The Probability of a Random Walk First Returning to the Origin at Time $t = 2n$

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What is the probability that a random walk, beginning at the origin, will return to the origin at time $t = 2n$? The walk can move up (+1) or down (-1) at any one step, with each movements having a probability of 1/2. The answer to this question involves probability theory, combinatorial identities, and generating functions.

1 Introduction: A Random Walk

(Note: The following discussion borrows from Chapter 12 of Grinstead and Snell's *Introduction to Probability* (Online Ed., 1997)[1] and Prof. Pitman's Online Lecture Notes[2])

Definition 1. Let$\{X_k\}_{k=1}^{\infty} = \{X_1, X_2, X_3, ..., X_k, ...\}$ be a sequence of independent and identically distributed (i.i.d) discrete random variables. For all $n \geq 1$, let $S_n = X_1 + X_2 + X_3 + \cdots + X_n$. The sequence of partial sums $\{S_n\}_{n=1}^{\infty}$, which also can be denoted as the series $\sum_{n=1}^{\infty} X_n$, is called a **random walk.**

In this discussion, we consider the case where the random variables X_i share the following distribution function:

$$f_X(x) = \begin{cases} \frac{1}{2}, & \text{if } x = \pm 1 \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

2 k-paths

Definition 2. When graphed on the Cartesian axis, we define a **k-path** to be the path a random walk can take up to its k-th step ($t = k$), the plot of a unique S_k.

Proposition 1. *The probability of a 2m-path returning to the origin is*

$$u_{2m} = \mathbb{P}_0(S_{2m} = 0) = \frac{\binom{2m}{m}}{2^{2m}} \tag{2}$$

The argument for this proposition is based on the properties of the binomial distribution. In this case, we have $2m$ trials and we want to know the probability of m succeses, with probabilities $p = 1/2$ (of a +1 movement) and $q = 1/2$ (of a -1 movement). Note that the number of +1 movements must equal the number of -1 movements, or in this case our X_is. We also conclude that the path can only return to the origin at an even time. Therefore,

$$\mathbb{P}(m \text{ successes in } 2m \text{ trials}) = \binom{2m}{m}\left(\frac{1}{2}\right)^{2m}$$

3 First Return

Definition 3. A random walk has a **first return** to the origin at its $2m$-th step if:

1. $m \geq 1$
2. $S_{2k} \neq 0 \quad \forall k < m$

We will express the probability of a random walk's first return at time $t = 2m$ as f_{2m}. Also, we define $f_0 = 0$.

[1] http://www.dartmouth.edu/˜chance/teaching_aids/books_articles/probability_book/pdf.html
[2] http://bibserver.berkeley.edu/150/lectures/lecture11/Lec11.pdf

Theorem 1. *For $n \geq 1$, $\{f_{2k}\}$ and $\{u_{2k}\}$ are related by the following equation:*

$$u_{2n} = f_0 u_{2n} + f_2 u_{2n-2} + \cdots + f_{2n} u_0 \tag{3}$$

Proof. We begin by noting that the expression $f_{2n}2^{2n}$ is equal to the number of $2n$-paths that only touch the origin at the endpoints, that is the on cartesian coordiantes $(0,0)$ and $(2n,0)$. Similarly, $u_{2n}2^{2n}$ is equal to the *total* number of $2n$-paths that end at the origin. The collection of these $2n$-paths can be partitioned into n sets, depending on their first return. For example, a path in this collection that has its first return at $t = 2k$, consists of a path from $(0,0)$ to $(2k,0)$ that only touches the origin at those endpoints and a path from $(2k,0)$ to $(2n,0)$ that has no restrictions other than the probablistic constraints that we gave the X_i's. Thus, the number of $2n$-paths that have their first return at $t = 2k$ is given by

$$f_{2k}2^{2k}u_{2n-2k}2^{2n-2k} = f_{2k}u_{2n-2k}2^{2n}$$

If we sum, the right hand side of the above equality, over k, we find that

$$u_{2n}2^{2n} = f_0 u_{2n}2^{2n} + f_2 u_{2n-2}2^{2n} + \cdots + f_{2n}u_0 2^{2n}$$

Dividing both sides by 2^{2n} gives (3). □

Given this relation, we should now try to express f_{2n} (unkown) in terms of u_{2n} (known). At this point, we use the properties of generating functions (power series) to help us simplify the relation given by (3).

4 Generating Functions

We define the following generating functions, as derived from u_{2m} and f_{2m},

$$U(x) = \sum_{m=0}^{\infty} u_{2m}x^m \quad \text{and} \quad F(x) = \sum_{m=1}^{\infty} f_{2m}x^m$$

A *convolution* argument can be simplified as follows

$$\begin{aligned} F(x)U(x) &= \left(\sum_{m=1}^{\infty} f_{2m}x^m\right)\left(\sum_{k=0}^{\infty} u_{2k}x^k\right) \\ &= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n} f_{2m}u_{2n-2m}\right)x^n \\ &= \sum_{n=1}^{\infty} u_{2n}x^n \\ &= U(x) - 1 \end{aligned}$$

Which implies that,

$$F(x) = \frac{U(x)-1}{U(x)} = 1 - \frac{1}{U(x)} \tag{4}$$

Therefore, if we can find a closed-form solution for $U(x)$, then we will have one for $F(x)$. We shift focus temporarily to establish some algebraic identities.

5 Algebra and Identities

By the *Binomial Theorem*

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \quad \forall n \geq 1 \tag{5}$$

this can be generalized to

$$(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k \quad \text{for } |x| < 1$$

Also, note that

$$\binom{a}{k} := \frac{a(a-1)\cdots(a-k+1)}{k!} \quad \forall a \in \mathbb{R} \tag{6}$$

These identities will help us find the closed-form solution of $U(x)$, we just need to prove one more claim.

Claim.

$$\binom{2n}{n} = 2^{2n}(-1)^n \binom{-\frac{1}{2}}{n} \tag{7}$$

Proof.

$$\begin{aligned}
\binom{2n}{n} &= \frac{1}{n!} \frac{2n(2n-1)\cdots(n+1)(n)(n-1)\cdots}{n(n-1)\cdots 1} \\
&= \frac{1}{n!} 2(2n-1)2(2n-3)2\cdots(5)2(3)2(1) \\
&= \frac{1}{n!} 2^n \cdot 1 \cdot 3 \cdots (2n-1) \\
&= \frac{1}{n!} 2^{2n} (\frac{1}{2})(\frac{1}{2}+1)(\frac{1}{2}+2)\cdots(\frac{1}{2}+n-1) \\
&= \frac{1}{n!} 2^{2n} (-1)^n (-\frac{1}{2})(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-n+1) \\
&= 2^{2n}(-1)^n \binom{-\frac{1}{2}}{n} && \text{by (6)}
\end{aligned}$$

□

6 Formulas for $U(x)$ and $F(x)$

We begin with the closed-form solution of $U(x)$:

$$\begin{aligned}
U(x) &= \sum_{n=0}^{\infty} u_{2n} x^n \\
&= \sum_{n=0}^{\infty} \binom{2n}{n} 2^{-2n} x^n && \text{by (2)} \\
&= \sum_{n=0}^{\infty} 2^{2n} (-1)^n \binom{-\frac{1}{2}}{n} 2^{-2n} x^n && \text{by (7)} \\
&= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x)^n \\
&= (1-x)^{-\frac{1}{2}} && \text{by (5)}
\end{aligned}$$

Recall that

$$F(x) = \sum_{m=1}^{\infty} f_{2m} x^m$$

where f_{2m} is the probability of the random walk's ($p = \frac{1}{2}, q = \frac{1}{2}$) first return at time $t = 2m$.We continue with an application of the binomial theorem on the results from above and (4).

$$\begin{aligned}
F(x) &= 1 - U(x)^{-1} \\
&= 1 - (1-x)^{\frac{1}{2}} \\
&= 1 - \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} (-x)^n \\
&= \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} (-1)^{n-1} (x)^n \\
&= \sum_{m=1}^{\infty} f_{2m} x^m
\end{aligned}$$

Comparing the coefficients we deduce that:

$$f_{2n} = (-1)^{n-1} \binom{\frac{1}{2}}{n} = \frac{\binom{2n}{n}}{(2n-1)2^{2n}} = \frac{u_{2n}}{2n-1}$$