Logarithm of multivector in real 3D Clifford algebras

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Abstract. Closed form expressions for a logarithm of general multivector (MV) in base-free form in real geometric algebras (GAs) $Cl_{p,q}$ are presented for all n = p + q = 3. In contrast to logarithm of complex numbers (isomorphic to $Cl_{0,1}$), 3D logarithmic functions, due to appearance of two double angle arc tangent functions, allow to include *two sets of sheets* characterized by discrete coefficients. Formulas for generic and special cases of individual blades and their combinations are provided.

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1. Introduction

Logarithm properties are well-known for real and complex numbers. Except the Hamilton quaternions which are isomorphic to $Cl_{0,2}$, the properties of logarithm in other 2D algebras (some partial formulas for 2D GAs are provided in [1, 2, 3, 4]) and higher dimensional Clifford algebras remain uninvestigated as yet. In general, GA logarithm properties are simplest for anti-Euclidean algebras $Cl_{0,n}$ [5, 6]. As in the complex algebra case we expect at least to have a principal logarithm and a part that makes the GA logarithm a multivalued function.

Recently in papers [7, 8], which will be the starting point for the present article, we have performed a detailed investigation of 3D exponential functions in real GAs. However, the GA logarithm is more difficult to analyze since one must take into account the multi-valuedness and the fact that in 3D algebras (except $Cl_{0,3}$) the logarithm may not exist for all MVs. Here, we have treated the logarithm as an inverse problem using for this purpose the *Mathematica* symbolic package, more precisely as an inverse GA function to exponential in separate 3D algebras

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 $Cl_{0,3}$, $Cl_{3,0}$, $Cl_{1,2}$, and $Cl_{2,1}$. The final GA logarithm formulas were checked symbolically as well as numerically. They are in complete agreement with more general formulas [9] suitable for computation on any function of diagonalizable multivector (MV). The exact logarithm formulas also have been applied to study convergence of series expansion MV logarithms.

In Sec. 2 the notation is introduced. Since the logarithm is closely related with a two argument arc tangent function $\arctan(x,y)$, its properties are summarized in this section as well. In Sec. 3 the logarithm of the simplest, namely $Cl_{0,3}$ algebra is considered. Since algebras $Cl_{3,0}$ and $Cl_{1,2}$ are isomorphic, in Sec. 4 the logarithms for both algebras are investigated simultaneously. In Sec. 5 the logarithm of the most difficult $Cl_{2,1}$ algebra is presented. Since GA logarithm may be applied to GA square root calculation [4, 10], in Sec. 6 the roots and arbitrary fractional powers of MV are discussed. In Sec. 7 the relations of the logarithm to GA inverse trigonometric and hyperbolic functions are presented. Finally, in Sec. 8 we summarize the obtained results.

2. Notation and general properties of GA logarithm

A general MV in GA space is expanded in the orthonormal basis in inverse degree lexicographic ordering: $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}, \mathbf{e}_{123} \equiv I\}$, where \mathbf{e}_i are basis vectors, \mathbf{e}_{ij} are the bivectors and I is the pseudoscalar. The number of subscripts indicates the grade. The scalar is a grade-0 element, the vectors \mathbf{e}_i are the grade-1 elements, etc. In the orthonormalized basis the geometric products of basis vectors satisfy the anticommutation relation, $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = \pm 2\delta_{ij}$. For $Cl_{3,0}$ and $Cl_{0,3}$ algebras the squares of basis vectors, correspondingly, are $\mathbf{e}_i^2 = +1$ and $\mathbf{e}_i^2 = -1$, where i = 1, 2, 3. For mixed signature algebras such as $Cl_{2,1}$ and $Cl_{1,2}$ the squares are $\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1$, $\mathbf{e}_3^2 = -1$ and $\mathbf{e}_1^2 = 1$, $\mathbf{e}_2^2 = \mathbf{e}_3^2 = -1$, respectively.

The general MV in real Clifford algebras $Cl_{p,q}$ for n = p + q = 3 is

$$A = a_0 + a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3 + a_{12} \mathbf{e}_{12} + a_{23} \mathbf{e}_{23} + a_{13} \mathbf{e}_{13} + a_{123} I$$

$$\equiv a_0 + \mathbf{a} + \mathcal{A} + a_{123} I = a_0 + \mathsf{A}_{1,2} + a_{123} I = \mathsf{A}_{0,1,2,3},$$
(2.1)

where a_i , a_{ij} and a_{123} are the real coefficients, and $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathcal{A} = a_{12}\mathbf{e}_{12} + a_{23}\mathbf{e}_{23} + a_{13}\mathbf{e}_{13}$ is, respectively, the vector and bivector. I is the pseudoscalar, $I = \mathbf{e}_{123}$. The double index in MV $A_{i,j}$ indicates the sum of MVs of grades i and j, i.e. $A_{i,j} = \langle A \rangle_i + \langle A \rangle_j$.

The main involutions, namely the reversion, grade inversion and Clifford conjugation denoted, respectively, by tilde, circumflex and their combination are defined by

$$\widetilde{A} = a_0 + \mathbf{a} - \mathcal{A} - a_{123}I, \quad \widehat{A} = a_0 - \mathbf{a} + \mathcal{A} - a_{123}I,$$

$$\widetilde{\widehat{A}} = a_0 - \mathbf{a} - \mathcal{A} + a_{123}I.$$
(2.2)

2.1. General properties of GA logarithm

The logarithm of MV is another MV that belongs to the same geometric algebra (GA). The defining equation for MV logarithm is $\log(e^{A}) = A$, where $A \in Cl_{p,q}$.

The GA logarithm is a multivalued function. In [1] it was suggested that "The principal value of the logarithm can be defined as the MV $M = \log(Y)$ with the smallest norm", where $Y \in Cl_{p,q}$. The natural norm for a MV is the determinant norm defined in subsection 2.2. The following properties hold for MV logarithm:

$$\begin{split} \log(\mathsf{AB}) &= \log(\mathsf{A}) + \log(\mathsf{B}) \quad \text{if } \mathsf{AB} = \mathsf{BA}, \\ e^{\log(\mathsf{A})} &= \mathsf{A}, \quad e^{-\log(\mathsf{A})} = \mathsf{A}^{-1}, \\ \widehat{\log(\mathsf{A})} &= \log(\widetilde{\mathsf{A}}), \quad \widehat{\log(\mathsf{A})} = \log(\widehat{\mathsf{A}}), \quad \widehat{\log(\mathsf{A})} = \log(\widehat{\widetilde{\mathsf{A}}}), \\ \mathsf{V} \log(\mathsf{A}) \mathsf{V}^{-1} &= \log(\mathsf{VAV}^{-1}). \end{split} \tag{2.3}$$

In the last expression the transformation V, for example the rotor, is pushed inside the logarithm.

2.2. GA logarithm series

In analogy with a definition of logarithm in complex plane for GA logarithm we can write

$$\log A = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (A-1)^k}{k}, \quad \text{if} \quad |A-1| < 1, \tag{2.4}$$

where |A-1| denotes the determinant norm. For arbitrary MV the determinant norm is defined as an absolute value of determinant Det(B) of MV B raised to fractional power 1/k, where $k = 2^{\lceil n/2 \rceil}$, i.e., $|\mathsf{B}| = (\mathsf{Det}(\mathsf{B}))^{1/k} > 0$. For algebras having negative determinant instead the semi-norm (aka pseudoscalar) is introduced $|B| = (abs(Det(B)))^{1/k} \ge 0$. The equality sign means that in case of seminorm the determinant may be zero although $B \neq 0$. In the following the same symbol will be used for both the norm and semi-norm. The (semi-)norm can be interpreted as a number of multipliers needed to define Det(B). In 3D algebras (n=3) we have $k=2^{\lceil 3/2 \rceil}=2^2=4$, which is the degree of characteristic [11] polynomial Det(B). In this way found integer k coincides with the number of multipliers in the 3D determinant: Det(B) = BBBB. The determinant norm for MV B in 3D algebras, therefore, is $|B| = \sqrt[4]{abs(Det(B))}$. It can be shown that for any GA that holds a basis element with property $e_i^2 = -1$ by adding a scalar one can construct a MV the norm of which may be identified with a module of a complex number. For example in $Cl_{3,0}$ the norm of $B=1+\mathbf{e}_{12}$ is $\sqrt{(1+\mathbf{e}_{12})(1-\mathbf{e}_{12})}=\sqrt{2}$ which coincides with $|B| = \sqrt[4]{abs(Det(B))} = \sqrt{2}$. (also refer to Example 1 below).

If the MV has a numerical form, to minimize the number of multiplications it is convenient to represent the logarithm in a nested form (aka Horner's rule). The logarithmic series [12] (also called Mercator series), if rewritten according to Horner's rule, assumes the following form,

$$\log B = B(1 + B(-\frac{1}{2} + B(\frac{1}{3} + B(-\frac{1}{4} + B(\frac{1}{5} + \cdots))))), \text{ where } B = A - 1.$$
 (2.5)

Example 1. MV equivalent to complex number. Let's take the MV $A = \frac{9}{10} - \frac{1}{3}e_3$

the determinant norm of which in $Cl_{0,3}$ is $|B| = |A - 1| = \frac{\sqrt{109}}{30} \approx 0.34801 < 1$.

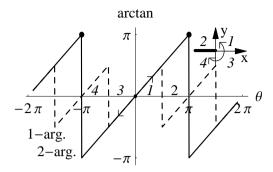


FIGURE 1. Graphical representation of single $\arctan(y/x) = \arctan(\sin\theta/\cos\theta)$ (dashed line) and double $\arctan(x,y) = \arctan(\cos\theta,\sin\theta)$ (solid line) argument tangent functions used by Mathematica. θ is an angle between x axis and vector (not shown) attached to the center of complex x-y plane. The vector may be rotated from x-axis anticlockwise, $\theta = (0...\pi]$, or clockwise, $\theta = [0...-\pi)$ as shown by arrows in the inset. In the inset also the numbering of the quadrants 1-4 and the branching, represented by thick line on the negative part of x axis, are shown.

Therefore, the standard series, Eq. (2.4), may be applied to find an approximate value (the result found by exact formula in Example 2 is $\log \frac{\sqrt{829}}{30} - \mathbf{e}_3 \arctan \frac{10}{27} \approx -0.0410873 - \mathbf{e}_3 0.354706$). Since $\mathbf{e}_3^2 = -1$ and it is the only basis vector in the considered MV, one may replace the MV by complex number $z = \frac{9}{10} - \mathrm{i}\frac{1}{3}$. The module is $|z-1| = \frac{\sqrt{109}}{30}$ which coincides with the MV determinant norm. Then, $\log z \approx -0.0410873 - \mathrm{i}\,0.354706$.

Now let's calculate the logarithm of $\mathsf{A}' = -\frac{9}{10} - \frac{1}{3}\mathbf{e}_3$ by the Horner series (2.5). Since $|\mathsf{A}' - 1| = \frac{\sqrt{3349}}{30} \approx 1.92902 > 1$ the series diverges. As shown in Example 2, the logarithm can be easily computed if exact GA logarithm formula obtained in the present paper is used. After replacement of the MV by complex number we obtain that |z' - 1| = 1.92902 which again coincides with the module of $z' = -\frac{9}{10} - \frac{1}{3}\mathbf{i}$. Computing the value of logarithm by Mathematica command FunctionExpand[Log[z']] we obtain $\log z' = \mathbf{i}(-\pi + \arctan(10/27)) - \log(30) + \log(829)/2$ which has the same numerical value as shown in the Example 2.

2.3. Double-argument arc tangent function

GA logarithm as we shall see, in its nature is a multi-valued function with period 2π . To account for quadrant sign in complex plane properly we shall need the double argument arc tangent function as given in the *Mathematica*, the properties of which are briefly mentioned below. Figure 1 shows the single and double argument arc tangent functions. The former has period π and its principal values lie in the interval $\theta = [-\pi/2, \pi/2)$, while the double argument arc tangent has, respectively, 2π period and principal values in $\theta = [-\pi, \pi)$. The inset on the right side of

Fig. 1 shows the quadrants 1-4 in x-y plane. Note that the anticlockwise rotation is done from quadrant 1 to quadrant 2, while clockwise rotation in order $3\rightarrow 4$, so that a jump in the double arc tangent value and associated branching occurs on the negative side of x-axis rather than on y axis as is in the standard single argument case. Also, in Fig. 1 note small points on vertical branching steps that indicate that respective arc tangent value on periodic line belong to upper rather than lower part, i.e., at $\theta=\pi$ we have $\arctan(\cos(\pi),\sin(\pi))=\arctan(-1,0)=\pi$, however, after addition of infinitesimal angle $\arctan(\cos(\pi+0_+),\sin(\pi+0_+))=-\pi$. Similarly, at $\theta=-\pi$ we have $\arctan(\cos(-\pi),\sin(-\pi))=\pi$, and, $\arctan(\cos(-\pi+0_+),\sin(-\pi+0_+))=-\pi$. If x,y were replaced by real numbers Mathematica will switch automatically to a single argument arc tangent in the first quadrant and principal values, for example, $\arctan(17,10)=\arctan(10/17)$, $\arctan(-17,10)=-\pi+\arctan(10/17)$, $\arctan(17,-10)=-\arctan(10/17)$, $\arctan(-17,-10)=-\pi+\arctan(10/17)$.

In the terms of a standard arc tangent function the argument of which is in the range $(-\pi/2, \pi/2)$, the double tangent principal values now in the range $(-\pi, \pi)$ can be expressed as follows:

$$\arctan(x,y) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0, \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y \ge 0, \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y < 0, \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0, \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$
 (2.6)

We will start from $Cl_{0,3}$ where the expanded exponential in a coordinate-form has the simplest MV coefficients and the logarithm exists for all MVs.

3. MV logarithms in $Cl_{0,3}$

3.1. Logarithm formula for generic MV

The term "generic" here will be understood as "not creating the problems". If for a given set of MV coefficients the generic formula is not applicable, for example, due to nullification of a denominator, or due to appearance of an undefined subexpression like $\arctan(0,0)$, we will refer to it as "special case". Special cases will be covered by more elaborate formula later.

Theorem 3.1 (Logarithm of multivector in $Cl_{0,3}$). The generic logarithm of $MV A = a_0 + \mathbf{a} + A + a_{123}I$ is the MV given by

$$\log(\mathsf{A}) = \frac{1}{2} \big(\mathsf{A}_{0_+} + \mathsf{A}_{0_-} + \mathsf{A}_{1,2_+} + \mathsf{A}_{1,2_-} + (\mathsf{A}_{0_+} - \mathsf{A}_{0_-})I \big), \tag{3.1}$$

with

$$A_{0_{+}} = \frac{1}{2} \log((a_{0} + a_{123})^{2} + a_{+}^{2}), \qquad a_{+} \neq 0, \quad (3.2)$$

$$A_{0_{-}} = \frac{1}{2} \log((a_0 - a_{123})^2 + a_{-}^2), \qquad a_{-} \neq 0, \quad (3.3)$$

$$A_{1,2_{+}} = \frac{1}{a_{+}} \left(\arctan(a_{0} + a_{123}, a_{+}) + 2\pi c_{1+} \right) \left(1 + I \right) \left(\mathbf{a} + \mathcal{A} \right), \quad a_{+} \neq 0, \quad (3.4)$$

$$\mathsf{A}_{1,2_{-}} = \frac{1}{a} \Big(\arctan(a_0 - a_{123}, a_{-}) + 2\pi c_{1-} \Big) \Big(1 - I \Big) \Big(\mathbf{a} + \mathcal{A} \Big), \quad a_{-} \neq 0. \quad (3.5)$$

The MVs $A_{0\pm}$, $A_{1,2\pm}$ and $A_{0\pm}I$ denote, respectively, the scalar, vector \pm bivector and the pseudoscalar components. $c_{1\pm}$, $c_{2\pm} \in \mathbb{Z}$ are arbitrary integers. The scalars $a_{+} \geq 0$ and $a_{-} \geq 0$ are given by expressions [7, 8],

$$a_{-} = \sqrt{-(\mathbf{a}\dot{\mathbf{a}} + \mathcal{A}\dot{\mathcal{A}}) + 2I\mathbf{a} \wedge \mathcal{A}}$$

$$= \sqrt{(a_3 + a_{12})^2 + (a_2 - a_{13})^2 + (a_1 + a_{23})^2},$$
(3.6)

$$a_{+} = \sqrt{-(\mathbf{a}\mathbf{a} + \mathcal{A}\mathcal{A}) - 2I\mathbf{a} \wedge \mathcal{A}}$$

$$= \sqrt{(a_{3} - a_{12})^{2} + (a_{2} + a_{13})^{2} + (a_{1} - a_{23})^{2}},$$
(3.7)

Proof. It is enough to check that after substitution of (3.1) into GA exponential formula (1) of [7] one will get the initial MV A.

The Theorem 3.1 gives the GA logarithm in a basis-free form. However, the derivation of above given generic logarithm formula at first was done in a coordinate form, from which the Theorem 3.1 follows (see Appendix A). The Theorem 3.1 ensures the existence of GA logarithm for all MVs with real coefficients in $Cl_{0,3}$, because in the mentioned algebra the zero determinant of MV (Det A = 0) occurs only if A = 0. As we shall see this property does not hold for remaining algebras.

3.2. Special cases

In Theorem 3.1 it was presumed that the both scalars a_- and a_+ do not vanish. This assumption is equivalent to the condition that the sum of vector and bivector must have non-zero-determinant¹, $\text{Det}(\mathbf{a} + \mathcal{A}) = a_+^2 a_-^2 \neq 0$. If either of scalars is zero then we have a special case. This situation is met in rare cases, for instance², when $a_1 = a_{23}$, $a_2 = -a_{13}$, $a_3 = a_{12}$. In such and similar cases the MVs $A_{0\pm}$, $A_{1,2\pm}$

 $^{^1}$ We shall remind that for $Cl_{0,3}$ algebra the determinant Det A = 0 means A = 0. This property makes the exponential/logarithm analysis in this algebra relatively simple. In remaining 3D algebras additional conditions are needed to include MVs with Det A = 0.

²Minus sign in $a_2 = -a_{13}$ comes from a strictly increasing order of numbers in the basis element indices, due to the so-called inverse degree lexicographic ordering [7].

and $A_{0+}I$ in the Theorem 3.1 must be supplemented by conditions:

$$\mathsf{A}_{0_{+}} = \begin{cases} \log(a_{0} + a_{123}) + 2\pi c_{2_{+}} \hat{\mathcal{U}}, & a_{+} = 0 \quad \text{and} \quad (a_{0} + a_{123}) > 0\\ \log(0_{+}), & a_{+} = 0 \quad \text{and} \quad (a_{0} + a_{123}) = 0\\ \log(-(a_{0} + a_{123}))\\ +(\pi + 2\pi c_{2_{+}})\hat{\mathbf{u}}, & a_{+} = 0 \quad \text{and} \quad (a_{0} + a_{123}) < 0, \end{cases}$$
(3.8)

$$\mathsf{A}_{0-} = \begin{cases} \log(a_0 - a_{123}) + 2\pi c_{2-}\hat{\mathcal{U}}, & a_{-} = 0 \text{ and } (a_0 - a_{123}) > 0\\ \log(0_+), & a_{-} = 0 \text{ and } (a_0 - a_{123}) = 0\\ \log(-(a_0 - a_{123}))\\ +(\pi + 2\pi c_{2-})\hat{\mathbf{u}}, & a_{-} = 0 \text{ and } (a_0 - a_{123}) < 0, \end{cases}$$
(3.9)

$$\mathsf{A}_{0_{\pm}}I \text{ in the Theorem 3.1 must be supplemented by conditions:}$$

$$\mathsf{A}_{0_{+}} = \begin{cases} \log(a_{0} + a_{123}) + 2\pi c_{2_{+}}\hat{\mathcal{U}}, & a_{+} = 0 & \text{and} & (a_{0} + a_{123}) > 0 \\ \log(0_{+}), & a_{+} = 0 & \text{and} & (a_{0} + a_{123}) = 0 \\ \log(-(a_{0} + a_{123})) \\ + (\pi + 2\pi c_{2_{+}})\hat{\mathbf{u}}, & a_{+} = 0 & \text{and} & (a_{0} + a_{123}) < 0, \end{cases}$$

$$\mathsf{A}_{0_{-}} = \begin{cases} \log(a_{0} - a_{123}) + 2\pi c_{2_{-}}\hat{\mathcal{U}}, & a_{-} = 0 & \text{and} & (a_{0} - a_{123}) > 0 \\ \log(0_{+}), & a_{-} = 0 & \text{and} & (a_{0} - a_{123}) = 0 \\ \log(-(a_{0} - a_{123})) \\ + (\pi + 2\pi c_{2_{-}})\hat{\mathbf{u}}, & a_{-} = 0 & \text{and} & (a_{0} - a_{123}) < 0, \end{cases}$$

$$\mathsf{A}_{1,2_{+}} = \begin{cases} (\frac{1}{a_{0} + a_{123}} + 2\pi c_{1_{+}}) \\ \times (1 + I)(\mathbf{a} + \mathcal{A}), & a_{+} = 0 & \text{and} & (a_{0} + a_{123}) > 0 \\ 0, & a_{+} = 0 & \text{and} & (a_{0} + a_{123}) = 0, \\ (\pi + 2\pi c_{1_{+}})(1 + I)(\mathbf{a} + \mathcal{A}), & a_{+} = 0 & \text{and} & (a_{0} + a_{123}) < 0, \end{cases}$$

$$(3.10)$$

$$\mathsf{A}_{1,2_{-}} = \begin{cases} \left(\frac{1}{a_{0} - a_{123}} + 2\pi c_{1_{-}}\right) \\ \times (1 - I)(\mathbf{a} + \mathcal{A}), & a_{-} = 0 \text{ and } (a_{0} - a_{123}) > 0 \\ 0, & a_{-} = 0 \text{ and } (a_{0} - a_{123}) = 0 \\ (\pi + 2\pi c_{1_{-}})(1 - I)(\mathbf{a} + \mathcal{A}), & a_{-} = 0 \text{ and } (a_{0} - a_{123}) < 0, \end{cases}$$
(3.11)

Here $c_{1_{\pm}}, c_{2_{\pm}} \in \mathbb{Z}$ are the arbitrary integers. The conditions for $(a_0 \pm a_{123})$ on the right-hand side take into account the case $\operatorname{Det}(\mathbf{a}+\mathcal{A})=0$. In scalars³ A_{0_+} and A_{0_-} , the symbols $\hat{\mathbf{u}}$ and $\hat{\mathcal{U}}$ represent any free unit vector or bivector, respectively, $\hat{\mathbf{u}}^2 =$ $\hat{\mathcal{U}}^2 = -1$. For example, the unit vector can be parametrized as $\hat{\mathbf{u}} = (u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 +$ $u_3\mathbf{e}_3/\sqrt{u_1^2+u_2^2+u_2^3}$. It should be noted that the term $1/(a_0\pm a_{123})$ in Eqs (3.10) and (3.11) represents the limit $\lim_{a_{\pm}\to 0}\arctan(a_0\pm a_{123},a_{\pm})/a_{\pm}=1/(a_0\pm a_{123})$ which is valid only when $a_0 \pm a_{123} > 0$. The notation of $\log(0_+)$ in expressions for A_{0+} and A_{0-} is explained in Example 6.

Interpretation of special conditions (3.8)-(3.11) in terms of the MV determinant [13, 14, 15] becomes more evident if one remembers that the determinant of MV A in $Cl_{0,3}$ can be expressed in a form $Det(A) = (a_-^2 + (a_0 - a_{123})^2)(a_+^2 + (a_0 + a_{123})^2)$ $(a_{123})^2$), whereas the condition $a_{\pm} = 0$ is equivalent to $\text{Det}(A_{12+}) = \text{Det}(\mathbf{a} + \mathcal{A}) = 0$ $a_{+}^{2}a_{-}^{2}$. All special cases therefore occur if $\mathrm{Det}(\mathbf{a}+\mathcal{A})=0$ and the condition are described by $a_0 \pm a_{123} \leq 0$. In conclusion, the symbolic expression for logarithm,

³The appearance of a free vector/bivector breaks the grade arrangement in the generic terms (3.2) and (3.3). The choice, however, results in a more simple final expression, since now it is enough to write only a single free \mathbf{u} or \mathcal{U} term (see Eq. (3.1)) instead of a pair \mathbf{u} and $\mathbf{u}I$, or \mathcal{U} and UI, if we would have chosen to move these terms to vector+bivector part by following up a strict grade notation convention.

has three special pieces (branches) $a_0 \pm a_{123} \leq 0$ provided the condition $a_{\pm} = 0$ is satisfied and the generic piece is characterized by $a_{\pm} \neq 0$.

3.3. Multivaluedness and free multivector

To include multivaluedness in GA logarithm we introduce a free multivector F by the following defining equation [4]

$$e^{\log(A)+F} = e^{\log(A)}e^F = e^{\log(A)}, \tag{3.12}$$

which implies two conditions the MV F must satisfy: the commutator $[\log(A), F] = 0$ and $e^F = 1$. As we shall see, for remaining n = 3 algebras the free MV F will play a similar role. One can check that the expression

$$F = \frac{\pi c_{1+}}{a_{+}} (1+I) (\mathbf{a} + \mathcal{A}) + \frac{\pi c_{1-}}{a_{-}} (1-I) (\mathbf{a} + \mathcal{A})$$
(3.13)

satisfies $e^{\mathsf{F}} = 1$, and that for a generic MV A, Eq. 3.1, the free term (3.13) commutes with $\log(\mathsf{A})$, i.e. $[\log(\mathsf{A}), \mathsf{F}] = 0$. The integer constants $c_{1_+}, c_{1_-} \in \mathbb{Z}$ in Eq. (3.13) add two free (discrete) parameters that may be used to shift the coefficients of vector and bivector in $\log(\mathsf{A})$ by some multiple of π . The sum $(\mathbf{a} + \mathcal{A})$ in (3.13) constitute vector+bivector part⁴ of the original MV A, therefore $(\mathbf{a} + \mathcal{A})$ automatically commutes with A. As a result only discrete free coefficients are possible in the logarithm generic formula. In special cases (see Eqs (A.5a) and (A.5b) in the Appendix A) the free MV F may also contain arbitrary unit vector $\hat{\mathbf{u}}$ and/or unit bivector $\hat{\mathcal{U}}$. In such cases one can include two additional continuous parameters interpreted as directions of $\hat{\mathbf{u}}$ or $\hat{\mathcal{U}}$.

Since $\arctan(x,y)$ has been defined in the range $(-\pi,\pi]$ (usually called the principal value or the main branch, Fig. 1), we can add to it any multiple of 2π . Therefore, the plus/minus instances of $\arctan(a_0 \pm a_{123}, a_{\pm})$ (see Eqs (A.5a) and (A.5b)) were replaced by more general expressions $\arctan(a_0 + a_{123}, a_{+}) + 2\pi c_{1_{+}}$ and $\arctan(a_0 - a_{123}, a_{-}) + 2\pi c_{1_{-}}$ in Eqs (3.4) and (3.5), respectively, which takes into account the multivaluedness of the argument. This explains the rationale behind the construction of the free MVs for GA logarithm.

In [16] the notion of principal logarithm (also called the principal value of logarithm) in case of matrices was introduced. In [1] it was suggested that the "logarithm principal value in GA can be defined as the MV M = log Y with the smallest norm". Formulas (3.2)-(3.5) and (3.8)-(3.11) might suggest that we could obtain the principal logarithm values after equating discrete free constants $c_{1\pm}$, $c_{2\pm}$ to zero. Unfortunately, extensive numerical checks revealed that this is not always the case.

Example 2. Logarithm of simple MV in $Cl_{0,3}$. For MV A = $\frac{9}{10} - \frac{1}{3}\mathbf{e}_3$ in the Example 1, Eqs (3.6) and (3.7) give $a_+ = a_- = 1/3$. The MVs in (3.1) then are $A_{0+} = A_{0-} = -\log(10/9) + \pi\mathbf{e}_3$, $A_{1,2\pm} = \frac{\pi}{3}(-\mathbf{e}_3 \pm \mathbf{e}_{12})$. The logarithm calculated by exact formula (3.1) is $\log(\frac{9}{10} - \frac{1}{3}\mathbf{e}_3) = -\frac{1}{2}\log\frac{900}{829} - \arctan(\frac{10}{27})\mathbf{e}_3 \approx -0.0410873 - 0.354706\mathbf{e}_3$ which coincides with result of Example 1. Now, let's

 $^{^4\}mathrm{In}$ 3D algebras, the scalar and pseudoscalar belong to algebra center and as a result they commute with all elements.

calculate GA logarithm of $\mathsf{A}' = -\frac{9}{10} - \frac{1}{3}\mathbf{e}_3$ that diverges when the series (2.5) is used. With exact formulas (3.2)–(3.7) we find: $a_+ = a_- = \frac{1}{3}$, $\mathsf{A}_{0+} = \mathsf{A}_{0-} = -\frac{1}{2}\log(900/829)$, $\mathsf{A}_{1,2+} = \left(\pi - \arctan(10/27)\right)(\mathsf{e}_{12} - \mathsf{e}_3)$, $\mathsf{A}_{1,2-} = \left(\arctan(10/27) - \pi\right)(\mathsf{e}_{12} + \mathsf{e}_3)$. Then Eq. (3.1) gives $\log(\mathsf{A}') = -\frac{1}{2}\log(\frac{900}{829}) + \left(\arctan(\frac{10}{27}) - \pi\right)\mathsf{e}_3 \approx -0.0410873 - 2.78689\mathsf{e}_3$. Exponentiation of the obtained logarithm gives initial MV, $\exp\left(\log(\mathsf{A}')\right) = \mathsf{A}'$. The result also can be checked by complex logarithm, because the initial MV consist of scalar and basis vector $\mathsf{e}_3^2 = -1$ only.

Example 3. Logarithm of generic MV in $Cl_{0,3}$. Let's compute the logarithm of $\mathsf{A} = -8 - 6\mathsf{e}_2 - 9\mathsf{e}_3 + 5\mathsf{e}_{12} - 5\mathsf{e}_{13} + 6\mathsf{e}_{23} - 4\mathsf{e}_{123}$. Then, $a_-^2 = 53$ and $a_+^2 = 353$. The Eqs (3.2)-(3.5) give $\mathsf{A}_{0_+} = \frac{1}{2}\log(497)$, $\mathsf{A}_{0_-} = \frac{1}{2}\log(69)$, $\mathsf{A}_{1,2_+} = (353)^{-1/2} \left(\pi - \arctan\left(\frac{\sqrt{353}}{12}\right) + 2\pi c_{1_+}\right) (1+I) \left(-6\mathsf{e}_2 - 9\mathsf{e}_3 + 5\mathsf{e}_{12} - 5\mathsf{e}_{13} + 6\mathsf{e}_{23}\right)$ and $\mathsf{A}_{1,2_-} = (53)^{-1/2} \left(\pi - \arctan\left(\frac{\sqrt{53}}{4}\right) + 2\pi c_{1_-}\right) (1-I) (-6\mathsf{e}_2 - 9\mathsf{e}_3 + 5\mathsf{e}_{12} - 5\mathsf{e}_{13} + 6\mathsf{e}_{23})$, where the free term F, Eq. (3.13), has been included via c_{1_\pm} . The logarithm is the sum of all above listed MVs: $\log(\mathsf{A}) = \frac{1}{2} \left(\mathsf{A}_{0_+} + \mathsf{A}_{0_-} + \mathsf{A}_{1,2_+} + \mathsf{A}_{1,2_-} + (\mathsf{A}_{0_+} - \mathsf{A}_{0_-})I\right)$. Using the exponential [7] one can check that the numerical logarithm $\log(\mathsf{A})$ indeed yields the initial MV A for arbitrary integer constants c_{1_+} .

Example 4. Logarithm of MV when $a_{+} = 0$ and $a_{0} + a_{123} > 0$. The MV that satisfies these conditions is $A = 1 + (3e_{1} - 2e_{2} + e_{3}) + (e_{12} + 2e_{13} + 3e_{23}) + 7e_{123} = 1 + a + A + 7e_{123}$. Equation (3.6) gives $a_{-} = \sqrt{56} = 2\sqrt{14}$ and $a_{0} - a_{123} = -6$. Eqs (3.8), (3.10) give $A_{0_{+}} = \log 8 + 2\pi c_{2_{+}}\hat{\mathcal{U}}$, $A_{1,2_{+}} = (\frac{1}{8} + 2\pi c_{1_{+}})(1+I)(\mathbf{a}+\mathbf{e}_{3}+\mathcal{A}) = 0$. Then from (3.3) and (3.5) we have $A_{0_{-}} = \frac{1}{2}\log 92$ and $A_{1,2_{-}} = \frac{\pi - \arctan(\sqrt{\frac{\sqrt{14}}{3}}) + 2\pi c_{1_{-}}}{2\sqrt{14}}(1-I)(\mathbf{a}+\mathcal{A})$. Finally, from Eq. (3.1) $\log(A) = \frac{1}{28}\left(7\left(\log 5888 - \log \frac{23}{16}\mathbf{e}_{123}\right) + \sqrt{14}\left((2c_{1_{-}} + 1)\pi - \arctan \frac{\sqrt{14}}{3}\right)(\mathbf{a}+\mathcal{A})\right) + (1+I)\pi c_{2_{+}}\hat{\mathcal{U}}$. After exponentiation of A the constants $c_{1_{-}}$ and $c_{2_{+}}$ and bivector $\hat{\mathcal{U}}$ simplify out.

Example 5. Logarithm of MV when $a_{-} = 0$ and $a_{0} - a_{123} < 0$. These conditions are satisfied by $A = 1 + (-3\mathbf{e}_{1} + 2\mathbf{e}_{2} - \mathbf{e}_{3}) + (\mathbf{e}_{12} + 2\mathbf{e}_{13} + 3\mathbf{e}_{23}) + 7\mathbf{e}_{123} = 1 + \mathbf{a} + \mathcal{A} + 7\mathbf{e}_{123}$. We have $a_{+}^{2} = 56$, $a_{0} + a_{123} = 9$ and $a_{0} - a_{123} = -6$. Then, Eq. (3.9) gives $A_{0_{-}} = \log 6 + (\pi + 2\pi c_{2_{-}})\hat{\mathbf{u}}$. The Eqs (3.2) and (3.4) give $A_{0_{+}} = \frac{1}{2}\log 120$, $A_{1,2_{+}} = \frac{1}{2\sqrt{14}}(\arctan(\frac{1}{2}\sqrt{7/2}) + 2\pi c_{1_{+}})(1 + I)(\mathbf{a} + \mathcal{A})$, and Eq. (3.11) $A_{1,2_{-}} = (\pi + 2\pi c_{1_{-}})(1 - I)(\mathbf{a} + \mathcal{A}) = 0$. Finally, $\log(A) = \frac{1}{2}(A_{0_{+}} + A_{0_{-}} + A_{1,2_{+}} + (A_{0_{+}} - A_{0_{-}})I)$.

Example 6. Logarithm with infinite subparts: The case $a_{+} = 0$ and $a_{0} + a_{123} = 0$. The example exhibits unusual and the most interesting case. In $Cl_{0,3}$, let's compute GA logarithm of $A = 1 + (-2e_{1} - 3e_{2} + 5e_{3}) + (5e_{12} + 3e_{13} - 2e_{23}) - e_{123} = 1 + a + A - e_{123}$. The remaining scalar is $a_{-} = 2\sqrt{38}$, $(a_{0} - a_{123}) = 2$. Then, Eq. (3.8) gives $A_{0_{+}} = \log(0_{+})$; Eq. (3.3) gives $A_{0_{-}} = \frac{1}{2}\log 156$; Eq. (3.10) gives $A_{1,2_{+}} = 0$; Eq. (3.5) gives $A_{1,2_{-}} = \frac{\arctan(\sqrt{38}) + 2\pi c_{1_{-}}}{\sqrt{38}}(a + A)$. Finally, the logarithm

of A is

$$\log(\mathsf{A}) = \frac{\arctan\left(\sqrt{38}\right) + 2\pi c_{1_{-}}}{2\sqrt{38}} \left(\mathbf{a} + \mathcal{A}\right) + \frac{1}{2} \left(\log(0_{+})\left(1 + \mathbf{e}_{123}\right) + \frac{1}{2}\log(156)\left(1 - \mathbf{e}_{123}\right)\right). \tag{3.14}$$

Note the factor $\log(0_+)$ in front of $(1 + \mathbf{e}_{123})$. If logarithm in this form is inserted into coordinate-free exponential [8] we will get

$$\left(\frac{1}{2}e^{\log(0_{+})}+1\right)+\mathbf{a}+\mathcal{A}+\left(\frac{1}{2}e^{\log(0_{+})}-1\right)\mathbf{e}_{123},$$
 (3.15)

which coincides with the initial MV if we assume that $\log(0_+) = -\infty$.

3.4. GA Logarithm of blades and their combinations in $Cl_{0,3}$

In this subsection, the logarithms for individual blades and their combinations that follow from generic logarithm (Theorem 3.1), and may be useful in practice are collected. The norms listed below are positive scalars.

Vector norm:
$$|\mathbf{a}| = \sqrt{\mathbf{a}\mathbf{\hat{a}}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$$
.
Paravector norm: $|a_0 + \mathbf{a}| = |A_{0,1}| = (A_{0,1}\widehat{A}_{0,1})^{\frac{1}{2}} = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$.
Bivector norm: $|\mathcal{A}| = (\mathcal{A}\widetilde{\mathcal{A}})^{\frac{1}{2}} = \sqrt{a_{12}^2 + a_{13}^2 + a_{23}^2}$.

Rotor norm: $|a_0 + \mathcal{A}| = |\mathsf{A}_{0,2}| = \left(\mathsf{A}_{0,2}\widetilde{\mathsf{A}}_{0,2}\right)^{\frac{1}{2}} = \sqrt{a_0^2 + a_{12}^2 + a_{13}^2 + a_{23}^2}$.

Logarithms of blades and their combinations.

Logarithm of vector $\mathbf{a} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3, c_i \in \mathbb{Z}$,

$$\log(\mathbf{a}) = \frac{1}{2}\log(|\mathbf{a}|^2) + \pi \frac{\mathbf{a}}{|\mathbf{a}|} (\frac{1}{2} + c_1(1+I) + c_2(1-I)), \quad |\mathbf{a}|^2 \neq 0.$$
 (3.16)

Logarithm of paravector $A_{0,1} = a_0 + \mathbf{a}$; $c_i \in \mathbb{Z}$ and $\hat{\mathbf{u}}^2 = -1$, $\hat{\mathcal{U}}^2 = -1$.

$$\log \mathsf{A}_{0,1} = \frac{1}{2} \log(|\mathsf{A}_{0,1}|^2) + \frac{\mathbf{a}}{|\mathbf{a}|} \left(\arctan(a_0, |\mathbf{a}|) + \pi(c_1(1+I) + c_2(1-I)) \right),$$
 $|\mathbf{a}| \neq 0.$ (3.17)

Logarithm of bivector $\mathcal{A} = a_{12}\mathbf{e}_{12} + a_{13}\mathbf{e}_{13} + a_{23}\mathbf{e}_{23}, c_i \in \mathbb{Z},$

$$\log(\mathcal{A}) = \frac{1}{2}\log(|\mathcal{A}|^2) + \pi \frac{\mathcal{A}}{|\mathcal{A}|} \left(\frac{1}{2} + c_1(1+I) + c_2(1-I)\right), \quad |\mathcal{A}|^2 \neq 0.$$
 (3.18)

Logarithm of parabivector and rotor $A_{0,2} = a_0 + A$,

$$\log A_{0,2} = \frac{1}{2} \log(|A_{0,2}|^2) + \frac{\mathcal{A}}{|\mathcal{A}|} \left(\arctan(a_0, |\mathcal{A}|) + \pi(c_1(1+I) + c_2(1-I)) \right),$$
(3.19)

⁵The statement can be made strict by considering the limit $\lim_{x\to 0_+} \exp(\log(x)) = 0$, where $x\to 0_+$ indicates that the limit is taken keeping x positive, i.e. from above.

Logarithm of center $A_{0.3} = a_0 + a_{123}I$,

$$\log \mathsf{A}_{0,3} = \begin{cases} \left(\frac{1}{2}\log(a_0 - a_{123}) + \pi c_1\hat{\mathcal{U}}_1\right) \left(1 - I\right) & (a_0 - a_{123}) > 0 \text{ and} \\ + \left(\frac{1}{2}\log(a_0 + a_{123}) + \pi c_2\hat{\mathcal{U}}_2\right) \left(1 + I\right), & (a_0 + a_{123}) > 0 \end{cases}$$

$$\left(\frac{1}{2}\log(a_0 - a_{123}) + \pi c_1\hat{\mathcal{U}}_1\right) \left(1 - I\right) & (a_0 - a_{123}) > 0 \text{ and} \\ + \left(\frac{1}{2}\log(-a_0 - a_{123}) + \pi(c_2 + \frac{1}{2})\hat{\mathbf{u}}_2\right) \left(1 + I\right), & (a_0 + a_{123}) < 0 \end{cases}$$

$$\left(\frac{1}{2}\log(-a_0 + a_{123}) + \pi(c_1 + \frac{1}{2})\hat{\mathbf{u}}_1\right) \left(1 - I\right) & (a_0 - a_{123}) < 0 \text{ and} \\ + \left(\frac{1}{2}\log(a_0 + a_{123}) + \pi c_2\hat{\mathcal{U}}_2\right) \left(1 + I\right), & (a_0 + a_{123}) > 0 \end{cases}$$

$$\left(\frac{1}{2}\log(-a_0 + a_{123}) + \pi(c_1 + \frac{1}{2})\hat{\mathbf{u}}_1\right) \left(1 - I\right) & (a_0 - a_{123}) < 0 \text{ and} \\ + \left(\frac{1}{2}\log(-a_0 - a_{123}) + \pi(c_2 + \frac{1}{2})\hat{\mathbf{u}}_2\right) \left(1 + I\right), & (a_0 + a_{123}) < 0 \end{cases}$$

$$(3.20)$$

where $\hat{\mathbf{u}}_i$ and $\hat{\mathcal{U}}_j$ are arbitrary non-commuting unit vector and bivector, respectively. If $(a_0 - a_{123}) = 0$ or $(a_0 + a_{123}) = 0$ some of subparts give $\log(0_+)$.

4. MV logarithms in $Cl_{3,0}$ and $Cl_{1,2}$

 $Cl_{3,0}$ and $Cl_{1,2}$ algebras are isomorphic. Their multiplication tables coincide, for example, after the following exchange of basis elements:

$$Cl_{3,0}$$
 {1, \mathbf{e}_1 , \mathbf{e}_2 , \mathbf{e}_3 , \mathbf{e}_{12} , \mathbf{e}_{13} , \mathbf{e}_{23} , \mathbf{e}_{123} } \downarrow $Cl_{1,2}$ {1, $-\mathbf{e}_1$, $-\mathbf{e}_{12}$, $-\mathbf{e}_{13}$, $-\mathbf{e}_2$, $-\mathbf{e}_3$, \mathbf{e}_{23} , $-\mathbf{e}_{123}$ }.

To find formulas for logarithm in coordinate form the same inverse solution method was used as described in the Appendix A for $Cl_{0,3}$ algebra. The logarithm in $Cl_{3,0}$ and $Cl_{1,2}$ exists for all MVs except of nonzero MVs of the form $A_{1,2} = \mathbf{a} + \mathcal{A}$ that satisfy the condition $Det(A_{1,2}) = (a_+^2 + a_-^2)^2 = 0$, i.e., for MVs that are the sums of vector and bivector and the determinant are equal to zero. These restrictions are the same as those for GA square root to exist (see [10] and Example 3 herein in case s = S = 0).

4.1. Logarithm formula for generic MV

Theorem 4.1 (Logarithm of multivector in $Cl_{3,0}$ and $Cl_{1,2}$). The logarithm of generic MV $A = a_0 + \mathbf{a} + A + a_{123}I$ is another MV

$$\log(A) = A_0 + A_{1,2_{\log}} + A_{1,2_{\arctan}} + A_I, \tag{4.1}$$

where

$$A_0 = \frac{1}{2} (\log k_+ + \log k_-), \qquad a_+^2 + a_-^2 \neq 0 \qquad (4.2)$$

$$A_{0} = \frac{1}{2} (\log k_{+} + \log k_{-}), \qquad a_{+}^{2} + a_{-}^{2} \neq 0 \qquad (4.2)$$

$$A_{1,2_{\log}} = \frac{1}{2} \frac{a_{+} - a_{-}I}{a_{-}^{2} + a_{+}^{2}} (\log k_{+} - \log k_{-}) (\mathbf{a} + \mathcal{A}), \qquad a_{+}^{2} + a_{-}^{2} \neq 0 \qquad (4.3)$$

$$A_{1,2_{\arctan}} = I \frac{a_{+} - a_{-}I}{a_{-}^{2} + a_{+}^{2}} (\mathbf{a} + \mathcal{A}) \left(\frac{1}{2} \arctan\left(-(a_{+}^{2} - a_{0}^{2}) - (a_{-}^{2} - a_{123}^{2}), \right. \right.$$

$$(a_{+} - a_{0})(a_{-} + a_{123}) - (a_{+} + a_{0})(a_{-} - a_{123}) + 2\pi c_{1} \right),$$

$$when \quad (a_{+}^{2} + a_{-}^{2} \neq 0) \quad and \quad (k_{-}k_{+} \neq 0), \quad (4.4)$$

$$A_{I} = I \arctan((a_{+} + a_{0})k_{-} - (a_{+} - a_{0})k_{+}, (a_{-} + a_{123})k_{-} - (a_{-} - a_{123})k_{+})$$

$$+ 2\pi c_{2}I, \qquad when \quad (a_{+}^{2} + a_{-}^{2} \neq 0) \quad and \ either$$

$$(a_{+} + a_{0})k_{-} - (a_{+} - a_{0})k_{+} \neq 0 \quad or \quad (a_{-} + a_{123})k_{-} - (a_{-} - a_{123})k_{+} \neq 0 \quad (4.5)$$

where scalar coefficients are

$$k_{-}^{2} = (a_{+} - a_{0})^{2} + (a_{-} - a_{123})^{2}, \qquad k_{+}^{2} = (a_{+} + a_{0})^{2} + (a_{-} + a_{123})^{2},$$
 (4.6)

and

$$a_{-} = \frac{-2I\mathbf{a} \wedge \mathcal{A}}{\sqrt{2}\sqrt{\mathbf{a}\dot{\mathbf{a}} + \mathcal{A}\dot{\mathcal{A}} + \sqrt{(\mathbf{a}\dot{\mathbf{a}} + \mathcal{A}\dot{\mathcal{A}})^{2} - 4(\mathbf{a} \wedge \mathcal{A})^{2}}}},$$

$$a_{+} = \frac{\sqrt{\mathbf{a}\dot{\mathbf{a}} + \mathcal{A}\dot{\mathcal{A}} + \sqrt{(\mathbf{a}\dot{\mathbf{a}} + \mathcal{A}\dot{\mathcal{A}})^{2} - 4(\mathbf{a} \wedge \mathcal{A})^{2}}}}{\sqrt{2}},$$

$$for \mathbf{a} \wedge \mathcal{A} \neq 0, \quad and$$

$$\begin{cases} a_{+} = \sqrt{\mathbf{a}\dot{\mathbf{a}} + \mathcal{A}\dot{\mathcal{A}}}, \quad a_{-} = 0, \quad \text{if } \mathbf{a}\dot{\mathbf{a}} + \mathcal{A}\dot{\mathcal{A}} \geq 0 \\ a_{+} = 0, \quad a_{-} = \sqrt{-(\mathbf{a}\dot{\mathbf{a}} + \mathcal{A}\dot{\mathcal{A}})}, \quad \text{if } \mathbf{a}\dot{\mathbf{a}} + \mathcal{A}\dot{\mathcal{A}} < 0, \\ when \quad \mathbf{a} \wedge \mathcal{A} = 0. \end{cases}$$

$$(4.7)$$

The constants c_1, c_2 are arbitrary integers.

Proof. It is enough to check that after substitution of log A expressions into exponential formula presented in [7] one gets the initial MV A. The factor $\frac{a_+ - a_- I}{a_-^2 + a_+^2}$ in the above formulas alternatively may be written as $(a_+ + a_- I)^{-1}$.

4.2. Special cases

When the conditions listed in Eqs (4.2)-(4.5) are not satisfied, we have special cases. In particular, the condition $k_{\pm}=0$ means that the MV determinant is zero, $\operatorname{Det}(\mathsf{A})=k_{-}^2k_{+}^2=0$. Similarly, the condition $a_{+}^2+a_{-}^2=0$ implies that determinant of vector+bivector part vanishes, $\operatorname{Det}(\mathsf{A}_{1,2})=(a_{+}^2+a_{-}^2)^2=0$. The specific relations $(a_{+}+a_{0})k_{-}-(a_{+}-a_{0})k_{+}\neq 0$ and $(a_{-}+a_{123})k_{-}-(a_{-}-a_{123})k_{+}\neq 0$ in Eq. (4.5) as well as the relation $k_{-}k_{+}\neq 0$ in Eq. (4.4) ensure that both arguments of $\operatorname{arctan}(x,y)$ do not nullify simultaneously.

When the generic formula is not applicable the expressions for A_0 , $A_{1,2_{log}}$, $A_{1,2_{arctan}}$ and A_I in Theorem 4.1 must be supplemented by following formulas

$$A_{0} = \begin{cases} \frac{1}{2} \log(a_{0}^{2} + a_{123}^{2}), & (a_{+}^{2} + a_{-}^{2} = 0) \wedge (a_{0}^{2} + a_{123}^{2} \neq 0), \\ \varnothing, & (a_{+}^{2} + a_{-}^{2} = 0) \wedge (a_{0}^{2} + a_{123}^{2} = 0) \end{cases}$$

$$(4.8)$$

$$\mathsf{A}_{1,2_{\text{log}}} = \begin{cases} 0, & (a_{+}^{2} + a_{-}^{2} = 0) \land (a_{0}^{2} + a_{123}^{2} \neq 0), \\ \varnothing, & (a_{+}^{2} + a_{-}^{2} = 0) \land (a_{0}^{2} + a_{123}^{2} = 0) \end{cases}$$
(4.9)

$$A_{1,2_{log}} = \begin{cases}
0, & (a_{+}^{2} + a_{-}^{2} = 0) \wedge (a_{0}^{2} + a_{123}^{2} = 0) \\
\emptyset, & (a_{+}^{2} + a_{-}^{2} = 0) \wedge (a_{0}^{2} + a_{123}^{2} \neq 0), \\
\emptyset, & (a_{+}^{2} + a_{-}^{2} = 0) \wedge (a_{0}^{2} + a_{123}^{2} = 0)
\end{cases}$$

$$A_{1,2_{arctan}} = \begin{cases}
\pi(\frac{1}{2} + 2c_{1})I \frac{a_{+} - a_{-}I}{a_{-}^{2} + a_{+}^{2}} (\mathbf{a} + \mathcal{A}), & (a_{+}^{2} + a_{-}^{2} \neq 0) \wedge (k_{-}k_{+} = 0), \\
\frac{a_{0} - a_{123}I}{a_{0}^{2} + a_{123}^{2}} (\mathbf{a} + \mathcal{A}) + \hat{\mathcal{F}}, & (a_{+}^{2} + a_{-}^{2} = 0) \wedge (a_{0}^{2} + a_{123}^{2} \neq 0), \\
\emptyset, & (a_{+}^{2} + a_{-}^{2} = 0) \wedge (a_{0}^{2} + a_{123}^{2} = 0)
\end{cases}$$

$$(4.9)$$

$$\mathsf{A}_{I} = \begin{cases} I\left(\arctan(-a_{-}, a_{+}) + 2\pi c_{2}\right), & (a_{+}^{2} + a_{-}^{2} \neq 0) \\ & \wedge ((a_{+} + a_{0})k_{-} - (a_{+} - a_{0})k_{+} = 0) \\ & \wedge ((a_{-} + a_{123})k_{-} - (a_{-} - a_{123})k_{+} = 0), \end{cases}$$

$$I\left(\arctan(a_{0}, a_{123}) + 2\pi c_{2}\right), & (a_{+}^{2} + a_{-}^{2} = 0) \wedge (a_{0}^{2} + a_{123}^{2} \neq 0),$$

$$\varnothing, & (a_{+}^{2} + a_{-}^{2} = 0) \wedge (a_{0}^{2} + a_{123}^{2} = 0) \end{cases}$$

$$(4.11)$$

Here the symbols \wedge and \vee in the conditions represent logical conjunction and disjunction, respectively. $\hat{\mathcal{F}} = \begin{cases} 2\pi c_1 \hat{\mathcal{U}}, & \text{if } \mathbf{a} + \mathcal{A} = 0 \\ 0, & \text{if } \mathbf{a} + \mathcal{A} \neq 0 \end{cases}$, where the free unit bivector

must satisfy $\hat{\mathcal{U}}^2 = -1$. After exponentiation it gives $\exp(\hat{\mathcal{U}}) = 1$ and represents continuous degree of freedom (a direction) in (4.10) and (4.11), and can be parameterized as

$$\hat{\mathcal{U}} = \begin{cases}
\frac{d_{12}\mathbf{e}_{12} + d_{13}\mathbf{e}_{13} + d_{23}\mathbf{e}_{23}}{\sqrt{d_{12}^2 + d_{13}^2 + d_{23}^2}}, & \text{for } Cl_{3,0} \\
\frac{d_{12}\mathbf{e}_{12} + d_{13}\mathbf{e}_{13} + d_{23}\mathbf{e}_{23}}{\sqrt{-d_{12}^2 - d_{13}^2 + d_{23}^2}}, & \text{for } Cl_{1,2}, \text{ when } -d_{12}^2 - d_{13}^2 + d_{23}^2 > 0
\end{cases} (4.12)$$

The cases $k_{\pm} = 0$ that represent MV with a vanishing determinant, Det(A) = $k_{-}^{2}k_{+}^{2}=0$, yield MVs with infinite coefficients (see Example 9 for details).

4.3. Multiveluedness and free multivector

In Eqs (4.4) and (4.5) we may add any multiple of 2π to both arc tangent functions, i.e. $\arctan(y_1, y_2) \to \arctan(y_1, y_2) + 2\pi c_i$. After collecting terms in front of free coefficients $c_1, c_2 \in \mathbb{Z}$, we obtain a free MV F that satisfies $\exp(\mathsf{F}) = 1$,

$$\mathsf{F} = \frac{2\pi c_1}{(a_-^2 + a_+^2)} \left(a_-(\mathbf{a} + \mathcal{A}) + a_+(\mathbf{a} + \mathcal{A})I \right) + 2\pi c_2 I, \tag{4.13}$$

where a_{\pm} are given by Eq. (4.7).

Example 7. Logarithm of generic MV in $Cl_{3,0}$. Let us take simple but representative MV: $A = -2 + \mathbf{e}_1 + \mathbf{e}_{23} - 3\mathbf{e}_{123}$. From Eqs (4.6) and (4.7) we have $k_{+}^{2} = 5$, $k_{-}^{2} = 25$ and $a_{+} = a_{-} = 1$. Then (4.2) and (4.3) yield $A_{0} = \frac{3 \log 5}{4}$ and $A_{1,2_{log}} = -\frac{\log 5}{8} (\mathbf{e}_1 + \mathbf{e}_{23})(1 - I)$. Next, the Eqs (4.4) and (4.5) give $A_{1,2_{arctan}} = -\frac{\log 5}{8} (\mathbf{e}_1 + \mathbf{e}_{23})(1 - I)$. $-\frac{1}{4}\left(-\arctan\frac{2}{11}+4\pi c_2\right)\left(\mathbf{e}_1+\mathbf{e}_{23}\right)(1+I)$ and $\mathsf{A}_I=\left(-\pi+\arctan\frac{-10-4\sqrt{5}}{-5-3\sqrt{5}}+2\pi c_1\right)\mathbf{e}_{123}$. Finally, after summation of all terms in (4.1) we obtain $\log(\mathsf{A})=\frac{\log 5}{4}\left(3-\mathbf{e}_1\right)+\frac{1}{2}\arctan\frac{2}{11}\mathbf{e}_{23}+\left(-\pi+\arctan\left(\frac{1}{2}(1+\sqrt{5})\right)\mathbf{e}_{123}+\mathsf{F}$, where the free MV $\mathsf{F}=2\pi\left(c_1\mathbf{e}_{123}-c_2\mathbf{e}_{23}\right)$. The coefficients $c_1,c_2\in\mathbb{Z}$ come from $\mathsf{A}_{1,2_{\arctan}}$ and A_I terms, respectively. Substitution of this result into exponential $\exp(\log(\mathsf{A}))$ returns the initial MV.

Example 8. Logarithm of center of $Cl_{3,0}$. A = 1 - 2e₁₂₃. Since $\mathbf{e}_{123}^2 = -1$ the MV is a counterpart of complex number logarithm. Eqs (4.6) and (4.7) give $a_+ = a_- = 0$ and $k_+^2 = k_-^2 = 5$. Then, Eq. (4.8) gives $\mathsf{A}_0 = \frac{\log 5}{2}$; Eq. (4.9) gives $\mathsf{A}_{1,2_{\log}} = 0$; Eq. (4.10) gives $\mathsf{A}_{1,2_{\arctan}} = 2\pi c_1 \hat{\mathcal{U}}$; Eq. (4.11) gives $\mathsf{A}_I = \left(-\arctan 2 + 2\pi c_2\right)\mathbf{e}_{123}$. Note that $\hat{\mathcal{U}}$ is the same free MV for both $\mathsf{A}_{1,2_{\arctan}}$. After summation of terms in (4.1) the final answer is $\log(\mathsf{A}) = \frac{\log 5}{2} + \left(-\arctan 2 + 2\pi c_2\right)\mathbf{e}_{123} + 2\pi c_1\hat{\mathcal{U}}$. On the other hand the complex number 1-2 i gives $\log(1-2$ i) $= \frac{1}{2}\log 5 - \arctan 2$, which coincides with $Cl_{3,0}$ algebra result if $c_1 = c_2 = 0$.

Example 9. Logarithm of singular MV when Det(A) = 0. This is the most intriguing and complicated case in $Cl_{3,0}$. Since $Det(A) = k_-^2 k_+^2$ we may have either $k_-^2 = 0$ or $k_+^2 = 0$. The case when $k_-^2 = k_+^2 = 0$ is trivial since it requires all MV components to vanish. Let's analyze the case when $k_+^2 \neq 0$ and $k_-^2 = 0$. It is represented, for example, by $A = 6 + (-8e_1 - 2e_3) + (-e_{12} + 10e_{13} + 10e_{23}) - 13e_{123} = 6 + a + A - 13e_{123}$. From Eq. (4.7) we find $a_+ = 6$, $a_- = -13$ and from (4.6) $k_+^2 = 820$, $k_-^2 = 0$. Then, Eq. (4.2) gives $A_0 = \frac{1}{2} (\log(2\sqrt{205}) + \log(0_+))$; Eq. (4.3) gives $A_{1,2_{log}} = \frac{1}{410} (\log(2\sqrt{205}) - \log(0_+)) (6 + 13e_{123}) (a + A) + 6(a + A))$; Eq. (4.10) gives $A_{1,2_{arctan}} = \frac{\pi}{205} (\frac{1}{2} + 2c_1) (-6 + 13e_{123}) (a + A) + 6(a + A))$; finally, Eq. (4.11) gives $A_I = (\arctan(\frac{6}{13}) + 2\pi c_2)e_{123}$. Summing up all terms we obtain the answer

$$\begin{split} \log \mathsf{A} = & \frac{1}{2} \left(\log(2\sqrt{205}) + \log(0_+) \right) + \left(\frac{1}{410} \left(\log(2\sqrt{205}) - \log(0_+) \right) \left(6 + 13 \mathbf{e}_{123} \right) \right. \\ & + \left. \frac{\pi}{205} (\frac{1}{2} + 2c_1) \left(-6 + 13 \mathbf{e}_{123} \right) \right) (\mathbf{a} + \mathcal{A}) + \left(\arctan(\frac{6}{13}) + 2\pi c_2 \right) \mathbf{e}_{123}. \end{split}$$

The result can be checked after replacement of $\log(0_+)$ by $\log(x)$ and substitution into exponential formula (4.1) of paper [7]. After simplification one can take the limit $\lim_{x\to 0_+} \exp(\log A)$, which returns the initial MV. This example demonstrates that the logarithm of MV with specific finite coefficients may yield MV with some of coefficients in the answer being infinite and which have to be understood as the limit $\lim_{x\to 0_+} \log(x)$. The answer, nevertheless, is meaningful since the substitution of the answer into exponential formula and computation of the limit reproduces the initial MV.

4.4. Logarithms of individual blades and their combinations

Below we use different norms for individual blades of $Cl_{3,0}$, since a positive scalar for vectors and bivectors is calculated differently. In particular, for a vector we will use $|\mathbf{a}| = \sqrt{\mathbf{a}\mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$, whereas for a bivector $|\mathcal{A}| = (\mathcal{A}\widetilde{\mathcal{A}})^{1/2} = (\mathcal{A}\widetilde{\mathcal{A}})^{1/2}$

 $\sqrt{a_{12}^2 + a_{13}^2 + a_{23}^2}. \text{ For a rotor } |a_0 + \mathcal{A}| = |\mathsf{A}_{0,2}| = \left(\mathsf{A}_{0,2}\widetilde{\mathsf{A}}_{0,2}\right)^{1/2} = \sqrt{a_0^2 + a_{12}^2 + a_{13}^2 + a_{23}^2},$ and for an element of center $|a_0 + a_{123}I| = |\mathsf{A}_{0,3}| = \left(\mathsf{A}_{0,3}\widehat{\mathsf{A}}_{0,3}\right)^{1/2} = \sqrt{a_0^2 + a_{123}^2 + a_{23}^2}.$

Logarithm of vector: $\mathbf{a} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3$,

$$\log(\mathbf{a}) = \frac{1}{2}\log(|\mathbf{a}|^2) - \pi(\frac{1}{2} + 2c_2)\frac{\mathbf{a}}{|\mathbf{a}|}I + \pi(\frac{1}{2} + 2c_1)I, \qquad |\mathbf{a}|^2 \neq 0.$$
 (4.14)

Logarithm of *bivector*: $A = a_{12}\mathbf{e}_{12} + a_{13}\mathbf{e}_{13} + a_{23}\mathbf{e}_{23}$,

$$\log(\mathcal{A}) = \frac{1}{2}\log(|\mathcal{A}|^2) - \pi(\frac{1}{2} + 2c_2)\frac{\mathcal{A}}{|\mathcal{A}|} + \pi(1 + 2c_1)I, \qquad |\mathcal{A}|^2 \neq 0.$$
 (4.15)

Logarithm of rotor: $A_{0,2} = a_0 + A$,

$$\log \mathsf{A}_{0,2} = \begin{cases} \frac{1}{2} \log(|\mathsf{A}_{0,2}|^2) + \left(\arctan(a_0,0) + 2\pi c_1\right)I \\ + \frac{\mathcal{A}}{|\mathcal{A}|} \left(2\pi c_2 - \frac{1}{2}\arctan(a_0^2 - |\mathcal{A}|^2, -2a_0|\mathcal{A}|)\right), & |\mathsf{A}_{0,2}| \neq 0 \\ \log a_0 + 2\pi c_1I, & |\mathcal{A}| = 0 \text{ and } a_0 \geq 0 \\ \log(-a_0) + 2\pi(c_1 + 1)I, & |\mathcal{A}| = 0 \text{ and } a_0 < 0 \\ \text{see bivector formula (4.15)}, & |\mathcal{A}| \neq 0 \text{ and } a_0 = 0, \\ & (4.16) \end{cases}$$

Logarithm of center: $A_{0,3} = a_0 + a_{123}e_{123} = a_0 + a_{123}I$, $|A_{0,3}|^2 = A_{0,3}\widetilde{A}_{0,3}$.

$$\log(\mathsf{A}_{0,3}) = \begin{cases} \frac{1}{2}\log(|\mathsf{A}_{0,3}|^2) + 2\pi c_2 \hat{\mathcal{U}} + \left(\arctan\left(a_0, a_{123}\right) + 4\pi c_1\right)I, & |\mathsf{A}_{0,3}|^2 \neq 0, \\ \log(0_+) + 2\pi c_2 \hat{\mathcal{U}}, & |\mathsf{A}_{0,3}|^2 = 0. \end{cases}$$

$$(4.17)$$

The paravector $A_{0,1} = a_0 + \mathbf{a}$ norm $|a_0 + \mathbf{a}|^2 \equiv |A_{0,1}|^2 = A_{0,1} \widehat{A}_{0,1} = a_0^2 - a_1^2 - a_2^2 - a_3^2$, contains coefficients with opposite signs. The logarithm formula, therefore, splits into many subcases and is impractical.

5. MV logarithms in $Cl_{2.1}$

Of all three algebras, the logarithm of $Cl_{2,1}$ appeared the most hard to recover. The logarithms in $Cl_{3,0}$ and $Cl_{1,2}$ algebras exist for almost all MVs except very small specific class of vectors and bivectors, $\mathbf{a} + \mathcal{A} \neq 0$, with the vanishing determinant Det $(\mathbf{a} + \mathcal{A}) = 0$. In $Cl_{2,1}$ algebra the logarithm does not exist for a large class of MVs. In contrast, in $Cl_{0,3}$ algebra the logarithm exists for all MVs.

Theorem 5.1. [Logarithm of multivector in $Cl_{2,1}$] The logarithm of multivector $A = a_0 + (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) + (a_{12}\mathbf{e}_{12} + a_{13}\mathbf{e}_{13} + a_{23}\mathbf{e}_{23}) + a_{123}I = a_0 + \mathbf{a} + \mathcal{A} + a_{123}I$ is the MV

$$\log(\mathsf{A}) = \begin{cases} \frac{1}{2} \left(\mathsf{A}_{0_{+}} + \mathsf{A}_{0_{-}} + \mathsf{A}_{1,2_{+}} + \mathsf{A}_{1,2_{-}} + (\mathsf{A}_{0_{+}} - \mathsf{A}_{0_{-}})I \right), & f_{\pm} \ge 0 \\ \varnothing, & f_{\pm} < 0 \end{cases}$$
(5.1)

where

$$f_{\pm} = (a_{0} \pm a_{123})^{2} + a_{\pm}^{2}, \qquad f_{\pm} \leq 0,$$

$$a_{-}^{(2)} = -(\mathbf{a}\dot{\mathbf{a}} + \mathcal{A}\dot{\mathcal{A}}) + 2I\mathbf{a} \wedge \mathcal{A}, \quad a_{-}^{(2)} \leq 0,$$

$$a_{+}^{(2)} = -(\mathbf{a}\dot{\mathbf{a}} + \mathcal{A}\dot{\mathcal{A}}) - 2I\mathbf{a} \wedge \mathcal{A}, \quad a_{+}^{(2)} \leq 0,$$
(5.2)

and

$$\mathsf{A}_{0\pm} = \begin{cases} \frac{1}{2}\log(f_{\pm}), & (a_{\pm}^{(2)} > 0) \\ \frac{1}{2}\log\left(a_{0} \pm a_{123} + \sqrt{-a_{\pm}^{(2)}}\right) & (a_{\pm}^{(2)} < 0) \land (a_{0} \pm a_{123} > 0) \\ + \frac{1}{2}\log\left(a_{0} \pm a_{123} - \sqrt{-a_{\pm}^{(2)}}\right), & (a_{\pm}^{(2)} = 0) \land (a_{0} \pm a_{123} > 0) \\ \log(a_{0} \pm a_{123}) + 2\pi c_{2\pm}\hat{\mathcal{F}}, & (a_{\pm}^{(2)} = 0) \land (a_{0} \pm a_{123} > 0) \\ \log(-(a_{0} \pm a_{123})) + (\pi + 2\pi c_{2\pm})\hat{\mathcal{U}}, & (a_{\pm}^{(2)} = 0) \land (a_{0} \pm a_{123} \leq 0) \\ & \land (\mathfrak{D} = \text{True}) \end{cases} \\ \varnothing, & ((a_{\pm}^{(2)} < 0) \land (a_{0} \pm a_{123} < 0)) \\ & \land (\mathfrak{D} = \text{False})) \end{cases}$$

$$(5.3)$$

$$\mathsf{A}_{1,2_{\pm}} = \begin{cases} \frac{1}{\sqrt{a_{\pm}^{(2)}}} \left(\arctan(a_0 \pm a_{123}, a_{\pm}) \right. \\ + 2\pi c_{1\pm} \right) (1 \pm I) \left(\mathbf{a} + \mathcal{A} \right), & (a_{\pm}^{(2)} > 0) \\ \frac{1}{\sqrt{-a_{\pm}^{(2)}}} \arctan\left(\frac{\sqrt{-a_{\pm}^{(2)}}}{a_0 \pm a_{123}} \right), & (a_{\pm}^{(2)} < 0) \land (a_0 \pm a_{123} > 0) \\ & \land \left(- a_{\pm}^{(2)} \neq (a_0 \pm a_{123}) \right) \end{cases} \\ \times (1 \pm I) \left(\mathbf{a} + \mathcal{A} \right), & (a_{\pm}^{(2)} < 0) \land (a_0 \pm a_{123} > 0) \\ & \land \left(- a_{\pm}^{(2)} \neq (a_0 \pm a_{123}) \right) \end{cases} \\ - \log\left(a_0 \pm a_{123} + \sqrt{-a_{\pm}^{(2)}}\right), & (a_{\pm}^{(2)} < 0) \land (a_0 \pm a_{123} > 0) \\ & \land \left(- a_{\pm}^{(2)} = (a_0 \pm a_{123}) \right) \end{cases} \\ \times \frac{1}{\sqrt{-a_{\pm}^{(2)}}} (1 \pm I) \left(\mathbf{a} + \mathcal{A} \right), & (a_{\pm}^{(2)} = 0) \land (a_0 \pm a_{123} > 0) \\ 0, & (a_{\pm}^{(2)} = 0) \land (a_0 \pm a_{123} < 0) \\ & \land \left(\mathfrak{D} = \text{True} \right) \end{cases} \\ \varnothing, & \left((a_{\pm}^{(2)} < 0) \land (a_0 \pm a_{123} < 0) \right) \\ \lor \left((a_{\pm}^{(2)} = 0) \land (a_0 \pm a_{123} < 0) \right) \\ & \land \left(\mathfrak{D} = \text{False} \right) \end{cases}$$

$$(5.4)$$

where the upper symbol in $a_{\pm}^{(2)}$ indicates that $a_{\pm}^{(2)}$ consists of the squared coefficients a_i^2 , a_{ij}^2 and $a_i a_{ij}$. $\hat{\mathcal{F}} = \begin{cases} \hat{\mathcal{U}}, & \text{if} \quad \mathfrak{D} = \text{True} \\ 0, & \text{if} \quad \mathfrak{D} = \text{False} \end{cases}$. The logical condition \mathfrak{D} is a conjunction of outcomes of three comparisons $\mathfrak{D}=(a_1=\pm a_{23})\wedge (a_2=\mp a_{13})\wedge (a_3=\pm a_{23})$

 $\mp a_{12}$) $\equiv ((a_1 = a_{23}) \wedge (a_2 = -a_{13}) \wedge (a_3 = -a_{12})) \vee ((a_1 = -a_{23}) \wedge (a_2 = a_{13}) \wedge (a_3 = a_{12}))$ that should be applied to $\mathsf{A}_{0\pm}$ and $\mathsf{A}_{1,2\pm}$ terms without paying attention to \pm signs in their subscripts. Unit bivector in $\mathsf{A}_{0\pm}$ may be parameterized as $\hat{\mathcal{U}} = \frac{d_{12}\mathsf{e}_{12} + d_{13}\mathsf{e}_{13} + d_{23}\mathsf{e}_{23}}{\sqrt{d_{12}^2 - d_{13}^2 - d_{23}^2}}$. The symbol \varnothing means that the solution set is empty. In all formulas the indices and conditions (except \mathfrak{D} as stated explicitly) must be included with either all upper or with all lower signs.

The case with $f_{\pm} \neq 0$ and $a_{\pm}^{(2)} > 0$ represents a generic instance. When either $f_{\pm} = 0$ or $a_{\pm}^{(2)} \leq 0$ we have the special case. Note that in Eq. (5.2) the condition $f_{\pm} = 0$ implies $a_{\pm}^{(2)} \leq 0$. Also, observe that the condition $f_{\pm} \geq 0$ ensures automatically that a less restrictive requirement $\text{Det}(A) = f_{-}f_{+} \geq 0$ is fulfilled automatically.

The equations (5.3)-(5.4) are similar to Eqs (3.8)-(3.11) in $Cl_{0,3}$ (see Sec. 3.2). Also, in (5.2) the expressions for scalar coefficients $a_{\pm} = \begin{cases} \sqrt{a_{\pm}^{(2)}}, & a_{\pm}^{(2)} \geq 0 \\ \sqrt{-a_{\pm}^{(2)}}, & a_{\pm}^{(2)} < 0 \end{cases}$ are similar to Eqs (3.6) and (3.7). The differences mainly arise at the parameter boundaries that define the existence of MV logarithm for $Cl_{2,1}$.

From our earlier calculations [10] we know the conditions that ensure an existence of MV square roots in $Cl_{2,1}$ algebra. Thus, we can rewrite and use here these conditions that limit the extent of the logarithm in Theorem 5.1. It appears that the quantities b_S and b_I in [10] may be expressed in terms of multipliers f_+ and f_- in the determinant D= Det $\mathsf{A}=f_-f_+$, where $f_\pm=(a_0\pm a_{123})^2+a_\pm^{(2)}$, in a form $b_I=\frac{1}{2}\big(f_+-f_-\big)$ and $b_S=\frac{1}{2}\big(f_++f_-\big)$. Now, note that f_\pm enter as arguments in logfunctions of Theorem 5.1, Eq. (5.4). Therefore, the square root existence condition $b_S - \sqrt{D} \geq 0$ in [10], in terms of the logarithm problem can be rewritten as a difference of the determinant factors, namely, $b_S - \sqrt{D} \Leftrightarrow \frac{1}{2} (\sqrt{f_-} - \sqrt{f_+})^2$. Now it becomes clear that this condition is always satisfied and therefore can be ignored, once we assume that the both factors satisfy $f_{-} \geq 0$ and $f_{+} \geq 0$. From all this we conclude that the requirement $f_{\pm} > 0$ constitutes one of the existence conditions of logarithm in Theorem 5.1. Also, $b_S - \sqrt{D} = 0$ is equivalent to $f_+ = f_-$. This restricts the maximal possible value of $a_{\pm}^{(2)}$. In particular, $|a_{\pm}^{(2)}| \leq (a_0 \pm a_{123})^2$. Remember, that notation $a_{\pm}^{(2)}$ (instead of a_{\pm}) was introduced to keep an analogy with $Cl_{0,3}$ case. It may be negative $a_{\pm}^{(2)} < 0$ (see definition (5.2)) and therefore the notation, in general, can't be interpreted as a square of scalar unless $a_{\pm}^{(2)} \geq 0$. When $a_{\pm}^{(2)} = 0$, an additional condition $a_0 \pm a_{123} \ge 0$ is required for logarithm to

Since $Cl_{2,1}$ algebra is rarely used we will not provide explicit formulas for pure blades (they can be found in the notebook ElementaryFunctions.nb in [17]). Also, because generic formulas are similar to those in $Cl_{0,3}$ the examples below are restricted to special cases only.

Example 10. Logarithm in $Cl_{2,1}$ when $a_{\pm}^{(2)} = 0$ and $a_0 \pm a_{123} > 0$. Let the MV be $\mathsf{A} = 7 + (2\mathbf{e}_1 + \mathbf{e}_2 + 3\mathbf{e}_3) + (2\mathbf{e}_{12} + 2\mathbf{e}_{13} - 2\mathbf{e}_{23}) + 5I = 7 + \mathbf{a} + \mathcal{A} + 5I$. From (5.2) we find $a_{+}^{(2)} = 0$, $f_{+} = 144$ and $a_{-}^{(2)} = 0$, $f_{-} = 144$. Since $a_0 \pm a_{123} = 7 \pm 5 > 0$ from (5.3) we have $\mathsf{A}_{0_{-}} = \log 2$, $\mathsf{A}_{0_{+}} = \log 12$ and from (5.4) $\mathsf{A}_{1,2_{-}} = \frac{1}{2}(1-I)(\mathbf{a} + \mathcal{A})$, $\mathsf{A}_{1,2_{+}} = \frac{1}{12}(1+I)(\mathbf{a} + \mathcal{A})$. Finally, $\log \mathsf{A} = \frac{1}{24}(12\log(24) + 24\mathbf{e}_1 + 17\mathbf{e}_2 + 31\mathbf{e}_3 + 29\mathbf{e}_{12} + 19\mathbf{e}_{13} - 24\mathbf{e}_{23} + 12\log(6)I)$. Note, because MV coefficients $a_3 \neq \pm a_{12}$ the condition $\mathfrak D$ is False, therefore the free MV in (5.3) is absent, $\hat{\mathcal F} = 0$.

Example 11. Logarithm when $a_{-}^{(2)}=0$, $a_0-a_{123}=0$ and $a_{+}^{(2)}>0$, $a_0+a_{123}<0$. In $Cl_{2,1}$ these properties are satisfied by MV A = $-2+(7\mathbf{e}_1+4\mathbf{e}_2+10\mathbf{e}_3)+(-10\mathbf{e}_{12}-4\mathbf{e}_{13}+7\mathbf{e}_{23})-2I=-2+\mathbf{a}+\mathcal{A}-2I$. From (5.2) we find $a_{+}^{(2)}=140$, $f_{+}=156$ and $a_{-}^{(2)}=0$, $f_{-}=0$. Then, because $a_0-a_{123}=-2-(-2)=0$ and $a_0+a_{123}=-2-2<0$, from (5.3) we have $A_{0-}=\log(0_+)+(\pi+2\pi c_{2-})\hat{\mathcal{U}}$, $A_{0+}=\frac{1}{2}\log(156)$. From (5.4) $A_{1,2-}=0$, $A_{1,2+}=\frac{2}{\sqrt{35}}(\pi+2\pi c_{1+}-\arctan(\sqrt{35}/2))(1+I)(\mathbf{a}+\mathcal{A})$. Then using (5.1) we obtain the final answer $\log A=\alpha_0+\alpha_1\mathbf{e}_1+\alpha_2\mathbf{e}_2+\alpha_3\mathbf{e}_3+\alpha_{12}\mathbf{e}_{12}+\alpha_{13}\mathbf{e}_{13}+\alpha_{23}\mathbf{e}_{23}+\alpha_{123}I$, where $\beta=\arctan(\sqrt{35}/2)$ and,

$$\begin{split} &\alpha_0 = \frac{1}{2}(\log 0_+ + \log \sqrt{156}), \quad \alpha_{123} = -\frac{1}{4}(2\log 0_+ - \log 156), \\ &\alpha_1 = -\frac{1}{20}\big(5\sqrt{3}\,\pi - 3\sqrt{35}\,\pi + 2\sqrt{35}\,\beta\big), \quad \alpha_2 = \big(\frac{\pi}{2\sqrt{3}} + \frac{2}{\sqrt{35}}\,\pi - \frac{2}{\sqrt{35}}\beta\big), \\ &\alpha_3 = \big(\sqrt{\frac{5}{7}}\,\pi + \frac{5\pi}{4\sqrt{3}} - \sqrt{\frac{5}{7}}\,\beta\big), \quad \alpha_{12} = \big(-\sqrt{\frac{5}{7}}\,\pi + \frac{5\pi}{4\sqrt{3}} + \sqrt{\frac{5}{7}}\,\beta\big), \\ &\alpha_{13} = \big(\frac{\pi}{2\sqrt{3}} - \frac{2}{\sqrt{35}}\,\pi + \frac{2}{\sqrt{35}}\beta\big), \quad \alpha_{23} = \frac{1}{20}\big(5\sqrt{3}\,\pi + 2\sqrt{35}\,\pi - 2\sqrt{35}\,\beta\big). \end{split}$$

For simplicity the constants $c_{i\pm}$ and $\hat{\mathcal{U}}$ were equated to zero. One can check that after replacement of $\log(0_+)$ by $\log(x)$ and substituting the final result into exponenial formula (23) in [8] and then computing the limit $x \to 0$ we recover the initial MV. To make the verification simple when $c_{i\pm}$ and $\hat{\mathcal{U}}$ are included, one may choose concrete values for arbitrary free constants $c_{i\pm}$ and arbitrary unit bivector $\hat{\mathcal{U}}^2 = -1$.

6. Roots and arbitrary powers of MV

If GA logarithm is known, the powers of a MV may be computed with $A^r = \exp(r \log A)$, i.e., by multiplying logarithm by power value r,which may be either an integer or a rational number, and then computing the exponential. In the preprint [10] we provided the algorithm how to obtain all possible square roots (r = 1/2) of MV for all n = 3 Clifford algebras. Here we want to show that the roots presented in [10] as numerical examples of algorithm are consistent with the above exp-log formula, thus actually we perform a cross check of 3D GA logarithm formulas by different methods. It should be stressed that the logarithm formula allows to find only a single⁶ square root, although there may exist, as shown in [10], many (up to 16 in case of $Cl_{2,1}$ algebra) roots. Thus, the GA logarithm

⁶More precisely two (plus/minus) roots.

function is not universal enough, although it may be sometimes useful if only a single fractional root, r = 1/n and $n \in \mathbb{N}$, is needed.

Example 12. $Cl_{3,0}$. Example 1 from [10]. Theorem 4.1 is used to calculate the root of MV A = $\mathbf{e}_1 - 2\mathbf{e}_{12}$. In $Cl_{3,0}$ algebra the logarithm is $\log A = \frac{\log 5}{2} - \frac{1}{2}\pi\mathbf{e}_{23} + \arctan(\frac{1}{2})I$. Then the square root \sqrt{A} is

$$\exp\left(\frac{1}{2}\log A\right) = \frac{\sqrt[4]{5}}{\sqrt{2}}\left(\cos\left(\frac{1}{2}\arctan\frac{1}{2}\right)\left(1 - \mathbf{e}_{23}\right) + \sin\left(\frac{1}{2}\arctan\frac{1}{2}\right)\left(\mathbf{e}_{1} + I\right)\right),$$

which after simplification coincides with root A_3 in Example 1 in [10].

Example 13. $Cl_{3,0}$. Example 2 from [10]. Logarithm of MV $A = -1 + \mathbf{e}_3 - \mathbf{e}_{12} + \frac{1}{2}I$ in $Cl_{3,0}$ is

$$\log \mathsf{A} = \log \left(\frac{\sqrt{5}}{2}\right) - \frac{\log 5}{2} \mathbf{e}_3 + \frac{1}{2} \left(\pi - \arctan \frac{4}{3}\right) \mathbf{e}_{12} + \left(-\pi + \arctan \frac{1}{2}\right) I,$$

Multiplication by $\frac{1}{2}$ and exponentiation gives the root $A_3 = \sqrt{A} = \frac{1}{2}(\mathbf{e}_3 + \mathbf{e}_{12}) - I$ which coincides with [10].

Example 14. $Cl_{3,0}$. Example 3 from [10]. Similarly, the logarithm of MV $A = -1 + \mathbf{e}_{123}$ in $Cl_{3,0}$ is found to be $\log A = \frac{\log 2}{2} + \frac{3}{4}\pi I$. Multiplication by $\frac{1}{2}$ and exponentiation gives the root A_1 of Example 3 [10], $\sqrt{A} = 2^{1/4} \left(\cos \frac{3\pi}{8} + I \sin \frac{3\pi}{8}\right) = \sqrt{-\frac{1}{2} + \frac{1}{\sqrt{2}}} + I\sqrt{\frac{1}{2} + \frac{1}{\sqrt{2}}}$. Likewise, an attempt to compute the logarithm of $A = \mathbf{e}_1 + \mathbf{e}_{12}$ yields empty set, i.e., the logarithm and as a result the square root do not exist.

Example 15. $Cl_{3,0}$. Quaternion. Example 4 from [10]. In $Cl_{3,0}$ algebra the even MV A = $1+\mathbf{e}_{12}-\mathbf{e}_{13}+\mathbf{e}_{23}$ is equivalent to Hamilton quaternion. The logarithm is $\log A = \log 2 + \frac{\pi}{3\sqrt{3}} (\mathbf{e}_{12} - \mathbf{e}_{13} + \mathbf{e}_{23})$. Multiplication by $\frac{1}{2}$ and exponentiation give the root A_3 in Example 4 [10], $\sqrt{A} = \frac{1}{\sqrt{6}}(3+\mathbf{e}_{12}-\mathbf{e}_{13}+\mathbf{e}_{23})$.

Example 16. $Cl_{0,3}$. Example 6 from [10]. To compute the logarithm of MV A = $\mathbf{e}_1 - 2\mathbf{e}_{23}$ in $Cl_{0,3}$ the Theorem 3.1 was applied which gives $\log A = \frac{\log 3}{2} - \frac{\pi}{2}\mathbf{e}_{23} + \frac{\log 3}{2}I$. Multiplication by $\frac{1}{2}$ and exponentiation gives the root A_3 of Example 6 in [10], $\sqrt{A} = \frac{1}{2}(d_2 + d_1\mathbf{e}_1 - d_2\mathbf{e}_{23} + I/d_2)$, where $d_1 = \sqrt{2 - \sqrt{3}}$ and $d_2 = \sqrt{2 + \sqrt{3}}$.

Example 17. $Cl_{0,3}$. Example 7 from [10]. The logarithm of MV A = $-\mathbf{e}_3$ + \mathbf{e}_{12} + 4I in $Cl_{0,3}$ algebra is computed by Theorem 3.1. The result is $\log \mathsf{A} = \frac{\log 320}{4} - \frac{1}{2}\arctan(\frac{1}{2})\mathbf{e}_3 + \frac{1}{2}\arctan(\frac{1}{2})\mathbf{e}_{12} + \frac{\pi}{2}(1-I)\hat{\mathbf{u}} + \frac{1}{4}\log\frac{5}{4}I$. We have assumed that the discrete free constants are equal to zero and retained only free unit vector $\hat{\mathbf{u}}$ that satisfies $\hat{\mathbf{u}}^2 = -1$. Multiplication by $\frac{1}{2}$ and exponentiation then gives $\sqrt{\mathsf{A}} = \frac{1}{2}c_2 + \frac{1}{2}c_1(\mathbf{e}_{12} - \mathbf{e}_3) + \frac{1}{2}c_2I + (1-I)\hat{\mathbf{u}}$, where $c_1 = \sqrt{\sqrt{5}-2}$ and $c_2 = \sqrt{2+\sqrt{5}}$, and corresponds to A_3 root in Example 7 in [10]. In particular, in order to obtain the numerical value corresponding to $V_2 = \frac{1}{2}, V_3 = 0$ of [10] we have to take $\hat{\mathbf{u}} = -\frac{1}{2}\sqrt{5-\sqrt{5}-2c_1}\mathbf{e}_1 + \frac{1}{2}(-1-c_1)\mathbf{e}_3$.

Example 18. $Cl_{2,1}$. Example 8 from [10]. With the Theorem 5.1 one may ascertain that the logarithm of MV $A = e_1 - 2e_{23}$ in $Cl_{2,1}$ algebra does not exist what is in agreement with square root absence of this MV in $Cl_{2,1}$. On the other hand, the logarithm of MV $A = 2 + e_1 + e_{13}$ is $\log A = \frac{\log 2}{2} - \frac{1}{\sqrt{2}} \operatorname{artanh}\left(\frac{1}{\sqrt{2}}\right)\left(e_1 + e_{13}\right)$. Multiplication by $\frac{1}{2}$ and exponentiation gives the root A_5 of example 8 in [10], $\sqrt{A} = \frac{1}{2}\left(\sqrt{2-\sqrt{2}}\left(e_1 + e_{13}\right) + \sqrt{2\left(2+\sqrt{2}\right)}\right)$.

Of course, after multiplication of logarithm by any integer or rational number and subsequent exponentiation we can obtain a corresponding power of the MV. For example, in $Cl_{3,0}$ the logarithm of $A = \mathbf{e}_1$ is $\log A = \frac{\pi}{2}\mathbf{e}_1$. Then, it is easy to check that after multiplication by $\frac{1}{3}$ and exponentiation we obtain the cubic root $\sqrt[3]{\mathbf{e}_1} = \frac{1}{2}(\sqrt{3} + \mathbf{e}_1)$.

7. Relations of the logarithm to GA inverse trigonometric and hyperbolic functions

Just like trigonometric and hyperbolic functions can be expressed by exponentials (Euler and de Moivre formulas), the inverse hyperbolic functions may be defined in terms of logarithms. Therefore, in GA we can use the following definitions to compute inverse hyperbolic and trigonometric functions of MV argument A. For hyperbolic inverse functions:

$$\operatorname{artanh} A = \frac{1}{2} (\log(1+A) - \log(1-A)),$$
 (7.1)

$$\operatorname{arcoth} \mathsf{A} = \begin{cases} \frac{1}{2} \left(\log(1 + \mathsf{A}^{-1}) - \log(1 - \mathsf{A}^{-1}) \right), & \mathsf{A} \neq 0, \\ \frac{\pi}{2} I, & \mathsf{A} = 0, \end{cases} \tag{7.2}$$

$$\operatorname{arcosh} A = \log(A + \sqrt{A - 1}\sqrt{A + 1}), \tag{7.3}$$

$$\operatorname{arsinh} A = \log(A + \sqrt{A^2 + 1}). \tag{7.4}$$

For inverse trigonometric functions:

$$\arcsin A = -I \log (AI + \sqrt{1 - A^2}), \tag{7.5}$$

$$\arccos A = \frac{\pi}{2} + I \log \left(AI + \sqrt{1 - A^2} \right), \tag{7.6}$$

$$\arctan A = \frac{I}{2} (\log(1 - IA) - \log(1 + IA)), \tag{7.7}$$

$$\operatorname{arccot} A = \begin{cases} \frac{1}{2} I \left(\log(1 - IA^{-1}) - \log(1 + IA^{-1}) \right), & A \neq 0, \\ \frac{\pi}{2}, & A = 0. \end{cases}$$
 (7.8)

These formulas are similar to those in the theory of real and complex functions except that instead of the imaginary unit the pseudoscalar appears in trigonometric functions. However, earlier we have found [10] that in GA the functions with the square root, in general, are multi-valued. Thus at a first sight it may appear that the listed above equations with square root are not valid in all circumstances. Nonetheless, our preliminary numerical experiments show that they, in fact, are

satisfied for all possible individual plus/minus pairs of square roots⁷ (see Example 8 in [8] and Example 19 below in this section).

With the above formulas for hyperbolic and trigonometric functions one can construct the following identities for generic $MVs:^8$

$$\sinh A = \frac{1}{2} \left(\exp(A) - \exp(-A) \right), \tag{7.9}$$

$$\cosh A = \frac{1}{2} \left(\exp(A) + \exp(-A) \right), \tag{7.10}$$

$$\tanh A = \sinh A (\cosh A)^{-1} = (\exp(A) - \exp(-A))(\exp(A) + \exp(-A))^{-1}, (7.11)$$

$$\coth A = \cosh A (\sinh A)^{-1} = (\exp(A) + \exp(-A))(\exp(A) - \exp(-A))^{-1}. \quad (7.12)$$

$$\sin A = \frac{1}{2}I^{-1}(\exp(IA) - \exp(-IA)),$$
 (7.13)

$$\cos A = \frac{1}{2} \left(\exp(IA) + \exp(-IA) \right), \tag{7.14}$$

$$\tan A = \sin A(\cos A)^{-1} = -I(\exp(IA) - \exp(-IA))(\exp(IA) + \exp(-IA))^{-1},$$
(7.15)

$$\cot A = \cos A(\sin A)^{-1} = I(\exp(IA) + \exp(-IA))(\exp(IA) - \exp(-IA))^{-1}.$$
(7.16)

We have not investigated how the presented formulas work in case when the MV square root or logarithm allows answer that depends on non-discrete free parameters and when the inverse MVs can't be be computed. Also, we have not considered MV logarithms that allow infinite coefficients at some of basis MVs.

Example 19. Inverse MV hyperbolic functions. To save space we will restrict ourselves to numerical examples only for $Cl_{3,0}$ generic MV A = $-1 - 5\mathbf{e}_1 + 7\mathbf{e}_2 - 9\mathbf{e}_3 + 7\mathbf{e}_{12} - 5\mathbf{e}_{13} + 9\mathbf{e}_{23} + 9I$. Then we find the following inverse hyperbolic functions,

$$\begin{array}{lll} \operatorname{artanh}\mathsf{A} = & 0.0544776 & -0.0683983\mathbf{e}_1 & -0.0034179\mathbf{e}_2 & +0.0712752\mathbf{e}_3 \\ & & -0.0259578\mathbf{e}_{12} - 0.0571283\mathbf{e}_{13} + 0.0036554\mathbf{e}_{23} + 1.5447402I, \\ \operatorname{arcoth}\mathsf{A} = & 0.0544776 & -0.0683983\mathbf{e}_1 & -0.0034179\mathbf{e}_2 & -0.0712752\mathbf{e}_3 \\ & & -0.0259578\mathbf{e}_{12} - 0.0571283\mathbf{e}_{13} + 0.0036555\mathbf{e}_{23} - 0.0260523I, \\ \operatorname{arcosh}\mathsf{A} = & 3.1995844 & +0.6349751\mathbf{e}_1 & +0.6477695\mathbf{e}_2 & +0.3396621\mathbf{e}_3 \\ & & +0.9970274\mathbf{e}_{12} + 0.4603461\mathbf{e}_{13} + 0.7081115\mathbf{e}_{23} + 1.0647020I. \end{array}$$

For identities that contain square roots $\sqrt{A \pm 1}$, for example arcosh A or arsinh A, all four roots are valid. Below they have been calculated by algorithm described

⁷This property does not allow us to write the equality sign between GA general expression $\log \sqrt{B}$ and $\frac{1}{2} \log B$.

⁸Trigonometric functions are defined only for algebras where the pseudoscalar 1) belongs to a center of an algebra, i.e. commutes commutative with remaining elements and 2) satisfy $I^2 = -1$. In the considered 3D algebras only for $Cl_{3,0}$ and $Cl_{1,2}$.

in [10],

Root 1 and 2:

$$\sqrt{\mathsf{A}-1} = \pm (-2.3936546 \quad -0.3144420\mathbf{e}_1 \quad -1.3708134\mathbf{e}_2 \quad +0.3806804\mathbf{e}_3 \\ -1.7824116\mathbf{e}_{12} -0.1086429\mathbf{e}_{13} -1.6154750\mathbf{e}_{23} -2.0134421I),$$

Root 3 and 4:

$$\begin{split} \sqrt{\mathsf{A}-1} &= \pm (-0.1660207 \quad +2.4324037 \mathbf{e}_1 \ +1.1337774 \mathbf{e}_2 \ +2.0055931 \mathbf{e}_3 \\ &\quad +2.1654007 \mathbf{e}_{12} +1.9165921 \mathbf{e}_{13} +1.0892769 \mathbf{e}_{23} +1.9243691I). \end{split}$$

And similarly for

$$\sqrt{\mathsf{A}+1} = \pm \{-2.6330243 \quad -0.1908183\mathbf{e}_1 \quad -1.3218252\mathbf{e}_2 \quad +0.4871255\mathbf{e}_3 \\ -1.6829550\mathbf{e}_{12} - 0.0102534\mathbf{e}_{13} -1.5705147\mathbf{e}_{23} -1.9117486I\},$$

$$\sqrt{\mathsf{A}+1} = \pm \{-0.2910283 \quad +2.6047118\mathbf{e}_1 \quad +1.0343473\mathbf{e}_2 \quad +2.2416242\mathbf{e}_3 \\ +2.0981982\mathbf{e}_{12} +2.0727864\mathbf{e}_{13} +0.9499284\mathbf{e}_{23} +1.8337753I\}.$$

It is important to stress that, in general, the individual formulas (arccos A, arcsin A and their hyperbolic analogues) that contain the sets of roots yield different function values for four different roots in the above listed sets.

$$\begin{aligned} \operatorname{arsinh} \mathsf{A} &= \pm (\ \ 3.2035891 \ \ \ +0.6313828 \mathbf{e}_1 \ +0.6490577 \mathbf{e}_2 \ +0.3351515 \mathbf{e}_3 \\ &+0.9974654 \mathbf{e}_{12} +0.4571790 \mathbf{e}_{13} +0.7100715 \mathbf{e}_{23} +1.0647010I). \end{aligned}$$

$$\begin{aligned} \operatorname{arsinh} \mathsf{A} &= \pm (0.4835482 + 2.5061943 \mathbf{e}_{1} - 0.7303556 \mathbf{e}_{2} + 3.0588480 \mathbf{e}_{3} \\ &- 0.0989201 \mathbf{e}_{12} + 2.1904765 \mathbf{e}_{13} - 1.1645414 \mathbf{e}_{23} + 2.5756463I). \end{aligned}$$

Example 20. Inverse trigonometric functions of MV. Numerical answers for $Cl_{3,0}$ generic MV A = $-1 - 5\mathbf{e}_1 + 7\mathbf{e}_2 - 9\mathbf{e}_3 + 7\mathbf{e}_{12} - 5\mathbf{e}_{13} + 9\mathbf{e}_{23} + 9I$ in a form of list for roots 1-4,

$$\begin{aligned} \arcsin \mathsf{A} &= \{ & 2.5745928 & +0.1233316\mathbf{e}_1 - 2.3715122\mathbf{e}_2 + 1.3713947\mathbf{e}_3 \\ & -2.8712504\mathbf{e}_{12} + 0.3732007\mathbf{e}_{13} - 2.8706092\mathbf{e}_{23} - 0.4882339I, \\ & 2.6354984 & +0.7081116\mathbf{e}_1 - 0.4603462\mathbf{e}_2 + 0.9970274\mathbf{e}_3 \\ & -0.3396621\mathbf{e}_{12} + 0.6477695\mathbf{e}_{13} - 0.6349751\mathbf{e}_{23} - 3.1995845I, \\ & +0.5669998 & -0.1233316\mathbf{e}_1 + 2.3715122\mathbf{e}_2 - 1.3713947\mathbf{e}_3 \\ & +2.8712504\mathbf{e}_{12} - 0.3732007\mathbf{e}_{13} + 2.8706092\mathbf{e}_{23} + 0.4882339I, \\ & 0.5060943 & -0.7081116\mathbf{e}_1 + 0.4603462\mathbf{e}_2 - 0.9970274\mathbf{e}_3 \\ & +0.3396621\mathbf{e}_{12} - 0.6477695\mathbf{e}_{13} + 0.6349751\mathbf{e}_{23} + 3.1995845I \}. \end{aligned}$$

Since formulas for arc sine and cosine also include square roots we obtain four different values for these functions too,

```
\begin{split} \arccos \mathsf{A} &= \pm \{-1.0037965 \quad -0.1233316 \mathbf{e}_1 \ +2.3715122 \mathbf{e}_2 \ -1.3713947 \mathbf{e}_3 \\ &\quad +2.8712504 \mathbf{e}_{12} -0.3732007 \mathbf{e}_{13} +2.8706092 \mathbf{e}_{23} +0.4882339I, \\ &\quad -1.0647021 \quad -0.7081116 \mathbf{e}_1 \ +0.4603462 \mathbf{e}_2 \ -0.9970274 \mathbf{e}_3 \\ &\quad +0.3396621 \mathbf{e}_{12} -0.6477695 \mathbf{e}_{13} +0.6349751 \mathbf{e}_{23} +3.1995845I\}. \end{split}
```

On the other hand the trigonometric tangent and cotangent have a single value since the square root here is absent, Eqs (7.7) and (7.8),

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\arctan \mathsf{A} = 1.5171201 + 0.0678435\mathbf{e}_1 + 0.0036019\mathbf{e}_2 + 0.0705863\mathbf{e}_3 \\ + 0.0260071\mathbf{e}_{12} + 0.0566409\mathbf{e}_{13} - 0.0033708\mathbf{e}_{23} + 0.0259164I, \operatorname{arccot} \mathsf{A} = 0.0536762 - 0.0678435\mathbf{e}_1 - 0.0036019\mathbf{e}_2 - 0.0705863\mathbf{e}_3 \\ - 0.0260071\mathbf{e}_{12} - 0.0566409\mathbf{e}_{13} + 0.0033708\mathbf{e}_{23} - 0.0259164I.
```

8. Discussion and conclusions

The logarithm together with the exponential [8, 7] and square root [10] are the most important functions in Clifford geometric algebra (GA). Starting from the respective exponential functions we presented here, as far as we know, for the first time the basis-free formulas for logarithms in all 3D GAs. The formulas for both the generic and special cases may be directly applied in GA programming. They were cross-checked using the basis-free GA exponential functions found in [8]. The derived formulas were implemented in *Mathematica* and tested with thousands of randomly generated multivectors [17]. In all cases the exponentiation of the logarithm was found to simplify to the initial MV.

Using numerical experiments [17] we observed that, in accord with the suggestion in [1], the principal value of the logarithm can be defined as a GA logarithm having the smallest determinant norm. In almost all cases the principal MV logarithm is attained by setting arbitrary integer parameters c_i in generic logarithms (Theorems 3.1, 4.1, 5.1) to zero. Exceptions from this rule, however, may occur in the case of simple specific MVs, for which commuting MVs may exist (Secs. 3.3 and 4.3), and therefore not restricted by free MVs (Eqs (3.13) and (4.13)). Apart from discrete parameters c_i , we have also found that continuous parameters represented by free unit vectors $\hat{\mathbf{u}}$ or bivectors $\hat{\mathcal{U}}$ may be included in special cases as well. The parameters vanish after exponentiation of the logarithm and do not contribute to the MV norm. Recently we have found that such free parameters may be also introduced into lower dimensional, quaternionic-type Clifford algebras [4]. However, more investigations are needed in this direction.

Also, the relation between the GA logarithm and square root of MV was investigated. The known formula $\sqrt{A} = \exp(\frac{1}{2}\log(A))$ served as an additional check correctness of GA logarithms. Unfortunately, the formula allows to compute only a single square root from many possible roots that may exist in GA [10].

Nevertheless, such a comparison was found to be very useful for testing purposes. Indeed, a test of square root of a MV is an algebraic problem since it reduces to a solution of system of algebraic equations. On the other hand, inversion of exponential used in finding the GA logarithm in the present paper requires solving a system of transcendental equations (Appendix A), a problem which is much more difficult (but at the same time more general) task. The mentioned exp-log relation also allows to check the condition whether the MV logarithm exists at all. Indeed, since we know how to calculate GA exponential [10] of arbitrary MV multiplied by factor $\frac{1}{2}$, from this follows that it is log(A) function which determines the existence condition for \sqrt{A} to exist. As a test, we have checked using our algorithm [10] that for each MV indeed there exists a single square root that is in agreement with the identity $\exp(\frac{1}{2}\log(A)) = \sqrt{A}$.

In conclusion, in the present paper the basis-free expressions have been found for GA logarithms in all 3D real algebras. The logarithm was found to exist for all MVs in case of real $Cl_{0,3}$ algebra. In Clifford algebra $Cl_{3,0}$ (and $Cl_{1,2}$) the logarithm exists for almost all MVs, except very small MV class which satisfies the condition $(a_+^2 + a_-^2 = 0) \wedge (a_0^2 + a_{123}^2 = 0)$. For example, the logarithm of MV $\mathbf{e}_1 \pm \mathbf{e}_{12}$ cannot be computed in Euclidean $Cl_{3,0}$ algebra. On the other hand in $Cl_{2,1}$ algebra the GA logarithm is absent in large sectors of a real coefficient space.

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Appendix A. Outline of derivation of generic logarithm formula in a coordinate form in $Cl_{0.3}$

The MV exponential exp $A = B = b_0 + b + B + b_{123}I$, where $b = b_1e_1 + b_2e_2 + b_3e_3$ and $B = b_{12}e_{12} + b_{13}e_{13} + b_{23}e_{23}$, in the coordinate form was constructed in [7]. For completeness, below we reproduce the expressions for scalar coefficients of B,

$$b_{0} = \frac{1}{2} e^{a_{0}} \left(e^{a_{123}} \cos a_{+} + e^{-a_{123}} \cos a_{-} \right),$$

$$b_{123} = \frac{1}{2} e^{a_{0}} \left(e^{a_{123}} \cos a_{+} - e^{-a_{123}} \cos a_{-} \right),$$

$$b_{1} = \frac{1}{2} e^{a_{0}} \left(e^{a_{123}} \left(a_{1} - a_{23} \right) \frac{\sin a_{+}}{a_{+}} + e^{-a_{123}} \left(a_{1} + a_{23} \right) \frac{\sin a_{-}}{a_{-}} \right),$$

$$b_{2} = \frac{1}{2} e^{a_{0}} \left(e^{a_{123}} \left(a_{2} + a_{13} \right) \frac{\sin a_{+}}{a_{+}} + e^{-a_{123}} \left(a_{2} - a_{13} \right) \frac{\sin a_{-}}{a_{-}} \right),$$

$$b_{3} = \frac{1}{2} e^{a_{0}} \left(e^{a_{123}} \left(a_{3} - a_{12} \right) \frac{\sin a_{+}}{a_{+}} + e^{-a_{123}} \left(a_{3} + a_{12} \right) \frac{\sin a_{-}}{a_{-}} \right),$$

$$b_{12} = \frac{1}{2} e^{a_{0}} \left(e^{a_{123}} \left(a_{3} - a_{12} \right) \frac{\sin a_{+}}{a_{+}} + e^{-a_{123}} \left(a_{3} + a_{12} \right) \frac{\sin a_{-}}{a_{-}} \right),$$

$$b_{13} = \frac{1}{2} e^{a_{0}} \left(e^{a_{123}} \left(a_{2} + a_{13} \right) \frac{\sin a_{+}}{a_{+}} - e^{-a_{123}} \left(a_{2} - a_{13} \right) \frac{\sin a_{-}}{a_{-}} \right),$$

$$b_{23} = \frac{1}{2} e^{a_{0}} \left(-e^{a_{123}} \left(a_{1} - a_{23} \right) \frac{\sin a_{+}}{a_{+}} + e^{-a_{123}} \left(a_{1} + a_{23} \right) \frac{\sin a_{-}}{a_{-}} \right),$$

$$(A.3)$$

where a_{-} and a_{+} are defined in Eqs (3.6) and (3.7), respectively. Explicit generic formula for a MV logarithm logB = A can be derived by inverting the equations (A.1)-(A.3). To this end, using (A.1)-(A.3) we will construct a system of eight

nonlinear trigonometric equations for coefficients at corresponding basis elements. Since two of the coefficients, a_0 and a_{123} , stand alone, i.e., they are not associated with remaining ones in (3.6) and (3.7), as a first step in this procedure we partially solve a pair of equations at scalar and pseudoscalar, b_0 and b_{123} , to get two new functions

$$a_0 = q_0'(b_0, b_{123}, a_-, a_+),$$
 (A.4a)

$$a_{123} = g'_{123}(b_0, b_{123}, a_-, a_+),$$
 (A.4b)

where the pair (a_-, a_+) includes only vector and bivector components as (3.6) and Eqs (3.7) show. After substituting new a_0 and a_{123} into remaining equations (A.2) and (A.3) we obtain a system of nonlinear trigonometric equations for coefficients at vector and bivector only which, unfortunately, cannot be solved by computer algebra system in this form. Further progress can be achieved by observing that the following relation between the coefficients holds in the generic case

$$a_{-} = \arctan(b_0 - b_{123}, b_{-}),$$
 (A.5a)

$$a_{+} = \arctan(b_0 + b_{123}, b_{+}),$$
 (A.5b)

where the pair (b_-, b_+) is given by Eqs (3.7) and (3.6) after replacement of \mathbf{a} and \mathcal{A} , by vector \mathbf{b} and bivector \mathcal{B} , respectively. Then, replacing all occurrences of (a_-, a_+) by (A.5a) and (A.5b) in (A.2) and (A.3) (with already replaced a_0 and a_{123} as was described above) we obtain a linear system for coefficients a_i, a_{ij} , that can be solved in a straightforward way. Finally, after substituting a_- and a_+ as given by (A.5a) and (A.5b) into (A.4a) and (A.4b) and performing simplifications we find the scalar coefficients of log $\mathbf{B} = \mathbf{A} = a_0 + \mathbf{a} + \mathcal{A} + a_{123}I$ expressed in terms of two-argument arc tangent functions (see Subsec. 2.1),

$$a_{0} = \frac{1}{4} \left(\log \left((b_{0} - b_{123})^{2} + b_{-}^{2} \right) + \log \left((b_{0} + b_{123})^{2} + b_{+}^{2} \right) \right),$$

$$a_{123} = \frac{1}{4} \left(\log \left((b_{0} + b_{123})^{2} + b_{+}^{2} \right) - \log \left((b_{0} - b_{123})^{2} + b_{-}^{2} \right) \right),$$
(A.6)

$$a_{1} = \frac{1}{2} \left(\frac{(b_{1} + b_{23}) \arctan(b_{0} - b_{123}, b_{-})}{b_{-}} + \frac{(b_{1} - b_{23}) \arctan(b_{0} + b_{123}, b_{+})}{b_{+}} \right),$$

$$a_{2} = \frac{1}{2} \left(\frac{(b_{2} - b_{13}) \arctan(b_{0} - b_{123}, b_{-})}{b_{-}} + \frac{(b_{2} + b_{13}) \arctan(b_{0} + b_{123}, b_{+})}{b_{+}} \right),$$

$$a_{3} = \frac{1}{2} \left(\frac{(b_{3} + b_{12}) \arctan(b_{0} - b_{123}, b_{-})}{b_{-}} + \frac{(b_{3} - b_{12}) \arctan(b_{0} + b_{123}, b_{+})}{b_{+}} \right),$$
(A.7)

$$a_{12} = \frac{1}{2} \left(\frac{(b_3 + b_{12}) \arctan(b_0 - b_{123}, b_-)}{b_-} + \frac{(b_{12} - b_3) \arctan(b_0 + b_{123}, b_+)}{b_+} \right),$$

$$a_{13} = \frac{1}{2} \left(\frac{(b_{13} - b_2) \arctan(b_0 - b_{123}, b_-)}{b_-} + \frac{(b_2 + b_{13}) \arctan(b_0 + b_{123}, b_+)}{b_+} \right),$$

$$a_{23} = \frac{1}{2} \left(\frac{(b_1 + b_{23}) \arctan(b_0 - b_{123}, b_-)}{b_-} + \frac{(b_{23} - b_1) \arctan(b_0 + b_{123}, b_+)}{b_+} \right).$$
(A.8)

Here b_+ and b_- are given by (3.7) and (3.6) after replacement of **a** and \mathcal{A} , respectively, by vector **b** and bivector \mathcal{B} . The equations (A.6)-(A.8) provide a generic solution of the inverse problem for MV logarithm in a coordinate form. It must be remembered that in the two-argument arc tangent functions the argument order and properties follow *Mathematica* convention (see Subsec. 2.1). To have a general GA logarithm formula of Theorem 4.1 in a basis-free form, it is enough to multiply the coefficients in (A.6)-(A.8) by respective basis elements, to add and regroup the resulting GA expression into coordinate-free form.

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