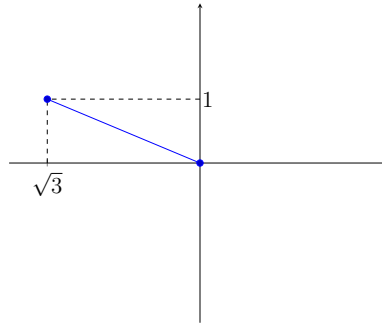


Further Complex Numbers

1 Expressions of complex numbers

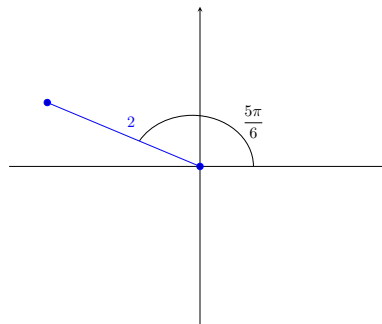
1.1 $x + iy$

This expresses the coordinate of the point at the end of the vector on the argand diagram.



1.2 $r(\cos \theta + i \sin \theta)$

This expresses the length of the line and the angle anticlockwise from the positive x axis



$$2 \left(\cos \left(\frac{5\pi}{6} \right) + i \sin \left(\frac{5\pi}{6} \right) \right)$$

1.3 $re^{i\theta}$

This uses the same parameters as $r(\cos \theta + i \sin \theta)$

2 Multiplying and dividing complex numbers

2.1 Multiplying

2.1.1 Trigonometric form

$$\begin{aligned} Z_1 Z_2 &= r_1(\cos \theta_1 + i \sin \theta_1) \times r_2(\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2 (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2) \end{aligned}$$

Apply the cos addition formula to the first two terms

$$= r_1 r_2 (\cos(\theta_1 + \theta_2) + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2)$$

Apply the sin addition formula to the last two terms

$$= r_1 r_2 (\cos(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2))$$

2.1.2 Exponential form

$$Z_1 Z_2 = r_1 e^{i\theta_1} \times r_2 e^{i\theta_2}$$

Apply laws of indices

$$Z_1 Z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

2.2 Dividing

2.2.1 Trigonometric form

$$\frac{Z_1}{Z_2} = \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)}$$

Multiply by the complex conjugate

$$\frac{Z_1}{Z_2} = \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} \times \frac{\cos \theta_2 - i \sin \theta_2}{\cos \theta_2 - i \sin \theta_2}$$

Expand

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \times \frac{\cos \theta_1 \cos \theta_2 - i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2}{\cos^2 \theta_2 - i \cos \theta_2 \sin \theta_2 + i \sin \theta_2 \cos \theta_2 + \sin^2 \theta_2}$$

Simplify

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \times (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))$$

2.2.2 Exponential form

$$\begin{aligned} \frac{Z_1}{Z_2} &= \frac{r_1}{r_2} \times \frac{e^{i\theta_1}}{e^{i\theta_2}} \\ \frac{Z_1}{Z_2} &= \frac{r_1}{r_2} \times e^{i(\theta_1 - \theta_2)} \end{aligned}$$

2.3 Comparison

Multiplying	Dividing
Multiply modulus, add arguments	Divide modulus, subtract arguments

3 De Moivre's Theorem

$$[r(\cos \theta + i \sin \theta)]^n = r^n(\cos(n\theta) + i \sin(n\theta))$$

3.1 Positive proof

Prove true for n=1

$$r(\cos \theta + i \sin \theta) = r(\cos \theta + i \sin \theta)$$

True for n=1

Assume true for n=k

$$[r(\cos \theta + i \sin \theta)]^k = r^k(\cos(k\theta) + i \sin(k\theta))$$

Prove true for n=k+1

$$\begin{aligned} & [r(\cos \theta + i \sin \theta)]^{k+1} \\ & [r(\cos \theta + i \sin \theta)]^k \times (r(\cos \theta + i \sin \theta))^1 \\ & r^k(\cos(k\theta) + i \sin(k\theta)) \times r(\cos \theta + i \sin \theta) \\ & r^k r(\cos(k\theta + \theta) + i \sin(k\theta + \theta)) \\ & r^{k+1}(\cos((k+1)\theta) + i \sin((k+1)\theta)) \end{aligned}$$

True

3.2 Negative proof

n=-m

$$[r(\cos \theta + i \sin \theta)]^{-m}$$

Multiply by complex conjugate

$$\frac{1}{[r(\cos \theta + i \sin \theta)]^m} \times \frac{[r(\cos \theta - i \sin \theta)]^m}{[r(\cos \theta - i \sin \theta)]^m}$$

Apply positive De Moivre's Theorem

$$\frac{r^m(\cos(m\theta) - i \sin(m\theta))}{r^m(\cos(m\theta) + i \sin(m\theta)) \times r^m(\cos(m\theta) - i \sin(m\theta))}$$

Simplify and expand

$$\frac{\cos(m\theta) - i \sin(m\theta)}{r^m(\cos^2 m\theta - i \cos m\theta \sin m\theta + i \cos m\theta \sin m\theta + \sin^2 m\theta)}$$

Simplify

$$\frac{\cos m\theta - i \sin m\theta}{r^m} = r^{-m}(\cos m\theta - i \sin m\theta)$$

Rewrite

$$r^{-m}(\cos(-m\theta) + i \sin(-m\theta))$$

Replace -m with n

$$r^n(\cos(n\theta) + i \sin(n\theta))$$

3.3 Applying De Moivres' Theorem

We can use the binomial expansion along with De Moivre's theorem to rewrite trigonometric expressions. This can be useful when integrating etc.

3.3.1 Example 1

Rewrite $\cos 5\theta$ in powers of $\cos \theta$

$$(\cos \theta + i \sin \theta)^5$$

Apply DM theorem

$$\cos 5\theta + i \sin 5\theta$$

Apply the binomial expansion to the initial expression

$$\cos^5 \theta + 5 \cos^4 \theta i \sin \theta + 10 \cos^3 \theta (i \sin \theta)^2 + 10 \cos^2 \theta (i \sin \theta)^3 + 5 \cos \theta (i \sin \theta)^4 + (i \sin \theta)^5$$

Simplify

$$\cos^5 \theta + 5i \cos^4 \theta \sin \theta - 10 \cos^3 \theta \sin^2 \theta - 10i \cos^2 \theta \sin^3 \theta + 5 \cos \theta \sin^4 \theta + i \sin^5 \theta$$

Equate the real parts of this to the real parts of the result from DM theorem

$$\cos 5\theta = \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta$$

Replace sin terms with cos

$$\cos 5\theta = \cos^5 \theta - 10 \cos^3 \theta (1 - \cos^2 \theta) + 5 \cos \theta (1 - \cos^2 \theta)^2$$

Simplify

$$\cos 5\theta = \cos^5 \theta - 10 \cos^3 \theta + 10 \cos^5 \theta + 5 \cos \theta (1 - \cos \cos^2 \theta + \cos^4 \theta)$$

Simplify further

$$\cos 5\theta = \cos^5 \theta - 10 \cos^3 \theta + 10 \cos^5 \theta + 5 \cos \theta - 10 \cos^3 \theta + 5 \cos^5 \theta$$

Collect terms

$$\cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta$$

3.3.2 Z formulas

If $z = \cos \theta + i \sin \theta$

$$z + \frac{1}{z} = 2 \cos \theta$$

$$z - \frac{1}{z} = 2i \sin \theta$$

$$z^n + \frac{1}{z^n} = 2 \cos n\theta$$

$$z^n - \frac{1}{z^n} = 2i \sin n\theta$$

3.3.2.1 Example

Express $\cos^5 \theta$ in the form $a \cos 5\theta + b \cos 3\theta + c \cos \theta$ Create the situation using the z formulas

$$\left(z + \frac{1}{z}\right)^5 = (2 \cos \theta)^5 = 32 \cos^5 \theta$$

Expand using the binomial

$$z^5 + 5z^4 \times \frac{1}{z} + 10z^3 \times \frac{1}{z^2} + 10z^2 \times \frac{1}{z^3} + 5z \times \frac{1}{z^4} + \frac{1}{z^5}$$

Combine like coloured term using z formula

$$32 \cos^5 \theta = 2 \cos \theta + 10 \cos 3\theta + 20 \cos \theta$$

$$\cos^5 \theta = \frac{\cos \theta}{16} + \frac{5 \cos 3\theta}{16} + \frac{5 \cos \theta}{8}$$

4 Solving complex equations

For any complex number, we generally define the argument to be between $-\pi$ and π . However we can add multiples of 2π to the argument to get equivalent answers.

We shall use this fact to find all solutions to complex equations.

Note: The number of solutions is equal to the order of the equation.

4.1 Example

Solve:

$$z^5 = i$$

$$r = 1 \quad \theta = \frac{\pi}{2}$$

$$z^5 = \cos\left(\frac{\pi}{2} + 2k\pi\right) + i \sin\left(\frac{\pi}{2} + 2k\pi\right)$$

$$z = \cos\left(\frac{\frac{\pi}{2} + 2k\pi}{5}\right) + i \sin\left(\frac{\frac{\pi}{2} + 2k\pi}{5}\right)$$

5 Solutions: $k = -2, -1, 0, 1, 2$

$$k = -2 \quad z = -0.588 - 0.809i$$

$$k = -1 \quad z = 0.588 - 0.809i$$

$$k = 0 \quad z = 0.951 + 0.309i$$

$$k = 1 \quad z = i$$

$$k = 2 \quad z = -0.951 + 0.309i$$

5 Loci on the complex plane

5.1 Lines

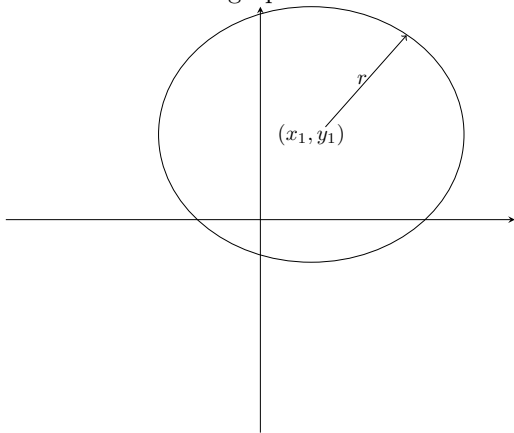
5.1.1 $|z - z_1| = r$

$|z - z_1| = r$ is represented by a circle, centre (x_1, y_1) with a radius r , where $z_1 = x_1 + iy_1$.

This is the same as the Cartesian equation:

$$(x - x_1)^2 + (y - y_1)^2 = r^2$$

This looks like the graph:

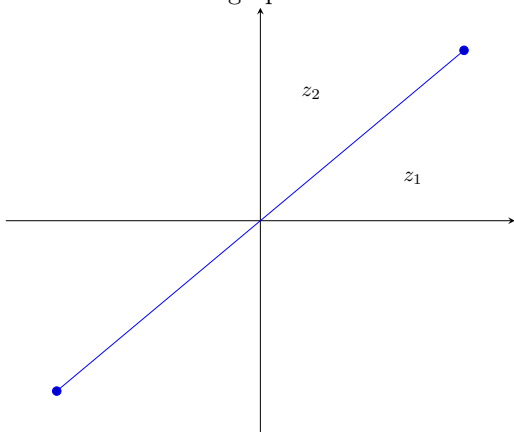


5.1.2 $|z - z_1| = |z - z_2|$

$|z - z_1| = |z - z_2|$ is represented by a perpendicular bisector of the line segment joining points z_1 and z_2 .

To find the Cartesian form, replace z with $x + iy$ and expand, squaring both sides to remove the modulus signs.

This looks like the graph:



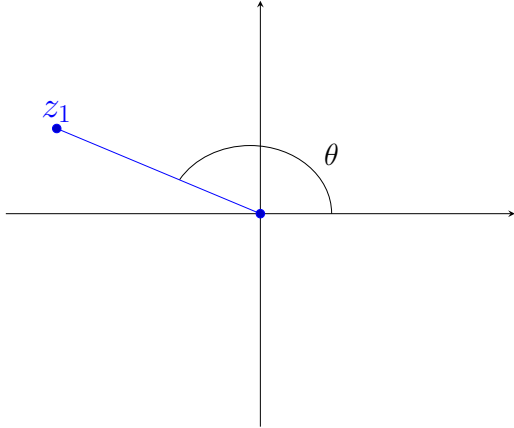
For cases where there is a constant in front of one of the modulus signs, use the algebraic method to find the Cartesian form, then plot that.

5.1.3 $\arg(\mathbf{z} - \mathbf{z}_1) = \theta$

$\arg(z - z_1) = \theta$ is represented by the half-line from the fixed point z_1 , making an angle θ with a line from the fixed point z_1 , parallel to the real axis.

To find the Cartesian form, replace z with $x + iy$ and separate into the real and imaginary parts. Then use the trigonometric identity $\tan \theta = \frac{\text{Opposite}}{\text{Adjacent}}$ and simplify.

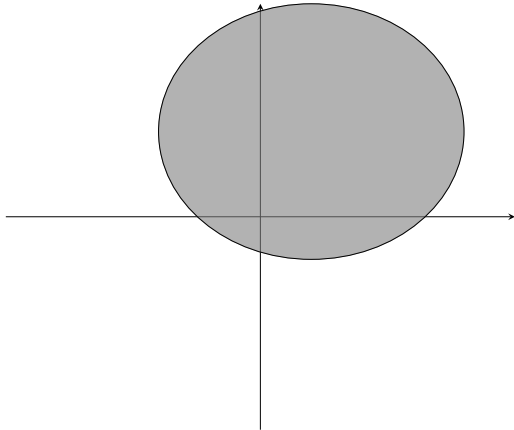
When plotting remember that θ is the angle anticlockwise.

**5.1.4** $\arg\left(\frac{\mathbf{z} - \mathbf{z}_1}{\mathbf{z} - \mathbf{z}_2}\right)$

$\arg\left(\frac{z - z_1}{z - z_2}\right)$ represents an arc anticlockwise between z_1 and z_2

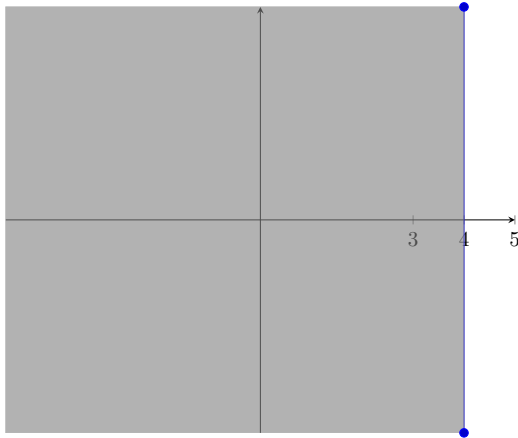
5.2 Regions**5.2.1** $|\mathbf{z} - (2 + 3\mathbf{i})| \leq 3$

This is the area inside the circle represented by $|z - (2 + 3i)| \leq 3$

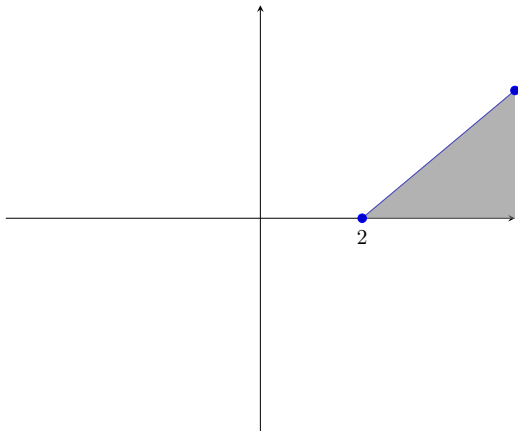


5.2.2 $|z - 3| < |z - 5|$

This is the area to one side of the perpendicular bisector represented equal to $|z - 3| = |z - 5|$

**5.2.3** $0 \leq \arg(z - 2) < \frac{\pi}{4}$

This is the area between the positive real axis and the half line equal to $\arg(z - 2) = \frac{\pi}{4}$

**6 Translations**

- $w = z + a + ib$ represents a translation with translation vector $\begin{pmatrix} a \\ b \end{pmatrix}$
- $w = kz$ represents an enlargement with scale factor k centre $(0, 0)$
- $w = kz + a + ib$ represents an enlargement scale factor k centre $(0, 0)$ followed by a translation with translation vector $\begin{pmatrix} a \\ b \end{pmatrix}$
- $w = z^2$ multiply a shape by itself, for example a circle of radius 4 would go to radius 16