

CONTROL SYSTEMS THEORY  
University of Florida  
Mechanical and Aerospace Engineering

**HW 6**

*You are encouraged to use MATLAB<sup>®</sup> to verify your answers whenever you can. However, unless specified, do not use MATLAB<sup>®</sup> to solve a problem.*

**Problem 1.** Find the transfer functions  $G_{u_1y}$  (from input  $u_1$  to the output  $y$ ) and  $G_{u_2y}$  (from input  $u_2$  to the output  $y$ ) of the following LTI system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

**Problem 2.** 1. Compute the eigenvalues of the following matrix (by hand):

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & -1 \\ 0 & 6 & 2 \end{bmatrix}$$

*Hint: This is a “block diagonal” matrix, one block being  $-1$  and the other block being  $\begin{bmatrix} -3 & -1 \\ 6 & 2 \end{bmatrix}$ . Use the fact that the eigenvalues of a block diagonal matrix is the union of the eigenvalues of the blocks that make up the matrix.*

2. Is the system  $\dot{x} = Ax$ , with  $A$  shown above, stable in the sense of Lyapunov, unstable, asymptotically stable, or “cannot say without information about Jordan blocks of the eigenvalues”?

**Problem 3.** A dynamic model of a car, where  $p(t)$  is the position and  $F(t)$  is the traction, is described by the following system of coupled ODEs:

$$m\ddot{p} = \alpha\dot{p} + F(t),$$

$$\ddot{F}(t) + a_1\dot{F}(t) + a_0F(t) = k\theta(t),$$

where  $m, \alpha, a_1, a_0, k$  are parameters of the system. The normalized throttle angle  $\theta$  (taking values between 0 and 1) is the input and the speed  $\dot{p}$  is the measured output.

1. Express the dynamic system in state space form, with position being one of the states.
2. Is it an LTI system?

**Problem 4** (with and without I.C.). Execute the following matlab script and answer the questions that follow:

```

clear all

A = [0.1 -2; 1 -3]; B = [0.1, -0.2]'; C = [1 1], D = 0;
P_ss = ss(A,B,C,D);
[num,den] = ss2tf(A,B,C,D);
P_tf = tf(num,den);

Ts = 0.001;
time = [0:Ts:6]';
u = sin(2*pi*2*time) - sin(2*pi*20*time - pi/3);
x0 = [-2; 3];

y_ss = lsim(P_ss,u,time,x0);
y_tf = lsim(P_tf,u,time);
figure
plot(time,y_tf,'b',time,y_ss,'r--');
ylabel('y'), xlabel('t');

```

1. “ $y_{ss}(t)$  and  $y_{tf}(t)$  are outputs of the same system in response to the same input”. True or false?
2. Why is  $y_{ss}(t)$  different from  $y_{tf}(t)$ ?
3. If we perform the simulation for longer time, will the two signals become the same for  $t$  sufficiently large? (Answer this question without running the simulation for a larger  $t$  but by analyzing the system.)

**Problem 5.** For each of the following state matrix  $A$ , determine if the corresponding LTI system  $\dot{x} = Ax$  is stable, asymptotically stable, or unstable in the sense of Lyapunov:

- 1.

$$A = \begin{bmatrix} 1 & 1 \\ -4 & -3 \end{bmatrix}$$

- 2.

$$A = PXP^{-1}, X = \begin{bmatrix} -1 - 5j & 1 & 23.5 \\ 0 & -1 + 5j & 2 \\ 0 & 0 & -0.01 \end{bmatrix}$$

where  $P$  is an invertible matrix. *Hint: notice that  $A$  and  $X$  are related by a similarity transform. Use the result that if two matrices are similar, they have the same eigenvalues.*

- 3.

$$A = PXP^{-1}, X = \begin{bmatrix} 5j & 1 & 0 & 0 \\ 0 & 5j & 0 & 0 \\ 0 & 0 & -5j & 1 \\ 0 & 0 & 0 & -5j \end{bmatrix}$$

where  $P$  is an invertible matrix.

**Problem 6** (Open loop control). Consider the LTI system  $\dot{x} = x + u$ , where both the state and the input are scalars. Let the initial condition  $x_0$ , and suppose the input signal defined below is used to drive the system, where  $T$  is a pre-specified constant:

$$u(t) = \frac{-2x_0}{1 - e^{-2T}} e^{-t}, \quad 0 \leq t \leq T$$

1. Verify that this control signal, when applied to the system during the time interval  $[0, T]$ , will bring the state from the initial condition  $x_0$  to the origin at time  $T$ . *Hint: use the “Variation of Constants” formula:  $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$  and plug in the given  $u$ .*
2. For the same initial condition, if the time interval  $T$  (within which the state is to be brought to the origin) is made shorter, does the required control effort increases or decreases? Does the answer make intuitive sense?

**Problem 7** (satellite). The equations of motion of a satellite, linearized around a steady state solution, are given by  $\dot{x} = Ax + Bu$ , where  $x_1$  and  $x_2$  denote perturbations in the radius and the radial velocity, respectively,  $x_3$  and  $x_4$  denote perturbations in the angle and angular velocity, and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The input vector consists of a radial thrust  $u_1$  and a tangential thrust  $u_2$ .

1. Show that the system is controllable for  $u$ . Is this true for every possible value of the angular speed  $\omega$ ? *You have to evaluate the rank of the controllability matrix by inspection. Rearrange the rows (or columns) does not change the rank.*
2. Can the system still be controllable if the radial thruster fails? What if the tangential thruster fails? *Hint: if the radial thruster fails then  $u_1(t) = 0$  for all  $t$ , in effect, the first component of  $u$  does not exist. Write down the new system  $\dot{x} = Ax + B'u'$  where  $u'$  is now a scale (= the command from the tangential thruster), and examine the controllability of this new system.*

(Hint: Review the linear algebra notes in e-learning about row operations and rank; that'll help in determining the rank of the controllability matrix.)

**Problem 8.** Consider the LTI system  $\dot{x} = Ax + Bu$ , where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -k_1 & -1 & 0 \\ 0 & k_2 & c \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

where  $k_1, k_2, c$  are constants.

1. Is  $(A, B)$  controllable? If the answer depends on the values of the constants  $k_1, k_2$  and  $c$ , state the dependency precisely.
2. Suppose  $k_2 = 0$ . Determine the set of points in  $\mathbb{R}^3$  that can be transferred to the origin in finite time by using appropriate control signals.

3. Suppose  $k_2 = 0$ . For what values of  $c$  is the following statement true: “it is possible to transfer every points in  $\mathbb{R}^3$  to the origin, if not in finite time then at least asymptotically (meaning, as  $t \rightarrow \infty$ , by using an appropriate input signal”.

**Problem 9.** You are asked to design a state feedback controller to stabilize the system  $\dot{x} = Ax + Bu$ , where  $A, B$  are given below,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

1. You are asked to design  $K$  so that the eigenvalues of  $A - BK$  are at  $-5 \pm 2j$ . Is it possible? Why or why not?
2. If possible, design such a  $K$  using the “place” command in MATLAB<sup>©</sup>.
3. Simulate the closed loop system for the initial condition  $[10, 10]^T$ . There is no external input, so you only have to compute the response of the system due to the initial condition. Use the `lsim` command in MATLAB<sup>©</sup>. Provide your matlab script and the resulting figure of the states vs. time. The script and figure should fit in one page. Please use different line styles and colors so that the two states are clearly visible even when printed on a B&W printer.

**Problem 10.** Consider the LTI system  $\dot{x} = Ax$  ( $x \in \mathbb{R}^n$ ) and suppose that there exists positive definite matrices  $P, Q \in \mathbb{R}^{n \times n}$  and a positive number  $\mu$  such that the equation

$$A^T P + PA + 2\mu P = -Q$$

holds. Prove that all eigenvalues of  $A$  have real part less than  $-\mu$ . *Hint: Use the result that all eigenvalues of  $A$  have real part less than  $-\mu$  if and only if all eigenvalues of  $A + \mu I$  have real part less than 0. (You proved one part of this statement in the previous homework. ) And use the Lyapunov test for stability.*

**Problem 11.** Consider a plant whose state-space description is  $\dot{x} = Ax + B(u + v), y = Cx$ , where  $v$  is a disturbance input

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1 \quad 1]$$

and the following state-feedback controller

$$u = -[k_1 \quad k_2] x$$

where  $k_1$  and  $k_2$  are constants to be chosen (designed).

1. Compute the closed loop state-space model with input  $v$  and output  $y$ , leaving your answer as a function of  $k_1$  and  $k_2$ .
2. Compute the closed loop transfer function from input  $v$  to output  $y$ , leaving your answer as a function of  $k_1$  and  $k_2$ .

3. Determine values of  $k_1$  and  $k_2$  so that the closed loop transfer function from  $v$  to  $y$  becomes<sup>1</sup>

$$\frac{1}{s+1}$$

Designing a controller to achieve a specific closed loop transfer function is called *model matching design*.

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<sup>1</sup>Hint: you will need to cancel one zero with a pole of the open loop transfer function you determined in part (2).