

Second Order Differential Equations

In FP2 we are interested in solving 2nd ODEs of the form:

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x) \quad a, b, c \text{ are constants}$$

We consider three distinct cases:

$$b^2 > 4ac \quad (\text{Two real solutions})$$

$$b^2 = 4ac \quad (\text{One repeated solution})$$

$$b^2 < 4ac \quad (\text{Two complex solutions})$$

To solve 2nd ODEs of this form we first consider solutions to:

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

The process of solving a 2nd ODE starts with a general solution to a 1st ODE of form:

$$b \frac{dy}{dx} + cy = 0$$

$$\int \frac{1}{b} dy = \int \frac{1}{-cy} dy$$

$$b \ln(y) = -cx + k$$

$$y = Ae^{-\frac{c}{b}x}$$

$$y = Ae^{mx}$$

This was suggested to be a solution to the 2nd ODE as well

We take $y = e^{mx}$ as a starting point for finding general solutions to:

$$(1) \quad a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

If $y = e^{mx}$ is a solution to (1):

$$\frac{dy}{dx} = me^{mx}$$

$$\frac{d^2 y}{dx^2} = m^2 e^{mx}$$

Then substitute this into (1)

$$am^2 e^{mx} + bme^{mx} + ce^{mx} = 0$$

Factor out e^{mx}

$$e^{mx}(am^2 + bm + c)$$

As e^x must be greater than zero $am^2 + bm + c = 0$

This is a solvable quadratic called the **Auxiliary equation**

1 Two real roots $b^2 > 4ac$

$$(1) \quad a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

The general solution to (1) is in the form:

$$y = Ae^{\alpha x} + Be^{\beta x}$$

Where A and B are constants and α and β are the roots to the AE

1.1 Example

$$(1) \quad 2 \frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} + 3y = 0$$

Find the auxiliary equation

$$2m^2 e^{mx} + 5m e^{mx} + 3e^{mx} = 0$$

$$e^{mx}(2m^2 + 5m + 3) = 0$$

$$m = -\frac{3}{2} \quad m = -1$$

$$\text{General solution: } y = Ae^{\alpha x} + Be^{\beta x}$$

$$GS = Ae^{-\frac{3}{2}x} + Be^{-x}$$

2 1 Real, Repeated root $b^2 = 4ac$

$$\text{General solution : } (A + bx)e^{\alpha x}$$

A and B are constants and α is the root of the AE

2.1 Example

Find the general solution of:

$$\frac{d^2 y}{dx^2} + 8 \frac{dy}{dx} + 16y = 0$$

Find the auxiliary equation

$$e^{mx}(m^2 + 8m + 16) = 0$$

Find the solution

$$m = -4$$

Substitute into the general solution formula

$$y = (A + Bx)e^{-4x}$$

3 Imaginary only roots $b^2 < 4ac$

This is when the AI has roots of form $\pm \alpha i$

$$\text{General solution: } y = A \cos(\alpha x) + B \sin(\alpha x)$$

4 Complex roots $b^2 < 4ac$

This is used when the root is in the form $\beta \pm \alpha i$

$$\text{General solution: } y = e^{\beta x}(A \cos(\alpha x) + B \sin(\alpha x))$$

However this can also be written in the standard form:

$$y = Ae^{(\beta+\alpha i)x} + Be^{(\beta-\alpha i)x}$$

4.1 Example

Find the general solution of:

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 34y = 0$$

Find the auxiliary equation

$$m^2 - 6m + 34 = 0$$

Solve to find roots

$$\text{Roots} = \frac{6 \pm \sqrt{36 - 4 \times 1 \times 34}}{2} = \frac{6 \pm 10i}{2} = 3 \pm 5i$$

Substitute into general solution formula

$$y = e^{3x}(A \cos(5x) + B \sin(5x))$$

5 Solving 2nd ODE = f(x)

Of the type:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

There are set forms of f(x)

The LHS will be solved in the standard way and the general solution of the LHS will be called the **complementary solution** (CS)

Solving the RHS will give us a **particular integral** (PI)

$$\text{Full general solution} = \text{Complementary function} + \text{Particular integral}$$

5.1 Standard forms of f(x)

$$f(x) = \lambda$$

$$f(x) = \lambda + \mu x$$

$$f(x) = \lambda + \mu x + \nu x^2$$

$$f(x) = ke^{px}$$

$$f(x) = m \cos \omega x$$

$$f(x) = m \sin \omega x$$

$$f(x) = m \cos \omega x \pm n \sin \omega x$$

5.2 Examples

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = f(x)$$

Find complementary function

$$m^2 - 5m + 6 = 0$$

$$m = 2 \quad m = 3$$

$$\text{Complementary function} = Ae^{3x} + Be^{2x}$$

5.2.1 2nd ODE= λ

$$f(x) = 3$$

Start with $y = \lambda$

$$y = \lambda$$

$$\frac{dy}{dx} = 0$$

$$\frac{d^2y}{dx^2} = 0$$

Substitute values into LHS

$$0 - 5 \times 0 + 6\lambda = 3$$

$$\lambda = \frac{1}{2}$$

Add Complementary function to particular integral

$$y = Ae^{3x} + Be^{2x} + \frac{1}{2}$$

5.2.2 2nd ODE= $\lambda + \mu x$

$$f(x) = 2x$$

Start with $y = \lambda + \mu x$

$$y = \lambda + \mu x$$

$$\frac{dy}{dx} = \mu$$

$$\frac{d^2y}{dx^2} = 0$$

Substitute into LHS

$$0 - 5\mu + 6(\lambda + \mu x) = 2x$$

Equate x terms

$$6\mu x = 2x$$

$$\mu = \frac{1}{3}$$

Equate constant terms

$$-\frac{5}{3} + 6\lambda = 0$$

$$6\lambda = \frac{5}{3}$$

$$\lambda = \frac{5}{18}$$

Substitute into form for the particular integral

$$y = \frac{1}{3}x + \frac{5}{18}$$

Add the PI and CF to find the general solution

$$y = Ae^{3x} + Be^{2x} + \frac{1}{3}x + \frac{5}{18}$$

5.2.3 2nd ODE $= \lambda + \mu x + \nu x^2$

$$f(x) = 3x^2$$

Start with $y = \lambda + \mu x + \nu x^2$

$$y = \lambda + \mu x + \nu x^2$$

$$\frac{dy}{dx} = \mu + 2\nu x$$

$$\frac{d^2y}{dx^2} = 2\nu$$

Substitute into the LHS

$$2\nu - 5(\mu + 2\nu x) + 6(\lambda + \mu x + \nu x^2) = 3x^2$$

Equate x^2 terms

$$6\nu = 3 \quad \nu = \frac{1}{2}$$

Equate x terms

$$-10 \times \frac{1}{2} \times x + 6\mu x = 0$$

$$-5 + 6\mu = 0$$

$$\mu = \frac{5}{6}$$

Equate constant coefficients

$$1 - 5 \times \frac{5}{6} + 6\lambda = 0$$

$$6\lambda = \frac{19}{6}$$

$$\lambda = \frac{19}{36}$$

Substitute into the particular integral form

$$y = \frac{1}{2}x^2 + \frac{5}{6}x + \frac{19}{36}$$

Add CF and PI to get the general solution

$$y = Ae^{3x} + Be^{2x} + \frac{1}{2}x^2 + \frac{5}{6}x + \frac{19}{36}$$

5.2.4 Trigonometric f(x)

General forms:

If $f(x) = m \cos \omega x$

$$PI : y = P \cos \omega x + Q \sin \omega x$$

If $f(x) = n \sin \omega x$

$$PI : y = P \cos \omega x + Q \sin \omega x$$

If $f(x) = m \cos \omega x \pm n \sin \omega x$

$$PI : y = P \cos \omega x + Q \sin \omega x$$

$f(x) = m \sin \omega x$ **or** $n \sin \omega x$ $f(x) = 13 \sin 4x$

$$PI : y = P \cos \omega x + Q \sin \omega x$$

$$\frac{dy}{dx} = -\omega P \sin \omega x + \omega Q \cos \omega x$$

$$\frac{d^2y}{dx^2} = -\omega^2 P \cos \omega x - \omega^2 Q \sin \omega x$$

$$\frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 6y = 13 \sin 3x$$

Substitute into LHS

$$-\omega^2 P \cos \omega x - \omega^2 Q \sin \omega x - 5(-\omega P \sin \omega x + \omega Q \cos \omega x) + 6(P \cos \omega x + Q \sin \omega x) = 13 \sin 3x$$

Equate cos terms

$$-\omega^2 P \cos \omega x - 5\omega Q \cos \omega x + 6P \cos \omega x = 0$$

Substitute $\omega = 3$ and divide by $\cos 3x$

$$-9P - 15Q + 6P = 0$$

Simplify

$$-3P - 15Q = 0 \quad (1)$$

Equate sin terms

$$-\omega^2 Q \sin \omega x + 5\omega P \sin \omega x + 6Q \sin \omega x = 13 \sin 3x$$

Substitute $\omega = 3$ and divide by $\sin 3x$

$$-9Q + 15P + 6Q = 13$$

Simplify

$$15P - 3Q = 13 \quad (2)$$

Multiply (1) by 5

$$-15P - 75Q = 0$$

Add the multiplied (1) and (2)

$$-78Q = 13$$

Simplify

$$Q = -\frac{1}{6}$$

Substitute to find P

$$15 \times \frac{1}{6} = -3P \quad P = \frac{5}{6}$$

Substitute and add to the CF to find the PI

$$y = Ae^{3x} + Be^{2x} + \frac{5}{6} \cos 3x - \frac{1}{6} \sin 3x$$

6 Clash of terms between CF and PI

Solve:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x}$$

Find the Complementary function

$$m^2 - 5m + 6 = 0$$

$$CF : y = Ae^{3x} + Be^{2x}$$

Here there will be a clash of terms between the CF and the PI so a different PI must be used, this will be given to you.

$$\text{Use PI } y = \lambda x e^{2x}$$

Differentiate twice

$$\frac{dy}{dx} = \lambda e^{2x} + 2\lambda x e^{2x}$$

$$\frac{d^2y}{dx^2} = 2\lambda e^{2x} + 2\lambda e^{2x} + 4\lambda x e^{2x}$$

Substitute

$$2\lambda e^{2x} + 2\lambda e^{2x} + 4\lambda x e^{2x} - 5(\lambda e^{2x} + 2\lambda x e^{2x}) + 6\lambda x e^{2x} = e^{2x}$$

$$\lambda = -1$$

Substitute

$$PI : y = -x e^{2x}$$

7 Applications of boundary conditions

If DE is in $\frac{d^2y}{dx^2}$ form then the numerical values for x,y and $\frac{dy}{dx}$ will be given for a value of x.

If (as is common in exams) DE is in $\frac{d^2x}{dt^2}$ form then numerical values of x,t and $\frac{dx}{dt}$ will be given for a value of x.

Used to find A and B in the CF. This gives a particular solution.

7.1 Example

When $y=1$ $x=0$ $\frac{dy}{dx} = 0$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 12e^x$$

Find the complementary function

$$y = Ae^{-3x} + Be^{-2x}$$

Find the general form of the particular integral

$$PI : \lambda e^x$$

Differentiate the particular integral twice

$$\frac{dy}{dx} = \lambda e^x$$

$$\frac{d^2y}{dx^2} = \lambda e^x$$

Substitute into the initial formula

$$\lambda e^5 + 5\lambda e^x + 6\lambda e^x = 12e^x$$

Simplify to find lambda

$$12\lambda e^x = 12e^x \quad \lambda = 1$$

Add this to the complementary function to find the general solution

$$GS : Ae^{-3x} + Be^{-2x} + e^x$$

Substitute $y=1$ and $x=0$ into the general solution

$$1 = A + B + 1$$

Simplify

$$A + B = 0$$

Substitute $\frac{dy}{dx} = 0$ and $x = 0$ into the differentiated form of the general solution
Differentiate the general solution

$$\frac{dy}{dx} = -3Ae^{-3x} - 2Be^{-2x} + e^x$$

Substitute values

$$0 = -3A - 2B + 1$$

Solve simultaneous equations

$$A = 1 \quad B = -1$$

Substitute into the general solution to find the particular solution

$$PS: y = e^{-3x} - e^{-2x} + e^x$$

8 Substitution

8.1 Example 1

Show that the substitution $x = e^u$ transforms

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0 \quad (1)$$

to

$$\frac{d^2 y}{du^2} + y = 0 \quad (2)$$

Find $\frac{dx}{du}$

$$\text{As } x = e^u, \quad \frac{dx}{du} = e^u = x$$

Find $\frac{dy}{du}$ using the chain rule

$$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du} = e^u \frac{dy}{dx} = x \frac{dy}{dx} \quad (3)$$

Find a simplification of $\frac{d^2 y}{du^2}$

$$\frac{d^2 y}{du^2} = \frac{d}{du} \left(\frac{dy}{du} \right)$$

Simplify using known expression for $\frac{dy}{du}$

$$\frac{d^2 y}{du^2} = \frac{d}{du} \left(e^u \frac{dy}{dx} \right)$$

Apply the product rule, using the chain rule to differentiate $\frac{dy}{dx}$ with respect to u

$$\frac{d^2 y}{du^2} = e^u \frac{dy}{dx} + e^u \frac{d^2 y}{dx^2} \frac{dx}{du}$$

Replace $e^u \frac{dy}{dx}$ with $\frac{dy}{du}$ and e^u and $\frac{dx}{du}$ each with x

$$\frac{d^2 y}{du^2} = \frac{dy}{du} + x^2 \frac{d^2 y}{dx^2}$$

Rearrange to find $x^2 \frac{d^2 y}{dx^2}$

$$x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{du^2} - \frac{dy}{du} \quad (4)$$

Substitute the results from (3) and (4) into (1).

$$\frac{d^2 y}{du^2} - \frac{dy}{du} + \frac{dy}{du} + y = 0$$

Simplify to find answer

$$\frac{d^2 y}{du^2} + y = 0$$

Solve this to find the general solution

$$m^2 + 1 = 0$$

$$m = \pm i$$

Substitute into the general form

$$y = A \cos u + B \sin u$$

Substitute $u = \ln x$ to give GS in terms of x

$$y = A \cos(\ln(x)) + B \sin(\ln(x))$$

8.2 Example 2

Show that the transformation $x = e^u$ transforms the differential equation

$$x^2 \frac{d^2 y}{dx^2} - 7x \frac{dy}{dx} + 16y = 2 \ln x, \quad x > 0 \quad (1)$$

into the differential equation

$$\frac{d^2 y}{du^2} - 8 \frac{dy}{du} + 16y = 2u \quad (2)$$

Find $\frac{dy}{dx}$ in terms of $\frac{dy}{du}$

$$x = e^u \quad \frac{dx}{du} = e^u \quad \frac{du}{dx} = e^{-u} \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^{-u} \frac{dy}{du}$$

Find $\frac{d^2 y}{dx^2}$ in terms of $\frac{d^2 y}{du^2}$ and $\frac{dy}{du}$

Find $\frac{dy}{du} \left(\frac{dy}{dx} \right)$ by applying the product rule

$$\frac{dy}{du} \left(\frac{dy}{dx} \right) = -e^{-u} \frac{dy}{du} + e^{-u} \frac{d^2 y}{du^2}$$

Multiply through by $\frac{du}{dx}$ to find the value of $\frac{d^2 y}{dx^2}$

$$\begin{aligned} \frac{d^2 y}{dx^2} &= \frac{d}{du} \left(\frac{dy}{dx} \right) \times \frac{du}{dx} \\ \frac{d^2 y}{dx^2} &= e^{-u} \left(-e^{-u} \frac{dy}{du} + e^{-u} \frac{d^2 y}{du^2} \right) \\ \frac{d^2 y}{dx^2} &= e^{-2u} \left(-\frac{dy}{du} + \frac{d^2 y}{du^2} \right) \end{aligned}$$

Substitute the values into the original formula, replacing x with e^u

$$\begin{aligned} e^{2u} \times e^{-2u} \left(-\frac{dy}{du} + \frac{d^2 y}{du^2} \right) - 7e^u \times e^{-u} \frac{dy}{du} + 16y &= 2u \\ -\frac{dy}{du} + \frac{d^2 y}{du^2} - 7 \frac{dy}{du} + 16y &= 2u \\ \frac{d^2 y}{du^2} - 8 \frac{dy}{du} + 16y &= 2u \end{aligned}$$