Partial fractions

1 Partial fractions

This requires us to split difficult algebraic fractions:

$$\frac{4}{(x+1)(x+2)} \rightarrow \frac{A}{x+1} + \frac{B}{x+2}$$

This allows us to:

- Do binomial expansion
- Integrate using difficult fractions

1.1 Normal Example

$$\frac{4}{(x+1)(x+2)} \rightarrow \frac{A}{x+1} + \frac{B}{x+2}$$

Recombine:

$$\frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$$

Equate:
$A(x+2)+B(x+1)=4$

Eliminate one:
Sub x=-1
$A(-1+2)=4, \underline{A=4}$

Sub x=-2
$B(-2+1), \underline{B=-4}$

Write as partial fractions:
$\frac{4}{x+1} - \frac{4}{x-1}$

1.2 Example with a cubic denominator

$\frac{6x^2+5x-2}{x(x+1)(2x+1)}$
Expand
$\frac{A}{x} + \frac{B}{x-1} + \frac{C}{2x+1}$
Equate
$A(x-1)(2x+1)+B(x)(2x+1)+C(x)(x-1)=6x^2+5x-2$
Eliminate one
Sub x=0
$-A=-2, \underline{A=2}$

Sub x=1
$3B=9$
$\underline{B=3}$

Sub $x=-\frac{1}{2}$
$-\frac{3}{4}C=-3, \underline{C=-4}$
Write as partial fractions
$\frac{2}{x} + \frac{3}{x-1} - \frac{4}{2x+1}$

1.3 Example with a Repeated root denominator

$\frac{6x^2-29x-29}{(x+1)(x-3)^2}$

Expand
$\frac{A}{x+1}+\frac{B}{x-3}+\frac{C}{(x-3)^2}$

Equate
$A(x-3)^2+B(x+1)(x-3)+C(x+1)=6x^2-29x-29$

Eliminate one
Sub x=-1
$A(-1-3)^2+B(-1+1)(-1-3)+C(-1+1)=6(-1)^2-29(-1)-29$
$A(-4)^2+B(0)(-1-3)+C(0)=6(-1)^2-29(-1)-29$
$16A=6, \underline{A=\frac{3}{8}}$

Sub x=3
$4C=-62, \underline{C=-\frac{31}{2}}$

Sub x=0
$-29=9\times\frac{3}{8}-3B-\frac{31}{2}$
$\underline{B=\frac{45}{8}}$

Write as partial fractions
$\frac{3}{8(x+1)}+\frac{45}{8(x-3)}-\frac{31}{2(x-3)^2}$

1.4 Example with partial fractions with same or higher denominator

$\frac{3x^2-3x-2}{(x-1)(x-2)}$

Long division to find remainder

$$
\begin{array}{r}
3 \\
x^2-3x+2 \overline{\big)\; 3x^2-3x-2} \\
\underline{-\;3x^2+9x-6} \\
6x-8
\end{array}
$$

Re-write with remainder
$3+\frac{6x-8}{(x-1)(x-2)}$

Expand
$\frac{A}{x-1}+\frac{B}{x-2}$

Equate
$A(x-2)+B(x-1)=6x-8$

Eliminate one
Sub x=1
$-A=-2$
$\underline{A=2}$

Sub x=2
$3B=12$
$\underline{B=4}$

Write as partial fractions
$3+\dfrac{2}{x-1}+\dfrac{4}{x-2}$

2 Binomial expansion

Expansion can be done using the $(1+x)^n$ expansion, including with $(1+ax)^n$

2.1 Negative powers

Example To expand $\dfrac{1}{1+x}$ turn it into $(1+x)^{-1}$ an use the formula from the book.
$1-x+x^2-x^3+x^4-x^5...$
As n is not a positive integer there will be no x coefficient equalling zero, meaning the expansion is infinite and convergent.
This gives valid values when $|x|<1$

2.2 Fractional powers

$\sqrt{1-3x}$
Simplify
$(1-3x)^{\frac{1}{2}}$

Find n and x
$n=\frac{1}{2}$
$x=-3x$

Substitute into the formula
$1+\frac{1}{2}\times -3x+\dfrac{\frac{1}{2}(\frac{1}{2}-1)}{1\times 2}\times(-3x)^2$

Simplify
$1-\frac{3}{2}x-\frac{9}{8}x^2$

Write conclusion
Convergent and infinite when: $|3x|<1$ $|x|<\frac{1}{3}$

2.3 Applying $(\mathbf{1+x})^{\mathbf{n}}$ to $(\mathbf{a\pm bx})^{\mathbf{n}}$

$(a\pm bx)^n$ can be rewritten as $a^n(1\pm\frac{b}{a}x)^n$

2.3.1 Example

Expand $\sqrt{4+x}$ *to the* x^3 *term*
Turn square root into power
$(4-x)^{\frac{1}{2}}$

Rewrite with a 1 in the bracket
$4^{\frac{1}{2}}(1+\frac{1}{4}x)^{\frac{1}{2}}$

Find n and x
n=$\dfrac{1}{2}$
x=$\frac{1}{4}x$

Substitute into the formula

$$2\left[1+\frac{1}{2}\times\frac{1}{4}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\left(\frac{1}{4}x\right)^2+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}\left(\frac{1}{4}x\right)^3\right]$$

Simplify
$2\left[1+\frac{x}{8}-\frac{x^2}{128}+\frac{x^3}{1024}\right]$
$2+\frac{x}{4}-\frac{x^2}{64}+\frac{x^3}{512}$

Write conclusion
Valid if $\left|\frac{x}{4}\right|<1$ so valid if $|x|<4$

2.4 Unknown coefficient type

$(a+bx)^{-2}$ can be approximated by
$a(1+\frac{b}{a}x)^{-2}$
$\frac{1}{a^2}(1-2\frac{b}{a}x)$

2.5 Fractional type

Expand up to x^3 $\frac{1+x}{2+x}$

Re-Write using powers
$(1+x)(2+x)^{-1}$

Ensure there is only a 1 in the bracket
$2(1+\frac{1}{2}x)^{-1}$

Find n and x
$n=-1$
$x=\frac{1}{2}x$

Substitute into the formula
$\frac{1}{2}\left(1+-1\times\frac{1}{2}x\right)+\frac{-1(-1-1)}{2!}\left(\frac{1}{2}(x)^2\right)^2+\frac{-1(-1-1)(-1-2}{3!}\left(\frac{1}{2}x\right)^3$

Simplify
$(1+x)\left(\frac{1}{2}-\frac{1}{4}x+\frac{1}{8}x^2-\frac{1}{16}x^3\right)$
$\frac{1}{2}+\frac{1}{4}x-\frac{1}{8}x^2+\frac{1}{16}x^3$

Write conclusion
Valid if $x\neq 2$

2.6 Approximating roots

Find the expansion of $\sqrt{1-2x}$ *up to* x^3
Re-Write using powers
$(1-2x)^{\frac{1}{2}}$

Find n and x
$n=\dfrac{1}{2}$
$x=-2x$

Substitute into the formula
$$1+\left(\frac{1}{2}\times -2x\right)+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\times(-2x)^2+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}\times(-2x)^3$$
Simplify $1-x+\dfrac{1}{2}x^2-\dfrac{1}{3}x^3$

By substituting $x=0.01$*, find a suitable approximation of* $\sqrt{2}$
Substitute values $\sqrt{1-\dfrac{2}{100}}=1-\dfrac{1}{100}-\dfrac{(\frac{1}{100})^2}{2}-\dfrac{(\frac{1}{100})^3}{2}$

Simplify $\sqrt{\dfrac{98}{100}}=\dfrac{\sqrt{98}}{10}=\dfrac{7\sqrt{2}}{10}$

Rearrange
$$\sqrt{2}\approx\frac{10}{7}\left(1-\frac{1}{100}-\frac{(\frac{1}{100})^2}{2}-\frac{(\frac{1}{100})^3}{2}\right)$$
$\sqrt{2}=1.414213571$