# Second Order Differential Equations

In FP2 we are interested in solving 2nd ODEs of the form:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x) \qquad \text{a,b,c are constants}$$

We consider three distinct cases:
$b^2>4ac$ (Two real solutions)
$b^2=4ac$ (One repeated solution)
$b^2<4ac$ (Two complex solutions)

To solve $2^{\text{nd}}$ ODEs of this form we first consider solutions to:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

The process of solving a $2^{\text{nd}}$ ODE starts with a general solution to a $1^{\text{st}}$ ODE of form:

$$b\frac{dy}{dx}+cy=0$$

$$\int\frac{1}{b}\,dy=\int\frac{1}{-cy}\,dy$$

$$b\ln(y)=-cx+k$$

$$y=Ae^{-\frac{c}{b}x}$$

$$y=Ae^{mx}$$

This was suggested to be a solution to the $2^{\text{nd}}$ ODE as well

We take $y=e^{mx}$ as a starting point for finding general solutions to:

$$(1) \quad a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

If $y=e^{mx}$ is a solution to (1):

$$\frac{dy}{dx}=me^{mx}$$

$$\frac{d^2y}{dx^2}=m^2e^{mx}$$

Then substitute this into (1)

$$am^2e^{mx}+bme^{mx}+ce^{mx}=0$$

Factor out $e^{mx}$

$$e^{mx}(am^2+bm+c)$$

As $e^x$ must be greater than zero $am^2+bm+c=0$
This is a solvable quadratic called the **Auxiliary equation**

# 1 Two real roots $\mathbf{b^2 > 4ac}$

$$(1) \quad a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

The general solution to (1) is in the form:

$$y = Ae^{\alpha x} + Be^{\beta x}$$

Where A and B are constants and $\alpha$ and $\beta$ are the roots to the AE

## 1.1 Example

$$(1) \quad 2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 3y = 0$$

Find the auxiliary equation

$$2m^2e^{mx} + 5me^{mx} + 3e^{mx} = 0$$

$$e^{mx}(2m^2 + 5m + 3 = 0$$

$m = -\frac{3}{2} \quad m = -1$

$$\text{General solution:} y = Ae^{\alpha x} + Be^{\beta x}$$

$$GS = Ae^{-\frac{3}{2}x} + Be^{-x}$$

# 2 1 Real, Repeated root $\mathbf{b^2 = 4ac}$

$$\text{General solution : } (A + bx)e^{\alpha x}$$

A and B are constants and $\alpha$ is the root of the AE

## 2.1 Example

*Find the general solution of:*

$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0$$

Find the auxiliary equation

$$e^{mx}(m^2 + 8m + 16) = 0$$

Find the solution

$$m = -4$$

Substitute into the general solution formula

$$y = (A + Bx)e^{-4x}$$

# 3 Imaginary only roots $\mathbf{b^2 < 4ac}$

This is when the AI has roots of form $\pm\alpha i$

$$\text{General solution:} \quad y = A\cos(\alpha x) + B\sin(\alpha x)$$

## 4 Complex roots $b^2 < 4ac$

This is used when the root is in the form $\beta \pm \alpha i$

$$\text{General solution:} \quad y = e^{\beta x}(A\cos(\alpha x) + B\sin(\alpha x))$$

### 4.1 Example

*Find the general solution of:*

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 34y = 0$$

Find the auxiliary equation

$$m^2 - 6m + 34 = 0$$

Solve to find roots

$$\text{Roots} = \frac{6 \pm \sqrt{36 - 4 \times 1 \times 34}}{2} = \frac{6 \pm 10i}{2} = 3 \pm 5i$$

Substitute into general solution formula

$$y = e^{3x}(A\cos(5x) + B\sin(5x))$$

## 5 Solving $2^{\text{nd}}$ ODE = f(x)

Of the type:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

There are set forms of f(x)

The LHS will be solved in the standard way and the general solution of the LHS will be called the **complementary solution** (CS)

Solving the RHS will give us a **particular integral** (PI)

Full general solution=Complementary function+Particular integral

### 5.1 Standard forms of f(x)

$f(x) = \lambda$
$f(x) = \lambda + \mu x$
$f(x) = \lambda + \mu x + \nu x^2$
$f(x) = ke^{px}$
$f(x) = m\cos\omega x$
$f(x) = m\sin\omega x$
$f(x) = m\cos\omega x \pm n\sin\omega x$

## 5.2 Examples

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = f(x)$$

Find complementary function

$$m^2 - 5m + 6 = 0$$

$$m = 2 \quad m = 2$$

$$\text{Complementary funtion} = Ae^{3x} + Be^{2x}$$

### 5.2.1 $2^{\text{nd}}$ ODE=$\lambda$

$f(x) = 3$
Start with $y = \lambda$

$$y = \lambda$$

$$\frac{dy}{dx} = 0$$

$$\frac{d^2y}{dx^2} = 0$$

Substitute values into LHS

$$0 - 5 \times 0 + 6\lambda = 3$$

$$\lambda = \frac{1}{2}$$

Add Complementary function to particular integral

$$y = Ae^{3x} + Be^{2x} + \frac{1}{2}$$

### 5.2.2 $2^{\text{nd}}$ ODE=$\lambda + \mu x$

$f(x) = 2x$
Start with $y = \lambda + \mu x$

$$y = \lambda + \mu x$$

$$\frac{dy}{dx} = \mu$$

$$\frac{d^2y}{dx^2} = 0$$

Substitute into LHS

$$0 - 5\mu + 6(\lambda + \mu x) = 2x$$

Equate x terms

$$6\mu x = 2x$$

$$\mu = \frac{1}{3}$$

Equate constant terms

$$-\frac{5}{3} + 6\lambda = 0$$

$$6\lambda = \frac{5}{3}$$

$$\lambda = \frac{5}{18}$$

Substitute into form for the particular integral

$$y = \frac{1}{3}x + \frac{5}{18}$$

Add the PI and CF to find the general solution

$$y = Ae^{3x} + Be^{2x} + \frac{1}{3}x + \frac{5}{18}$$

### 5.2.3 $2^{\text{nd}}$ ODE=$\lambda + \mu x + \nu x^2$

$f(x) = 3x^2$
Start with $y = \lambda + \mu x + \nu x^2$

$$y = \lambda + \mu x + \nu x^2$$

$$\frac{dy}{dx} = \mu + 2\nu x$$

$$\frac{d^2y}{dx^2} = 2\nu$$

Substitute into the LHS

$$2\nu - 5(\mu + 2\nu x) + 6(\lambda + \mu x + \nu x^2) = 3x^2$$

Equate $x^2$ terms

$$6\nu = 3 \quad \nu = \frac{1}{2}$$

Equate x terms

$$-10 \times \frac{1}{2} \times x + 6\mu x = 0$$

$$-5 + 6\mu = 0$$

$$\mu = \frac{5}{6}$$

Equate constant coefficients

$$1 - 5 \times \frac{5}{6} + 6\lambda = 0$$

$$6\lambda = \frac{19}{6}$$

$$\lambda = \frac{19}{36}$$

Substitute into the particular integral form

$$y = \frac{1}{2}x^2 + \frac{5}{6}x + \frac{19}{36}$$

Add CF and PI to get the general solution

$$y = Ae^{3x} + Be^{2x} + \frac{1}{2}x^2 + \frac{5}{6}x + \frac{19}{36}$$

### 5.2.4 Trigonometric f(x)

General forms:
If $f(x) = m\cos\omega x$

$$PI: \; y = P\cos\omega x + Q\sin\omega x$$

If $f(x) = n\sin\omega x$

$$PI: \; y = P\cos\omega x + Q\sin\omega x$$

If $f(x) = m\cos\omega x \pm n\sin\omega x$

$$PI: \; y = P\cos\omega x + Q\sin\omega x$$

#### 5.2.4.1 $f(x) = m\sin\omega x$ or $n\sin\omega x$

$f(x) = 13\sin 4x$

$$PI: \; y = P\cos\omega x + Q\sin\omega x$$

$$\frac{dy}{dx} = -\omega P\sin\omega x + \omega Q\cos\omega x$$

$$\frac{d^2y}{dx^2} = -\omega^2 P\cos\omega x - \omega^2 Q\sin\omega x$$

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 13\sin 3x$$

Substitute into LHS

$$-\omega^2 P\cos\omega x - \omega^2 Q\sin\omega x - 5(-\omega P\sin\omega x + \omega Q\cos\omega x) + 6(P\cos\omega x + Q\sin\omega x) = 13\sin 3x$$

Equate cos terms

$$-\omega^2 P\cos\omega x - 5\omega Q\cos\omega x + 6P\cos\omega x = 0$$

Substitute $\omega = 3$ and divide by $\cos 3x$

$$-9P - 15Q + 6P = 0$$

Simplify

$$-3P - 15Q = 0 \qquad [1]$$

Equate sin terms

$$-\omega^2 Q\sin\omega x + 5\omega P\sin\omega x + 6Q\sin\omega x = 13\sin 3x$$

Substitute $\omega = 3$ and divide by $\sin 3x$

$$-9Q + 15P + 6Q = 13$$

Simplify

$$15P - 3Q = 13 \qquad [2]$$

Multiply [1] by 5

$$-15P - 75Q = 0$$

Add the multiplied [1] and [2]

$$-78Q = 13$$

Simplify

$$Q = -\frac{1}{6}$$

Substitute to find P

$$15 \times \frac{1}{6} = -3P \quad P = \frac{5}{6}$$

Substitute and add to the CF to find the PI

$$y = Ae^{3x} + Be^{2x} + \frac{5}{6}\cos 3x - \frac{1}{6}\sin 3x$$

## 6 Clash of terms between CF and PI

*Solve:*

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x}$$

Find the Complementary function

$$m^2 - 5m + 6 = 0$$

$$CF: \ y = Ae^{3x} + Be^{2x}$$

Here there will be a clash of terms between the CF and the PI so a different PI must be used, this will be given to you.

$$\text{Use PI } y = \lambda xe^{2x}$$

Differentiate twice

$$\frac{dy}{dx} = \lambda e^{2x} + 2\lambda xe^{2x}$$

$$\frac{d^2y}{dx^2} = 2\lambda e^{2x} + 2\lambda e^{2x} + 4\lambda xe^{2x}$$

Substitute

$$2\lambda e^{2x} + 2\lambda e^{2x} + 4\lambda xe^{2x} - 5(\lambda e^{2x} + 2\lambda xe^{2x}) + 6\lambda xe^{2x} = e^{2x}$$

$$\lambda = -1$$

Substitute

$$PI: \ y = -xe^{2x}$$

## 7 Applications of boundary conditions

If DE is in $\frac{d^2y}{dx^2}$ form then the numerical values for x,y and $\frac{dy}{dx}$ will be given for a value of x.

If (as is common in exams) DE is in $\frac{d^2x}{dt^2}$ form then numerical values of x,t and $\frac{dx}{dt}$ will be given for a value of x.

Used to find A and B in the CF. This gives a particular solution.