

# Continuous Random Variables

## 1 Probability density function (PDF)

The PDF describes how the probabilities are distributed over the range of values.

$f(x)$  must satisfy these basic properties:

- $f(x) \geq 0$  for all values of  $x$ , so that no probabilities are negative.

- $\int_{-\infty}^{\infty} f(x) dx = 1$  (The sum of all probabilities is 1)

- $P(a < x < b) = \int_a^b f(x) dx$

For continuous random variables  $P(x < z) = P(x \leq z)$

**Example**

$$f(x) = \begin{cases} \frac{1}{2}(x-3), & \text{for } 3 \leq x \leq 5 \\ 0, & \text{otherwise} \end{cases}$$

$$\int_3^5 \frac{1}{2}x - \frac{3}{2} dx = \left[ \frac{x^2}{4} - \frac{3x}{2} \right]_3^5 = \left( \frac{5^2}{4} - \frac{3 \times 5}{2} \right) - \left( \frac{3^2}{4} - \frac{3 \times 3}{2} \right) = 1$$

Find  $P(x < 4)$

$$\int_3^4 \frac{1}{2}x - \frac{3}{2} dx = \left[ \frac{x^2}{4} - \frac{3x}{2} \right]_3^4 = 0.25$$

**Example 2**

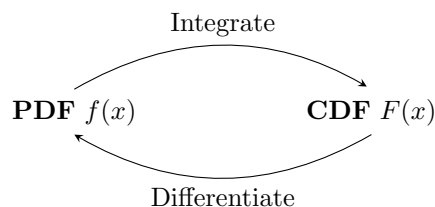
$$f(x) = \begin{cases} x, & \text{for } 0 \leq x \leq 1 \\ 2-x, & \text{for } 1 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

$$P(0.5 < x < 1.3) = \int_{0.5}^1 x dx + \int_1^{1.3} 2-x dx = \left[ \frac{1}{2}x^2 \right]_{0.5}^1 + \left[ 2x - \frac{1}{2}x^2 \right]_{0.5}^1 = 0.63$$

## 2 Cumulative Distribution Function

The CDF is  $P(X \leq x)$

To find it, integrate the **PDF** between the lower limit and  $x$ .



**Example**

$$f(x) = \begin{cases} \frac{1}{2}(x-3), & \text{for } 3 \leq x \leq 5 \\ 0, & \text{otherwise} \end{cases}$$

$$\int_3^{x_0} \frac{1}{2}x - \frac{3}{2} dx = \left[ \frac{x^2}{4} - \frac{3x}{2} \right]_{0.5}^1 = \left( \frac{x^2}{4} - \frac{3x}{2} \right) - \left( \frac{3^2}{4} - \frac{3 \times 3}{2} \right) = \frac{x^2}{4} - \frac{3x}{2} + \frac{9}{4}$$

$$F(x) = \begin{cases} 0, & \text{for } x < 3 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{9}{4}, & \text{for } 3 \leq x \leq 5 \\ 1, & \text{for } x > 5 \end{cases}$$

**Example 2**

$$f(x) = \begin{cases} x, & \text{for } 0 \leq x \leq 1 \\ 2-x, & \text{for } 1 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the CDF

$$\int_0^x x \, dx = \left[ \frac{x^2}{2} \right]_0^x = \frac{x^2}{2}$$

$$F(1) + \int_1^x 2-x \, dx = \frac{1^2}{2} + \left[ 2x - \frac{x^2}{2} \right]_1^x = \frac{1}{2} + \left( \left( 2x - \frac{x^2}{2} \right) - \left( 2 - \frac{1}{2} \right) \right) = 2x - \frac{x^2}{2} - 1$$

$$f(x) = \begin{cases} 0, & \text{for } x < 0 \\ \frac{x^2}{2}, & \text{for } 0 \leq x \leq 1 \\ 2x - \frac{x^2}{2} - 1, & \text{for } 1 \leq x \leq 2 \\ 1, & \text{for } x > 2 \end{cases}$$

**3 Mean and variance of continuous random variables**

$$\text{Mean} = \mu = E(x) = \int_a^b x f(x) \, dx \text{ for } a \leq x \leq b$$

$$\text{Var}(x) = E(x^2) - (E(x))^2$$

$$E(x^2) = \int_a^b x^2 f(x) \, dx$$