Control Systems Theory
University of Florida
Mechanical and Aerospace Engineering

HW 5

You are encouraged to use MATLAB© to verify your answers whenever you can. However, unless specified, do not use MATLAB© to solve a problem.

Problem 1. Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$$

1. Determine the range space of the matrix A above and show it pictorially. (hint: you should be able to determine the linearly independent columns by inspection. If you cannot, reduce it to row-echelon form)
2. What is the rank of this matrix? Verify your answer by using the `rank` command in MATLAB©.
3. Determine the null space of the matrix A above show it pictorially.

Problem 2. 1. Find a basis for $\mathcal{R}(A)$, where $A = \begin{bmatrix} 1 & 3 & -1 & -3 \\ 2 & 4 & 1 & 2 \end{bmatrix}$. (hint: you should be able to do this by inspection). Show $\mathcal{R}(A)$ pictorially.

2. What is the rank of this matrix? Verify your answer by using the `rank` command in MATLAB©.
3. Compute an orthonormal basis of $\mathcal{R}(A)$ for the A given in part 1 by using the `orth` command in MATLAB©. Show the span of this basis pictorially. (The MATLAB© answer may look different from your answer at first glance, but the picture will tell you they are the same)

Problem 3. 1. Compute a row-echelon form of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix} \tag{1}$$

2. What is the $\mathcal{R}(A)$? (write it as the span of basic columns of A)
3. Determine the null space, $\mathcal{N}(A)$ (write it as the span of a basis of the null space)
4. What is the rank of A?

Problem 4. Consider the following system of linear equations:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ 2x_1 + 4x_2 &= 2 \\ 3x_1 + 6x_2 + x_3 &= 4 \end{aligned}$$

1. Does it have no solution, an unique solution, or infinite number of solutions?
2. If no solutions, say so. If there is an infinite number of solutions, find one. If there is an unique solution, find that one.

Problem 5. Consider the system of linear equations $Ax = y$, where:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 0 \\ 3 & 6 & 1 \end{bmatrix}, y = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

1. Does it have no solution, an unique solution, or infinite number of solutions?
2. If no solutions, say so. If there is an infinite number of solutions, find one. If there is an unique solution, find that one.

Problem 6. 1. Compute the eigenvalues and eigenvectors of $\begin{bmatrix} 1 & 0.5 \\ -0.5 & 0 \end{bmatrix}$ by hand, and then verify your answer by using the `eig` command in MATLAB©.

2. Is this matrix diagonalizable? (hint: compute the eigenvectors)

Problem 7. Consider the matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

What is the rank of this matrix? 3, since it has exactly 3 non-zero eigenvalues.

Prove the following generalization of this observation: If a real symmetric $n \times n$ matrix A has m zero eigenvalues and $m < n$ (meaning, it has $n - m$ non-zero eigenvalues), the rank of the matrix is $n - m$. (Hint: use diagonalization, and then use the following result: if $X, Y, Z \in \mathbb{R}^{n \times n}$ and X, Z are invertible, then the rank of the product matrix XYZ is equal to the rank of Y.)

Problem 8. Suppose $a \in \mathbb{R}^n$. Prove that the $n \times n$ matrix aa^T has only one non-zero eigenvalue, which is given by $a^T a$. What is the corresponding eigenvector? (Hint: That there is only one non-zero eigenvector follows from a rank argument that uses the result from the previous problem. Call that λ. if v is the corresponding eigenvector, then $Mv = \lambda v$. Pre multiply both sides by a^T and recognize that $v^T a = a^T v$ because $v^T a$ is a scalar so it is equal to its transpose.)

Problem 9. Consider the two square matrices A and $A - \mu I$, where μ is an arbitrary constant. Prove that

1. if λ is an eigenvalue of A, then $\lambda - \mu$ is an eigenvalue of $A - \mu I$,
2. The two matrices have the same set of eigenvectors.

Problem 10. [Similarity preserves eigenvalues] Two matrices A and B are called *similar* if there exists an invertible matrix V such that $V^{-1}AV = B$. Prove that if two matrices are similar, they have the same set of eigenvalues[1].

[1]This result will be useful later to show that a transfer function can be realized by an infinite number of state space models

Problem 11 (A fun(?) exercise. There nothing to submit; will not be graded)**.** Run the following MATLAB© code segment to visualize the range space of the A matrix provided in the code.

```
clear all

A = [1 2; 2 -3; 3 5];

fh = figure
allxs = randn(2,1000)*10;
allys = A*allxs;
figure(fh); hold on;
plot3(allys(1,:),allys(2,:),allys(3,:),'b.');
plot3(0,0,0,'r*');
grid on;
set(gca,'CameraPosition', [316.5072 -1.7181e+03 1.4311e+03]);
set(gca,'CameraTarget', [-10 -11.8747 0]);
set(gca,'CameraUpVector',[0 0 1]);
set(gca,'CameraViewAngle',9.4228);
```