

Binomial Expansion

1 Introduction to Binomial expansion

Expansion can be done using the $(1+x)^n$ expansion, including with $(1+ax)^n$

2 Negative powers

Example To expand $\frac{1}{1+x}$ turn it into $(1+x)^{-1}$ and use the formula from the book.

$$1 - x + x^2 - x^3 + x^4 - x^5 \dots$$

As n is not a positive integer there will be no x coefficient equalling zero, meaning the expansion is infinite and convergent.

This gives valid values when $|x| < 1$

3 Fractional powers

$$\sqrt{1-3x}$$

Simplify

$$(1-3x)^{\frac{1}{2}}$$

Find n and x

$$n = \frac{1}{2}$$

$$x = -3x$$

Substitute into the formula

$$1 + \frac{1}{2} \times -3x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1 \times 2} \times (-3x)^2$$

Simplify

$$1 - \frac{3}{2}x - \frac{9}{8}x^2$$

Write conclusion

Convergent and infinite when: $|3x| < 1$ $|x| < \frac{1}{3}$

4 Applying $(1+x)^n$ to $(a \pm bx)^n$

$(a \pm bx)^n$ can be rewritten as $a^n(1 \pm \frac{b}{a}x)^n$

4.1 Example

Expand $\sqrt{4+x}$ to the x^3 term

Turn square root into power

$$(4-x)^{\frac{1}{2}}$$

Rewrite with a 1 in the bracket

$$4^{\frac{1}{2}}(1 + \frac{1}{4}x)^{\frac{1}{2}}$$

Find n and x

$$n = \frac{1}{2}$$

$$x = \frac{1}{4}x$$

Substitute into the formula

$$2 \left[1 + \frac{1}{2} \times \frac{1}{4}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} \left(\frac{1}{4}x \right)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!} \left(\frac{1}{4}x \right)^3 \right]$$

Simplify

$$2 \left[1 + \frac{x}{8} - \frac{x^2}{128} + \frac{x^3}{1024} \right]$$

$$2 + \frac{x}{4} - \frac{x^2}{64} + \frac{x^3}{512}$$

Write conclusionValid if $\left| \frac{x}{4} \right| < 1$ so valid if $|x| < 4$

5 Unknown coefficient type

 $(a + bx)^{-2}$ can be approximated by

$$a(1 + \frac{b}{a}x)^{-2}$$

$$\frac{1}{a^2}(1 - 2\frac{b}{a}x)$$

6 Fractional type

Expand up to $x^3 \frac{1+x}{2+x}$ **Re-Write using powers**

$$(1+x)(2+x)^{-1}$$

Ensure there is only a 1 in the bracket

$$2(1 + \frac{1}{2}x)^{-1}$$

Find n and x

$$n = -1$$

$$x = \frac{1}{2}x$$

Substitute into the formula

$$\frac{1}{2} \left(1 + -1 \times \frac{1}{2}x \right) + \frac{-1(-1-1)}{2!} \left(\frac{1}{2}(x)^2 \right)^2 + \frac{-1(-1-1)(-1-2)}{3!} \left(\frac{1}{2}x \right)^3$$

Simplify

$$(1+x) \left(\frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3 \right)$$

$$\frac{1}{2} + \frac{1}{4}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$$

Write conclusionValid if $x \neq 2$

7 Approximating roots

Find the expansion of $\sqrt{1-2x}$ up to x^3

Re-Write using powers

$$(1-2x)^{\frac{1}{2}}$$

Find n and x

$$n = \frac{1}{2}$$

$$x = -2x$$

Substitute into the formula

$$1 + \left(\frac{1}{2} \times -2x\right) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} \times (-2x)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!} \times (-2x)^3$$

Simplify

$$1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3$$

By substituting $x = 0.01$, find a suitable approximation of $\sqrt{2}$

Substitute values

$$\sqrt{1 - \frac{2}{100}} = 1 - \frac{1}{100} - \frac{(\frac{1}{100})^2}{2} - \frac{(\frac{1}{100})^3}{2}$$

Simplify

$$\sqrt{\frac{98}{100}} = \frac{\sqrt{98}}{10} = \frac{7\sqrt{2}}{10}$$

Rearrange

$$\sqrt{2} \approx \frac{10}{7} \left(1 - \frac{1}{100} - \frac{(\frac{1}{100})^2}{2} - \frac{(\frac{1}{100})^3}{2} \right)$$

$$\sqrt{2} = 1.414213571$$