

# Estimation, confidence intervals and tests

## 1 Estimators

A statistic that is used to estimate a population parameter is an **estimator**.

A particular individual value is an **estimate**.

**Bias** is how far the estimator is from the true value

If a statistic  $T$  is used as an estimator for a population parameter  $\theta$  then the bias is:

$$E(T) - \theta$$

If  $E(T) = \theta$  then the statistic is unbiased

### 1.1 Proving $E(\bar{X}) = \mu$

*Prove that  $\bar{X}$  is an unbiased estimator for  $\mu$  when the population is normally distributed*

A random sample  $X_1, X_2, X_3 \dots X_N$  is taken for a population with  $X \sim N(\mu, \sigma^2)$

$$\bar{X} = \frac{1}{n} \sum X$$

$$E(\bar{X}) = E\left(\frac{1}{n} \sum X\right)$$

$$E(\bar{X}) = \frac{1}{n} (E(X_1) + E(X_2) + E(X_3) \dots + E(X_N))$$

$$E(\bar{X}) = \frac{1}{n} (\mu + \mu + \mu \dots + \mu)$$

$$E(\bar{X}) = \frac{1}{n} (n\mu)$$

$$E(\bar{X}) = \mu$$

### 1.2 Variance

$$Var(\bar{X}) = \frac{\sigma^2}{\text{Sample size}}$$

We use  $S^2$  as an estimator for  $\sigma^2$

$$S^2 = \frac{1}{n-1} (\sum X^2 - n\bar{X}^2)$$

### 1.3 Uniform Distributions

*Random variable  $X$  is continuously uniform  $[0, \alpha]$ . A sample  $X_1, X_2 \dots X_N$  is taken*

*Show that  $\bar{X}$  is a biased estimate and state the bias*

$$\bar{X} = \frac{0 + \alpha}{2} = \frac{\alpha}{2}$$

$$E(\bar{X}) = E\left(\frac{\alpha}{2}\right) = \frac{1}{2} E(\alpha) = \frac{\alpha}{2}$$

$$\text{Bias} = \frac{\alpha}{2} - \alpha = -\frac{\alpha}{2}$$

## 2 Standard error

$$\text{Standard error} = \frac{\sigma}{\sqrt{n}} \text{ or } \frac{s}{\sqrt{n}}$$

## 3 Central limit theorem

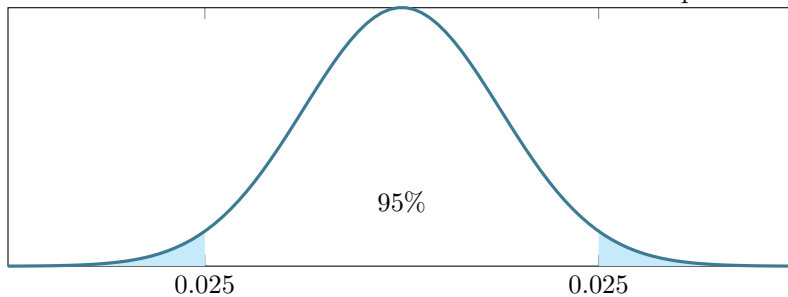
The central limit theorem (C.L.T.) states that if  $X_1, X_2, X_3 \dots$  is a random sample, from any distribution with mean  $\mu$  and variance  $\sigma^2$ . Then the sample means,  $\bar{X}$  are distributed with a normal distribution.

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

For this, the sample size must be greater than 50

## 4 Confidence intervals

A 95% confidence interval is two values in which there is a probability of 0.95 that it will contain the population mean.



Look up the value of one of the tails on the percentage points table and substitute into the formula.

For example:

At a 95% confidence interval each of the tails is 0.025 and the corresponding z value is 1.96.

$$\bar{x} \pm 1.96 \times \frac{\sigma}{\sqrt{n}}$$

## 5 Hypothesis testing

The test statistic for the population mean  $\mu$  is  $Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$

A hypothesis test can be done quicker by finding the critical values from the percentage points table.

### 5.1 Example

$$\mu = 0.58 \quad \sigma = 0.015$$

Measured

$$n = 50 \quad \mu = 0.577 \quad \sigma = 0.015$$

$$H_0 : \mu = 0.50$$

$$H_1 : \mu \neq 0.58$$

$$\begin{aligned} \bar{D} &\sim N\left(0.58, \frac{0.015^2}{50}\right) \\ Z &= \frac{0.577 - 0.58}{\sqrt{\frac{0.015^2}{50}}} = -1.41 \end{aligned}$$

At 0.5% per level  $Z=2.5758$

$$-1.41 > -2.5758$$

Not in critical region so not significant, no evidence to reject  $H_0$ , the diameter of the bolts is not changed.