# Second Order Differential Equations

In FP2 we are interested in solving 2nd ODEs of the form:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x) \qquad \text{a,b,c are constants}$$

We consider three distinct cases:
$b^2>4ac$ (Two real solutions)
$b^2=4ac$ (One repeated solution)
$b^2<4ac$ (Two complex solutions)

To solve $2^{\text{nd}}$ ODEs of this form we first consider solutions to:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

The process of solving a $2^{\text{nd}}$ ODE starts with a general solution to a $1^{\text{st}}$ ODE of form:

$$b\frac{dy}{dx}+cy=0$$

$$\int\frac{1}{b}\;dy=\int\frac{1}{-cy}\;dy$$

$$b\ln(y)=-cx+k$$

$$y=Ae^{-\frac{c}{b}x}$$

$$y=Ae^{mx}$$

This was suggested to be a solution to the $2^{\text{nd}}$ ODE as well

We take $y=e^{mx}$ as a starting point for finding general solutions to:

$$(1) \quad a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

If $y=e^{mx}$ is a solution to (1):

$$\frac{dy}{dx}=me^{mx}$$

$$\frac{d^2y}{dx^2}=m^2e^{mx}$$

Then substitute this into (1)

$$am^2e^{mx}+bme^{mx}+ce^{mx}=0$$

Factor out $e^{mx}$

$$e^{mx}(am^2+bm+c)$$

As $e^x$ must be greater than zero $am^2+bm+c=0$
This is a solvable quadratic called the **Auxiliary equation**

# 1 Two real roots $\mathbf{b^2 > 4ac}$

$$(1) \quad a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

The general solution to (1) is in the form:

$$y = Ae^{\alpha x} + Be^{\beta x}$$

Where A and B are constants and $\alpha$ and $\beta$ are the roots to the AE

## 1.1 Example

$$(1) \quad 2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 3y = 0$$

Find the auxiliary equation

$$2m^2e^{mx} + 5me^{mx} + 3e^{mx} = 0$$

$$e^{mx}(2m^2 + 5m + 3 = 0$$

$m = -\frac{3}{2} \quad m = -1$

$$\text{General solution:} y = Ae^{\alpha x} + Be^{\beta x}$$

$$GS = Ae^{-\frac{3}{2}x} + Be^{-x}$$

# 2 1 Real, Repeated root $\mathbf{b^2 = 4ac}$

$$\text{General solution : } (A + bx)e^{\alpha x}$$

A and B are constants and $\alpha$ is the root of the AE

## 2.1 Example

*Find the general solution of:*

$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0$$

Find the auxiliary equation

$$e^{mx}(m^2 + 8m + 16) = 0$$

Find the solution

$$m = -4$$

Substitute into the general solution formula

$$y = (A + Bx)e^{-4x}$$

# 3 Imaginary only roots $\mathbf{b^2 < 4ac}$

This is when the AI has roots of form $\pm\alpha i$

$$\text{General solution:} \quad y = A\cos(\alpha x) + B\sin(\alpha x)$$

# 4 Complex roots $b^2 < 4ac$

This is used when the root is in the form $\beta \pm \alpha i$

$$\text{General solution:} \quad y = e^{\beta x}(A\cos(\alpha x) + B\sin(\alpha x))$$

## 4.1 Example

*Find the general solution of:*

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 34y = 0$$

Find the auxiliary equation

$$m^2 - 6m + 34 = 0$$

Solve to find roots

$$\text{Roots} = \frac{6 \pm \sqrt{36 - 4 \times 1 \times 34}}{2} = \frac{6 \pm 10i}{2} = 3 \pm 5i$$

Substitute into general solution formula

$$y = e^{3x}(A\cos(5x) + B\sin(5x))$$