the areas required to make the area, half the shape and use symmetry to make it easier

$$C_2 : 0 \leqslant \theta \leqslant \arccos\left(\frac{3}{4}\right)$$

$$C_1 : \arccos\left(\frac{3}{4}\right) \leqslant \theta \leqslant \frac{\pi}{2}$$

Write the integrals required using the formula $A = \frac{1}{2}\int r^2 \, d\theta$

$$A = 2\Big(\frac{1}{2}\int_{\alpha}^{\frac{\pi}{2}} (7\cos\theta)^2 \, d\theta + \frac{1}{2}\int_{0}^{\alpha} (3(1+\cos\theta))^2 \, d\theta \Big)$$

Multiply through by 2 and expand the brackets

$$A = 49\int_{\alpha}^{\frac{\pi}{2}} \cos^2\theta \, d\theta + 9\int_{0}^{\alpha} \cos^2\theta + 2\cos\theta + 1 \, d\theta$$

Do the integrals

$$A = 49\left[\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta\right]_{\alpha}^{\frac{\pi}{2}} + 9\left[\frac{1}{2}\sin\theta\cos\theta + \frac{3}{2}\theta + 2\sin\theta\right]_{0}^{\alpha}$$

Find the values

$$A = 8.62 + 23.84 = 32.46$$