# Second Order Differential Equations

In FP2 we are interested in solving 2nd ODEs of the form:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x) \qquad \text{a,b,c are constants}$$

We consider three distinct cases:
$b^2>4ac$ (Two real solutions)
$b^2=4ac$ (One repeated solution)
$b^2<4ac$ (Two complex solutions)

To solve 2$^{\text{nd}}$ ODEs of this form we first consider solutions to:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

The process of solving a 2$^{\text{nd}}$ ODE starts with a general solution to a 1$^{\text{st}}$ ODE of form:

$$b\frac{dy}{dx}+cy=0$$

$$\int\frac{1}{b}\,dy=\int\frac{1}{-cy}\,dy$$

$$b\ln(y)=-cx+k$$

$$y=Ae^{-\frac{c}{b}x}$$

$$y=Ae^{mx}$$

This was suggested to be a solution to the 2$^{\text{nd}}$ ODE as well

We take $y=e^{mx}$ as a starting point for finding general solutions to:

$$(1) \quad a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

If $y=e^{mx}$ is a solution to (1):

$$\frac{dy}{dx}=me^{mx}$$

$$\frac{d^2y}{dx^2}=m^2e^{mx}$$

Then substitute this into (1)

$$am^2e^{mx}+bme^{mx}+ce^{mx}=0$$

Factor out $e^{mx}$

$$e^{mx}(am^2+bm+c)$$

As $e^x$ must be greater than zero $am^2+bm+c=0$
This is a solvable quadratic called the **Auxiliary equation**

# 1 Two real roots $b^2 > 4ac$

$$(1) \quad a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

The general solution to (1) is in the form:

$$y = Ae^{\alpha x} + Be^{\beta x}$$

Where A and B are constants and $\alpha$ and $\beta$ are the roots to the AE

## 1.1 Example

$$(1) \quad 2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 3y = 0$$

Find the auxiliary equation

$$2m^2e^{mx} + 5me^{mx} + 3e^{mx} = 0$$

$$e^{mx}(2m^2 + 5m + 3 = 0$$

$m = -\frac{3}{2} \quad m = -1$

$$\text{General solution:} y = Ae^{\alpha x} + Be^{\beta x}$$

$$GS = Ae^{-\frac{3}{2}x} + Be^{-x}$$

# 2 1 Real, Repeated root $b^2 = 4ac$

$$\text{General solution : } (A + bx)e^{\alpha x}$$

A and B are constants and $\alpha$ is the root of the AE

## 2.1 Example

*Find the general solution of:*

$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0$$

Find the auxiliary equation

$$e^{mx}(m^2 + 8m + 16) = 0$$

Find the solution

$$m = -4$$

Substitute into the general solution formula

$$y = (A + Bx)e^{-4x}$$

# 3 Imaginary only roots $b^2 < 4ac$

This is when the AI has roots of form $\pm\alpha i$

$$\text{General solution:} \quad y = A\cos(\alpha x) + B\sin(\alpha x)$$

# 4 Complex roots $b^2 < 4ac$

This is used when the root is in the form $\beta \pm \alpha i$

$$\text{General solution:} \quad y = e^{\beta x}(A\cos(\alpha x) + B\sin(\alpha x))$$

However this can also be written in the standard form:

$$y = Ae^{(\beta+\alpha i)x} + Be^{(\beta-\alpha i)x}$$

## 4.1 Example

*Find the general solution of:*

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 34y = 0$$

Find the auxiliary equation

$$m^2 - 6m + 34 = 0$$

Solve to find roots

$$\text{Roots} = \frac{6 \pm \sqrt{36 - 4 \times 1 \times 34}}{2} = \frac{6 \pm 10i}{2} = 3 \pm 5i$$

Substitute into general solution formula

$$y = e^{3x}(A\cos(5x) + B\sin(5x))$$

# 5 Solving $2^{\text{nd}}$ ODE = f(x)

Of the type:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

There are set forms of f(x)

The LHS will be solved in the standard way and the general solution of the LHS will be called the **complementary solution** (CS)

Solving the RHS will give us a **particular integral** (PI)

Full general solution=Complementary function+Particular integral

## 5.1 Standard forms of f(x)

$f(x) = \lambda$
$f(x) = \lambda + \mu x$
$f(x) = \lambda + \mu x + \nu x^2$
$f(x) = ke^{px}$
$f(x) = m\cos\omega x$
$f(x) = m\sin\omega x$
$f(x) = m\cos\omega x \pm n\sin\omega x$

## 5.2 Examples

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = f(x)$$

Find complementary function

$$m^2 - 5m + 6 = 0$$

$$m = 2 \quad m = 2$$

$$\text{Complementary funtion} = Ae^{3x} + Be^{2x}$$

### 5.2.1 2nd ODE=$\lambda$

$f(x) = 3$
Start with $y = \lambda$

$$y = \lambda$$

$$\frac{dy}{dx} = 0$$

$$\frac{d^2y}{dx^2} = 0$$

Substitute values into LHS

$$0 - 5 \times 0 + 6\lambda = 3$$

$$\lambda = \frac{1}{2}$$

Add Complementary function to particular integral

$$y = Ae^{3x} + Be^{2x} + \frac{1}{2}$$

### 5.2.2 2nd ODE=$\lambda + \mu x$

$f(x) = 2x$
Start with $y = \lambda + \mu x$

$$y = \lambda + \mu x$$

$$\frac{dy}{dx} = \mu$$

$$\frac{d^2y}{dx^2} = 0$$

Substitute into LHS

$$0 - 5\mu + 6(\lambda + \mu x) = 2x$$

Equate x terms

$$6\mu x = 2x$$

$$\mu = \frac{1}{3}$$

Equate constant terms

$$-\frac{5}{3} + 6\lambda = 0$$

$$6\lambda = \frac{5}{3}$$

$$\lambda = \frac{5}{18}$$

Substitute into form for the particular integral

$$y = \frac{1}{3}x + \frac{5}{18}$$

Add the PI and CF to find the general solution

$$y = Ae^{3x} + Be^{2x} + \frac{1}{3}x + \frac{5}{18}$$

### 5.2.3 $2^{\text{nd}}$ ODE=$\lambda + \mu x + \nu x^2$

$f(x) = 3x^2$
Start with $y = \lambda + \mu x + \nu x^2$

$$y = \lambda + \mu x + \nu x^2$$
$$\frac{dy}{dx} = \mu + 2\nu x$$
$$\frac{d^2y}{dx^2} = 2\nu$$

Substitute into the LHS

$$2\nu - 5(\mu + 2\nu x) + 6(\lambda + \mu x + \nu x^2) = 3x^2$$

Equate $x^2$ terms

$$6\nu = 3 \quad \nu = \frac{1}{2}$$

Equate x terms

$$-10 \times \frac{1}{2} \times x + 6\mu x = 0$$
$$-5 + 6\mu = 0$$
$$\mu = \frac{5}{6}$$

Equate constant coefficients

$$1 - 5 \times \frac{5}{6} + 6\lambda = 0$$
$$6\lambda = \frac{19}{6}$$
$$\lambda = \frac{19}{36}$$

Substitute into the particular integral form

$$y = \frac{1}{2}x^2 + \frac{5}{6}x + \frac{19}{36}$$

Add CF and PI to get the general solution

$$y = Ae^{3x} + Be^{2x} + \frac{1}{2}x^2 + \frac{5}{6}x + \frac{19}{36}$$

### 5.2.4 Trigonometric f(x)

General forms:
If $f(x) = m\cos\omega x$

$$PI: \ y = P\cos\omega x + Q\sin\omega x$$

If $f(x) = n\sin\omega x$

$$PI: \ y = P\cos\omega x + Q\sin\omega x$$

If $f(x) = m\cos\omega x \pm n\sin\omega x$

$$PI: \ y = P\cos\omega x + Q\sin\omega x$$

$f(x) = m\sin\omega x$ **or** $n\sin\omega x$ $\quad f(x) = 13\sin 4x$

$$PI: \ y = P\cos\omega x + Q\sin\omega x$$

$$\frac{dy}{dx} = -\omega P\sin\omega x + \omega Q\cos\omega x$$

$$\frac{d^2y}{dx^2} = -\omega^2 P\cos\omega x - \omega^2 Q\sin\omega x$$

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 13\sin 3x$$

Substitute into LHS

$$-\omega^2 P\cos\omega x - \omega^2 Q\sin\omega x - 5(-\omega P\sin\omega x + \omega Q\cos\omega x) + 6(P\cos\omega x + Q\sin\omega x) = 13\sin 3x$$

Equate cos terms

$$-\omega^2 P\cos\omega x - 5\omega Q\cos\omega x + 6P\cos\omega x = 0$$

Substitute $\omega = 3$ and divide by $\cos 3x$

$$-9P - 15Q + 6P = 0$$

Simplify

$$-3P - 15Q = 0 \tag{1}$$

Equate sin terms

$$-\omega^2 Q\sin\omega x + 5\omega P\sin\omega x + 6Q\sin\omega x = 13\sin 3x$$

Substitute $\omega = 3$ and divide by $\sin 3x$

$$-9Q + 15P + 6Q = 13$$

Simplify

$$15P - 3Q = 13 \tag{2}$$

Multiply (1) by 5

$$-15P - 75Q = 0$$

Add the multiplied (1) and (2)

$$-78Q = 13$$

Simplify

$$Q = -\frac{1}{6}$$

Substitute to find P

$$15 \times \frac{1}{6} = -3P \quad P = \frac{5}{6}$$

Substitute and add to the CF to find the PI

$$y = Ae^{3x} + Be^{2x} + \frac{5}{6}\cos 3x - \frac{1}{6}\sin 3x$$

# 6 Clash of terms between CF and PI

*Solve:*

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x}$$

Find the Complementary function

$$m^2 - 5m + 6 = 0$$

$$CF: \ y = Ae^{3x} + Be^{2x}$$

Here there will be a clash of terms between the CF and the PI so a different PI must be used, this will be given to you.

$$\text{Use PI } y = \lambda xe^{2x}$$

Differentiate twice

$$\frac{dy}{dx} = \lambda e^{2x} + 2\lambda xe^{2x}$$

$$\frac{d^2y}{dx^2} = 2\lambda e^{2x} + 2\lambda e^{2x} + 4\lambda xe^{2x}$$

Substitute

$$2\lambda e^{2x} + 2\lambda e^{2x} + 4\lambda xe^{2x} - 5(\lambda e^{2x} + 2\lambda xe^{2x}) + 6\lambda xe^{2x} = e^{2x}$$

$$\lambda = -1$$

Substitute

$$PI: \ y = -xe^{2x}$$

# 7 Applications of boundary conditions

If DE is in $\frac{d^2y}{dx^2}$ form then the numerical values for x,y and $\frac{dy}{dx}$ will be given for a value of x.

If (as is common in exams) DE is in $\frac{d^2x}{dt^2}$ form then numerical values of x,t and $\frac{dx}{dt}$ will be given for a value of x.

Used to find A and B in the CF. This gives a particular solution.

## 7.1 Example

*When y=1 x=0* $\frac{dy}{dx} = 0$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 12e^{x}$$

Find the complementary function

$$y = Ae^{-3x} + Be^{-2x}$$

Find the general form of the particular integral

$$PI: \ \lambda e^{x}$$

Differentiate the particular integral twice

$$\frac{dy}{dx} = \lambda e^{x}$$

$$\frac{d^2y}{dx^2} = \lambda e^{x}$$

Substitute into the initial formula

$$\lambda e^{5} + 5\lambda e^{x} + 6\lambda e^{x} = 12e^{x}$$

Simplify to find lambda

$$12\lambda e^{x} = 12e^{x} \quad \lambda = 1$$

Add this to the complementary function to find the general solution

$$GS: \ Ae^{-3x} + Be^{-2x} + e^{x}$$

Substitute y=1 and x=0 into the general solution

$$1 = A + B + 1$$

Simplify

$$A + B = 0$$

Substitute $\frac{dy}{dx} = 0$ and $x = 0$ into the differentiated form of the general solution
Differentiate the general solution

$$\frac{dy}{dx} = -3Ae^{-3x} - 2Be^{-2x} + e^x$$

Substitute values

$$0 = -3A - 2B + 1$$

Solve simultaneous equations

$$A = 1 \quad B = -1$$

Substitute into the general solution to find the particular solution

$$PS: \ y = e^{-3x} - e^{-2x} + e^x$$

# 8 Substitution

## 8.1 Example 1

*Show that the substitution $x = e^u$ transforms*

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \tag{1}$$

to

$$\frac{d^2y}{du^2} + y = 0 \tag{2}$$

Find $\frac{dx}{du}$

$$\text{As } x = e^u,\ \frac{dx}{du} = e^u = x$$

Find $\frac{dy}{du}$ using the chain rule

$$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du} = e^u\frac{dy}{dx} = x\frac{dy}{dx} \tag{3}$$

Find a simplification of $\frac{d^2y}{du^2}$

$$\frac{d^2y}{du^2} = \frac{d}{du}\left(\frac{dy}{du}\right)$$

Simplify using known expression for $\frac{dy}{du}$

$$\frac{d^2y}{du^2} = \frac{d}{du}\left(e^u\frac{dy}{dx}\right)$$

Apply the product rule,using the chain rule to differentiate $\frac{dy}{dx}$ with respect to u

$$\frac{d^2y}{dx^2} = e^u\frac{dy}{dx} + e^u\frac{d^2y}{dx^2}\frac{dx}{du}$$

Replace $e^u\frac{dy}{dx}$ with $\frac{dy}{du}$ and $e^u$ and $\frac{dx}{du}$ each with x

$$\frac{d^2y}{dx^2} = \frac{dy}{du} + x^2\frac{d^2y}{dx^2}$$

Rearrange to find $x^2\frac{d^2y}{dx^2}$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du} \tag{4}$$

Substitute the results from (3) and (4) into (1).

$$\frac{d^2y}{du^2} - \frac{dy}{du} + \frac{dy}{du} + y = 0$$

Simplify to find answer

$$\frac{d^2y}{du^2} + y = 0$$

*Solve this to find the general solution*

$$m^2 + 1 = 0$$

$$m = \pm i$$

Substitute into the general form

$$y = A\cos u + B\sin u$$

Substitute $u = \ln x$ to give GS in terms of x

$$y = A\cos(\ln(x)) + B\sin(\ln(x))$$

### 8.2 Example 2

*Show that the transformation $x = e^u$ transforms the differential equation*

$$x^2\frac{d^2y}{dx^2} - 7x\frac{dy}{dx} + 16y = 2\ln x,\ x > 0 \tag{1}$$

into the differential equation

$$\frac{d^2y}{du^2} - 8\frac{dy}{du} + 16y = 2u \tag{2}$$

Find $\frac{dy}{dx}$ in terms of $\frac{dy}{du}$

$$x = e^u \quad \frac{dx}{du} = e^u \quad \frac{du}{dx} = e^{-u} \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \mathbf{e^{-u}\frac{dy}{du}}$$

Find $\frac{d^2y}{dx^2}$ in terms of $\frac{d^2y}{du^2}$ and $\frac{dy}{du}$

Find $\frac{dy}{du}\left(\frac{dy}{dx}\right)$ by applying the product rule

$$\frac{dy}{du}\left(\frac{dy}{dx}\right) = -e^{-u}\frac{dy}{du} + e^{-u}\frac{d^2y}{du^2}$$

Multiply through by $\frac{du}{dx}$ to find the value of $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = \frac{dy}{du}\left(\frac{dy}{dx}\right) \times \frac{du}{dx}$$

$$\frac{d^2y}{dx^2} = e^{-u}\left(-e^{-u}\frac{dy}{du} + e^{-u}\frac{d^2y}{du^2}\right)$$

$$\frac{d^2y}{dx^2} = \mathbf{e^{-2u}\left(-\frac{dy}{du} + \frac{d^2y}{du^2}\right)}$$

Substitute the values into the original formula, replacing $x$ with $e^u$

$$e^{2u} \times e^{-2u}\left(-\frac{dy}{du} + \frac{d^2y}{du^2}\right) - 7e^u \times e^{-u}\frac{dy}{du} + 16y = 2u$$

$$-\frac{dy}{du} + \frac{d^2y}{du^2} - 7\frac{dy}{du} + 16y = 2u$$

$$\frac{d^2y}{du^2} - 8\frac{dy}{du} + 16y = 2u$$