Differentiation

1 Implicit differentiation

An implicit function is in terms of x and y and cannot easily be written as $y =?$

1.1 Formula

$\frac{d}{dx}(y^n) = ny^{n-1} \times \frac{dy}{dx}$
This is a formula you need to remember for the exam

1.2 Example

$x^3 + x + y^3 + 3y = 6$
$\frac{d}{dx}(x^3 + x + y^3 + 3y) = \frac{d}{dx}(6)$

Differentiate each term
$3x^2 + 1 + 3y^2\frac{dy}{dx} + 3\frac{dy}{dx} = 0$

Collect terms
$\frac{dy}{dx}(3y^2 + 3) = -3x^2 - 1$

Isolate $\frac{\mathbf{dy}}{\mathbf{dx}}$
$\frac{dy}{dx} = \frac{-3x^2 - 1}{3y^2 + 3}$

1.3 Example with combined xy terms

$\frac{d}{dx}(x, y) = x'y + xy'$

Example
$\frac{d}{dx}xy$

$\frac{d}{dx}x = 1$
$\frac{d}{dx}y = \frac{dy}{dx}$

$\frac{d}{dx} = 1 \times y + x \times \frac{dy}{dx}$

Example 2
Find $\frac{dy}{dx}$ *of* $4xy^2 + \frac{6x^2}{y}$

Simplify
$4xy^3 + 6x^2 - 10y = 0$

Differentiate each term
$\frac{d}{dx} = \left[1 \times 4y^3 + 4x \times 3y^2\frac{dy}{dx}\right] + 12x - 10\frac{dy}{dx}$

Collect terms
$\frac{dy}{dx}(12xy^2 - 10) = -(4y^3 + 12x)$

Isolate $\frac{\mathbf{dy}}{\mathbf{dx}}$
$\frac{dy}{dx} = \frac{-(4y^3 + 12x)}{12xy^2 - 10}$

2 Differentiating $\mathbf{y = a^x}$

$y = a^x$

Write in terms of logs
$\ln(y) = x\ln(a)$

Differentiate both sides
$\frac{d}{dx}\ln(y) = \frac{d}{dx}x\ln(a)$

$\frac{1}{y}\frac{dy}{dx} = \ln(a)$

Simplify
$\frac{dy}{dx} = y\ln(a)$

Sub y with $\mathbf{a^x}$
$\frac{dy}{dx} = a^x\ln(a)$

2.1 Differentiating $\mathbf{y = a^{f(x)}}$

$\frac{dy}{dx} = a^{f(x)}f'(x)\ln(a)$

3 Related rates of change

You can use the chain rule to connect related rates of change
For example:
$\frac{dx}{dt} = \frac{dV}{dt} \times \frac{dx}{dV}$
Where x is side length, v is volume and t is time.

We need to be able to interpret numerical information as a rate of change, for example:
$2cm^2s^-1 = \frac{dA}{dt}$

Example 1
A cylinder is expanding under heat
After t seconds:
The radius is x cm
The length is 5x cm
The cross sectional area is increasing at a constant rate of $0.037cm^2s^{-1}$

Find $\frac{dx}{dt}$ when r=4

Find the area in terms of x
$A = \pi x^2$

Differentiate the area with respect to x
$\frac{dA}{dx} = 2\pi x$

Re write the rate of change

$\frac{dA}{dt} = 0.037$

Use the chain rule $\frac{dx}{dt} = \frac{dx}{dA} \times \frac{dA}{dt}$

$\frac{dx}{dt} = \frac{0.037}{2\pi x}$

Sub x=4
$\frac{dx}{dt} = 0.00147cms^{-1}$

Find the rate of change of the volume when x=4
$V = 5\pi x^3$

Differentiate the volume with respect to x
$\frac{dV}{dx} = 15\pi x^2$

Use the chain rule
$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt} = 15\pi x^2 \times 0.00147$

Sub x=4
$1.11cm^3s^{-1}$