EML 6350 — Fall 2020

Nonlinear Control Systems.

① Given dynamics

$$(m_x+m_\phi \sin^2(\phi))\ddot{x} = -c_x\dot{x} + m_\phi g\cos(\phi)\sin(\phi) + m_\phi l\dot{\phi}^2\sin(\phi) + u_x$$
$$\dot{u}_x = -a_x u_x + \mathcal{U}_x.$$
$$m_\phi l^2\ddot{\phi} = -c_\phi\dot{\phi} - m_\phi lg\sin(\phi) - m_\phi l\cos(\phi)\ddot{x} + u_\phi$$
$$\dot{u}_\phi = -a_\phi u_\phi + \mathcal{U}_\phi.$$

For the 1st set of dynamics
tracking error

$$e = x_d - x$$
$$\dot{e} = \dot{x}_d - \dot{x}$$
$$\ddot{e} = \ddot{x}_d - \ddot{x}$$

multiply both sides by $(m_x+m_\phi \sin^2(\phi))$.

$$(m_x+m_\phi \sin^2(\phi))\ddot{e} = (m_x+m_\phi\sin^2\phi)\ddot{x}_d - (m_x+m_\phi\sin^2\phi)\ddot{x} \quad -①$$

Reference tracking error.

$$r = \dot{e} + \alpha e.$$
$$\Rightarrow \dot{r} = \ddot{e} + \alpha\dot{e}$$

multiply both sides by $(m_x+m_\phi\sin^2\phi)$.

$$(m_x+m_\phi\sin^2\phi)\dot{r} = (m_x+m_\phi\sin^2\phi)\ddot{e} + (m_x+m_\phi\sin^2\phi)\alpha\dot{e}.$$

Substituting equ ①.

$$(m_x+m_\phi\sin^2\phi)\dot{r} = (m_x+m_\phi\sin^2\phi)\ddot{x}_d - (m_x+m_\phi\sin^2\phi)\ddot{x} + (m_x+m_\phi\sin^2\phi)\alpha\dot{e}.$$

$$(m_x+m_\phi\sin^2\phi)\dot{r} = (m_x+m_\phi\sin^2\phi)\ddot{x}_d - (-c_x\dot{x} + m_\phi g\cos(\phi)\sin\phi + m_\phi l\dot{\phi}^2\sin\phi - \tilde{u}_x + u_{dx}) + (m_x+m_\phi\sin^2\phi)\alpha\dot{e} + r m_\phi\sin\phi\cos\phi\,\dot{\phi} - r m_\phi\sin\phi\cos\phi\,\dot{\phi}$$

$$Y = \left[(\ddot{x}_d+\alpha\dot{e}),\ (\alpha\dot{e}\sin^2(\phi) + \sin^2\phi\,\ddot{x}_d - g\cos\phi\sin\phi + \sin\phi\cos\phi(\dot{\phi}\,r)),\ (-\dot{\phi}^2\sin\phi),\ \dot{x}\right]$$

$$\theta = \begin{bmatrix} m_x \\ m_\phi \\ m_\phi l \\ C_x \end{bmatrix}$$

$$\Rightarrow (m_x + m_\phi \sin^2\phi)\dot{r} = Y\theta - m_\phi \sin\phi\cos\phi(\dot{\phi})r + \tilde{u}_x - u_{xd} \quad -②$$

Let $\mathcal{E}_1 = \begin{bmatrix} e \\ r \\ \tilde{\theta} \\ \tilde{u} \\ \tilde{a} \end{bmatrix}$

Lyapunov function.

$$V = \frac{1}{2}e^2 + \frac{1}{2}(m_x + m_\phi \sin^2\phi)r^2 + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} + \frac{1}{2}\tilde{u}^2 + \frac{1}{2\gamma_a}\tilde{a}^2.$$

$$\dot{V} = e\dot{e} + \frac{1}{2}(m_x + m_\phi \sin^2\phi)(2r\dot{r}) + \frac{1}{2}(m_\phi(2\sin\phi\cos\phi\,\dot{\phi})r^2 + \tilde{\theta}\Gamma\dot{\tilde{\theta}} + \tilde{u}\dot{\tilde{u}} + \frac{\tilde{a}\dot{\tilde{a}}}{\gamma_a}.$$

substitute equ ②.

$$\dot{V} = e(r - \alpha e) + r(Y\theta - \cancel{m_\phi \sin\phi\cos\phi\,\dot{\phi}r} + \tilde{u}_x - u_{xd}) + \cancel{m_\phi \sin\phi\cos\phi\,\dot{\phi}r^2} + \tilde{\theta}^T\Gamma\dot{\tilde{\theta}} + \tilde{u}\dot{\tilde{u}} + \frac{\tilde{a}\dot{\tilde{a}}}{\gamma_a}.$$

we know that.

$\dot{\tilde{\theta}} = -\dot{\hat{\theta}}$ and $\dot{\tilde{a}} = -\dot{\hat{a}}$

as $\tilde{\theta} = \theta - \hat{\theta}$ $\quad$ $\tilde{a} = a - \hat{a}$

$$\dot{V} = e(r - \alpha e) + r(Y\theta + \tilde{u}_x - u_{xd}) + \tilde{u}_x(u_{xd} + a_x u_x - U_x) - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} - \frac{\tilde{a}\dot{\hat{a}}}{\gamma_a}.$$

Design $u_{xd} = Y\hat{\Theta} + \beta r + e$.

$$\dot{V} = \cancel{er} - \alpha e^2 + r_x\left(Y\hat{\Theta} + \tilde{u}_x - (Y\hat{\Theta} + \beta r + e)\right) + \tilde{u}_x\left(\dot{u}_{xd} + a_x u_x - U_x\right) - \tilde{\Theta}^{*}\Gamma^{-1}\dot{\hat{\Theta}} - \frac{\tilde{a}\dot{\hat{a}}}{\gamma_a}.$$

$$\dot{V} = -\alpha e^2 + rY\tilde{\Theta} - \beta r^2 + r\tilde{u}_x + \tilde{u}_x\left(\dot{u}_{xd} + a_x u_x - U_x\right) - \tilde{\Theta}^{*}\Gamma^{-1}\dot{\hat{\Theta}} - \frac{\tilde{a}\dot{\hat{a}}}{\gamma_a}.$$

Design $U_x = \hat{a}_x u_x + \dot{u}_{dx} + r + \gamma\tilde{u}_x$

$$\dot{V} = -\alpha e^2 + rY\tilde{\Theta} - \beta r^2 + r\tilde{u}_x + \tilde{u}_x\left(\dot{u}_{xd} + a_x u_x - (\hat{a}_x u_x + \dot{u}_{dx} + r + \gamma\tilde{u}_x)\right) - \tilde{\Theta}^{*}\Gamma^{-1}\dot{\hat{\Theta}} - \frac{\tilde{a}\dot{\hat{a}}}{\gamma_a}.$$

$$\dot{V} = -\alpha e^2 + rY\tilde{\Theta} - \beta r^2 + \cancel{r\tilde{u}_x} + \tilde{a}_x u_x \tilde{u}_x - \cancel{r\tilde{u}_x} - \gamma\tilde{u}_x^2 - \tilde{\Theta}^{T}\Gamma^{-1}\dot{\hat{\Theta}} - \frac{\tilde{a}\dot{\hat{a}}}{\gamma_a}$$

$$\dot{V} = -\alpha e^2 - \beta r^2 - \gamma\tilde{u}_x^2 + rY\tilde{\Theta} + \tilde{a}_x u_x \tilde{u}_x - \tilde{\Theta}^{*}\Gamma^{-1}\dot{\hat{\Theta}} - \frac{\tilde{a}\dot{\hat{a}}}{\gamma_a}.$$

Design $\dot{\hat{a}} = \gamma_a(\tilde{u}_x u_x)$

$$\dot{V} = -\alpha e^2 - \beta r^2 - \gamma\tilde{u}_x^2 + rY\tilde{\Theta} + \cancel{a_x u_x \tilde{u}_x} - \tilde{\Theta}^{T}\Gamma^{-1}\dot{\hat{\Theta}} - \cancel{\frac{\tilde{a}(\gamma_a(\tilde{u}_x u_x))}{\gamma_a}}.$$

$$\dot{V} = -\alpha e^2 - \beta r^2 - \gamma\tilde{u}_x^2 + rY\tilde{\Theta} - \tilde{\Theta}^{*}\Gamma^{-1}\dot{\hat{\Theta}}$$

Design $\dot{\hat{\Theta}} = \Gamma Y^T r + \Gamma K_\Theta(y^T U - y^T y\hat{\Theta})$ for $t < T$

$$\dot{V} = -\alpha e^2 - \beta r^2 - \gamma\tilde{u}_x^2 + \underbrace{\cancel{rY\tilde{\Theta}}}_{\tilde{\Theta}^{*}Y^T r} - \tilde{\Theta}^{*}\Gamma^{-1}\left(\cancel{\Gamma Y^T r} + \Gamma K_\Theta(y^T U - y^T y\hat{\Theta})\right)$$

$$\dot{V} = -\alpha e^2 - \beta r^2 - \gamma\tilde{u}_x^2 - \tilde{\Theta}^{*}\cancel{\Gamma^{-1}}\left(\cancel{\Gamma}K_\Theta(y^T y\tilde{\Theta})\right)$$

$$\dot{V} = -\alpha e^2 - \beta r^2 - \gamma\tilde{u}_x^2 - \tilde{\Theta}^{*}K_\Theta y^T y\tilde{\Theta}.$$

For $t > T$.

design $\dot{\hat{\theta}} = \Gamma Y^T r + \Gamma K_\theta (y^T U - y^T y \hat{\theta}) + \Gamma K_{CL} \sum_{i=1}^{j} y^T(t_j) y(t_j) \tilde{\theta}$

$$\dot{V} = -\alpha e^2 - \beta r^2 - \gamma \tilde{u}^2 + r Y \tilde{\theta} + \tilde{\theta}^T \Gamma^{-1} \big(\Gamma Y^T r + \Gamma K_\theta (y^T U - y^T y \hat{\theta}) + \Gamma K_{CL} \sum_{i=1}^{j} y^T(t_j) y(t_j) \tilde{\theta} \big).$$

$$\dot{V} = -\alpha e^2 - \beta r^2 - \gamma \tilde{u}^2 + \tilde{\theta}^T \Gamma^{-1} \big(\cancel{\Gamma} K_\theta y^T y \tilde{\theta} + \cancel{\Gamma} K_{CL} \sum_{i=1}^{j} y^T(t_j) y(t_j) \tilde{\theta} \big)$$

$$\dot{V} = -\alpha e^2 - \beta r^2 - \gamma \tilde{u}^2 + \tilde{\theta}^T \big(K_\theta y^T y \cancel{\tilde{\theta}} + K_{CL} \sum_{i=1}^{j} y^T(t_j) y(t_j) \big) \tilde{\theta}$$

For time $t < T$.

$$\dot{V} = -\alpha e^2 - \beta r^2 - \gamma \tilde{u}^2 + \tilde{\theta}^T (K_\theta y^T y) \tilde{\theta}$$

→ all we know is PSD.

$$\dot{V} \leq -\alpha e^2 - \beta r^2 - \gamma \tilde{u}^2$$

~~from Barballet's Lemma.~~

$$\int_0^t \dot{V}_j(\mathcal{E}(\sigma,\sigma),\sigma)\, d\sigma \leq -\int_0^t \alpha e^2(\sigma) + \beta r^2(\sigma) + \gamma \tilde{u}^2(\sigma)\, d\sigma$$

$$V(\xi(0), 0) - V(\xi(t), t) \geq \int_0^t \alpha e^2(\sigma) + \beta r^2(\sigma) + \gamma \tilde{u}^2(\sigma)\, d\sigma.$$

since $V(0), V(t) \in L_\infty \Rightarrow e, r, \tilde{u} \in L_2$.

$e, r \in L_\infty \Rightarrow \dot{e} \in L_\infty$ $(r = \dot{e} + \alpha e)$.

e is Uniformly continuous.

$e, \dot{e}, x_d, \ddot{x}_d \in L_\infty \Rightarrow x, \dot{x} \in L_\infty$.

$x, \dot{x}, \ddot{x}_d, \dot{e}, \dot{\phi} \in L_\infty \Rightarrow Y \in L_\infty$.

$\tilde{\theta} \in L_\infty \Rightarrow \hat{\theta} \in L_\infty$.

$e, r, Y, \tilde{\theta} \in L_\infty \Rightarrow \tilde{u} \in L_\infty$.

$$\dot{V} \leq -\alpha e^2 - \beta r^2 - \gamma \tilde{u}^2 .$$

therefore, all we can say about the system at $t < T$ is that it is localy stable.

For $t > T$

$$\dot{V} = -\alpha e^2 - \beta r^2 - \gamma \tilde{u}^2 + \tilde{\theta}\big(\underset{\text{still PSD}}{K_\theta y^T y} + K_{CL} \underset{\text{not PSD}}{\sum_{i=1}^{j} y^T(t_i) y(t_i)}\big)\tilde{\theta}$$

$$\dot{V} \leq \alpha e^2 - \beta r^2 - \gamma \tilde{u}^2 + \tilde{\theta}\Big(K_{cL} \sum_{i=1}^{j} y^T(t_i) y(t_i)\Big)\tilde{\theta}$$

Using Barballet's Lemma.

$$\int_0^{t} \dot{V}(\xi(\theta), \theta), \theta)\, \delta\theta \leq -\int_0^{t} \alpha e^2(\theta) + \beta r^2(\theta) + \gamma \tilde{u}^2(\theta)\, \partial\theta + \sum_{i=1}^{j} y^T(t_i) y(t_i) \tilde{\theta}^2(\theta)$$

$$V(\xi(0), 0) - V(\xi(t), t) \geq \int_0^{t} \alpha e^2(\theta) + \beta r^2(\theta) + \gamma \tilde{u}^2(\theta) + \sum_{i=1}^{j} y^T(t_i) y(t_i) \tilde{\theta}^2(\theta)\, \partial\theta.$$

Since $V(0), V(t) \in L_\infty \Rightarrow e, r, \tilde{u} \in L_2$

$e, r \in L_\infty \Rightarrow \dot{e} \in L_\infty$ $(\because r = \dot{e} + \alpha e)$.

e is Uniformly continous.

$e, \dot{e}, x_d, \dot{x}_d \in L_\infty, \Rightarrow x, \dot{x} \in L_\infty$

$\phi, \dot{\phi}, x, \dot{x}, \ddot{x}_d, \dot{e}, \dot{\phi} \in L_\infty \Rightarrow Y \in L_\infty$

$\tilde{\theta} \in L_\infty \Rightarrow \hat{\theta} \in L_\infty$.

$e, r, Y, \tilde{\theta} \in L_\infty \Rightarrow \tilde{a} \in L_\infty$.

$r, \tilde{u}, \tilde{\theta}$ are all uniformly continous.

Therefore, from Barballet's Lemma we can prove global asymptotic tracking (GAT).

For 2^{nd} set of dynamics

$$m_\phi l^2 \ddot{\phi} = -c_\phi \dot{\phi} - m_\phi l g \sin(\phi) - m_\phi l \cos\phi \ddot{x} + u_\phi$$

$$\dot{u}_\phi = -a_\phi u_\phi + \mu_\phi$$

error tracking.

$$e_\phi = \phi_d - \phi$$

$$\dot{e}_\phi = \dot{\phi}_d - \dot{\phi}$$

$$\ddot{e}_\phi = \ddot{\phi}_d - \ddot{\phi}$$

multiply both sides by $m_\phi l^2$.

$$m_\phi l^2 \ddot{e}_\phi = m_\phi l^2 \ddot{\phi}_d - m_\phi l^2 \ddot{\phi} \quad — ③.$$

Reference tracking error.

$$r = \dot{e} + \alpha e.$$

$$\dot{r} = \ddot{e} + \alpha \dot{e}$$

multiplying both sides by $m_\phi l^2$.

$$m_\phi l^2 \dot{r} = m_\phi l^2 \ddot{e} + \alpha m_\phi l^2 \dot{e}$$

substituting equ ③

$$m_\phi l^2 \dot{r} = m_\phi l^2 \ddot{\phi}_d - m_\phi l^2 \ddot{\phi} + \alpha m_\phi l^2 \dot{e}$$

$$m_\phi l^2 \dot{r} = m_\phi l^2 \ddot{\phi}_d - (-c_\phi \dot{\phi} - m_\phi l g \sin\phi - m_\phi l \cos\phi \ddot{x} + \overbrace{u_\phi - u_{\phi d}}^{-\tilde{u}_\phi} + u_{\phi d}) + \alpha m_\phi l^2 \dot{e}.$$

$$m_\phi l^2 \dot{r} = m_\phi l^2 \ddot{\phi}_d + c_\phi \dot{\phi} + m_\phi l g \sin\phi + m_\phi l \cos\phi \ddot{x} + \tilde{u}_\phi - u_{\phi d} + \alpha m_\phi l^2 \dot{e}$$

$$m_\phi l^2 \dot{r} = \underbrace{\begin{bmatrix} (g\sin\phi + \cos\phi \ddot{x}) & (\ddot{\phi}_d + \alpha \dot{e}_\phi) & \dot{\phi} \end{bmatrix}}_{Y} \underbrace{\begin{bmatrix} m_\phi l \\ m_\phi l^2 \\ c_\phi \end{bmatrix}}_{\theta} - u_{\phi d} + \tilde{u}_\phi \quad — ④$$

Let $\xi_3 = \begin{bmatrix} e \\ r \\ \tilde{\theta} \\ \tilde{u} \\ \tilde{a} \end{bmatrix}$

Lyapunov candidate.

$$V = \frac{1}{2}e^2 + \frac{1}{2} m_\phi l^2 \cancel{\phi}\, r^2 + \frac{1}{2}\tilde{\theta}^T \Gamma^{-1} \tilde{\theta} + \frac{1}{2}\tilde{u}^2 + \frac{1}{2\gamma_a}\tilde{a}^2.$$

$$\dot{V} = e\dot{e} + \frac{1}{\cancel{2}}(m_\phi l^2)\, r\dot{r}\cancel{(2)} + \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}} + \tilde{u}\dot{\tilde{u}} + \frac{\tilde{a}\dot{\tilde{a}}}{\gamma_a}.$$

we know $\tilde{u}_\phi = u_{d\phi} - u_\phi \Rightarrow \dot{\tilde{u}} = \dot{u}_d - \dot{u}$

$$\dot{V} = e(r - \alpha e) + \tilde{u}(\dot{u}_d + a_\phi u - \mathcal{U}) - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} + (m_\phi l^2)(r\dot{r}) + \frac{\tilde{a}\dot{\tilde{a}}}{\gamma_a}.$$

substitute eqn (4).

$$\dot{V} = e(r - \alpha e) + \tilde{u}(\dot{u}_d + a_\phi u - \mathcal{U}) - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} - \frac{\tilde{a}\dot{\hat{a}}}{\gamma_a}$$

$$+ r(Y\theta - u_{d\phi} + \tilde{u}_\phi)$$

Design $u_{d\phi} = Y\hat{\theta} + \beta r + e$.

$$\dot{V} = e(\cancel{r} - \alpha e) + \tilde{u}(\dot{u}_d + a_\phi u - \mathcal{U}) - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} + r(\overbrace{Y\theta + \tilde{u}_\phi - (Y\hat{\theta}}^{Y\tilde{\theta}} + \beta r + \cancel{e})) - \frac{\tilde{a}\dot{\hat{a}}}{\gamma_a}$$

$$\dot{V} = -\alpha e^2 + \tilde{u}(\dot{u}_d + a_\phi u - \mathcal{U}) - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} - rY\tilde{\theta} + \tilde{u}r - \beta r^2 - \frac{\tilde{a}\dot{\hat{a}}}{\gamma_a}.$$

$$\dot{V} = -\alpha e^2 + \tilde{u}(\dot{u}_d + a_\phi u - \mathcal{U}) - \tilde{\theta}\Gamma^{-1}\dot{\hat{\theta}} - rY\tilde{\theta} + \tilde{u}r - \beta r^2 - \frac{\tilde{a}\dot{\hat{a}}}{\gamma_a}.$$

Design $\mathcal{U} = \hat{a}_\phi u + \dot{u}_d + r + \gamma\tilde{u}$.

$$\dot{V} = -\alpha e^2 + \tilde{u}(\cancel{\dot{u}_d} + \overbrace{a_\phi u - (\hat{a}_\phi u}^{\tilde{a}_\phi u} + \cancel{\dot{u}_d} + \cancel{r} + \gamma\tilde{u})) - \tilde{\theta}\Gamma^{-1}\dot{\hat{\theta}} - rY\tilde{\theta} - \cancel{\tilde{u}r} - \beta r^2 - \frac{\tilde{a}\dot{\hat{a}}}{\gamma_a}$$

$$\dot{V} = -\alpha e^2 + \tilde{a}u\tilde{u} - \gamma\tilde{u}^2 - \beta r^2 - rY\tilde{\theta} - \tilde{\theta}\Gamma^{-1}\dot{\hat{\theta}} - \frac{\tilde{a}\dot{\hat{a}}}{\gamma_a}.$$

Design $\dot{\hat{a}} = \gamma_a u \tilde{u}$

$$\dot{V} = -\alpha e^2 + \cancel{\tilde{a} u \tilde{u}} - \gamma \tilde{u}^2 - \beta r^2 - r Y \tilde{\theta} - \tilde{\theta}\Gamma^{-1}\dot{\hat{\theta}} - \cancel{\frac{\tilde{a}(\gamma_a u \tilde{u})}{\gamma_a}}$$

$$\dot{V} = -\alpha e^2 - \gamma\tilde{u}^2 - \beta r^2 - rY\tilde{\theta} - \tilde{\theta}\Gamma^{-1}\dot{\hat{\theta}}$$

Design $\dot{\hat{\theta}} = \Gamma Y^T r + \Gamma K_\theta (y^T U - y^T y \hat{\theta})$ $\boxed{\text{for } t < T}$

$$\dot{V} = -\alpha e^2 - \beta r^2 - \gamma \tilde{u}_\phi^2 + \cancel{rY\tilde{\theta}} - \tilde{\theta}\Gamma^{-1}(\cancel{\Gamma Y^T r} + \Gamma K_\theta(y^T U - y^T y\hat{\theta}))$$

$$\dot{V} = -\alpha e^2 - \beta r^2 - \gamma\tilde{u}_\phi^2 - \tilde{\theta}\cancel{\Gamma^{-1}}(\cancel{\Gamma} K_\theta(y^T y \tilde{\theta})).$$

Design $\dot{\hat{\theta}} = \Gamma Y^T r + \Gamma K_\theta(y^T u - y^T y\hat{\theta}) + \Gamma K_a \sum_{i=1}^{j} y^T(t_j)\, y(t_j)\tilde{\theta}$ $\boxed{\text{for } t > T}$

$$\dot{V} = -\alpha e^2 - \beta r^2 - \gamma\tilde{u}^2 + \tilde{\theta}\Gamma^{-1}(\cancel{\Gamma Y^T r} + \Gamma K_\theta(y^T u - y^T y\hat{\theta}) + \Gamma K_a \sum_{i=1}^{j} y^T(t_j)\, y(t_j)\tilde{\theta} \quad \cancel{+ rY\tilde{\theta}}$$

$$\dot{V} = -\alpha e^2 - \beta r^2 - \gamma\tilde{u}^2 + \tilde{\theta}(K_\theta y^T y\theta + K_a \sum_{i=1}^{j} y^T(t_j)\, y(t_j))\tilde{\theta}$$

For $t < T$

$$\dot{V} = -\alpha e^2 - \beta r^2 - \gamma\tilde{u}_\phi^2 - \tilde{\theta}\underbrace{K_\theta y^T y}_{\text{all we know is PSD}}\tilde{\theta}.$$

So,

$$\dot{V} \le -\alpha e^2 - \beta r^2 - \gamma\tilde{u}_\phi^2$$

Therefore, all we can prove is local stability for $t < T$.

For $t > T$

$$\dot{V} = -\alpha e^2 - \beta r^2 - \gamma\tilde{u}^2 + \tilde{\theta}(\underbrace{K_\theta y^T y\theta}_{\text{all we know is still PSD}} + \underbrace{K_a \sum_{i=1}^{j} y^T(t_j)\, y(t_j)}_{\text{not PSD but PD}})\tilde{\theta}$$

So,

$$\dot{V} \leq -\alpha e^2 - \beta r^2 - \gamma \tilde{u}^2 + \tilde{\Theta}\left(K_{CL} \sum_{i=1}^{j} y^T(t_i) y(t_i)\right)\tilde{\Theta}.$$

Using Barballet's Lemma.

$$\int_0^t \dot{V}(\xi(\sigma),\sigma),\sigma)\, \delta\sigma \leq -\int_0^t \alpha e^2(\sigma) + \beta r^2(\sigma) + \gamma \tilde{u}^2(\sigma) + \sum_{i=1}^{j} y^T(t_i) y(t_i) \tilde{\Theta}^2(\sigma).$$

$$V(\xi(0),0) - V(\xi(t),t) \geq \int_0^t \alpha e^2(\sigma) + \beta r^2(\sigma) + \gamma \tilde{u}^2(\sigma) + \sum_{i=1}^{j} y^T(t_i) y(t_i) \tilde{\Theta}^2(\sigma)\, \partial\sigma.$$

Since,

$V(0), V(t) \in L_\infty \Rightarrow e, r, \tilde{u}, \tilde{\Theta} \in L_2$.

$e, r \in L_\infty \Rightarrow \dot{e} \in L_\infty$ $(\because r = \dot{e} + \alpha e)$.

e is uniformly continous.

$e, \dot{e}, x_d, \dot{x}_d \in L_\infty \Rightarrow x, \dot{x} \in L_\infty$.

$\phi, \dot{\phi}, x, \dot{x}, x_d, \phi_d, \dot{e}, \dot{\phi} \in L_\infty \Rightarrow Y \in L_\infty$.

$\tilde{\Theta} \in L_\infty \Rightarrow \hat{\Theta} \in L_\infty$

$e, r, Y, \tilde{\Theta} \in L_\infty \Rightarrow \tilde{u} \in L_\infty$.

$r, \tilde{u}, \tilde{\Theta}$, are all UC.

Therefore, from Barballet's Lemma we can prove global Asymptotic tracking (GAT).

(b)(ii) Used Concurrent Learning and backstepping to design my controller as suggested theoretically analysis, I was able to ~~ao~~ achieve GAT Using concurrent learning term "k_{cl}" designed in my theta_hat_dot, I was able to retain the theta term in my analysis which enabled me to prove this GAT using Barbalat's Lemma.

A different design term k_{cl} was added once enough data was collected to attain sufficient excitation.

ie $\sum_{i=1}^{j} y^T(t_i)\, y(t_i) > \underline{\lambda}$

for a small enough $\underline{\lambda}$ value.