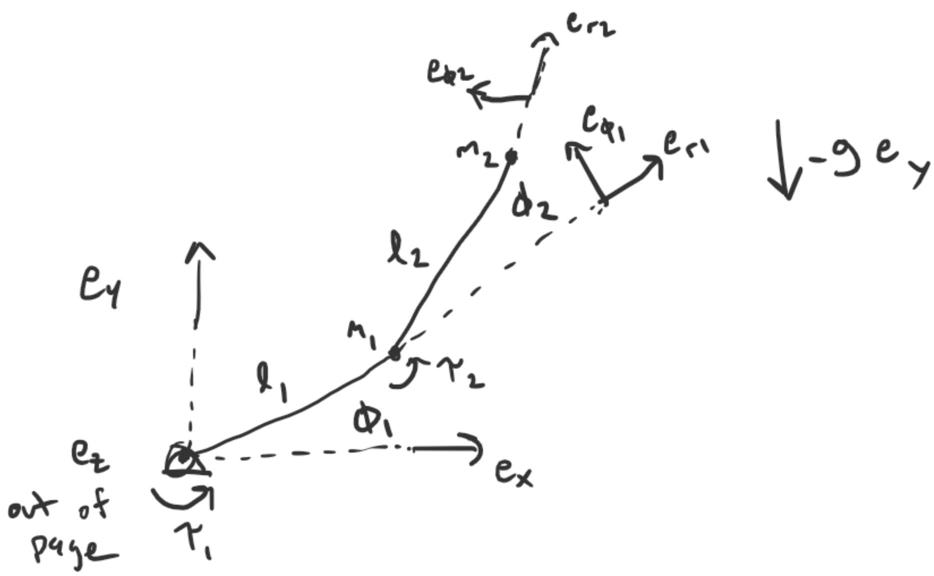
Two-Link Dynamics



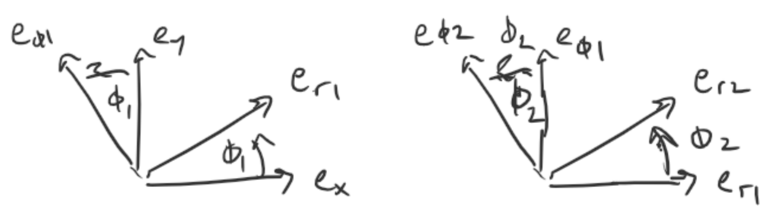
I. Two-Link Parameters

The two-link serial arm above has two links of lengths l_1 and l_2 and a mass at each link, labeled m_1 and m_2. Motors are at the base joint of each link that apply the torque τ_1 and τ_2. The inertial coordinate system consists of the axes $\{e_x, e_y, e_z\}$ which are to the right, up, and out of page, respectively, so a positive rotation is counter-clockwise and gravity of magnitude g is assumed to at downwards. The basis is represented as a simplified column matrix (dropping the e_z since all dynamics are about this axis)

$$e_x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$e_y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Each link has an attached coordinate system consisting of the axes $\{e_{ri}, e_{\phi i}, e_z\}$ which are equivalent to $\{e_x, e_y, e_z\}$ while the angles $\phi_1 = 0$ and $\phi_2 = 0$, which measure the angles between e_{r1} and $e_x$axes and the e_{r2} and e_{r1} axes, respectively.

II. Kinematics



The above diagrams describe the relationship between each coordinate system to aid in the development of the kinematics. Which shows for the first link

$$e_{r1} = c_1 e_x + s_1 e_y = \begin{bmatrix} c_1 \\ s_1 \end{bmatrix}$$
$$e_{\phi 1} = -s_1 e_x + c_1 e_y = \begin{bmatrix} -s_1 \\ c_1 \end{bmatrix}$$

where $c_1 \triangleq \cos(\phi_1)$ and $s_1 \triangleq \sin(\phi_1)$. For the second link

$$e_{r2} = c_2 e_{r1} + s_2 e_{\phi 1}$$
$$e_{\phi 2} = -s_2 e_{r1} + c_2 e_{\phi 1}$$

where we can use the first link relationships to yield

$$e_{r2} = c_2 \begin{bmatrix} c_1 \\ s_1 \end{bmatrix} + s_2 \begin{bmatrix} -s_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} c_1 c_2 - s_1 s_2 \\ s_1 c_2 + c_1 s_2 \end{bmatrix}$$
$$e_{r2} = \begin{bmatrix} c_{12} \\ s_{12} \end{bmatrix}$$

with $c_{12} \triangleq \cos(\phi_1 + \phi_2)$ and $s_{12} \triangleq \sin(\phi_1 + \phi_2)$.

A. *First Link Kinematics*

The position of m_1 is

$$p_1 = l_1 e_{r1} = l_1 \begin{bmatrix} c_1 \\ s_1 \end{bmatrix}.$$

The velocity is then

$$v_1 = \frac{d}{dt} p_1 = l_1 \begin{bmatrix} -s_1 \\ c_1 \end{bmatrix} \dot{\phi}_1.$$

B. *Second Link Kinematics*

The position of m_2 is

$$p_2 = l_1 e_{r1} + l_2 e_{r2} = l_1 \begin{bmatrix} c_1 \\ s_1 \end{bmatrix} + l_2 \begin{bmatrix} c_{12} \\ s_{12} \end{bmatrix}$$
$$p_2 = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \end{bmatrix}$$

The velocity is then

$$v_2 = \frac{d}{dt} p_2 = \begin{bmatrix} -l_1 s_1 \dot{\phi}_1 - l_2 s_{12} \left(\dot{\phi}_1 + \dot{\phi}_2 \right) \\ l_1 c_1 \dot{\phi}_1 + l_2 c_{12} \left(\dot{\phi}_1 + \dot{\phi}_2 \right) \end{bmatrix}$$
$$v_2 = \begin{bmatrix} -\left(l_1 s_1 + l_2 s_{12} \right) \dot{\phi}_1 - l_2 s_{12} \dot{\phi}_2 \\ \left(l_1 c_1 + l_2 c_{12} \right) \dot{\phi}_1 + l_2 c_{12} \dot{\phi}_2 \end{bmatrix}$$
$$v_2 = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}$$
$$v_2 = B_2 \dot{\phi}$$

where $B_2 \triangleq \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$ and $\dot{\phi} \triangleq \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}$.

III. Dynamics using Lagrangian Energy Method

To determine dynamic relationship of the system, we will use the Lagrangian approach

$$\tau_i = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} - \frac{\partial \mathcal{L}}{\partial \phi_i}$$

where the Lagrangian $\mathcal{L}$ is equivalent to

$$\mathcal{L} = \sum_i \mathcal{K}_i - \mathcal{P}_i$$

with the ith kinetic energy $\mathcal{K}_i$ and potential energy $\mathcal{P}_i$

$$\begin{aligned}
\mathcal{K}_i &= \frac{1}{2} m_i v_i^\top v_i, \\
\mathcal{P}_i &= -f_{gi}^\top p_i, \\
f_{gi} &= -m_i g e_y = -m_i g \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
\mathcal{P}_i &= -\left(-m_i g \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)^\top p_i \\
\mathcal{P}_i &= m_i g \begin{bmatrix} 0 & 1 \end{bmatrix} p_i,
\end{aligned}$$

wheref_{gi} is the force of gravity on mass i, and p_i, v_i are the position and velocity of mass i.

A. *First Link Energy*

The kinetic energy for the first link is

$$\begin{aligned}
\mathcal{K}_1 &= \frac{1}{2} m_1 v_1^\top v_1 \\
\mathcal{K}_1 &= \frac{1}{2} m_1 \left(l_1 \begin{bmatrix} -s_1 \\ c_1 \end{bmatrix} \dot{\phi}_1 \right)^\top \left(l_1 \begin{bmatrix} -s_1 \\ c_1 \end{bmatrix} \dot{\phi}_1 \right) \\
\mathcal{K}_1 &= \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 \begin{bmatrix} -s_1 & c_1 \end{bmatrix} \begin{bmatrix} -s_1 \\ c_1 \end{bmatrix} \\
\mathcal{K}_1 &= \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 \left(s_1^2 + c_1^2 \right) \\
\left(s_1^2 + c_1^2 \right) &= 1 \\
\implies \mathcal{K}_1 &= \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2.
\end{aligned}$$

The potential energy for the first link is

$$\begin{aligned}
\mathcal{P}_1 &= m_1 g \begin{bmatrix} 0 & 1 \end{bmatrix} p_1 \\
\mathcal{P}_1 &= m_1 g \begin{bmatrix} 0 & 1 \end{bmatrix} \left(l_1 \begin{bmatrix} c_1 \\ s_1 \end{bmatrix} \right) \\
\mathcal{P}_1 &= m_1 g l_1 \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ s_1 \end{bmatrix} \\
\mathcal{P}_1 &= m_1 g l_1 s_1.
\end{aligned}$$

B. *Second Link Energy*

The kinetic energy for the second link is

$$\begin{aligned}
\mathcal{K}_2 &= \frac{1}{2} m_2 v_2^\top v_2 \\
\mathcal{K}_2 &= \frac{1}{2} m_2 \left(B_2 \dot{\phi} \right)^\top \left(B_2 \dot{\phi} \right) \\
\mathcal{K}_2 &= \frac{1}{2} m_2 \dot{\phi}^\top B_2^\top B_2 \dot{\phi}.
\end{aligned}$$

To simplify this development let

$$B_2 \triangleq \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{aligned}
B_2^\top B_2 &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}^\top \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
B_2^\top B_2 &= \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
B_2^\top B_2 &= \begin{bmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{bmatrix}.
\end{aligned}$$

Now determine $a^2 + c^2$

$$
\begin{aligned}
a^2 + c^2 &= \left(-l_1 s_1 - l_2 s_{12}\right)^2 + \left(l_1 c_1 + l_2 c_{12}\right)^2 \\
a^2 + c^2 &= l_1^2 s_1^2 + 2 l_1 l_2 s_1 s_{12} + l_2^2 s_{12}^2 + l_1^2 c_1^2 + 2 l_1 l_2 c_1 c_{12} + l_2^2 c_{12}^2 \\
a^2 + c^2 &= l_1^2 \left(s_1^2 + c_1^2\right) + 2 l_1 l_2 \left(s_1 s_{12} + c_1 c_{12}\right) + l_2^2 \left(s_{12}^2 + c_{12}^2\right) \\
\left(s_1^2 + c_1^2\right) &= 1 \\
\left(s_1 s_{12} + c_1 c_{12}\right) &= \left(s_1 \left(s_1 c_2 + c_1 s_2\right) + c_1 \left(c_1 c_2 - s_1 s_2\right)\right) \\
\left(s_1 s_{12} + c_1 c_{12}\right) &= \left(s_1^2 c_2 + c_1 s_1 s_2 + c_1^2 c_2 - c_1 s_1 s_2\right) \\
\left(s_1 s_{12} + c_1 c_{12}\right) &= \left(\left(s_1^2 + c_1^2\right) c_2\right) \\
\left(s_1 s_{12} + c_1 c_{12}\right) &= c_2 \\
\left(s_{12}^2 + c_{12}^2\right) &= 1 \\
\implies a^2 + c^2 &= l_1^2 + 2 l_1 l_2 c_2 + l_2^2
\end{aligned}
$$

Now determine $ab + cd$

$$
\begin{aligned}
ab + cd &= \left(-l_1 s_1 - l_2 s_{12}\right)\left(-l_2 s_{12}\right) + \left(l_1 c_1 + l_2 c_{12}\right)\left(l_2 c_{12}\right) \\
ab + cd &= l_1 l_2 s_1 s_{12} + l_2^2 s_{12}^2 + l_1 l_2 c_1 c_{12} + l_2^2 c_{12}^2 \\
ab + cd &= l_1 l_2 \left(s_1 s_{12} + c_1 c_{12}\right) + l_2^2 \left(s_{12}^2 + c_{12}^2\right) \\
\left(s_1 s_{12} + c_1 c_{12}\right) &= c_2 \\
\left(s_{12}^2 + c_{12}^2\right) &= 1 \\
\implies ab + cd &= l_1 l_2 c_2 + l_2^2
\end{aligned}
$$

Now determine $b^2 + d^2$

$$
\begin{aligned}
b^2 + d^2 &= \left(-l_2 s_{12}\right)^2 + \left(l_2 c_{12}\right)^2 \\
b^2 + d^2 &= l_2^2 s_{12}^2 + l_2^2 c_{12}^2 \\
b^2 + d^2 &= l_2^2 \left(s_{12}^2 + c_{12}^2\right) \\
\left(s_{12}^2 + c_{12}^2\right) &= 1 \\
\implies b^2 + d^2 &= l_2^2
\end{aligned}
$$

Using these yields

$$
\begin{aligned}
\mathcal{K}_2 &= \frac{1}{2} m_2 \begin{bmatrix} \dot{\phi}_1 & \dot{\phi}_2 \end{bmatrix} \begin{bmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} \\
\mathcal{K}_2 &= \frac{1}{2} m_2 \begin{bmatrix} \left(a^2 + c^2\right) \dot{\phi}_1 + \left(ab + cd\right) \dot{\phi}_2 & \left(ab + cd\right) \dot{\phi}_1 + \left(b^2 + d^2\right) \dot{\phi}_2 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} \\
\mathcal{K}_2 &= \frac{1}{2} m_2 \left(\left(a^2 + c^2\right) \dot{\phi}_1^2 + \left(ab + cd\right) \dot{\phi}_1 \dot{\phi}_2 + \left(ab + cd\right) \dot{\phi}_1 \dot{\phi}_2 + \left(b^2 + d^2\right) \dot{\phi}_2^2\right) \\
\mathcal{K}_2 &= \frac{1}{2} m_2 \left(\left(a^2 + c^2\right) \dot{\phi}_1^2 + 2\left(ab + cd\right) \dot{\phi}_1 \dot{\phi}_2 + \left(b^2 + d^2\right) \dot{\phi}_2^2\right) \\
a^2 + c^2 &= l_1^2 + 2 l_1 l_2 c_2 + l_2^2 \\
ab + cd &= l_1 l_2 c_2 + l_2^2 \\
b^2 + d^2 &= l_2^2 \\
\implies \mathcal{K}_2 &= \frac{1}{2} m_2 \left(\left(l_1^2 + 2 l_1 l_2 c_2 + l_2^2\right) \dot{\phi}_1^2 + 2\left(l_1 l_2 c_2 + l_2^2\right) \dot{\phi}_1 \dot{\phi}_2 + l_2^2 \dot{\phi}_2^2\right).
\end{aligned}
$$

The potential energy for the second link is

$$
\begin{aligned}
\mathcal{P}_2 &= m_2 g \begin{bmatrix} 0 & 1 \end{bmatrix} p_2 \\
\mathcal{P}_2 &= m_2 g \begin{bmatrix} 0 & 1 \end{bmatrix} \left(\begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \end{bmatrix}\right) \\
\mathcal{P}_2 &= m_2 g \left(l_1 s_1 + l_2 s_{12}\right)
\end{aligned}
$$

C. Lagrangian

Using the energy for each link, the Lagrangian is

$$\mathcal{L} = \sum_i \mathcal{K}_i - \mathcal{P}_i$$
$$\mathcal{L} = \mathcal{K}_1 - \mathcal{P}_1 + \mathcal{K}_2 - \mathcal{P}_2$$
$$\mathcal{L} = \frac{1}{2}m_1 l_1^2 \dot{\phi}_1^2 - m_1 g l_1 s_1 + \frac{1}{2}m_2\left(\left(l_1^2 + 2l_1l_2c_2 + l_2^2\right)\dot{\phi}_1^2 + 2\left(l_1l_2c_2 + l_2^2\right)\dot{\phi}_1\dot{\phi}_2 + l_2^2\dot{\phi}_2^2\right) - m_2 g\left(l_1 s_1 + l_2 s_{12}\right)$$

and to determine the torques we can use

$$\tau_i = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} - \frac{\partial \mathcal{L}}{\partial \phi_i}.$$

D. First Link Torque

The first link torque is

$$\tau_1 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} - \frac{\partial \mathcal{L}}{\partial \phi_1}$$
$$\tau_1 = \frac{d}{dt}\frac{\partial \mathcal{K}_1}{\partial \dot{\phi}_1} - \frac{d}{dt}\frac{\partial \mathcal{P}_1}{\partial \dot{\phi}_1} + \frac{d}{dt}\frac{\partial \mathcal{K}_2}{\partial \dot{\phi}_1} - \frac{d}{dt}\frac{\partial \mathcal{P}_2}{\partial \dot{\phi}_1} - \left(\frac{\partial \mathcal{K}_1}{\partial \phi_1} - \frac{\partial \mathcal{P}_1}{\partial \phi_1} + \frac{\partial \mathcal{K}_2}{\partial \phi_1} - \frac{\partial \mathcal{P}_2}{\partial \phi_1}\right)$$
$$\tau_1 = \frac{d}{dt}\frac{\partial \mathcal{K}_1}{\partial \dot{\phi}_1} - \frac{d}{dt}\frac{\partial \mathcal{P}_1}{\partial \dot{\phi}_1} + \frac{d}{dt}\frac{\partial \mathcal{K}_2}{\partial \dot{\phi}_1} - \frac{d}{dt}\frac{\partial \mathcal{P}_2}{\partial \dot{\phi}_1} - \frac{\partial \mathcal{K}_1}{\partial \phi_1} + \frac{\partial \mathcal{P}_1}{\partial \phi_1} - \frac{\partial \mathcal{K}_2}{\partial \phi_1} + \frac{\partial \mathcal{P}_2}{\partial \phi_1}$$

For $\mathcal{K}_1 = \frac{1}{2}m_1 l_1^2 \dot{\phi}_1^2$

$$\frac{d}{dt}\frac{\partial \mathcal{K}_1}{\partial \dot{\phi}_1} = \frac{d}{dt}\frac{\partial}{\partial \dot{\phi}_1}\left(\frac{1}{2}m_1 l_1^2 \dot{\phi}_1^2\right)$$
$$\frac{d}{dt}\frac{\partial \mathcal{K}_1}{\partial \dot{\phi}_1} = \frac{d}{dt}\left(2\frac{1}{2}m_1 l_1^2 \dot{\phi}_1\right) = m_1 l_1^2 \ddot{\phi}_1$$

$$\frac{\partial \mathcal{K}_1}{\partial \phi_1} = \frac{\partial}{\partial \phi_1}\left(\frac{1}{2}m_1 l_1^2 \dot{\phi}_1^2\right) = 0$$

For $\mathcal{K}_2 = \frac{1}{2}m_2\left(\left(l_1^2 + 2l_1l_2c_2 + l_2^2\right)\dot{\phi}_1^2 + 2\left(l_1l_2c_2 + l_2^2\right)\dot{\phi}_1\dot{\phi}_2 + l_2^2\dot{\phi}_2^2\right)$

$$\frac{d}{dt}\frac{\partial \mathcal{K}_2}{\partial \dot{\phi}_1} = \frac{d}{dt}\frac{\partial}{\partial \dot{\phi}_1}\left(\frac{1}{2}m_2\left(\left(l_1^2 + 2l_1l_2c_2 + l_2^2\right)\dot{\phi}_1^2 + 2\left(l_1l_2c_2 + l_2^2\right)\dot{\phi}_1\dot{\phi}_2 + l_2^2\dot{\phi}_2^2\right)\right)$$
$$\frac{d}{dt}\frac{\partial \mathcal{K}_2}{\partial \dot{\phi}_1} = \frac{d}{dt}\left(\frac{1}{2}m_2\left(2\left(l_1^2 + 2l_1l_2c_2 + l_2^2\right)\dot{\phi}_1 + 2\left(l_1l_2c_2 + l_2^2\right)\dot{\phi}_2\right)\right)$$
$$\frac{d}{dt}\frac{\partial \mathcal{K}_2}{\partial \dot{\phi}_1} = m_2\left(\frac{d}{dt}\left(l_1^2 + 2l_1l_2c_2 + l_2^2\right)\dot{\phi}_1 + \left(l_1^2 + 2l_1l_2c_2 + l_2^2\right)\frac{d}{dt}\dot{\phi}_1 + \frac{d}{dt}\left(l_1l_2c_2 + l_2^2\right)\dot{\phi}_2 + \left(l_1l_2c_2 + l_2^2\right)\frac{d}{dt}\dot{\phi}_2\right)$$
$$\frac{d}{dt}\frac{\partial \mathcal{K}_2}{\partial \dot{\phi}_1} = m_2\left(\left(-2l_1l_2s_2\dot{\phi}_2\right)\dot{\phi}_1 + \left(l_1^2 + 2l_1l_2c_2 + l_2^2\right)\ddot{\phi}_1 + \left(-l_1l_2s_2\dot{\phi}_2\right)\dot{\phi}_2 + \left(l_1l_2c_2 + l_2^2\right)\ddot{\phi}_2\right)$$
$$\frac{d}{dt}\frac{\partial \mathcal{K}_2}{\partial \dot{\phi}_1} = m_2\left(-2l_1l_2s_2\dot{\phi}_1\dot{\phi}_2 + \left(l_1^2 + 2l_1l_2c_2 + l_2^2\right)\ddot{\phi}_1 - l_1l_2s_2\dot{\phi}_2^2 + \left(l_1l_2c_2 + l_2^2\right)\ddot{\phi}_2\right)$$

$$\frac{\partial \mathcal{K}_2}{\partial \phi_1} = \frac{\partial}{\partial \phi_1}\left(\frac{1}{2}m_2\left(\left(l_1^2 + 2l_1l_2c_2 + l_2^2\right)\dot{\phi}_1^2 + 2\left(l_1l_2c_2 + l_2^2\right)\dot{\phi}_1\dot{\phi}_2 + l_2^2\dot{\phi}_2^2\right)\right) = 0$$

For $\mathcal{P}_1 = m_1 g l_1 s_1$

$$\frac{d}{dt}\frac{\partial \mathcal{P}_1}{\partial \dot{\phi}_1} = \frac{d}{dt}\frac{\partial}{\partial \dot{\phi}_1}\left(m_1 g l_1 s_1\right) = 0$$

$$\frac{\partial \mathcal{P}_1}{\partial \phi_1} = \frac{\partial}{\partial \phi_1}\left(m_1 g l_1 s_1\right) = m_1 g l_1 c_1$$

For $\mathcal{P}_2 = m_2 g\left(l_1 s_1 + l_2 s_{12}\right)$

$$\frac{d}{dt}\frac{\partial \mathcal{P}_2}{\partial \dot{\phi}_1} = \frac{d}{dt}\frac{\partial}{\partial \dot{\phi}_1}\left(m_2 g\left(l_1 s_1 + l_2 s_{12}\right)\right) = 0$$

$$\begin{aligned}
\frac{\partial \mathcal{P}_2}{\partial \phi_1} &= \frac{\partial}{\partial \phi_1}\left(m_2 g\left(l_1 s_1 + l_2 s_{12}\right)\right)\\
\frac{\partial \mathcal{P}_2}{\partial \phi_1} &= \frac{\partial}{\partial \phi_1}\left(m_2 g\left(l_1 s_1 + l_2\left(s_1 c_2 + c_1 s_2\right)\right)\right)\\
\frac{\partial \mathcal{P}_2}{\partial \phi_1} &= m_2 g\left(l_1 c_1 + l_2\left(c_1 c_2 - s_1 s_2\right)\right)\\
\left(c_1 c_2 - s_1 s_2\right) &= c_{12}\\
\frac{\partial \mathcal{P}_2}{\partial \phi_1} &= m_2 g\left(l_1 c_1 + l_2 c_{12}\right)
\end{aligned}$$

Substituting these in yields

$$\begin{aligned}
\tau_1 &= m_1 l_1^2 \ddot{\phi}_1 - 0 + m_2\left(-2 l_1 l_2 s_2 \dot{\phi}_1 \dot{\phi}_2 + \left(l_1^2 + 2 l_1 l_2 c_2 + l_2^2\right)\ddot{\phi}_1 - l_1 l_2 s_2 \dot{\phi}_2^2 + \left(l_1 l_2 c_2 + l_2^2\right)\ddot{\phi}_2\right)\\
&\quad - 0 - 0 + m_1 g l_1 c_1 - 0 + m_2 g\left(l_1 c_1 + l_2 c_{12}\right)
\end{aligned}$$

$$\begin{aligned}
\tau_1 &= \left(m_1 l_1^2 + m_2\left(l_1^2 + 2 l_1 l_2 c_2 + l_2^2\right)\right)\ddot{\phi}_1 + m_2\left(l_1 l_2 c_2 + l_2^2\right)\ddot{\phi}_2\\
&\quad - 2 m_2 l_1 l_2 s_2 \dot{\phi}_1 \dot{\phi}_2\\
&\quad - m_2 l_1 l_2 s_2 \dot{\phi}_2^2\\
&\quad + \left(m_1 + m_2\right) g l_1 c_1 + m_2 g l_2 c_{12}
\end{aligned}$$

E. Second Link Torque

The second link torque is

$$\begin{aligned}
\tau_2 &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_2} - \frac{\partial \mathcal{L}}{\partial \phi_2}\\
\tau_2 &= \frac{d}{dt}\frac{\partial \mathcal{K}_1}{\partial \dot{\phi}_2} - \frac{d}{dt}\frac{\partial \mathcal{P}_1}{\partial \dot{\phi}_2} + \frac{d}{dt}\frac{\partial \mathcal{K}_2}{\partial \dot{\phi}_2} - \frac{d}{dt}\frac{\partial \mathcal{P}_2}{\partial \dot{\phi}_2} - \left(\frac{\partial \mathcal{K}_1}{\partial \phi_2} - \frac{\partial \mathcal{P}_1}{\partial \phi_2} + \frac{\partial \mathcal{K}_2}{\partial \phi_2} - \frac{\partial \mathcal{P}_2}{\partial \phi_2}\right)\\
\tau_2 &= \frac{d}{dt}\frac{\partial \mathcal{K}_1}{\partial \dot{\phi}_2} - \frac{d}{dt}\frac{\partial \mathcal{P}_1}{\partial \dot{\phi}_2} + \frac{d}{dt}\frac{\partial \mathcal{K}_2}{\partial \dot{\phi}_2} - \frac{d}{dt}\frac{\partial \mathcal{P}_2}{\partial \dot{\phi}_2} - \frac{\partial \mathcal{K}_1}{\partial \phi_2} + \frac{\partial \mathcal{P}_1}{\partial \phi_2} - \frac{\partial \mathcal{K}_2}{\partial \phi_2} + \frac{\partial \mathcal{P}_2}{\partial \phi_2}
\end{aligned}$$

For $\mathcal{K}_1 = \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2$

$$\frac{d}{dt}\frac{\partial \mathcal{K}_1}{\partial \dot{\phi}_2} = \frac{d}{dt}\frac{\partial}{\partial \dot{\phi}_2}\left(\frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2\right) = 0$$

$$\frac{\partial \mathcal{K}_1}{\partial \phi_2} = \frac{\partial}{\partial \phi_2}\left(\frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2\right) = 0$$

For $\mathcal{K}_2 = \frac{1}{2}m_2\left(\left(l_1^2 + 2l_1l_2c_2 + l_2^2\right)\dot{\phi}_1^2 + 2\left(l_1l_2c_2 + l_2^2\right)\dot{\phi}_1\dot{\phi}_2 + l_2^2\dot{\phi}_2^2\right)$

$$\begin{aligned}
\frac{d}{dt}\frac{\partial\mathcal{K}_2}{\partial\dot{\phi}_2} &= \frac{d}{dt}\frac{\partial}{\partial\dot{\phi}_2}\left(\frac{1}{2}m_2\left(\left(l_1^2 + 2l_1l_2c_2 + l_2^2\right)\dot{\phi}_1^2 + 2\left(l_1l_2c_2 + l_2^2\right)\dot{\phi}_1\dot{\phi}_2 + l_2^2\dot{\phi}_2^2\right)\right)\\
\frac{d}{dt}\frac{\partial\mathcal{K}_2}{\partial\dot{\phi}_2} &= \frac{d}{dt}\left(\frac{1}{2}m_2\left(2\left(l_1l_2c_2 + l_2^2\right)\dot{\phi}_1 + 2l_2^2\dot{\phi}_2\right)\right)\\
\frac{d}{dt}\frac{\partial\mathcal{K}_2}{\partial\dot{\phi}_2} &= m_2\left(\frac{d}{dt}\left(l_1l_2c_2 + l_2^2\right)\dot{\phi}_1 + \left(l_1l_2c_2 + l_2^2\right)\frac{d}{dt}\dot{\phi}_1 + l_2^2\frac{d}{dt}\dot{\phi}_2\right)\\
\frac{d}{dt}\frac{\partial\mathcal{K}_2}{\partial\dot{\phi}_2} &= m_2\left(\left(-l_1l_2s_2\dot{\phi}_2\right)\dot{\phi}_1 + \left(l_1l_2c_2 + l_2^2\right)\ddot{\phi}_1 + l_2^2\ddot{\phi}_2\right)\\
\frac{d}{dt}\frac{\partial\mathcal{K}_2}{\partial\dot{\phi}_2} &= m_2\left(-l_1l_2s_2\dot{\phi}_1\dot{\phi}_2 + \left(l_1l_2c_2 + l_2^2\right)\ddot{\phi}_1 + l_2^2\ddot{\phi}_2\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\mathcal{K}_2}{\partial\phi_2} &= \frac{\partial}{\partial\phi_2}\left(\frac{1}{2}m_2\left(\left(l_1^2 + 2l_1l_2c_2 + l_2^2\right)\dot{\phi}_1^2 + 2\left(l_1l_2c_2 + l_2^2\right)\dot{\phi}_1\dot{\phi}_2 + l_2^2\dot{\phi}_2^2\right)\right)\\
\frac{\partial\mathcal{K}_2}{\partial\phi_2} &= \frac{1}{2}m_2\left(\left(-2l_1l_2s_2\right)\dot{\phi}_1^2 + 2\left(-l_1l_2s_2\right)\dot{\phi}_1\dot{\phi}_2\right)\\
\frac{\partial\mathcal{K}_2}{\partial\phi_2} &= m_2\left(-l_1l_2s_2\dot{\phi}_1^2 - l_1l_2s_2\dot{\phi}_1\dot{\phi}_2\right)
\end{aligned}$$

For $\mathcal{P}_1 = m_1gl_1s_1$

$$\frac{d}{dt}\frac{\partial\mathcal{P}_1}{\partial\dot{\phi}_2} = \frac{d}{dt}\frac{\partial}{\partial\dot{\phi}_2}\left(m_1gl_1s_1\right) = 0$$

$$\frac{\partial\mathcal{P}_1}{\partial\phi_2} = \frac{\partial}{\partial\phi_2}\left(m_1gl_1s_1\right) = 0$$

For $\mathcal{P}_2 = m_2g\left(l_1s_1 + l_2s_{12}\right)$

$$\frac{d}{dt}\frac{\partial\mathcal{P}_2}{\partial\dot{\phi}_2} = \frac{d}{dt}\frac{\partial}{\partial\dot{\phi}_2}\left(m_2g\left(l_1s_1 + l_2s_{12}\right)\right) = 0$$

$$\begin{aligned}
\frac{\partial\mathcal{P}_2}{\partial\phi_2} &= \frac{\partial}{\partial\phi_2}\left(m_2g\left(l_1s_1 + l_2s_{12}\right)\right)\\
\frac{\partial\mathcal{P}_2}{\partial\phi_2} &= \frac{\partial}{\partial\phi_2}\left(m_2g\left(l_1s_1 + l_2\left(s_1c_2 + c_1s_2\right)\right)\right)\\
\frac{\partial\mathcal{P}_2}{\partial\phi_2} &= m_2gl_2\left(-s_1s_2 + c_1c_2\right)\\
\left(c_1c_2 - s_1s_2\right) &= c_{12}\\
\frac{\partial\mathcal{P}_2}{\partial\phi_1} &= m_2gl_2c_{12}
\end{aligned}$$

Substituting these in yields

$$\begin{aligned}\tau_2 &= 0 - 0 + m_2\left(-l_1 l_2 s_2 \dot{\phi}_1 \dot{\phi}_2 + \left(l_1 l_2 c_2 + l_2^2\right)\ddot{\phi}_1 + l_2^2 \ddot{\phi}_2\right) - 0\\ &- 0 + 0 - m_2\left(-l_1 l_2 s_2 \dot{\phi}_1^2 - l_1 l_2 s_2 \dot{\phi}_1 \dot{\phi}_2\right) + m_2 g l_2 c_{12}\end{aligned}$$

$$\begin{aligned}\tau_2 &= m_2\left(l_1 l_2 c_2 + l_2^2\right)\ddot{\phi}_1 + m_2 l_2^2 \ddot{\phi}_2\\ &- m_2 l_1 l_2 s_2 \dot{\phi}_1 \dot{\phi}_2 + m_2 l_1 l_2 s_2 \dot{\phi}_1 \dot{\phi}_2\\ &+ m_2 l_1 l_2 s_2 \dot{\phi}_1^2\\ &+ m_2 g l_2 c_{12}\end{aligned}$$

$$\begin{aligned}\tau_2 &= m_2\left(l_1 l_2 c_2 + l_2^2\right)\ddot{\phi}_1 + m_2 l_2^2 \ddot{\phi}_2\\ &+ m_2 l_1 l_2 s_2 \dot{\phi}_1^2\\ &+ m_2 g l_2 c_{12}\end{aligned}$$

F. Combined System

We can now combine the dynamics into a single system

$$M(\phi)\ddot{\phi} + C\left(\phi, \dot{\phi}\right) + G(\phi) = \tau$$

where the inertia terms $M(\phi)$, the Coriolis and centripetal terms $C\left(\phi, \dot{\phi}\right)$, and the gravity terms $G(\phi)$ are defined as

$$\begin{aligned}M(\phi) &\triangleq \begin{bmatrix}\left(m_1 l_1^2 + m_2\left(l_1^2 + 2 l_1 l_2 c_2 + l_2^2\right)\right) & m_2\left(l_1 l_2 c_2 + l_2^2\right)\\ m_2\left(l_1 l_2 c_2 + l_2^2\right) & m_2 l_2^2\end{bmatrix}\\ C\left(\phi, \dot{\phi}\right) &\triangleq \begin{bmatrix}-2 m_2 l_1 l_2 s_2 \dot{\phi}_1 \dot{\phi}_2\\ 0\end{bmatrix} + \begin{bmatrix}-m_2 l_1 l_2 s_2 \dot{\phi}_2^2\\ m_2 l_1 l_2 s_2 \dot{\phi}_1^2\end{bmatrix}\\ G(\phi) &\triangleq \begin{bmatrix}\left(m_1 + m_2\right) g l_1 c_1 + m_2 g l_2 c_{12}\\ m_2 g l_2 c_{12}\end{bmatrix}.\end{aligned}$$