Wealth Distributions in Models of Capital Exchange

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A dynamical model of capital exchange is introduced in which a specified amount of capital is exchanged between two individuals when they meet. The resulting time dependent wealth distributions are determined for a variety of exchange rules. For "greedy" exchange, an interaction between a rich and a poor individual results in the rich taking a specified amount of capital from the poor. When this amount is independent of the capitals of the two traders, a mean-field analysis yields a Fermi-like scaled wealth distribution in the long-time limit. This same distribution also arises in greedier exchange processes, where the interaction rate is an increasing function of the capital difference of the two traders. The wealth distribution in multiplicative processes, where the amount of capital exchanged is a finite fraction of the capital of one of the traders, are also discussed. For random multiplicative exchange, a steady state wealth distribution is reached, while in greedy multiplicative exchange a non-steady power law wealth distribution arises, in which the support of the distribution continuously increases. Finally, extensions of our results to arbitrary spatial dimension and to growth processes, where capital is created in an interaction, are presented.

I. INTRODUCTION

Economics underlies many of our day-to-day activities, and yet, remarkably, resists axiomatization and first-principles explanations. However, the recent applications of ideas developed in statistical physics, such as scaling, self-organization, landscape paradigms, *etc.* may help establish a conceptual framework for the scientific analysis of economic activities [1–4]. In this spirit of simplicity and concreteness, we introduce capital exchange models as an attempt to account for the wealth distribution of a population. While there is considerable data available on this phenomenon [5], there does not appear to be a simple and causal explanation of the observed distributions. The basis of our modeling is that the elemental kernel of economic activity in a dynamic economy is the interaction between two individuals which results in the rearrangement of their capital. Through repeated two-body interactions between pairs of traders, a global wealth distribution develops and we wish to understand how generic features of this distribution depend on the nature of the two-body interaction.

Our models have the obvious shortcomings of being oversimplified and on focusing on only one mechanism among the myriad of factors that influence individual wealth. Based on everyday experience, however, it may be argued that at an elemental level much economic activity is essentially the trading of capital between two individuals. The potential utility of our models is that they yield realistic wealth distributions for certain exchange rules, as well suggesting avenues for potentially fruitful developments.

In the next section, we first treat "additive" processes in which a fixed amount of capital is exchanged in an interaction between two individuals, independent of their initial capital [6]. While the restriction of fixed capital is not realistic, the resulting models are soluble and pro-vide a natural starting point for economically-motivated generalizations. Within a (mean-field) rate equation description, we find that the wealth distribution generally exhibits scaling, from which both the time dependence of the average wealth and the shape of the scaled wealth distribution can be obtained. As a preliminary, we give an exact solution for random exchange, where either trader is equally likely to profit in an interaction. The scaling approach is then applied to "greedy" exchange, in which the richer person always wins in an interaction. For this system, the resulting wealth distribution closely resembles the occupancy distribution of an ideal Fermi gas.

We next consider "multiplicative" processes in Sec. III, in which the amount of capital exchanged is a fixed fraction of the current capital of one of the traders. The motivation for this model is the observation that fractional exchange typically underlies many economic transactions – for example, a loan at the end of its repayment period. For a broad range of model parameters, realistic forms of the wealth distribution result. In the interesting case of greedy multiplicative exchange, the distribution is a power with a cutoff at small wealth that decays exponentially in time and a large wealth cutoff that grows linearly. In Sec. IV, we then discuss the behavior of our wealth exchange models in arbitrary spatial dimension, when diffusion is the transport mechanism which brings trading partners together. As might be anticipated, when the spatial dimension $d < 2$, dimension-dependent behavior is found for the time evolution of the wealth distribution. Finally, in Sec. V, we summarize and also briefly mention a toy model for economic growth in which capital is produced in an elemental two-body interaction. The relation between capital production and capital conserving models are also outlined.

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II. ADDITIVE CAPITAL EXCHANGE

Consider a population of traders, each of which possesses a certain amount of capital which is assumed to be quantized in units of minimal capital. Taking this latter quantity as the basic unit, the fortune of an individual is restricted to the integers. The wealth of the population evolves by the repeated interaction of random pairs of traders. In each interaction, one unit of capital is transferred between the trading partners. To complete the description, we specify that if a poorest individual (with one unit of capital) loses all remaining capital by virtue of a "loss", the bankrupt individual is considered to be economically dead and no longer participates in economic activity.

In the following subsections, we consider three specific realizations of additive capital exchange. In "random" exchange, the direction of the capital exchange is independent of the relative capital of the traders. While this rule has little economic basis, the resulting model is completely soluble and thus provides a helpful pedagogical starting point. We next consider "greedy" exchange in which a richer person takes one unit of capital from a poorer person in a trade. Such a rule is a reasonable starting point for describing exploitive economic activity. Finally, we consider a more heartless version – "very greedy" exchange – in which the rate of exchange is proportional to the difference in capital between the two interacting individuals. These latter two cases can be solved by a scaling approach. The primary result is that the scaled wealth distribution resembles a finite-temperature Fermi distribution, with an effective temperature that goes to zero in the long-time limit.

A. Random Exchange

In this process, one unit of capital is exchanged between trading partners, as represented by the reaction scheme $(j,k) \rightarrow (j \pm 1, k \mp 1)$. Let $c_k(t)$ be the density of individuals with capital k. Within a mean-field description, $c_k(t)$ evolves according to

$$\frac{dc_k(t)}{dt} = N(t) \left[c_{k+1}(t) + c_{k-1}(t) - 2c_k(t) \right], \qquad (1)$$

with $N(t) \equiv M_0(t) = \sum_{k=1}^{\infty} c_k(t)$ the population density. The first two terms account for the gain in $c_k(t)$ due to the interactions $(j, k+1) \rightarrow (j+1, k)$ and $(j, k-1) \rightarrow (j-1, k)$, respectively, while the last term accounts for the loss in $c_k(t)$ due to the interactions $(j, k) \rightarrow (j \pm 1, k \mp 1)$

By defining a modified time variable,

$$T = \int_0^t dt' N(t'), \qquad (2)$$

Eq. (1) is reduced to the discrete diffusion equation

$$\frac{dc_k(T)}{dT} = c_{k+1}(T) + c_{k-1}(T) - 2c_k(T). \qquad (3)$$

The rate equation for the poorest density has the slightly different form, $dc_1/dT = c_2 - 2c_1$, but may be written in the same form as Eq. (3) if we impose the boundary condition $c_0(T) = 0$.

Eq. (3) may be readily solved for arbitrary initial conditions [7]. For illustrative purposes, let us assume that initially all individuals have one unit of capital, $c_k(0) = \delta_{k1}$. The solution to Eq. (3) subject to these initial and boundary conditions is

$$c_k(T) = e^{-2T} \left[I_{k-1}(2T) - I_{k+1}(2T) \right], \qquad (4)$$

where I_n denotes the modified Bessel function of order n [8]. Consequently, the total density $N(T)$ is

$$N(T) = e^{-2T} \left[I_0(2T) + I_1(2T) \right]. \qquad (5)$$

To re-express this exact solution in terms of the physical time t, we first invert Eq. (2) to obtain $t(T) = \int_0^T dT'/N(T')$, and then eliminate T in favor of t in the solution for $c_k(T)$. For simplicity and concreteness, let us consider the long-time limit. From Eq. (4),

$$c_k(T) \simeq \frac{k}{\sqrt{4\pi T^3}} \exp\left(-\frac{k^2}{4T} \right), \qquad (6)$$

and from Eq. (5),

$$N(T) \simeq (\pi T)^{-1/2}. \qquad (7)$$

Eq. (7) also implies $t \simeq \frac{2}{3}\sqrt{\pi T^3}$ which gives

$$N(t) \simeq \left(\frac{2}{3\pi t} \right)^{1/3}, \qquad (8)$$

and

$$c_k(t) \simeq \frac{k}{3t} \exp\left[-\left(\frac{\pi}{144} \right)^{1/3} \frac{k^2}{t^{2/3}} \right]. \qquad (9)$$

Note that this latter expression may be written in the scaling form $c_k(t) \propto N^2 x\, e^{-x^2}$, with the scaling variable $x \propto kN$. One can also confirm that the scaling solution represents the basin of attraction for almost all exact solutions. Indeed, for any initial condition with $c_k(0)$ decaying faster than k^{-2}, the system reaches the scaling limit $c_k(t) \propto N^2 x\, e^{-x^2}$. On the other hand, if $c_k(0) \sim k^{-1-\alpha}$, with $0 < \alpha < 1$, such an initial state converges to an alternative scaling limit which depends on α, as discussed, e. g., in Ref. [9]. These solutions exhibit a slower decay of the total density, $N \sim t^{-\alpha/(1+\alpha)}$, while the scaling form of the wealth distribution is

$$c_k(t) \sim N^{2/\alpha} \mathcal{C}_\alpha(x), \quad x \propto kN^{1/\alpha}, \qquad (10)$$

with the scaling function

$$\mathcal{C}_\alpha(x) = e^{-x^2} \int_0^\infty du \, \frac{e^{-u^2} \sinh(2ux)}{u^{1+\alpha}}. \qquad (11)$$

Evaluating the integral by the Laplace method gives an asymptotic distribution which exhibits the same $x^{-1-\alpha}$ as the initial distribution. This anomalous scaling in the solution to the diffusion equation is a direct consequence of the extended initial condition. This latter case is not physically relevant, however, since the extended initial distribution leads to a divergent initial wealth density.

B. Greedy Exchange

In greedy exchange, when two individuals meet, the richer person takes one unit of capital from the poorer person, as represented by the reaction scheme $(j, k) \rightarrow (j+1, k-1)$ for $j \geq k$. In the rate equation approximation, the densities $c_k(t)$ now evolve according to

$$\frac{dc_k}{dt} = c_{k-1} \sum_{j=1}^{k-1} c_j + c_{k+1} \sum_{j=k+1}^\infty c_j - c_k N - c_k^2. \quad (12)$$

The first two terms account for the gain in $c_k(t)$ due to the interaction between pairs of individuals of capitals $(j, k-1)$, with $j < k$ and $(j, k+1)$ with $j > k$, respectively. The last two terms correspondingly account for the loss of $c_k(t)$. One can check that the wealth density $M_1 \equiv \sum_{k=1}^\infty k \, c_k(t)$ is conserved and that the population density obeys

$$\frac{dN}{dt} = -c_1 N. \qquad (13)$$

Eqs. (12) are conceptually similar to the Smoluchowski equations for aggregation with a constant reaction rate [10]. Mathematically, however, they appear to be more complex and we have been unable to solve them analytically. Fortunately, Eq. (12) is amenable to a scaling solution [11]. For this purpose, we first re-write Eq. (12) as

$$\frac{dc_k}{dt} = -c_k(c_k + c_{k+1}) + N(c_{k-1} - c_k)$$
$$+ (c_{k+1} - c_{k-1}) \sum_{j=k}^\infty c_j. \qquad (14)$$

Taking the continuum limit and substituting the scaling ansatz

$$c_k(t) \simeq N^2 \mathcal{C}(x), \quad \text{with} \quad x = kN, \qquad (15)$$

transforms Eqs. (13) and (14) to

$$\frac{dN}{dt} = -\mathcal{C}(0)N^3, \qquad (16)$$

and

$$\mathcal{C}(0)[2\mathcal{C} + x\mathcal{C}'] = 2\mathcal{C}^2 + \mathcal{C}' \left[1 - 2 \int_x^\infty dy \mathcal{C}(y) \right], \quad (17)$$

where $\mathcal{C}' = d\mathcal{C}/dx$. Note also that the scaling function must obey the integral relations

$$\int_0^\infty dx \, \mathcal{C}(x) = 1, \quad \text{and} \quad \int_0^\infty dx \, x \, \mathcal{C}(x) = 1. \quad (18)$$

The former follows from the definition of the density, $N = \sum c_k(t) \simeq N \int dx \, \mathcal{C}(x)$, while the latter follows if we set, without loss of generality, the (conserved) wealth density equal to unity, $\sum_k k c_k(t) = 1$.

Introducing $\mathcal{B}(x) = \int_0^x dy \, \mathcal{C}(y)$ recasts Eq. (17) into $\mathcal{C}(0)[2\mathcal{B}' + x\mathcal{B}''] = 2\mathcal{B}'^2 + \mathcal{B}''[2\mathcal{B} - 1]$. Integrating twice gives $[\mathcal{C}(0)x - \mathcal{B}][\mathcal{B} - 1] = 0$, with solution $\mathcal{B}(x) = \mathcal{C}(0)x$ for $x < x_f$ and $\mathcal{B}(x) = 1$ for $x \geq x_f$, from which we conclude that the scaled wealth distribution $\mathcal{C}(x) = \mathcal{B}'(x)$ coincides with the zero-temperature Fermi distribution;

$$\mathcal{C}(x) = \begin{cases} \mathcal{C}(0), & x < x_f; \\ 0, & x \geq x_f. \end{cases} \qquad (19)$$

Hence the scaled profile has a sharp front at $x = x_f$, with $x_f = 1/\mathcal{C}(0)$ found by matching the two branches of the solution for $\mathcal{B}(x)$. Making use of the second integral relation (18) gives $\mathcal{C}(0) = 1/2$ and thereby closes the solution. Thus the unscaled wealth distribution $c_k(t)$ reads

$$c_k(t) = \begin{cases} 1/(2t), & k < 2\sqrt{t}; \\ 0, & k \geq 2\sqrt{t}; \end{cases} \qquad (20)$$

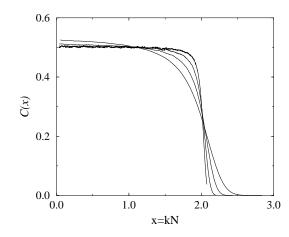and the total density is $N(t) = t^{-1/2}$.

FIG. 1. Simulation results for the wealth distribution in greedy additive exchange based on 2500 configurations for 10^6 traders. Shown are the scaled distributions $\mathcal{C}(x)$ versus $x = kN$ for $t = 1.5^n$, with $n = 18, 24, 30,$ and 36; these steepen with increasing time. Each data set has been averaged over a range of $\approx 3\%$ of the data points to reduce fluctuations.

We checked these predictions by numerical simulations (Fig. 1). In the simulation, two individuals are randomly chosen to undergo greedy exchange and this process is repeated. When an individual reaches zero capital he is eliminated from the system, and the number of active traders is reduced by one. After each reaction, the time is incremented by the inverse of the number of active traders. While the mean-field predictions are substantially corroborated, the scaled wealth distribution for finite time actually resembles a finite-temperature Fermi distribution (Fig. 1). As time increases, the wealth distribution becomes sharper and approaches Eq. (20). In analogy with the Fermi distribution, the relative width of the front may be viewed as an effective temperature. Thus the wealth distribution is characterized by two scales; one of order $\sqrt{t}$ characterizes the typical wealth of active traders and a second, smaller scale which characterizes the width of the front [12].

To quantify the spreading of the front, let us include the next corrections in the continuum limit of the rate equations, Eq. (14). This gives,

$$\frac{\partial c}{\partial t} = 2\frac{\partial}{\partial k}\left[c\int_k^\infty dj\, c(j)\right] - c\frac{\partial c}{\partial k} - N\frac{\partial c}{\partial k} + \frac{N}{2}\frac{\partial^2 c}{\partial k^2}. \quad (21)$$

Here the second and fourth terms on the right-hand side represent the next corrections. Since the "convective" (third) term determines the location of the front to be at $k_f = 2\sqrt{t}$, it is natural to expect that the (diffusive) fourth term describes the spreading of the front. The term $c\frac{\partial c}{\partial k}$ turns out to be negligible in comparison to the diffusive spreading term and is henceforth neglected.

The dominant convective term can be removed by transforming to a frame of reference which moves with the front, namely, $k \to K = k - 2\sqrt{t}$. Among the remaining terms in the transformed rate equation, the width of the front region W can now be determined by demanding that the diffusion term has the same order of magnitude as the reactive terms, that is, $N\frac{\partial^2 c}{\partial k^2} \sim c^2$. This implies $W \sim \sqrt{N/c}$. Combining this with $N = t^{-1/2}$ and $c \sim t^{-1}$ gives $W \sim t^{1/4}$, or a relative width $w = W/k_f \sim t^{-1/4}$. This suggests the appropriate scaling ansatz for the front region is

$$c_k(t) = \frac{1}{t}X(\xi), \quad \xi = \frac{k - 2\sqrt{t}}{t^{1/4}}. \quad (22)$$

Substituting this ansatz into Eq. (21) gives a non-linear single variable integro-differential equation for the scaling function $X(\xi)$. Together with the appropriate boundary conditions, this represents, in principle, a more complete solution to the wealth distribution. However, the essential scaling behavior of the finite-time spreading of the front is already described by Eq. (22), so that solving for $X(\xi)$ itself does not provide additional scaling information. Analysis of our data by several rudimentary approaches gives $w \sim t^{-\alpha}$ with $\alpha \approx 1/5$. We attribute

this discrepancy to the fact that w is obtained by differentiating $\mathcal{C}(x)$, an operation which generally leads to an increase in numerical errors.

C. Very Greedy Exchange

We now consider the variation in which events occur at a rate equal to the difference in capital of the two traders. That is, an individual is more likely to take capital from a much poorer person rather than from someone of slightly less wealth. For this "very greedy" exchange, the corresponding rate equations are

$$\frac{dc_k}{dt} = c_{k-1}\sum_{j=1}^{k-1}(k-1-j)c_j + c_{k+1}\sum_{j=k+1}^{\infty}(j-k-1)c_j$$
$$- c_k\sum_{j=1}^{\infty}|k-j|c_j, \quad (23)$$

while the total density obeys

$$\frac{dN}{dt} = -c_1(1-N), \quad (24)$$

under the assumption that the (conserved) total wealth density is set equal to one, $\sum kc_k = 1$.

These rate equations can be solved by again applying scaling. For this purpose, it is first expedient to rewrite the rate equations as

$$\frac{dc_k}{dt} = (c_{k-1} - c_k)\sum_{j=1}^{k-1}(k-j)c_j - c_{k-1}\sum_{j=1}^{k-1}c_j \quad (25)$$
$$+ (c_{k+1} - c_k)\sum_{j=k+1}^{\infty}(j-k)c_j - c_{k+1}\sum_{j=k+1}^{\infty}c_j.$$

Taking the continuum limit gives

$$\frac{\partial c}{\partial t} = \frac{\partial c}{\partial k} - N\frac{\partial}{\partial k}(kc). \quad (26)$$

We now substitute the scaling ansatz, Eq. (15), to yield

$$\mathcal{C}(0)[2\mathcal{C} + x\mathcal{C}'] = (x-1)\mathcal{C}' + \mathcal{C}, \quad (27)$$

and

$$\frac{dN}{dt} = -\mathcal{C}(0)N^2. \quad (28)$$

Solving the above equations gives $N \simeq [\mathcal{C}(0)t]^{-1}$ and

$$\mathcal{C}(x) = (1+\mu)(1+\mu x)^{-2-1/\mu}, \quad (29)$$

with $\mu = \mathcal{C}(0) - 1$. It may readily be verified that this expression for $\mathcal{C}(x)$ satisfies both integral relations of Eq. (18). The scaling approach has thus found a family of solutions which are parameterized by μ, and additional information is needed to determine which of these solutions is appropriate for our system. For this purpose,

note that Eq. (29) exhibits different behaviors depending on the sign of μ. When $\mu > 0$, there is an extended non-universal power-law distribution, while for $\mu = 0$ the solution is the pure exponential, $\mathcal{C}(x) = e^{-x}$. These solutions may be rejected because the wealth distribution cannot extend over an unbounded domain if the initial wealth extends over a finite range.

The accessible solutions therefore correspond to $-1 < \mu < 0$, where the distribution is compact and finite, with $\mathcal{C}(x) \equiv 0$ for $x \geq x_{\mathrm{f}} = -\mu^{-1}$. To determine the true solution, let us re-examine the continuum form of the rate equation, Eq. (26). From naive power counting, the first two terms are asymptotically dominant and they give a propagating front with k_{f} exactly equal to t. Consequently, the scaled location of the front is given by $x_{\mathrm{f}} = N k_{\mathrm{f}}$. Now the result $N \simeq [\mathcal{C}(0)t]^{-1}$ gives $x_{\mathrm{f}} = 1/\mathcal{C}(0)$. Comparing this expression with the corresponding value from the scaling approach, $x_{\mathrm{f}} = [1 - \mathcal{C}(0)]^{-1}$, selects the value $\mathcal{C}(0) = 1/2$. Remarkably, this scaling solution coincides with the Fermi distribution that found for the case of constant interaction rate. Finally, in terms of the unscaled variables k and t, the wealth distribution is

$$c_k(t) = \begin{cases} 4/t^2, & k < t; \\ 0, & k \geq t. \end{cases} \qquad (30)$$

Following the same reasoning as the previous section, this discontinuity is smoothed out by diffusive spreading.

Another interesting feature is that if the interaction rate is sufficiently greedy, "gelation" occurs [13], whereby a finite fraction of the total capital is possessed by a single individual. For interaction rates, or kernels $K(j,k)$ between individuals of capital j and k which do not give rise to gelation, the total density typically varies as a power law in time, while for gelling kernels $N(t)$ goes to zero at some finite time. At the border between these regimes $N(t)$ typically decays exponentially in time [11,13]. We seek a similar transition in behavior for the capital exchange model by considering the rate equation for the density

$$\frac{dN}{dt} = -c_1 \sum_{k=1}^{\infty} K(1,k)c_k. \qquad (31)$$

For the family of kernels with $K(1,k) \sim k^\nu$ as $k \to \infty$, substitution of the scaling ansatz gives $\dot{N} \sim -N^{3-\nu}$. Thus $N(t)$ exhibits a power-law behavior $N \sim t^{1/(2-\nu)}$ for $\nu < 2$ and an exponential behavior for $\nu = 2$. Thus gelation should arise for $\nu > 2$.

III. MULTIPLICATIVE CAPITAL EXCHANGE

We have thus far focused on additive processes in which the amount of capital transferred in a two-body interaction is fixed. This leads to the unrealistic feature of a vanishing density of active traders in the long time limit, as an individual who possesses the minimal amount of capital loses all assets in an unfavorable interaction. In many economic transactions, however, the amount of capital transferred is a finite fraction of the initial capital of one of the participants. This observation motivates us to consider capital exchange models with exactly this multiplicative property. A simple realization which preserves both the number of participants and the total capital is the reaction scheme $(x,y) \to (x-\alpha x, y+\alpha x)$. Here $0 < \alpha < 1$ represents the fraction of loser's capital which is gained by the winner. In this process, the capital of any individual remains non-zero, although it can become vanishingly small.

In the following, we consider the cases of random exchange, where the winner may equally likely be the richer or the poorer of the two traders, and greedy exchange, where only the richer of the two traders profits in the interaction. The former system quickly reaches a steady state, while the latter gives rise to a non-stationary power-law distribution of wealth.

A. Random Exchange

To determine the rate equation for random multiplicative exchange, it is expedient to first write an integral form of the equation, for which the origin of the various terms is clear. This rate equation is

$$\frac{\partial c(x)}{\partial t} = \frac{1}{2} \int \int dy\, dz\, c(y)c(z) \times [-\delta(x-z) - \delta(x-y)$$
$$+ \delta(y(1-\alpha) - x) + \delta(z + \alpha y - x)]. \qquad (32)$$

The first two terms account for the loss of $c(x)$ due to the interaction of an individual of capital x. The next term accounts for the gain in $c(x)$ by the losing interaction $(x/(1-\alpha), y) \to (x, y + \alpha x/(1-\alpha))$. The last term also accounts for gain in $c(x)$ by the profitable interaction $(y, x-\alpha y) \to (y(1-\alpha), x)$. By integrating over the delta functions, this rate equation reduces to

$$\frac{\partial c(x)}{\partial t} = -c(x) + \frac{1}{2(1-\alpha)} c\left(\frac{x}{1-\alpha}\right)$$
$$+ \frac{1}{2\alpha} \int_0^x dy\, c(y)\, c\left(\frac{x-y}{\alpha}\right), \qquad (33)$$

where the total density is set equal to one. In this form, the rate equation describes a diffusive-like process on a logarithmic scale, except that the (third) term, which describes hopping to the right, is non-local and two-body in character.

To help understand the nature of the resulting wealth distribution, let us first consider the moments, $M_n(t) \equiv \int_0^\infty dx\, x^n c(x,t)$. From Eq. (33) one can straightforwardly verify that the first two moments, M_0 and M_1, the population and wealth densities, respectively, are conserved. Without loss of generality, we choose $M_0 = 1$ and $M_1 = M$. The equation of motion for the second moment is

$$\frac{dM_2(t)}{dt} = -\alpha(1-\alpha)M_2(t) + \alpha M^2 \qquad (34)$$

with solution

$$M_2(t) = \frac{M^2}{1-\alpha} + \left[M_2(0) - \frac{M^2}{1-\alpha}\right]e^{-\alpha(1-\alpha)t}. \qquad (35)$$

Similarly, higher moments also exhibit exponential convergence to constant values, so that the wealth distribution approaches a steady state. The mechanism for this steady state is simply that the typical size of a profitable interaction is likely to be much smaller than an unprofitable interaction for a rich individual, while the opposite holds for a poor individual. This bias prevents the unlimited spread of the wealth distribution and stabilizes a steady state.

To determine the steady state wealth distribution, we substitute simple "test" solutions into the rate equations. By this approach, we find that the exponential form $c(x) = Be^{-bx}$ satisfies the steady-state version of the rate equation, Eq. (33), *iff* $\alpha = \frac{1}{2}$ and $B = b$. Thus when the winner receives one half the capital of the loser, the exact steady wealth distribution is a simple exponential $c(x) = M^{-1}\exp(-x/M)$. For general $0 < \alpha < 1$, the large-α tail is again an exponential, $c(x) \simeq 2b(1-\alpha)e^{-bx}$. However, for $x \ll 1$, we find, by substitution and applying dominant balance, that $c(x) \sim x^\lambda$ is the asymptotic solution, with exponent $\lambda = -1 - \ln 2/\ln(1-\alpha)$.
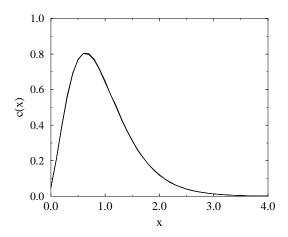

FIG. 2. Representative results for the wealth distribution in random multiplicative exchange for the case $\alpha = 0.25$ based on simulation of 10 configurations of 10^5 traders. Shown are the steady-state wealth distributions $c(x)$ as a function of wealth x for $t = 1.5^n$, with $n = 6, 8, 10$, and 12. The various curves are indistinguishable. The predicted $x^{1.409\ldots}$ small-x tail is not resolvable because of the coarseness of the data binning.

Interestingly, λ is positive when $\alpha < 1/2$, so that the density of the poor is vanishingly small. A heuristic justification for this phenomenon is that for $\alpha < 1/2$ an unfavorable interaction leads to a relatively small capital loss, and this loss is more than compensated for by favorable interactions so that a poor individual has the possibility of climbing out of poverty. In the opposite case of $\alpha > 1/2$, unprofitable interactions are sufficiently devastating that a large and persistent underclass is formed, with a power-law divergence in the number of poor in the limit of vanishing wealth.

Our simulations substantially confirm these results (Fig. 2). The scope of the simulations is less than that in additive processes, since the number of traders remains fixed, so that CPU time scales linearly in the simulation time. In contrast, for additive exchange, the CPU time scale as $\int^t dt' N(t')$, which can be much smaller than t. Numerically, we find that the moments $M_n(t)$ quickly converge to equilibrium values. The resulting wealth distribution is clearly a simple exponential for $\alpha = 1/2$ and exhibits either a power-law divergence or a power-law zero as $x \to 0$ for $\alpha > 1/2$ and $\alpha < 1/2$, respectively, in agreement with our analytical results.

B. Greedy Exchange

Parallel to our discussion of additive processes, we now investigate greedy multiplicative exchange, where only the richer trader profits, as represented by the reaction $(x, y) \to (x - \alpha x, y + \alpha x)$ for $x < y$. Following the same reasoning as that used in the previous subsection, the rate equation for greedy multiplicative exchange is

$$\frac{\partial c(x)}{\partial t} = -c(x) + \frac{1}{1-\alpha}\, c\left(\frac{x}{1-\alpha}\right) N\left(\frac{x}{1-\alpha}\right)$$
$$+ \frac{1}{\alpha}\int_{x/(1+\alpha)}^x dy\, c(y)\, c\left(\frac{x-y}{\alpha}\right), \qquad (36)$$

where $N(x) = \int_x^\infty dz\, c(z)$ is the population density whose wealth exceeds x.

Numerical simulations of this system show that the wealth distribution evolves *ad infinitum* and that the most of the population eventually becomes impoverished (Fig. 3). Note that the discreteness of our linear data binning lumps the poorest into a single bin at the origin which is not visible on the double logarithmic scale. The pervasive impoverishment arises because greedy exchange causes the poor to become poorer and the rich to become richer, but wealth conservation implies that there must be many more poor than rich individuals. In the long-time limit therefore, a small fraction of the population possesses most of the wealth.

To understand these features analytically, first consider the extreme case of $\alpha = 1$ which reduces to classical constant kernel aggregation [10], except for the added feature that individuals of zero wealth are now included in the distribution. Consequently, the scaling form of the wealth distribution in the long time limit is

$$c(x, t) \simeq t^{-2}e^{-x/t} + (1 - t^{-1})\delta(x). \qquad (37)$$

The first term is just the scaling solution to constant-kernel aggregation [10], so that the delta function term represents the population with zero wealth.
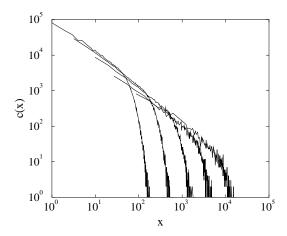

FIG. 3. The unnormalized wealth distribution in greedy multiplicative exchange for the case $\alpha = 0.5$ based on simulation of 10 configurations of 10^5 traders. Shown on a double logarithmic scale are the wealth distributions $c(x)$ as a function of wealth x for $t = 1.5^n$, with $n = 7$, 10, 13, and 16.

For general $0 < \alpha < 1$ a qualitatively related distribution can be anticipated which consists of a large impoverished class of negligible wealth and a much smaller and widely distributed population of wealthy. To determine this distribution, it is expedient to re-write Eq. (36) as

$$\frac{\partial c(x)}{\partial t} = \int_0^{x/(1+\alpha)} dz\, c(z)\left[c(x - \alpha z) - c(x)\right] \qquad (38)$$
$$-c(x)N\left(\frac{x}{1+\alpha}\right) + \frac{1}{1-\alpha}c\left(\frac{x}{1-\alpha}\right)N\left(\frac{x}{1-\alpha}\right).$$

Since Fig. 3 indicates that the wealth distribution is a power law, we substitute such a form in Eq. (38) and find that

$$c(x,t) = \frac{A}{xt}, \quad A = -\frac{1}{\ln(1-\alpha)} \qquad (39)$$

is a solution. However, this exact form cannot be realized starting from any initial condition. Thus Eq. (39) should be regarded as an attractor for the family of solutions to Eq. (38) which begin from a specified initial condition. Another pathology of Eq. (39) is that all moments, $M_n(t) = \int_0^\infty dx\, x^n c(x,t)$, are divergent. Note also that the last two terms on the right-hand side of Eq. (38) diverge (although their difference is regularized to the finite value $\frac{A^2}{xt^2}\ln\frac{1+\alpha}{1-\alpha}$). These observations suggest that a true solution to Eq. (38) converges to Eq. (39) only in the scaling region $x_1(t) < x < x_2(t)$, while a true solution has not yet had time to extend outside this domain.

To estimate these cutoffs for the scaling region, we evaluate the moments

$$M_0(t) \sim \int_{x_1}^{x_2} dx\, c(x,t) \sim \frac{A\ln(x_2/x_1)}{t}$$
$$M_1(t) \sim \int_{x_1}^{x_2} dx\, xc(x,t) \sim \frac{Ax_2}{t} \qquad (40)$$

Since $M_0 = 1$ and $M_1 = M$ are constant, one obtains

$$x_1(t) \sim e^{-t/A} = (1-\alpha)^t, \quad x_2(t) \propto t. \qquad (41)$$

The factor $x_1(t)$ clearly gives the wealth of the poorest at time t. Since a losing interaction leads to a reduction of capital by the factor $(1 - \alpha)$, the poorest individual at time t will have capital $(1-\alpha)^t$ for the monodisperse initial condition, $c_0(x) = \delta(x - 1)$, and a constant reaction rate. To understand the upper cutoff, suppose that $x \gg t$. In this case, the last two terms on the right-hand side of (38) are negligible. In the remaining term, the expression in the square brackets may be replaced by $\alpha z \frac{\partial c(x,t)}{\partial x}$, so that the resulting integral is simply equal to M. With these simplifications, the rate equation reduces to $c_t + \alpha M c_x = 0$. This linear wave equation admits the general solution $c(x,t) = c_0(x - \alpha M t)$ and suggests the upper cutoff $x_2(t) \simeq \alpha M t$, consistent with Eq. (41).
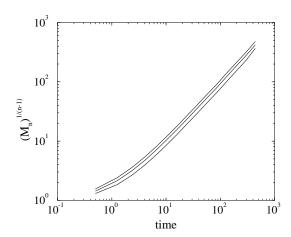

FIG. 4. The reduced moments $M_n^{1/(n-1)}$ versus time for greedy multiplicative exchange for the case $\alpha = 0.5$ based on simulation of 10 configurations of 10^5 traders. These reduced moments are predicted to increase linearly in time (see text).

Note, however, that there is inconsistency in our reasoning, as starting with the assumption $x \gg t$ leads to an upper cutoff of order t. In spite of this logical shortcoming, we have verified many of the resulting quantitative characteristics. For example, numerical simulation clearly yields the $1/x$ power-law tail of the wealth distribution. Furthermore, if one defines the wealthy as those whose capital exceeds some threshold ϵ, then Eqs. (39) and (41) give the density of wealthy proportional to $\ln(t/\epsilon)/t$. It is in this sense that we can view

the wealth distribution as consisting of two components: the wealthy density which is proportional to $\ln(t/\epsilon)/t$ and a complementary density of the poor. Using Eq. (39) one can also readily determine the behavior of the moments to be $M_n(t) \sim t^{-1} \int_0^t dx\, x^{n-1} \sim t^{n-1}$, for $n > 1$, or equivalently $M_n^{1/(n-1)}$ should grow linearly with time, as is observed in our simulations (Fig. 4). Least-square fits to the data with the first few data points deleted clearly indicate a growth rate which is very nearly linear.

IV. ARBITRARY SPATIAL DIMENSION

We now consider the role of spatial dimensionality on the asymptotic wealth distribution in additive exchange processes. While the rate equations apply for perfectly mixed traders or for diffusing traders in infinite spatial dimension, $d = \infty$, deviations from the resulting mean-field predictions are expected when d is below an upper critical dimension d_c. In arbitrary spatial dimension, we assume that an interaction occurs whenever two diffusing individuals meet. This transport mechanism should not be interpreted as diffusion in real space, but rather in a space of economic activity. For example, if economic activity were confined to a one-dimensional road, we would use $d = 1$. More generally, an economic network may have an effective dimension $d > 1$, with $d \to \infty$ in the modern global economy.

We further make the simplifying assumption that the diffusivity is independent of an individual's capital. Then an interaction occurs when $\mathcal{N} \cdot N \approx 1$, where $\mathcal{N}(\tau)$ is the average number of distinct sites visited by a random walk in a time interval τ. This quantity scales as [7]

$$\mathcal{N}(\tau) \sim \begin{cases} \tau^{d/2}, & d < 2; \\ \tau/\ln\tau, & d = 2; \\ \tau, & d > 2; \end{cases} \qquad (42)$$

as $\tau \to \infty$ and thus gives the following estimates for the density dependence of the time interval between events

$$\tau \sim \begin{cases} N^{-2/d}, & d < 2; \\ N^{-1}\ln(1/N), & d = 2; \\ N^{-1}, & d > 2. \end{cases} \qquad (43)$$

Since the total density decreases only in events which involve the poorest individuals, we have

$$\frac{dN}{dt} \sim -\frac{c_1}{\tau} \qquad (44)$$

Since we already know how the collision time τ depends on N, we need to express c_1 on N to solve Eq. (44) and complete the solution.

For random exchange, rate equations similar to Eq. (1) should apply, except for the obvious change of the collision rate N by the dimension-dependent rate τ^{-1} from Eq. (43). Thus introducing the modified time variable

$$T = \int_0^t \frac{dt'}{\tau(t')} \qquad (45)$$

reduces the governing equations to the pure diffusion equation, as in Sec. II. Combining Eqs. (7), (43), and (45), we find

$$N(t) \sim \begin{cases} t^{-d/2(d+1)}, & d < 2; \\ (t/\ln t)^{-1/3}, & d = 2; \\ t^{-1/3}, & d > 2. \end{cases} \qquad (46)$$

The wealth distribution is given by $c_k(t) \sim N^2 x\, e^{-x^2}$ with $N(t)$ given by Eq. (46). In particular, the density of the poorest individuals is proportional to N^3.

For greedy exchange in $d < 2$, we assume that the scaling ansatz, Eq. (15), still applies, but with the slightly stronger addition condition $\mathcal{C}(0) > 0$. This immediately gives $c_1 \sim N^2$. Using this result, together with Eq. (43) in Eq. (44), gives

$$N \sim \begin{cases} t^{-d/(d+2)}, & d < 2; \\ (t/\ln t)^{-1/2}, & d = 2; \\ t^{-1/2}, & d > 2. \end{cases} \qquad (47)$$

Our results for $N(t)$ given in Eqs. (46) and (47) also indicate that $d_c = 2$ is the upper critical dimension for additive capital exchange since it demarcates dimension-independent and dimension-dependent kinetics.

V. SUMMARY AND DISCUSSION

We have investigated the dynamical behavior of wealth distributions in simple capital exchange models. For additive processes, the amount of capital exchanged in all transactions is fixed, with the sense of the exchange being either random – "random" exchange – or favoring the rich – "greedy" exchange. The former leads to a Gaussian wealth distribution, while the latter gives rise to a Fermi-like distribution. In both cases, the number of economically viable individuals decays as a power law in time and their average wealth correspondingly increases. A "very greedy" process was also introduced in which the trading rate between two individuals is an increasing function of their capital difference. This case also gives rise to a Fermi-like wealth distribution, but with a faster decay in the density of active traders and a concomitant faster growth in their wealth.

We next considered multiplicative processes in which a finite fraction of the capital of one of the traders is exchanged in a transaction, namely, $(x, y) \to (x - \alpha x, y + \alpha x)$. From our naive viewpoint, this multiplicative rule, especially for relatively small α, appears to provide a plausible description of real economic transactions. This process gives rise to non-local rate equations which we have been unable to solve in closed form. Nevertheless, considerable insight was gained by analysis of the moments and asymptotics of the wealth distribution. For

random exchange, the basic interaction biases individuals with extreme wealth or extreme poverty towards the center of the distribution and a steady state is quickly reached. The large wealth tail of the distribution is exponential, while at small wealth there is a power-law tail which may be either divergent of vanishing, for $\alpha > 1/2$ or $\alpha < 1/2$, respectively.

For greedy exchange, the wealth distribution assumes a power law form $c(x,t) \simeq 1/(xt)$ for wealth in the range $(1-\alpha)^t < x < t$. The two cutoffs correspond to the wealth levels of the poorest and richest individuals, levels which continue to evolve with time. This evolving power law form conforms to our uninformed view of the wealth distribution in certain countries with developing economies. If we define the rich as those whose capital is greater than the average initial capital, their fraction decays as $\ln t/t$ and their average wealth grows as $t/\ln t$

Formally, it would be interesting to investigate limiting situations where well-known pathologies in the underlying capital exchange process can occur. For example, as briefly mentioned in Sec. II, there is the possibility of one individual acquiring a finite fraction of the total capital in a finite time. This feature corresponds to gelation in polymerization processes [13]. Mathematically, this singularity should manifest itself in the violation of wealth conservation, where the loss of wealth in the population of finitely wealthy individuals signals the appearance of an infinitely rich individual. For multiplicative exchange processes which possess two conservation laws – total density and wealth – there is also the possibility of losing population of individuals of finite wealth to a "dust" phase which consists of a finite fraction of all individuals who possess no wealth. This latter phenomenon is analogous to the "shattering" transition in fragmentation processes [14].

At a practical level, a potentially fruitful direction would be to incorporate additional realistic elements into capital exchange models. For example, in all societies, there is some form of wealth redistribution by taxation. It would be worthwhile to determine how various types of taxation algorithms – graduated tax, "flat" tax, $etc.$, in concert with capital exchange – influence the asymptotic wealth distribution. Conversely, it would be interesting to understand how financial "safety nets", such as welfare, affect the density and wealth of the poorest individuals. Another aspect worth pursuing is heterogeneous exchange rules, where the factor α is either different for each individual or depends on some other aspect of a trading event. One might hope to find universality in the wealth distributions with respect to this heterogeneity.

More generally, the focus on conserved exchange of capital is deficient because there is no mechanism for economic growth. Pure exchange might be appropriate for short time scales during which economic growth is negligible. However, pure exchange cannot be expected to be suitable on longer time scales where long-term economic growth is an overriding influence. Thus a two-body in-

teraction rule with a net increase in the capital might be a more appropriate long-time description of an economically interacting system.

Perhaps the simplest realization is random capital growth, $(j,k) \to (j+1,k)$ or $(j,k+1)$. Within the rate equation approach, this model is readily solved to find the asymptotic wealth distribution $c_k(t) \sim t^{-1/2} \exp[-(t-k)^2/2t]$. Hence the average wealth grows linearly in time, $\langle k \rangle \simeq t$, while the relative fluctuation $\sqrt{\langle (\Delta k)^2 \rangle}/\langle k \rangle$ decreases as $t^{-1/2}$. Thus an (economically) fair society with a sharp wealth distribution arises, albeit with an unrealistic linear, rather than exponential, growth of the average wealth. Such an exponential growth can be achieved, $e.~g.$, by an interaction rate which equals the sum of capitals of the participants, $K(i,j) = i + j$. the total wealth density $M = \sum k c_k$ now obeys $\dot{M} = 2NM$ and thus grows exponentially. This model fulfills the Marxist dream of fast wealth growth $\langle k \rangle \sim e^{2Nt}$ with all participating equally in the prosperity (the relative fluctuation decreasing as $\sqrt{t}e^{-Nt}$). For an interaction rate which increases even more rapidly with wealth, there is a pathology analogous to gelation – infinite prosperity in a finite time. For example, for the product kernel, $K(i,j) = ij$, the wealth distribution for a monodisperse initial condition is $c_k(t) = (1-t)t^{k-1}$. Interestingly, the Marxist ideal lies on the boundary between algebraic wealth growth and the pathology of a finite-time wealth divergence, a feature similar to the "life on the edge of chaos" advocated by Kauffman as a generic property of complex systems [3].

Related behavior occurs in a multiplicative growth process where $(x,y) \to (x, y + \alpha x)$. Naively, a single multiplicative interaction which changes the capital by an amount proportional to the existing capital k is roughly equivalent to k successive additive interactions. Consequently, this simple version of multiplicative growth should exhibit exponential wealth growth. For the specific case that we were able to solve, $\alpha = 1$, the wealth distribution approaches the scaling form $c_k(t) \sim e^{-t} \exp(-xe^{-t})$. However, the wealth distribution is broad, a feature generic of multiplicative growth processes for general values of α.

More realistically, an overall growth process should incorporate the possibility that in a single trade: both traders profit, one profits, or neither profits, but with an overall gain in capital when averaged over many trading events. It would be interesting to determine the nature of the wealth distribution in such a multiplicative system.

VI. ACKNOWLEDGMENTS

Added Note: After this work was completed, we learned on a related work [15] in which a model similar to greedy multiplicative exchange was employed to describe the distribution of company sizes. We thank H. Takayasu for kindly informing us of this development.

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