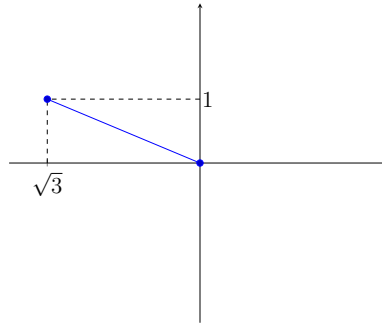


# Further Complex Numbers

## 1 Expressions of complex numbers

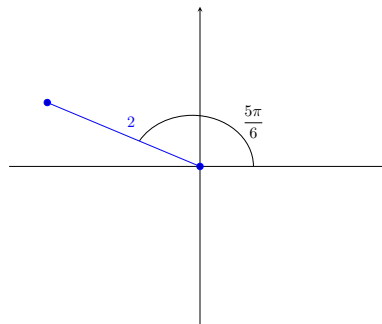
### 1.1 $x + iy$

This expresses the coordinate of the point at the end of the vector on the argand diagram.



### 1.2 $r(\cos \theta + i \sin \theta)$

This expresses the length of the line and the angle anticlockwise from the positive x axis



$$2 \left( \cos \left( \frac{5\pi}{6} \right) + i \sin \left( \frac{5\pi}{6} \right) \right)$$

### 1.3 $re^{i\theta}$

This uses the same parameters as  $r(\cos \theta + i \sin \theta)$

## 2 Multiplying and dividing complex numbers

### 2.1 Multiplying

#### 2.1.1 Trigonometric form

$$\begin{aligned} Z_1 Z_2 &= r_1(\cos \theta_1 + i \sin \theta_1) \times r_2(\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2 (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2) \end{aligned}$$

Apply the cos addition formula to the first two terms

$$= r_1 r_2 (\cos(\theta_1 + \theta_2) + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2)$$

Apply the sin addition formula to the last two terms

$$= r_1 r_2 (\cos(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2))$$

#### 2.1.2 Exponential form

$$Z_1 Z_2 = r_1 e^{i\theta_1} \times r_2 e^{i\theta_2}$$

Apply laws of indices

$$Z_1 Z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

### 2.2 Dividing

#### 2.2.1 Trigonometric form

$$\frac{Z_1}{Z_2} = \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)}$$

Multiply by the complex conjugate

$$\frac{Z_1}{Z_2} = \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} \times \frac{\cos \theta_2 - i \sin \theta_2}{\cos \theta_2 - i \sin \theta_2}$$

Expand

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \times \frac{\cos \theta_1 \cos \theta_2 - i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2}{\cos^2 \theta_2 - i \cos \theta_2 \sin \theta_2 + i \sin \theta_2 \cos \theta_2 + \sin^2 \theta_2}$$

Simplify

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \times (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))$$

#### 2.2.2 Exponential form

$$\begin{aligned} \frac{Z_1}{Z_2} &= \frac{r_1}{r_2} \times \frac{e^{i\theta_1}}{e^{i\theta_2}} \\ \frac{Z_1}{Z_2} &= \frac{r_1}{r_2} \times e^{i(\theta_1 - \theta_2)} \end{aligned}$$

### 2.3 Comparison

Multiplying	Dividing
Multiply modulus, add arguments	Divide modulus, subtract arguments

### 3 De Moivre's Theorem

$$[r(\cos \theta + i \sin \theta)]^n = r^n(\cos(n\theta) + i \sin(n\theta))$$

#### 3.1 Positive proof

Prove true for n=1

$$r(\cos \theta + i \sin \theta) = r(\cos \theta + i \sin \theta)$$

True for n=1

Assume true for n=k

$$[r(\cos \theta + i \sin \theta)]^k = r^k(\cos(k\theta) + i \sin(k\theta))$$

Prove true for n=k+1

$$\begin{aligned} & [r(\cos \theta + i \sin \theta)]^{k+1} \\ & [r(\cos \theta + i \sin \theta)]^k \times (r(\cos \theta + i \sin \theta))^1 \\ & r^k(\cos(k\theta) + i \sin(k\theta)) \times r(\cos \theta + i \sin \theta) \\ & r^k r(\cos(k\theta + \theta) + i \sin(k\theta + \theta)) \\ & r^{k+1}(\cos((k+1)\theta) + i \sin((k+1)\theta)) \end{aligned}$$

True

#### 3.2 Negative proof

n=-m

$$[r(\cos \theta + i \sin \theta)]^{-m}$$

Multiply by complex conjugate

$$\frac{1}{[r(\cos \theta + i \sin \theta)]^m} \times \frac{[r(\cos \theta - i \sin \theta)]^m}{[r(\cos \theta - i \sin \theta)]^m}$$

Apply positive De Moivre's Theorem

$$\frac{r^m(\cos(m\theta) - i \sin(m\theta))}{r^m(\cos(m\theta) + i \sin(m\theta)) \times r^m(\cos(m\theta) - i \sin(m\theta))}$$

Simplify and expand

$$\frac{\cos(m\theta) - i \sin(m\theta)}{r^m(\cos^2 m\theta - i \cos m\theta \sin m\theta + i \cos m\theta \sin m\theta + \sin^2 m\theta)}$$

Simplify

$$\frac{\cos m\theta - i \sin m\theta}{r^m} = r^{-m}(\cos m\theta - i \sin m\theta)$$

Rewrite

$$r^{-m}(\cos(-m\theta) + i \sin(-m\theta))$$

Replace -m with n

$$r^n(\cos(n\theta) + i \sin(n\theta))$$

### 3.3 Applying De Moivres' Theorem

We can use the binomial expansion along with De Moivre's theorem to rewrite trigonometric expressions. This can be useful when integrating etc.

#### 3.3.1 Example 1

Rewrite  $\cos 5\theta$  in powers of  $\cos \theta$

$$(\cos \theta + i \sin \theta)^5$$

Apply DM theorem

$$\cos 5\theta + i \sin 5\theta$$

Apply the binomial expansion to the initial expression

$$\cos^5 \theta + 5 \cos^4 \theta i \sin \theta + 10 \cos^3 \theta (i \sin \theta)^2 + 10 \cos^2 \theta (i \sin \theta)^3 + 5 \cos \theta (i \sin \theta)^4 + (i \sin \theta)^5$$

Simplify

$$\cos^5 \theta + 5i \cos^4 \theta \sin \theta - 10 \cos^3 \theta \sin^2 \theta - 10i \cos^2 \theta \sin^3 \theta + 5 \cos \theta \sin^4 \theta + i \sin^5 \theta$$

Equate the real parts of this to the real parts of the result from DM theorem

$$\cos 5\theta = \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta$$

Replace sin terms with cos

$$\cos 5\theta = \cos^5 \theta - 10 \cos^3 \theta (1 - \cos^2 \theta) + 5 \cos \theta (1 - \cos^2 \theta)^2$$

Simplify

$$\cos 5\theta = \cos^5 \theta - 10 \cos^3 \theta + 10 \cos^5 \theta + 5 \cos \theta (1 - \cos \cos^2 \theta + \cos^4 \theta)$$

Simplify further

$$\cos 5\theta = \cos^5 \theta - 10 \cos^3 \theta + 10 \cos^5 \theta + 5 \cos \theta - 10 \cos^3 \theta + 5 \cos^5 \theta$$

Collect terms

$$\cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta$$

### 3.3.2 Z formulas

If  $z = \cos \theta + i \sin \theta$

$$z + \frac{1}{z} = 2 \cos \theta$$

$$z - \frac{1}{z} = 2i \sin \theta$$

$$z^n + \frac{1}{z^n} = 2 \cos n\theta$$

$$z^n - \frac{1}{z^n} = 2i \sin n\theta$$

#### 3.3.2.1 Example

Express  $\cos^5 \theta$  in the form  $a \cos 5\theta + b \cos 3\theta + c \cos \theta$  Create the situation using the z formulas

$$\left(z + \frac{1}{z}\right)^5 = (2 \cos \theta)^5 = 32 \cos^5 \theta$$

Expand using the binomial

$$z^5 + 5z^4 \times \frac{1}{z} + 10z^3 \times \frac{1}{z^2} + 10z^2 \times \frac{1}{z^3} + 5z \times \frac{1}{z^4} + \frac{1}{z^5}$$

Combine like coloured term using z formula

$$32 \cos^5 \theta = 2 \cos \theta + 10 \cos 3\theta + 20 \cos \theta$$

$$\cos^5 \theta = \frac{\cos \theta}{16} + \frac{5 \cos 3\theta}{16} + \frac{5 \cos \theta}{8}$$

## 4 Solving complex equations

For any complex number, we generally define the argument to be between  $-\pi$  and  $\pi$ . However we can add multiples of  $2\pi$  to the argument to get equivalent answers.

We shall use this fact to find all solutions to complex equations.

Note: The number of solutions is equal to the order of the equation.

### 4.1 Example

Solve:

$$z^5 = i$$

$$r = 1 \quad \theta = \frac{\pi}{2}$$

$$z^5 = \cos\left(\frac{\pi}{2} + 2k\pi\right) + i \sin\left(\frac{\pi}{2} + 2k\pi\right)$$

$$z = \cos\left(\frac{\frac{\pi}{2} + 2k\pi}{5}\right) + i \sin\left(\frac{\frac{\pi}{2} + 2k\pi}{5}\right)$$

5 Solutions:  $k = -2, -1, 0, 1, 2$

$$k = -2 \quad z = -0.588 - 0.809i$$

$$k = -1 \quad z = 0.588 - 0.809i$$

$$k = 0 \quad z = 0.951 + 0.309i$$

$$k = 1 \quad z = i$$

$$k = 2 \quad z = -0.951 + 0.309i$$

## 5 Loci on the complex plane

### 5.1 Lines

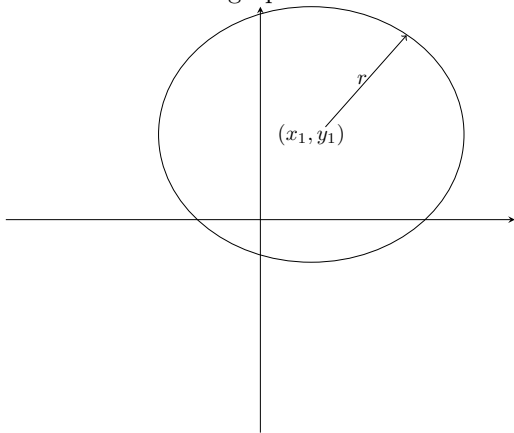
#### 5.1.1 $|z - z_1| = r$

$|z - z_1| = r$  is represented by a circle, centre  $(x_1, y_1)$  with a radius  $r$ , where  $z_1 = x_1 + iy_1$ .

This is the same as the Cartesian equation:

$$(x - x_1)^2 + (y - y_1)^2 = r^2$$

This looks like the graph:

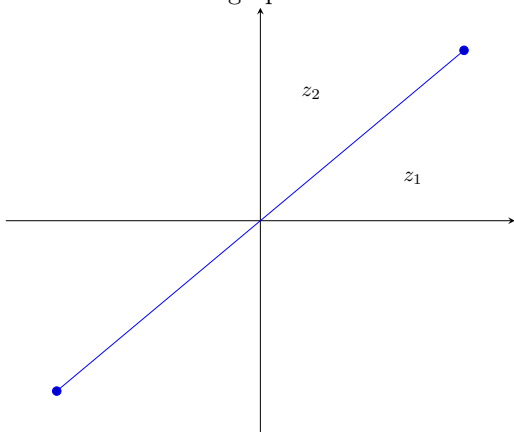


#### 5.1.2 $|z - z_1| = |z - z_2|$

$|z - z_1| = |z - z_2|$  is represented by a perpendicular bisector of the line segment joining points  $z_1$  and  $z_2$ .

To find the Cartesian form, replace  $z$  with  $x + iy$  and expand, squaring both sides to remove the modulus signs.

This looks like the graph:



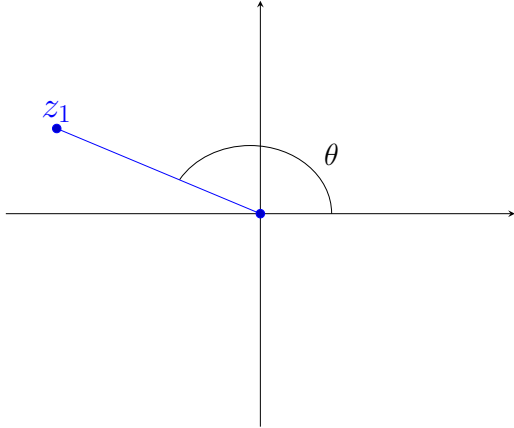
For cases where there is a constant in front of one of the modulus signs, use the algebraic method to find the Cartesian form, then plot that.

**5.1.3**  $\arg(\mathbf{z} - \mathbf{z}_1) = \theta$ 

$\arg(z - z_1) = \theta$  is represented by the half-line from the fixed point  $z_1$ , making an angle  $\theta$  with a line from the fixed point  $z_1$ , parallel to the real axis.

To find the Cartesian form, replace  $z$  with  $x + iy$  and separate into the real and imaginary parts. Then use the trigonometric identity  $\tan \theta = \frac{\text{Opposite}}{\text{Adjacent}}$  and simplify.

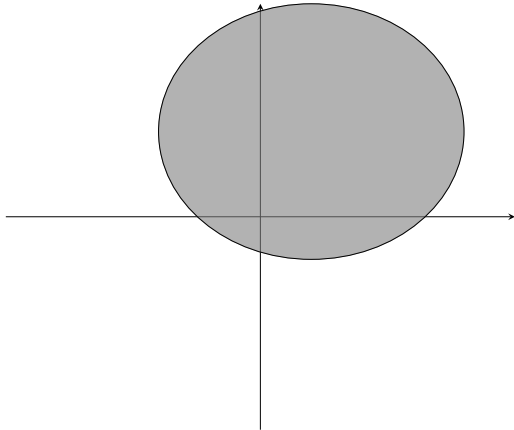
When plotting remember that  $\theta$  is the angle anticlockwise.

**5.1.4**  $\arg\left(\frac{\mathbf{z} - \mathbf{z}_1}{\mathbf{z} - \mathbf{z}_2}\right)$ 

$\arg\left(\frac{z - z_1}{z - z_2}\right)$  represents an arc anticlockwise between  $z_1$  and  $z_2$

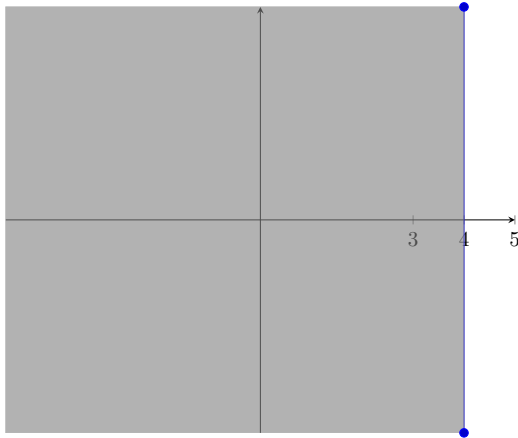
**5.2 Regions****5.2.1**  $|z - (2 + 3i)| \leq 3$ 

This is the area inside the circle represented by  $|z - (2 + 3i)| \leq 3$

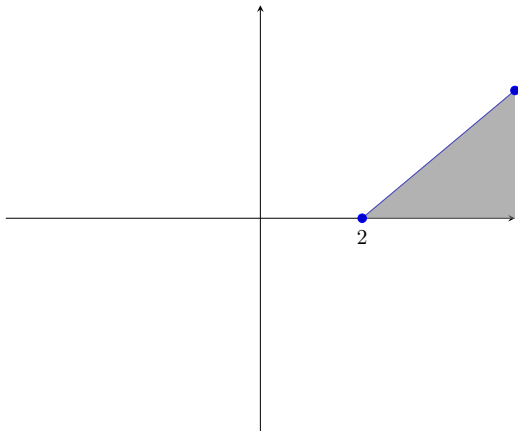


**5.2.2**  $|z - 3| < |z - 5|$ 

This is the area to one side of the perpendicular bisector represented equal to  $|z - 3| = |z - 5|$

**5.2.3**  $0 \leq \arg(z - 2) < \frac{\pi}{4}$ 

This is the area between the positive real axis and the half line equal to  $\arg(z - 2) = \frac{\pi}{4}$

**6 Translations**

- $w = z + a + ib$  represents a translation with translation vector  $\begin{pmatrix} a \\ b \end{pmatrix}$
- $w = kz$  represents an enlargement with scale factor  $k$  centre  $(0, 0)$
- $w = kz + a + ib$  represents an enlargement scale factor  $k$  centre  $(0, 0)$  followed by a translation with translation vector  $\begin{pmatrix} a \\ b \end{pmatrix}$
- $w = z^2$  multiply a shape by itself, for example a circle of radius 4 would go to radius 16