C3

1 Algebraic fractions

Any polynomial $F(x)$ can be put in the form:
$F(x) = Q(x) \times$divisor+remainder
Where Q(x) is the quotient

2 Functions

2.1 Range and domain

The input to a function is the **Domain**

The output from a function is the **Range**

2.2 Function mapping

One-to-one function: One element in the domain maps to one element in the range
Many-to-one function: To elements of the domain maps to one element in the range
Not a function: One input maps to two outputs

2.3 Mappings to functions by changing the domain

Consider $y = \sqrt{x}$
If the domain is all the real numbers $x \in \Re$ then it is not a function as values less than 0 don't get mapped anywhere.
The domain must be restricted to $x \geq 0$

2.4 Combining functions

$fg(x)$ means apply g to x, then apply f
$f^2(x)$ means $ff(x)$

2.5 Inverse functions

The inverse of $f(x)$ is written as $f^{-1}x$
The domain of $f(x)$ is the range of $f^{-1}x$
The range of $f(x)$ is the domain of $f^{-1}x$
Example, find the inverse function of $y = 2x^2 - 7$:
$y + 7 = 2x^2$

$\dfrac{y+7}{2} = x^2$

$x = \sqrt{\dfrac{y+7}{2}}$

$f^{-1}x = \sqrt{\dfrac{x+7}{2}}$

When finding the graph of an inverse function, reflect $f(x)$ in the line y=x.

3 The exponential and log functions

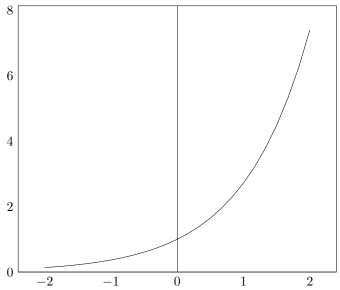
3.1 Exponential functions

Exponential functions are in the form $y = a^x$, graphs of these functions all pass through (0,1) as $a^0 = 1$ for any value of a.

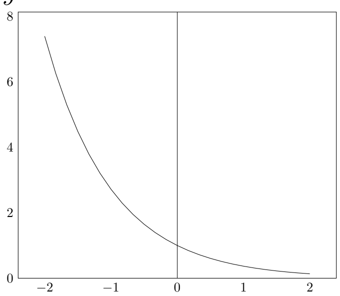
3.2 Functions including e

The function $y = e^x$ is the function where the gradient is identical to the function.
$y = e^x \dfrac{dy}{dx} = e^x$

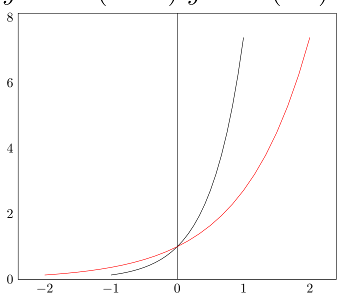
Below is the graph of $y = e^x$



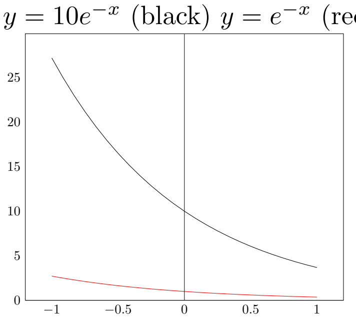
$y = e^{-x}$



$y = e^{2x}$ (black) $y = e^x$ (red)



$y = 10e^{-x}$ (black) $y = e^{-x}$ (red)



3.3 Formulas for exponential growth or decay

Example:

$P = 16000e^{-\frac{t}{10}}$

Where P is the Price in £s and t is the years from new

What was the price when new?

Substitute t=0

$$P = 16000e^{-\frac{0}{10}}$$

$$P = 16000 \times 1$$

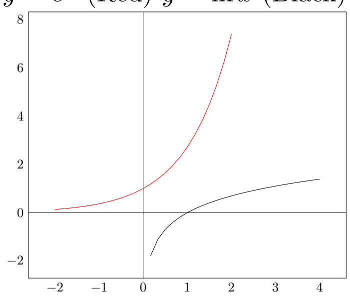
What is the value at 5 years old
Substitute t=5

$P = 16000e^{-\frac{5}{10}}$
$P = £9704.49$
What does the model say about the eventual value of the car
As $t \to \infty$, $e^{-\frac{t}{10}} \to \infty$
Therefore $P \to 16000 \times 0 = 0$
The eventual value is zero.

3.4 The inverse of the exponential function

The inverse of e^x is $\log_e x$ (also written as $\ln x$)
Examples:
If $e^x = 3$ then $x = \ln 3$
If $\ln x = 4$ then $x = e^4$

$y = e^x$ (Red) $y = \ln x$ (Black)



The function $f(x) = \ln x$ has domain $\{x \in \mathbb{R},\, x > 0\}$ and range $\{f(x) \in \mathbb{R}\}$

4 Numerical methods

4.1 Approximations for roots based on graphs

Approximations for roots can be found graphically by plotting the function and finding where the line crosses the x axis. This value is one of the roots of the function.
If trying to find a range in which a root can be found, substitute the values at the extreme of the range, and if there is a change in sign between the two results, there will be a root in the range.
The exception to this rule is $f(x) = \dfrac{1}{x}$ and transformations of this as there is a discontinuity at x=0. The function changes sign in the interval that includes x=0, but there is not a root.

4.2 Iteration for finding approximations of roots

To solve an equation of the form $f(x) = 0$ by an iterative method, rearrange $f(x) = 0$ into a for $x = g(x)$ and use the iterative formula $x_{n+1} = g(x_n)$.

Example, find a root of the equation $x^2 - 4x + 1 = 0$
Re-write as $x = 4 - \dfrac{1}{x}$
Create the formula $x_{n+1} = 4 - \dfrac{1}{x_n}$
You get given a rough approximation, $x_0 = 3$
Substitute
$x_1 = 4 - \dfrac{1}{x_0}$
$x_1 = 4 - \dfrac{1}{3}$
$x_1 = \dfrac{11}{3}$

$x_2 = 4 - \dfrac{1}{\frac{11}{3}}$

$x_2 = \frac{41}{11}$
Continuing this increases the accuracy of the result.
This may not work and will not converge to a root.

5 Transforming graphs of functions

5.1 $\mathbf{y = |f(x)|}$ Graphs

The modulus of a number is written as $|a|$, this is the positive numerical value.
When $f(x) \geq 0, |f(x)| = f(x)$
When $f(x) < 0, |f(x)| = -f(x)$

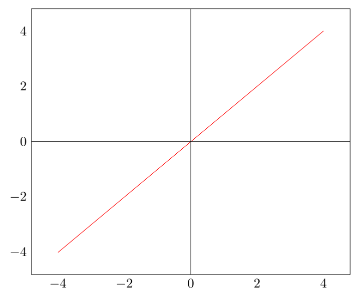


Figure 1: $y = x$

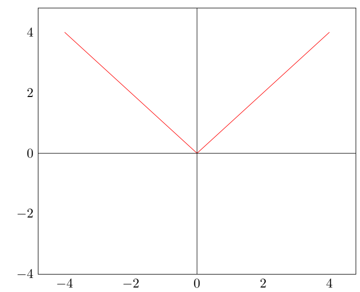


Figure 2: $y = |x|$

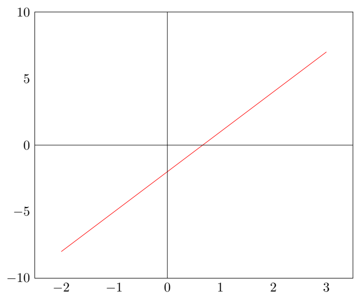


Figure 3: $y = 3x - 2$

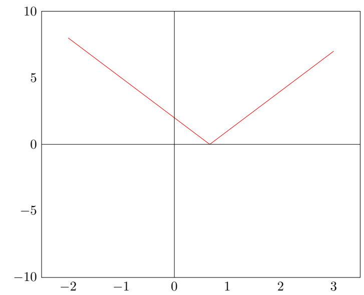


Figure 4: $y = |3x - 2|$

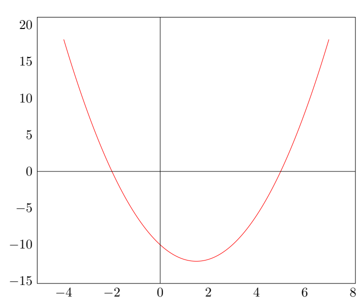


Figure 5: $y = x^2 - 3x - 10$

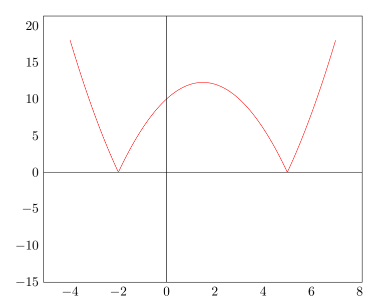


Figure 6: $y = |x^2 - 3x - 10|$

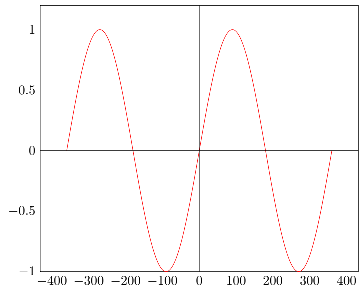


Figure 7: $y = sin(x)$

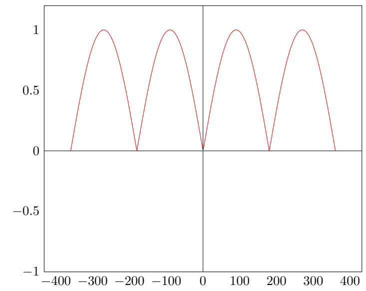
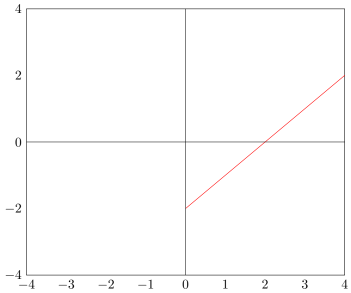


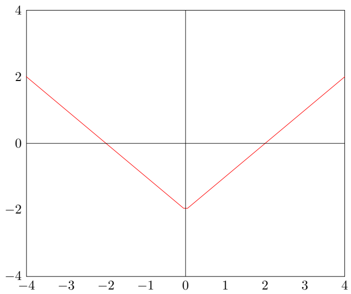
Figure 8: $y = |sin(x)|$

5.2 $\mathbf{y = f(|x|)}$ Graphs

To plot the graph of $y = |x| - 2$, first sketch the graph of $y = x - 2$ for $x \geq 0$:



Then reflect that graph in the y axis



Examples:

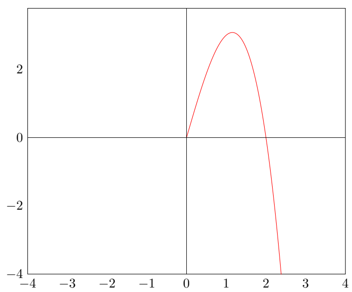


Figure 9: $y = 4x - x^3$

Figure 10: $y = 4|x| - |x|^3$

5.3 Graph transformations

- $f(x + a)$ - Horizontal translation of $-\mathbf{a}$
- $f(x) + a$ - Vertical translation of $+\mathbf{a}$
- $f(ax)$ - Horizontal stretch of scale factor $\mathbf{\frac{1}{a}}$
- $f(-x)$ - Reflection in the **y axis**

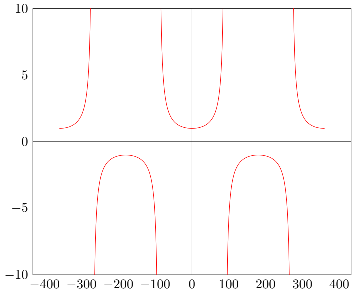
- $af(x)$ - Vertical stretch of scale factor **a**
- $-f(x)$ - Reflection in the **x axis**

6 Trigonometry

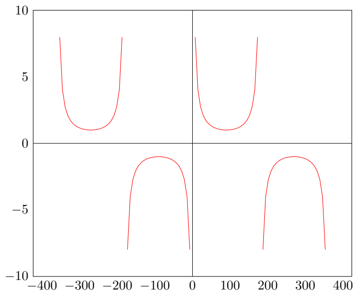
- $\sec\theta = \dfrac{1}{\cos\theta}$
- $\csc\theta = \dfrac{1}{\sin\theta} = cosec\theta$
- $\cot\theta = \dfrac{1}{\tan\theta}$
- $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

6.1 Graphs of the new functions

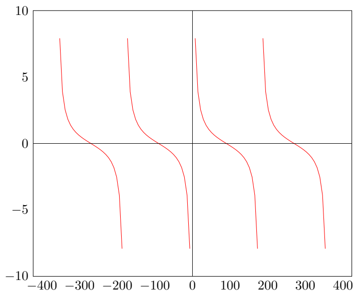
$\sec(x)$



$\csc(x)$



$\cot(x)$

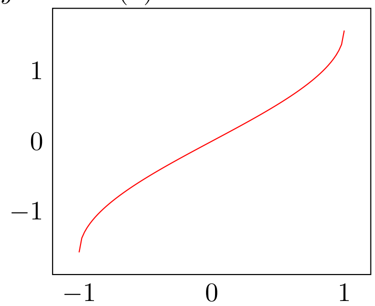


6.2 New identities

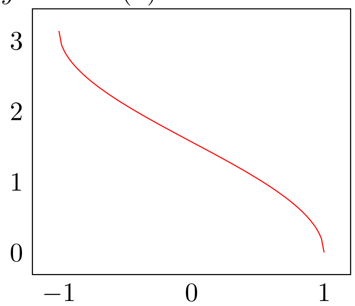
- $1 + \tan^2\theta = \sec^2\theta$
- $1 + \cot^2\theta = \csc^2\theta$

6.3 Graphs of inverse functions

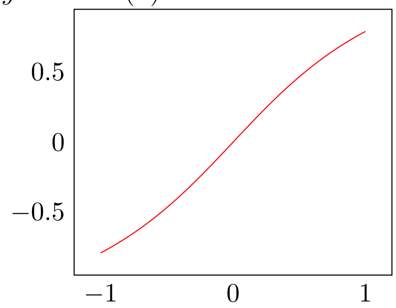
$y = \arcsin(x)$



$y = \arccos(x)$



$y = \arctan(x)$



7 Further trigonometric identities and their applications

- $\sin(a \pm b) \equiv \sin A \sin B \pm \cos A \sin B$
- $\cos(a \pm b) \equiv \cos A \cos B \mp \sin A \sin B$
- $\tan(a \pm b) \equiv \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

7.1 Double angle formulae

- $\sin 2A \equiv 2 \sin A \cos A$
- $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$
- $\tan 2A \equiv \dfrac{2 \tan A}{1 - \tan^2 A}$

7.2 The R formula

For positive values of a and b
$a \sin\theta \pm b \cos\theta$ can be expressed in the form $R\sin(\theta \pm \alpha)$, where $0 < \alpha < 90$
$a \cos\theta \pm b \sin\theta$ can be expressed in the form $R\cos(\theta \mp \alpha)$, where $0 < \alpha < 90$
$R\cos\alpha = a$, $R\sin\alpha = b$
$R = \sqrt{a^2 + b^2}$

8 Differentiation

8.1 The chain rule

If $y = [f(x)]^n$ then $\dfrac{dy}{dx} = n[f(x)]^{n-1}f'x)$

If $y = f[g(x)]$ then $\dfrac{dy}{dx} = f'[g(x)]g'(x)$

Example

$f(x) = (3x^4 + x)^5$

$f'(x) = 12x^3 + 1$

$\dfrac{dy}{dx} = 5(3x^4 + x)^4(12x^2 + 1)$

8.1.1 Another form of the chain rule

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

Example

$y = (x^2 - 7x)^4$

$u = (x^2 - 7x)^4$

$y = u^4$

$\dfrac{du}{dx} = 2x - 7$

$\dfrac{dy}{du} = 4u^3$

Using the chain rule:

$\dfrac{dy}{dx} = 4u^3 \times (2x - 7)$

$\dfrac{dy}{dx} = 4(2x - 7)(x^2 - 7x)^3$

8.2 The product rule

The product rule is used to differentiate the product of two functions.

If $y = uv$ then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

Example:

$f(x) = x^2\sqrt{3x - 1}$

$u = x^2$, $v = (3x - 1)^{\frac{1}{2}}$

$\dfrac{du}{dx} = 2x$ and $\dfrac{dv}{dx} = \dfrac{3}{2}(3x - 1)^{-\frac{1}{2}}$

$f'(x) = x^2 \times \frac{3}{2}(3x - 1)^{-\frac{1}{2}} + \sqrt{3x - 1} \times 2x$

$f'(x) = \dfrac{15x^2 - 4x}{2\sqrt{3x - 1}}$

$f'(x) = \dfrac{x(15x - 4)}{2\sqrt{3x - 1}}$

8.3 The quotient rule

If $y = \dfrac{u(x)}{v(x)}$ then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Example

$y = \dfrac{x}{2x + 5}$

$u = x, v = 2x + 5$

$\dfrac{du}{dx} = 1$, $\dfrac{dv}{dx} = 2$

$\dfrac{dy}{dx} = \dfrac{(2x + 5) \times 1 - x \times 2}{(2x + 5)^2}$

$\frac{dy}{dx} = \frac{5}{(2x+5)^2}$

8.4 The exponential function

If $y = e^x$ then $\frac{dy}{dx} = e^x$
If $y = e^{f(x)}$ then $\frac{dy}{dx} = f'(x)e^{f(x)}$
Example
$y = e^{2x+3}$
$\frac{dy}{dx}2x + 3 = 2$
$\frac{dy}{dx}e^{2x+3} = 2e^{2x+3}$

8.5 The logarithmic function

If $y = \ln(x)$ then $\frac{dy}{dx} = \frac{1}{x}$
If $y = \ln[f(x)]$ then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
Example
$y = \ln(6x - 1)$
$\frac{dy}{dx}6x - 1 = 6$
$\frac{dy}{dx}\ln(6x - 1) = \frac{6}{6x - 1}$

8.6 Trig functions

8.6.1 Sin

If $y = \sin(x)$ then $\frac{dy}{dx} = \cos(x)$
If $y = \sin f(x)$ then $\frac{dy}{dx} = f'(x)\cos f(x)$

8.6.2 Cos

If $y = \cos(x)$ then $\frac{dy}{dx} = -\sin(x)$
If $y = \cos f(x)$ then $\frac{dy}{dx} = -f'(x)\sin f(x)$

8.6.3 Tan

If $y = \tan(x)$ then $\frac{dy}{dx} = \sec^2(x)$
If $y = \tan f(x)$ then $\frac{dy}{dx} = f'(x)\sec^2 f(x)$

8.6.4 Csc

If $y = \csc(x)$ then $\frac{dy}{dx} = -\csc(x)\cot(x)$
If $y = \csc f(x)$ then $\frac{dy}{dx} = -f'(x)\csc f(x)\cot f(x)$

8.6.5 Sec

If $y = \sec(x)$ then $\dfrac{dy}{dx} = \sec(x)\tan(x)$
If $y = \sec f(x)$ then $\dfrac{dy}{dx} = f'(x)\sec f(x)\tan f(x)$

8.6.6 Cot

If $y = \cot(x)$ then $\dfrac{dy}{dx} = -\csc^2(x)$
If $y = \cot f(x)$ then $\dfrac{dy}{dx} = -f'(x)\csc^2 f(x)$

Differentiation Table

y	$\frac{dy}{dx}$
$[f(x)]^n$	$n(f(x))^{n-1}f'(x)$
uv	$u\frac{dv}{dx}+v\frac{du}{dx}$
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$
$e^{f(x)}$	$f'(x)e^{f(x)}$
$\ln\lvert f(x)\rvert$	$\frac{f'(x)}{f(x)}$
$y=\sin f(x)$	$\frac{dy}{dx}=f'(x)\cos f(x)$
$y=\cos f(x)$	$\frac{dy}{dx}=-f'(x)\sin f(x)$
$y=\tan f(x)$	$\frac{dy}{dx}=f'(x)\sec^2 f(x)$
$y=\csc f(x)$	$\frac{dy}{dx}=-f'(x)\csc f(x)\cot f(x)$
$y=\sec f(x)$	$\frac{dy}{dx}=f'(x)\sec f(x)\tan f(x)$
$y=\cot f(x)$	$\frac{dy}{dx}=-f'(x)\csc^2 f(x)$