

# C3

## 1 Algebraic fractions

Any polynomial  $F(x)$  can be put in the form:

$$F(x) = Q(x) \times \text{divisor} + \text{remainder}$$

Where  $Q(x)$  is the quotient

## 2 Functions

### 2.1 Range and domain

The input to a function is the **Domain**

The output from a function is the **Range**

### 2.2 Function mapping

One-to-one function: One element in the domain maps to one element in the range

Many-to-one function: To elements of the domain maps to one element in the range

Not a function: One input maps to two outputs

### 2.3 Mappings to functions by changing the domain

Consider  $y = \sqrt{x}$

If the domain is all the real numbers  $x \in \mathbb{R}$  then it is not a function as values less than 0 don't get mapped anywhere.

The domain must be restricted to  $x \geq 0$

### 2.4 Combining functions

$fg(x)$  means apply  $g$  to  $x$ , then apply  $f$

$f^2(x)$  means  $ff(x)$

### 2.5 Inverse functions

The inverse of  $f(x)$  is written as  $f^{-1}x$

The domain of  $f(x)$  is the range of  $f^{-1}x$

The range of  $f(x)$  is the domain of  $f^{-1}x$

Example, find the inverse function of  $y = 2x^2 - 7$ :

$$y + 7 = 2x^2$$

$$\frac{y + 7}{2} = x^2$$

$$x = \sqrt{\frac{y + 7}{2}}$$

$$f^{-1}x = \sqrt{\frac{y + 7}{2}}$$

When finding the graph of an inverse function, reflect  $f(x)$  in the line  $y=x$ .

## 3 The exponential and log functions

### 3.1 Exponential functions

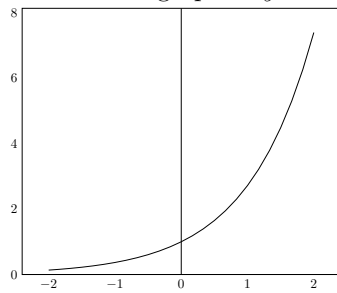
Exponential functions are in the form  $y = a^x$ , graphs of these functions all pass through (0,1) as  $a^0 = 1$  for any value of  $a$ .

### 3.2 Functions including $e$

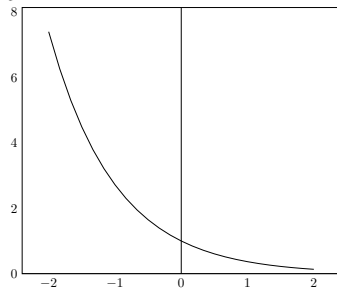
The function  $y = e^x$  is the function where the gradient is identical to the function.

$$y = e^x \frac{dy}{dx} = e^x$$

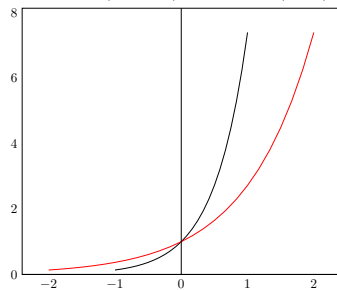
Below is the graph of  $y = e^x$



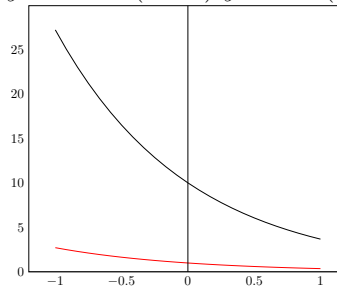
$y = e^{-x}$



$y = e^{2x}$  (black)  $y = e^x$  (red)



$y = 10e^{-x}$  (black)  $y = e^{-x}$  (red)



### 3.3 Formulas for exponential growth or decay

Example:

$$P = 16000e^{-\frac{t}{10}}$$

Where P is the Price in £s and t is the years from new

*What was the price when new?*

Substitute  $t=0$

$$P = 16000e^{-\frac{0}{10}}$$

$$P = 16000 \times 1$$

What is the value at 5 years old

Substitute  $t=5$

$$P = 16000e^{-\frac{5}{10}}$$

$$P = \pounds 9704.49$$

What does the model say about the eventual value of the car

As  $t \rightarrow \infty$ ,  $e^{-\frac{t}{10}} \rightarrow 0$

Therefore  $P \rightarrow 16000 \times 0 = 0$

The eventual value is zero.

### 3.4 The inverse of the exponential function

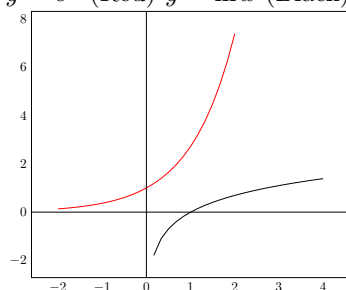
The inverse of  $e^x$  is  $\log_e x$  (also written as  $\ln x$ )

Examples:

If  $e^x = 3$  then  $x = \ln 3$

If  $\ln x = 4$  then  $x = e^4$

$y = e^x$  (Red)  $y = \ln x$  (Black)



The function  $f(x) = \ln x$  has domain  $\{x \in \mathbb{R}, x > 0\}$  and range  $\{f(x) \in \mathbb{R}\}$

## 4 Numerical methods

### 4.1 Approximations for roots based on graphs

Approximations for roots can be found graphically by plotting the function and finding where the line crosses the x axis. This value is one of the roots of the function.

If trying to find a range in which a root can be found, substitute the values at the extreme of the range, and if there is a change in sign between the two results, there will be a root in the range.

The exception to this rule is  $f(x) = \frac{1}{x}$  and transformations of this as there is a discontinuity at  $x=0$ . The function changes sign in the interval that includes  $x=0$ , but there is not a root.

### 4.2 Iteration for finding approximations of roots

To solve an equation of the form  $f(x) = 0$  by an iterative method, rearrange  $f(x) = 0$  into a form  $x = g(x)$  and use the iterative formula  $x_{n+1} = g(x_n)$ .

Example, find a root of the equation  $x^2 - 4x + 1 = 0$

Re-write as  $x = 4 - \frac{1}{x}$

Create the formula  $x_{n+1} = 4 - \frac{1}{x_n}$

You get given a rough approximation,  $x_0 = 3$

Substitute

$$x_1 = 4 - \frac{1}{x_0}$$

$$x_1 = 4 - \frac{1}{3}$$

$$x_1 = \frac{11}{3}$$

$$x_2 = 4 - \frac{1}{\frac{11}{3}}$$

$$x_2 = \frac{41}{11}$$

Continuing this increases the accuracy of the result.

This may not work and will not converge to a root.

## 5 Transforming graphs of functions

### 5.1 $y = |f(x)|$ Graphs

The modulus of a number is written as  $|a|$ , this is the positive numerical value.

When  $f(x) \geq 0$ ,  $|f(x)| = f(x)$

When  $f(x) < 0$ ,  $|f(x)| = -f(x)$

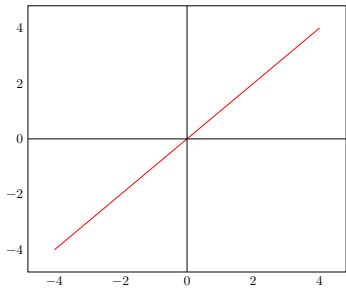


Figure 1:  $y = x$

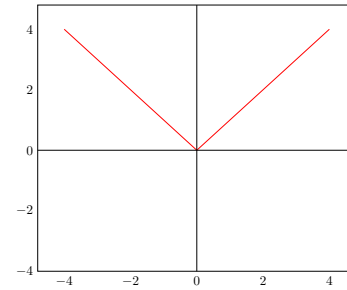


Figure 2:  $y = |x|$

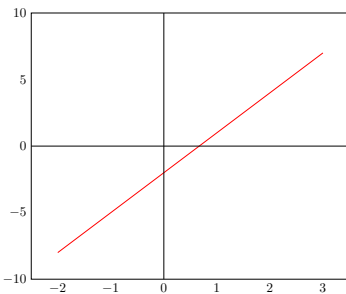


Figure 3:  $y = 3x - 2$

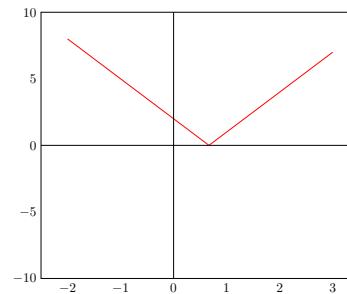


Figure 4:  $y = |3x - 2|$

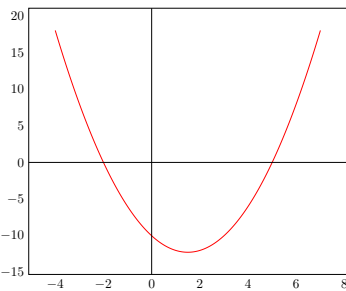


Figure 5:  $y = x^2 - 3x - 10$

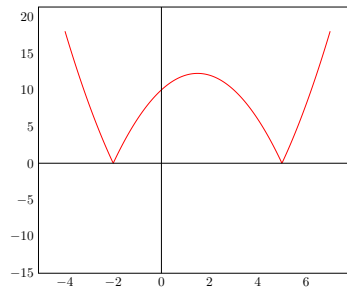
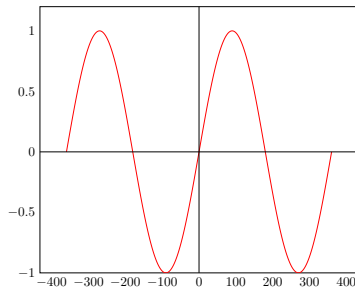
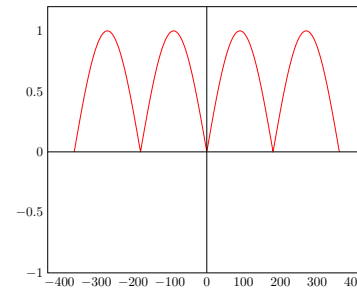
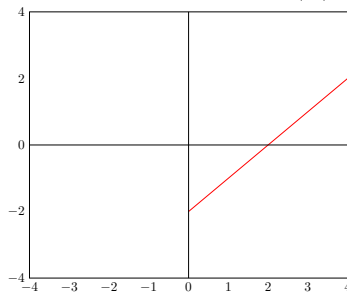


Figure 6:  $y = |x^2 - 3x - 10|$

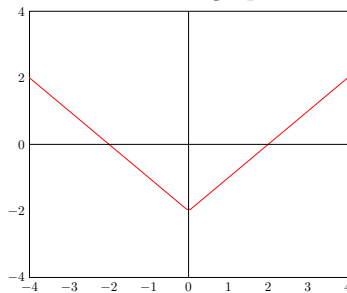
Figure 7:  $y = \sin(x)$ Figure 8:  $y = |\sin(x)|$ 

## 5.2 $y = f(|x|)$ Graphs

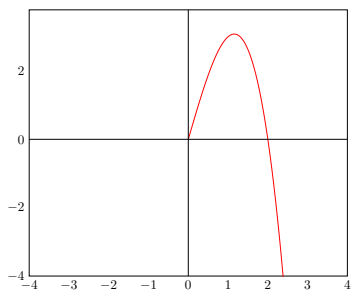
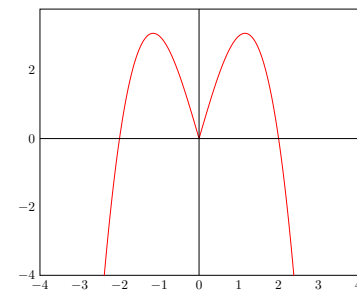
To plot the graph of  $y = |x| - 2$ , first sketch the graph of  $y = x - 2$  for  $x \geq 0$ :



Then reflect that graph in the y axis



Examples:

Figure 9:  $y = 4x - x^3$ Figure 10:  $y = 4|x| - |x|^3$ 

## 5.3 Graph transformations

- $f(x + a)$  - Horizontal translation of  $-a$
- $f(x) + a$  - Vertical translation of  $+a$
- $f(ax)$  - Horizontal stretch of scale factor  $\frac{1}{a}$
- $f(-x)$  - Reflection in the **y** axis

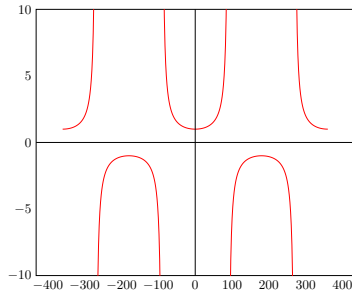
- $af(x)$  - Vertical stretch of scale factor **a**
- $-f(x)$  - Reflection in the **x axis**

## 6 Trigonometry

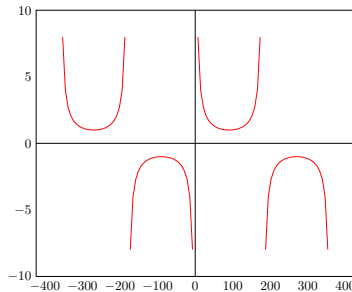
- $\sec \theta = \frac{1}{\cos \theta}$
- $\csc \theta = \frac{1}{\sin \theta} = \operatorname{cosec} \theta$
- $\cot \theta = \frac{1}{\tan \theta}$
- $\cot \theta = \frac{\cos \theta}{\sin \theta}$

### 6.1 Graphs of the new functions

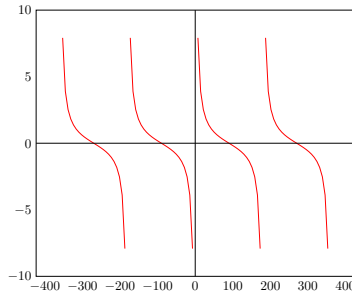
$\sec(x)$



$\csc(x)$



$\cot(x)$

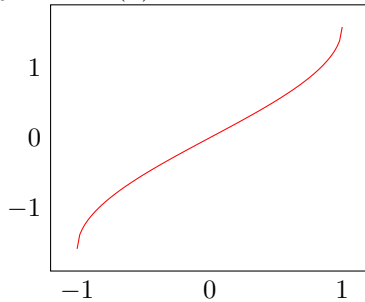


### 6.2 New identities

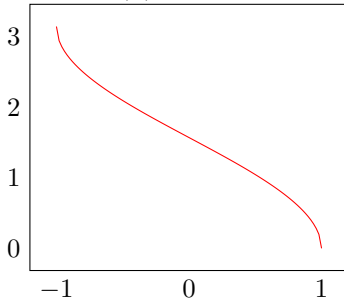
- $1 + \tan^2 \theta = \sec^2 \theta$
- $1 + \cot^2 \theta = \csc^2 \theta$

### 6.3 Graphs of inverse functions

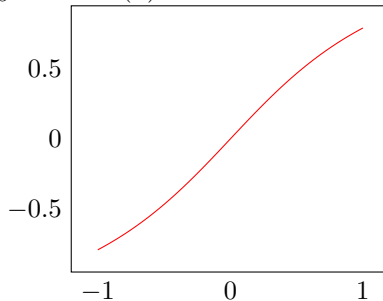
$$y = \arcsin(x)$$



$$y = \arccos(x)$$



$$y = \arctan(x)$$



## 7 Further trigonometric identities and their applications

- $\sin(a \pm b) \equiv \sin A \cos B \pm \cos A \sin B$
- $\cos(a \pm b) \equiv \cos A \cos B \mp \sin A \sin B$
- $\tan(a \pm b) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

### 7.1 Double angle formulae

- $\sin 2A \equiv 2 \sin A \cos A$
- $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2 \cos^2 A - 1 \equiv 1 - 2 \sin^2 A$
- $\tan 2A \equiv \frac{2 \tan A}{1 - \tan^2 A}$

### 7.2 The R formula

For positive values of  $a$  and  $b$

$a \sin \theta \pm b \cos \theta$  can be expressed in the form  $R \sin(\theta \pm \alpha)$ , where  $0 < \alpha < 90$

$a \cos \theta \pm b \sin \theta$  can be expressed in the form  $R \cos(\theta \mp \alpha)$ , where  $0 < \alpha < 90$

$R \cos \alpha = a$ ,  $R \sin \alpha = b$

$R = \sqrt{a^2 + b^2}$

## 8 Differentiation

### 8.1 The chain rule

If  $y = [f(x)]^n$  then  $\frac{dy}{dx} = n[f(x)]^{n-1} f'(x)$

If  $y = f[g(x)]$  then  $\frac{dy}{dx} = f'[g(x)]g'(x)$

**Example**

$$f(x) = 3x^4 + x$$

$$f'(x) = 12x^3 + 1$$

$$\frac{dy}{dx} = 5(3x^4 + x)^4(12x^3 + 1)$$

#### 8.1.1 Another form of the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

**Example**

$$y = (x^2 - 7x)^4$$

$$u = (x^2 - 7x)^4$$

$$y = u^4$$

$$\frac{du}{dx} = 2x - 7$$

$$\frac{dy}{du} = 4u^3$$

Using the chain rule:

$$\frac{dy}{dx} = 4u^3 \times (2x - 7)$$

$$\frac{dy}{dx} = 4(2x - 7)(x^2 - 7x)^3$$

### 8.2 The product rule

The product rule is used to differentiate the product of two functions.

If  $y = uv$  then  $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

**Example:**

$$f(x) = x^2 \sqrt{3x - 1}$$

$$u = x^2, v = (3x - 1)^{\frac{1}{2}}$$

$$\frac{du}{dx} = 2x \text{ and } \frac{dv}{dx} = \frac{3}{2}(3x - 1)^{-\frac{1}{2}}$$

$$f'(x) = x^2 \times \frac{3}{2}(3x - 1)^{-\frac{1}{2}} + \sqrt{3x - 1} \times 2x$$

$$f'(x) = \frac{15x^2 - 4x}{2\sqrt{3x - 1}}$$

$$f'(x) = \frac{x(15x - 4)}{2\sqrt{3x - 1}}$$

### 8.3 The quotient rule

If  $y = \frac{u(x)}{v(x)}$  then  $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

**Example**

$$y = \frac{x}{2x + 5}$$

$$u = x, v = 2x + 5$$

$$\frac{du}{dx} = 1, \frac{dv}{dx} = 2$$

$$\frac{dy}{dx} = \frac{(2x + 5) \times 1 - x \times 2}{(2x + 5)^2}$$



$$\frac{dy}{dx} = \frac{5}{(2x+5)^2}$$

## 8.4 The exponential function

If  $y = e^x$  then  $\frac{dy}{dx} = e^x$

If  $y = e^{f(x)}$  then  $\frac{dy}{dx} = f'(x)e^{f(x)}$

### Example

$$y = e^{2x+3}$$

$$\frac{dy}{dx} 2x + 3 = 2$$

$$\frac{dy}{dx} e^{2x+3} = 2e^{2x+3}$$

## 8.5 The logarithmic function

If  $y = \ln(x)$  then  $\frac{dy}{dx} = \frac{1}{x}$

If  $y = \ln[f(x)]$  then  $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

### Example

$$y = \ln(6x - 1)$$

$$\frac{dy}{dx} 6x - 1 = 6$$

$$\frac{dy}{dx} \ln(6x - 1) = \frac{6}{6x - 1}$$

## 8.6 Trig functions

### 8.6.1 Sin

If  $y = \sin(x)$  then  $\frac{dy}{dx} = \cos(x)$

If  $y = \sin f(x)$  then  $\frac{dy}{dx} = f'(x) \cos f(x)$

### 8.6.2 Cos

If  $y = \cos(x)$  then  $\frac{dy}{dx} = -\sin(x)$

If  $y = \cos f(x)$  then  $\frac{dy}{dx} = -f'(x) \sin f(x)$

### 8.6.3 Tan

If  $y = \tan(x)$  then  $\frac{dy}{dx} = \sec^2(x)$

If  $y = \tan f(x)$  then  $\frac{dy}{dx} = f'(x) \sec^2 f(x)$

### 8.6.4 Csc

If  $y = \csc(x)$  then  $\frac{dy}{dx} = -\csc(x) \cot(x)$

If  $y = \csc f(x)$  then  $\frac{dy}{dx} = -f'(x) \csc f(x) \cot f(x)$

**8.6.5 Sec**

If  $y = \sec(x)$  then  $\frac{dy}{dx} = \sec(x) \tan(x)$

If  $y = \sec f(x)$  then  $\frac{dy}{dx} = f'(x) \sec f(x) \tan f(x)$

**8.6.6 Cot**

If  $y = \cot(x)$  then  $\frac{dy}{dx} = -\csc^2(x)$

If  $y = \cot f(x)$  then  $\frac{dy}{dx} = -f'(x) \csc^2 f(x)$

## Differentiation Table

$y$	$\frac{dy}{dx}$
$[f(x)]^n$	$n(f(x))^{n-1}f'(x)$
$uv$	$u\frac{dv}{dx} + v\frac{du}{dx}$
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
$e^{f(x)}$	$f'(x)e^{f(x)}$
$\ln  f(x) $	$\frac{f'(x)}{f(x)}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x) \cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x) \sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x) \sec^2 f(x)$
$y = \csc f(x)$	$\frac{dy}{dx} = -f'(x) \csc f(x) \cot f(x)$
$y = \sec f(x)$	$\frac{dy}{dx} = f'(x) \sec f(x) \tan f(x)$
$y = \cot f(x)$	$\frac{dy}{dx} = -f'(x) \csc^2 f(x)$