

Partial fractions

1 Partial fractions

This requires us to split difficult algebraic fractions:

$$\frac{4}{(x+1)(x+2)} \rightarrow \frac{A}{x+1} + \frac{B}{x+2}$$

This allows us to:

- Do binomial expansion
- Integrate using difficult fractions

1.1 Normal Example

$$\frac{4}{(x+1)(x+2)} \rightarrow \frac{A}{x+1} + \frac{B}{x+2}$$

Recombine:

$$\frac{A(x+2) + B(x+1)}{(x+1)(x+2)}$$

Equate:

$$A(x+2) + B(x+1) = 4$$

Eliminate one:

Sub $x=-1$

$$A(-1+2) = 4, \underline{A = 4}$$

Sub $x=-2$

$$B(-2+1), \underline{B = -4}$$

Write as partial fractions:

$$\frac{4}{x+1} - \frac{4}{x+2}$$

1.2 Example with a cubic denominator

$$\frac{6x^2 + 5x - 2}{x(x+1)(2x+1)}$$

Expand

$$\frac{A}{x} + \frac{B}{x+1} + \frac{C}{2x+1}$$

Equate

$$A(x+1)(2x+1) + B(x)(2x+1) + C(x)(x-1) = 6x^2 + 5x - 2$$

Eliminate one

Sub $x=0$

$$-A = -2, \underline{A = 2}$$

Sub $x=1$

$$3B = 9$$

$$\underline{B = 3}$$

Sub $x = -\frac{1}{2}$

$$-\frac{3}{4}C = -3, \underline{C = 4}$$

Write as partial fractions

$$\frac{2}{x} + \frac{3}{x+1} - \frac{4}{2x+1}$$

1.3 Example with a Repeated root denominator

$$\frac{6x^2 - 29x - 29}{(x+1)(x-3)^2}$$

Expand

$$\frac{A}{x+1} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

Equate

$$A(x-3)^2 + B(x+1)(x-3) + C(x+1) = 6x^2 - 29x - 29$$

Eliminate one

Sub $x=-1$

$$A(-1-3)^2 + B(-1+1)(-1-3) + C(-1+1) = 6(-1)^2 - 29(-1) - 29$$

$$A(-4)^2 + B(0)(-1-3) + C(0) = 6(-1)^2 - 29(-1) - 29$$

$$16A = 6, \underline{A = \frac{3}{8}}$$

Sub $x=3$

$$4C = -62, \underline{C = -\frac{31}{2}}$$

Sub $x=0$

$$-29 = 9 \times \frac{3}{8} - 3B - \frac{31}{2}$$

$$\underline{B = \frac{45}{8}}$$

Write as partial fractions

$$\frac{3}{8(x+1)} + \frac{45}{8(x-3)} - \frac{31}{2(x-3)^2}$$

1.4 Example with partial fractions with same or higher denominator

$$\frac{3x^2 - 3x - 2}{(x-1)(x-2)}$$

Long division to find remainder

$$\begin{array}{r} 3 \\ x^2 - 3x + 2 \overline{) 3x^2 - 3x - 2} \\ \underline{-3x^2 + 9x - 6} \\ 6x - 8 \end{array}$$

Re-write with remainder

$$3 + \frac{6x-8}{(x-1)(x-2)}$$

Expand

$$\frac{A}{x-1} + \frac{B}{x-2}$$

Equate

$$A(x-2) + B(x-1) = 6x-8$$

Eliminate one

Sub $x=1$

$$-A = -2$$

$$\underline{A = 2}$$

Sub $x=2$

$$3B = 12$$

$$\underline{B = 4}$$

Write as partial fractions

$$3 + \frac{2}{x-1} + \frac{4}{x-2}$$

2 Binomial expansion

Expansion can be done using the $(1+x)^n$ expansion, including with $(1+ax)^n$

2.1 Negative powers

Example To expand $\frac{1}{1+x}$ turn it into $(1+x)^{-1}$ and use the formula from the book.

$$1 - x + x^2 - x^3 + x^4 - x^5 \dots$$

As n is not a positive integer there will be no x coefficient equalling zero, meaning the expansion is infinite and convergent.

This gives valid values when $|x| < 1$

2.2 Fractional powers

$$\sqrt{1-3x}$$

Simplify

$$(1-3x)^{\frac{1}{2}}$$

Find n and x

$$n = \frac{1}{2}$$

$$x = -3x$$

Substitute into the formula

$$1 + \frac{1}{2} \times -3x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1 \times 2} \times (-3x)^2$$

Simplify

$$1 - \frac{3}{2}x - \frac{9}{8}x^2$$

Write conclusion

Convergent and infinite when: $|3x| < 1$ $|x| < \frac{1}{3}$

2.3 Applying $(1+x)^n$ to $(a \pm bx)^n$

$(a \pm bx)^n$ can be rewritten as $a^n(1 \pm \frac{b}{a}x)^n$

2.3.1 Example

Expand $\sqrt{4+x}$ to the x^3 term

Turn square root into power

$$(4-x)^{\frac{1}{2}}$$

Rewrite with a 1 in the bracket

$$4^{\frac{1}{2}}(1 + \frac{1}{4}x)^{\frac{1}{2}}$$

Find n and x

$$n = \frac{1}{2}$$

$$x = \frac{1}{4}x$$

Substitute into the formula

$$2 \left[1 + \frac{1}{2} \times \frac{1}{4}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} \left(\frac{1}{4}x \right)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!} \left(\frac{1}{4}x \right)^3 \right]$$

Simplify

$$2 \left[1 + \frac{x}{8} - \frac{x^2}{128} + \frac{x^3}{1024} \right]$$

$$2 + \frac{x}{4} - \frac{x^2}{64} + \frac{x^3}{512}$$

Write conclusionValid if $\left| \frac{x}{4} \right| < 1$ so valid if $|x| < 4$ **2.4 Unknown coefficient type** $(a + bx)^{-2}$ can be approximated by

$$a \left(1 + \frac{b}{a}x \right)^{-2}$$

$$\frac{1}{a^2} \left(1 - 2\frac{b}{a}x \right)$$

2.5 Fractional typeExpand up to x^3 $\frac{1+x}{2+x}$ **Re-Write using powers**

$$(1+x)(2+x)^{-1}$$

Ensure there is only a 1 in the bracket

$$2 \left(1 + \frac{1}{2}x \right)^{-1}$$

Find n and x

$$n = -1$$

$$x = \frac{1}{2}x$$

Substitute into the formula

$$\frac{1}{2} \left(1 + -1 \times \frac{1}{2}x \right) + \frac{-1(-1-1)}{2!} \left(\frac{1}{2}(x)^2 \right)^2 + \frac{-1(-1-1)(-1-2)}{3!} \left(\frac{1}{2}x \right)^3$$

Simplify

$$(1+x) \left(\frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3 \right)$$

$$\frac{1}{2} + \frac{1}{4}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$$

Write conclusionValid if $x \neq 2$

2.6 Approximating roots

Find the expansion of $\sqrt{1-2x}$ up to x^3

Re-Write using powers

$$(1-2x)^{\frac{1}{2}}$$

Find n and x

$$n = \frac{1}{2}$$

$$x = -2x$$

Substitute into the formula

$$1 + \left(\frac{1}{2} \times -2x\right) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} \times (-2x)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!} \times (-2x)^3$$

Simplify $1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3$

By substituting $x = 0.01$, find a suitable approximation of $\sqrt{2}$

Substitute values $\sqrt{1 - \frac{2}{100}} = 1 - \frac{1}{100} - \frac{(\frac{1}{100})^2}{2} - \frac{(\frac{1}{100})^3}{2}$

Simplify $\sqrt{\frac{98}{100}} = \frac{\sqrt{98}}{10} = \frac{7\sqrt{2}}{10}$

Rearrange

$$\sqrt{2} \approx \frac{10}{7} \left(1 - \frac{1}{100} - \frac{(\frac{1}{100})^2}{2} - \frac{(\frac{1}{100})^3}{2} \right)$$

$$\sqrt{2} = 1.414213571$$