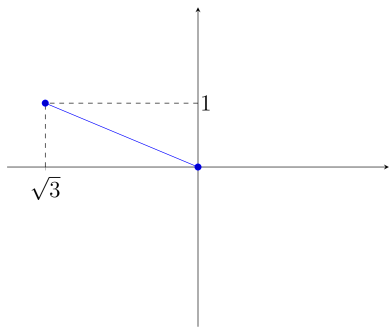
Further Complex Numbers

1 Expressions of complex numbers

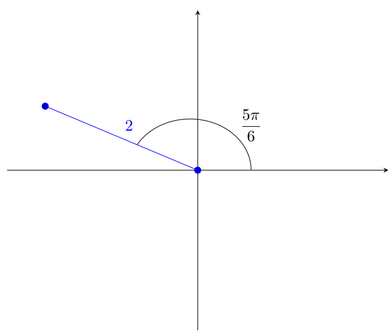
1.1 $\mathbf{x + iy}$

This expresses the coordinate of the point at the end of the vector on the argand diagram.



1.2 $\mathbf{r}(\cos\theta + \mathbf{i}\sin\theta)$

This expresses the length of the line and the angle anticlockwise from the positive x axis



$$2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$$

1.3 $re^{i\theta}$

This uses the same parameters as $r(\cos\theta + i\sin\theta)$

2 Absolute square

When squaring $|z|$, where $z = x + iy$, first find $|z|$, which is $\sqrt{x^2 + y^2}$, then square it to get $x^2 + y^2$.

3 Multiplying and dividing complex numbers

3.1 Multiplying

3.1.1 Trigonometric form

$$Z_1Z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2)$$

Apply the cos addition formula to the first two terms

$$= r_1r_2(\cos(\theta_1+\theta_2) + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2)$$

Apply the sin addition formula to the last two terms

$$= r_1r_2(\cos(\theta_1+\theta_2) + \sin(\theta_1+\theta_2))$$

3.1.2 Exponential form

$$Z_1Z_2 = r_1e^{i\theta_1} \times r_2e^{i\theta_2}$$

Apply laws of indices

$$Z_1Z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$$

3.2 Dividing

3.2.1 Trigonometric form

$$\frac{Z_1}{Z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

Multiply by the complex conjugate

$$\frac{Z_1}{Z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$$

Expand

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \times \frac{cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}{\cos^2\theta_2 - i\cos\theta_2\sin\theta_2 + i\sin\theta_2\cos\theta_2 + \sin^2\theta_2}$$

Simplify

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \times \Big(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\Big)$$

3.2.2 Exponential form

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \times \frac{e^{i\theta_1}}{e^{i\theta_2}}$$
$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \times e^{i(\theta_1-\theta_2)}$$

3.3 Comparison

Multiplying	Dividing
Multiply modulus, add arguments	Divide modulus, subtract arguments

4 De Moivre's Theorem

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

4.1 Positive proof

Prove true for n=1

$$r(\cos\theta + i\sin\theta) = r(\cos\theta + i\sin\theta)$$

True for n=1

Assume true for n=k

$$[r(\cos\theta + i\sin\theta)]^k = r^k(\cos(k\theta) + i\sin(k\theta))$$

Prove true for n=k+1

$$[r(\cos\theta + i\sin\theta)]^{k+1}$$
$$[r(\cos\theta + i\sin\theta)]^k \times (r(\cos\theta + i\sin\theta))^1$$
$$r^k(\cos(k\theta) + i\sin(k\theta)) \times r(\cos\theta + i\sin\theta)$$
$$r^k r(\cos(k\theta + \theta) + i\sin(k\theta + \theta))$$
$$r^{k+1}(\cos((k+1)\theta) + i\sin((k+1)\theta))$$

True

4.2 Negative proof

n=-m

$$[r(\cos\theta + i\sin\theta)]^{-m}$$

Multiply by complex conjugate

$$\frac{1}{[r(\cos\theta + i\sin\theta)]^m} \times \frac{[r(\cos\theta - i\sin\theta)]^m}{[r(\cos\theta - i\sin\theta)]^m}$$

Apply positive De Moivre's Theorem

$$\frac{r^m(\cos(m\theta) - i\sin(m\theta))}{r^m(\cos(m\theta) + i\sin(m\theta)) \times r^m(\cos(m\theta) - i\sin(m\theta))}$$

Simplify and expand

$$\frac{\cos(m\theta) - i\sin(m\theta)}{r^m(\cos^2 m\theta - i\cos m\theta\sin m\theta + i\cos m\theta\sin m\theta + \sin^2 m\theta}$$

Simplify

$$\frac{\cos m\theta - i\sin m\theta}{r^m} = r^{-m}(\cos m\theta - i\sin m\theta)$$

Rewrite

$$r^{-m}(\cos(-m\theta) + i\sin(-m\theta))$$

Replace -m with n

$$r^n(\cos(n\theta) + i\sin(n\theta))$$

4.3 Applying De Moivres' Theorem

We can use the binomial expansion along with De Moivre's theorem to rewrite trigonometric expressions. This can be useful when integrating etc.

4.3.1 Example 1

Rewrite $\cos 5\theta$ *in powers of* $\cos\theta$

$$(\cos\theta + i\sin\theta)^5$$

Apply DM theorem

$$\cos 5\theta + i\sin 5\theta$$

Apply the binomial expansion to the initial expression

$$\cos^5\theta + 5\cos^4\theta i\sin\theta + 10\cos^3\theta(i\sin\theta)^2 + 10\cos^2\theta(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$$

Simplify

$$\cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$$

Equate the real parts of this to the real parts of the result from DM theorem

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

Replace sin terms with cos

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$$

Simplify

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1 - \cos\cos^2\theta + cos^4\theta)$$

Simplify further

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos 63\theta + 5\cos^5\theta$$

Collect terms

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

4.3.2 Z formulas

If $z = \cos\theta + i\sin\theta$

$$z + \frac{1}{z} = 2\cos\theta$$

$$z - \frac{1}{z} = 2i\sin\theta$$

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

$$z^n - \frac{1}{z^n} = 2i\sin n\theta$$

4.3.2.1 Example

Express $\cos^5\theta$ *in the form* $a\cos 5\theta + b\cos 3\theta + c\cos\theta$ Create the situation using the z formulas

$$\left(z + \frac{1}{z}\right)^5 = (2\cos\theta)^5 = 32\cos^5\theta$$

Expand using the binomial

$$z^5 + 5z^4 \times \frac{1}{z} + 10z^3 \times \frac{1}{z^2} + 10z^2 \times \frac{1}{z^3} + 5z \times \frac{1}{z^4} + \frac{1}{z^5}$$

Combine like coloured term using z formula

$$32\cos^5\theta = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$$

$$\cos^5\theta = \frac{\cos 5\theta}{16} + \frac{5\cos 3\theta}{16} + \frac{5\cos\theta}{8}$$

5 Solving complex equations

For any complex number, we generally define the argument to be between $-\pi$ and π. However we can add multiples of 2π to the argument to get equivalent answers.

We shall use this fact to find all solutions to complex equations.

Note: The number of solutions is equal to the order of the equation.

5.1 Example

Solve:
$z^5 = i$

$$r = 1 \quad \theta = \frac{\theta}{2}$$

$$z^5 = \cos(\frac{\pi}{2} + 2k\pi) + i\sin(\frac{\pi}{2} + 2k\pi)$$

$$z = \cos\left(\frac{\frac{\pi}{2} + 2k\pi}{5}\right) + i\sin\left(\frac{\frac{\pi}{2} + 2k\pi}{5}\right)$$

5 Solutions: k=-2,-1,0,1,2
$k = -2 \quad z = -0.588 - 0.809i$
$k = -1 \quad z = 0.588 - 0.809i$
$k = 0 \quad z = 0.951 + 0.309i$
$k = 1 \quad z = i$
$k = 2 \quad z = -0.951 + 0.309i$

6 Loci on the complex plane

6.1 Lines

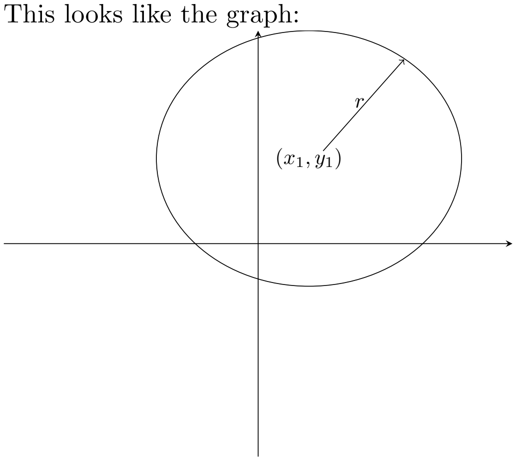
6.1.1 $|\mathbf{z}-\mathbf{z_1}|=\mathbf{r}$

$|z-z_1|=r$ is represented by a circle, centre (x_1,y_1) with a radius r, where $z_1=x_1+iy_1$.

This is the same as the Cartesian equation:

$$(x-x_1)^2+(y-y_1)^2=r^2$$

This looks like the graph:

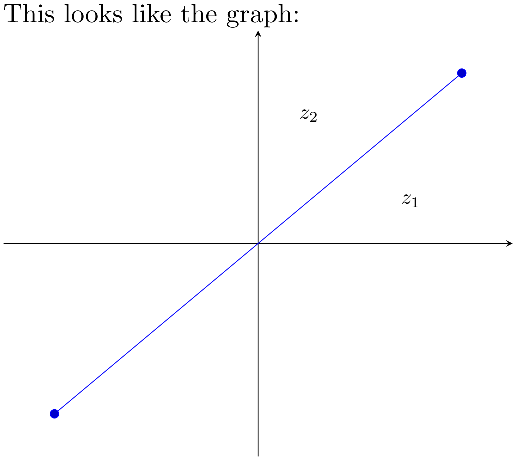


6.1.2 $|\mathbf{z}-\mathbf{z_1}|=|\mathbf{z}-\mathbf{z_2}|$

$|z-z_1|=|z-z_2|$ is represented by a perpendicular bisector of the line segment joining points z_1 and z_2

To find the Cartesian form, replace z with $x+iy$ and expand, squaring both sides to remove the modulus signs.

This looks like the graph:



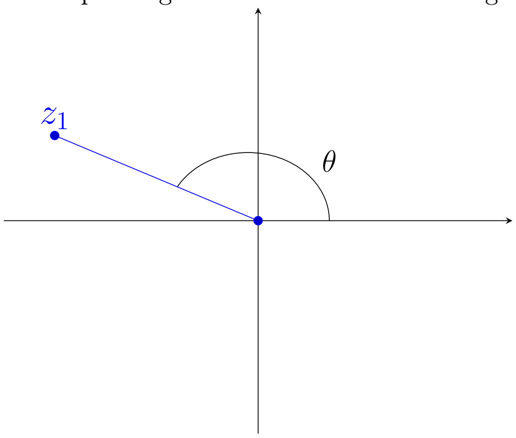
For cases where there is a constant in front of one of the modulus signs, use the algebraic method to find the Cartesian form, then plot that.

6.1.3 $\arg(\mathbf{z}-\mathbf{z_1})=\theta$

$\arg(z-z_1)=\theta$ is represented by the half-line from the fixed point z_1, making an angle θ with a line from the fixed point z_1, parallel to the real axis.

To find the Cartesian form, replace z with $x+iy$ and separate into the real and imaginary parts. Then use the trigonometric identity $\tan\theta=\dfrac{\text{Opposite}}{\text{Adjacent}}$ and simplify.

When plotting remember that θ is the angle anticlockwise.



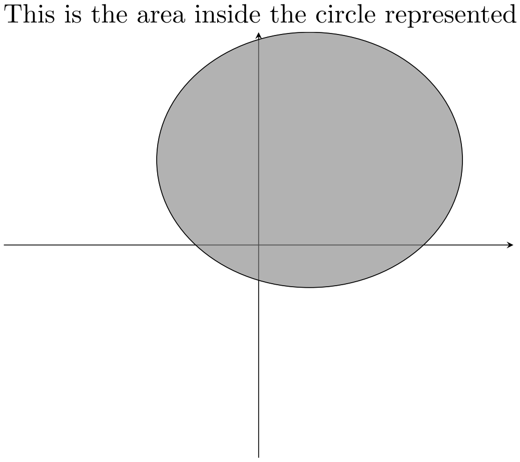
6.1.4 $\arg\left(\dfrac{\mathbf{z}-\mathbf{z_1}}{\mathbf{z}-\mathbf{z_2}}\right)$

$\arg\left(\dfrac{z-z_1}{z-z_2}\right)$ represents an arc anticlockwise between z_1 and z_2

6.2 Regions

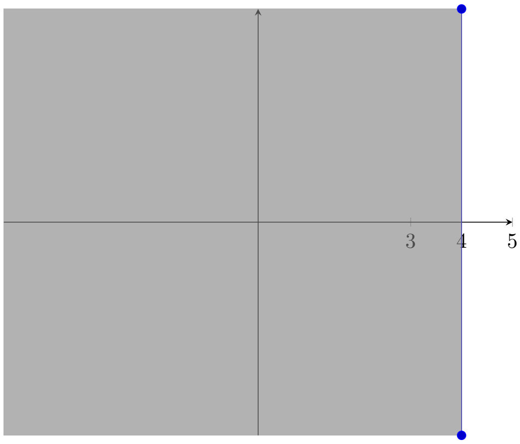
6.2.1 $|\mathbf{z}-(\mathbf{2}+\mathbf{3i})|\leqslant\mathbf{3}$

This is the area inside the circle represented by $|z-(2+3i)|\leqslant 3$



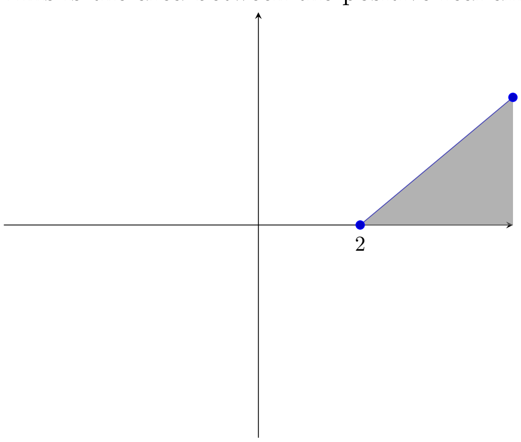
6.2.2 $|\mathbf{z-3}| < |\mathbf{z-5}|$

This is the area to one side of the perpendicular bisector represented equal to $|z-3| = |z-5|$



6.2.3 $\mathbf{0} \leqslant \arg(\mathbf{z-2}) < \frac{\pi}{4}$

This is the area between the positive real axis and the half line equal to $\arg(z-2) = \frac{\pi}{4}$



7 Translations

- $w = z + a + ib$ represents a translation with translation vector $\begin{pmatrix} a \\ b \end{pmatrix}$
- $w = kz$ represents an enlargement with scale factor k centre $(0,0)$
- $w = kz + a + ib$ represents an enlargement scale factor k centre $(0,0)$ followed by a translation with translation vector $\begin{pmatrix} a \\ b \end{pmatrix}$
- $w = z^2$ multiply a shape by itself, for example a circle of radius 4 would go to radius 16

When doing translations, the input is $z = x + iy$ and the output is $w = u + iv$, unless otherwise stated.