Maclaurin and Taylor Series

1 Maclaurin's expansion

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + ... + f^r(0)\frac{x^r}{r!}...$$

For the continuous function, f, given by $f : x \Rightarrow f(x)$(where x is real), then providing $f(0), f'(0), f''(0)$ etc all have finite values. This is an infinite series.

1.1 Example

Given that $f(x) = e^x$ can be written as an infinite series in the form:

$$f(x) = e^x = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_rx^4 + ...$$

And that it is valid to differentiate an infinite series term by term, show that:

$$e^2 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^r}{r!} + ...$$

Find up to the third differential of $f(x)$ and the value of zero for each

$f(x) = e^x$ $\quad f(0) = 1$
$f'(x) = e^x$ $\quad f''(0) = 1$
$f''(x) = e^x$ $\quad f''(0) = 1$
$f'''(x) = e^x$ $\quad f'''(0) = 1$

$$f(x) = 1 + 1 \times x + 1 \times \frac{x^2}{2!} + 1 \times \frac{x^3}{3!}$$

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^r}{r!} + ...$$

1.2 Standard results

Standard results are given on the data sheet, these can then be used for adapted forms of the results also. Remember to consider the limits where appropriate.

2 Taylor expansion

The conditions of the Maclaurin expansion mean that some functions, such as $\ln x$ cannot be expanded as a series in ascending powers of x.

The construction of the Maclaurin expansion focuses on $x = 0$ and values of x very close to zero. The Taylor expansion focuses on $x = a$.

Considering the functions f and g, where $f(x + a) \equiv g(x)$ then:
$f^r(a) = g^r(0)$

Turning the Maclaurin expansion for g from:

$$g(x) = g(0) + g'(0)x + \frac{g''(0)}{2!}x^2...$$

Into

$$f(x + a) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + ... + \frac{f^r(a)}{r!}x^r$$

Replacing x by x-a gives

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + ... + \frac{f^r(a)}{r!}(x - a)^r$$

These are the two forms of the Taylor expansion, when a=0, they both become the Maclaurin expansion.