Maclaurin and Taylor Series

1 Maclaurin's expansion

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + ... + f^r(0)\frac{x^r}{r!}...$$

For the continuous function, f, given by $f : x \Rightarrow f(x)$(where x is real), then providing $f(0), f'(0), f''(0)$ etc all have finite values. This is an infinite series.

1.1 Example

Given that $f(x) = e^x$ can be written as an infinite series in the form:

$$f(x) = e^x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... + a_r x^4 + ...$$

And that it is valid to differentiate an infinite series term by term, show that:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^r}{r!} + ...$$

Find up to the third differential of $f(x)$ and the value of zero for each

$f(x) = e^x$ $f(0) = 1$
$f'(x) = e^x$ $f''(0) = 1$
$f''(x) = e^x$ $f''(0) = 1$
$f'''(x) = e^x$ $f'''(0) = 1$

$$f(x) = 1 + 1 \times x + 1 \times \frac{x^2}{2!} + 1 \times \frac{x^3}{3!}$$

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^r}{r!} + ...$$

1.2 Standard results

Standard results are given on the data sheet, these can then be used for adapted forms of the results also. Remember to consider the limits where appropriate.

2 Taylor expansion

The conditions of the Maclaurin expansion mean that some functions, such as $\ln x$ cannot be expanded as a series in ascending powers of x.

The construction of the Maclaurin expansion focuses on $x = 0$ and values of x very close to zero. The Taylor expansion focuses on $x = a$.

Considering the functions f and g, where $f(x + a) \equiv g(x)$ then:
$f^r(a) = g^r(0)$

Turning the Maclaurin expansion for g from:

$$g(x) = g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 ...$$

Into

$$f(x + a) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + ... + \frac{f^r(a)}{r!}x^r$$

Replacing x by x-a gives

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + ... + \frac{f^r(a)}{r!}(x - a)^r$$

These are the two forms of the Taylor expansion, when a=0, they both become the Maclaurin expansion.

2.1 Example

Find the Taylor expansion of $\cos 2x$*, in ascending powers of* $(x - \frac{\pi}{4})$ *up to* $(x - \frac{\pi}{4})^5$ Find the differentials of $f(x)$ up to the fifth derivative, and the associated values when $x = \frac{\pi}{4}$

$f(x) = \cos 2x$ $\quad f(\frac{\pi}{4}) = 0$
$f'(x) = -2 \sin 2x$ $\quad f(\frac{\pi}{4}) = -2$
$f''(x) = -4 \cos 2x$ $\quad f(\frac{\pi}{4}) = 0$
$f'''(x) = 8 \sin 2x$ $\quad f(\frac{\pi}{4}) = 8$
$f''''(x) = 16 \cos 2x$ $\quad f(\frac{\pi}{4}) = 0$
$f'''''(x) = -32 \sin 2x$ $\quad f(\frac{\pi}{4}) = -32$

Substitute in the associated values into the formula

$$\cos 2x = 0 - 2\left(x - \frac{\pi}{4}\right) + 0 + \frac{8}{3!}\left(x - \frac{\pi}{4}\right)^3 + 0 - \frac{32}{5!}\left(x - \frac{\pi}{4}\right)^5$$

$$\cos 2x = -2\left(x - \frac{\pi}{4}\right) + \frac{4}{5}\left(x - \frac{\pi}{4}\right)^3 - \frac{4}{15}\left(x - \frac{\pi}{4}\right)^5$$

3 Finding the solution, in the form of a series to a differential equation using the Taylor series method

Suppose you have a first order differential equation of the form $\frac{dy}{dx} = f(x, y)$ and you know the initial condition that at $x = x_0$,$y = y_0$, then you can calculate $\left(\frac{dy}{dx}\right)_{x_0}$ by substituting x_0 and y_0 into the original differential equation.

By successive differentiation of the original differential equation, the values of $\left(\frac{d^2y}{dx^2}\right)_{x_0}$ and $\left(\frac{d^3y}{dx^3}\right)_{x_0}$ and so on can be found by substituting previous results into the derived equations.

The series solution to the differential equation is found using the Taylor series in the form:

$$y = y_0 + (x - x_0)\left(\frac{dy}{dx}\right)_{x_0} + \frac{(x - x_0)^2}{2!}\left(\frac{d^2y}{dx^2}\right)_{x_0} + \frac{(x - x_0)^3}{3!}\left(\frac{d^3y}{dx^3}\right)_{x_0} + \ldots$$

In the common situation where $x_0 = 0$ then this reduces to the Maclaurin series

$$y = y_0 + x\left(\frac{dy}{dx}\right)_{x_0} + \frac{x^2}{2!}\left(\frac{d^2y}{dx^2}\right)_{x_0} + \frac{x^3}{3!}\left(\frac{d^3y}{dx^3}\right)_{x_0} + \ldots$$

3.1 Example

Using the Taylor method to find a series solution, in ascending powers of x up to and including the term in x^3, of:

$$\frac{d^2y}{dx^2} = y - \sin x$$

Given that when $x = 0$,$y = 1$ and $\frac{dy}{dx} = 2$

Use the formula to find $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = 1 - \sin(0) = 1$$

$$\mathbf{\frac{d^2y}{dx^2} = 1}$$

Differentiate the formula to obtain a formula for $\frac{d^3y}{dx^3}$

$$\frac{d^3y}{dx^3} = \frac{dy}{dx} - \cos x = 2 - \cos(0) = 1$$

$$\mathbf{\frac{d^3y}{dx^3} = 1}$$

Substitute the known values into the Maclaurin formula

$$y = 1 + x \times 2 + \frac{x^2}{2!} \times 1 + \frac{x^3}{3!} \times 1$$

$$y = 1 + 2x + \frac{x^2}{2} + \frac{x^3}{6}$$