Brachistochrone Optimal Control Problem

The objective of this project is to provide experience with formulating the first-order optimality conditions of an optimal control problem and to solve these optimality conditions using a simple shooting method.

As stated, the first-order optimality conditions for each problem are solved using a shooting method. A shooting method is formulated as follows. Consider the following two-point boundary-value problem. Determine the trajectory $y(t) \in R^{n_y}$ on the time interval $t \in [t_0, t_f]$ that satisfies the system of ordinary differential equations

$$y(t) = f(t, y(t)) \quad (1)$$

subject to the boundary conditions

$$b(y(t_0), t_0, y(t_f), t_f) = 0 \quad (2)$$

where,

$$b : R^n \times R \times R^n \times R \longrightarrow R^{n_b}.$$

Note that the function b given in Eq. (2) may be such that some of the boundary conditions are given at the initial time, t_0, while other boundary conditions may be given at the terminal time t_f. A shooting method for solving the aforementioned boundary value problem is given as follows:
(1) Choose an initial guess for any or all components of $y(t_0)$ that are necessary to provide a full set of initial conditions.
(2) Choose an initial guess for the initial time, t_0, and/or the terminal time, t_f.
(3) Numerically solve the differential equations given in Eq. (??) on the time interval $t \in [t_0, t_f]$.
(4) Using the value of $y(t_f)$ together will all other required quantities, evaluate the boundary conditions given in Eq. (2).
(5) Update the unknown values of $y(t_0)$, t_0 and/or t_f and return the Step (3) until the boundary conditions in Eq. (2) are satisfied to within a given tolerance.
Note that the iterative procedure given in Steps (1)–(5) above together form a shooting method.
In this assignment the boundary-value problem will be solved using a shooting method that incorporates the MATLAB algebraic equation solver fsolve together with the differential equation ode113.

The objective is to determine the trajectory $(x(t), y(t), v(t))$ and the control $\theta(t)$ that minimize the objective functional

$$J = t_f$$

subject to the differential equation constraints

$$\dot{x} = v \sin \theta,$$
$$\dot{y} = v \cos \theta,$$
$$\dot{v} = g \cos \theta,$$

the boundary conditions

$$x(0) = x_0 = 0,$$
$$y(0) = y_0 = 0,$$

$$v(0) = v_0 = 0,$$
$$x(t_f) = x_f = 2,$$
$$y(t_f) = y_f = 2,$$
$$v(t_f) = Free,$$

and the parameter $g = 10$.

Note: For this problem it is not possible to obtain a well-defined analytic expression for the control. Instead, the control itself must be obtained by solving a root-finding problem from the condition $\partial H/\partial\theta = 0$.

We know,

$$H = L + \lambda^T f$$
$$= \lambda_x v \sin\theta + \lambda_v g \cos\theta + \lambda_y v \cos\theta$$
$$H = \lambda_x v \sin\theta + (\lambda_v g + \lambda_y v)\cos\theta$$

Costate equations

$$\dot{\lambda}_x = -H_x = 0$$
$$\dot{\lambda}_y = -H_y = 0$$
$$\dot{\lambda}_v = -H_v = -\lambda_x \sin\theta - \lambda_y \cos\theta$$

We know,
Optimal Control $H_\theta = 0$

$$H_\theta = \lambda_x v \cos\theta - (\lambda_v g + \lambda_y v)\sin\theta = 0$$
$$\lambda_x v \cos\theta = (\lambda_v g + \lambda_y v)\sin\theta$$
$$\tan\theta = \frac{\lambda_x v}{(\lambda_v g + \lambda_y v)}$$

But, we cannot find the control angle by taking atan of the above equation.

Using, transversality condition to find all constraints.
We know, $t_0 = 0 \rightarrow \delta t_0 = 0$ ie. Transversality condition on H does not apply.

$t_f = free$ -> $\delta t_f \neq 0$ ie. Transversality condition of H does apply

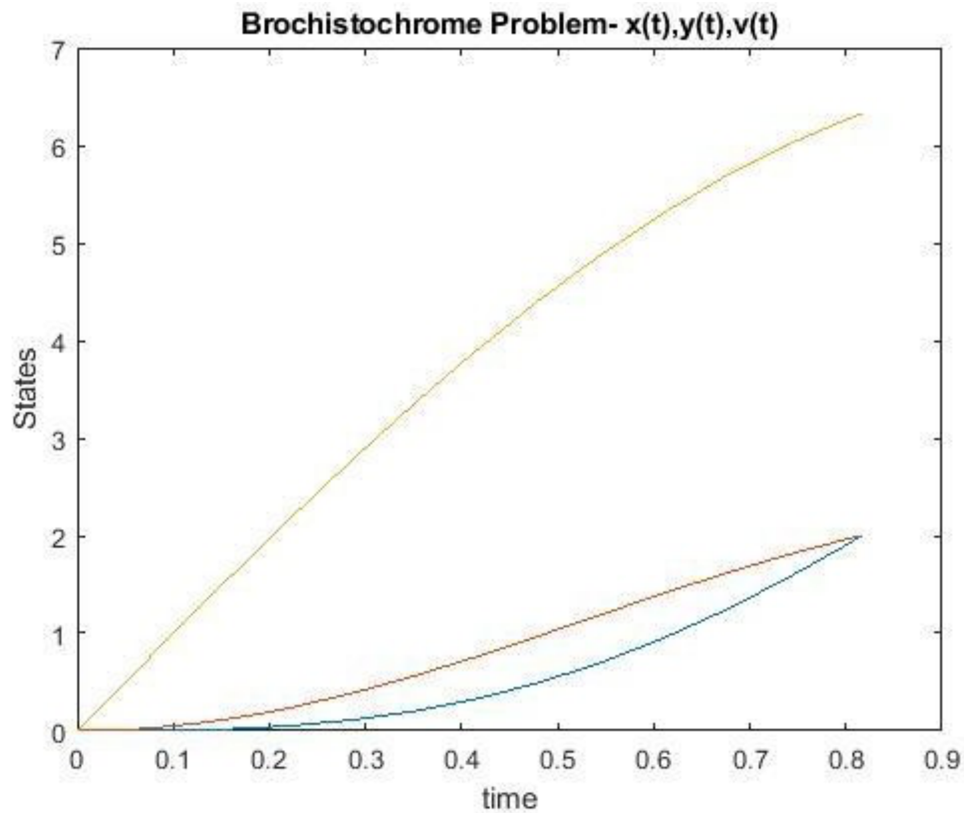
$$H(t_f) + \frac{\partial M}{\partial t_f} - v^T \frac{\partial b}{\partial t_f} = H(t_f) + 1 = 0$$
$$H(t_f) = -1$$

We have, (x(0), y(0), v(0)) = (0,0,0) -> $\delta x_0 = 0$
Transversality condition on $\lambda(t_0)$ does not apply
But, (x(t_f), y(t_f), v(t_f)) = (xf,yf,free)
this implies that the transversality condition on δx_f applies.

$$\lambda_v(t_f) - \frac{\partial M}{\partial v(t_f)} + v^T \frac{\partial b}{\partial v(t_f)} = 0$$
$$\delta\lambda_v(t_f) = 0$$

Solution obtained from single indirect shooting method.

1. Output



Workspace

Name	Value
E	[2.9221e-13;-6.4171e-13;-1.0905e-14;1.1990e-14]
g	10
lambda_0	[-0.1954;-0.0746;-0.1323;0.8165]
lambda_v_0	0.2000
lambda_v_tf	0
lambda_x_0	0.2000
lambda_y_0	0.2000
options	*1x1 struct*
P	*24x7 double*
P_guess	[0.2000;0.2000;0.2000;2]
t	*24x1 double*
t_f	0.8165
t_f_guess	2
v_0	0
x_0	0
x_tf	2
y_0	0
y_tf	2

Conclusion

The optimal final time was calculated to be lambda_0(4) = 0.8165 seconds, when my initial guess of final time was 2 seconds. And the ode solver fsolve failed at a higher guess of final time t_f.