

Control System Theory

HW-7

Problem 1 - Given -

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

 $n=3$ .

$$C = [B \ AB \ A^2B]$$

$$AB = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$$

$$e_0 = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & 3 \\ -1 & 0 & 3 \end{bmatrix}$$

Finding Row-echelon form of C.

$$R_3 \rightarrow R_3 + \frac{1}{3}R_2 \quad R_3 \rightarrow R_3 + R_1$$

$$C = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & 3 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{Rank}(C) = 3 \quad (\text{the system is controllable}).$$

The controllability subspace has a dimension 2.

Observability matrix =  $\Theta = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$

$$CA = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -3 \\ 0 & -2 & 3 \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} 0 & 2 & -3 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -6 \\ 0 & 1 & 6 \end{bmatrix}$$

$$\Theta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & -3 \\ 0 & -2 & 3 \\ 0 & -1 & -6 \\ 0 & 1 & 6 \end{bmatrix} \quad \Theta^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & -2 & -1 & 1 \\ 0 & -1 & -3 & 3 & 3 & -6 \\ 0 & 1 & 6 & -6 & 6 & 0 \end{bmatrix}$$

Calculating the Row-Echelon form of  $\Theta^T$   
 $R_1 \leftrightarrow R_2 ; R_2 \leftrightarrow R_3$ .

$$\Theta^T = \begin{bmatrix} 1 & -1 & 2 & -2 & -1 & 1 \\ 0 & -1 & -3 & 3 & -6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \rightarrow -R_2 ; R_1 \rightarrow R_1 + R_2$$

$$\Theta^T = \begin{bmatrix} 1 & 0 & 5 & -5 & 5 & -5 \\ 0 & 1 & 3 & -1 & 6 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Rank}(\Theta) = \text{Rank}(\Theta^T) = 2 \neq n. \text{ (System is not observable)}$$

The system has a 2 dimensional observable subspace and a one dimensional unobservable subspace.

2. Since the system has 3 inputs equal to the number of dimension  $n$  of matrix  $A$  the system is controllable. But since the number of output is not equal to the value of  $n$ , the system is not stable.

### Problem 2 -

1) The given system is not observable when

$$[c_1, c_2, c_3] = [10 \ 0 \ 5]$$

2) The system is observable when

$$[c_1, c_2, c_3] = [10 \ 2 \ 5].$$

3). p:  $c$ 's do not take up the eigen values of  $A$ .

q: the system is observable.

$$P \Rightarrow q.$$

P is the sufficient condition for q. and  
q is a necessary condition for P.

### Problem -3 -

Given -  $\hat{x} = A\hat{x} + Bu + L(y - C\hat{x} - Du)$

$$1. \Rightarrow \dot{\hat{x}} = A\hat{x} - LC\hat{x} + Ly + Bu - LDu \\ = (A - LC)\hat{x} + Ly + (B - LD)u$$

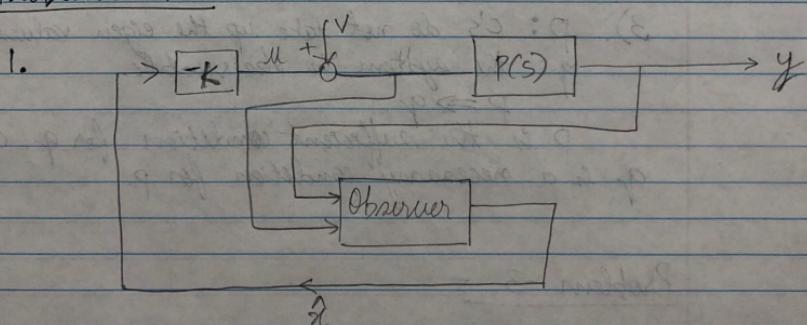
$$\dot{\hat{x}} = (A - LC)\hat{x} + [L \quad B - LD] \begin{bmatrix} y \\ u \end{bmatrix}$$

2. MATLAB problem.

3. The Observer poles have to be faster than controller poles.

If the eigenvalues of  $A - LC$  are chosen so that the poles of observer are further away from the controller poles, the transient state of the system increases or hence, the system reaches steady state slower.

#### Problem - 4



2. We know that,

$$\begin{aligned} c &\equiv x - \hat{x} \\ \Rightarrow \dot{c} &= \dot{x} - \dot{\hat{x}} \\ &= Ax + Bu - [A\hat{x} + B(\hat{x} - (Cx - Du))] \\ &= Ax - A\hat{x} - L((Cx + Du) - (\hat{x} - Du)) \\ \dot{c} &= Ae - LCe \\ c &= (A - LC)e. \end{aligned}$$

and

$$\begin{aligned} \dot{x} &= Ax + B(-Kx) + Bu \\ &= Ax - BK\hat{x} + Bu \\ &= Ax - BK(x - e) + Bu \\ &= (A - BK)x + BKe + Bu. \end{aligned}$$

$$\Rightarrow \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u.$$

$$\begin{aligned} y &= Cx + Du \\ y &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + Du \end{aligned}$$

3. Real Time = 0.0974.

(MATLAB)

4. (MATLAB & SIMULINK).

Initially I started out by defining the  $(A, B, C, D)$  matrices and then by computing the eigenvalues of  $A$  I was able to determine the controllable poles as

slow as possible (ie very close to the origin). Then I made the observer poles 5-10 times slower than my controller poles. From this I was able to calculate K and L required to design the determine the (A, B, C, D) matrices to of the closed loop. And along with the given initial conditions the desired result was reached.

### Problem - 5

$$\text{Given} - \ddot{y} + 2\dot{y} + 5y = f(t), y(0) = -4, \dot{y}(0) = 5.$$

$$\text{Let } y = x_1$$

$$\dot{y} = x_2$$

$$x_1 = x_2$$

$$\dot{x}_1 = u - 2x_2 - 5x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = Cx + Du$$

1. (MATLAB).

2. (MATLAB).

3. It is clear from the previous exam problem that the predicted solutions fit accurately to the state of the actual system when the sampling period ( $T_s$ ) are small. Hence, my advise to the engineer would be to make have a small sampling period.

### Problem - 6

$$1. \text{ Given } - A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} -3 & 1 \end{bmatrix}, D = 0.$$

$$H(s) = C(sI - A)^{-1}B + D \\ = \begin{bmatrix} -3 & 1 \end{bmatrix} \left[ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \right]^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0.$$

$$= \begin{bmatrix} -3 & 1 \end{bmatrix} \begin{bmatrix} s-1 \\ -3s+2 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 1 \end{bmatrix} \frac{1}{(s(s-2)-3)} \begin{bmatrix} s-2 & 1 \\ 3 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$H(s) = \begin{bmatrix} -3 & 1 \end{bmatrix} \left( \frac{1}{s(s-2)-3} \right) \begin{bmatrix} 1 \\ s \end{bmatrix} = \left( \frac{1}{s^2-2s-3} \right) (-3+s)$$

$$\Rightarrow H(s) = \frac{s-3}{s^2-2s-3} = \frac{s-3}{(s-3)(s+1)}$$

$$H(s) = \underline{\underline{\frac{1}{s+1}}}$$

2. Finding eigen values of A.  
Characteristic equation =  $\det(A - \lambda I) = 0$ .

$$\det \left( \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = -2\lambda + \lambda^2 - 3.$$

$$\Rightarrow -2\lambda + \lambda^2 - 3 = 0 \Rightarrow \lambda = 3 \text{ and } -1.$$

Hence, the eigen value of A is the same as the pole of the transfer function.

- (3) The system is ~~not~~ BIBO stable as all it's poles are in SLHP.

$$\text{ie } s+1 \Rightarrow s = -1.$$

But the system is unstable in the sense of Lyapunov since the eigen value of matrix A. ie  $\lambda=3$  is not  $\leq 0$  in SLHP.

### Problem - 7

$$\text{float } a11 = 0.9397$$

$$\text{float } a12 = 0$$

$$\text{float } a21 = 0.06034$$

$$\text{float } a22 = -1$$

$$\text{float } b11 = 1.931$$

$$\text{float } b21 = 0.0607$$

$$\text{float } c11 = 13.75$$

$$\text{float } c12 = 7.813$$

$$\text{float } d = 10.$$

(MATLAB)

Problem - 8

$$\text{Given } P(s) = \frac{0.5}{(s+0.5)(s-1)}$$

$$(a) H_{2R} = \frac{CP}{1+CP} \quad (C(s)) = \frac{s-1}{s+2}$$

$$= \frac{(s-1)(0.5)}{(s+2)(s+0.5)(s-1)} \\ 1 + \frac{(s-1)(0.5)}{(s+2)(s+0.5)(s-1)} \\ = \frac{(s-1)(0.5)}{(s+2)(s+0.5)(s-1) + (s-1)(0.5)}$$

$$= \frac{(s-1)(0.5)}{(s-1)((s+2)(s+0.5) + 0.5)} = \frac{0.5}{(s+2)(s+0.5) + 0.5}$$

Since, all the poles are in SLHP as  $\Rightarrow$  BIBO stable.

$$(b). H_{YR} = \frac{PC}{1+PC}$$

$$\text{since } H_{YR}^{(s)} = H_{2R}^{(s)}$$

$H_{YR}(s)$  is BIBO stable as all its poles are in SLHP.

$$(c) H_{2W} = \frac{P}{1+CP} = \frac{0.5}{(s+0.5)(s-1)} \\ 1 + \frac{(s-1)(0.5)}{(s+2)(s+0.5)(s-1)}$$

$$H_{2W} = \frac{0.5(s+2)}{(s+0.5)(s-1)(s+2) + 0.5(s-1)}$$

$$H_{zw} = \frac{0.5(s+2)}{(s-1)((s+0.5)(s+2)+0.5)}$$

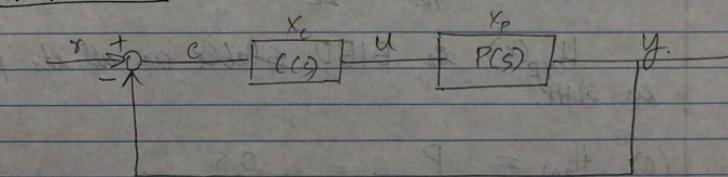
$H_{zw}(s)$  is not BIBO stable as it has ~~does not~~ poles in the SLHP. (ie it has positive poles).

$$(d) H_{ur} = \frac{C(s)}{1 + C(s)P(s)} = \frac{(s-1)}{(s+2)} = \frac{(s-1)^2(s+0.5)}{(s+0.5)(s-1)(s+2)}$$

$$H_{ur}(s) = \frac{(s-1)(s+0.5)}{(s+0.5)(s+2) + 0.5}$$

$H_{ur}(s)$  is BIBO stable as all the poles are in SLHP.

Problem 9 -



Plant

$$\dot{X_p} = A_p X_p + B_p u$$

$$y = C_p X_p + D_p u$$

Controller

$$\dot{X_c} = A_c X_c + B_c e$$

$$u = C_c X_c + D_c e$$

$$y = C_p X_p + D_p (C_c X_c + D_c e)$$

$$= C_p X_p + D_p (C_c X_c + D_c (r - y))$$

$$= C_p X_p + D_p (C_c X_c + D_c r - D_c y)$$

$$y = C_p X_p + D_p C_c X_c + D_p D_c r - D_p D_c y$$

$$y(1 + D_p D_c) = C_p X_p + D_p C_c X_c + D_p D_c r$$

$$y = \left[ \begin{array}{cc} C_p & D_p C_c \\ \frac{1}{1+D_p D_c} & \frac{-1}{1+D_p D_c} \end{array} \right] \begin{bmatrix} X_p \\ X_c \end{bmatrix} + \left[ \begin{array}{c} D_p D_c \\ \frac{1}{1+D_p D_c} \end{array} \right] r(t).$$

$\underbrace{\quad\quad\quad}_{C_{cl}}$        $\underbrace{\quad\quad\quad}_{D_{cl}}$

$$\dot{X_p} = A_p X_p + B_p (C_c X_c + D_c (r - y))$$

$$= A_p X_p + B_p C_c X_c + B_p D_c r - B_p D_c y$$

$$= \begin{bmatrix} A_p & B_p C_c \end{bmatrix} \begin{bmatrix} X_p \\ X_c \end{bmatrix} + B_p D_c r - B_p D_c \left[ \begin{bmatrix} C_p & D_p C_c \\ \frac{1}{1+D_p D_c} & \frac{-1}{1+D_p D_c} \end{bmatrix} \begin{bmatrix} X_p \\ X_c \end{bmatrix} \right]$$

$$\dot{X_p} = \begin{bmatrix} A_p - \frac{B_p D_c C_p}{1+D_p D_c} & B_p C_c - \frac{B_p D_c D_c}{1+D_p D_c} \end{bmatrix} \begin{bmatrix} X_p \\ X_c \end{bmatrix} + \left[ \begin{bmatrix} D_p D_c \\ \frac{1}{1+D_p D_c} \end{bmatrix} r(t) + \begin{bmatrix} B_p D_c + \frac{D_p D_c}{1+D_p D_c} \\ \frac{-1}{1+D_p D_c} \end{bmatrix} y(t) \right]$$

$$\dot{x}_c = A_c x_c + B_c (r - y).$$

$$\begin{aligned} &= A_c x_c + B_c r - B_c y = A_c x_c + B_c r - B_c \left[ \frac{C_p}{1+D_c D_p} \frac{D_p C_c}{1+D_c D_p} \right] x_p + \left[ \frac{D_p D_c}{1+D_p D_c} \right] r(t) \end{aligned}$$

$$\dot{x}_i = \left( \frac{-B_c C_p}{1+D_c D_p} \frac{A_c - B_c D_p C_c}{1+D_c D_p} \right) \begin{bmatrix} x_p \\ x_c \end{bmatrix} + \left[ B_c + \frac{D_p D_c}{1+D_c D_p} \right] r(t).$$

Assuming  $(1+D_c D_p)$  is invertible.

$$\alpha = (1+D_p D_c)^{-1}.$$

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A_p - \alpha B_p C_p D_c & \alpha B_p C_c \\ -\alpha B_c C_p & A_c - \alpha B_c C_c D_p \end{bmatrix} \begin{bmatrix} x_p \\ x_c \end{bmatrix} + \begin{bmatrix} \alpha B_p D_c \\ -\alpha B_c \end{bmatrix} r.$$

$$y = \begin{bmatrix} \alpha C_p & \alpha D_p C_c \end{bmatrix} \begin{bmatrix} x_p \\ x_c \end{bmatrix} + \alpha D_c D_p r.$$

$$z = [x_p^T, x_c^T]^T.$$

$$\Rightarrow z = \begin{bmatrix} A_p - \frac{B_p D_c C_p}{1+D_p D_c} & B_p C_c - \frac{B_p D_c D_p C_c}{1+D_p D_c} \\ -\frac{B_c C_p}{1+D_p D_c} & A_c - \frac{B_c D_p C_c}{1+D_p D_c} \end{bmatrix} z + \begin{bmatrix} \frac{B_p D_c - B_p C_p D_c}{1+D_p D_c} \\ \frac{B_c - B_c D_p C_c}{1+D_p D_c} \end{bmatrix} r$$

$$\Rightarrow y = \begin{bmatrix} C_p & D_p C_c \\ (1+D_p D_c) & (1+D_p D_c) \end{bmatrix} z + \frac{D_p D_c}{1+D_p D_c} r.$$

### Problem - 10

$$\text{Given: } P(s) = \frac{0.5}{(s+0.5)(s-1)} \quad C(s) = \frac{s-1}{s+2}$$

$$P(s) = \frac{0.5}{s^2 + 0.5s - 0.5} = \frac{0.5}{s^2 - 0.5s - 0.5}$$

$$(s^2 - 0.5s - 0.5) C(s) = 0.5 R(s)$$

with zero initial conditions,

we have,

$$\ddot{c} - 0.5\dot{c} - 0.5c = 0.5r$$

$$\text{let } c = x_1$$

$$\dot{c} = x_2$$

$$\ddot{x}_1 = x_2$$

$$\ddot{x}_2 = 0.5r + 0.5x_1 + 0.5x_2$$

$$\Rightarrow \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ 0.5 & 0.5 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} r$$

1. The closed loop is not stable in the sense of Lyapunov as the eigenvalues of  $A$  are not in  $\mathbb{C} \setminus \text{SLHP}$ .
2. Problem 8 showed that some of the external properties of the system were not BIBO stable. And Lyapunov stability is a condition or a form of internal stability of the system. This result shows that zero-pole cancellation of an unstable pole only hides the fact that the system has external instability.

---

```

n=5;
m=1;
p=2;
A=[-14.5 -59.5 -149.5 -38.5 65
    1 0 0 0 0
    0 1 0 0 0
    0 0 1 0 0
    0 0 0 1 0];
B=[ 1; 0; 0; 0; 0];
C=[ 0 0 0 100 500];
D=zeros(size(C,1), size(B,2));
OB=obsv(A,C);
rank_OB=rank(OB);
j=sqrt(-1);
cont_poles=[-15 -30 -50 -15+3j -15-3j];
obsv_poles =[-15+0j -7+3j -7-3j -6+0j 5.5+0j];
K2=place(A',C',obsv_poles);
K1=place(A,B,cont_poles);
L=K2';
A_obs = (A-L*C);
B_obs = [L (B - L*D)];
C_obs = eye(n); %why?
D_obs = zeros(size(C_obs,1),size(B_obs,2));

A_cl=[A-(B*K1) (B*K1)
      zeros(size(A)) A-L*C];
B_cl=[B; zeros(size(B))];
C_cl=[C zeros(size(C))];
D_cl=zeros(size(C_cl,1),size(B_cl,2));

x0=[1 1 1 1 1]';
x0_hat=[0 0 0 0 0]';
SYS=ss(A_cl,B_cl,C_cl,D_cl);
figure(1)
step(SYS)
hold on;
% figure(2)
stepinfo(SYS,'SettlingTimeThreshold',0.02)
Ts = 0.1;
t_final = 15;
time = [0:Ts:t_final]';

% sim('observer1')

```

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```
clc;
clear;
A=[0 1;-5 -2];
B=[ 0; 1];
C=rand(1,2);
D=zeros(size(C,1),size(B,2));
SS=ss(A,B,C,D);
t=[0:0.25:5];
x0=[-4 5];
U=2*sin(2*t)+5;
lsim(SS,U,t,x0);
```

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---

```

clc;
clear;
A=[0 1;-5 -2];
B=[0; 1];
C=[1 0];
D=zeros(size(C,1),size(B,2));
SS=ss(A,B,C,D);
% time=[0:0.25:10];

Ts = 0.5;%time step
time = [0:Ts:10];%array of time points

x_init=[-4; 5];

U=2*sin(2*time)+5;
junk = find(time>=1); index1sec = junk(1);

u = zeros(1,length(time));
u(index1sec:end) = 2*sin(2*time(index1sec:end))+5;
x_euler = zeros(2,length(time)+1);%intialize for storage
x_euler(:,1) = x_init;
x_now = x_init;
x_euler(:,1) = x_init;
for t_index = 1:length(time)
    u_now = u(t_index); %read u(t)
x_next = (eye(2)+Ts*A)*x_now + Ts*B*u_now; % Euler approx
x_euler(:,t_index+1) = x_next; %store state
x_now = x_next; %update state
end;
y_euler = C*x_euler(:,1:length(time)); % y = Cx+Du
figure
h=lsim(SS,U,time,x_init);
plot(time,h,'go:');
hold on;
plot(time,y_euler,'bo:');


```

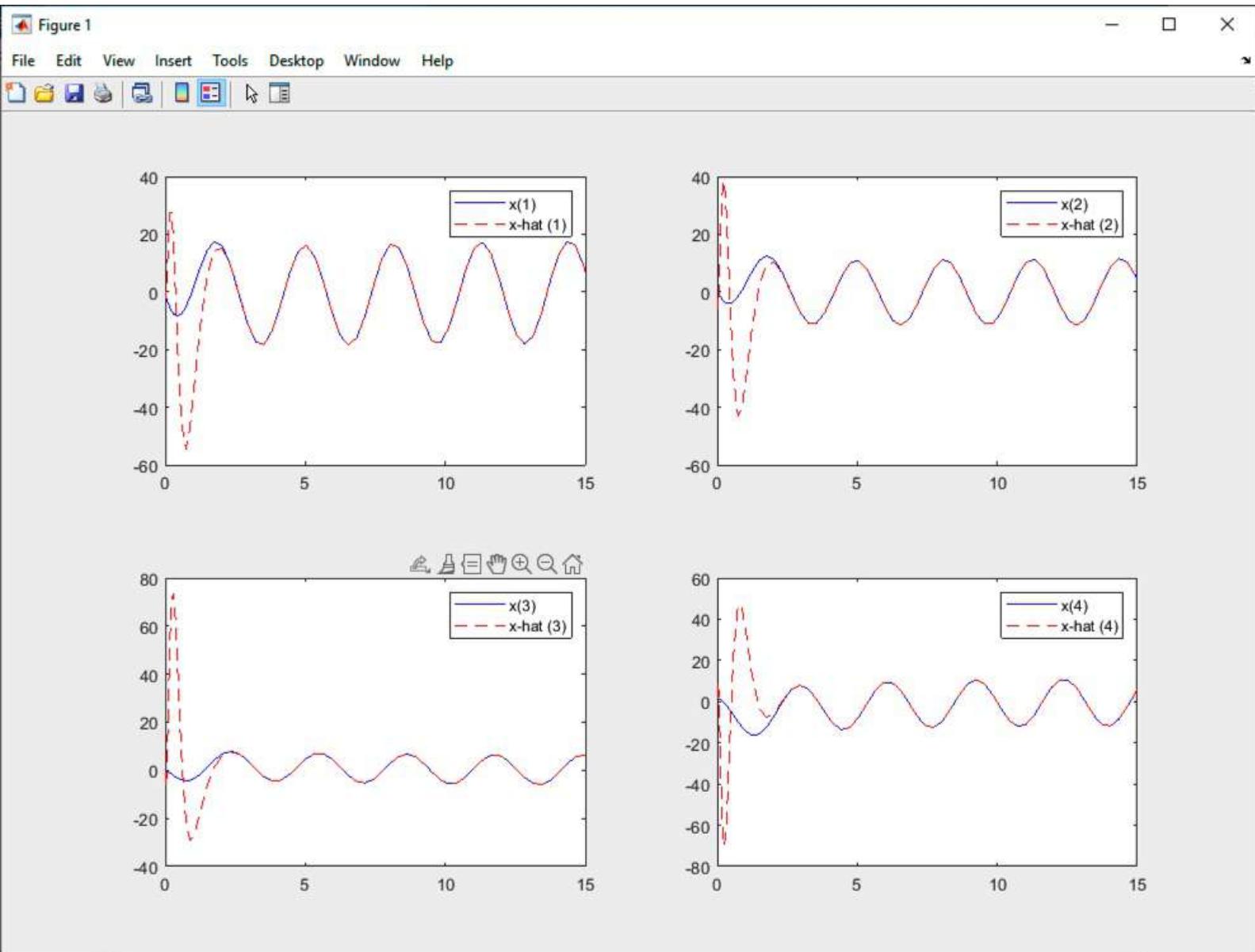
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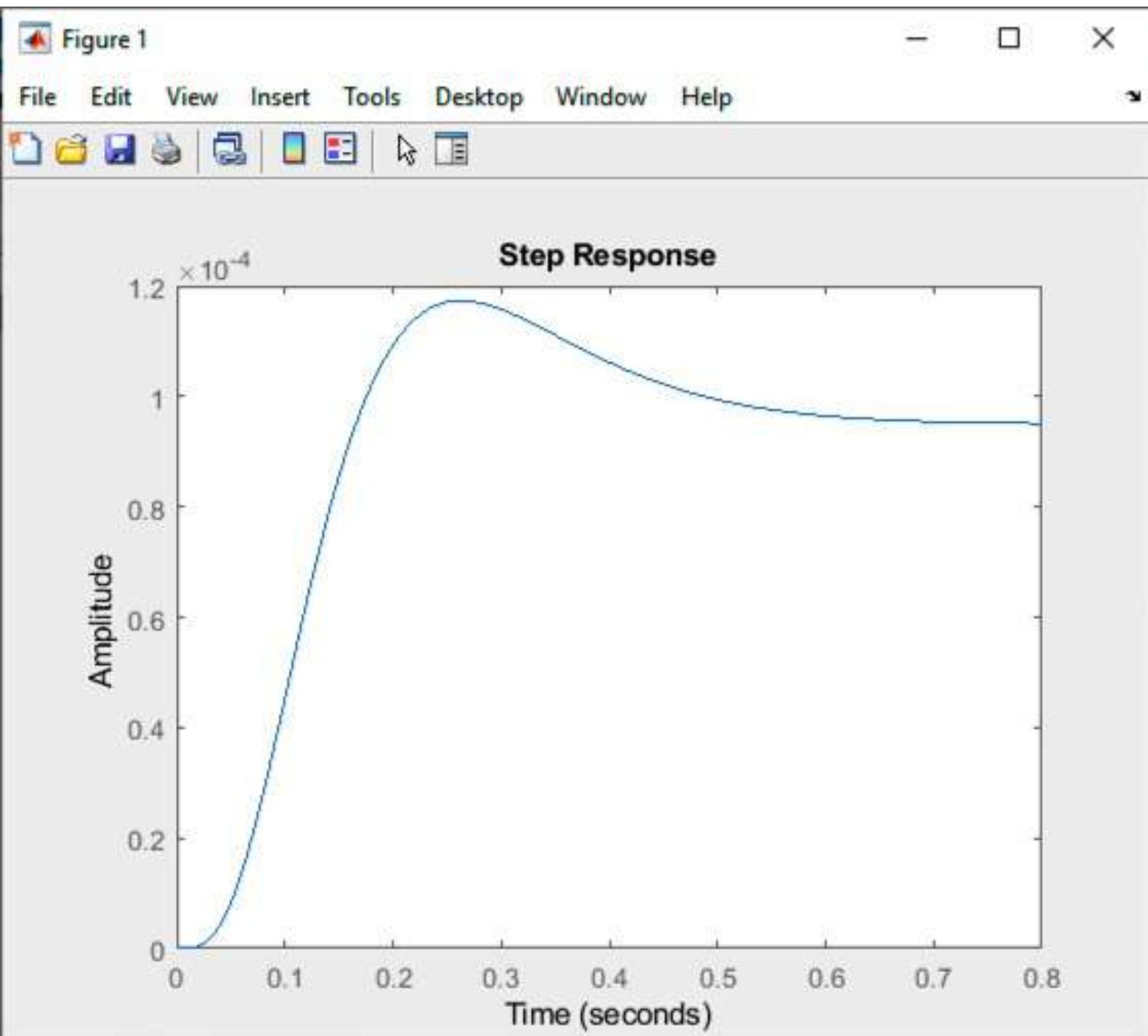
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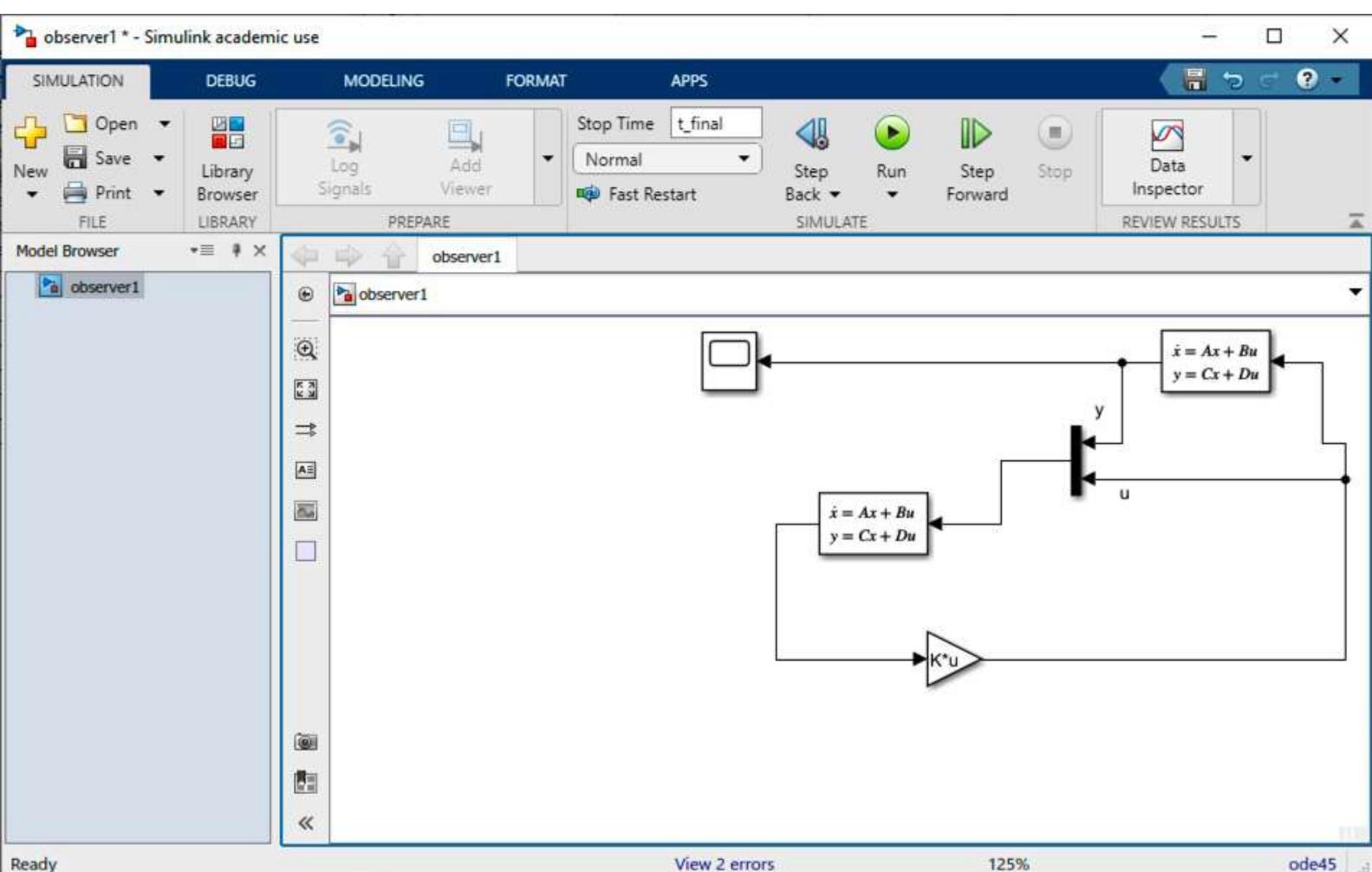
```
TF=tf([10 450 250],[1 1 0])
SYS=ss(TF)
TS=1.25*exp(-3)
SYSD=c2d(SYS,TS,'zoh')
```

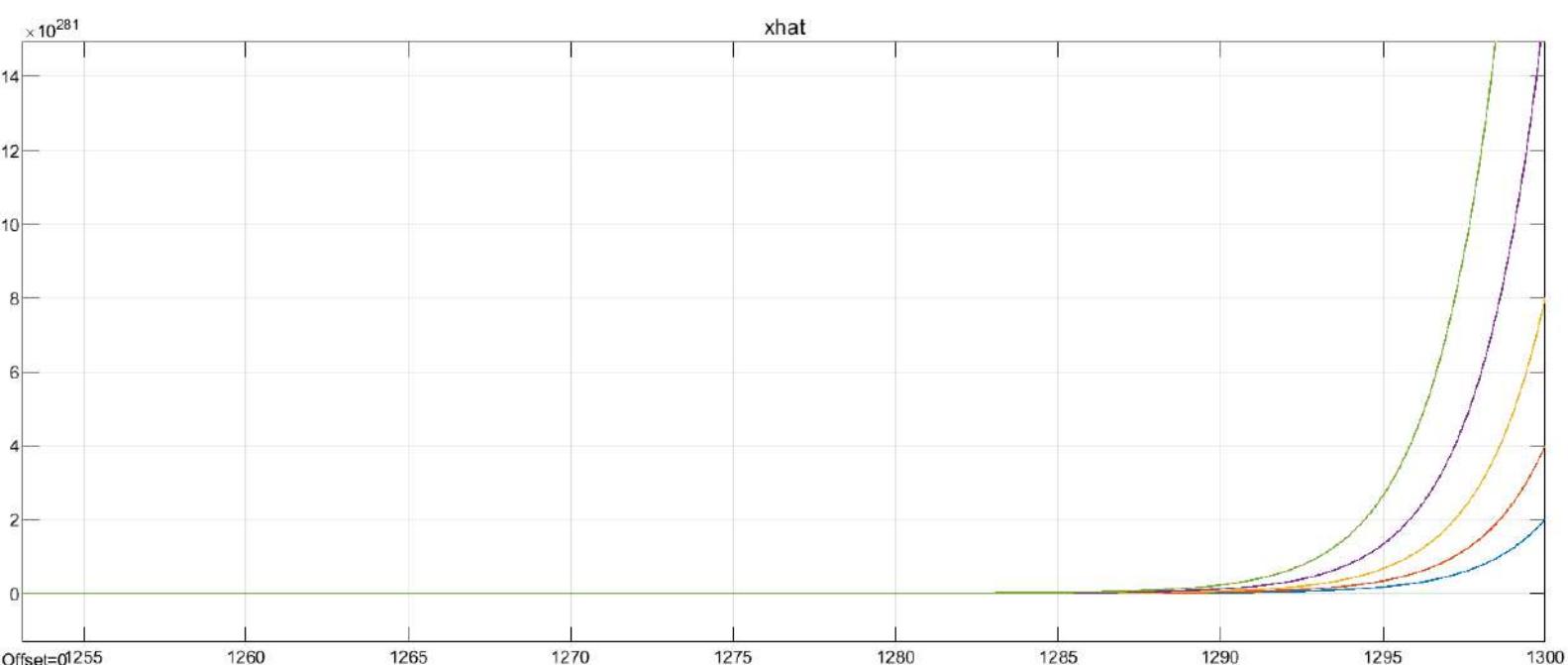
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```
31
32 %% this is the part that you have to do!
33 %
34 % design observer gain L
35 - j=sqrt(-1);
36 - obsv_poles =[-5+2j -5-2j -6+0j -5.2+0j];
37 - K=place(A',C',obsv_poles);
38 - L=K';%choose eigenvalues of A-LC
39 %compute L by using 'place' command here
40
41 % initialize state of the observer
42 - xhat_init = randn(4,1); %or use something else if you prefer
43
44 %define system matrices for the observer
45 - A_obs = (A-L*C);
46 - B_obs = [L (B - L*D)];
47 - C_obs = eye(n); %why?
48 - D_obs = zeros(size(C_obs,1),size(B_obs,2));
49 %
50
51
52
53
54 %% define plant inputs - right now I am choosing it arbitrarily:
55 % you are welcome to change this to something else
```









```
ans =  
  
struct with fields:  
  
    RiseTime: 0.0974  
    SettlingTime: 0.5784  
    SettlingMin: 8.5781e-05  
    SettlingMax: 1.1732e-04  
    Overshoot: 23.5373  
    Undershoot: 0  
    Peak: 1.1732e-04  
    PeakTime: 0.2640
```

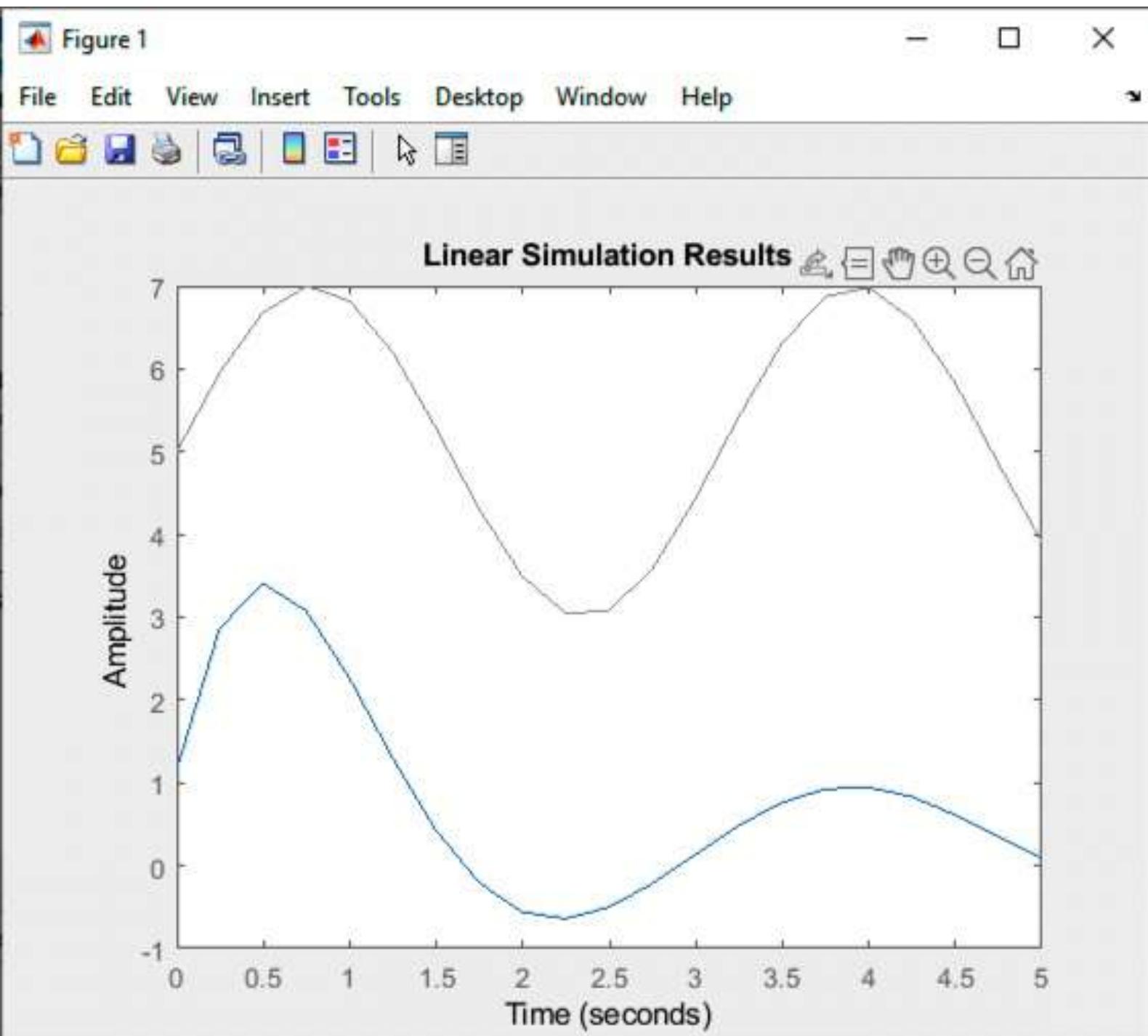


Figure 1

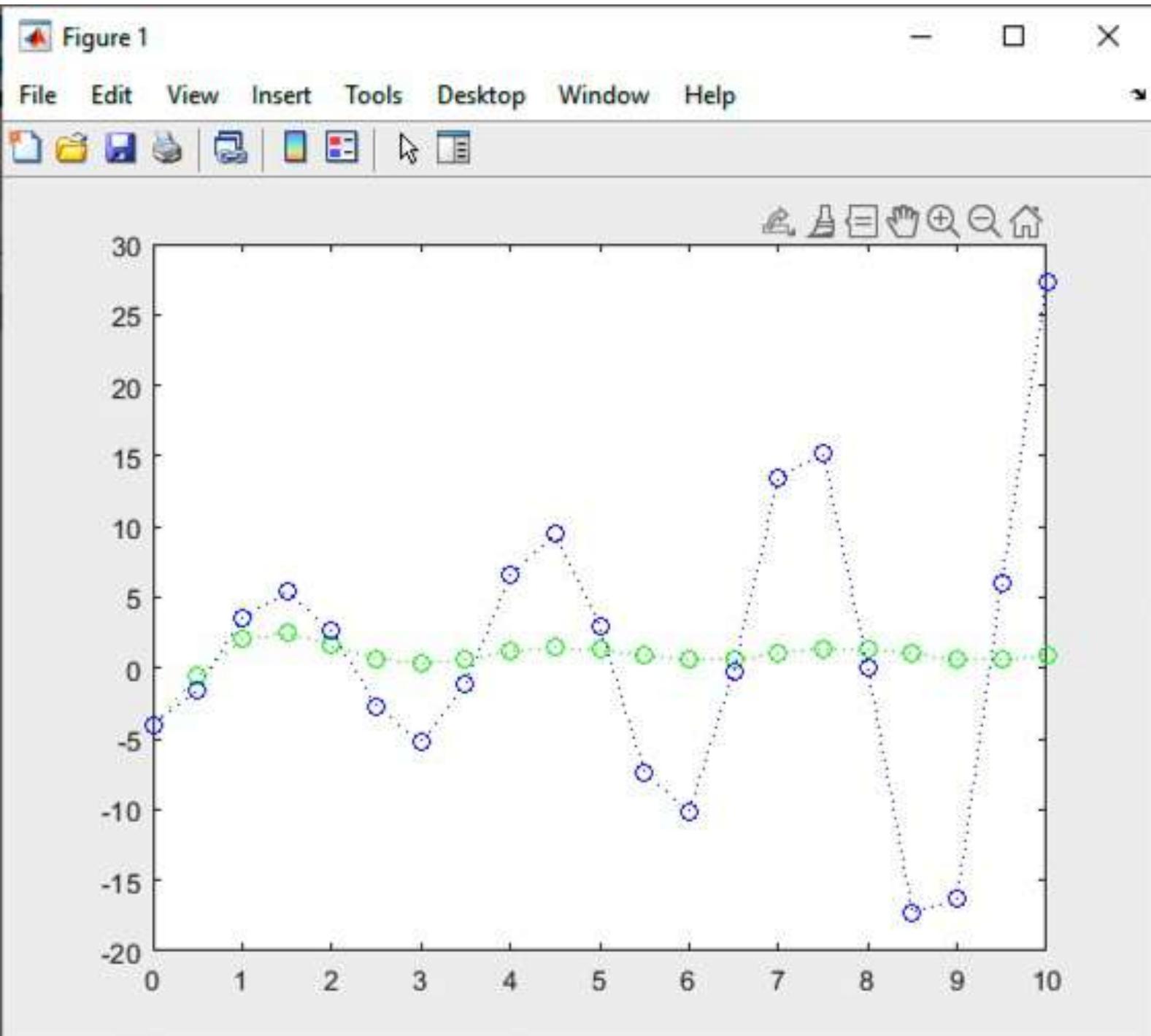
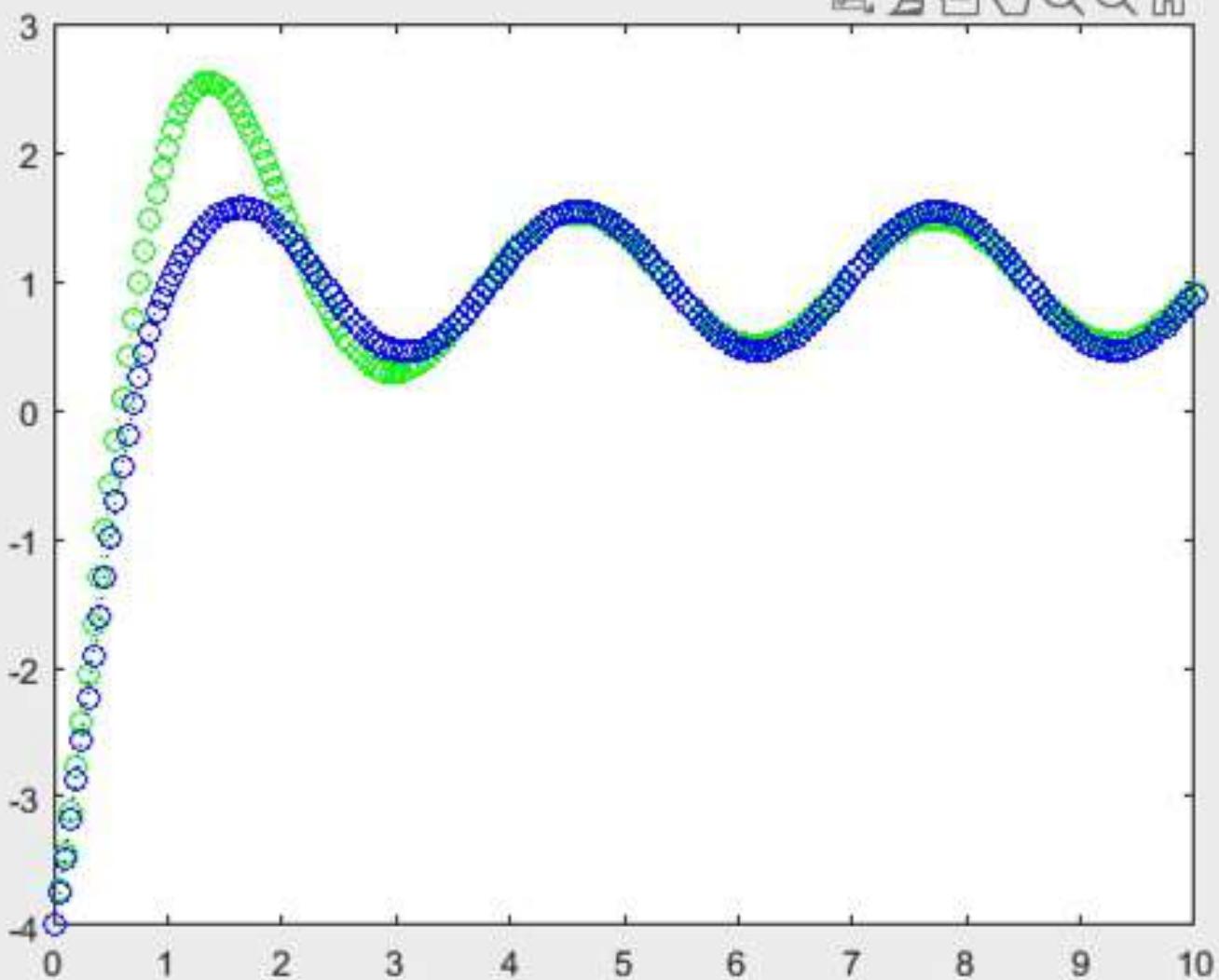


Figure 1

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**Figure 1**

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