EML 5311

# Control System Theory

## HW-5

**Problem 1 :-**

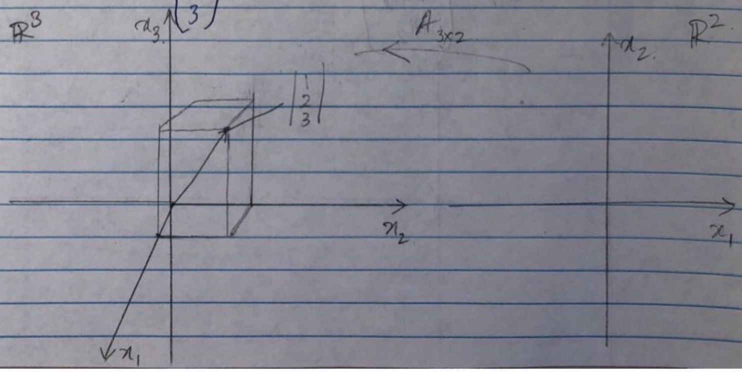
① $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$ $R(A) = ?$

$$Ax = y \implies \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} x_2.$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} x_1 + 2x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} (x_1 + 2x_2).$$

The vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ determines the range space of A $(R(A))$.

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2. Rank of a matrix = no of linearly independent columns of A.

$\Rightarrow$ rank (A) = 1. ($\because$ there is only one LI column).

(MATLAB)

3. N(A) = ?.

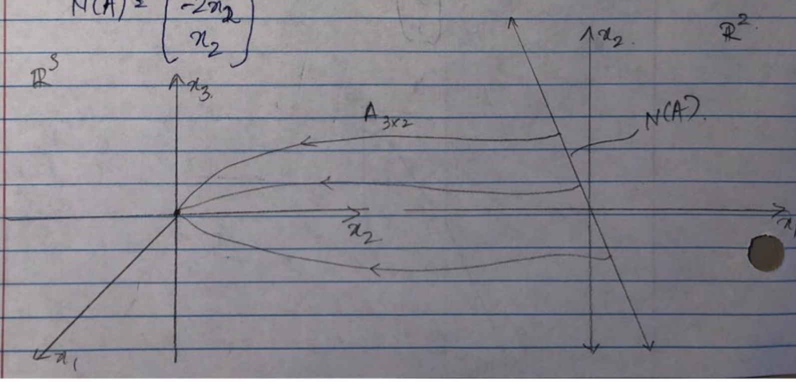
$$Ax = 0$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$$\Rightarrow \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} x_2 = 0.$$

$$x_1 + 2x_2 = 0. \quad \Rightarrow x_1 = -2x_2.$$
$$2x_1 + 4x_2 = 0.$$
$$3x_1 + 6x_2 = 0$$

$$N(A) = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix}$$



```
A=[1 2; 2 4; 3 6];
rank(A)
```

```
ans =

     1
```

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Problem - 2

① Finding the a basis of R(A) using row-echelon form.

$$A = \begin{bmatrix} 1 & 3 & -1 & -3 \\ 2 & 4 & 1 & 2 \end{bmatrix}.$$

$R_2 - 2R_1 \to R_2$.

$$A = \begin{bmatrix} 1 & 3 & -1 & -3. \\ 0 & -2 & 3 & 8 \end{bmatrix}$$

$R_2 \to (-\frac{1}{2}) R_2$.

$$A = \begin{bmatrix} 1 & 3 & -1 & -3 \\ 0 & 1 & \frac{-3}{2} & -4 \end{bmatrix}.$$

$R_1 - 3R_2 \to R_1$.

$$A = \begin{bmatrix} 1 & 0 & 7/2 & 9 \\ 0 & 1 & -3/2 & -4 \end{bmatrix} = E_A.$$

Hence, the basic columns of $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ also form ~~that~~ a basis of R(A).

a basis of $R(A) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.



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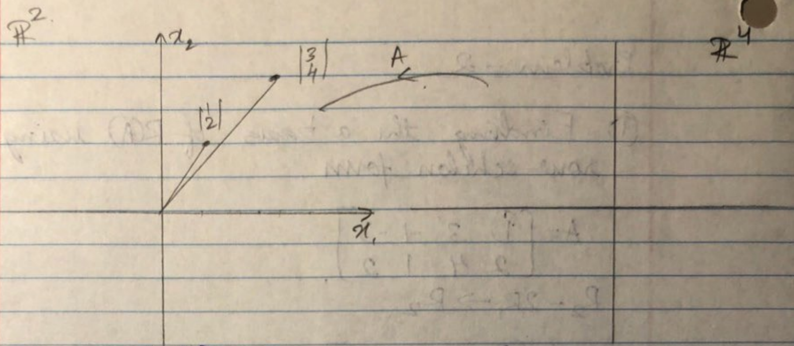
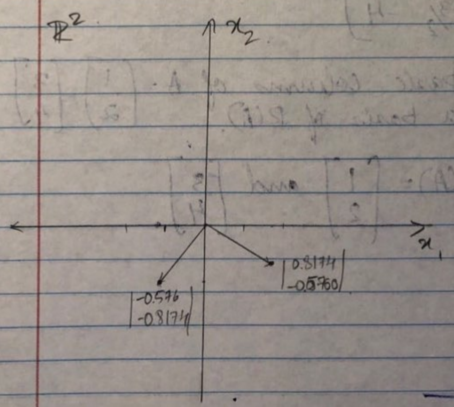



2. rank $(A) = 2$. ($\because$ rank is equal to the no of non zero rows in $E_A$).

(MATLAB).

3. Orthonormal bases of $R(A)$ is

$$\text{orth}(A) = \begin{bmatrix}-0.5760\\-0.8174\end{bmatrix}, \begin{bmatrix}0.8174\\-0.5760\end{bmatrix}.$$

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```
A=[1 3 -1 -3; 2 4 1 2];
rank(A)
orth(A)
```

```
ans =

     2


ans =

   -0.5760    0.8174
   -0.8174   -0.5760
```

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Problem 3 -

1. Row - echelon form.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix}$$

$R_2 - 2R_1 \to R_2$ and $R_3 - 3R_1 \to R_3$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{bmatrix}$$

$R_2 \to (-1/2) R_2$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$R_3 + 2R \to R_3$ and $R_1 - R_2 \to R_1$.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = E_A.$$

2. $R(A) =$ span (basic colums of $E_A$).

$$R(A) = \text{span}\left( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$



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3. $N(A) = N(E_A)$.

$E_A x = 0$.

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$x_2$ : free.

$x_1 + 2x_2 = 0$.
$x_3 = 0$.

$$N(A) = \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{matrix} -2\beta \\ \beta \\ 0 \end{matrix} \quad \beta \in \mathbb{R} \right\}.$$

$$\begin{bmatrix} -2\beta \\ \beta \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \beta \qquad N(A) = \operatorname{span}\left( \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right) //$$

4. $\operatorname{rank}(A) = 2$. ($\because$ there are 2 non zero rows in $E_A$).

## Problem - 4

From the given set of equations. we get.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix} \qquad y = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}.$$

Checking consistency of linear equations.



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$$[A/y] = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 0 & 2 \\ 3 & 6 & 1 & 4 \end{bmatrix} \quad \& \quad Z = \begin{bmatrix} 2 \\ 2 \\ 4 \\ -1 \end{bmatrix}$$

To find the ~~argument matrix~~ row echelon form of the argument matrix.

$R_2 - 2R_1 \rightarrow R_2$ and $R_3 - 3R_1 \rightarrow R_3$.

$$E_{Ag} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & -2 & -2 \end{bmatrix}$$

$R_2 \rightarrow (-\frac{1}{2}) R_2$

$$E_{Ag} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & -2 \end{bmatrix}$$

$R_3 + 2R_2 \rightarrow R_3$ and $R_1 - R_2 \rightarrow R_1$.

$$E_{Ag} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_{Ag} X = 0 \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ -1 \end{bmatrix} = 0.$$

since $0(-1) = 0$ A solution exists for the set of equations.

$\Rightarrow$ The equations have infinite number of solutions

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2. From the previous step we get

$$x_1 + 2x_2 + 0(x_3) - 1 = 0.$$
$$\Rightarrow \quad x_1 + 2x_2 = 1$$

and

$$x_3 - 1 = 0 \Rightarrow x_3 = 1.$$

This implies that the set of equation have infinite number of solutions and one such solution is.

$$N(A) = \begin{bmatrix} 1/2 \\ 1/4 \\ 1 \end{bmatrix}$$ which satisfies all the above conditions

Problem 5 -

(i) $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 0 \\ 3 & 6 & 1 \end{bmatrix}, \quad y = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}.$

$$[A/y] = A_g = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 5 & 0 & 2 \\ 3 & 6 & 1 & 4 \end{bmatrix}$$

Writting $A_g$ in its row-echelon form,

$R_2 - 2R_1 \rightarrow R_2$ and $R_3 - 3R_1 \rightarrow R_3$.

$$A_g = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -2 & -2 \end{bmatrix}$$

$R_1 - 2R_2 \rightarrow R_1$

$$A_g = \begin{bmatrix} 1 & 0 & 5 & 6 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -2 & -2 \end{bmatrix}$$



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$R_3 \to \left(-\frac{1}{2}\right)R_3$.

$$\begin{bmatrix} 1 & 0 & 5 & 6 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} = E_{Ag}.$$

$R_2 \to R_2 + 2R_3$ and $R_1 \to R_1 - 5R_3$.

$$E_{Ag} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$E_{Ag} X = 0$.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ -1 \end{bmatrix} = 0.$$

we get

$x_1 - 1 = 0 \Rightarrow x_1 = 1$

$x_2 = 0$.

$x_3 - 1 = 0. \Rightarrow x_3 = 1$.

(2) therefore, the system of linear equation has an unique solution.

where $x_1 = 1$, $x_2 = 0$ and $x_3 = 1$

$N(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$ //

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Problem - 6

$$A = \begin{bmatrix} 1 & 0.5 \\ -0.5 & 0 \end{bmatrix}.$$

then the characteristic equation is.

$$|A - \lambda I| = 0.$$

$$\Rightarrow \left| \begin{bmatrix} 1 & 0.5 \\ -0.5 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1-\lambda & 0.5 \\ -0.5 & -\lambda \end{bmatrix} \right| = 0.$$

$$\Rightarrow (1-\lambda)(-\lambda) - (0.5)(-0.5) = 0.$$

$$-\lambda + \lambda^2 + 0.25 = 0.$$

$$\lambda^2 - \lambda + 0.25 = 0.$$

$$(\lambda - 0.5)^2 = 0.$$

$$\lambda_1 = 0.5, \text{and } 0.5 = \lambda_2.$$

The 2 eigenvalues are.

$$\lambda_1 = 0.5 \quad \text{and} \quad \lambda_2 = 0.5 \,_{//}.$$

To find eigenvectors, when $\lambda_1 = 0.5$

$$A v_1 = \lambda_1 v_1$$

$$(A - \lambda_1) v_1 = 0$$

$$\cancel{\begin{bmatrix} 0.5 & 0 \\ -1 & -0.5 \end{bmatrix}} \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & -0.5 \end{bmatrix} \cdot v_1 = 0.$$

$$\begin{bmatrix} 0.5 & 0.5 \\ -0.5 & -0.5 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0.$$

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$$\Rightarrow 0.5v_{11} + 0.5v_{12} = 0.$$
$$0.5v_{11} = -0.5v_{12}.$$
$$v_{11} = -v_{12}.$$

$\|^{ly}$ $$-0.5v_{11} - 0.5v_{12} = 0.$$
$$v_{11} = -v_{12}.$$

$$V_1 = K_1 \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$ where $K_1$ = arbitary constant.

$$V_2 = K_2 \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$

(MATLAB)

eigenvector 'v' = $V = \begin{bmatrix} 0.7071 & -0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}$.

eigenvalues = $d = \begin{bmatrix} 0.500 & 0 \\ 0 & 0.500 \end{bmatrix}$

2. It is diagonizable if.

$A = PDP^{-1}$.

where $P = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$D = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$.

$P^{-1} = \frac{1}{1-1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. = Does not exist.

therefore, the matrix A is not diagonalizable.



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```
A=[1 0.5; -0.5 0];
[v,d]=eig(A)
% B=(v*d)*(inv(v));
```

```
v =

    0.7071   -0.7071
   -0.7071    0.7071


d =

    0.5000         0
         0    0.5000
```

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Problem -7

From the equation of diagonalization

$$PAP^{-1} = \Lambda \longrightarrow \text{diagonal matrix}$$

$\Rightarrow$ $\text{rank}(PAP^{-1}) = \text{rank}(\Lambda)$

we know that,

$$\text{rank}(XYZ) = \text{rank}(Y)$$

therefore,

$$\text{rank}(A) = \text{rank}(\Lambda)$$

since $\Lambda$ is a diagonal matrix

$$\text{rank}(A) = \text{rank}(\Lambda)$$
$$= \# \text{ of non zero rows}$$
$$= n-m$$

Thus,

$$\text{rank}(A) = n-m$$

Problem 8-

Let $\lambda, v$ be the non zero eigenvalue and eigen vector of $aa^T$. Then,

$(aa^T)v = \lambda v$

$a^T(aa^T)v = a^T \lambda v$ ($\because$ pre multiplying $a^T$ on both sides).

$(a^Ta)a^Tv = \lambda(a^Tv)$

$\Rightarrow$ $a^Ta = \lambda$ $\Rightarrow$ $a^Ta = a^Ta$.

. Hence proved.



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To determine the eigenvector 'V' we have to find out what value of v satisfies the equation,

$$aa^T v = a^T a v.$$

from this we can see that the equation holds only when $v = a$.

Problem 9 -

Let A and $A - \mu I$ be a square $n \times n$ matrix, where $\mu$ is an arbitrary constant.

(1) From the definition of eigenpairs we have.

$$Av = \lambda_1 v \quad - (1)$$

and

$$(A - \mu I) v = \lambda_2 v.$$

$$Av - \mu I v = \lambda_2 v.$$

$$\lambda_1 v - \mu v = \lambda_2 v.$$

$$(\lambda_1 - \mu) v = \lambda_2 v.$$

$\Rightarrow$ Hence proved.

(2) Since,

$$(A - \mu I) v = (\lambda - \mu) v$$

the two vectors have the same set of eigenvector, v.



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## Problem -10

To show that similarity preserves eigenvalues.

Let $\lambda, v$ be the eigenpair of a square $n \times n$ matrix $A$, and let $V$ be some invertible real matrix of the same dimension as $A$. and

$$B \triangleq V^{-1} A V.$$

then,

$$Av = \lambda v$$

$$V^{-1} A v = V^{-1} \lambda v \qquad (\because \text{ pre multiplying both sides by } V^{-1})$$

$$V^{-1} I A v = V^{-1} \lambda v.$$

$$\Rightarrow \quad V^{-1} A (V V^{-1}) v = V^{-1} \lambda v$$

$$B V^{-1} v = \lambda V^{-1} v$$

$$\Rightarrow \quad B \tilde{v} = \underline{\lambda \tilde{v}} \qquad (\tilde{v} \triangleq V^{-1} v).$$

Hence, it is clear that $\lambda, \tilde{v}$ is also an eigenpair of $B$.



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