Hyper-Sensitive Optimal Control Problem

To determine the trajectories x(t) and the control input u(t) on the interval $t \in [t_0, t_f]$ where, $t_0 = 0$ with the objective to minimize the cost function.

$$J = \frac{1}{2}\int_0^{t_f}\left(x^2(t) + u^2(t)\right)dt$$

Subject to constraints

$$\dot{x} = -x + u \;\; \rightarrow \;\; f(x) - \dot{x} = 0$$

and the boundary conditions

$$x(0) = x_0 = 1$$
$$x(t_f) = x_f = 1$$
$$t_f = fixed$$

Let $\lambda(t) = Legrange\ multiplier\ Associated\ with\ the\ dynamics\ constraints.$

Let,

$$J_a = J + \int_0^{t_f} \lambda(f(x,u) - \dot{x})dt$$
$$= \frac{1}{2}\int_0^{t_f}\left(x^2(t) + u^2(t)\right)dt + \int_0^{t_f} \lambda(f(x,u) - \dot{x})dt$$
$$= \int_0^{t_f} L + \lambda(f - \dot{x})dt$$
$$J_a = \int_0^{t_f} H - \lambda\dot{x}dt$$

We know that the optimal solution of $J_a = J^*{}_a$ should be equal to the optimal solution.

For the 1st Variation of J_a

$$\delta J_a = \int_0^{t_f} \delta H - \lambda\delta\dot{x} - \delta\lambda\dot{x}\, dt$$

we know that $H = H\,(x, u, \lambda)$

$$\delta H = \frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial u}\,\delta u + \frac{\partial H}{\partial \lambda}\delta\lambda$$

So,

$$\delta J_a = \int_0^{t_f} \frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial u}\,\delta u + \frac{\partial H}{\partial \lambda}\delta\lambda - \lambda\delta\dot{x} - \delta\lambda\dot{x}\, dt$$
$$= -\lambda\delta x_0^{t_f} + \int_0^{t_f} \frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial u}\,\delta u + \frac{\partial H}{\partial \lambda}\delta\lambda - \lambda\delta\dot{x} + \delta x\dot{\lambda}\, dt$$
$$\delta J_a = \int_0^{t_f} (\frac{\partial H}{\partial x} + \dot{\lambda})\delta x + \frac{\partial H}{\partial u}\,\delta u + \left(\frac{\partial H}{\partial \lambda} - \dot{x}\right)\delta\lambda\, dt$$

For $\delta J_a = 0$ we need,

$$\frac{\partial H}{\partial x} + \dot{\lambda} = 0$$
$$\frac{\partial H}{\partial u} = 0$$
$$\left(\frac{\partial H}{\partial \lambda} - \dot{x}\right) = 0$$

Therefore,

$$\dot{x} = \frac{\partial H}{\partial \lambda} = f$$
$$\dot{\lambda} = -\frac{\partial H}{\partial x}$$
$$\frac{\partial H}{\partial u} = 0$$

we know, $H = L + \lambda f = \frac{1}{2}(x^2 + u^2) + \lambda(-x + u)$

This implies,

$$\frac{\partial H}{\partial \lambda} = -x + u \rightarrow \dot{x} = -x + u$$

$$\frac{\partial H}{\partial x} = x - \lambda \rightarrow \dot{\lambda} = -x + \lambda$$
$$\frac{\partial H}{\partial u} = u + \lambda \rightarrow u = -\lambda$$

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

$x(0) = x_0$ is specified

- the transversality conditions on the costate does not apply
- Since the start and end times are fixed the transversality condition on the costates does not apply.
- Transversality conditions on the Hamiltonian at t_0 and t_f do not apply

The highlighted equation represents the first order differential equations governing this process.

Solution from Single Shooting Method.

The roots of the function are at lambda_0 = 0.4142

The following are my plots of state 'x' when simulated at time tf ∈ [10, 20, 30, 40, 50].

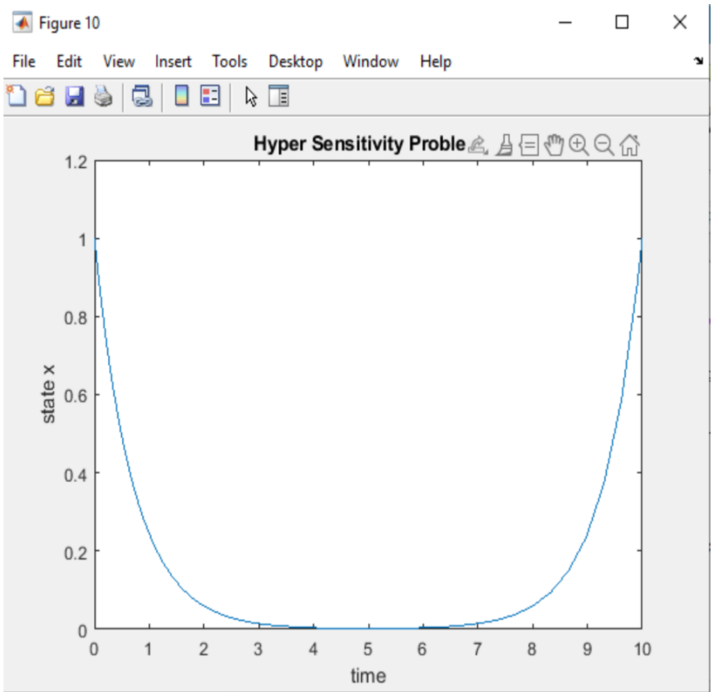
Figure 10
File Edit View Insert Tools Desktop Window Help
Hyper Sensitivity Proble
1.2
1
0.8
0.6
0.4
0.2
0
state x
0 1 2 3 4 5 6 7 8 9 10
time

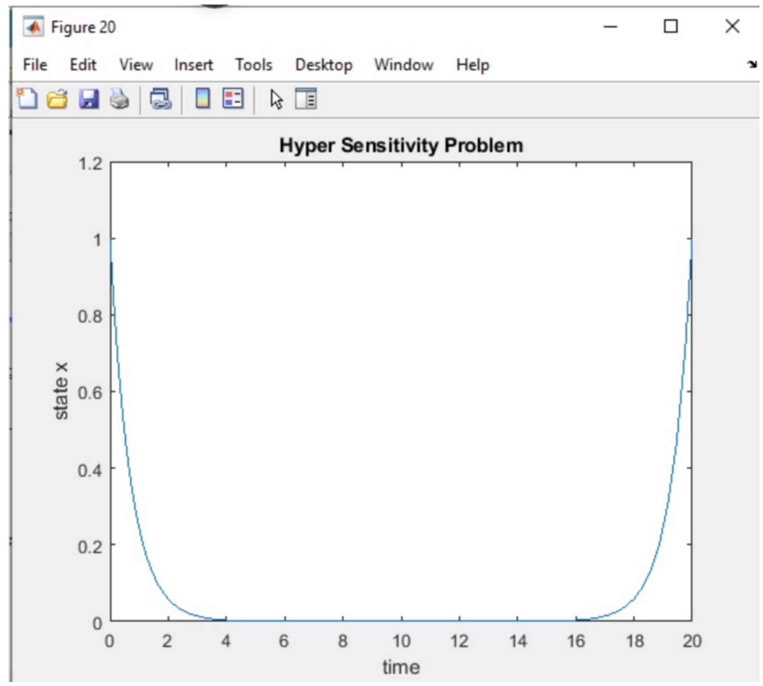
Figure 20
File Edit View Insert Tools Desktop Window Help
Hyper Sensitivity Problem
1.2
1
0.8
0.6
0.4
0.2
0
state x
0 2 4 6 8 10 12 14 16 18 20
time

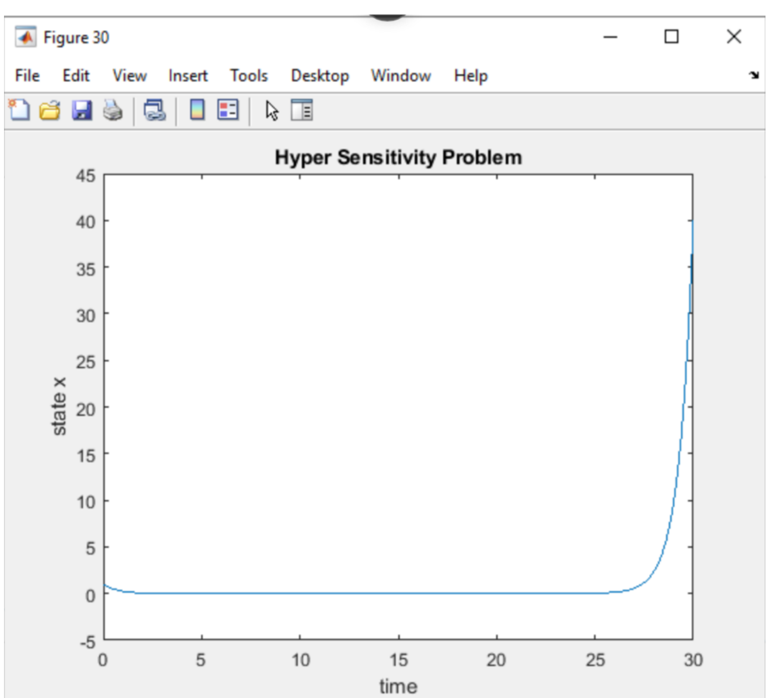
Figure 30
File Edit View Insert Tools Desktop Window Help
Hyper Sensitivity Problem
45
40
35
30
25
20
15
10
5
0
-5
state x
0 5 10 15 20 25 30
time

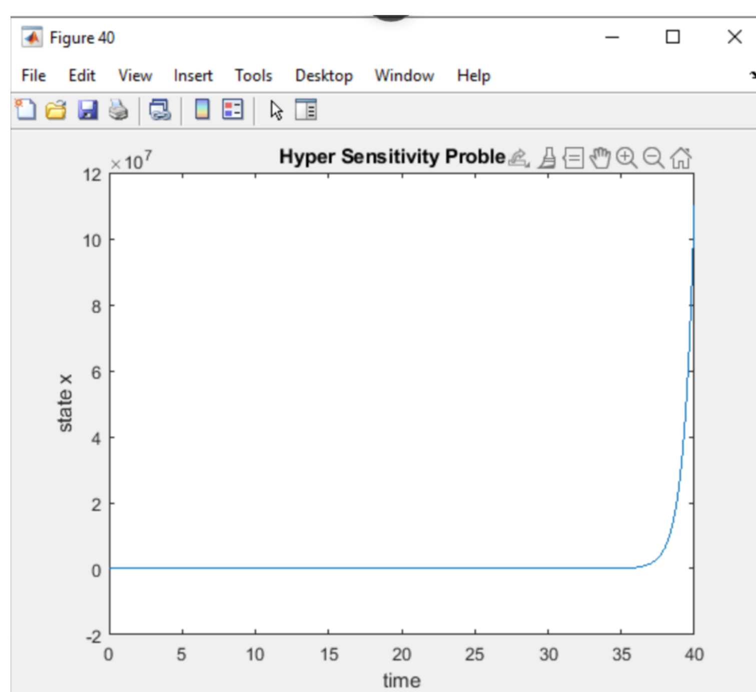
Figure 40
File Edit View Insert Tools Desktop Window Help
Hyper Sensitivity Proble
×10^7
12
10
8
6
4
2
0
-2
state x
0 5 10 15 20 25 30 35 40
time

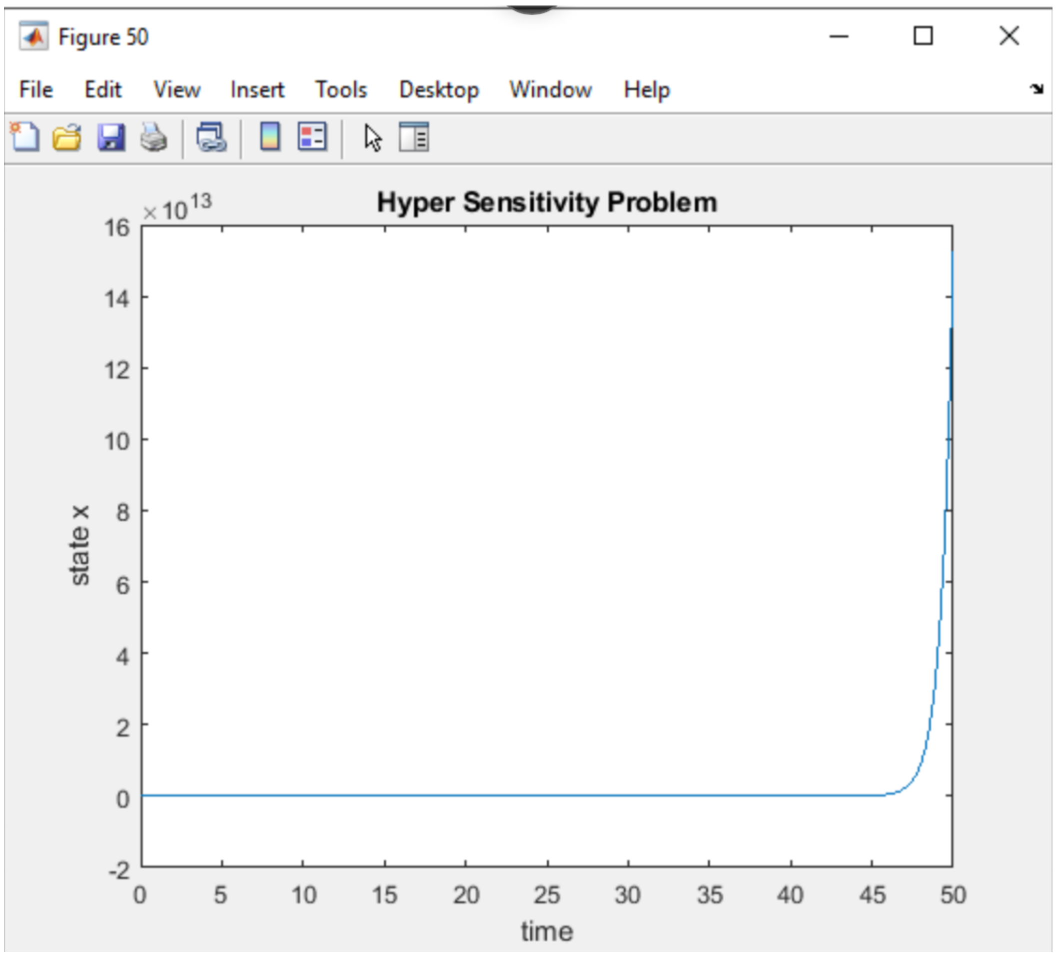
Figure 50
File Edit View Insert Tools Desktop Window Help
Hyper Sensitivity Problem
×10^13
16
14
12
10
8
6
4
2
0
-2
state x
0 5 10 15 20 25 30 35 40 45 50
time