MTH 311

Introduction
to
Higher Mathematics

Spring 2015

Outline & Problem Sets

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0. Preface

This booklet is an outline of the material presented in MTH 311 Introduction to Higher Mathematics, together with problem sets on each section. It is designed to be used in conjunction with the lecture material in MTH 311. The lectures provide examples, motivation, and proofs. This outline provides precisely worded definitions and statements of results: the most basic level of material that you need to learn.

Notice the emphasis on *precision*. Mathematics is a precise language. It matters what the inverse image $f^{-1}(Y)$ of a set *means*; students will not be able to do any of the problems involving the inverse image of a set if they do not know what the inverse image is.

To get the most from this course, you need to accomplish at least the following 2 tasks:

(1) LEARN THE DEFINITIONS THOROUGHLY
- Memorize each definition.
- Go over each definition, giving examples and non-examples, until you understand the idea behind the definition.
- Review each definition, its meaning, its examples, and its non-examples, until you can recite all this information *in your sleep*.

(2) LEARN THE PROPOSITIONS AND THE IDEAS BEHIND THEM THOROUGHLY
- Memorize the statement of each proposition (theorem, lemma, corollary).
- Go over each proposition until you understand the central idea behind it.
- Learn the flow of the proof: what comes first, what comes last.
- Memorize the central ideas behind the proofs. Do not memorize proofs.
- Work through a proof of each proposition *while* referring to your lecture notes.
- Work through a proof of each proposition *without* referring to your lecture notes.

Neither of these tasks is easy. You may find that flashcards of definitions and propositions are a useful tool. Practice and repetition will make the tasks easier. The purpose of this outline is to provide a place to start. Together with the lecture notes, it specifies what you need to know.

The problems vary from easy (follows directly from a definition or proposition) to sophisticated (requires an idea you have to come up with on your own). Don't expect to look at a problem and solve it immediately, as you may have done with most calculus problems. If you can't solve a problem immediately, come back to it after a few hours or the next day. In the meanwhile, go on to the next problems; often they will not be linked together. Note this means that you need to start doing the homework assignments early, to give yourself enough time to think.

Contents

1. A LITTLE LOGIC

Definition 1.1 (Statement)**.** A *statement* P is a sentence that is either true or false (but not both).

Definition 1.2 (Negation)**.** If P is a statement, then the *negation* of P is $\neg P$, read "not P." The negation of P is defined to be true if P is false and false if P is true.

Definition 1.3 (Conjunction)**.** If P and Q are statements, then their *conjunction* $P \wedge Q$, read "P and Q," is true if P and Q are both true; otherwise their conjunction is false.

Note the similarity between the symbol "$\wedge$" and the letter "A" for "and".

Definition 1.4 (Disjunction)**.** If P and Q are statements, then their *disjunction* $P \vee Q$, read "P or Q," is true if either P or Q or both are true; otherwise their disjunction is false.

Definition 1.5 (Existential Quantifier)**.** The *existential quantifier* is denoted by the symbol $\exists$, read "there exists."

Definition 1.6 (Universal quantifier)**.** The *universal quantifier* is denoted by the symbol $\forall$, read "for all."

Definition 1.7 (Formal implication)**.** If P and Q are statements, then the *implication* $P \Rightarrow Q$, read "if P then Q" or "P implies Q" or "P only if Q" or "Q if P," is the statement that if P is true then Q is true. P is called the *premise* and Q is called the *conclusion*.

Note that implication in this formal sense is only concerned with the truth or falsity of P and Q as statements, not with the meaning of P and Q or with a chain of reasoning between P and Q.

For technical reasons, we say $P \Rightarrow Q$ is true when P is false, regardless of whether Q is true.

We sometimes write $Q \Leftarrow P$, read "Q is implied by P," instead of $P \Rightarrow Q$, just as we sometimes write $y < x$ instead of $x > y$.

Definition 1.8 (Converse)**.** The *converse* of an implication $P \Rightarrow Q$ is the implication $Q \Rightarrow P$.

Note that the converse of a true implication could be false and the converse of a false implication could be true.

Definition 1.9 (Equivalence)**.** If P and Q are statements, then P and Q are *equivalent*, written $P \Leftrightarrow Q$ and read "P if and only if Q" or "P is equivalent to Q", if $P \Rightarrow Q$ and $Q \Rightarrow P$.

Definition 1.10 (Contrapositive)**.** The *contrapositive* of the implication $P \Rightarrow Q$ is the implication $\neg Q \Rightarrow \neg P$.

The following proposition is often useful:

Proposition 1.11 (An implication and its contrapositive are equivalent)**.**

$$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)\,.$$

Definition 1.12. A sentence whose truth value depends on the values of some variable is an *open sentence*.

Note. We write $P(x), P(x, y)$, etc. for open sentences depending on the variables x or x and y, etc.

Proposition 1.13. *Let $P(x)$ and $Q(x)$ be open sentences. Then*

(1) $\neg(\forall x, P(x)) \Leftrightarrow (\exists x, \neg P(x))$
(2) $\neg(\exists x, P(x)) \Leftrightarrow (\forall x, \neg P(x))$
(3) $\neg(P(x) \Rightarrow Q(x)) \Leftrightarrow (\exists x, (P(x) \wedge \neg Q(x)))$

PROBLEMS 1.

Problem 1.1. For each of the following, state whether it is true or false and give a short reason for your answer:

(1) Today is January 1, 2020 and red is a color.
(2) Today is January 1, 2020 or the sun is a star.
(3) Whales are mammals and red is a color.
(4) Whales are mammals or the sun is a star.

Problem 1.2. For each of the following statements, find P and Q so that the statement is equivalent to the implication $P \Rightarrow Q$:

(1) I am happy if I am listening to music.
(2) I am happy only if I am listening to music.

Problem 1.3. For each of the following statements, write the negation of the statement in standard English:

(1) All cats have 9 lives.
(2) There exists a cat with 9 lives.

Problem 1.4.

(1) Let P be the statement "$x^2+2 = 11$ for all real numbers x such that $x^3+32 = 5$." Is P true? Why or why not?
(2) Let Q be the statement "$x^3+32 = 5$ for all real numbers x such that $x^2+2 = 11$." Is Q true? Why or why not?
(3) Let R be the statement "$x^3+32 = 5$ for all real numbers x such that $x \cdot 0 = 9$." Is R true? Why or why not?

2. AXIOMS FOR THE INTEGERS

The set $\mathbf{Z}$ of integers

(1) has two operations: addition (denoted '+') and multiplication (denoted '$\cdot$')
(2) contains a subset $\mathbf{N}$ (the natural numbers, the positive integers)

satisfying the following axioms:

Z1: $\mathbf{Z}$ is *closed* under addition and multiplication. That is, if a and b are in $\mathbf{Z}$, then $a+b$ and $a \cdot b$ are in $\mathbf{Z}$.
Z2: Addition is *associative*. That is, $a+(b+c) = (a+b)+c$ for all a, b, and c in $\mathbf{Z}$.
Z3: Addition is *commutative*. That is, $a+b = b+a$ for all a and b in $\mathbf{Z}$.
Z4: $\mathbf{Z}$ contains an *additive identity* 0, such that $0+a = a+0 = a$ for all integers a in $\mathbf{Z}$.
Z5: Each integer a in $\mathbf{Z}$ has an *additive inverse* $-a$ in $\mathbf{Z}$, such that $a+(-a) = (-a)+a = 0$.
Z6: Multiplication is associative. That is, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all a, b, and c in $\mathbf{Z}$.
Z7: Multiplication is commutative. That is, $a \cdot b = b \cdot a$ for all a and b in $\mathbf{Z}$.
Z8: $\mathbf{Z}$ contains a *multiplicative identity* 1, $1 \neq 0$, such that $1 \cdot a = a \cdot 1 = a$ for all integers a in $\mathbf{Z}$
Z9: Multiplication *distributes over addition*: $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(a+b) \cdot c = a \cdot c + b \cdot c$.
Z10: $\mathbf{N}$ is closed under addition and multiplication.
Z11: For each integer a in $\mathbf{Z}$, exactly one of the following conditions is true: a is in $\mathbf{N}$, or $-a$ is in $\mathbf{N}$, or $a = 0$. [Note that 0 is not a natural number.]
Z12: *Well-Ordering Principle for Integers:* If A is a non-empty subset of $\mathbf{N}$, then A has a smallest element.

Note. As part of what it means for '+' and '$\cdot$' to be operations, you may use that if $a = b$ and $c = d$, then $a+c = b+d$ and $a \cdot c = b \cdot d$.

Notation 2.1. We often write

$a - b$ to mean $a + (-b)$
ab to mean $a \cdot b$
$a+b+c$ to mean $(a+b)+c$ (same as $a+(b+c)$) and abc to mean $(ab)c$ (same as $a(bc)$).
$a < b$ to mean $b - a$ is in $\mathbf{N}$; similar notation for $a \leq b$, $b > a$, etc.

In particular, with this notation Z11 implies that for every two integers a and b exactly one of the following holds: $a > b$, or $a < b$, or $a = b$.

Definition 2.2. A set A of integers has a smallest element if there exists an integer a in A such that $a \leq b$ for all integers b in A.

Note 2.3. We denote $1+1$ by 2, $2+1$ by 3, ..., $9+1$ by 10. We denote 10 by 10^1, $10 \cdot 10$ by 10^2, $10 \cdot 10^2$ by 10^3, etc. Finally we denote, for example, $4 \cdot 10^3 + 9 \cdot 10^2 + 8 \cdot 10^1 + 7$ by 4987. We will show later that every positive integer can be written in this form.

BASIC PROPERTIES OF THE INTEGERS:

Note that a, b, c, d all denote integers

P1: *Cancellation of addition:* If $a + c = b + c$ then $a = b$. Similarly, if $a + b = a + c$ then $b = c$.

P2: 0 is the unique additive identity.

P3: 1 is the unique multiplicative identity.

P4: For each a in $\mathbf{Z}$, $-a$ is unique.

P5: $-(-a) = a$.

P6: $0 \cdot a = 0$.

P7: $(-1) \cdot a = -a$.

P8: $(-a) \cdot b = a \cdot (-b) = -(ab)$.

P9: $(-a)(-b) = ab$.

P10: Multiplication *distributes over subtraction*: $a \cdot (b - c) = a \cdot b - a \cdot c$ and $(a - b) \cdot c = a \cdot c - b \cdot c$.

P11: 1 is in $\mathbf{N}$.

P12: If a and b are non-zero integers, then $a \cdot b$ is non-zero.

P13: *Cancellation of multiplication:* If $a \neq 0$, then $a \cdot b = a \cdot c$ implies $b = c$.

P14: If $a < b$ and $b < c$, then $a < c$.

P15: If $a < b$, then $a + c < b + c$ and $a - c < b - c$.

P16: If $a + c < b + c$, then $a < b$; if $a - c < b - c$ then $a < b$.

P17: If $c > 0$ and $a < b$, then $ac < bc$.

P18: If $c < 0$ and $a < b$, then $ac > bc$.

P19: If $c > 0$ and $ac < bc$, then $a < b$.

P20: If $c < 0$ and $ac < bc$, then $a > b$.

P21: If $0 < a < b$ and $0 < c < d$, then $a \cdot c < b \cdot d$.

P22: 1 is the smallest natural number.

Proposition 2.4. *Let k be an integer and let A be a non-empty subset of integers such that $a > k$ for each a in A. Then A has a smallest element. The same is true if $a \geq k$ for each a in A.*

Proposition 2.5. *Let k be an integer and let A be a non-empty subset of integers such that $a < k$ for each a in A. Then A has a largest element. The same is true if $a \leq k$ for each a in A.*

PROBLEMS 2.

Problem 2.1. Using any of the axioms Z1-Z9 and properties listed before the property you are to prove, prove the following properties: 1) P3; 2) P5; 3) P7; 4) P8; 5) P9; 6) P10.

Problem 2.2. Using any of the axioms Z1-Z12 and properties listed before the property you are to prove, prove the following properties: 1) P12 (break the proof up into cases. Can you prove for a, b both positive, for example?); 2) P13; 3) P14; 4) P15; 5) P16; 6) P17; 7) P18; 8) P19; 9) P20; 10) P21.

Problem 2.3. Prove the last part of Proposition 2.4.

Problem 2.4. Prove Proposition 2.5.

3. DIVISIBILITY

Definition 3.1. Let n be an integer and let a be a non-zero integer. We say that a divides n , written $a \mid n$, if there exists an integer b such that $n = a \cdot b$.

Note. The following are all equivalent ways of writing the same thing:

(1) a divides n
(2) a is a factor of n
(3) a is a divisor of n
(4) n is a multiple of a

Note. If $a \mid n$ then $-a \mid n$ and $a \mid -n$.

Proposition 3.2. *If a and n are positive integers such that $a \mid n$, then $a \leq n$.*

Definition 3.3. A positive integer n is *prime* if $n > 1$ and the only positive factors of n are 1 and n. *Note:* for technical reasons 1 is not a prime.

Definition 3.4 (Greatest common divisor)**.** Let m and n be non-zero integers. The greatest common divisor of m and n, denoted $\gcd(m, n)$, is the largest integer that divides both m and n.

Note. The gcd is always positive.

Proposition 3.5. *Let m and n be non-zero integers. Then they have a greatest common divisor.*

Definition 3.6. Let m and n be non-zero integers. We say that m and n are *relatively prime* if $\gcd(m, n) = 1$.

Note. m and n are relatively prime is the same as they have no common divisors other than 1 and -1.

Proposition 3.7. *Let m and n be non-zero integers and let d be $\gcd(m, n)$. Let $m = d \cdot a$ and $n = d \cdot b$ for integers a and b. Then a and b are relatively prime.*

Definition 3.8. An even integer is an integer that can be written in the form $2m$, where m is an integer.

Definition 3.9. An odd integer is an integer that can be written in the form $2m+1$, where m is an integer.

Proposition 3.10. *The sum of two even integers or two odd integers is even. The sum of an even integer and an odd integer is odd.*

Proposition 3.11. *The product of an even integer with any integer (even or odd) is even.*

Proposition 3.12. *The product of two odd integers is odd.*

Proposition 3.13. *Every integer is either even or odd.*

Proposition 3.14. *Each positive integer a can be written uniquely in the form $a = b_n 10^n + b_{n-1} 10^{n-1} + \dots b_2 10^2 + b_1 10 + b_0$, where $b_0, \dots, b_n$ are integers from 0 to 9 and $b_n \neq 0$. [Uses Problems 3.13 and 3.14]*

PROBLEMS 3. Use the axioms and properties of integers to do the following problems.

Problem 3.1. If a is a non-zero integer, prove that $a \mid a$.

Problem 3.2. If a, b, and c are non-zero integers, prove that $a \mid b$ and $b \mid c$ implies $a \mid c$.

Problem 3.3. Prove that the only integers that divide 1 are 1 and -1.

Problem 3.4. Let a and b be non-zero integers. If $a \mid b$ and $b \mid a$, prove that $b = a$ or $b = -a$.

Problem 3.5. Is 0 even or is it odd? Justify your answer.

Problem 3.6. Show that if n is an even integer, then $-n$ is an even integer. If n is an odd integer, then $-n$ is an odd integer.

Problem 3.7. Prove Proposition 3.11

Problem 3.8. Prove Proposition 3.12

Problem 3.9. Let k be a positive integer. Prove that if $k^2 + 3$ is prime, then k is even.

Problem 3.10. Is it true that if k is even then $k^2 + 3$ is prime? Why or why not?

Problem 3.11. Use the method of Proposition 3.13 to prove that every integer is what Marty Weissman at Berkeley calls "threeven", "throdd", or "thrugly", that is, can be written in the form $3m$, $3m + 1$, or $3m + 2$, where m is an integer.

Problem 3.12. Show that the representation in Problem 3.11 of an integer in the form $3m$, $3m + 1$, or $3m + 2$ is unique. That is, if an integer n can be written in the form $3\ell + r$ and can also be written in the form $3m + s$, where $\ell, r, m, s \in \mathbf{Z}$ satisfy $0 \leq r < 3$ and $0 \leq s < 3$, then $\ell = m$ and $r = s$. Hint: Try to set things up so that you can use Proposition 3.2. *Note: this implies that an integer cannot be both threeven and throdd, or both threeven and thrugly, or both throdd and thrugly.*

Problem 3.13. Use the method of Proposition 3.13 to prove that every integer can be written in the form $10m, 10m + 1, \ldots$ or $10m + 9$, where m is an integer.

Problem 3.14. Show that the representation of an integer in the form $10m, 10m + 1, \ldots$ or $10m + 9$, where m is an integer, is unique. (Similar to Problem 3.12.)

4. AXIOMS FOR THE REAL NUMBERS

The set $\mathbf{R}$ of real numbers

(1) has two operations: addition (denoted '$+$') and multiplication (denoted '$\cdot$')
(2) contains a subset $\mathbf{R}^+$ (the positive real numbers)

satisfying the following axioms:

R1: $\mathbf{R}$ is closed under addition and multiplication.
R2: Addition is associative.
R3: Addition is commutative.
R4: $\mathbf{R}$ contains an additive identity 0.
R5: Each real number x has an additive inverse $-x$.
R6: Multiplication is associative.
R7: Multiplication is commutative.
R8: $\mathbf{R}$ contains a multiplicative identity 1, $1 \neq 0$.
R9: Each nonzero real number x has a *multiplicative inverse* $\frac{1}{x}$ such that $x \cdot \frac{1}{x} = \frac{1}{x} \cdot x = 1$.
R10: Multiplication distributes over addition.
R11: $\mathbf{R}^+$ is closed under addition and multiplication.
R12: For each real number x , exactly one of the following conditions is true: x is in $\mathbf{R}^+$, or $-x$ is in $\mathbf{R}^+$, or $x = 0$.
R13: The *Least Upper Bound Axiom*(will be discussed Section 14).
R14: The set of integers $\mathbf{Z}$ is a subset of $\mathbf{R}$. The additive and multiplicative identities of $\mathbf{Z}$ and $\mathbf{R}$ are the same, and if a and b are integers, then $a + b$ is the same whether a and b are viewed as integers or as real numbers, and similarly for $a \cdot b$.

Notation: we often write
$x - y$ to mean $x + (-y)$
xy to mean $x \cdot y$
$\frac{x}{y}$ to mean $x \cdot \frac{1}{y}$
$x + y + z$ to mean $(x + y) + z$ (same as $x + (y + z)$) and xyz to mean $(xy)z$ (same as $x(yz)$).
$x < y$ to mean $y - x$ is in $\mathbf{R}^+$; similar notation for $x \leq y$, $y > x$, etc.
In particular, with this notation R12 implies that for every two real numbers x and y exactly one of the following holds: $x > y$, or $x < y$ or $x = y$.

If a positive integer is denoted by n (as in Note 2.3), and if $n = 1 + 1 + \cdots + 1$, then we say that n is obtained by *adding* 1 *to itself* n *times*. If x is a real number, and n is a positive integer, then $nx = (1 + 1 + \cdots + 1) \cdot x = 1 \cdot x + 1 \cdot x + \cdots + 1 \cdot x$, so that multiplying x by n is the same as *adding* x *to itself* n *times*.

Properties P1-P21 for the integers were proved using axioms Z1-Z11 for the integers (only property P22, that 1 is the smallest natural number, required the Well-Ordering Principle (Axiom Z12)). Since $\mathbf{R}$ also satisfies these axioms (with $\mathbf{Z}$ replaced by $\mathbf{R}$, $\mathbf{N}$ replaced by $\mathbf{R}^+$, and "integer" replaced by "real number"), properties P1-P21 also hold for the real numbers (again with $\mathbf{Z}$ replaced by $\mathbf{R}$, $\mathbf{N}$ replaced by $\mathbf{R}^+$, and "integer" replaced by "real number"). (The proofs of P11, P13, P19 and P20 can be simplified by using Axiom R9.)

In addition, the real numbers also have the property:

P23: For each nonzero real number a, its multiplicative inverse $\frac{1}{a}$ is unique.

It follows from Proposition 3.14 that every positive integer can be written as the number 1 added to itself a finite number of times. Since both $\mathbf{N}$ and $\mathbf{R}^+$ are closed under addition, and since 1 is in $\mathbf{R}^+$ (property P10 for the real numbers), it follows that $\mathbf{N}$ is a subset of $\mathbf{R}^+$.

The following useful result will be proved later using the Least Upper Bound Axiom.

Proposition 4.1 (Archimedean Principle)**.** *Let x and y be positive real numbers. Then there exists a positive integer n such that $nx > y$.*

Definition 4.2. Let x be a real number. Then the *greatest integer* in x, denoted $\lfloor x \rfloor$, is the largest integer n such that $n \leq x$.

Corollary 4.3. *If x is a real number then $\lfloor x \rfloor$ exists.*

Note: if x is a real number and n is an integer such that $n \leq x < n + 1$, then $n = \lfloor x \rfloor$.

Another consequence of the Least Upper Bound Axiom that we shall use without proof is the following.

Proposition 4.4. *If x is a positive real number and n is a positive integer, then x has a real n-th root. That is, there exists a real number y such that $y^n = x$.*

Note: $y^1 = y, y^2 = y \cdot y, y^3 = y \cdot y^2, \ldots$.

PROBLEMS 4.

Problem 4.1. Prove property P23 of the real numbers.

Problem 4.2. Let x be a non-zero real number. Prove that $\frac{1}{x}$ is non-zero and that $\frac{1}{(\frac{1}{x})} = x$.

Problem 4.3. Let x and y be non-zero real numbers. Prove that $\frac{1}{xy} = \frac{1}{x} \cdot \frac{1}{y}$.

Problem 4.4. Let x and y be real numbers such that $0 < x < y$. Prove that $\frac{1}{y} < \frac{1}{x}$.

Problem 4.5. Let x, y, and z be real numbers such that y and z are non-zero. Prove *Cancellation for Division:* $\frac{xz}{yz} = \frac{x}{y}$.

Problem 4.6. Let a, b, c, and d be real numbers such that b and d are non-zero. Prove that $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ and $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

Problem 4.7. Let x and y be real numbers, with $x < y$ and $y - x < 1$. Then there is at most one integer n strictly between x and y.

Problem 4.8. Let x and y denote real numbers. In terms of the greatest integer:

a) Show that $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor$.
b) Give an example where $\lfloor x \rfloor + \lfloor y \rfloor < \lfloor x + y \rfloor$.
c) Show that $x + y < \lfloor x \rfloor + \lfloor y \rfloor + 2$.
d) Give an example where $x + y = \lfloor x \rfloor + \lfloor y \rfloor + 1.99$.

Problem 4.9. Prove that the following three results are equivalent (that is, show that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a)):

(a) (Archimedean principle) Let x and y be positive real numbers. Then there exists a positive integer n such that $nx > y$.
(b) Let z be a positive real number. Then there exists a positive integer n such that $n > z$.
(c) Let ε be a positive real number. Then there exists a positive integer n such that $0 < \frac{1}{n} < \varepsilon$.

Hint: To prove that (a) $\Rightarrow$ (b), you assume that (a) is true and try to prove (b). How can you choose x and y (depending on z) so that (a) gives you the result you want about z in (b)?

Note: Traditionally the Greek letter ε (epsilon) has been used to designate a potentially small quantity. Although in result (c) ε is not necessarily small, the only time this result is useful is when ε is small, so this is a clue to the user of the result.

5. RATIONAL AND IRRATIONAL NUMBERS

Definition 5.1 (Rational Number)**.** A *rational number* r is a number which can be written in the form $r = \frac{m}{n}$, where m and n are integers and $n \neq 0$.

Proposition 5.2. *The sum of two rational numbers is rational.*

Proposition 5.3. *The product of two rational numbers is rational.*

Note 5.4. The set of all rational numbers is denoted by $\mathbf{Q}$. In fact, $\mathbf{Q}$ satisfies all the axioms for the real numbers except for the Least Upper Bound Axiom.

Definition 5.5 (Irrational Number)**.** An *irrational number* x is a real number which is not rational.

Proposition 5.6. *The sum of a rational number and an irrational number is irrational.*

Proposition 5.7. $\sqrt{2}$ *is irrational.*

Proposition 5.8. *Between each two distinct real numbers there is a rational and an irrational.*

Definition 5.9 (Dense)**.** Let S be a set of real numbers. Then S is *"dense" in the real numbers* if between each two distinct real numbers there is an element of S.

Note 5.10. There is no general agreement on what to call the property in Definition 5.9. Some write that 'S is dense in R', some write that 'S is interleaved (or interwoven) in R'. To further complicate matters, there *is* a standard definition of what it means for a set A to be dense in a certain kind of set B. This definition is not the same as the one in Definition 5.9, but turns out to be equivalent to it when $B = \mathbf{R}$.

Corollary 5.11. *The set of rationals and the set of irrationals are each "dense" in the real numbers.*

Definition 5.12. A decimal expansion $\pm b_n \ldots b_0.a_1a_2a_3\ldots$ is *terminating* if $a_i = 0$ for all $i \geq N$, where N is a positive integer; it is *repeating* if $a_{i+p} = a_i$ for all $i \geq N$, where N and p are positive integers.

Proposition 5.13. *A repeating decimal represents a rational number.*

Proposition 5.14. *A rational number is represented by a repeating decimal.*

PROBLEMS 5.

Problem 5.1. State and prove a correct proposition, similar to Proposition 5.3, about the quotient of rational numbers.

Problem 5.2. State and prove a correct proposition, similar to Proposition 5.6, about the product of a rational and an irrational number. Be careful.

Problem 5.3. Give a counter-example to the following statement: The sum of irrational numbers is irrational.

A counter-example is a specific example which shows that a statement is not true. So you have to write down two actual irrational numbers whose sum is not irrational.

Problem 5.4. By Problem 3.11 every integer is *threeven, throdd,* or *thrugly,* that is, of the form $3m$, $3m+1$, or $3m+2$. These three types play the role for 3 that the even and odd integers play for 2 (every integer is of the form $2m$ or $2m+1$).

(1) Show that the product of a threeven integer and an integer is threeven.
(2) Show that the product of two throdd integers is throdd.
(3) Show that the product of two thrugly integers is throdd.

Problem 5.5. Use Problem 5.4 to prove that $\sqrt{3}$ is irrational. Be sure to justify carefully the essential points of your proof.

Problem 5.6. Find the decimal expansion of $7/31$ and use your work to explain why the decimal expansion of every rational number is repeating.

Problem 5.7. Give an actual, explicit example of a non-repeating (non-terminating) decimal. Your answer should be such that you could determine without a computer what the thousandth decimal place would be. "The decimal expansion of π" or "The decimal expansion of $\sqrt{2}$" is not an answer, since you can't describe these decimal expansions explicitly (what is the thousandth decimal place of $\sqrt{2}$?).

In the following problems, use the (possibly non-standard) definition of "dense" given in Definition 5.9.

Problem 5.8. Let S be a finite set of real numbers, $S := \{x_1, \dots, x_n\}$, where $x_1 < \cdots < x_n$. Prove that S is not "dense" in the reals.

Problem 5.9. Give an example of an infinite set of numbers which is not "dense" in the real numbers.

Problem 5.10. Prove that $\mathbf{Q}-\{0\}$ is "dense" in the real numbers, where $\mathbf{Q}-\{0\} := \{r \in \mathbf{Q} \mid r \neq 0\}$.

Problem 5.11. Let $r_1 < \ldots < r_n$ be n rational numbers. Prove that $\mathbf{Q}-\{r_1, \dots, r_n\}$ is "dense" in the reals ($\mathbf{Q} - \{r_1, \dots, r_n\}$ is the set of rational numbers not equal to $\{r_1, \dots, r_n\}$).

Problem 5.12. Let S be the set of all rational numbers of the form $p/(2n)$ where p is an odd integer and n is a positive integer. Prove that S is "dense" in the reals.

6. INDUCTION

Recall the **Well-ordering Principle for Integers:** If A is a non-empty subset of $\mathbf{N}$, then A has a smallest element.

Proposition 6.1 (Principle of Mathematical Induction)**.** *Let $P(n)$ be a statement for each integer $n \geq 1$. If both of the following hold:*

*(1) $P(1)$ is true (*the base case*)*
*(2) For each $k \geq 1$, $P(k+1)$ is true under the assumption that $P(k)$ is true (*the induction step*)*

then $P(n)$ is true for all $n \geq 1$.

Note 6.2. Given the properties of the integers that we have developed to this point, we can convert any proof by induction to a proof by well-ordering and most proofs by well-ordering to proofs by induction.

Note 6.3. If x is a real number, we have previously defined x^n so that $x^1 := x$ and $x^{n+1} := x^n \cdot x$ for all integers $n \geq 1$.

Proposition 6.4. *Let x and y be real numbers. Then $(xy)^n = x^n y^n$ for all positive integers n.*

Proposition 6.5. *Let x be a real number. Then $x^m x^n = x^{m+n}$ for all positive integers m and n.*

Proposition 6.6. *Let x be a real number. Then $(x^m)^n = x^{mn}$ for all positive integers m and n.*

Proposition 6.7. *Let S be a set consisting of exactly n distinct real numbers $x_1, \ldots,$ x_n, where $n \geq 1$. Then S contains a smallest element and a largest element.*

Definition 6.8 (Binomial coefficients)**.** Let k and n be integers, $0 \leq k \leq n$. Define the *binomial coefficient* $\binom{n}{k}$, by:

$$\binom{n}{k} := \frac{n!}{k!(n-k)!},$$

where $0! := 1$.

Lemma 6.9. *If n is an integer, $n \geq 0$, then $\binom{n}{0} = \binom{n}{n} = 1$.*

Proposition 6.10 (Pascal's Triangle)**.** *If $0 < k < n$, then $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.*

Proposition 6.11. *Let k and n be integers, $0 \leq k \leq n$, then $\binom{n}{k}$ is an integer.*

Proposition 6.12. *Binomial Theorem Let x and y be real numbers and let n be a positive integer. Then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proposition 6.13 (Induction with base m). *Let $P(n)$ be a statement for each integer $n \geq m$. If both of the following hold:*

*(1) $P(m)$ is true (*the base case*)*
*(2) For each $k \geq m$, $P(k+1)$ is true under the assumption that $P(k)$ is true (*the induction step*)*

then $P(n)$ is true for all $n \geq m$.

Proposition 6.14 (Complete Induction). *Let $Q(n)$ be a statement for each integer $n \geq 1$. If both of the following hold:*

*(1) $Q(1)$ is true (*the base case*)*
*(2) For each $k \geq 1$, $Q(k+1)$ is true under the assumption that $Q(1), \ldots, Q(k)$ are all true (*the complete induction step*)*

then $Q(n)$ is true for all $n \geq 1$.

PROBLEMS 6.

Problem 6.1. Use induction to prove that $2^n > n$ for all integers $n \geq 1$.

Problem 6.2. Guess at a formula for $1 + 3 + 5 + \ldots + (2n - 1)$, and prove your result by induction for $n \geq 1$.

Problem 6.3. Use induction to prove that every positive integer n is of the form $3m$, $3m + 1$ or $3m + 2$. (This same problem was proved by well-ordering in Problem 3.11.)

Problem 6.4. Use induction to prove that $1^3 + \ldots + n^3 = (1 + \ldots + n)^2$.

Problem 6.5. Use induction to prove that $n^3 - n$ is divisible by 6 for all $n \geq 0$.

Problem 6.6. Let S be a set consisting of exactly n distinct real numbers, where $n \geq 1$. Prove that S has a largest element. (This is part of the proof of Proposition 6.7.)

Problem 6.7. Use induction to prove that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} \leq 2 - \frac{2}{n+1}$$

for all $n \geq 1$.

Problem 6.8. Use induction to prove:

(1) Proposition 6.4
(2) Proposition 6.5
(3) Proposition 6.6

Problem 6.9. Prove Lemma 6.9

Problem 6.10. Prove Proposition 6.10

Problem 6.11. Use Lemma 6.9 and Proposition 6.10 and induction on n to prove Proposition 6.11

Problem 6.12. The Fibonacci numbers are: $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$. If F_n denotes the n-th Fibonacci number, then $F_1 := 1, F_2 := 1$, and $F_{n+2} := F_n + F_{n+1}$ for $n \geq 1$. Prove that $F_n < 2^n$.

Prove that the Principle of Induction and the Principle of Complete Induction are equivalent, as follows:

Problem 6.13. Use the Principle of Complete Induction to prove the Principle of Induction.

Hint: Assume the hypotheses of the Principle of Induction apply to the statements $P(n)$. Let $Q(n) := P(n)$. Show that the $Q(n)$ satisfy the hypotheses of the Principle of Complete Induction. What does the conclusion of the Principle of Complete Induction tell you about $P(n)$?

Problem 6.14. Use the Principle of Induction to prove the Principle of Complete Induction.

Hint: Assume the hypotheses of the Principle of Complete Induction apply to the statements $Q(n)$. Let $P(n)$ be the statement that $Q(1), \ldots, Q(n)$ are all true. Show that the $P(n)$ satisfy the hypotheses of the Principle of Induction. What does the conclusion of the Principle of Induction tell you about $Q(n)$?

Problem 6.15. Let

$$S_n^0 = 1 \quad \text{for all } n \geq 1 \tag{6.15.1}$$

$$S_n^i = \sum_{j=1}^{n} S_j^{i-1} \quad \text{for all } n \geq 1, \quad \text{for} \quad i \geq 1 \tag{6.15.2}$$

Prove that

$$S_n^i = \binom{n+i-1}{i}$$

for all $n \geq 1, i \geq 0$.

Hint: Use double induction. That is, induct on i first. To prove the induction step for i, induct on n. Note that for $n > 1$, 6.15.2 can be written as $S_n^i = S_{n-1}^i + S_n^{i-1}$.

7. SETS

Definition 7.1 (Subset)**.** Let A and B be sets. We say A is *contained in* B or A is a *subset* of B, written $A \subset B$, if whenever $x \in A$, then $x \in B$. As an alternative, we sometimes say that B *contains* A, written $B \supset A$.

Note: some people write $A \subseteq B$ (which is consistent with writing $x \leq y$) instead of $A \subset B$. You may use whichever you would like, but understand that writing $A \subset B$ means that A could possibly equal B (unlike writing $x < y$, which definitely means that x is not equal to y). In the $A \subset B$ notation, to indicate that A is a subset but is *not* equal to B, we write $A \subsetneq B$.

Definition 7.2 (Equality of Sets)**.** We say that A *equals* B, written $A = B$, if $A \subset B$ and $A \supset B$.

Definition 7.3 (Union and intersection)**.** The *union* $A \cup B$ of two sets A and B is defined by $A \cup B := \{x \mid x \in A \text{ or } x \in B\}$. Note that 'or' in mathematics means 'and/or.' The *intersection* $A \cap B$ is defined by $A \cap B := \{x \mid x \in A \text{ and } x \in B\}$.

Note 7.4. Union and intersection can be defined for sets $A_1, A_2, \ldots, A_n$ and for sets $A_1, A_2, \ldots$ in a similar fashion: $\bigcup_{i=1}^{n} A_i$ is the set of all x for which $x \in A_i$ for some i, $1 \leq i \leq n$; $\bigcup_{i=1}^{\infty} A_i$ is the set of all x for which $x \in A_i$ for some $i \geq 1$; $\bigcap_{i=1}^{n} A_i$ is the set of all x for which $x \in A_i$ for all i, $1 \leq i \leq n$; $\bigcap_{i=1}^{\infty} A_i$ is the set of all x for which $x \in A_i$ for all $i \geq 1$.

Definition 7.5 (Disjoint)**.** Sets A and B are *disjoint* if $A \cap B = \varnothing$.

Definition 7.6 (Power set)**.** The *power set* of a set A, denoted 2^A, is the set of all subsets of A. That is, $2^A := \{X \mid X \subset A\}$.

Proposition 7.7 (Basic properties of sets)**.** *If A, B, C are sets, then*

(1) $\varnothing \cap A = \varnothing$; $\varnothing \cup A = A$
(2) $A \cap B \subset A$
(3) $A \subset A \cup B$
(4) $A \cup B = B \cup A$; $A \cap B = B \cap A$
(5) $A \cup (B \cup C) = (A \cup B) \cup C$
(6) $A \cup A = A \cap A = A$
(7) If $A \subset B$, then $A \cup C \subset B \cup C$ and $A \cap C \subset B \cap C$

Proposition 7.8 (Distributive Rules)**.** *If A, B, C are sets, then*

(1) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(2) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Definition 7.9 (Complement)**.** If X and A are sets, the *complement* $X - A$ of A in X is defined by $X - A := \{x \in X \mid x \notin A\}$

Proposition 7.10. *If A and B are sets, then $A \cup B = A \cup (B - A)$.*

Proposition 7.11 (de Morgan's laws)**.** *If A, B and X are sets then*

(1) $X - (A \cup B) = (X - A) \cap (X - B)$
(2) $X - (A \cap B) = (X - A) \cup (X - B)$

Definition 7.12 (Formal definition of ordered pair)**.** Let A and B be sets, let $a \in A$, $b \in B$. Then we define the *ordered pair* (a, b) by $(a, b) := \{\{a\}, \{a, b\}\}$.

Proposition 7.13 (Fundamental property of ordered pairs)**.** *Let a and c be in A; and b and d be in B. Then $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.*

Definition 7.14 (Cartesian Product)**.** The *Cartesian product* $A \times B$ of sets A and B is defined by $A \times B := \{(a, b) \mid a \in A \text{ and } b \in B\}$

Note: writing $(a, b) \in A \times B$ implies that $a \in A$ and $b \in B$.

PROBLEMS 7.

Problem 7.1. Prove each of the following, where A and B are sets
a) $\varnothing \cap A = \varnothing$; $\varnothing \cup A = A$.
b) $A \cap B \subset A$.
c) $A \subset A \cup B$.
d) $A \cup B = B \cup A$; $A \cap B = B \cap A$.
e) $A \cup (B \cup C) = (A \cup B) \cup C$.
f) $A \cup A = A \cap A = A$.

Problem 7.2. Prove that if $A \subset B$ and $C \subset D$ then
a) $A \cup C \subset B \cup D$.
b) $A \cap C \subset B \cap D$.

Problem 7.3. Prove that $A \cup B = A$ if and only if $B \subset A$.

Problem 7.4. If A and B are sets, then $A \cup B = A \cup (B - A)$.

Problem 7.5. Prove that if A is a subset of X, then
a) $A \cap (X - A) = \varnothing$.
b) $A \cup (X - A) = X$.
c) $X - (X - A) = A$. Hint: use a) and b).

Problem 7.6. Prove that if A and B are subsets of X, then
a) $A - B = A \cap (X - B)$.
b) $X - (A \cap B) = (X - A) \cup (X - B)$.
c) $A \subset B$ if and only if $X - B \subset X - A$.

Problem 7.7. Let A be a set.

i) If $A := \{a, b\}$ is a set with 2 distinct elements, list all the elements of 2^A.

ii) Do the same for $A := \{a, b, c\}$, a set with 3 distinct elements.

iii) If A is a set with exactly n distinct elements $a_1, \ldots, a_n$, use induction to prove that 2^A has exactly 2^n elements.

Do not use the formal definition of ordered pair in exercises 7.8-7.11; just use the properties of ordered pairs.

Problem 7.8. Let A and B be sets. Prove that $A \times \emptyset = \emptyset \times B = \emptyset$.

Problem 7.9. Let A and B be non-empty sets. Prove that $A \times B = B \times A$ if and only if $A = B$. Where do you use that A and B are non-empty?

Problem 7.10. Let A, B, C be non-empty sets, with $B \subset C$. Prove that $A \times B \subset A \times C$.

Problem 7.11. Let A, B, C be non-empty sets. Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$ and that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

8. FUNCTIONS

Definition 8.1 (Modern definition of function)**.** A *function* f from a set A to a set B, denoted $f\colon A \to B$, is a subset Γ_f of $A \times B$ with the following properties:

(1) If $x \in A$, then there exists $y \in B$ such that (x, y) is in Γ_f *(f is defined for all x in A)*
(2) If $(x, y) \in \Gamma_f$ and $(x, z) \in \Gamma_f$, then $y = z$ *(the vertical line test)*.

Definition 8.2. If f is a function from A to B, then A is called the *domain* of f and B is called the *codomain* of f.

Note 8.3. If $(x, y) \in \Gamma_f$ then we denote y by $f(x)$.

Definition 8.4 (Composition)**.** If $f\colon A \to B$ and $g\colon B \to C$ then the *composition* $g \circ f$ of g with f is defined by $(g \circ f)(x) := g(f(x))$.

Proposition 8.5 (Gluing functions together)**.** *Let $f\colon A \to C$ and $g\colon B \to D$ be functions, with $f(x) = g(x)$ for all $x \in A \cap B$. Let $h\colon A \cup B \to C \cup D$ be defined by* gluing together *f and g as follows:*

$$h(x) := \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

Then h is a function from $A \cup B$ to $C \cup D$.

Definition 8.6 (Image)**.** Let $f\colon A \to B$ be a function and let X be a subset of A. Then the *image of X under f*, denoted $f(X)$, is defined by $f(X) := \{f(a) \mid a \in X\}$. Alternatively, $f(X) := \{y \in B \mid y = f(x) \text{ for some } x \in X\}$.

Note 8.7. The *image* of f is short for the image of A under f, often called the *range* of f. That is, the image of f and the range of f are each $\{f(a) \mid a \in A\}$. Warning: the "image of f" and "the image of X under f" are not generally the same; the first is $f(A)$, the second is $f(X)$—where X is not necessarily equal to A.

Definition 8.8 (Inverse image)**.** Let $f\colon A \to B$ be a function and let Y be a subset of B. Then the *inverse image of Y under f*, denoted $f^{-1}(Y)$, is defined by $f^{-1}(Y) := \{x \in A \mid f(x) \in Y\}$.

Proposition 8.9 (Image of inverse image is a subset)**.** *$f(f^{-1}(Y)) \subset Y$ for all $Y \subset B$.*

Definition 8.10 (Binary Operation)**.** A binary operation on a set A is a function from $A \times A \to A$.

Note 8.11. The term 'operation' used in the axioms for the integers and the real numbers is short for 'binary operation.' In particular, if $a = c$ and $b = d$, then $(a, b) = (c, d)$, so the result of an operation performed on a and b is the same as the result of the same operation performed on c and d. (That is why $a + b = c + d$ and $a \cdot b = c \cdot d$ if $a = c$ and $b = d$, where a, b, c, d are integers or real numbers.)

PROBLEMS 8.

Problem 8.1. Let $A = \{1, 2, 3, 4\}$ and $B = \mathbf{R}$. Using the formal definition of function, check whether the following subsets of $A \times B$ correspond to functions of A to B. Give your reasons why these are or are not functions

i) $\Gamma_f = \{(1, \pi), (2, e), (1, \sqrt{2}), (3, 1.8717171\ldots), (4, -1/13)\}$

ii) $\Gamma_f = \{(1, 73.555), (2, -14), (3, 1.882)\}$

iii) $\Gamma_f = \{(1, 2^3), (2, \pi), (1, 8), (3, -55), (4, 10^{1000})\}$

Problem 8.2. Let $A = \{1, 2, 3, 4\}$ and $B = \mathbf{Q}$. Using the formal definition of function, check whether the following subset of $A \times B$ corresponds to a function of A to B. Give your reasons why this is or is not a function:

$$\Gamma_f = \{(1, 3/5), (2, \sqrt{2}), (3, 1.8717171\ldots), (4, -1/13)\}$$

Problem 8.3. Let $A = B = \mathbf{R}$. Using the formal definition of function, check whether the following subsets of $A \times B$ correspond to functions of A to B. Give your reasons why these are or are not functions:

i) $\Gamma_f = \{(x, y) \mid x, y \in \mathbf{R} \quad \text{and} \quad x = y^4\}$

ii) $\Gamma_f = \{(x, y) \mid x, y \in \mathbf{R} \quad \text{and} \quad x = y^5 + 4y^3 + 7\}$ (You will need to use calculus to solve this problem)

Problem 8.4. Let $A = \{1\}$ and $B = \{1, 2, 3\}$. Write down all functions from A to B.

Problem 8.5. Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$. Write down all functions from A to B.

Problem 8.6. Let A be a finite set with m elements, and B be a finite set with n elements. Find a formula expressing the number of different functions from A to B. Prove your result. What notation does this suggest for "the set of all functions from A to B".

Problem 8.7. If A is a finite set, then the set of all functions from A to $\{0, 1\}$ has the same number of elements as the power set of A, by Problem 8.6 above and by Problem 7.7 in the previous section. So let A be a set (finite or not). Show that each function of A to $\{0, 1\}$ determines a subset of A and vice-versa.

Problem 8.8. Prove Proposition 8.5.

Problem 8.9. Let $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ be sets and let $f_i \colon A_i \to B_i$ be a function for each $i \geq 1$. If $f_i(x) = f_j(x)$ for all $x \in A_i \cap A_j$ for each $i, j \geq 1$, show that $h \colon \bigcup_{i=1}^{\infty} A_i \to \bigcup_{i=1}^{\infty} B_i$ defined by $h(x) := f_i(x)$ for all $x \in A_i$ for each $i \geq 1$, is a function.

Problem 8.10. Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5\}$. Define a function $f : A \to B$ by $f(1) = 2, f(2) = 2, f(3) = 5, f(4) = 4$.

i. Find the image of f.

ii. Find $f^{-1}(\{3, 4\})$.
iii. Find $f(\{1, 2, 4\})$.
iv. Find $f^{-1}(\{3\})$.
v. Find $f(f^{-1}(\{2, 3\}))$.

Problem 8.11. Let $A = \{1, \ldots, 5\}, B = \{1, \ldots, 6\}, C = \{1, \ldots, 4\}$. Define $f : A \to B$ by $1, 2, 3, 4, 5 \to 2, 4, 3, 6, 1$ and $g : B \to C$ by $1, 2, 3, 4, 5, 6 \to 4, 4, 1, 3, 2, 2$. Find $g \circ f$.

Problem 8.12. Let $f, g : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^3 + 1$ for all $x \in \mathbf{R}$ and $g(t) = 1 - t$ for all $t \in \mathbf{R}$. Find $(f \circ g)(3)$ and $(g \circ f)(3)$.

Problem 8.13. Let $f : A \to B$ be a function.

i. Let $Y \subset B$ be a subset. Prove that $f(f^{-1}(Y)) = Y$ if and only if $Y \subset \operatorname{image} f$.
ii. Prove that $X \subset f^{-1}(f(X))$ for all $X \subset A$.

Problem 8.14. Let $f : A \to B$ be a function. Let X_1 and X_2 be subsets of A and Y_1 and Y_2 be subsets of B. Prove the following:

i) $f(X_1 \cup X_2) = f(X_1) \cup f(X_2)$
ii) $f(X_1 \cap X_2) \subset f(X_1) \cap f(X_2)$
iii) $f^{-1}(Y_1 \cup Y_2) = f^{-1}(Y_1) \cup f^{-1}(Y_2)$
iv) $f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$
v) Give an example where $f(X_1 \cap X_2) \neq f(X_1) \cap f(X_2)$.

Problem 8.15. Let A be a set and let $f : A \to \{1, 2, 3, 4\}$ be a function. Let $A_i = f^{-1}(\{i\})$ for $i = 1, 2, 3$, and 4. Note: f is not assumed to be one to one or onto.

i) Prove that $A_i \cap A_j = \emptyset$ for $i \neq j$.
ii) Prove that $A = A_1 \cup \ldots \cup A_4$

9. INVERSE FUNCTIONS

Definition 9.1 (One to one)**.** A function $f\colon A \to B$ is *one to one (injective)* if $f(x_1) = f(x_2)$ for some x_1 and x_2 in A always implies that $x_1 = x_2$.

Definition 9.2 (Onto)**.** A function $f\colon A \to B$ is *onto (surjective)* if for each $y \in B$ there is an $x \in A$ such that $f(x) = y$.

Note 9.3. f is onto if and only if $f(A) = B$.

Definition 9.4 (Restriction)**.** Let $f\colon A \to B$ be a function and let X be a subset of A. Define $f|_X : X \to A$, the *restriction* of f to X, by $f|_X(x) := f(x)$ for all $x \in X$.

Definition 9.5 (Identity function)**.** The *identity function* $id_A\colon A \to A$ is defined by $id_A(x) := x$ for all $x \in A$.

Definition 9.6 (Inverse)**.** Let $f\colon A \to B$ be a function. Define $\Gamma_{f^{-1}} := \{(b, a) \in B \times A \mid (a, b) \in \Gamma_f\}$.

Proposition 9.7. *$\Gamma_{f^{-1}}$ is a function from B to A if and only if f is one to one and onto.*

Definition 9.8. If f is one to one and onto, then the *inverse function* f^{-1} from B to A is defined by $f^{-1}(y) := x$ for $y \in B$ if and only if $y = f(x)$ for $x \in A$.

Proposition 9.9. *Let $f : A \to B$ and $g : B \to C$ be one to one and onto functions. Then $g \circ f : A \to C$ is one to one and onto; and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

PROBLEMS 9.

Problem 9.1. Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5\}$. Define a function $f : A \to B$ by $f(1) = 2, f(2) = 2, f(3) = 5, f(4) = 4$.

i. Is f one to one? Why or why not?

ii. Is f onto? Why or why not?

Problem 9.2. Show that the function $f : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ given by $f(m, n) := 2^m 3^n$ is one to one.

Problem 9.3. Let $f \colon \mathbf{R} \to \mathbf{R}$ be defined by $f(x) := x^2$. Let $X = \{x \in \mathbf{R} \mid x \geq 0\}$. Is f one to one? Is the restriction $f|_X$ one to one? Justify your answers.

Problem 9.4. Let A, B, C be non-empty sets and let $f \colon A \to B$ and $g \colon B \to C$ be functions.

a) If f and g are one to one, prove that $g \circ f$ is one to one.

b) If If f and g are onto, prove that $g \circ f$ is onto.

Problem 9.5. Let $A_1, A_2, \dots$ and $B_1, B_2, \dots$ be sets and let $f_i \colon A_i \to B_i$ be an onto function for each $i \geq 1$. If $f_i(x) = f_j(x)$ for all $x \in A_i \cap A_j$ for each $i, j \geq 1$, show that $h \colon \bigcup_{i=1}^{\infty} A_i \to \bigcup_{i=1}^{\infty} B_i$ defined by $h(x) := f_i(x)$ for $x \in A_i$ for each $i \geq 1$, is onto.

Problem 9.6. Let $A_1, A_2, \dots$ and $B_1, B_2, \dots$ be sets and let $f_i \colon A_i \to B_i$ be a one to one function for each $i \geq 1$. If $A_i \cap A_j = \varnothing$ for each $i, j \geq 1$, and if $B_i \cap B_j = \varnothing$ for each $i, j \geq 1$ show that $h \colon \bigcup_{i=1}^{\infty} A_i \to \bigcup_{i=1}^{\infty} B_i$ defined by $h(x) := f_i(x)$ for $x \in A_i$ for each $i \geq 1$, is one to one.

Problem 9.7. Let $f : A \to B$ be a one to one function. Let $X \subset A$ be a subset. Let x be an element of A. Prove that if $f(x) \in f(X)$ then $x \in X$.

Problem 9.8. Let $f : A \to B$ be a function. Prove that $X = f^{-1}(f(X))$ for all $X \subset A$ if and only if f is one to one. Note: part of the problem was done in Problem 8.13.ii of Section 8.

Problem 9.9. Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 3, 4, 5\}$. Define $f : A \to B$ by $1, 2, 3, 4 \to 4, 2, 5, 3$. Check that f is one to one and onto and find the inverse function f^{-1}.

Problem 9.10. Let $f : A \to B$ and $g : B \to A$ be functions.

i) Prove that $g \circ f = id_A$ implies f is one to one.

ii) Prove that $f \circ g = id_B$ implies f is onto.

Problem 9.11. Let $f : A \to B$ be one to one and onto, and let $g : B \to A$ be a function.

i) If $g \circ f = id_A$, prove that $g = f^{-1}$.

ii) If $f \circ g = id_B$, prove that $g = f^{-1}$.

Problem 9.12. Let $f : A \to B$ be a function, where A and B are non-empty sets. If f is one to one, prove there exists a function $g : B \to A$ such that $g \circ f = id_A$. *Hint: define g on $f(A)$ to make $g \circ f = id_A$. Then define g on $B - f(A)$. Does it matter how you define g on $B - f(A)$?*

Problem 9.13. Prove Proposition 9.9.

10. EQUIVALENCE OF SETS

Definition 10.1 (Correspondence)**.** A function $f\colon A \to B$ is a *one to one correspondence* if f is one to one and onto.

Definition 10.2 (Equivalent sets)**.** Sets A and B are *equivalent*, written $A \sim B$, if there exists a one to one correspondence $f\colon A \to B$.

Definition 10.3 (Cardinality)**.** Sets A and B *have the same cardinality* if they are equivalent.

Proposition 10.4. *Equivalence of sets is:*

(1) reflexive ($A \sim A$)
(2) symmetric ($A \sim B$ implies $B \sim A$)
(3) transitive ($A \sim B$ and $B \sim C$ implies $A \sim C$).

Note 10.5. If A and B are sets, then the two main ways to show that $A \sim B$:

(1) Find a function from A to B that is one to one and onto, or
(2) Find a set C such that $A \sim C$ and $B \sim C$.

Proposition 10.6. *Let A and B be sets. Then $(A \times B) \sim (B \times A)$.*

Lemma 10.7 (Gluing Together Lemma)**.** *If $A \sim C$ and $B \sim D$, where A and B are disjoint and C and D are disjoint, then $\big(A \cup B\big) \sim \big(C \cup D\big)$.*

Lemma 10.8 (One Point Reduction Lemma)**.** *Let A and B be sets with $a \in A$ and $b \in B$, such that $A \neq \{a\}$ and $B \neq \{b\}$. If $A \sim B$, then $A - \{a\} \sim B - \{b\}$.*

Definition 10.9 (Finite set)**.** A set is *finite* if either

(1) the set is empty, or
(2) the set is equivalent to $\{1, \ldots, n\}$ for some positive integer n.

Proposition 10.10. *The sets $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$ are not equivalent for $m \neq n$.*

Proposition 10.11. *A subset of a finite set is finite.*

Lemma 10.12. *If A and B are disjoint finite sets, then $A \cup B$ is finite.*

Proposition 10.13. *If A and B are finite sets, then $A \cup B$ is finite.*

Definition 10.14 (Infinite set)**.** A set is *infinite* if it is not finite.

Proposition 10.15. $\mathbf{N}$ *is infinite.*

Definition 10.16 (Number of elements)**.**

$$\text{A set } A \text{ has } \begin{cases} 0 \text{ elements if } A = \emptyset \\ n \text{ elements if } A \sim \{1, \ldots, n\}, \text{ for } n \text{ a positive integer} \\ \text{infinitely many elements if } A \text{ is infinite.} \end{cases}$$

PROBLEMS 10.

Problem 10.1. Prove Proposition 10.4.

Problem 10.2. The sets $\mathbf{N}$ and $\mathbf{Z}$ are equivalent via the function f which sends $1, 2, 3, 4, 5, 6, 7, 8 \ldots$ to $0, -1, 1, -2, 2, -3, 3, -4 \ldots$. Write this function in closed form (that is, find a formula for $f(n)$). Use the closed form expression for f to prove that f is a one to one correspondence.

Problem 10.3. Let m be an integer and let n be a positive integer. Show that $\{1, \ldots, n\} \sim \{m+1, \ldots, m+n\}$.

Problem 10.4. Let m be an integer. Show that $\mathbf{N} \sim \{n \in \mathbf{Z} \mid n \geq m\}$.

Problem 10.5. Prove (without using calculus) that the following sets have the same cardinality:

i) The set of positive integers which are perfect squares and the set of positive integers which are perfect cubes.

ii) The open intervals $(0, 2)$ and $(0, 11)$.

iii) Any two open intervals (a, b) and (c, d) where $a < b, c < d$ and a, b, c, d are all (finite) real numbers.

iv) Any two closed intervals $[a, b]$ and $[c, d]$ *satisfying the same conditions as iii).*

v) The open interval $(-1, 1)$ and $\mathbf{R}$. *Hint: use* $x/(1-x^2)$.

vi) The open interval $(0, 1)$ and $\mathbf{R}$. Hint: use v).

Problem 10.6. Let B be a set, with $b, c \in B$, where $b \neq c$. Define $g \colon B \to B$ by $g(b) := c$, $g(c) := b$, and $g(y) := y$ for $y \in B$, $y \neq b, c$. Show that g is one to one and onto.

Problem 10.7. Let A and B be equivalent sets. Prove that the power sets 2^A and 2^B are equivalent.

Problem 10.8. Let $A_1, \ldots, A_n$ be non-empty sets. Prove

$$(A_1 \times \ldots \times A_{n-1}) \times A_n \sim A_1 \times \ldots \times A_n.$$

Problem 10.9 (Gluing together finitely many equivalent sets)**.** Let $A_1, A_2, \ldots, A_k$ be disjoint sets ($A_n \cap A_m = \emptyset$ for $n \neq m$), with $A = \bigcup_{n=1}^{k} A_n$ (that is, $x \in A$ if and only if $x \in A_n$ for some integer $1 \leq n \leq k$). Similarly, let $B_1, B_2, \ldots, B_k$ be disjoint sets, with $B = \bigcup_{n=1}^{k} B_n$. Prove that if $A_n \sim B_n$ for each $1 \leq n \leq k$, then $A \sim B$.

Problem 10.10 (Gluing together countably many equivalent sets)**.** Let $A_1, A_2, \ldots$ be disjoint sets, with $A = \bigcup_{n=1}^{\infty} A_n$ (that is, $x \in A$ if and only if $x \in A_n$ for some integer $n \geq 1$). Similarly, let $B_1, B_2, \ldots$ be disjoint sets, with $B = \bigcup_{n=1}^{\infty} B_n$. Prove that if $A_n \sim B_n$ for each $n \geq 1$, then $A \sim B$.

11. COUNTABILITY

Definition 11.1 (Countable)**.** A set A is *countable* if either A is finite or $\mathbf{N} \sim A$. In the latter case, A is *countably infinite.*

Definition 11.2 (Uncountable)**.** A set A is *uncountable* if it is not countable.

Proposition 11.3. *The set* $\mathbf{Q}$ *of rational numbers is countable.*

Proposition 11.4. *The union of a countably infinite set and a finite set is countable.*

Proposition 11.5. *The union of two disjoint countably infinite sets is countable.*

Proposition 11.6. *If* A *and* B *are disjoint countable sets, then* $A \cup B$ *is countable.*

Lemma 11.7. *An infinite subset of* $\mathbf{N}$ *is countable.*

Proposition 11.8. *A subset of a countable set is countable.*

Corollary 11.9. *If* A *and* B *are countable sets, then* $A \cup B$ *is countable.*

Proposition 11.10. *Let* A *be a non-empty set. Then*
(1) A *is countable if and only if*
(2) $\exists g : \mathbf{N} \to A$ *such that* g *is onto if and only if*
(3) $\exists f : A \to \mathbf{N}$ *such that* f *is one to one.*

Proposition 11.11. *The union of a countable collection of countable sets is countable.*

Proposition 11.12. *The set* $\mathbf{R}$ *of real numbers is uncountable.*

Corollary 11.13. *The set of irrational numbers is uncountable.*

Proposition 11.14. *Let* A *be a set. Then* A *and* 2^A *are not equivalent.*

PROBLEMS 11.

Problem 11.1. Prove that if A and B are countably infinite sets, then $A \times B$ is countably infinite. *Hint: arrange the product so it can be counted.*

Problem 11.2. Prove that if $A_1, \ldots, A_n$ are countably infinite sets then so is $A_1 \times \ldots \times A_n$. Hint: use problem 11.1 and problem 10.8 of Section 10 to induct.

Problem 11.3. For each of the following give an example of finite sets $A_1, A_2, \ldots$ such that: $\bigcup_{i=1}^{\infty} A_i = \emptyset$; $\bigcup_{i=1}^{\infty} A_i$ is finite; $\bigcup_{i=1}^{\infty} A_i$ is infinite.

Problem 11.4. Prove Proposition 11.6.

Problem 11.5. Prove that the intervals $(0,1)$ and $[0,1]$ are equivalent as follows: Let $B_1 := \{1/2, 1/3, 1/4, \ldots\}$. Let $A_1 := A_2 := (0,1) - B_1$. Let $B_2 := \{0,1\} \cup B_1$. Check that $(0,1) = A_1 \cup B_1$ and that $[0,1] := A_2 \cup B_2$. Now use the Gluing Together Lemma (Proposition 10.7) to prove that $(0,1)$ and $[0,1]$ are equivalent.

Problem 11.6. Let $f : A \to 2^A$ be a function. We say that x in A is 'self-included' if $x \in f(x)$.

Let $A = \{1, \ldots, 20\}$. Define $f : A \to 2^A$ by

$$f(n) = \{m^2 \mid m \in \mathbf{N}, m^2 \in A, m \text{ divides } n \text{ and } m \neq 1, n\}.$$

Find $S = \{x \in A \mid x \text{ is self included}\}$.

Problem 11.7. Denote by B^A the set of all functions from A to B

$$B^A = \{f \mid f : A \to B\}$$

Use the Cantor diagonalization process to prove that $\{0,1\}^{\mathbf{N}}$ is uncountable.

12. ALGEBRAIC AND TRANSCENDENTAL NUMBERS

Definition 12.1. A real number a is *algebraic* if it is the root of a non-zero polynomial with *integer* coefficients.

Proposition 12.2. *The set of all real algebraic numbers is countable.*

Definition 12.3. A real number a is *transcendental* if it is not algebraic.

Proposition 12.4. *The set of real transcendental numbers is uncountable.*

PROBLEMS 12.

Problem 12.1. i. Show that $\sqrt{3}$ is algebraic. Hint: what quadratic polynomial with integer coefficients has $\sqrt{3}$ as a root?

ii. Prove that each rational number is algebraic. Hint: find a linear polynomial $a+bx$ for which p/q is a root.

Problem 12.2. Let n be a positive integer. Prove there are only countably many non-zero polynomials of degree at most n with integer coefficients. Hint: show the set of polynomials of degree at most n is equivalent to $\mathbf{Z}\times\ldots\times\mathbf{Z}$ ($n+1$ times). What does this say about the countability of the set of *non-zero* polynomials of degree at most n?

Problem 12.3. Since each non-zero polynomial of degree at most n has at most n real roots, prove that there are only countably many roots of all non-zero polynomials with integer coefficients and degree at most n.

Problem 12.4. Prove Proposition 12.2

Remark: It is possible, but very difficult, to prove that e and π are transcendental. (Note that π is a root of a polynomial with real coefficients, for example $x^2-3\pi x+2\pi^2$, but not a root of a polynomial with integer coefficients).

Problem 12.5. Prove Proposition 12.4. (So you have proved there are infinitely many transcendental numbers, despite the fact that you probably could only find two of them). Hint: $\mathbf{R}$ is the union of the algebraic and transcendental numbers.

13. INFINITE SEQUENCES AND LIMITS

Definition 13.1 (Absolute value)**.** If x is a real number, then the absolute value of x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

Alternatively,

$$|x| = \sqrt{x^2}.$$

Proposition 13.2 (Properties of Absolute Value)**.** *Absolute value satisfies the following properties, where a, x, and y denote real numbers:*

(1) $|xy| = |x||y|$
(2) $-|x| \leq x \leq |x|$
(3) For $r > 0$, $|x - a| < r$ *if and only if* $a - r < x < a + r$
(4) The triangle inequality: $|x + y| \leq |x| + |y|$

Definition 13.3 (Infinite Sequence)**.** An infinite sequence is a function $A\colon \mathbf{N} \to \mathbf{R}$. Note: Instead of writing $A(n)$ for the value of the function A at a positive integer n, we usually write a_n.

Note: the sequence itself is usually represented by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

Definition 13.4. If $\{a_n\}$ is an infinite sequence and c is a real number, then $\{ca_n\}$ is the sequence that takes n to ca_n for each $n \in \mathbf{N}$.

Definition 13.5. If $\{a_n\}$ and $\{b_n\}$ are infinite sequences, then $\{a_n + b_n\}$ is the sequence that takes n to $a_n + b_n$ for each $n \in \mathbf{N}$. The sequences $\{a_n - b_n\}$, $\{a_n b_n\}$ and $\{a_n/b_n\}$ are defined similarly (the quotient sequence is only defined if $b_n \neq 0$ for all n).

Definition 13.6 (Limit)**.** An infinite sequence $\{a_n\}$ has *limit* L, written $\lim\limits_{n\to\infty} a_n = L$, if for each $\varepsilon > 0$ there exists a positive integer N such that $|a_n - L| < \varepsilon$ whenever $n \geq N$.

Definition 13.7 (Convergence)**.** An infinite sequence $\{a_n\}$ *converges* to a real number L, written $a_n \to L$, if $\lim\limits_{n\to\infty} a_n = L$. The sequence $\{a_n\}$ *converges* if there exists a real number L such that $a_n \to L$.

Definition 13.8 (Divergence)**.** An infinite sequence $\{a_n\}$ *diverges* if it does not converge.

Lemma 13.9 (Limit of a constant sequence)**.** *Let $\{a_n\}$ be an infinite sequence such that $a_n = c$ for all n. Then $a_n \to c$. [See Problems.]*

Lemma 13.10. *Let $\{a_n\}$ be an infinite sequence. Then $a_n \to L$ if and only if $a_n - L \to 0$. [See Problems.]*

Proposition 13.11 (An infinite sequence has at most one limit)**.** *If $a_n \to L$ and $a_n \to M$, then $L = M$.*

Proposition 13.12. *Let $\{a_n\}$ be an infinite sequence and let c be a real number. If $a_n \to a$, then $ca_n \to ca$.*

Proposition 13.13 (Limit Theorems)**.** *Let $\{a_n\}$ and $\{b_n\}$ be sequences. If $a_n \to a$ and $b_n \to b$ then*

(1) $a_n + b_n \to a + b$ (Limit of sum is sum of limits)
(2) $a_n b_n \to ab$ (Limit of product is product of limits)
(3) If $b \neq 0$ and $b_n \neq 0$ for all $n \in \mathbf{N}$, then $a_n/b_n \to a/b$ (Limit of quotient is quotient of limits if denominator does not approach 0).

Corollary 13.14. *Let $\{a_n\}$ be an infinite sequence and let m be a positive integer. Then $a_n \to a$ implies that $(a_n)^m \to a^m$.*

Proposition 13.15. *Let $\{a_n\}$ and $\{b_n\}$ be sequences, $b_n \neq 0$ for all $n \in \mathbf{N}$. If $a_n \to a$ and $b_n \to 0$, where $a \neq 0$, then a_n/b_n diverges.*

Note 13.16. The only case combining convergent sequences that is not covered by the limit theorems is for the quotient when $a_n \to 0$, $b_n \to 0$, and $b_n \neq 0$ for all $n \in \mathbf{N}$. The limit of a_n/b_n is said to be *indeterminate* (that is, the limit theorems do not give an immediate result) and convergence or divergence must actually be worked out.

Lemma 13.17. *If $a_n \geq 0$ and $a_n \to a$, then $a \geq 0$.*

Note 13.18. If $a_n > 0$ and $a_n \to a$, then $a > 0$ is not necessarily true; all we know is that $a \geq 0$.

Lemma 13.19. *If $a_n \to a$ and $a > 0$, then there exists $N \in \mathbf{N}$ such that $a_n > 0$ for all $n \geq N$.*

Note 13.20. If $a_n \to a$ and $a \geq 0$, then a_n might not satisfy $a_n \geq 0$ for even one $n \in \mathbf{N}$.

Proposition 13.21 (Squeeze Lemma)**.** *Let $\{a_n\}, \{b_n\},$ and $\{c_n\}$ be infinite sequences such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbf{N}$. If $a_n \to L$ and $c_n \to L$ (same limit for both), then $b_n \to L$.*

PROBLEMS 13.

Problem 13.1. Use the limit theorems to find $\lim_{n\to\infty} 4 - \frac{5}{n^3}$. Use the definition of limit, not the limit theorems, to check that your answer is correct.

Problem 13.2. Use the limit theorems to find $\lim_{n\to\infty} \frac{6n+4}{8n+1}$. Use the definition of limit, not the limit theorems, to check that your answer is correct.

Problem 13.3. Give an example of divergent sequences $\{a_n\}$ and $\{b_n\}$, such that the sequence $\{a_n + b_n\}$ converges.

Problem 13.4. Let $a_n \to 0$. Use the definition of limit, not the limit theorems, to check that $(a_n)^2 \to 0$.

Problem 13.5. Prove Lemma 13.9.

Problem 13.6. Prove Proposition 13.2, parts (1) and (2).

Problem 13.7. Prove Lemma 13.10.

Problem 13.8. Let $\{a_n\}$ and $\{b_n\}$ be sequences. If $a_n \to a$ and $b_n \to b$ prove that $a_n - b_n \to a - b$ (Limit of difference is difference of limits).

Problem 13.9. Prove Corollary 13.14.

Problem 13.10. If $a_n \to a$, $b_n \to b$ and $a_n \leq b_n$, prove that $a \leq b$.

Problem 13.11. Let a and b be real numbers, with $a < b$. If $a_n \to a$, prove there exists $N \in \mathbf{N}$ such that $a_n < b$ for all $n \geq N$.

Problem 13.12. Prove Proposition 13.21.

14. LEAST UPPER BOUND AXIOM

Definition 14.1. Let A be a set of real numbers. Then A is *bounded above* if there exists $u \in \mathbf{R}$ such that $x \leq u$ for all $x \in A$. Such a u is called an *upper bound* for A.

Note: it is often easier to show that u is an upper bound for A by using the following:

Lemma 14.2. *u is an upper bound for A if and only if $x > u$ implies that x is not in A.*

Definition 14.3. Let A be a set of real numbers. Then A is *bounded below* if there is $b \in \mathbf{R}$ such that $x \geq b$ for all $x \in A$. Such a b is called an *lower bound* for A.

Lemma 14.4. *b is an lower bound for A if and only if $x < b$ implies that x is not in A.*

Definition 14.5. Let A be a set of real numbers. Then l is a *least upper bound* for A (LUB for short) if

1. l is an upper bound for A, and
2. $l \leq u$ for every upper bound u for A.

Note: it is often easier to show that l is a least upper bound for A by using the following:

Lemma 14.6. *l is a least upper bound for A if and only if*

(1) $x > l$ implies x is not in A, and
(2) for each real number u less than l there exists an $x \in A$ such that $u < x \leq l$.

Proposition 14.7. *Let A be a set of real numbers. Then A has at most one least upper bound.*

Note 14.8. If A has a least upper bound l, then $l \in A$ and $l \notin A$ are each possible.

Definition 14.9 (sup/inf)**.** Let A be a non-empty set of real numbers. The *supremum* of A, denoted $\sup A$, is defined by

$$\sup A := \begin{cases} \text{the LUB of } A, & \text{if } A \text{ is bounded above} \\ \infty, & \text{if } A \text{ is not bounded above.} \end{cases}$$

Similarly, the *infimum* of A, denoted $\inf A$, is is defined by

$$\inf A := \begin{cases} \text{the GLB of } A, & \text{if } A \text{ is bounded below} \\ -\infty, & \text{if } A \text{ is not bounded below.} \end{cases}$$

Axiom 14.10 (Least Upper Bound Axiom)**.** *Let A be a set of real numbers. If*

(1) A is non-empty and
(2) A is bounded above

then A has a least upper bound.

Note 14.11. The set of rational numbers does not satisfy the Least Upper Bound Axiom. That is, there exists a non-empty set A of rational numbers such that A is bounded above, but for which there is no rational number that is a least upper bound.

We can now prove two results we presented without proof earlier. The first is the Archimedean Principle.

Proposition 14.12 (Archimedean Principle)**.** *If x and y are positive real numbers, then there exists a positive integer n such that $nx > y$.*

The second result left over from before is the existence of an n-th root. First we prove a Lemma.

Lemma 14.13. *Let $a > 0$ be a real number and let $A := \{x > 0 \mid x^n < a\}$. Let $u > 0$ be a real number. Then*

(1) u is an upper bound for A if and only if $u^n \geq a$.
(2) If $u^n > a$, then u is not a least upper bound for A.

Note 14.14. That the set A of Lemma 14.13 is non-empty and bounded above is an exercise.

Proposition 14.15. *If a is a positive real number and n is a positive integer, then a has a real n-th root. That is, there exists a real number b such that $b^n = a$.*

PROBLEMS 14.

Problem 14.1. Let $A := \{x \in R \mid x^3 - 10^8 x^2 < 30\}$. Prove carefully that A has a least upper bound. Hint: factor the left side of the inequality.

Problem 14.2. Prove that the set A of Lemma 14.13 is non-empty and bounded above. Hint: break into cases: $a < 1$, $a = 1$, and $a > 1$.

Problem 14.3. Let A and B be sets of real numbers such that $A \subset B$, A is non-empty, and B is bounded above. Prove that A and B each have a least upper bound and that $\sup A \leq \sup B$.

Problem 14.4. Let A be a non-empty set of real numbers. Let u be an upper bound for A and let b be a lower bound for A. Prove that $b \leq u$. Is this still true without the condition on A that it be non-empty? Justify your answer.

Problem 14.5. Formulate conditions similar to those in Lemma 14.6 for g to be the greatest lower bound of a set A.

Problem 14.6. Prove that the Least Upper Bound Axiom implies the Greatest Lower Bound Axiom, namely that every non-empty set which is bounded below has a greatest lower bound (GLB).

Problem 14.7. If a set A of real numbers has both a least upper bound and a greatest lower bound, show that $\inf A \leq \sup A$.

Problem 14.8. Let A and B be non-empty subsets of $\mathbf{R}$ with the following property: for each $a \in A$ and for each $b \in B$, the inequality $a \leq b$ is always true.

i. Prove that $\sup A \leq b$ for each $b \in B$.

ii. Prove that $\sup A \leq \inf B$.

15. MONOTONE SEQUENCE PROPERTY

Definition 15.1. A sequence $\{a_n\}$ is *bounded above* if there exists $M \in \mathbf{R}$ such that $a_n \leq M$ for all $n \in \mathbf{N}$. Similarly, the sequence is *bounded below* if there exists $m \in \mathbf{R}$ such that $a_n \geq m$ for all $n \in \mathbf{N}$. The sequence is *bounded* if it is bounded above and below.

Proposition 15.2. *A convergent sequence is bounded.*

Definition 15.3. A sequence $\{a_n\}$ is *increasing* if $a_{n+1} \geq a_n$ for all $n \in \mathbf{N}$. Similarly, the sequence is *decreasing* if $a_{n+1} \leq a_n$ for all $n \in \mathbf{N}$. A sequence is *monotone* if either it is increasing or decreasing.

In this terminology, a sequence is *strictly increasing* if $a_{n+1} > a_n$ for all $n \in \mathbf{N}$. Some call this 'increasing' and call $a_{n+1} \geq a_n$ for all n 'non-decreasing'. Similarly, we call $a_{n+1} < a_n$ for all $n \in \mathbf{N}$ *strictly decreasing*. Some call this 'decreasing' and call $a_{n+1} \leq a_n$ for all n 'non-increasing'.

Proposition 15.4 (The Monotone Sequence Property)**.** *An increasing sequence that is bounded above converges. A decreasing sequence that is bounded below converges.*

Proposition 15.5. *Let $\{a_n\}$ be a bounded sequence. Let $m_n := \inf\{a_k \mid k \geq n\}$ and let $M_n := \sup\{a_k \mid k \geq n\}$. Then*

(1) $\{m_n\}$ is increasing and $\{M_n\}$ is decreasing, and
(2) $m_i \leq M_j$ for all positive integers i and j.

In particular, $\{m_n\}$ and $\{M_n\}$ each converge and $\lim_{n\to\infty} m_n \leq \lim_{n\to\infty} M_n$.

Definition 15.6 (lim sup/lim inf)**.** Let $\{a_n\}$ be a bounded sequence. With the notation from the proposition above, we define $\liminf a_n$ and $\limsup a_n$ by

$$\liminf a_n := \lim_{n\to\infty} m_n \quad \text{and}$$
$$\limsup a_n := \lim_{n\to\infty} M_n.$$

Proposition 15.7. *Let $\{a_n\}$ be a bounded sequence. Then $a_n \to L$ if and only if $\liminf a_n = L$ and $\limsup a_n = L$.*

Proposition 15.8. *Let A and x_1 be positive real numbers. Define a sequence by $x_n := \frac{1}{2}(x_{n-1} + \frac{A}{x_{n-1}})$ for $n \geq 2$. Then $x_n \to \sqrt{A}$.*

PROBLEMS 15.

Problem 15.1. Let $a_n := 1/1 + 1/2 + \ldots + 1/n - \ln n$. Show that $\{a_n\}$ is decreasing. Note that $\ln n := \int_1^n 1/x \, dx$. (You may find it useful to look at the picture accompanying the proof of the integral test in any calculus book.)

Problem 15.2. (continuation of Problem 15.1) Let $b_n := 1/1 + 1/2 + \ldots + 1/(n-1) - \ln n$ for $n \geq 2$. Show that $b_n \geq 0$ for all $n \geq 2$. Hint: this is similar to the integral test. Use the result on b_n to conclude that $\{a_n\}$ is bounded below.

Problem 15.3. (continuation of Problem 15.1) Show that $\{a_n\}$ converges. The limit is called Euler's Constant or the Euler-Mascheroni Constant and is denoted by γ. The value of γ is approximately $.5772157\ldots$; it is unknown whether γ is irrational.

16. SERIES

Definition 16.1 (Formal Definition of Series)**.** Let $\{a_n\}$ be a sequence. Define a sequence $\{S_n\}$ by $S_n := a_1 + \ldots + a_n$. Then the series $\sum_{n=1}^{\infty} a_n$ is defined to be the sequence $\{S_n\}$, and S_n is called the n-th partial sum of the series.

Definition 16.2 (Convergence of Series)**.** A series $\sum_{n=1}^{\infty} a_n$ converges to L, written $\sum_{n=1}^{\infty} a_n = L$, if $S_n \to L$ as $n \to \infty$, where S_n is the n-th partial sum. The series $\sum_{n=1}^{\infty} a_n$ converges if there exists an L such that $S_n \to L$.

Definition 16.3 (Divergence of Series)**.** A series $\sum_{n=1}^{\infty} a_n$ diverges if it does not converge.

Proposition 16.4 (Convergence Properties)**.** *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series, and let c be a constant. Then $\sum_{n=1}^{\infty} ca_n$, $\sum_{n=1}^{\infty} a_n + b_n$ and $\sum_{n=1}^{\infty} a_n - b_n$ converge, and*

(1) $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$
(2) $\sum_{n=1}^{\infty} a_n + b_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$
(3) $\sum_{n=1}^{\infty} a_n - b_n = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Note 16.5. There are no general results for $\sum_{n=1}^{\infty} a_n \cdot b_n$ or for $\sum_{n=1}^{\infty} a_n/b_n$.

Proposition 16.6 (Divergence Test)**.** *If a_n does not converge to 0, then the series $\sum_{n=1}^{\infty} a_n$ diverges. WARNING: if $a_n \to 0$, this test is indeterminate.*

Proposition 16.7. *Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n \geq 0$. If the partial sums are bounded above, then $\sum_{n=1}^{\infty} a_n$ converges.*

Proposition 16.8 (Comparison Test)**.** *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with $a_n \geq 0$, $b_n \geq 0$, and $a_n \leq b_n$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges (and if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges).*

Proposition 16.9. *The series $\sum_{n=1}^{\infty} 1/n$ diverges and the series $\sum_{n=1}^{\infty} 1/n^2$ converges.*

Note 16.10. Sometimes a sequence is indexed beginning with k instead of 1 (so we think of the sequence as a function from the set of integers $n \geq k$ to $\mathbf{R}$ rather than from $\mathbf{N}$ to $\mathbf{R}$). The sequence is then denoted by $\{a_n\}_{n=k}^{\infty}$. The definition of limit is the same as before except we now require $N \geq k$ in the definition. The corresponding series is $\sum_{n=k}^{\infty} a_n$, where the n-th partial sum S_n is now defined by $S_n := a_k + \ldots + a_n$ for $n \geq k$. So for example, if a sequence starts with a_0, then the series $\sum_{n=0}^{\infty} a_n$ is by definition the sequence $\{S_n\}_{n=0}^{\infty}$ and the series converges if $\lim_{n\to\infty} S_n$ exists, as before.

Proposition 16.11. $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ *if $-1 < r < 1$ and diverges for $r \geq 1$ or $r \leq -1$. Note that the sum starts with $n = 0$.*

Proposition 16.12 (Limit Comparison Test)**.** *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with $a_n > 0$ and $b_n > 0$ for all $n \geq 1$. If $a_n/b_n \to L \neq 0, \infty$, then either both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or both diverge.*

Proposition 16.13 (Ratio Test). *Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n > 0$ for all $n \geq 1$. If $\lim_{n\to\infty} a_{n+1}/a_n = \rho$, then the series $\sum_{n=1}^{\infty} a_n$*

(1) converges if $\rho < 1$
(2) diverges if $\rho > 1$
(3) is indeterminate if $\rho = 1$

PROBLEMS 16.

Problem 16.1. Prove the Convergence Properties of Proposition 16.4.

Problem 16.2. (Continuation of Proposition 16.8.) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series, where $0 \le a_n \le b_n$ for all $n \ge 1$. If $\sum_{n=1}^{\infty} b_n$ converges, prove that $0 \le \sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} b_n$.

Problem 16.3. Give an example of divergent series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ such that $\sum_{n=1}^{\infty} a_n + b_n$ converges.

Problem 16.4. Does $\sum_{n=1}^{\infty} 1/(n^2 - 1/2)$ converge? Justify your answer.

Problem 16.5. Does $\sum_{n=1}^{\infty} n^4/(n^6 - 1/3)$ converge? Justify your answer.

Problem 16.6. Does $\sum_{n=1}^{\infty} n^4/(n^5 + 1/3)$ converge? Justify your answer.

Problem 16.7. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence and let m be a positive integer. Show that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=m}^{\infty} a_n$ converges. Furthermore, show that if $\sum_{n=m}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n = a_1 + \ldots + a_{m-1} + \sum_{n=m}^{\infty} a_n$. Note: convergence only depends on convergence from m on is often referred to as "the tail of a series determines convergence."

Problem 16.8. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series, where $a_n \ge 0$. Prove that $\sum_{n=1}^{\infty} {a_n}^2$ converges. Hint: first show that $a_n \le 1$ for n large enough.

Problem 16.9. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Show that the *telescoping series*

$$\sum_{n=1}^{\infty} (a_n - a_{n+1})$$

converges if and only if the sequence $\{a_n\}$ converges. If $a_n \to L$, evaluate the series $\sum_{n=1}^{\infty} (a_n - a_{n+1})$.