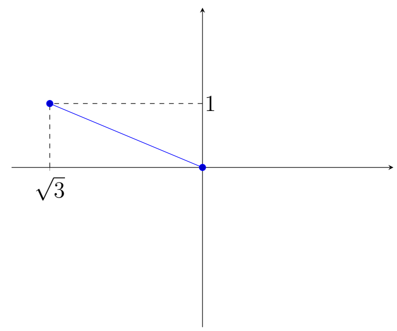
Further Complex Numbers

1 Expressions of complex numbers

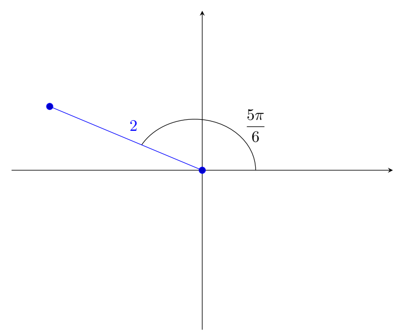
1.1 $\mathbf{x + iy}$

This expresses the coordinate of the point at the end of the vector on the argand diagram.



1.2 $\mathbf{r}(\cos\theta + \mathbf{i}\sin\theta)$

This expresses the length of the line and the angle anticlockwise from the positive x axis



$$2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$$

1.3 $re^{i\theta}$

This uses the same parameters as $r(\cos\theta + i\sin\theta)$

2 Multiplying and dividing complex numbers

2.1 Multiplying

2.1.1 Trigonometric form

$$Z_1Z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2)$$

Apply the cos addition formula to the first two terms

$$= r_1r_2(\cos(\theta_1+\theta_2) + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2)$$

Apply the sin addition formula to the last two terms

$$= r_1r_2(\cos(\theta_1+\theta_2) + \sin(\theta_1+\theta_2))$$

2.1.2 Exponential form

$$Z_1Z_2 = r_1e^{i\theta_1} \times r_2e^{i\theta_2}$$

Apply laws of indices

$$Z_1Z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$$

2.2 Dividing

2.2.1 Trigonometric form

$$\frac{Z_1}{Z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

Multiply by the complex conjugate

$$\frac{Z_1}{Z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$$

Expand

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \times \frac{cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}{\cos^2\theta_2 - i\cos\theta_2\sin\theta_2 + i\sin\theta_2\cos\theta_2 + \sin^2\theta_2}$$

Simplify

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \times \Big(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\Big)$$

2.2.2 Exponential form

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \times \frac{e^{i\theta_1}}{e^{i\theta_2}}$$
$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \times e^{i(\theta_1-\theta_2)}$$

2.3 Comparison

Multiplying	Dividing
Multiply modulus, add arguments	Divide modulus, subtract arguments

3 De Moivre's Theorem

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

3.1 Positive proof

Prove true for n=1

$$r(\cos\theta + i\sin\theta) = r(\cos\theta + i\sin\theta)$$

True for n=1

Assume true for n=k

$$[r(\cos\theta + i\sin\theta)]^k = r^k(\cos(k\theta) + i\sin(k\theta))$$

Prove true for n=k+1

$$[r(\cos\theta + i\sin\theta)]^{k+1}$$

$$[r(\cos\theta + i\sin\theta)]^k \times (r(\cos\theta + i\sin\theta))^1$$

$$r^k(\cos(k\theta) + i\sin(k\theta)) \times r(\cos\theta + i\sin\theta)$$

$$r^k r(\cos(k\theta + \theta) + i\sin(k\theta + \theta))$$

$$r^{k+1}(\cos((k+1)\theta) + i\sin((k+1)\theta))$$

True

3.2 Negative proof

n=-m

$$[r(\cos\theta + i\sin\theta)]^{-m}$$

Multiply by complex conjugate

$$\frac{1}{[r(\cos\theta + i\sin\theta)]^m} \times \frac{[r(\cos\theta - i\sin\theta)]^m}{[r(\cos\theta - i\sin\theta)]^m}$$

Apply positive De Moivre's Theorem

$$\frac{r^m(\cos(m\theta) - i\sin(m\theta))}{r^m(\cos(m\theta) + i\sin(m\theta)) \times r^m(\cos(m\theta) - i\sin(m\theta))}$$

Simplify and expand

$$\frac{\cos(m\theta) - i\sin(m\theta)}{r^m(\cos^2 m\theta - i\cos m\theta\sin m\theta + i\cos m\theta\sin m\theta + \sin^2 m\theta}$$

Simplify

$$\frac{\cos m\theta - i\sin m\theta}{r^m} = r^{-m}(\cos m\theta - i\sin m\theta)$$

Rewrite

$$r^{-m}(\cos(-m\theta) + i\sin(-m\theta))$$

Replace -m with n

$$r^n(\cos(n\theta) + i\sin(n\theta))$$