

# Partial fractions

## 1 Partial fractions

This requires us to split difficult algebraic fractions:

$$\frac{4}{(x+1)(x+2)} \rightarrow \frac{A}{x+1} + \frac{B}{x+2}$$

This allows us to:

- Do binomial expansion
- Integrate using difficult fractions

### 1.1 Normal Example

$$\frac{4}{(x+1)(x+2)} \rightarrow \frac{A}{x+1} + \frac{B}{x+2}$$

**Recombine:**

$$\frac{A(x+2) + B(x+1)}{(x+1)(x+2)}$$

**Equate:**

$$A(x+2) + B(x+1) = 4$$

**Eliminate one:**

*Sub*  $x=-1$

$$A(-1+2) = 4, \underline{A = 4}$$

*Sub*  $x=-2$

$$B(-2+1), \underline{B = -4}$$

**Write as partial fractions:**

$$\frac{4}{x+1} - \frac{4}{x+2}$$

### 1.2 Example with a cubic denominator

$$\frac{6x^2 + 5x - 2}{x(x+1)(2x+1)}$$

**Expand**

$$\frac{A}{x} + \frac{B}{x+1} + \frac{C}{2x+1}$$

**Equate**

$$A(x+1)(2x+1) + B(x)(2x+1) + C(x)(x-1) = 6x^2 + 5x - 2$$

**Eliminate one**

*Sub*  $x=0$

$$-A = -2, \underline{A = 2}$$

*Sub*  $x=1$

$$3B = 9$$

$$\underline{B = 3}$$

*Sub*  $x = -\frac{1}{2}$

$$-\frac{3}{4}C = -3, \underline{C = 4}$$

**Write as partial fractions**

$$\frac{2}{x} + \frac{3}{x+1} - \frac{4}{2x+1}$$

### 1.3 Example with a Repeated root denominator

$$\frac{6x^2 - 29x - 29}{(x+1)(x-3)^2}$$

**Expand**

$$\frac{A}{x+1} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

**Equate**

$$A(x-3)^2 + B(x+1)(x-3) + C(x+1) = 6x^2 - 29x - 29$$

**Eliminate one**

*Sub  $x=-1$*

$$A(-1-3)^2 + B(-1+1)(-1-3) + C(-1+1) = 6(-1)^2 - 29(-1) - 29$$

$$A(-4)^2 + B(0)(-1-3) + C(0) = 6(-1)^2 - 29(-1) - 29$$

$$16A = 6, \underline{A = \frac{3}{8}}$$

*Sub  $x=3$*

$$4C = -62, \underline{C = -\frac{31}{2}}$$

*Sub  $x=0$*

$$-29 = 9 \times \frac{3}{8} - 3B - \frac{31}{2}$$

$$\underline{B = \frac{45}{8}}$$

**Write as partial fractions**

$$\frac{3}{8(x+1)} + \frac{45}{8(x-3)} - \frac{31}{2(x-3)^2}$$

### 1.4 Example with partial fractions with same or higher denominator

$$\frac{3x^2 - 3x - 2}{(x-1)(x-2)}$$

**Long division to find remainder**

$$\begin{array}{r} 3 \\ x^2 - 3x + 2 \overline{) 3x^2 - 3x - 2} \\ \underline{- 3x^2 + 9x - 6} \phantom{2} \\ 6x - 8 \end{array}$$

**Re-write with remainder**

$$3 + \frac{6x-8}{(x-1)(x-2)}$$

**Expand**

$$\frac{A}{x-1} + \frac{B}{x-2}$$

**Equate**

$$A(x-2) + B(x-1) = 6x-8$$

**Eliminate one**

*Sub  $x=1$*

$$-A = -2$$

$$\underline{A = 2}$$

*Sub  $x=2$*

$$3B = 12$$

$$\underline{B = 4}$$

Write as partial fractions

$$3 + \frac{2}{x-1} + \frac{4}{x-2}$$

## 2 Binomial expansion

Expansion can be done using the  $(1+x)^n$  expansion, including with  $(1+ax)^n$

### 2.1 Negative powers

**Example** To expand  $\frac{1}{1+x}$  turn it into  $(1+x)^{-1}$  and use the formula from the book.

$$1 - x + x^2 - x^3 + x^4 - x^5 \dots$$

As  $n$  is not a positive integer there will be no  $x$  coefficient equalling zero, meaning the expansion is infinite and convergent.

This gives valid values when  $|x| < 1$

### 2.2 Fractional powers

$$\sqrt{1-3x}$$

**Simplify**

$$(1-3x)^{\frac{1}{2}}$$

**Find  $n$  and  $x$**

$$n = \frac{1}{2}$$

$$x = -3x$$

**Substitute into the formula**

$$1 + \frac{1}{2} \times -3x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1 \times 2} \times (-3x)^2$$

**Simplify**

$$1 - \frac{3}{2}x - \frac{9}{8}x^2$$

**Write conclusion**

Convergent and infinite when:  $|3x| < 1$   $|x| < \frac{1}{3}$

### 2.3 Applying $(1+x)^n$ to $(a \pm bx)^n$

$(a \pm bx)^n$  can be rewritten as  $a^n(1 \pm \frac{b}{a}x)^n$

#### 2.3.1 Example

Expand  $\sqrt{4+x}$  to the  $x^3$  term

**Turn square root into power**

$$(4-x)^{\frac{1}{2}}$$

**Rewrite with a 1 in the bracket**

$$4^{\frac{1}{2}}(1 + \frac{1}{4}x)^{\frac{1}{2}}$$

**Find  $n$  and  $x$**

$$n = \frac{1}{2}$$

$$x = \frac{1}{4}x$$

**Substitute into the formula**

$$2 \left[ 1 + \frac{1}{2} \times \frac{1}{4}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} \left( \frac{1}{4}x \right)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!} \left( \frac{1}{4}x \right)^3 \right]$$

**Simplify**

$$2 \left[ 1 + \frac{x}{8} - \frac{x^2}{128} + \frac{x^3}{1024} \right]$$

$$2 + \frac{x}{4} - \frac{x^2}{64} + \frac{x^3}{512}$$

**Write conclusion**Valid if  $\left| \frac{x}{4} \right| < 1$  so valid if  $|x| < 4$ **2.4 Unknown coefficient type** $(a + bx)^{-2}$  can be approximated by

$$a \left( 1 + \frac{b}{a}x \right)^{-2}$$

$$\frac{1}{a^2} \left( 1 - 2\frac{b}{a}x \right)$$

**2.5 Fractional type**Expand up to  $x^3$   $\frac{1+x}{2+x}$ **Re-Write using powers**

$$(1+x)(2+x)^{-1}$$

**Ensure there is only a 1 in the bracket**

$$2 \left( 1 + \frac{1}{2}x \right)^{-1}$$

**Find n and x**

$$n = -1$$

$$x = \frac{1}{2}x$$

**Substitute into the formula**

$$\frac{1}{2} \left( 1 + -1 \times \frac{1}{2}x \right) + \frac{-1(-1-1)}{2!} \left( \frac{1}{2}(x)^2 \right) + \frac{-1(-1-1)(-1-2)}{3!} \left( \frac{1}{2}x \right)^3$$

**Simplify**

$$(1+x) \left( \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3 \right)$$

$$\frac{1}{2} + \frac{1}{4}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$$

**Write conclusion**Valid if  $x \neq 2$

## 2.6 Approximating roots

Find the expansion of  $\sqrt{1-2x}$  up to  $x^3$

**Re-Write using powers**

$$(1-2x)^{\frac{1}{2}}$$

**Find n and x**

$$n = \frac{1}{2}$$

$$x = -2x$$

**Substitute into the formula**

$$1 + \left(\frac{1}{2} \times -2x\right) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} \times (-2x)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!} \times (-2x)^3$$

**Simplify**  $1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3$

By substituting  $x = 0.01$ , find a suitable approximation of  $\sqrt{2}$

**Substitute values**  $\sqrt{1 - \frac{2}{100}} = 1 - \frac{1}{100} - \frac{(\frac{1}{100})^2}{2} - \frac{(\frac{1}{100})^3}{2}$

**Simplify**  $\sqrt{\frac{98}{100}} = \frac{\sqrt{98}}{10} = \frac{7\sqrt{2}}{10}$

**Rearrange**

$$\sqrt{2} \approx \frac{10}{7} \left( 1 - \frac{1}{100} - \frac{(\frac{1}{100})^2}{2} - \frac{(\frac{1}{100})^3}{2} \right)$$

$$\sqrt{2} = 1.414213571$$