Integration

1 Standard Results

Standard results not on formula book:

$\mathbf{f(x)}$	$\int \mathbf{f(x)\ dx}$
x^n	$\frac{x^{n+1}}{n+1}+c$
e^x	e^x+c
$\frac{1}{x}$	$\ln\|x\|+c$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
a^x	$\frac{a^x}{\ln\|a\|}+c$

2 Integration by substitution

Example

$\int (3x+2)^4\ dx$

Find a value for u

$u=3x+2$

Substitute the value of u into the initial question

$\int u^4\ dx$

Find dx in terms of du

$\frac{du}{dx}=3$

$du=3dx$

$dx=\frac{1}{3}du$

Note - if an x term remains with the dx there may be a way to replace both the x term and dx

Replace dx with du into the modified initial equation

$\frac{1}{3}\int u^4\ du$

Integrate using normal integration rules

$\frac{1}{3}\int u^4\ du=\frac{1}{15}u^5$

Substitute the value for u back in

$\frac{1}{15}u^5=\frac{1}{15}(3x+2)^5$

3 Parametric integration

$\int y\ dx$ gives the area under the function y.

For parametric equations we use:
$\int y \frac{dx}{dt} dt$

We have to integrate with respect to t or θ as it is the parameter.
Limits will sometimes need to be changed from x to t or θ.

3.1 Example

$x = 5t^2$
$y = t^3$
Find $\int_1^2 y\ dt$

Find $\mathbf{\frac{dx}{dt}}$
$\frac{dx}{dt} = 10t$

Substitute into the model $\int \mathbf{y \frac{dx}{dt}}$ dt
$\int_1^2 t^3 \times 10t\ dt = \int_1^2 10t^4\ dt = \left[2t^5\right]_1^2 = 2 \times 2^5 - 2 \times 1^5 = 64 - 2 = 62$

4 Integration using trig identities

4.1 Example 1 - sinf(x)cosf(x)

$\int \sin(3x)\cos(3x)\ dx$

Use addition formula to simplify
$\sin 2x = 2\cos x \sin x$
$\sin 3x \cos 3x = \frac{1}{2}\sin 6x$
$\frac{1}{2}\int \sin 6x\ dx$

Find a value for u
$u = 6x$

Find dx in terms of du
$\frac{du}{dx} = 6$
$dx = \frac{1}{6}du$

Substitute two conversions from x to u
$\frac{1}{2} \times \frac{1}{6}\int \sin u\ du$

Use integration of sin formula
$\frac{1}{12}\int \sin u\ du = -\frac{1}{12}\cos 6x + c$

4.2 Example 2 - sinf(x)cosg(x)

$\int \sin(3x)\cos(2x)\ dx$

Add together sin addition formula and sin subtraction formula
$\sin(A+B) = \sin A\cos B + \cos A\sin B$
$\sin(A-B) = \sin A\cos B - \cos A\sin B$
$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$
$\sin 3x\cos 2x = \sin(3x+2x) + \sin(3x-2x)$
$\frac{1}{2}\int \sin 5x + \sin x\ dx = \frac{1}{2}\int \sin 5x\ dx + \frac{1}{2}\int \sin x\ dx$

Integrate using sin standard result
$-\frac{1}{10}\cos 5x - \frac{1}{1}\cos x + c$

4.3 Example 3 - $\sin^2 f(x)\cos^2 f(x)$

$\int \sin^2(x)cos^2(x)\ dx$
Give $\mathbf{sin^2(x)cos^2(x)}$ in terms of $\mathbf{sin^2(x)}$
$2\sin(x)\cos(x) = \sin 2x$
$4\sin^2(x)\cos^2(x) = \sin^2 2x$
$\cos^2 x\sin^2 x = \frac{1}{4}\sin^2 2x$
Sub simplified version
$\int \frac{1}{4}\sin^2 2x\ dx$
Give $\mathbf{sin^2 x}$ in terms of cosx
$2\sin^2 2x = \sin^2 2x + (1-\cos^2 2x)$
$2\sin^2 2x = 1 - (\cos^2 2x - \sin^2 2x)$
$2\sin^2 2x = 1 - \cos 4x$ - Use of cos subtraction formula
$\sin^2 2x = \frac{1}{2} - \frac{1}{2}\cos 4x$
Sub simplified version
$\frac{1}{4}\int \frac{1}{2} - \frac{1}{2}\cos 4x\ dx$
$\frac{1}{8}\int 1 - \cos 4x\ dx$
Use cos integration standard result
$\frac{1}{8}(x - \frac{1}{4}\sin 4x) + c$

$\frac{1}{8}x - \frac{1}{32}\sin 4x + c$

5 Integration of partial fractions

5.1 Simple example

$$\int \frac{x-5}{(x+1)(x-2)}$$

Write as separate fractions

$$\frac{A}{x+1} + \frac{B}{x-2}$$

Multiply out

$A(x-2) + B(x+1) = x-5$

Solve

B=-1

A=2

Substitute

$$\int \frac{2}{x+1} - \frac{1}{x-2}\ dx$$

Separate and simplify

$$2\int \frac{1}{x+1}\ dx - \int \frac{1}{x-2}\ dx$$

Use integration of $\mathbf{\frac{1}{x}}$ standard result

$2\ln(x+1) - \ln(x-2) + c$

Simplify using laws of logs

$$\ln\left|\frac{(x+1)^2}{x-2}\right| + c$$

6 Integration by substitution - Fractional type

$\int \frac{f'(x)}{f(x)}\ dx$ can be written as $\ln f(x) + c$

7 Integration by parts

This is the integration equivalent of the product rule

$$\frac{d}{dx}(uv) = \frac{du}{dx} \times V + \frac{dV}{dx} \times u$$

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - \frac{du}{dx}v$$

$$\int u\frac{dv}{dx}dx = \int \frac{d}{dx}(uv)dx - \int \frac{du}{dx}v\ dx$$

$$\int u\frac{dv}{dx}dx = uv - \int \frac{du}{dx}v\ dx$$

Easier to read version:

$$\int uv'\ dx = uv - \int u'v\ dx$$

7.1 Example

$\int x \cos x \; dx$
$u = x$ - can be eliminated by differentiating
$v' = \cos x$

$u' = 1$
$v = \sin x$

$\int x \cos x \; dx = x \sin x - \int 1 \times \sin x \; dx$
$\int x \cos x \; dx = x \sin x + \cos x + c$

7.2 Double application

When integrating by parts once gives an integral that cannot be solved simply, a second application of integration is needed.

A double application can sometimes lead to the integral having the negative version of the integral on the other side. In this case, add this integral to both sides, allowing both sides to be divided by 2, eliminating a term.

8 Trapezium rule

In C4 we may be asked to find the % error between the trapezium rule approximation and an exact value.

$$\% \text{ error} = \frac{\text{Approx-Actual}}{\text{Actual}} \times 100$$

9 Volume of revolutions

This finds the volume of a graph when rotated round the x axis.

$$\text{Volume} = \pi \int_a^b y^2 \; dx$$

9.1 Example

$y = \sin 2x$
Find the volume between $x = 0$ and $x = \frac{\pi}{2}$ when rotated round the x axis by 2π

$\text{Volume} = \pi \int_0^{\frac{\pi}{2}} \sin^2 2x \; dx$

$\text{Volume} = \pi \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos 4x) \; dx$

$\text{Volume} = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} 1 \; dx - \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \cos 4x \; dx$

$\text{Volume} = \left[\frac{\pi}{2} x\right]_0^{\frac{\pi}{2}} - \left[\frac{\pi}{8} \sin 4x\right]_0^{\frac{\pi}{2}}$

$\text{Volume} = \frac{\pi^2}{4}$

10 Differential equations

$\frac{dy}{dx} = f(x) \cdot g(y)$

Separate variables

$\frac{dy}{g(y)} = f(x)\ dx$

Solve using integration

$\int \frac{1}{g(y)}\ dy = \int f(x)\ dx$

10.1 General solution

General solutions have an unknown constant term

10.2 Example

$\frac{dy}{dx} = (1+y)(1-2x)$

$\int \frac{1}{1+y}\ dy = \int 1-2x\ dx$

$\ln(1+y) = x - x^2 + c$

$1 + y = e^{x-x^2} \times k$

$y = ke^{x(1-x)} - 1$

10.3 Boundary conditions

We can find a "particular solution" to a differential equation if we are given boundary conditions.

You start by finding the general solution, then apply the boundary conditions.

10.3.1 Example

The rate of change of the volume of a container is with respect to time.