Continuous Random Variables

1 Probability density function (PDF)

The PDF describes how the probabilities are distributed over the range of values.
f(x) must satisfy these basic properties:

- $f(x) \geqslant 0$ for all values of x, so that no probabilities are negative.
- $\int_{\infty}^{-\infty} f(x)dx = 1$ (The sum of all probabilities is 1)
- $P(a < x < b) = \int_{a}^{b} f(x)dx$

For continuous random variables $P(x < z) = P(x \leqslant z)$

Example

$$f(x) = \begin{cases} \frac{1}{2}(x-3), & \text{for } 3 \leq x \leq 5 \\ 0, & \text{otherwise} \end{cases}$$

$$\int_{3}^{5} \tfrac{1}{2}x - \tfrac{3}{2}\ dx = \left[\frac{x^2}{4} - \frac{3x}{2}\right]_{3}^{5} = \left(\frac{5^2}{4} - \frac{3 \times 5}{2}\right) - \left(\frac{3^2}{4} - \frac{3 \times 3}{2}\right) = 1$$

Find $P(x < 4)$

$$\int_{3}^{4} \tfrac{1}{2}x - \tfrac{3}{2}\ dx = \left[\frac{x^2}{4} - \frac{3x}{2}\right]_{3}^{4} = 0.25$$

Example 2

$$f(x) = \begin{cases} x, & \text{for } 0 \leqslant x \leqslant 1 \\ 2-x, & \text{for } 1 \leqslant x \leqslant 2 \\ 0, & \text{otherwise} \end{cases}$$

$$P(0.5 < x < 1.3) = \int_{0.5}^{1} x\ dx + \int_{1}^{1.3} 2 - x\ dx = \left[\tfrac{1}{2}x^2\right]_{0.5}^{1} + \left[2x - \tfrac{1}{2}x^2\right]_{0.5}^{1} = 0.63$$

2 Cumulative Distribution Function

The CDF is $P(X \leqslant x)$
To find it, integrate the **PDF** between the lower limit and x.

$$\begin{tikzcd} \textbf{PDF } f(x) \arrow[r, bend left, "\text{Integrate}"] & \textbf{CDF } F(x) \arrow[l, bend left, "\text{Differentiate}"] \end{tikzcd}$$

Example

$$f(x) = \begin{cases} \frac{1}{2}(x-3), & \text{for } 3 \leqslant x \leqslant 5 \\ 0, & \text{otherwise} \end{cases}$$

$$\int_{3}^{x_0} \frac{1}{2}x - \frac{3}{2}\ dx = \left[\frac{x^2}{4} - \frac{3x}{2}\right]_{0.5}^{1} = \left(\frac{x^2}{4} - \frac{3x}{2}\right) - \left(\frac{3^2}{4} - \frac{3 \times 3}{2}\right) = \frac{x^2}{4} - \frac{3x}{2} + \frac{9}{4}$$

$$F(x) = \begin{cases} 0, & \text{for } x < 3 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{9}{4}, & \text{for } 3 \leqslant x \leqslant 5 \\ 1, & \text{for } x > 5 \end{cases}$$

Example 2

$f(x) = \begin{cases} x, & \text{for } 0 \leqslant x \leqslant 1 \\ 2-x, & \text{for } 1 \leqslant x \leqslant 2 \\ 0, & \text{otherwise} \end{cases}$

Find the CDF

$$\int_0^x x\ dx = \left[\frac{x^2}{2}\right]_0^x = \frac{x^2}{2}$$

$$F(1) + \int_1^x 2-x\ dx = \frac{1^2}{2} + \left[2x - \frac{x^2}{2}\right]_1^x = \frac{1}{2} + \Big((2x - \frac{x^2}{2}) - (2 - \frac{1}{2})\Big) = 2x - \frac{x^2}{2} - 1$$

$$f(x) = \begin{cases} 0, & \text{for } x < 0 \\ \frac{x^2}{2}, & \text{for } 0 \leqslant x \leqslant 1 \\ 2x - \frac{x^2}{2} - 1, & \text{for } 1 \leqslant x \leqslant 2 \\ 1, & \text{for } x > 2 \end{cases}$$

3 Mean and variance of continuous random variables

Mean$= \mu = E(x) = \displaystyle\int_a^b xf(x)\ dx$ for $a \leqslant x \leqslant b$

$\text{Var}(x) = E(x^2) - (E(x))^2$

$E(x^2) = \displaystyle\int_a^b x^2 f(x)\ dx$