First Order Differential Equations

1 Families of curves

General solutions with a constant of integration C will give rise to a family of curves.

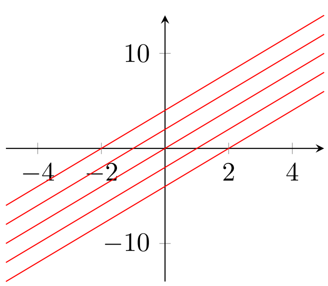
If given boundary conditions we can find specific solutions.

1.1 Example

Find the solution to:

$\frac{dy}{dx} = 2$

$y = 2x + c$



2 Introduction to first order differential equations

Implicitly differentiate:
$x^3y \quad \text{wrt} x$
$3x^2y + x^3\frac{dy}{dx}$

We can use this method in reverse to solve first order DEs

2.1 Example 1

$$x^3\frac{dy}{dx} + 3x^2y = \sin(x)$$

We look for standard patterns in the LHS and look to rewrite using the reverse implicit product rule

$$\frac{d}{dx}(x^3y) = \sin(x)$$
$$x^3y = -\cos(x) + c$$
$$y = \frac{-\cos(x) + c}{x^3}$$

If given an (x,y) point a particular solution can be found

In general:

$$f(x)\frac{dy}{dx} + f'(x)y = \frac{d}{dx}(f(x)y)$$

3 Solving first order DE using an integrating factor

Solving $\frac{dy}{dx} + P(x)y = Q(x)$

IF(Integrating factor) is found by finding $e^{\int p(x)\ dx}$ And multiplying the DE by the IF.

This will result in the DE being in the form:

$$f(x)\frac{dy}{dx} + f'(x)y$$

This form can then be shortened by integrating:

$$\int f'(x)g(x) + f(x)g'(x)dx = f(x)g(x) + c$$

Integrate both sides then simplify

3.1 Example

$$p\frac{dx}{dt} + qx = r \qquad \text{Where p,q and r are constants}$$

Given that $x = 0$ when $t = 0$
Find x in terms of t

Divide through by p to ensure $\frac{dx}{dt}$ has no multiplier

$$\frac{dx}{dt} + \frac{q}{p}x = \frac{r}{p}$$

Find the integrating factor (IF)

$$IF : e^{\int \frac{q}{p}\ dt} = e^{\frac{qt}{p}}$$

Multiply through by the IF to make the equation in the right form

$$e^{\frac{qt}{p}}\frac{dx}{dt} + \frac{q}{t}xe^{\frac{qt}{p}} = \frac{r}{p}e^{\frac{qt}{p}}$$

Integrate both sides, simplify the LHS

$$xe^{\frac{qt}{p}} = \int \frac{r}{p}e^{\frac{qt}{p}}\ dt$$

Do the integration on the RHS and simplify

$$xe^{\frac{qt}{p}} = \frac{r}{q}e^{\frac{qt}{p}} + c$$

Substitute in x=0 and t=0 to find the value of c

$$0 = \frac{r}{q} + c$$

$$c = -\frac{r}{q}$$

Substitute in the value of c and divide through by the IF

$$xe^{\frac{qt}{p}} = \frac{r}{q}e^{\frac{qt}{p}} - \frac{r}{q}$$

$$x = \frac{r}{q} - \frac{r}{q}e^{-\frac{qt}{p}}$$

4 First order DE with given substitution

Type 1 reduces to separation of variables

Type 2 reduces to $\frac{d}{dx}(f(x)y)$ form

4.1 Type 1

Show that the substitution $z = \frac{y}{x}$ transforms

$$(1) \quad \frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$$

to

$$(2) \quad x\frac{dz}{dx} = \frac{1+z^2}{2z}$$

We need to find $\frac{dz}{dx}$ and replace y in (1)

$$y = xz \qquad \frac{dy}{dx} = z + x\frac{dz}{dx}$$

$$x\frac{dz}{dx} = \frac{x^2+3(xz)^2}{2x(xz)} - z$$

$$x\frac{dz}{dx} = \frac{1+3z^2}{2z} - z$$

$$x\frac{dz}{dx} = \frac{1+z^2}{2z}$$

Solve (2) to find z as a function of x

Check if SoV is possible

$$\int \frac{2z}{1+z^2}\,dz = \int \frac{1}{x}\,dx$$

$$\ln|1+z^2| = \ln|x| + c$$

$$1+z^2 = A|x|$$

Substitute to obtain y in terms of x

$$\left(\frac{y}{x}\right) = kx - 1$$

$$\frac{y^2}{x^2} = kx - 1$$

$$y^2 = kx^3 - x^2$$