Hyper-Sensitive Optimal Control Problem

The objective of this project is to provide experience with formulating the first-order optimality conditions of an optimal control problem and to solve these optimality conditions using a simple shooting method.

As stated, the first-order optimality conditions for each problem are solved using a shooting method. A shooting method is formulated as follows. Consider the following two-point boundary-value problem. Determine the trajectory $y(t) \in R_{ny}$ on the time interval $t \in [t_0, t_f]$ that satisfies the system of ordinary differential equations

$$y(t) = f(t, y(t)) \quad (1)$$

subject to the boundary conditions

$$b(y(t_0), t_0, y(t_f), t_f) = 0 \quad (2)$$

where,

$$b : R_n \times R \times R_n \times R \longrightarrow R_{nb}.$$

Note that the function b given in Eq. (2) may be such that some of the boundary conditions are given at the initial time, t_0, while other boundary conditions may be given at the terminal time t_f. A shooting method for solving the aforementioned boundary value problem is given as follows:

(1) Choose an initial guess for any or all components of $y(t_0)$ that are necessary to provide a full set of initial conditions.

(2) Choose an initial guess for the initial time, t_0, and/or the terminal time, t_f.

(3) Numerically solve the differential equations given in Eq. (??) on the time interval $t \in [t_0, t_f]$.

(4) Using the value of $y(t_f)$ together will all other required quantities, evaluate the boundary conditions given in Eq. (2).

(5) Update the unknown values of $y(t_0)$, t_0 and/or t_f and return the Step (3) until the boundary conditions in Eq. (2) are satisfied to within a given tolerance.

Note that the iterative procedure given in Steps (1)–(5) above together form a shooting method.

In this assignment the boundary-value problem will be solved using a shooting method that incorporates the MATLAB algebraic equation solver fsolve together with the differential equation ode113.

To determine the trajectories x(t) and the control input u(t) on the interval $t \in [t_0, t_f]$ where, $t_0 = 0$ with the objective to minimize the cost function.

$$J = \frac{1}{2}\int_0^{t_f} \big(x^2(t) + u^2(t)\big)dt$$

Subject to constraints

$$\dot{x} = -x + u \;\; \rightarrow \;\; f(x) - \dot{x} = 0$$

and the boundary conditions

$$x(0) = x_0 = 1$$
$$x(t_f) = x_f = 1$$
$$t_f = fixed$$

Let $\lambda(t) = Legrange\ multiplier\ Associated\ with\ the\ dynamics\ constraints.$

Let,

$$J_a = J + \int_0^{t_f} \lambda(f(x,u) - \dot{x})dt$$
$$= \frac{1}{2}\int_0^{t_f} \big(x^2(t) + u^2(t)\big)dt + \int_0^{t_f} \lambda(f(x,u) - \dot{x})dt$$
$$= \int_0^{t_f} L + \lambda(f - \dot{x})dt$$
$$J_a = \int_0^{t_f} H - \lambda\dot{x}dt$$

We know that the optimal solution of $J_a = J^*{}_a$ should be equal to the optimal solution.

For the 1st Variation of J_a

$$\delta J_a = \int_0^{t_f} \delta H - \lambda\delta\dot{x} - \delta\lambda\dot{x}\, dt$$

we know that $H = H\,(x, u, \lambda)$

$$\delta H = \frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial u}\,\delta u + \frac{\partial H}{\partial \lambda}\delta\lambda$$

So,

$$\delta J_a = \int_0^{t_f} \frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial u}\,\delta u + \frac{\partial H}{\partial \lambda}\delta\lambda - \lambda\delta\dot{x} - \delta\lambda\dot{x}\, dt$$
$$= -\lambda\delta x_0^{t_f} + \int_0^{t_f} \frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial u}\,\delta u + \frac{\partial H}{\partial \lambda}\delta\lambda - \lambda\delta\dot{x} + \delta x\dot{\lambda}\, dt$$
$$\delta J_a = \int_0^{t_f} (\frac{\partial H}{\partial x} + \dot{\lambda})\delta x + \frac{\partial H}{\partial u}\,\delta u + \left(\frac{\partial H}{\partial \lambda} - \dot{x}\right)\delta\lambda\, dt$$

For $\delta J_a = 0$ we need,

$$\frac{\partial H}{\partial x} + \dot{\lambda} = 0$$

$$\frac{\partial H}{\partial u} = 0$$
$$\left(\frac{\partial H}{\partial \lambda} - \dot{x}\right) = 0$$
Therefore,

$$\dot{x} = \frac{\partial H}{\partial \lambda} = f$$
$$\dot{\lambda} = -\frac{\partial H}{\partial x}$$
$$\frac{\partial H}{\partial u} = 0$$
we know, $H = L + \lambda f = \frac{1}{2}(x^2 + u^2) + \lambda(-x + u)$

This implies,
$$\frac{\partial H}{\partial \lambda} = -x + u \rightarrow \dot{x} = -x + u$$

$$\frac{\partial H}{\partial x} = x - \lambda \rightarrow \dot{\lambda} = -x + \lambda$$
$$\frac{\partial H}{\partial u} = u + \lambda \rightarrow u = -\lambda$$

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$
$x(0) = x_0$ is specified
- the transversality conditions on the costate does not apply
- Since the start and end times are fixed the transversality condition on the costates does not apply.
- Transversality conditions on the Hamiltonian at $t_0 \; and \; t_f$ do not apply

The highlighted equation represents the first order differential equations governing this process.

Solution from Single Shooting Method.

The roots of the function are at lambda_0 = 0.4142

The following are my plots of state 'x' when simulated at time tf ∈ [10, 20, 30, 40, 50].

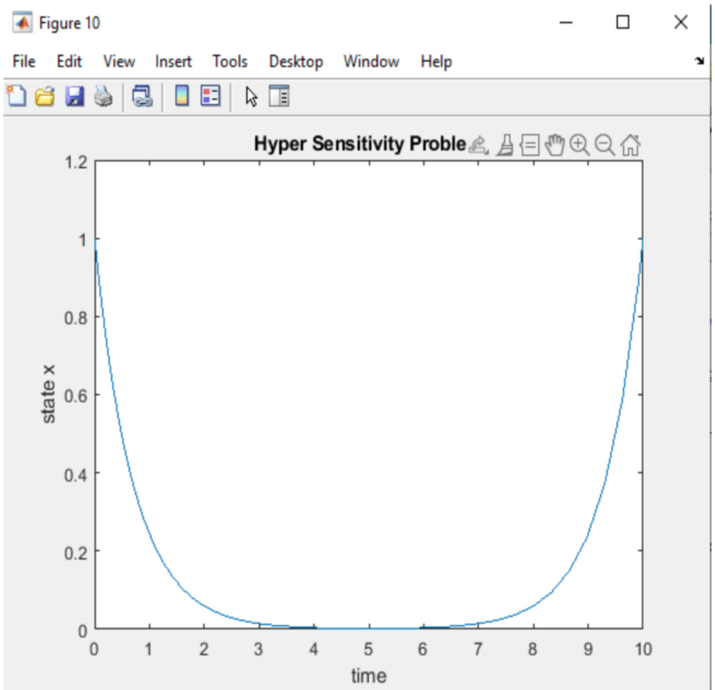
Figure 10
File Edit View Insert Tools Desktop Window Help
Hyper Sensitivity Proble
state x
1.2 1 0.8 0.6 0.4 0.2 0
0 1 2 3 4 5 6 7 8 9 10
time

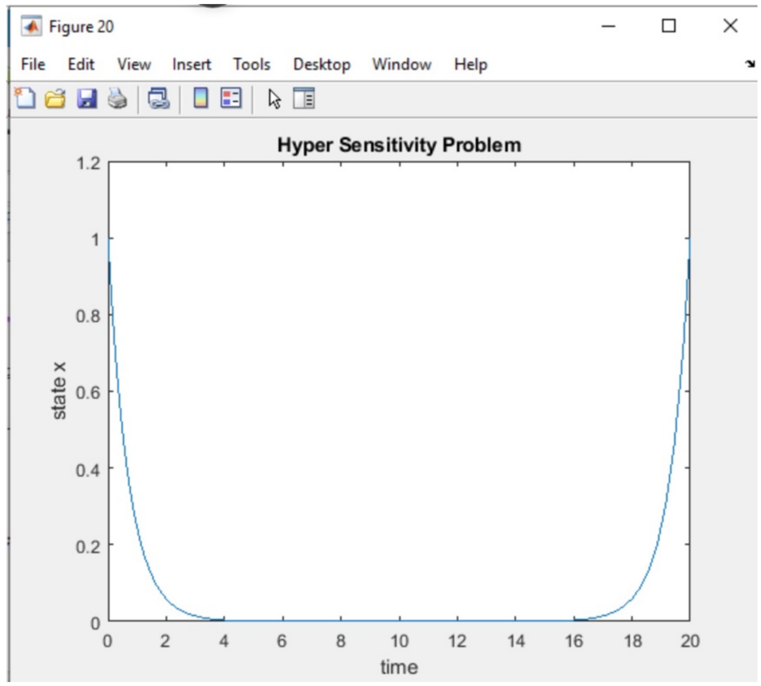
Figure 20
File Edit View Insert Tools Desktop Window Help
Hyper Sensitivity Problem
state x
1.2 1 0.8 0.6 0.4 0.2 0
0 2 4 6 8 10 12 14 16 18 20
time

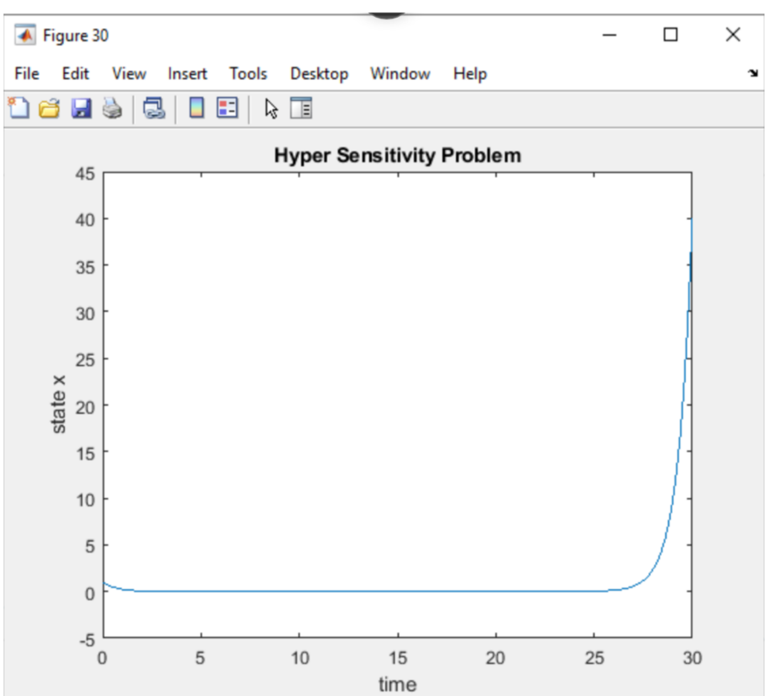
Figure 30
File Edit View Insert Tools Desktop Window Help
Hyper Sensitivity Problem
state x
45 40 35 30 25 20 15 10 5 0 -5
0 5 10 15 20 25 30
time

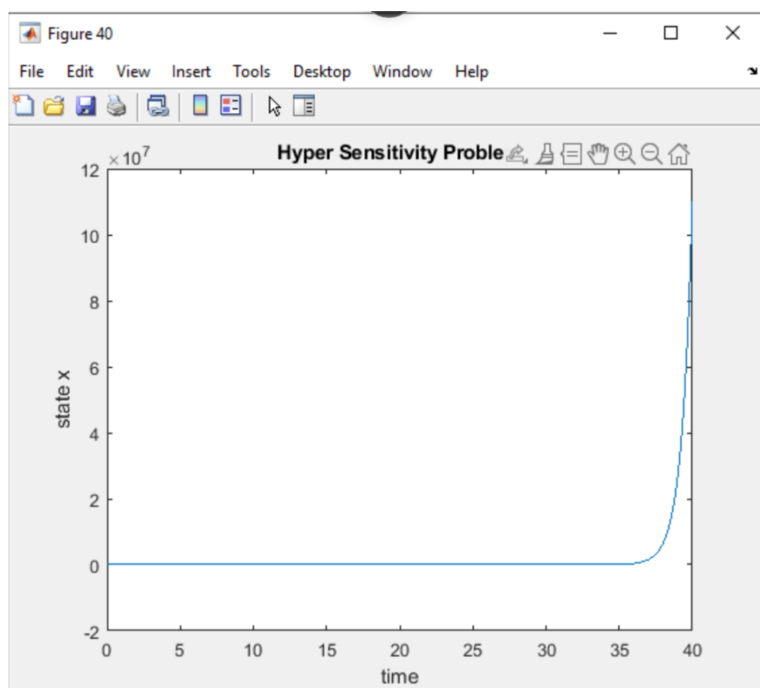
Figure 40
File Edit View Insert Tools Desktop Window Help
Hyper Sensitivity Proble
×10^7
state x
12 10 8 6 4 2 0 -2
0 5 10 15 20 25 30 35 40
time

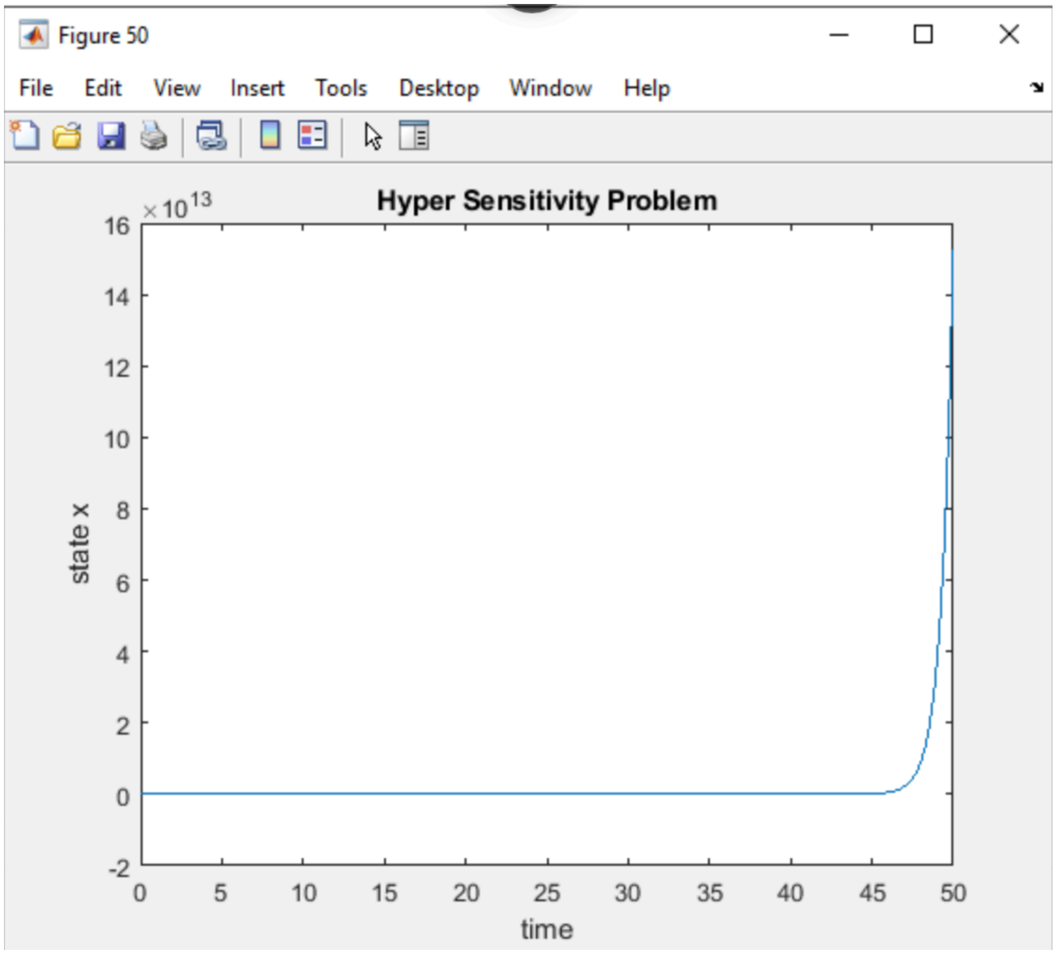
Figure 50
File Edit View Insert Tools Desktop Window Help
Hyper Sensitivity Problem
× 10^13
16
14
12
10
8
6
4
2
0
-2
state x
0
5
10
15
20
25
30
35
40
45
50
time