Projective Geometric Algebra $\mathcal{G}_{3,0,1}$

projectivegeometricalgebra.org

Basis Elements

Type	Values	Grade / Antigrade
Scalar	1	0 / 4
Vectors	$\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ $\mathbf{e}_4$	1 / 3
Bivectors	$\mathbf{e}_{23} = \mathbf{e}_2 \wedge \mathbf{e}_3$ $\mathbf{e}_{31} = \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$ $\mathbf{e}_{43} = \mathbf{e}_4 \wedge \mathbf{e}_3$ $\mathbf{e}_{42} = \mathbf{e}_4 \wedge \mathbf{e}_2$ $\mathbf{e}_{41} = \mathbf{e}_4 \wedge \mathbf{e}_1$	2 / 2
Trivectors / Antivectors	$\mathbf{e}_{321} = \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1$ $\mathbf{e}_{412} = \mathbf{e}_4 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2$ $\mathbf{e}_{431} = \mathbf{e}_4 \wedge \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{423} = \mathbf{e}_4 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$	3 / 1
Antiscalar	$\mathbb{1} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$	4 / 0

Exterior Products

Notation	Description
$\mathbf{a} \wedge \mathbf{b}$	• Exterior product • Wedge product • **a** "wedge" **b**
$\mathbf{a} \vee \mathbf{b}$	• Exterior antiproduct • Antiwedge product • **a** "antiwedge" **b**

Inner Products

Notation	Description
$\mathbf{a} \bullet \mathbf{b}$	• Inner product • Dot product • **a** "dot" **b**
$\mathbf{a} \circ \mathbf{b}$	• Inner antiproduct • Antidot product • **a** "antidot" **b**

Geometric Products

Notation	Description
$\mathbf{a} \mathbin{⟑} \mathbf{b}$	• Geometric product • **a** "wedge-dot" **b** • Identity is scalar **1**
$\mathbf{a} \mathbin{⟇} \mathbf{b}$	• Geometric antiproduct • **a** "antiwedge-dot" **b** • Identity is antiscalar $\mathbb{1}$

Commutators

Notation	Definition
$[\mathbf{a},\mathbf{b}]^{⟑}_{-}$	$[\mathbf{a},\mathbf{b}]^{⟑}_{-} = \frac{1}{2}(\mathbf{a} \mathbin{⟑} \mathbf{b} - \mathbf{b} \mathbin{⟑} \mathbf{a})$
$[\mathbf{a},\mathbf{b}]^{⟑}_{+}$	$[\mathbf{a},\mathbf{b}]^{⟑}_{+} = \frac{1}{2}(\mathbf{a} \mathbin{⟑} \mathbf{b} + \mathbf{b} \mathbin{⟑} \mathbf{a})$
$[\mathbf{a},\mathbf{b}]^{⟇}_{-}$	$[\mathbf{a},\mathbf{b}]^{⟇}_{-} = \frac{1}{2}(\mathbf{a} \mathbin{⟇} \mathbf{b} - \mathbf{b} \mathbin{⟇} \mathbf{a})$
$[\mathbf{a},\mathbf{b}]^{⟇}_{+}$	$[\mathbf{a},\mathbf{b}]^{⟇}_{+} = \frac{1}{2}(\mathbf{a} \mathbin{⟇} \mathbf{b} + \mathbf{b} \mathbin{⟇} \mathbf{a})$

Interior Products

Notation	Description	Definition
$\mathbf{a} \vdash \mathbf{b}$	Right interior product	$\mathbf{a} \vdash \mathbf{b} = \mathbf{a} \vee \overline{\mathbf{b}}$
$\mathbf{a} \dashv \mathbf{b}$	Left interior product	$\mathbf{a} \dashv \mathbf{b} = \underline{\mathbf{a}} \vee \mathbf{b}$
$\mathbf{a} \vDash \mathbf{b}$	Right interior antiproduct	$\mathbf{a} \vDash \mathbf{b} = \mathbf{a} \wedge \overline{\mathbf{b}}$
$\mathbf{a} \Dashv \mathbf{b}$	Left interior antiproduct	$\mathbf{a} \Dashv \mathbf{b} = \underline{\mathbf{a}} \wedge \mathbf{b}$

Unary Operations

Basis element $\mathbf{a}$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$\mathbf{e}_{321}$	$\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$\mathbb{1}$
Right complement $\overline{\mathbf{a}}$	$\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_4$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_1$	1
Left complement $\underline{\mathbf{a}}$	$\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$\mathbf{e}_4$	$\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_1$	1
Double complement $\overline{\overline{\mathbf{a}}}$ or $\underline{\underline{\mathbf{a}}}$	1	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_4$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$\mathbb{1}$
Reverse $\tilde{\mathbf{a}}$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$\mathbb{1}$
Antireverse $\underset{\sim}{\mathbf{a}}$	1	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_4$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{321}$	$\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$\mathbb{1}$

Geometric Product $\mathbf{a} \mathbin{⟑} \mathbf{b}$

$\mathbf{a} \mathbin{⟑} \mathbf{b} = \mathbf{a} \wedge \mathbf{b}$ $\mathbf{a} \mathbin{⟑} \mathbf{b} = \mathbf{a} \bullet \mathbf{b}$

a \ b	1	e1	e2	e3	e4	e23	e31	e12	e43	e42	e41	e321	e412	e431	e423	𝟙
1	1	e1	e2	e3	e4	e23	e31	e12	e43	e42	e41	e321	e412	e431	e423	𝟙
e1	e1	1	e12	−e31	−e41	−e321	−e3	e2	e431	−e412	−e4	−e23	−e42	e43	𝟙	e423
e2	e2	−e12	1	e23	−e42	e3	−e321	−e1	−e423	−e4	e412	−e31	e41	𝟙	−e43	e431
e3	e3	e31	−e23	1	−e43	−e2	e1	−e321	−e4	e423	−e431	−e12	𝟙	−e41	e42	e412
e4	e4	e41	e42	e43	0	e423	e431	e412	0	0	0	𝟙	0	0	0	0
e23	e23	−e321	−e3	e2	e423	−1	−e12	e31	e42	−e43	−𝟙	e1	e431	−e412	−e4	e41
e31	e31	e3	−e321	−e1	e431	e12	−1	−e23	−e41	−𝟙	e43	e2	−e423	−e4	e412	e42
e12	e12	−e2	e1	−e321	e412	−e31	e23	−1	−𝟙	e41	−e42	e3	−e4	e423	−e431	e43
e43	e43	e431	−e423	e4	0	−e42	e41	−𝟙	0	0	0	−e412	0	0	0	0
e42	e42	−e412	e4	e423	0	e43	−𝟙	−e41	0	0	0	−e431	0	0	0	0
e41	e41	e4	e412	−e431	0	−𝟙	−e43	e42	0	0	0	−e423	0	0	0	0
e321	e321	−e23	−e31	−e12	−𝟙	e1	e2	e3	e412	e431	e423	−1	−e43	−e42	−e41	e4
e412	e412	−e42	e41	−𝟙	0	−e431	e423	−e4	0	0	0	e43	0	0	0	0
e431	e431	e43	−𝟙	−e41	0	e412	−e4	−e423	0	0	0	e42	0	0	0	0
e423	e423	−𝟙	−e43	e42	0	−e4	−e412	e431	0	0	0	e41	0	0	0	0
𝟙	𝟙	−e423	−e431	−e412	0	e41	e42	e43	0	0	0	−e4	0	0	0	0

Geometric Antiproduct $\mathbf{a} \mathbin{⟇} \mathbf{b}$

$\mathbf{a} \mathbin{⟇} \mathbf{b} = \mathbf{a} \vee \mathbf{b}$ $\mathbf{a} \mathbin{⟇} \mathbf{b} = \mathbf{a} \circ \mathbf{b}$

a \ b	1	e1	e2	e3	e4	e23	e31	e12	e43	e42	e41	e321	e412	e431	e423	𝟙
1	0	0	0	0	e321	0	0	0	e12	e31	e23	0	e3	e2	e1	1
e1	0	0	0	0	−e23	0	0	0	−e2	e3	−e321	0	e31	−e12	1	e1
e2	0	0	0	0	−e31	0	0	0	e1	−e321	−e3	0	−e23	1	e12	e2
e3	0	0	0	0	−e12	0	0	0	−e321	−e1	e2	0	1	e23	−e31	e3
e4	−e321	e23	e31	e12	−𝟙	−e1	−e2	−e3	e412	e431	e423	1	−e43	−e42	−e41	e4
e23	0	0	0	0	e1	0	0	0	−e31	e12	−1	0	−e2	e3	−e321	e23
e31	0	0	0	0	e2	0	0	0	e23	−1	−e12	0	e1	−e321	−e3	e31
e12	0	0	0	0	e3	0	0	0	−1	−e23	e31	0	−e321	−e1	e2	e12
e43	e12	e2	−e1	−e321	e412	e31	−e23	−1	−𝟙	−e41	e42	e3	−e4	−e423	e431	e43
e42	e31	−e3	−e321	e1	e431	−e12	−1	e23	e41	−𝟙	−e43	e2	e423	−e4	−e412	e42
e41	e23	−e321	e3	−e2	e423	−1	e12	−e31	−e42	e43	−𝟙	e1	−e431	e412	−e4	e41
e321	0	0	0	0	−1	0	0	0	e3	e2	e1	0	−e12	−e31	−e23	e321
e412	−e3	e31	−e23	−1	−e43	−e2	e1	e321	−e4	−e423	e431	e12	𝟙	e41	−e42	e412
e431	−e2	−e12	−1	e23	−e42	e3	e321	−e1	e423	−e4	−e412	e31	−e41	𝟙	e43	e431
e423	−e1	−1	e12	−e31	−e41	e321	−e3	e2	−e431	e412	−e4	e23	e42	−e43	𝟙	e423
𝟙	1	e1	e2	e3	e4	e23	e31	e12	e43	e42	e41	e321	e412	e431	e423	𝟙

Interior Products

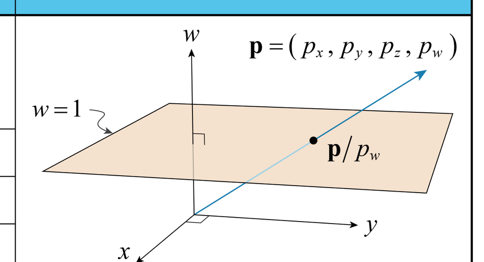
$\mathbf{a} \vdash \mathbf{b}$ $\mathbf{a} \dashv \mathbf{b}$

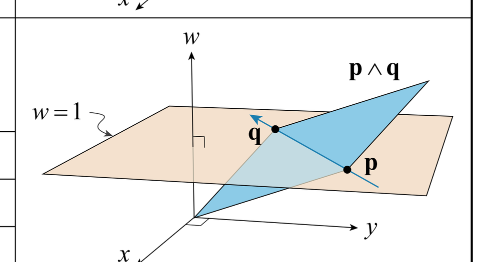
a \ b	1	e1	e2	e3	e4	e23	e31	e12	e43	e42	e41	e321	e412	e431	e423	𝟙
1	1	e1	e2	e3	e4	e23	e31	e12	e43	e42	e41	e321	e412	e431	e423	𝟙
e1	e1	1	0	0	0	0	e3	−e2	0	0	e4	−e23	−e42	e43	0	−e423
e2	e2	0	1	0	0	−e3	0	e1	0	e4	0	−e31	e41	0	−e43	−e431
e3	e3	0	0	1	0	e2	−e1	0	e4	0	0	−e12	0	−e41	e42	−e412
e4	e4	0	0	0	1	0	0	0	−e3	−e2	−e1	0	e12	e31	e23	−e321
e23	e23	0	e3	−e2	0	1	0	0	0	0	0	−e1	0	0	e4	−e41
e31	e31	−e3	0	e1	0	0	1	0	0	0	0	−e2	0	e4	0	−e42
e12	e12	e2	−e1	0	0	0	0	1	0	0	0	−e3	e4	0	0	−e43
e43	e43	0	0	−e4	e3	0	0	0	1	0	0	0	0	e1	−e2	−e12
e42	e42	0	−e4	0	e2	0	0	0	0	1	0	0	−e1	0	e3	−e31
e41	e41	−e4	0	0	e1	0	0	0	0	0	1	0	−e3	e2	0	−e23
e321	e321	−e23	−e31	−e12	0	−e1	−e2	−e3	0	0	0	1	0	0	0	e4
e412	e412	−e42	e41	0	e12	0	0	e4	0	−e1	e2	0	1	0	0	e3
e431	e431	e43	0	−e41	e31	0	e4	0	e1	0	−e3	0	0	1	0	e2
e423	e423	0	−e43	e42	e23	e4	0	0	−e2	e3	0	0	0	0	1	e1
𝟙	𝟙	e423	e431	e412	e321	−e41	−e42	−e43	−e12	−e31	−e23	−e4	−e3	−e2	−e1	1

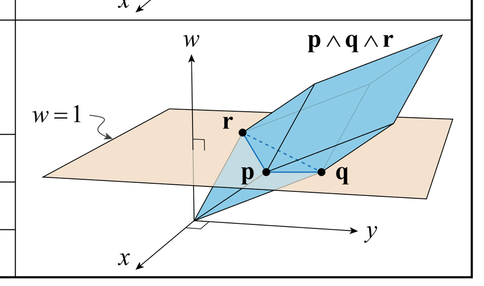
Projective Geometries

Type	Representation	
Point $\mathbf{p}$ (Vector) **0D**	$\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4$	
	Bulk	$\mathbf{p}_\bullet = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$
	Weight	$\mathbf{p}_\circ = p_w\mathbf{e}_4$
	Unitization	$p_w^2 = 1$
Line $\mathbf{L}$ (Bivector) **1D**	$\mathbf{L} = v_x\mathbf{e}_{41} + v_y\mathbf{e}_{42} + v_z\mathbf{e}_{43} + m_x\mathbf{e}_{23} + m_y\mathbf{e}_{31} + m_z\mathbf{e}_{12}$ $\mathbf{L} \mathbin{⟑} \tilde{\mathbf{L}} = \mathbf{L} \bullet \tilde{\mathbf{L}} \implies v_x m_x + v_y m_y + v_z m_z = 0$	
	Bulk	$\mathbf{L}_\bullet = m_x\mathbf{e}_{23} + m_y\mathbf{e}_{31} + m_z\mathbf{e}_{12}$ (moment)
	Weight	$\mathbf{L}_\circ = v_x\mathbf{e}_{41} + v_y\mathbf{e}_{42} + v_z\mathbf{e}_{43}$ (direction)
	Unitization	$v_x^2 + v_y^2 + v_z^2 = 1$
Plane $\mathbf{f}$ (Trivector) **2D**	$\mathbf{f} = f_x\mathbf{e}_{423} + f_y\mathbf{e}_{431} + f_z\mathbf{e}_{412} + f_w\mathbf{e}_{321}$	
	Bulk	$\mathbf{f}_\bullet = f_w\mathbf{e}_{321}$
	Weight	$\mathbf{f}_\circ = f_x\mathbf{e}_{423} + f_y\mathbf{e}_{431} + f_z\mathbf{e}_{412}$ (normal)
	Unitization	$f_x^2 + f_y^2 + f_z^2 = 1$

Illustration







Bulk and Weight

Notation	Definition
$\mathbf{a}_\bullet$	Bulk of element **a**. All components without factor of $\mathbf{e}_4$. $1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{12}, \mathbf{e}_{321}$
$\mathbf{a}_\circ$	Weight of element **a**. All components with factor of $\mathbf{e}_4$. $\mathbf{e}_4, \mathbf{e}_{41}, \mathbf{e}_{42}, \mathbf{e}_{43}, \mathbf{e}_{423}, \mathbf{e}_{431}, \mathbf{e}_{412}, \mathbb{1}$

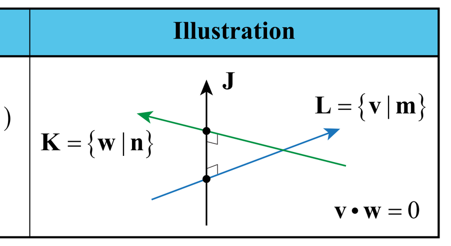
Dualization

$\overline{\mathbf{a}_\bullet} = \tilde{\mathbf{a}} \mathbin{⟑} \mathbb{1}$	Bulk right complement of **a**.
$\underline{\mathbf{a}_\bullet} = \underset{\sim}{\mathbf{a}} \mathbin{⟑} \mathbb{1}$	Bulk left complement of **a**.
$\overline{\mathbf{a}_\circ} = \mathbf{1} \mathbin{⟇} \tilde{\mathbf{a}}$	Weight right complement of **a**.
$\underline{\mathbf{a}_\circ} = \mathbf{1} \mathbin{⟇} \underset{\sim}{\mathbf{a}}$	Weight left complement of **a**.
$\overline{\mathbf{a}} = \tilde{\mathbf{a}} \mathbin{⟑} \mathbb{1} + \mathbf{1} \mathbin{⟇} \tilde{\mathbf{a}}$	Right complement of **a**.
$\underline{\mathbf{a}} = \underset{\sim}{\mathbf{a}} \mathbin{⟑} \mathbb{1} + \mathbf{1} \mathbin{⟇} \underset{\sim}{\mathbf{a}}$	Left complement of **a**.

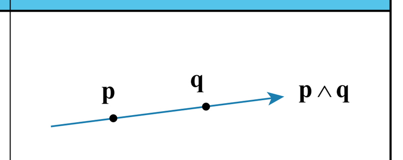
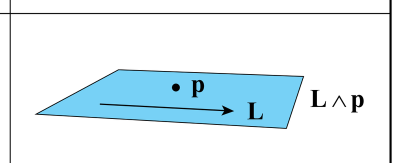
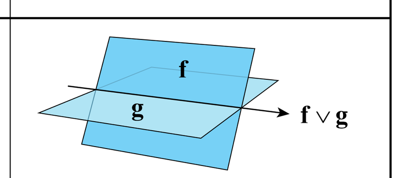
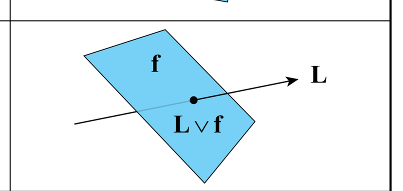
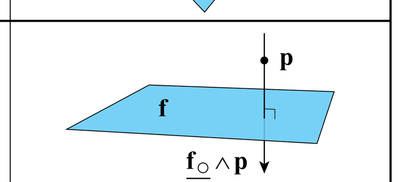
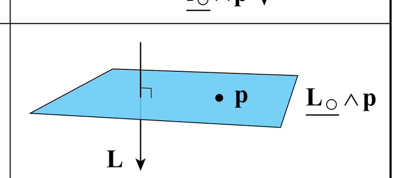
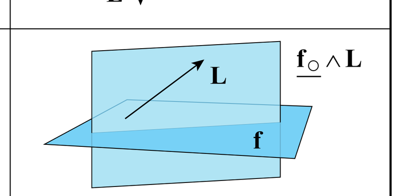
Attitude Extraction

Formula	Interpretation
$\underline{\mathbf{p}_\circ} = -p_w\mathbf{e}_{321}$	Plane at infinity.
$\underline{\mathbf{L}_\circ} = -v_x\mathbf{e}_{23} - v_y\mathbf{e}_{31} - v_z\mathbf{e}_{12}$	Line at infinity perpendicular to line **L**.
$\underline{\mathbf{f}_\circ} = f_x\mathbf{e}_1 + f_y\mathbf{e}_2 + f_z\mathbf{e}_3$	Normal vector of plane **f**.
$\underline{\mathbf{Q}_\circ} = -r_x\mathbf{e}_{23} - r_y\mathbf{e}_{31} - r_z\mathbf{e}_{12} + r_w$	Quaternion, directional part of motor **Q**. $\mathbf{q} = (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})\sin\phi + \cos\phi$ $\mathbf{q} \mathbin{⟑} \mathbf{x} \mathbin{⟑} \tilde{\mathbf{q}}$ = rotation about origin.
$\underline{\mathbf{G}_\circ} = h_x\mathbf{e}_1 + h_y\mathbf{e}_2 + h_z\mathbf{e}_3 - s_w\mathbf{e}_{321}$	Directional part of flector **G**. $\mathbf{g} = (a_x\mathbf{e}_1 + a_y\mathbf{e}_2 + a_z\mathbf{e}_3)\cos\phi + \mathbf{e}_{321}\sin\phi$ $-\mathbf{g} \mathbin{⟑} \mathbf{x} \mathbin{⟑} \tilde{\mathbf{g}}$ = rotoreflection about origin.

Skew Lines

Formula	Illustration
$\mathbf{J} = [\mathbf{L},\mathbf{K}]^{⟇}_{-} = (v_y w_z - v_z w_y)\mathbf{e}_{41}$ $+ (v_z w_x - v_x w_z)\mathbf{e}_{42} + (v_x w_y - v_y w_z\mathbf{e}_{43})$ $+ (v_y n_z - v_z n_y + m_y w_z - m_z w_y)\mathbf{e}_{23}$ $+ (v_z n_x - v_x n_z + m_z w_x - m_x w_z)\mathbf{e}_{31}$ $+ (v_x n_y - v_y n_x + m_x w_y - m_y w_x)\mathbf{e}_{12}$	

Join and Meet

Formula	Description	Illustration
$\mathbf{p} \wedge \mathbf{q} = (q_x p_w - p_x q_w)\mathbf{e}_{41} + (q_y p_w - p_y q_w)\mathbf{e}_{42} + (q_z p_w - p_z q_w)\mathbf{e}_{43}$ $+ (p_y q_z - p_z q_y)\mathbf{e}_{23} + (p_z q_x - p_x q_z)\mathbf{e}_{31} + (p_x q_y - p_y q_x)\mathbf{e}_{12}$ $= [\mathbf{p},\mathbf{q}]^{⟑}_{-}$	Line containing points **p** and **q**. Zero if **p** and **q** are coincident.	
$\mathbf{L} \wedge \mathbf{p} = (v_y p_z - v_z p_y + m_x p_w)\mathbf{e}_{423} + (v_z p_x - v_x p_z + m_y p_w)\mathbf{e}_{431}$ $+ (v_x p_y - v_y p_x + m_z p_w)\mathbf{e}_{412} - (m_x p_x + m_y p_y + m_z p_z)\mathbf{e}_{321}$ $= [\mathbf{L},\mathbf{p}]^{⟑}_{+}$	Plane containing line **L** and point **p**. Normal is zero if **p** lies in **L**.	
$\mathbf{f} \vee \mathbf{g} = (f_z g_y - f_y g_z)\mathbf{e}_{41} + (f_x g_z - f_z g_x)\mathbf{e}_{42} + (f_y g_x - f_x g_y)\mathbf{e}_{43}$ $+ (f_x g_w - g_x f_w)\mathbf{e}_{23} + (f_y g_w - g_y f_w)\mathbf{e}_{31} + (f_z g_w - g_z f_w)\mathbf{e}_{12}$ $= [\mathbf{f},\mathbf{g}]^{⟇}_{-}$	Line where planes **f** and **g** intersect. Direction is zero if **f** and **g** are parallel.	
$\mathbf{L} \vee \mathbf{f} = (m_y f_z - m_z f_y + v_x f_w)\mathbf{e}_1 + (m_z f_x - m_x f_z + v_y f_w)\mathbf{e}_2$ $+ (m_x f_y - m_y f_x + v_z f_w)\mathbf{e}_3 - (v_x f_x + v_y f_y + v_z f_z)\mathbf{e}_4$ $= [\mathbf{L},\mathbf{f}]^{⟇}_{+}$	Point where line **L** intersects plane **f**. Weight is zero if **L** and **f** are parallel.	
$\underline{\mathbf{f}_\circ} \wedge \mathbf{p} = -f_x p_w\mathbf{e}_{41} - f_y p_w\mathbf{e}_{42} - f_z p_w\mathbf{e}_{43}$ $+ (f_y p_z - f_z p_y)\mathbf{e}_{23} + (f_z p_x - f_x p_z)\mathbf{e}_{31} + (f_x p_y - f_y p_x)\mathbf{e}_{12}$ $= [\mathbf{p},\mathbf{f}]^{⟇}_{+}$	Line perpendicular to plane **f** and passing through point **p**.	
$\underline{\mathbf{L}_\circ} \wedge \mathbf{p} = -v_x p_w\mathbf{e}_{423} - v_y p_w\mathbf{e}_{431} - v_z p_w\mathbf{e}_{412}$ $+ (v_x p_x + v_y p_y + v_z p_z)\mathbf{e}_{321}$ $= -[\mathbf{p},\mathbf{L}]^{⟇}_{+}$	Plane perpendicular to line **L** and containing point **p**.	
$\underline{\mathbf{f}_\circ} \wedge \mathbf{L} = (v_y f_z - v_z f_y)\mathbf{e}_{423} + (v_z f_x - v_x f_z)\mathbf{e}_{431} + (v_x f_y - v_y f_x)\mathbf{e}_{412}$ $- (m_x f_x + m_y f_y + m_z f_z)\mathbf{e}_{321}$ $= [\mathbf{L},\mathbf{f}]^{⟇}_{-}$	Plane perpendicular to plane **f** and containing line **L**. Zero if **L** is perpendicular to **f**.	

Motors (Rigid Motion Operators)

Representation of Proper Euclidean Isometry

$$\mathbf{Q} = r_x\mathbf{e}_{41} + r_y\mathbf{e}_{42} + r_z\mathbf{e}_{43} + r_w\mathbb{1} + u_x\mathbf{e}_{23} + u_y\mathbf{e}_{31} + u_z\mathbf{e}_{12} + u_w$$

$$\mathbf{Q} \mathbin{⟑} \tilde{\mathbf{Q}} = \mathbf{Q} \bullet \tilde{\mathbf{Q}} \implies r_x u_x + r_y u_y + r_z u_z + r_w u_w = 0$$

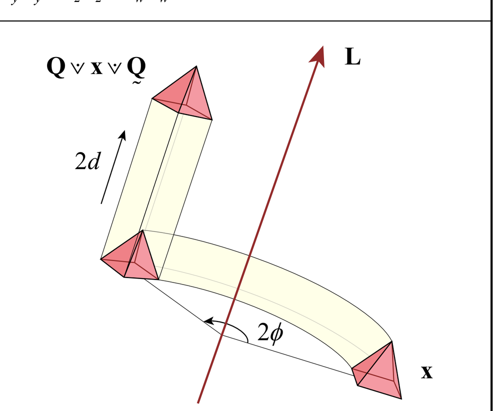
Bulk	$\mathbf{Q}_\bullet = u_x\mathbf{e}_{23} + u_y\mathbf{e}_{31} + u_z\mathbf{e}_{12} + u_w$
Weight	$\mathbf{Q}_\circ = r_x\mathbf{e}_{41} + r_y\mathbf{e}_{42} + r_z\mathbf{e}_{43} + r_w\mathbb{1}$
Unitization	$r_x^2 + r_y^2 + r_z^2 + r_w^2 = 1$

$\mathbf{Q} = e_{\vee}^{(d+\phi\mathbb{1})\vee\mathbf{L}} = \cos_{\vee}(d+\phi\mathbb{1}) + \sin_{\vee}(d+\phi\mathbb{1}) \vee \mathbf{L}$

$\mathbf{Q} = \mathbf{L}\sin\phi + \mathbb{1}\cos\phi + (d \vee \mathbf{L})\cos\phi - d\sin\phi$

$\mathbf{Q} = (v_x\mathbf{e}_{41} + v_y\mathbf{e}_{42} + v_z\mathbf{e}_{43})\sin\phi + \mathbb{1}\cos\phi - d\sin\phi$
$+ (dv_x\mathbf{e}_{23} + dv_y\mathbf{e}_{31} + dv_z\mathbf{e}_{12})\cos\phi$
$+ (m_x\mathbf{e}_{23} + m_y\mathbf{e}_{31} + m_z\mathbf{e}_{12})\sin\phi$

$\mathbf{Q} \mathbin{⟇} \mathbf{x} \mathbin{⟇} \underset{\sim}{\mathbf{Q}}$ rotates object **x** through angle 2ϕ about line **L** and translates by distance $2d$ along direction of line **L**.



$$\sqrt{\mathbf{Q}} = \frac{\mathbf{Q} + \mathbb{1}}{\sqrt{2 + 2Q_{\mathbb{1}}}} \mathbin{⟇} \left(\mathbb{1} - \frac{Q_{\mathbf{1}}}{2 + 2Q_{\mathbb{1}}}\right)$$

Simplified Motors

$\mathbf{R} = (a_x\mathbf{e}_{41} + a_y\mathbf{e}_{42} + a_z\mathbf{e}_{43})\sin\phi + \mathbb{1}\cos\phi$	$\mathbf{R} \mathbin{⟇} \mathbf{x} \mathbin{⟇} \underset{\sim}{\mathbf{R}}$ rotates object **x** through angle 2ϕ about unit axis **a** through origin.
$\mathbf{S} = \mathbf{L}\sin\phi + \mathbb{1}\cos\phi$	$\mathbf{S} \mathbin{⟇} \mathbf{x} \mathbin{⟇} \underset{\sim}{\mathbf{S}}$ rotates object **x** through angle 2ϕ about unitized line **L**.
$\mathbf{T} = t_x\mathbf{e}_{23} + t_y\mathbf{e}_{31} + t_z\mathbf{e}_{12} + \mathbb{1}$	$\mathbf{T} \mathbin{⟇} \mathbf{x} \mathbin{⟇} \underset{\sim}{\mathbf{T}}$ translates object **x** by vector $2\mathbf{t}$.
Line **L**	$\mathbf{L} \mathbin{⟇} \mathbf{x} \mathbin{⟇} \underset{\sim}{\mathbf{L}}$ rotates object **x** through 180° about line **L**.

Matrix Conversion $\mathbf{M} = \mathbf{A} + \mathbf{B}$ $\mathbf{M}^{-1} = \mathbf{A} - \mathbf{B}$

$$\mathbf{A} = \begin{bmatrix} 1-2(r_y^2+r_z^2) & 2r_x r_y & 2r_z r_x & 2(r_y u_z - r_z u_y) \\ 2r_x r_y & 1-2(r_z^2+r_x^2) & 2r_y r_z & 2(r_z u_x - r_x u_z) \\ 2r_z r_x & 2r_y r_z & 1-2(r_x^2+r_y^2) & 2(r_x u_y - r_y u_x) \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & -2r_z r_w & 2r_y r_w & 2(r_w u_x - r_x u_w) \\ 2r_z r_w & 0 & -2r_x r_w & 2(r_w u_y - r_y u_w) \\ -2r_y r_w & 2r_x r_w & 0 & 2(r_w u_z - r_z u_w) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Flectors (Reflection Operators)

Representation of Improper Euclidean Isometry

$$\mathbf{G} = s_x\mathbf{e}_1 + s_y\mathbf{e}_2 + s_z\mathbf{e}_3 + s_w\mathbf{e}_4 + h_x\mathbf{e}_{423} + h_y\mathbf{e}_{431} + h_z\mathbf{e}_{412} + h_w\mathbf{e}_{321}$$

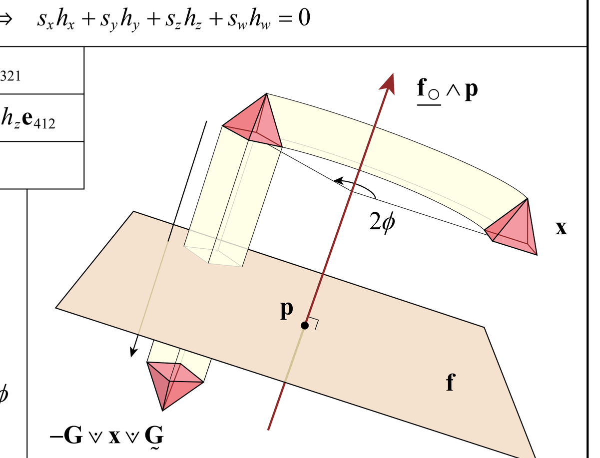
$$\mathbf{G} \mathbin{⟑} \tilde{\mathbf{G}} = \mathbf{G} \bullet \tilde{\mathbf{G}} \implies s_x h_x + s_y h_y + s_z h_z + s_w h_w = 0$$

Bulk	$\mathbf{G}_\bullet = s_x\mathbf{e}_1 + s_y\mathbf{e}_2 + s_z\mathbf{e}_3 + h_w\mathbf{e}_{321}$
Weight	$\mathbf{G}_\circ = s_w\mathbf{e}_4 + h_x\mathbf{e}_{423} + h_y\mathbf{e}_{431} + h_z\mathbf{e}_{412}$
Unitization	$h_x^2 + h_y^2 + h_z^2 + s_w^2 = 1$

$\mathbf{G} = \mathbf{p}\sin\phi + \mathbf{f}\cos\phi$

$\mathbf{G} = (p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4)\sin\phi$
$+ (f_x\mathbf{e}_{423} + f_y\mathbf{e}_{431} + f_z\mathbf{e}_{412} + f_w\mathbf{e}_{321})\cos\phi$

$-\mathbf{G} \mathbin{⟇} \mathbf{x} \mathbin{⟇} \underset{\sim}{\mathbf{G}}$ rotates object **x** through angle 2ϕ about line perpendicular to plane **f** passing through point **p** and reflects across plane **f**.



Simplified Flectors

$\mathbf{H} = s_x\mathbf{e}_1 + s_y\mathbf{e}_2 + s_z\mathbf{e}_3 + \mathbf{h}$	$-\mathbf{H} \mathbin{⟇} \mathbf{x} \mathbin{⟇} \underset{\sim}{\mathbf{H}}$ translates object **x** by vector $2(h_x, h_y, h_z) \times (s_x, s_y, s_z)$ and reflects across unitized plane **h** (transflection).
Point **p**	$-\mathbf{p} \mathbin{⟇} \mathbf{x} \mathbin{⟇} \underset{\sim}{\mathbf{p}}$ inverts object **x** through point **p**.
Plane **f**	$-\mathbf{f} \mathbin{⟇} \mathbf{x} \mathbin{⟇} \underset{\sim}{\mathbf{f}}$ reflects object **x** across plane **f**.

Matrix Conversion $\mathbf{M} = \mathbf{A} + \mathbf{B}$ $\mathbf{M}^{-1} = \mathbf{A} - \mathbf{B}$

$$\mathbf{A} = \begin{bmatrix} 2(h_y^2+h_z^2)-1 & -2h_x h_y & -2h_z h_x & 2(s_x s_w - h_x h_w) \\ -2h_x h_y & 2(h_z^2+h_x^2)-1 & -2h_y h_z & 2(s_y s_w - h_y h_w) \\ -2h_z h_x & -2h_y h_z & 2(h_x^2+h_y^2)-1 & 2(s_z s_w - h_z h_w) \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 2h_z s_w & -2h_y s_w & 2(h_y s_z - h_z s_y) \\ -2h_z s_w & 0 & 2h_x s_w & 2(h_z s_x - h_x s_z) \\ 2h_y s_w & -2h_x s_w & 0 & 2(h_x s_y - h_y s_x) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Norms

Definitions

$\lVert\mathbf{a}\rVert_\bullet = \sqrt{\mathbf{a} \bullet \tilde{\mathbf{a}}}$ Bulk norm of **a**.	$\lVert\mathbf{a}\rVert_\circ = \sqrt{\mathbf{a} \circ \underset{\sim}{\mathbf{a}}}$ Weight norm of **a**.	$\lVert\widehat{\mathbf{a}}\rVert = \dfrac{\lVert\mathbf{a}\rVert_\bullet}{\lVert\mathbf{a}\rVert_\circ} = \dfrac{\sqrt{\mathbf{a} \bullet \tilde{\mathbf{a}}}}{\sqrt{\mathbf{a} \circ \underset{\sim}{\mathbf{a}}}}$ Projected geometric norm of **a**.
$\lVert\mathbf{a}\rVert = \lVert\mathbf{a}\rVert_\bullet + \lVert\mathbf{a}\rVert_\circ = \sqrt{\mathbf{a} \bullet \tilde{\mathbf{a}}} + \sqrt{\mathbf{a} \circ \underset{\sim}{\mathbf{a}}}$ Geometric norm of **a**.		

Type	Bulk and Weight Norms	Projected Geometric Norm	Interpretation
Point **p**	$\lVert\mathbf{p}\rVert_\bullet = \sqrt{p_x^2 + p_y^2 + p_z^2}$ $\lVert\mathbf{p}\rVert_\circ = \lvert p_w\rvert\mathbb{1}$	$\lVert\widehat{\mathbf{p}}\rVert = \dfrac{\sqrt{p_x^2 + p_y^2 + p_z^2}}{\lvert p_w\rvert}$	Distance from origin to point **p**. Half distance that origin is moved by flector **p**.
Line **L**	$\lVert\mathbf{L}\rVert_\bullet = \sqrt{m_x^2 + m_y^2 + m_z^2}$ $\lVert\mathbf{L}\rVert_\circ = \mathbb{1}\sqrt{v_x^2 + v_y^2 + v_z^2}$	$\lVert\widehat{\mathbf{L}}\rVert = \sqrt{\dfrac{m_x^2 + m_y^2 + m_z^2}{v_x^2 + v_y^2 + v_z^2}}$	Perpendicular distance from origin to line **L**. Half distance that origin is moved by motor **L**.
Plane **f**	$\lVert\mathbf{f}\rVert_\bullet = \lvert f_w\rvert$ $\lVert\mathbf{f}\rVert_\circ = \mathbb{1}\sqrt{f_x^2 + f_y^2 + f_z^2}$	$\lVert\widehat{\mathbf{f}}\rVert = \dfrac{\lvert f_w\rvert}{\sqrt{f_x^2 + f_y^2 + f_z^2}}$	Perpendicular distance from origin to plane **f**. Half distance that origin is moved by flector **f**.
Motor **Q**	$\lVert\mathbf{Q}\rVert_\bullet = \sqrt{u_x^2 + u_y^2 + u_z^2 + u_w^2}$ $\lVert\mathbf{Q}\rVert_\circ = \mathbb{1}\sqrt{r_x^2 + r_y^2 + r_z^2 + r_w^2}$	$\lVert\widehat{\mathbf{Q}}\rVert = \sqrt{\dfrac{u_x^2 + u_y^2 + u_z^2 + u_w^2}{r_x^2 + r_y^2 + r_z^2 + r_w^2}}$	Half distance that origin is moved by motor **Q**.
Flector **G**	$\lVert\mathbf{G}\rVert_\bullet = \sqrt{s_x^2 + s_y^2 + s_z^2 + h_w^2}$ $\lVert\mathbf{G}\rVert_\circ = \mathbb{1}\sqrt{h_x^2 + h_y^2 + h_z^2 + s_w^2}$	$\lVert\widehat{\mathbf{G}}\rVert = \sqrt{\dfrac{s_x^2 + s_y^2 + s_z^2 + h_w^2}{h_x^2 + h_y^2 + h_z^2 + s_w^2}}$	Half distance that origin is moved by flector **G**.

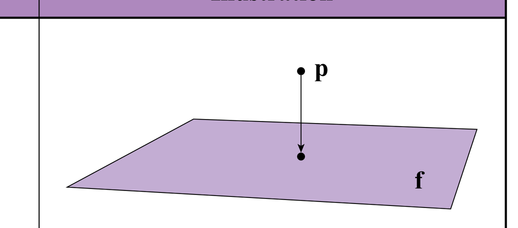
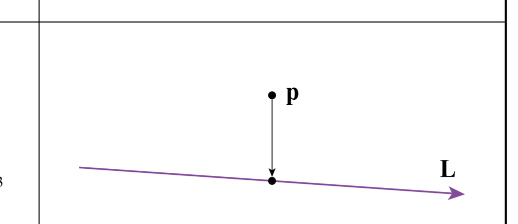
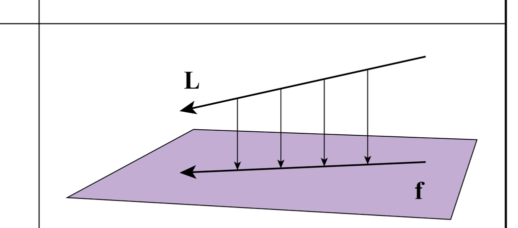
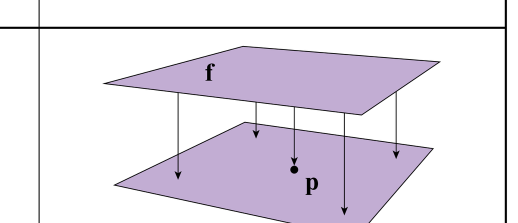
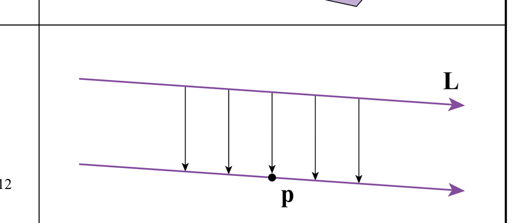
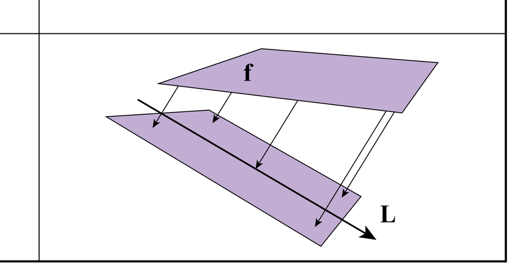
Euclidean Distances

Formula	Interpretation
$\dfrac{\lVert[\mathbf{p},\mathbf{q}]^{⟑}_{-}\rVert_\circ}{\lVert[\mathbf{p},\mathbf{q}]^{⟇}_{+}\rVert_\circ} = \dfrac{\sqrt{(q_x p_w - p_x q_w)^2 + (q_y p_w - p_y q_w)^2 + (q_z p_w - p_z q_w)^2}}{\lvert p_w q_w\rvert}$	Distance between points **p** and **q**.
$\dfrac{\lVert[\mathbf{p},\mathbf{L}]^{⟑}_{+}\rVert_\circ}{\lVert[\mathbf{p},\mathbf{L}]^{⟇}_{+}\rVert_\circ} = \dfrac{\sqrt{(v_y p_z - v_z p_y + m_x p_w)^2 + (v_z p_x - v_x p_z + m_y p_w)^2 + (v_x p_y - v_y p_x + m_z p_w)^2}}{\lvert p_w\rvert\sqrt{v_x^2 + v_y^2 + v_z^2}}$	Perpendicular distance between point **p** and line **L**.
$\dfrac{\lVert[\mathbf{p},\mathbf{f}]^{⟑}_{-}\rVert_\circ}{\lVert[\mathbf{p},\mathbf{f}]^{⟇}_{+}\rVert_\circ} = \dfrac{\lvert p_x f_x + p_y f_y + p_z f_z + p_w f_w\rvert}{\lvert p_w\rvert\sqrt{f_x^2 + f_y^2 + f_z^2}}$	Perpendicular distance between point **p** and plane **f**.
$\dfrac{\lVert[\mathbf{L},\mathbf{K}]^{⟑}_{+}\rVert_\circ}{\lVert[\mathbf{L},\mathbf{K}]^{⟇}_{-}\rVert_\circ} = \dfrac{\lvert v_x n_x + v_y n_y + v_z n_z + w_x m_x + w_y m_y + w_z m_z\rvert}{\sqrt{(v_y w_z - v_z w_y)^2 + (v_z w_x - v_x w_z)^2 + (v_x w_y - v_y w_x)^2}}$ $\mathbf{L} = \{\mathbf{v} \mid \mathbf{m}\}$ $\mathbf{K} = \{\mathbf{w} \mid \mathbf{n}\}$	Perpendicular distance between lines **L** and **K**.

Projections

Operations

$(\mathbf{b}_\circ \vdash \mathbf{a}) \dashv \mathbf{b} = (\underline{\mathbf{b}_\circ} \wedge \mathbf{a}) \vee \mathbf{b}$	General projection of **a** onto **b**.
$(\mathbf{b}_\circ \vDash \mathbf{a}) \Dashv \mathbf{b} = (\underline{\mathbf{b}_\circ} \vee \mathbf{a}) \wedge \mathbf{b}$	General antiprojection of **a** onto **b**.

Formula	Illustration
Projection of point **p** onto plane **f**. $(\underline{\mathbf{f}_\circ} \wedge \mathbf{p}) \vee \mathbf{f} = (f_x^2 + f_y^2 + f_z^2)\mathbf{p}$ $- (f_x p_x + f_y p_y + f_z p_z + f_w p_w)(f_x\mathbf{e}_1 + f_y\mathbf{e}_2 + f_z\mathbf{e}_3)$	
Projection of point **p** onto line **L**. $(\underline{\mathbf{L}_\circ} \wedge \mathbf{p}) \vee \mathbf{L} = (v_x p_x + v_y p_y + v_z p_z)\mathbf{v} + (v_x^2 + v_y^2 + v_z^2)p_w\mathbf{e}_4$ $+ (v_y m_z - v_z m_y)p_w\mathbf{e}_1 + (v_z m_x - v_x m_z)p_w\mathbf{e}_2 + (v_x m_y - v_y m_x)p_w\mathbf{e}_3$	
Projection of line **L** onto plane **f**. $(\underline{\mathbf{f}_\circ} \wedge \mathbf{L}) \vee \mathbf{f} = (f_x^2 + f_y^2 + f_z^2)(v_x\mathbf{e}_{41} + v_y\mathbf{e}_{42} + v_z\mathbf{e}_{43})$ $- (f_x v_x + f_y v_y + f_z v_z)(f_x\mathbf{e}_{41} + f_y\mathbf{e}_{42} + f_z\mathbf{e}_{43})$ $+ (f_x m_x + f_y m_y + f_z m_z)(f_x\mathbf{e}_{23} + f_y\mathbf{e}_{31} + f_z\mathbf{e}_{12})$ $+ (v_y f_z - v_z f_y)f_w\mathbf{e}_{23} + (v_z f_x - v_x f_z)f_w\mathbf{e}_{31} + (v_x f_y - v_y f_x)f_w\mathbf{e}_{12}$	
Antiprojection of plane **f** onto point **p**. $(\underline{\mathbf{p}_\circ} \vee \mathbf{f}) \wedge \mathbf{p} = f_x p_w^2\mathbf{e}_{423} + f_y p_w^2\mathbf{e}_{431} + f_z p_w^2\mathbf{e}_{412} - (f_x p_x + f_y p_y + f_z p_z)p_w\mathbf{e}_{321}$	
Antiprojection of line **L** onto point **p**. $(\underline{\mathbf{p}_\circ} \vee \mathbf{L}) \wedge \mathbf{p} = v_x p_w^2\mathbf{e}_{41} + v_y p_w^2\mathbf{e}_{42} + v_z p_w^2\mathbf{e}_{43}$ $+ (p_y v_z - p_z v_y)p_w\mathbf{e}_{23} + (p_z v_x - p_x v_z)p_w\mathbf{e}_{31} + (p_x x_y - p_y v_x)p_w\mathbf{e}_{12}$	
Antiprojection of plane **f** onto line **L**. $(\underline{\mathbf{L}_\circ} \vee \mathbf{f}) \wedge \mathbf{L} = (v_x^2 + v_y^2 + v_z^2)(f_x\mathbf{e}_{423} + f_y\mathbf{e}_{431} + f_z\mathbf{e}_{412})$ $- (f_x v_x + f_y v_y + f_z v_z)(v_x\mathbf{e}_{423} + v_y\mathbf{e}_{431} + v_z\mathbf{e}_{412})$ $+ (f_x m_y v_z - f_x m_z v_y + f_y m_z v_x - f_y m_x v_z + f_z m_x v_y - f_z m_y v_x)\mathbf{e}_{321}$	

By Eric Lengyel