EML 6351

Nonlinear Control - 2 - Adaptive Control.

Project - 1.

Dynamics.

$$M(\phi)\ddot{\phi} + C(\phi,\dot{\phi}) + G(\phi) = \tau$$

where $\phi = \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}$ $\dot{\phi} = \begin{bmatrix}\dot{\phi}_1\\ \dot{\phi}_2\end{bmatrix}$ $\ddot{\phi} = \begin{bmatrix}\ddot{\phi}_1\\ \ddot{\phi}_2\end{bmatrix}$. $\tau = \begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix}$

$$M(\phi) = \begin{bmatrix} (m_1 l_1^2 + m_2(l_1^2 + 2l_1l_2c_2 + l_2^2) & m_2(l_1l_2c_2 + l_2^2) \\ m_2(l_1l_2c_2 + l_2^2) & m_2 l_2^2 \end{bmatrix}$$

$$C(\phi,\dot{\phi}) = \begin{bmatrix} -2m_2l_1l_2s_2\dot{\phi}_1\dot{\phi}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -m_2l_1l_2s_2\dot{\phi}_2^2 \\ m_2l_1l_2s_2\dot{\phi}_1^2 \end{bmatrix}$$

$$G(\phi) = \begin{bmatrix} (m_1+m_2)gl_1c_1 + m_2gl_2c_{12} \\ m_2gl_2c_{12} \end{bmatrix}$$

Defining tracking error.

$$e = \phi_d - \phi.$$

$$\dot{e} = \dot{\phi}_d - \dot{\phi}$$

$$\ddot{e} = \ddot{\phi}_d - \ddot{\phi}.$$

$M(\phi)\ddot{e} = M(\phi)\ddot{\phi}_d - M(\phi)\ddot{\phi}$ — ①

filtered tracking error.

$r = \dot{e} + \alpha e$

$\dot{r} = \ddot{e} + \alpha\dot{e}$.

$\Rightarrow M(\phi)\dot{r} = M(\phi)\ddot{e} + M(\phi)\alpha\dot{e}$.

Sub equ ① we get.

$M(\phi)\dot{r} = \left(M(\phi)\ddot{\phi}_d - M(\phi)\ddot{\phi}\right) + M(\phi)\alpha\dot{e}$.

From dynamics.

$M(\phi)\ddot{\phi} = \tau - C(\phi,\dot{\phi}) - G(\phi)$.

$\Rightarrow M(\phi)\dot{r} = M(\phi)\ddot{\phi}_d - \left(\tau - C(\phi,\dot{\phi}) - G(\phi)\right) + M(\phi)\alpha\dot{e}$

$= \underbrace{M(\phi)(\ddot{\phi}_d + \alpha\dot{e}) + C(\phi,\dot{\phi}) + G(\phi)}_{Y\theta} - \tau$

Define $\theta \in \mathbb{R}^p$ where p = no of terms in θ

Let $\tilde{\theta} = \theta - \hat{\theta}$.

$\Rightarrow \dot{\tilde{\theta}} = -\dot{\hat{\theta}}$ ($\because \theta$ is a constant).

$\dot{\hat{\theta}}(t) \in \mathbb{R}^p$ is the parametric update law.

Let $\xi_1 = \begin{bmatrix} e \\ r \\ \tilde{\theta} \end{bmatrix}$ be the stacked errors.

Lyapunov candidate.

$$V(\xi_1, t) = \frac{1}{2} e^T e + \frac{1}{2} r^T M(\phi) r + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}.$$

$$\dot{V} = e^T \dot{e} + \frac{1}{2} r^T \dot{M}(\phi) r + \frac{1}{2} r^T M(\phi) \dot{r} + \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}}$$

we know. $\dot{\tilde{\theta}} = -\dot{\hat{\theta}}$ and $\dot{e} = r - \alpha e$.

$$\dot{V} = e^T (r - \alpha e) + \frac{1}{2} r^T \dot{M}(\phi) r + \frac{1}{2} r^T M(\phi) \dot{r} - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}}$$

$$\dot{M}(\phi) = \frac{\partial}{\partial t} M(\phi).$$

$$= \frac{\partial}{\partial t} \begin{bmatrix} (m_1 l_1^2 + m_2 (l_1^2 + 2 l_1 l_2 c_2 + l_2^2) & m_2 (l_1 l_2 c_2 + l_2^2) \\ m_2 (l_1 l_2 c_2 + l_2^2) & m_2 l_2^2 \end{bmatrix}$$

$$= \begin{bmatrix} -m_2 (2 l_1 l_2 s_2) \dot{\phi}_2 & -m_2 l_1 l_2 s_2 \dot{\phi}_2 \\ -m_2 l_1 l_2 s_2 \dot{\phi}_2 & 0. \end{bmatrix}$$

and $M(\phi)\dot{r} = M(\phi)(\ddot{\phi}_0 + \alpha \dot{e}) + C(\phi, \dot{\phi}) + G(\phi) - \tau.$

$$\dot{V} = e^T (r - \alpha e) + \frac{1}{2} r^T \dot{M}(\phi) r + \cancel{\frac{1}{2} r^T M(\phi) \dot{r}}$$

$$+ \frac{1}{2} r^T (M(\phi)(\ddot{\phi}_0 + \alpha e) + C(\phi, \dot{\phi}) + G(\phi) - \tau) - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}}$$

$$\dot V = -e^T\alpha e + e^T r - \tilde\theta^T \Gamma^{-1}\dot{\hat\theta}$$
$$+ r^T\left(M(\phi)(\ddot\phi_d + \alpha\dot e) + C(\phi,\dot\phi) + G(\phi) + \tfrac{1}{2}\dot M(\phi) r - \tau\right)$$

To find the regressor.
we want.

$$M(\phi)(\underbrace{\ddot\phi_d + \alpha\dot e}_{\sim}) + C(\phi,\dot\phi) + G(\phi) + \tfrac{1}{2}\dot M(\phi) r = Y\theta$$

to get the minimum realization of θ.

$$M(\phi)\sim = \begin{bmatrix} \underbrace{m_1 l_1^2 + m_2 l_1^2 + m_2 l_2^2}_{\theta_1} + 2\underbrace{l_1 l_2}_{\theta_2} c_2 m_2 & \underbrace{m_2 l_1 l_2}_{\theta_2} c_2 + \underbrace{m_2 l_2^2}_{\theta_3} \\ \underbrace{m_2 l_1 l_2 c_2}_{\theta_2} + \underbrace{m_2 l_2^2}_{\theta_3} & \underbrace{m_2 l_2^2}_{\theta_3} \end{bmatrix} \sim$$

$$M(\phi)\sim = \begin{bmatrix} \theta_1 + 2\theta_2 c_2 & c_2\theta_2 + \theta_3 \\ c_2\theta_2 + \theta_3 & \theta_3 \end{bmatrix} \begin{bmatrix} \sim_1 \\ \sim_2 \end{bmatrix}.$$

$$\cancel{M(\phi)\sim =} \begin{bmatrix} \cancel{(\theta_1+\theta_2)\sim_1 + (\theta_2+\theta_3)\sim_2} \\ \cancel{(\theta_2+\theta_3)\sim_1 + \theta_3\sim_2} \end{bmatrix}$$

$$\cancel{C(\phi,\dot\phi) =} \begin{bmatrix} \\ \end{bmatrix}$$

$$M(\phi)\sim = \begin{bmatrix} \cancel{\sim_1 \quad \sim_1 + \sim_2} \\ \\ \end{bmatrix}$$

$$M(\phi)\nu = \begin{bmatrix} (\theta_1 + 2\theta_2 c_2)\nu_1 + (\theta_2 c_2 + \theta_3)\nu_2 \\ (c_2\theta_2 + \theta_3)\nu_1 \quad \theta_3 \nu_2 \end{bmatrix}$$

$$M(\phi)\nu = \begin{bmatrix} \nu_1 & 2\nu_1 c_2 + \nu_2 c_2 & \nu_2 \\ 0 & c_2 \nu_1 & \nu_1 + \nu_2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

Now, looking at $C(\phi, \dot\phi)$.

$$C(\phi,\dot\phi) = \begin{bmatrix} -2\overbrace{m_2 l_1 l_2}^{\theta_2} s_2 \dot\phi_1 \dot\phi_2 - \overbrace{m_2 l_1 l_2}^{\theta_2} s_2 \dot\phi_2^2 \\ \underbrace{m_2 l_1 l_2}_{\theta_2} s_2 \dot\phi_1^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2s_2\dot\phi_1\dot\phi_2 - s_2\dot\phi_2^2 & 0 \\ 0 & s_2\dot\phi_1^2 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

Now looking at $G(\phi)$.

$$G(\phi) = \begin{bmatrix} \underbrace{(m_1+m_2)}_{\theta_4} g l_1 c_1 + \overbrace{m_2 g l_2}^{\theta_5} c_{12} \\ \underbrace{m_2 g l_2}_{\theta_5} c_{12} \end{bmatrix}$$

$$= \begin{bmatrix} g c_1 & g c_{12} \\ 0 & g c_{12} \end{bmatrix} \begin{bmatrix} \theta_4 \\ \theta_5 \end{bmatrix}$$

Now $\dot{M}(\phi,\dot{\phi})r$

$$\dot{M}(\phi,\dot{\phi})r = \begin{bmatrix} -2\overbrace{m_2 l_1 l_2}^{\theta_2} s_2\dot{\phi}_2 & -\overbrace{m_2 l_1 l_2}^{\theta_2} s_2\dot{\phi}_2 \\ -\underbrace{m_2 l_1 l_2}_{\theta_2} s_2\dot{\phi}_2 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$= \begin{bmatrix} -2\theta_2\dot{\phi}_2 r_1 - \theta_2 s_2\dot{\phi}_2 r_2 \\ -\theta_2 s_2\dot{\phi}_2 r_1 \end{bmatrix}$$

$$\dot{M}(\phi,\dot{\phi})r = \begin{bmatrix} 0 & -2\dot{\phi}_2 r_1 - s_2\dot{\phi}_2 r_2 & 0 \\ 0 & -s_2\dot{\phi}_2 r_1 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}.$$

Writing all the above

$$M(\phi)\nu = \underbrace{\begin{bmatrix} \nu_1 & 2c_1\nu_1 + c_2\nu_2 & \nu_2 & 0 & 0 \\ 0 & c_2\nu_1 & \nu_1+\nu_2 & 0 & 0 \end{bmatrix}}_{Y_m(\phi)} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix}$$

$$C(\phi,\dot{\phi}) = \underbrace{\begin{bmatrix} 0 & -2s_2\dot{\phi}_1\dot{\phi}_2 - s_2\dot{\phi}_2^2 & 0 & 0 & 0 \\ 0 & s_2\dot{\phi}_1^2 & 0 & 0 & 0 \end{bmatrix}}_{Y_c(\phi,\dot{\phi})} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix}$$

$$G(\phi) = \underbrace{\begin{bmatrix} 0 & 0 & 0 & gc_1 & gc_{12} \\ 0 & 0 & 0 & 0 & gc_{12} \end{bmatrix}}_{Y_G(\phi)} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix}$$

$$\dot{M}(\phi,\dot{\phi})\, r = \underbrace{\begin{bmatrix} 0 & -2S_1\dot{\phi}_2 r_1 - S_2\dot{\phi}_2 r_2 & 0 & 0 & 0 \\ 0 & -S_2\dot{\phi}_2 r_1 & 0 & 0 & 0 \end{bmatrix}}_{Y_{\dot{m}}(\phi,\dot{\phi},r)} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix}$$

$$Y\theta = \left(Y_m(\phi,N) + Y_c(\phi,\dot{\phi}) + Y_G(\phi) + Y_{\dot{m}}(\phi,\dot{\phi},r)\right)\theta.$$

Now,

$$\dot{V}(\xi,t) = -e^T\alpha e + \underbrace{e^T r}_{r^T e} + r^T(Y\theta - \tau) - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}}$$

Design, $\tau = Y\hat{\theta} + e + \beta r.$

$$\dot{V} = -e^T\alpha e + r^T(Y\theta + e - (Y\hat{\theta} + e + \beta r)) - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}}$$

$$\dot{V} = -e^T\alpha e - r^T\beta r + \underbrace{r^T Y\tilde{\theta}}_{\tilde{\theta}^T Y^T r} - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}}$$

design $\dot{\hat{\theta}} = \Gamma Y^T r.$

$$\dot{V} = -e^T\alpha e - r^T\beta r + \cancel{\tilde{\theta}^T Y^T r} - \cancel{\tilde{\theta}^T\Gamma^{-1}(\Gamma Y^T r)}$$

$$\dot{V} = -e^T\alpha e - r^T\beta r.$$

$$\Rightarrow \dot{V} \leq -\underline{\alpha}\|e\|^2 - \underline{\beta}\|r\|^2.$$

where $\underline{\alpha}$ and $\underline{\beta}$ are min eigen values of e and r.

Using Barbalat's Lemma, we can show that

as $t \to \infty$ $e \to 0$

$\Rightarrow r = \dot{e} + \alpha e \Rightarrow r \to 0$

$\Rightarrow \phi_d + \alpha \dot{e} \in L_\infty$ as ϕ_d is a constant.

$\Rightarrow N \in L_\infty$.

$\Rightarrow \phi \in L_\infty$ as $\phi_d, e \in L_\infty$, $\dot{\phi}, \ddot{\phi} \in L_\infty$.

$\Rightarrow Y_m(\phi), Y_c(\phi, \dot{\phi}), Y_G(\phi), Y_{\dot{m}}(\phi, \dot{\phi}) \in L_\infty$.

$\Rightarrow Y \in L_\infty$.

$\dot{\hat{\theta}} = \Gamma Y r$ implies $\dot{\hat{\theta}} \in L_\infty$.

$\tau = Y\hat{\theta} + \beta r + e$. $\Rightarrow \tau \in L_\infty$.

$\dot{r} \in L_\infty$ and r is uniformly continous.

$\Rightarrow \lim_{t \to \infty} r(t) = 0$.

e and r have Global asymptotic tracking.

Since $r, e, \in L_2$ & are Uniformly Continous.

Therefore, the system has GAT using Barbalat's Lemma.