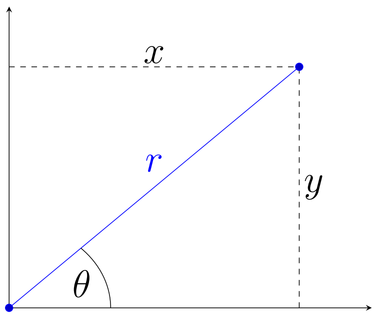
Polar Coordinates



Usual conventions are either $-\pi < \theta \leqslant \pi$ or $0 \leqslant \theta < \pi$

$x = r\cos\theta$
$y = r\sin\theta$
$r^2 = x^2 + y^2$
$\theta = \arctan\left(\frac{y}{x}\right)$

1 Converting between polar and Cartesian form

1.1 Example 1

Find the Cartesian equation of: $r = 5$

$$\sqrt{x^2+y^2} = 5$$

$$x^2 + y^2 = 25$$

1.2 Example 2

Find the Cartesian equation of:

$$r = 2 + \cos 2\theta$$

Replace r and convert $\cos 2\theta$

$$\sqrt{x^2+y^2} = 2 + \cos^2\theta - \sin^2\theta$$

Convert $\sin^2\theta$ and $\cos^2\theta$

$$\sqrt{x^2+y^2} = 2 + \frac{x^2}{x^2+y^2} - \frac{y^2}{x^2+y^2}$$

Multiply all terms by $x^2 + y^2$

$$(x^2+y^2)^{\frac{3}{2}} = 3x^2 + y^2$$

2 Sketching polar curves

To plot less standard types we look for axes intercepts and max and min values

2.1 Standard types

2.1.1 r=a

A circle, centre (0,0) with a radius of a

2.1.2 $\theta = \alpha$

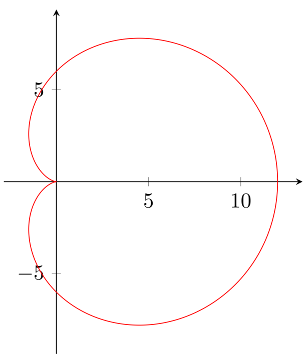
A half line starting from (0,0) making an angle of α with the initial line (positive x axis)

2.1.3 $r = a\theta$

A spiral starting at the origin

2.2 Cardioid type

2.2.1 $r = a(1 + \cos\theta)$

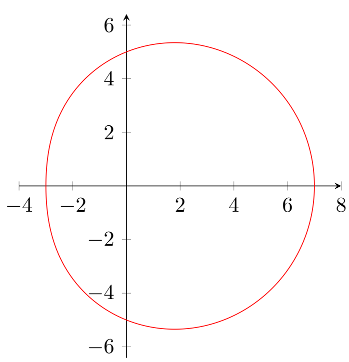


2.2.2 $r = a(p + q\cos\theta)$

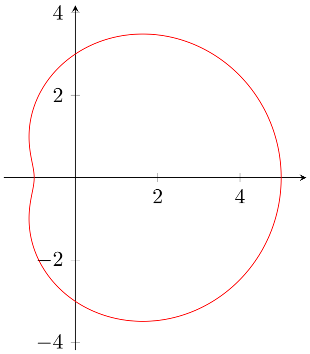
2.2.2.1 p=q

Factor out the value of p and plot as normal

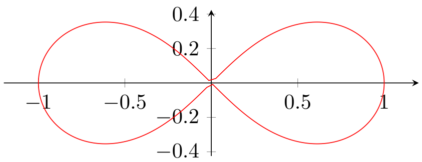
2.2.2.2 $p \geqslant 2q$



2.2.2.3 $q \leqslant p < 2q$

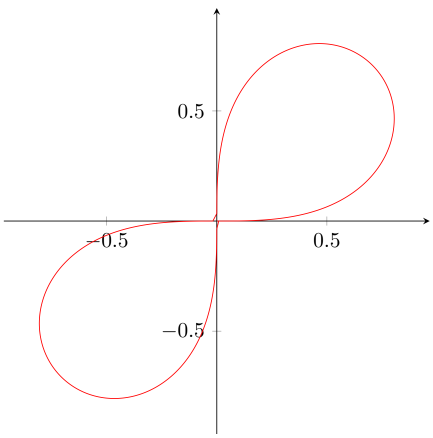


2.2.3 $r^2 = a^2 \cos 2\theta$



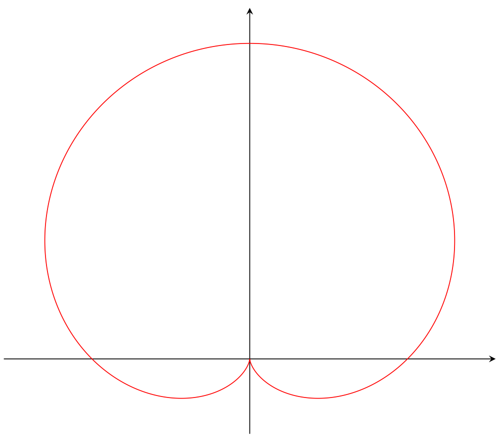
The 4 asymptotes are half lines at $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

2.3 $r^2 = a^2 \sin 2\theta$



This is an anticlockwise rotation of $r^2 = a^2 \cos 2\theta$ by $\frac{\pi}{4}$, it is this not $\frac{\pi}{2}$ as the phase difference between $\cos 2\theta$ and $\sin 2\theta$ is $\frac{\pi}{4}$ compared to $\frac{\pi}{2}$ with one θ.

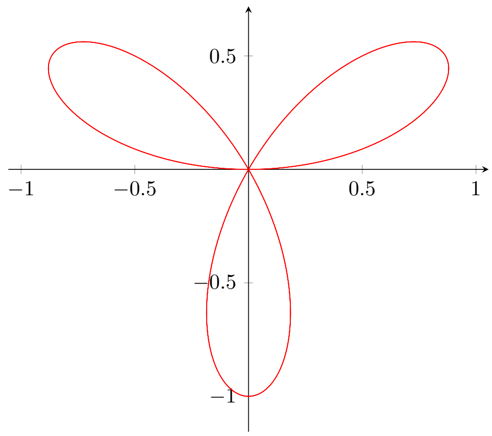
2.3.1 $r = a(p + q \sin \theta)$



This is a rotation of $r = a(p + q \cos \theta)$ anticlockwise by $\frac{\pi}{2}$

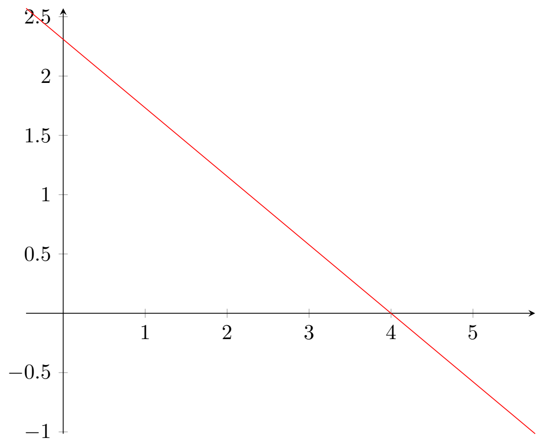
2.4 Other types

2.4.1 $r = a\sin 3\theta$



Asymptotes occur when $r = 0$

2.4.2 $r = 2\sec(\theta - \frac{\pi}{3})$



3 Integration

$$\text{Area} = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \ d\theta$$

Where the area of the sector is bounded by the half lines $\theta = \alpha$ and $\theta = \beta$, the value of θ must always be in radians.

The the radius in this sector must not drop to zero, choose angles so this doesn't happen. Or choose multiple sections and add together, or use symmetry.

4 Differentiating Polar Equations to find tangents

We only find tangents parallel and perpendicular to the initial line.

$x = r\cos\theta$
$y = r\sin\theta$

Using the chain rule:

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

Parallel: $\frac{dy}{d\theta} = 0$
Perpendicular: $\frac{dx}{d\theta} = 0$

4.1 Example

$r = a(1 + \cos\theta)$

Find the coordinates where the tangents are parallel to the initial line with $\theta = 0$

$$y = r\sin\theta$$

Replace r with $a(1+\cos\theta)$

$$y = a(\sin\theta + \sin\theta\cos\theta)$$

Now we can find $\frac{dy}{d\theta}$

$$\frac{dy}{d\theta} = a(\cos\theta + \cos^2\theta - \sin^2\theta)$$

Simplify and set equal to zero

$$2\cos^2\theta + \cos\theta - 1$$

$$0 = (2\cos\theta - 1)(\cos\theta + 1)$$

Find $\cos\theta$, θ and r

$$\cos\theta \quad \cos\theta = -1$$

$$\theta = \pm\frac{\pi}{3} \quad \theta = \pi$$

$$r = \frac{3a}{2} \quad r = 0$$

4.1.1 Finding the equations of tangents

$$\text{Cartesian equation: } y = r\sin\theta$$

$$\text{Polar equation: } r = y\csc\theta = r\sin\theta\csc\theta$$