# Integration

## 1 Standard Results

Standard results not on formula book:

| **f(x)** | $\int \mathbf{f(x)\ dx}$ |
|---|---|
| $x^n$ | $\frac{x^{n+1}}{n+1}+c$ |
| $e^x$ | $e^x+c$ |
| $\frac{1}{x}$ | $\ln\lvert x\rvert+c$ |
| $\sin x$ | $-\cos x$ |
| $\cos x$ | $\sin x$ |
| $a^x$ | $\frac{a^x}{\ln\lvert a\rvert}+c$ |

## 2 Integration by substitution

**Example**

$\int (3x+2)^4\ dx$

**Find a value for u**

$u = 3x+2$

**Substitute the value of u into the initial question**

$\int u^4\ dx$

**Find dx in terms of du**

$\frac{du}{dx} = 3$

$du = 3dx$

$dx = \frac{1}{3}du$

Note - if an x term remains with the dx there may be a way to replace both the x term and dx

**Replace dx with du into the modified initial equation**

$\frac{1}{3}\int u^4\ du$

**Integrate using normal integration rules**

$\frac{1}{3}\int u^4\ du = \frac{1}{15}u^5$

**Substitute the value for u back in**

$\frac{1}{15}u^5 = \frac{1}{15}(3x+2)^5$

# 3 Parametric integration

$\int y\ dx$ gives the area under the function y.

For parametric equations we use:
$\int y \frac{dx}{dt} dt$

We have to integrate with respect to t or $\theta$ as it is the parameter.
Limits will sometimes need to be changed from x to t or $\theta$.

## 3.1 Example

$x = 5t^2$
$y = t^3$
Find $\int_1^2 y\ dt$

**Find $\mathbf{\frac{dx}{dt}}$**
$\frac{dx}{dt} = 10t$

**Substitute into the model $\int \mathbf{y \frac{dx}{dt}}$ dt**
$\int_1^2 t^3 \times 10t\ dt = \int_1^2 10t^4\ dt = \left[2t^5\right]_1^2 = 2 \times 2^5 - 2 \times 1^5 = 64 - 2 = 62$

# 4 Integration using trig identities

## 4.1 Example 1 - sinf(x)cosf(x)

$\int \sin(3x)\cos(3x)\ dx$

**Use addition formula to simplify**
$\sin 2x = 2\cos x \sin x$
$\sin 3x \cos 3x = \frac{1}{2}\sin 6x$
$\frac{1}{2}\int \sin 6x\ dx$

**Find a value for u**
$u = 6x$

**Find dx in terms of du**
$\frac{du}{dx} = 6$
$dx = \frac{1}{6}du$

**Substitute two conversions from x to u**
$\frac{1}{2} \times \frac{1}{6}\int \sin u\ du$

**Use integration of sin formula**
$\frac{1}{12}\int \sin u\ du = -\frac{1}{12}\cos 6x + c$

## 4.2 Example 2 - sinf(x)cosg(x)

$\int \sin(3x)\cos(2x)\ dx$

**Add together sin addition formula and sin subtraction formula**

$\sin(A+B) = \sin A\cos B + \cos A\sin B$
$\sin(A-B) = \sin A\cos B - \cos A\sin B$
$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$
$\sin 3x\cos 2x = \sin(3x+2x) + \sin(3x-2x)$
$\frac{1}{2}\int \sin 5x + \sin x\ dx = \frac{1}{2}\int \sin 5x\ dx + \frac{1}{2}\int \sin x\ dx$

**Integrate using sin standard result**

$-\frac{1}{10}\cos 5x - \frac{1}{1}\cos x + c$

## 4.3 Example 3 - $\sin^2 f(x)\cos^2 f(x)$

$\int \sin^2(x)cos^2(x)\ dx$

**Give $\mathbf{sin^2(x)cos^2(x)}$ in terms of $\mathbf{sin^2(x)}$**

$2\sin(x)\cos(x) = \sin 2x$
$4\sin^2(x)\cos^2(x) = \sin^2 2x$
$\cos^2 x\sin^2 x = \frac{1}{4}\sin^2 2x$

**Sub simplified version**

$\int \frac{1}{4}\sin^2 2x\ dx$

**Give $\mathbf{sin^2x}$ in terms of cosx**

$2\sin^2 2x = \sin^2 2x + (1-\cos^2 2x)$
$2\sin^2 2x = 1 - (\cos^2 2x - \sin^2 2x)$
$2\sin^2 2x = 1 - \cos 4x$ - Use of cos subtraction formula
$\sin^2 2x = \frac{1}{2} - \frac{1}{2}\cos 4x$

**Sub simplified version**

$\frac{1}{4}\int \frac{1}{2} - \frac{1}{2}\cos 4x\ dx$
$\frac{1}{8}\int 1 - \cos 4x\ dx$

**Use cos integration standard result**

$\frac{1}{8}(x - \frac{1}{4}\sin 4x) + c$

$\frac{1}{8}x - \frac{1}{32}\sin 4x + c$

# 5 Integration of partial fractions

## 5.1 Simple example

$\int \frac{x-5}{(x+1)(x-2)}$

**Write as separate fractions**

$\frac{A}{x+1}+\frac{B}{x-2}$

**Multiply out**

$A(x-2)+B(x+1)=x-5$

**Solve**

B=-1
A=2

**Substitute**

$\int \frac{2}{x+1}-\frac{1}{x-2}\ dx$

**Separate and simplify**

$2\int \frac{1}{x+1}\ dx-\int \frac{1}{x-2}\ dx$

**Use integration of $\mathbf{\frac{1}{x}}$ standard result**

$2\ln(x+1)-\ln(x-2)+c$

**Simplify using laws of logs**

$\ln\left|\frac{(x+1)^2}{x-2}\right|+c$

# 6 Integration by substitution - Fractional type

$\int \frac{f'(x)}{f(x)}\ dx$ can be written as $\ln f(x)+c$