

Differentiation

1 Implicit differentiation

An implicit function is in terms of x and y and cannot easily be written as $y = ?$

1.1 Formula

$$\frac{d}{dx}(y^n) = ny^{n-1} \times \frac{dy}{dx}$$

This is a formula you need to remember for the exam

1.2 Example

$$x^3 + x + y^3 + 3y = 6$$

$$\frac{d}{dx}(x^3 + x + y^3 + 3y) = \frac{d}{dx}(6)$$

Differentiate each term

$$3x^2 + 1 + 3y^2 \frac{dy}{dx} + 3 \frac{dy}{dx} = 0$$

Collect terms

$$\frac{dy}{dx}(3y^2 + 3) = -3x^2 - 1$$

Isolate $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-3x^2 - 1}{3y^2 + 3}$$

1.3 Example with combined xy terms

$$\frac{d}{dx}(x, y) = x'y + xy'$$

Example

$$\frac{d}{dx}xy$$

$$\frac{d}{dx}x = 1$$

$$\frac{d}{dx}y = \frac{dy}{dx}$$

$$\frac{d}{dx} = 1 \times y + x \times \frac{dy}{dx}$$

Example 2

$$\text{Find } \frac{dy}{dx} \text{ of } 4xy^2 + \frac{6x^2}{y}$$

Simplify

$$4xy^3 + 6x^2 - 10y = 0$$

Differentiate each term

$$\frac{d}{dx} = \left[1 \times 4y^3 + 4x \times 3y^2 \frac{dy}{dx} \right] + 12x - 10 \frac{dy}{dx}$$

Collect terms

$$\frac{dy}{dx}(12xy^2 - 10) = -(4y^3 + 12x)$$

Isolate $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-(4y^3 + 12x)}{12xy^2 - 10}$$

2 Differentiating $y = a^x$

$$y = a^x$$

Write in terms of logs

$$\ln(y) = x \ln(a)$$

Differentiate both sides

$$\frac{d}{dx} \ln(y) = \frac{d}{dx} x \ln(a)$$

$$\frac{1}{y} \frac{dy}{dx} = \ln(a)$$

Simplify

$$\frac{dy}{dx} = y \ln(a)$$

Sub y with a^x

$$\frac{dy}{dx} = a^x \ln(a)$$

2.1 Differentiating $y = a^{f(x)}$

$$\frac{dy}{dx} = a^{f(x)} f'(x) \ln(a)$$

3 Related rates of change

You can use the chain rule to connect related rates of change

For example:

$$\frac{dx}{dt} = \frac{dV}{dt} \times \frac{dx}{dV}$$

Where x is side length, v is volume and t is time.

We need to be able to interpret numerical information as a rate of change, for example:

$$2cm^2s^{-1} = \frac{dA}{dt}$$

Example 1

A cylinder is expanding under heat

After t seconds:

The radius is x cm

The length is 5x cm

The cross sectional area is increasing at a constant rate of $0.037cm^2s^{-1}$

Find $\frac{dx}{dt}$ when $r=4$

Find the area in terms of x

$$A = \pi x^2$$

Differentiate the area with respect to x

$$\frac{dA}{dx} = 2\pi x$$

Re write the rate of change

$$\frac{dA}{dt} = 0.037$$

Use the chain rule $\frac{dx}{dt} = \frac{dx}{dA} \times \frac{dA}{dt}$

$$\frac{dx}{dt} = \frac{0.037}{2\pi x}$$

Sub x=4

$$\frac{dx}{dt} = 0.00147 \text{ cm s}^{-1}$$

Find the rate of change of the volume when x=4

$$V = 5\pi x^3$$

Differentiate the volume with respect to x

$$\frac{dV}{dx} = 15\pi x^2$$

Use the chain rule

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt} = 15\pi x^2 \times 0.00147$$

Sub x=4

$$1.11 \text{ cm}^3 \text{ s}^{-1}$$