

# Binomial and Poisson Distributions

## 1 The Binomial Distribution

### 1.1 Introduction to the binomial distribution

The binomial distribution is a **discrete** distribution.

Conditions for a binomial distribution:

- There are a fixed number of trials, **n**
- There are two outcomes (success and failure)
- Each trial is independent
- The probability of success is constant, **p**

Formula:

For  $X \sim B(n, p)$

$$P(X = r) = \binom{n}{r} p^r (1 - p)^{n-r}$$

Example:

If 25 dice are thrown, find the probability three sixes are obtained.

$X \sim B(25, \frac{1}{6})$

$$P(X = 3) = \binom{25}{3} (\frac{1}{6})^3 (\frac{5}{6})^{22} = 0.1929$$

### 1.2 Use of tables

Tables give "less than or equal to" probabilities.

Example:

$X \sim B(5, 0.35)$

$$P(x \leq 3) = 0.9640 \text{ (in the table where } n=5, p=0.35 \text{ and } x=3)$$

$$P(x < 4) = 0.9640 \text{ (the data is discrete)}$$

$$P(x = 3) = P(x \leq 3) - P(x \leq 2)$$

$$P(x = 3) = 0.9640 - 0.7648$$

$$P(x \geq 3) = 1 - P(X \leq 2)$$

$$P(x \geq 3) = 1 - 0.7648$$

$$P(x \geq 3) = 0.2352$$

### 1.3 Dealing with $P > \frac{1}{2}$

Example:

In the production of a car it is found that 85 % are without defects.

The cars are produced in batches of 50

*Find the probability there are at least 40 defect-free cars in a batch*

$X$  = Number of cars without defects

$X \sim B(50, 0.85)$

$$P(X \geq 40)$$

$Y$  = Number of cars with defects

$Y \sim B(50, 0.15)$

$$P(Y \leq 10) = 0.8801 = P(x \geq 40)$$

### 1.4 Mean and variance of the binomial distribution

$$E(X) = np$$

$$Var(x) = np(1 - p)$$

$$\sigma(x) = \sqrt{Var(x)}$$

Example 1:

$X \sim B(80, 0.4)$  Find  $E(x)$  and  $\sigma$

$$E(x) = 80 \times 0.4 = 32$$

$$\sigma = 80 \times 0.4(1 - 0.4) = 4.38$$

Example 2:

$$E(x) = 8$$

$$Var(x) = 6.4$$

Find  $n$  and  $p$

$$np - np^2 = 6.4$$

$$8 - np^2 = 6.4$$

$$np^2 = 1.6$$

$$p = \frac{np^2}{np} = \frac{1.6}{8} = 0.2$$

$$n = \frac{np}{p} = \frac{8}{0.2} = 40$$

## 2 The Poisson Distribution

### 2.1 Introduction to the Poisson Distribution

The conditions for a poisson distribution are:

- Events occur at random
- Events occur independently of each other
- The average rate of occurrences remains constant
- There is zero probability of simultaneous occurrences

$$P(x = r) = \frac{e^{-\lambda} \lambda^r}{r!}$$

$\lambda$  represents the mean number of occurrences in the time period.

**Example**

$$x \sim P_o(4)$$

$$P(x = 2) = \frac{e^{-4} \times 4^2}{2!} = 0.1465$$

### 2.2 Mean and variance of the poisson distribution

If  $X \sim P_o(\lambda)$  then  $E(x) = \mu = \lambda$  and  $Var(x) = \lambda$

### 2.3 Using the Poisson as an approximation of the binomial

You can use the Poisson distribution as an approximation if:

- $p$  is small ( $< 0.1$ )
- $n$  is large ( $> 50$ )

$$\lambda = np$$

$$X \sim B(n, p) \approx Y \sim P_o(np)$$

**Example**

$$X \sim B(250, 0.01) \approx X \sim P_o(2.5)$$