

C3

1 Algebraic fractions

Any polynomial $F(x)$ can be put in the form:

$$F(x) = Q(x) \times \text{divisor} + \text{remainder}$$

Where $Q(x)$ is the quotient

2 Functions

2.1 Range and domain

The input to a function is the **Domain**

The output from a function is the **Range**

2.2 Function mapping

One-to-one function: One element in the domain maps to one element in the range

Many-to-one function: To elements of the domain maps to one element in the range

Not a function: One input maps to two outputs

2.3 Mappings to functions by changing the domain

Consider $y = \sqrt{x}$

If the domain is all the real numbers $x \in \mathbb{R}$ then it is not a function as values less than 0 don't get mapped anywhere.

The domain must be restricted to $x \geq 0$

2.4 Combining functions

$fg(x)$ means apply g to x , then apply f

$f^2(x)$ means $ff(x)$

2.5 Inverse functions

The inverse of $f(x)$ is written as $f^{-1}x$

The domain of $f(x)$ is the range of $f^{-1}x$

The range of $f(x)$ is the domain of $f^{-1}x$

Example, find the inverse function of $y = 2x^2 - 7$:

$$y + 7 = 2x^2$$

$$\frac{y + 7}{2} = x^2$$

$$x = \sqrt{\frac{y + 7}{2}}$$

$$f^{-1}x = \sqrt{\frac{x + 7}{2}}$$

When finding the graph of an inverse function, reflect $f(x)$ in the line $y=x$.

3 The exponential and log functions

3.1 Exponential functions

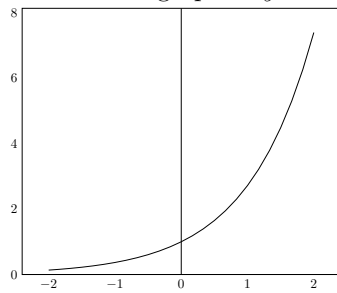
Exponential functions are in the form $y = a^x$, graphs of these functions all pass through (0,1) as $a^0 = 1$ for any value of a .

3.2 Functions including e

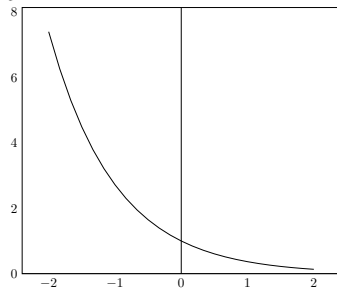
The function $y = e^x$ is the function where the gradient is identical to the function.

$$y = e^x \frac{dy}{dx} = e^x$$

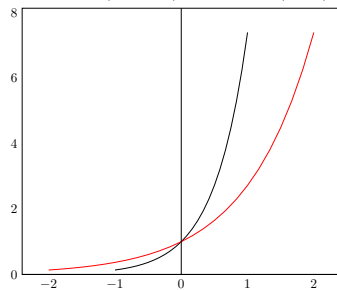
Below is the graph of $y = e^x$



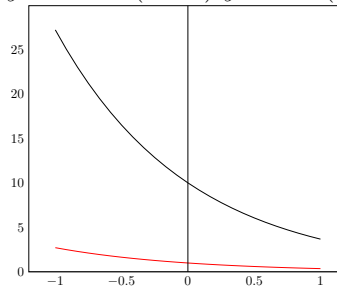
$y = e^{-x}$



$y = e^{2x}$ (black) $y = e^x$ (red)



$y = 10e^{-x}$ (black) $y = e^{-x}$ (red)



3.3 Formulas for exponential growth or decay

Example:

$$P = 16000e^{-\frac{t}{10}}$$

Where P is the Price in £s and t is the years from new

What was the price when new?

Substitute $t=0$

$$P = 16000e^{-\frac{0}{10}}$$

$$P = 16000 \times 1$$

What is the value at 5 years old

Substitute $t=5$

$$P = 16000e^{-\frac{5}{10}}$$

$$P = \pounds 9704.49$$

What does the model say about the eventual value of the car

As $t \rightarrow \infty$, $e^{-\frac{t}{10}} \rightarrow 0$

Therefore $P \rightarrow 16000 \times 0 = 0$

The eventual value is zero.

3.4 The inverse of the exponential function

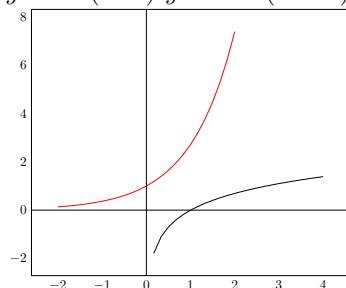
The inverse of e^x is $\log_e x$ (also written as $\ln x$)

Examples:

If $e^x = 3$ then $x = \ln 3$

If $\ln x = 4$ then $x = e^4$

$y = e^x$ (Red) $y = \ln x$ (Black)



The function $f(x) = \ln x$ has domain $\{x \in \mathbb{R}, x > 0\}$ and range $\{f(x) \in \mathbb{R}\}$

4 Numerical methods

4.1 Approximations for roots based on graphs

Approximations for roots can be found graphically by plotting the function and finding where the line crosses the x axis. This value is one of the roots of the function.

If trying to find a range in which a root can be found, substitute the values at the extreme of the range, and if there is a change in sign between the two results, there will be a root in the range.

The exception to this rule is $f(x) = \frac{1}{x}$ and transformations of this as there is a discontinuity at $x=0$. The function changes sign in the interval that includes $x=0$, but there is not a root.

4.2 Iteration for finding approximations of roots

To solve an equation of the form $f(x) = 0$ by an iterative method, rearrange $f(x) = 0$ into a form $x = g(x)$ and use the iterative formula $x_{n+1} = g(x_n)$.

Example, find a root of the equation $x^2 - 4x + 1 = 0$

Re-write as $x = 4 - \frac{1}{x}$

Create the formula $x_{n+1} = 4 - \frac{1}{x_n}$

You get given a rough approximation, $x_0 = 3$

Substitute

$$x_1 = 4 - \frac{1}{x_0}$$

$$x_1 = 4 - \frac{1}{3}$$

$$x_1 = \frac{11}{3}$$

$$x_2 = 4 - \frac{1}{\frac{11}{3}}$$

$$x_2 = \frac{41}{11}$$

Continuing this increases the accuracy of the result.

This may not work and will not converge to a root.

5 Transforming graphs of functions

5.1 $y = |f(x)|$ Graphs

The modulus of a number is written as $|a|$, this is the positive numerical value.

When $f(x) \geq 0$, $|f(x)| = f(x)$

When $f(x) < 0$, $|f(x)| = -f(x)$

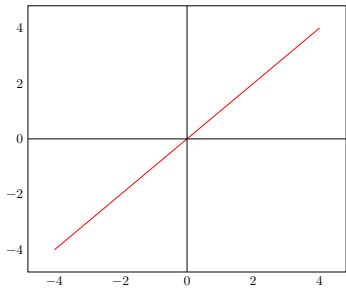


Figure 1: $y = x$

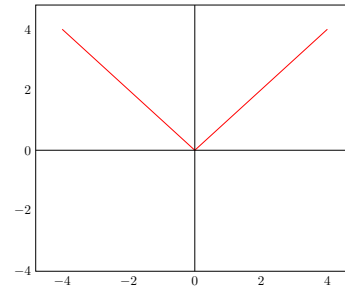


Figure 2: $y = |x|$

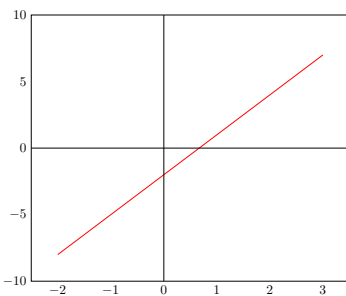


Figure 3: $y = 3x - 2$

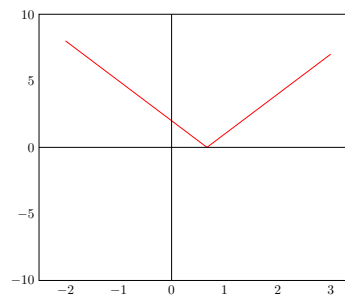


Figure 4: $y = |3x - 2|$

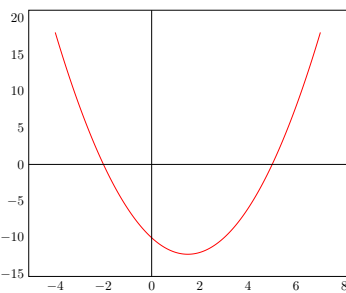


Figure 5: $y = x^2 - 3x - 10$

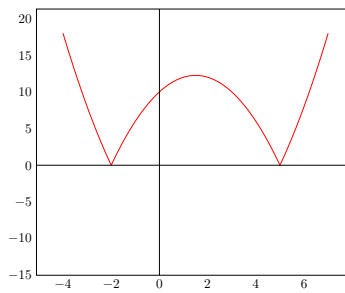
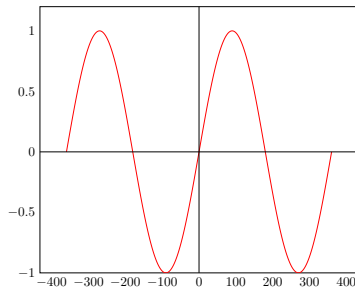
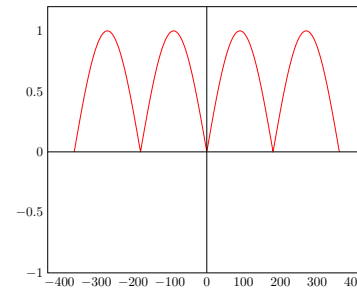
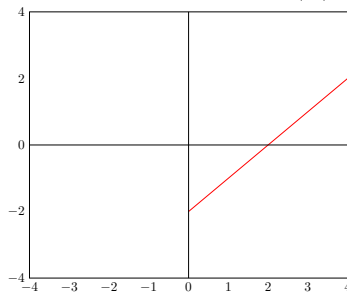


Figure 6: $y = |x^2 - 3x - 10|$

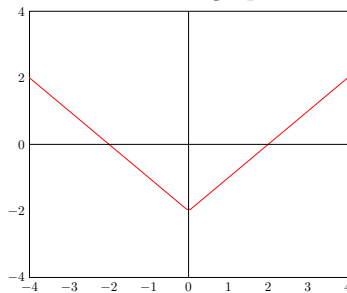
Figure 7: $y = \sin(x)$ Figure 8: $y = |\sin(x)|$

5.2 $y = f(|x|)$ Graphs

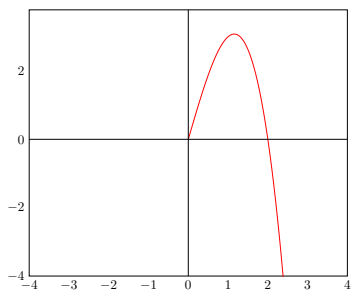
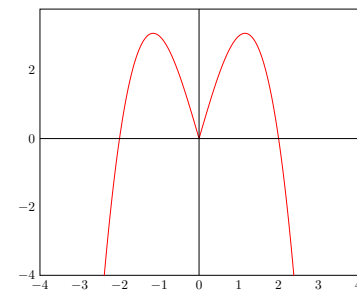
To plot the graph of $y = |x| - 2$, first sketch the graph of $y = x - 2$ for $x \geq 0$:



Then reflect that graph in the y axis



Examples:

Figure 9: $y = 4x - x^3$ Figure 10: $y = 4|x| - |x|^3$

5.3 Graph transformations

- $f(x + a)$ - Horizontal translation of $-a$
- $f(x) + a$ - Vertical translation of $+a$
- $f(ax)$ - Horizontal stretch of scale factor $\frac{1}{a}$
- $f(-x)$ - Reflection in the **y** axis

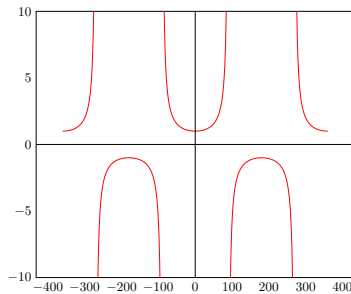
- $af(x)$ - Vertical stretch of scale factor **a**
- $-f(x)$ - Reflection in the **x axis**

6 Trigonometry

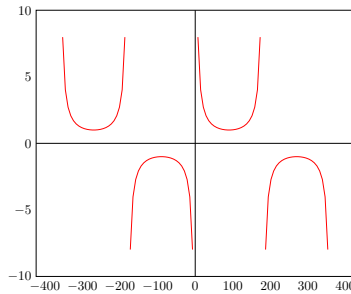
- $\sec \theta = \frac{1}{\cos \theta}$
- $\csc \theta = \frac{1}{\sin \theta} = \operatorname{cosec} \theta$
- $\cot \theta = \frac{1}{\tan \theta}$
- $\cot \theta = \frac{\cos \theta}{\sin \theta}$

6.1 Graphs of the new functions

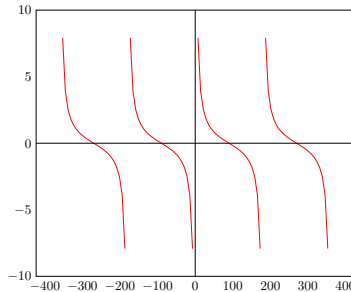
$\sec(x)$



$\csc(x)$



$\cot(x)$

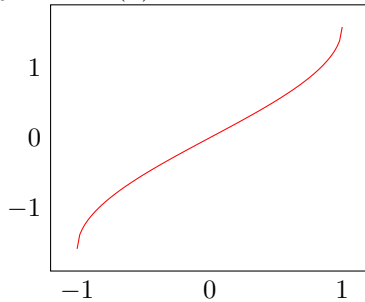


6.2 New identities

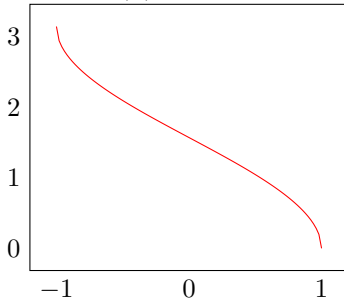
- $1 + \tan^2 \theta = \sec^2 \theta$
- $1 + \cot^2 \theta = \csc^2 \theta$

6.3 Graphs of inverse functions

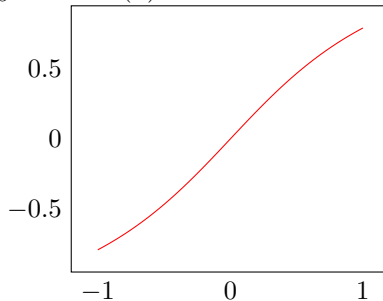
$$y = \arcsin(x)$$



$$y = \arccos(x)$$



$$y = \arctan(x)$$



7 Further trigonometric identities and their applications

- $\sin(a \pm b) \equiv \sin A \cos B \pm \cos A \sin B$
- $\cos(a \pm b) \equiv \cos A \cos B \mp \sin A \sin B$
- $\tan(a \pm b) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

7.1 Double angle formulae

- $\sin 2A \equiv 2 \sin A \cos A$
- $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2 \cos^2 A - 1 \equiv 1 - 2 \sin^2 A$
- $\tan 2A \equiv \frac{2 \tan A}{1 - \tan^2 A}$

7.2 The R formula

For positive values of a and b

$a \sin \theta \pm b \cos \theta$ can be expressed in the form $R \sin(\theta \pm \alpha)$, where $0 < \alpha < 90$

$a \cos \theta \pm b \sin \theta$ can be expressed in the form $R \cos(\theta \mp \alpha)$, where $0 < \alpha < 90$

$R \cos \alpha = a$, $R \sin \alpha = b$

$R = \sqrt{a^2 + b^2}$

8 Differentiation

8.1 The chain rule

If $y = [f(x)]^n$ then $\frac{dy}{dx} = n[f(x)]^{n-1} f'(x)$

If $y = f[g(x)]$ then $\frac{dy}{dx} = f'[g(x)]g'(x)$

Example

$$f(x) = (3x^4 + x)^5$$

$$f'(x) = 12x^3 + 1$$

$$\frac{dy}{dx} = 5(3x^4 + x)^4(12x^3 + 1)$$

8.1.1 Another form of the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Example

$$y = (x^2 - 7x)^4$$

$$u = (x^2 - 7x)^4$$

$$y = u^4$$

$$\frac{du}{dx} = 2x - 7$$

$$\frac{dy}{du} = 4u^3$$

Using the chain rule:

$$\frac{dy}{dx} = 4u^3 \times (2x - 7)$$

$$\frac{dy}{dx} = 4(2x - 7)(x^2 - 7x)^3$$

8.2 The product rule

The product rule is used to differentiate the product of two functions.

If $y = uv$ then $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

Example:

$$f(x) = x^2 \sqrt{3x - 1}$$

$$u = x^2, v = (3x - 1)^{\frac{1}{2}}$$

$$\frac{du}{dx} = 2x \text{ and } \frac{dv}{dx} = \frac{3}{2}(3x - 1)^{-\frac{1}{2}}$$

$$f'(x) = x^2 \times \frac{3}{2}(3x - 1)^{-\frac{1}{2}} + \sqrt{3x - 1} \times 2x$$

$$f'(x) = \frac{15x^2 - 4x}{2\sqrt{3x - 1}}$$

$$f'(x) = \frac{x(15x - 4)}{2\sqrt{3x - 1}}$$

8.3 The quotient rule

If $y = \frac{u(x)}{v(x)}$ then $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

Example

$$y = \frac{x}{2x + 5}$$

$$u = x, v = 2x + 5$$

$$\frac{du}{dx} = 1, \frac{dv}{dx} = 2$$

$$\frac{dy}{dx} = \frac{(2x + 5) \times 1 - x \times 2}{(2x + 5)^2}$$

$$\frac{dy}{dx} = \frac{5}{(2x+5)^2}$$

8.4 The exponential function

If $y = e^x$ then $\frac{dy}{dx} = e^x$

If $y = e^{f(x)}$ then $\frac{dy}{dx} = f'(x)e^{f(x)}$

Example

$$y = e^{2x+3}$$

$$\frac{dy}{dx} 2x + 3 = 2$$

$$\frac{dy}{dx} e^{2x+3} = 2e^{2x+3}$$

8.5 The logarithmic function

If $y = \ln(x)$ then $\frac{dy}{dx} = \frac{1}{x}$

If $y = \ln[f(x)]$ then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

Example

$$y = \ln(6x - 1)$$

$$\frac{dy}{dx} 6x - 1 = 6$$

$$\frac{dy}{dx} \ln(6x - 1) = \frac{6}{6x - 1}$$

8.6 Trig functions

8.6.1 Sin

If $y = \sin(x)$ then $\frac{dy}{dx} = \cos(x)$

If $y = \sin f(x)$ then $\frac{dy}{dx} = f'(x) \cos f(x)$

8.6.2 Cos

If $y = \cos(x)$ then $\frac{dy}{dx} = -\sin(x)$

If $y = \cos f(x)$ then $\frac{dy}{dx} = -f'(x) \sin f(x)$

8.6.3 Tan

If $y = \tan(x)$ then $\frac{dy}{dx} = \sec^2(x)$

If $y = \tan f(x)$ then $\frac{dy}{dx} = f'(x) \sec^2 f(x)$

8.6.4 Csc

If $y = \csc(x)$ then $\frac{dy}{dx} = -\csc(x) \cot(x)$

If $y = \csc f(x)$ then $\frac{dy}{dx} = -f'(x) \csc f(x) \cot f(x)$

8.6.5 Sec

If $y = \sec(x)$ then $\frac{dy}{dx} = \sec(x) \tan(x)$

If $y = \sec f(x)$ then $\frac{dy}{dx} = f'(x) \sec f(x) \tan f(x)$

8.6.6 Cot

If $y = \cot(x)$ then $\frac{dy}{dx} = -\csc^2(x)$

If $y = \cot f(x)$ then $\frac{dy}{dx} = -f'(x) \csc^2 f(x)$

Differentiation Table

y	$\frac{dy}{dx}$
$[f(x)]^n$	$n(f(x))^{n-1}f'(x)$
uv	$u\frac{dv}{dx} + v\frac{du}{dx}$
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
$e^{f(x)}$	$f'(x)e^{f(x)}$
$\ln f(x) $	$\frac{f'(x)}{f(x)}$
$\sin f(x)$	$f'(x) \cos f(x)$
$\cos f(x)$	$-f'(x) \sin f(x)$
$\tan f(x)$	$f'(x) \sec^2 f(x)$
$\csc f(x)$	$-f'(x) \csc f(x) \cot f(x)$
$\sec f(x)$	$f'(x) \sec f(x) \tan f(x)$
$\cot f(x)$	$-f'(x) \csc^2 f(x)$