Notes for Chaos, Fractals, Complexity, Self-Organization and Emergence
Adult Learning Class - Fall 2016 at Agnes Scott College - Bradley Observatory

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Contents

Chapter 1

Chaos

1.1. What is Chaos?

Let us make a first, quick pass over many ideas related to the subject of **chaos**. We will cover all of these ideas again in more detail later in these notes. Our first pass is desgined to let us see the big picture.

Let me first illustrate(in class) some real systems, namely the *chaotic calculator* and the *Barnsley game*. Both will seem like silly exercises, but we will see later that there is very deep stuff happening in these systems.

The irregular aspects of nature, the discontinuous or erratic side, such as *fractal structures* I will illustrate in class, or the behavior of simple equations such as the *logistic map*, the *roots of three*, the *Mandelbrot set*, or the *Lorenz map* (we will illustrate these in class), have always been regarded as puzzles or even monstrosities; this also includes things like turbulence, fluctuations, etc.

These phenomena were always regarded as hopelessly **"chaotic"** (without any understanding of the meaning of the term) and thus, not possible to understand.

The study of chaos has heavily used computer calculations and computer graphics in an effort to better understand the underlying complexities; we will use these tools to demonstrate various phenomena in this class and to run simulations.

All kinds of new words and a new language has appeared such as *fractals, bifurcations, limit cycles, attractors, etc* - we will attempt to understand them all and learn to **"speak"** this language.

Chaos seems (now) to be everywhere - smoke swirls, flag waving, dripping faucets, cars on a highway, fluid flow, etc.

All of these phenomena are seemingly very different, but, as we will see, they all obey the same set of new laws.

Chaos is a science of the **GLOBAL** nature of systems - it crosses over the traditional boundaries between disciplines and to understand it we will need to mix disciplines.

The studies we will be carrying out will show that one needs less specialization in education - specialized training is too narrow to deal with the systems we will investigate. I do not mean to imply that one should not have extensive knowledge of a field - what I am saying is that your training should encompass more than one field(in depth).

Accepted methods will fail ....

Complex, chaotic systems have universal(common) aspects - universality will appear.

Scientistis that study chaos look for patterns, especially patterns that are independent of scale - they explore randomness and complexity, jagged edges and sudden changes(discontinuities).

> **Relativity** eliminated the Newtonian illusion of absolute space and time - that space and time were independent.
>
> **Quantum Mechanics** eliminated the Newtonian dream of a controllable measurement process.
>
> **Chaos**, as we will see, eliminates the Laplacian fantasy of deterministic predictability.

The mainstream study of physics in the past century has really been the study of the **"building blocks of matter"**. At higher and higher energies the investigations correspond to smaller and smaller scales and to shorter and shorter times - the hope is to find the ultimate building blocks (fundamental entities) with which everything else can be built. We note there has been only slow, excruciating progress (if any) for decades.

Suppose that we did understand the fundamental laws or rules (hopefully they will be simple rules at the lowest level) of nature. This leaves unanswered, however, the question of how to apply the rules to anything beyond the most simple systems involving only a few particles. One wonders how they might be applied to a bathtub full of water or the weather, etc.

The physics developed in the 20th century, which has often been described as a *search for the theory of everything*, cannot even be used to answer the most fundamental questions about nature - How did life begin? What is turbulence?

How does order arise? and so on.

Even phenomena thought to be well understood (simple fluids, mechanical systems like the pendulum, etc) were not understood, as we will see!

The simplest systems, once assumed to be completely understood, have extraordinarily complex behavior related to *unpredictability*.

When we look (investigate) carefully, however, a new kind of order arises spontaneously(emerges) in these systems - chaos and order intermingled.

Traditionally, when physicists saw complex results, they looked for complex causes, i.e., the turbulence at the bottom of a waterfall is a good example - one might ask the question - what causes the complex, turbulent behavior at the bottom of the waterfall? It seemed clear to traditional physicists that the bottom of the waterfall required a very complex theory for its understand - it does not!

If physicists saw a random relationship between input and output for a system, then they would build into the theory some randomness by artificially adding noise or error to their initial conditions. This was the wrong approach.

What is the correct way?

Let us look at some simple (mathematical and physical) systems (a first pass with more later). We will find in all cases that very small input(initial) differences can lead to what we will call *chaotic* output differences. We will find that simple equations (laws or rules) lead to the turbulence(chaotic behavior) at the bottom of a waterfall.

Let us begin by talking about something that you have probably heard about - the *butterfly effect*.

Long ago, many meteorologists thought that weather forecasting was a lot of guesswork and helped by experience. They also felt that the computer with its ability to crunch numbers very fast, would allow then to prove the Newtonian idea that the weather follows a deterministic path - like planets, comets, eclipses, tides, etc, i.e., that we could make predictions from known initial conditions and known physical laws even if the the system was complex.

We should be able to make weather calculations so accurate that they will effectively be forecasts. The weather is complicated, but is is governed by known laws. Thus, a powerful computer with accurate initial conditions should be able to make forecasts.

This was the view of science in the 1960's. There was one small fly in the ointment, however. We would need to know the initial conditions **EXACTLY**. This might be done with a grid of measuring instruments all over the surface of the earth, on/under the oceans, in the atmosphere, etc.

In actuality, however, this can only represent approximate initial conditions since sensors cannot be everywhere. Science, however, has always assumed that approximate initial conditions imply approximate behavior of the system, i.e., if we have small errors or a set of sensor data that is, for all practical purposes, complete, then we can neglects the small errors or small amounts of missing data - it was thought that arbitrarily small influences (that might be missed) do not generate arbitrarily large effects!

This always worked and was firmly believed to be true in all systems - approximate (to some degree) input always give approximate(to the same degree) output!

In 1963, a strange thing happened while Edward Lorenz was working on his weather model. He was running a computer simulation using his model and after many days, when it was just beginning to get interesting, the computer crashed. Suppose one of the weather variables he was calculating looked like the graph below (as a function of time).

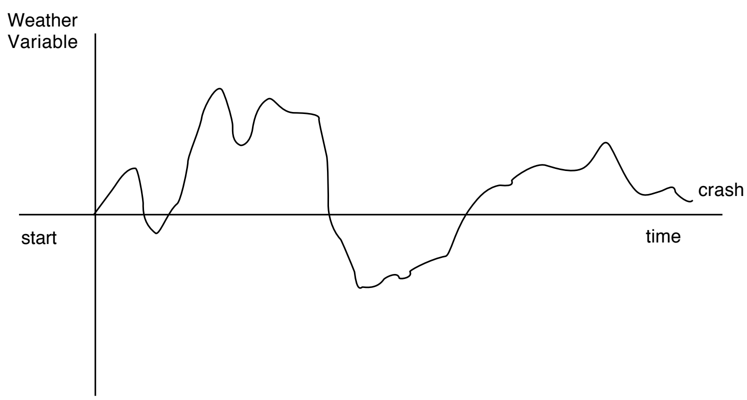


Figure 1: Lorenz Computer Model - Run 1

He then did the "standard thing". He looked at the printouts and used the data from some time before the computer crash to restart the simulation. The values he used for the initial conditions were accurate to 6 decimal places and he assumed that the simulation would be identical up to the crash and then all new calculations would be the same as if the first run had not crashed.

This, however, was not what happened. As shown in the graph below

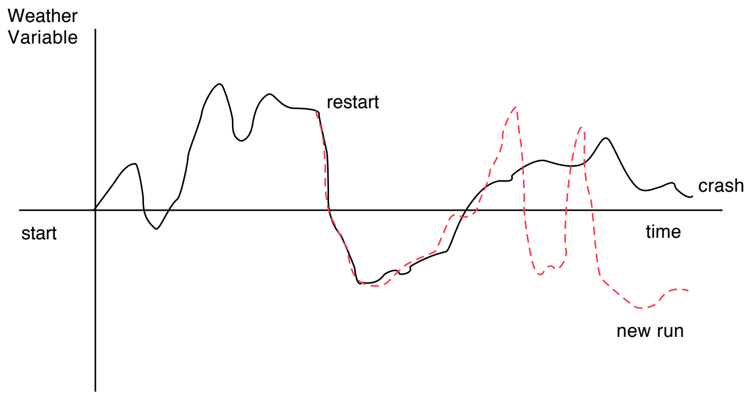


Figure 2: Lorenz Computer Model - Run 2

the second run was identical to the first run for a short time but then deviated and after some time they were totally different - the weather behavior at large times for the two runs did not resemble one another in any way!

The initial values for the second run could only have been in error by 1 part in 1,000,000 and this should not have led to such a dramatic deviation in the results.

When using a completely deterministic system of equations with approximate initial conditions, it was thought that slight differences scattered around should cancel out and thus have no significant effect on large scale features of the weather - but in actuality one observes catastrophic differences in such systems, as seen above.

Therefore, while the equations do imply a weather prediction that probably occurs somewhere at some time, it is likely that it is not where and when the model predicts, thus implying that any form of prediction is doomed! It seems that any physical system that has non-periodic behavior will be unpredictable. Again we can see this in a class demo of the Lorenz effect with many starting points.

The phenomenon is called the **butterfly effect**.

The sensor grid with missing values hides fluctuations and leads to approximate initial conditions. No prediction is possible - a butterfly might have flapped its wings in a gap!

It seemed that predictability was giving way to randomness (no predictability).

The Lorenz results, however, seemed to imply more than randomness. They seemed to imply some kind a geometrical structure at a fine scale - some kind of order masquerading as randomness.

This led to studies of so-called *aperiodic* systems or systems that almost repeated themselves but never quite succeeded. These studies implied that there must be a link between aperiodicity and unpredictability.

The butterfly effect is no accident! The studies implied that it is necessary, i.e., suppose that small perturbations remain small instead of generating the observed chaotic effects. This would imply that when the weather came arbitrarily close to a state it had passed earlier, it would stay arbitrarily close to all the patterns that followed that earlier state. This means that we could predict future weather patters and cycles, which we cannot - thus, the butterfly effect must exist!

The butterfly effect is equivalent to *sensitivity to initial conditions*.

But how does it happen? How could such unpredictability - such chaos - arise from a simple deterministic system of equation(there were 12 equations in the Lorenz weather model). The simpler Lorenz model I have been using for illustrations has only three equations. They are non-linear equations (we will define them in detail later). Non-linear equations are not generally soluble analytically. All real systems are non-linear although the non-linear terms are usually neglected (assumed to be small effects that that would not influence large scale behavior) - thus, the butterfly effect was removed from the models and everyone was then surprised when it appeared!

Let us look at a simple example model - the Lorentzian water wheel as shown below.

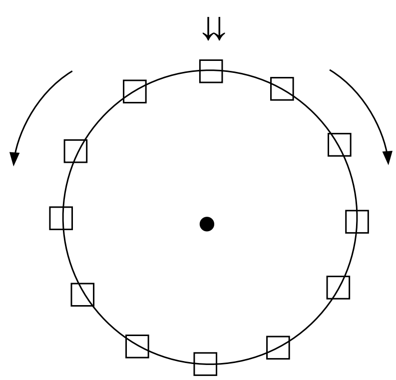


Figure 3: Lorentzian Water Wheel

As shown, we have a wheel that can rotate in either direction. Buckets are attached to wheel. Water flows into a bucket at the top and the buckets leak water. The steady flow of water into the buckets correspond to energy inflow into the system - it means that the system is *driven* by an *external* system (supplies the water). The leakage of water means that the system is losing energy also - this is called *dissipation*.

> Case #1: The rate of inflow is very low. This implies that the top bucket never fills up. Thus there is no movement.
>
> Case #2: We have a slightly faster rate of inflow until steady motion sets in (corresponds to net torque =0 and all water leaked out by time buckets reaches the bottom).
>
> Case #3: If inflow rate is increased, the rotational motion becomes chaotic - due to non-linear effects that have been built into the system - the rate of spin is proportional to the amount of inflow into the bucket which is proportional to the spin and so on....i.e., if the spin is large, buckets have little time to fill up - also buckets can pass the bottom and start up the sides before emptying. This causes a slowing down and possibly a reversal of the spin direction. In fact, over long periods, the spin reverses many times, never settling down to any kind of steady-state and never repeating itself in any predictable manner.

Videos: `https://www.youtube.com/watch?v=RG-MbYDjpGM`

A pre-chaos physicist's intuition would imply that over the long term - if the rate never varied - a *steady-state* would evolve, i.e, either the wheel would rotate steadily or it would oscillate steadily back and forth at constant intervals.

But that is not what is observed for this system. To understand the observations and to hint at future discussions consider the following ideas.

We assume the system has three equations and three variables. To visualize how the system evolves in time we could plot each variable as a function of time (called a *time series*) as shown below.

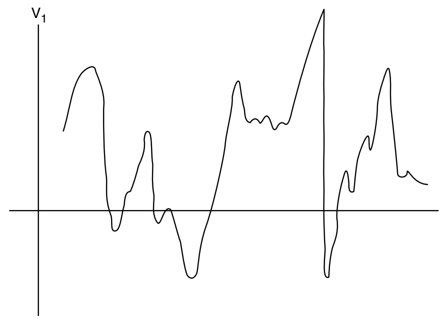


Figure 4: 1 Variable Time Series

Alternatively we could plot the evolution in a 3-dimensional space as shown below (called a *phase space*).

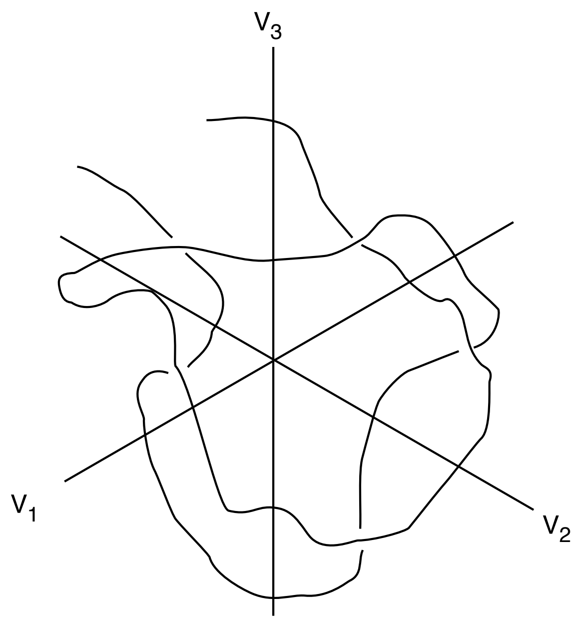


Figure 5: 3 Variable Phase Space

Here the path also represents how the variables are changing with time - each point on curve corresponds to a given time t and the point has coordinates $(V_1(t), V_2(t), V_3(t))$.

The meanings of different types of paths in *phase space* are as follows: If the path stops a a point, this corresponds to the variables remaining constant or a steady-state situation. If the path loops (closed path), then this implies a periodic (repeating) motion.

Some examples are;

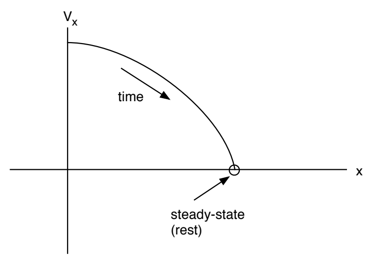


Figure 6: Particle with friction

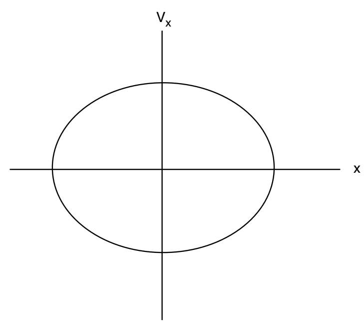


Figure 7: Simple periodic motion

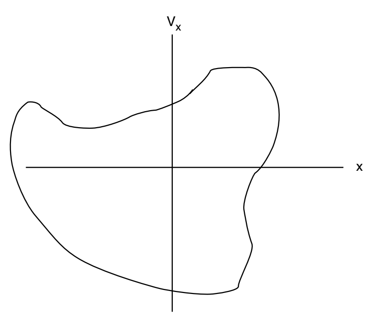


Figure 8: Not so simple periodic motion

What did Lorenz see?

He saw no steady states and no loops implying periodicity. He only saw what he called *infinite complexity*. In particular, he found that

1. The path stayed in a finite volume (within well-defined bounds).

2. The path never repeated itself.

3. The path traveled along a strange, distinctive shape - a kind of double spiral in 3D space (like a pair of butterfly wings (as we saw earlier)). This implied pure disorder, since no points ever repeated!

However, it contained a *new* kind of order.

Why didn't scientists see this **"chaos"** if it was everywhere? It was because typical scientific training leads to narrow, discipline oriented studies - to compartmentalized knowledge. Each discipline has its own narrow methods applicable beautifully to the problems of that particular discipline. Each discipline also tends to ignore problems that cannot be solved using their standard approaches and methods. They were not seeing a large, cross-disciplinary picture! But is was there and was seen by Lorenz!

1.1.1. Revolution

Science is not an orderly process. It does not progress by simple accretion of knowledge. Most scientists are not innovators, but only solvers of puzzles. The problem is that in order to advance in one's discipline, the puzzles that one attempts to solve are the ones they *believe* can be stated and solved *within* the conventional wisdom of the field (the so-called orthodox approach).

Then there are **revolutions**.

Revolutions lead to new science.

Revolutions usually occur when scientists cross boundaries (of the disciplines).

Revolutions usually use **"illegitimate"** or **"unorthodox"** modes of inquiry.

This is what happened in the study of chaos ... the early investigators were shunned, ridiculed, ...

Chaos has become more than a theory and more than a set of beliefs. It has its own technique of using computers via flexible interaction, i.e.,

> mathematic becomes an experimental science
>
> computers replace laboratories
>
> computer graphics becomes important because of the unique ability of the human eye-brain interface to synthesize.

1.1.2. Studying Chaos

What can we study to see chaos?

What about the *rock of Gibraltar* of deterministic physics, namely, the **pendulum**, which is the prime example of constrained action, the epitome of clockwork regularity. Of course, no "real", "top-ranked", physicist would have bothered to study the pendulum in the 20th century - it was *completely understood* - it was the *classic deterministic system.*

But that was a mistake. For within the study of the pendulum were some hidden surprises - hidden because the standard investigatory methods were not designed to looked for them!

Certainly, there is amazing regularity (predictability) in the pendulum system, but investigators saw that easily in many experiments because that is what their approximate theories predicted - the approximate theory was powerful - so powerful that it saw a regularity that *does not exist.*

Maybe experimentalists should not know the theory prior to doing experiments!

A better theory, i.e., one that includes some non-linearity due to the size of the initial displacement angle leads to some irregularities; scientists, however, assumed that small non-linear inputs would only lead to small output effects!

To get an *absolutely* regular system, we must ignore all non-linearities in the system. We must ignore, friction, air resistance and not displace the system from its rest position (equilibrium) very much.

In the real world, however, pendulums do not remain regular - they **stop**. The two figures below (in phase space) illustrate this dilemma.

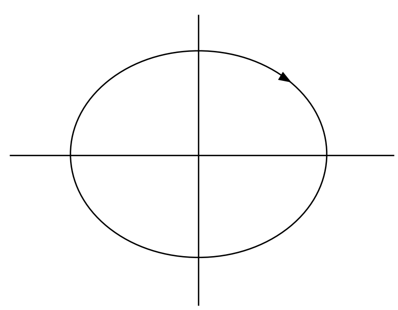


Figure 9: Periodic, regular motion

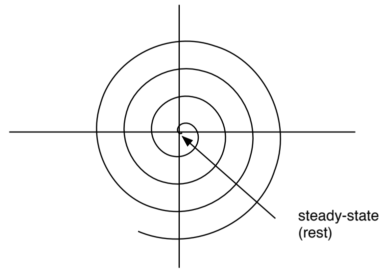


Figure 10: Real motion

Dissipative systems change and this defeats regularity. On top of this, other non-linear effects lead to systems that are not solvable exactly(analytically) - they can be approximated if the non-linearities are small, but that defeats the investigation of realistic systems.

No one suspected chaos lurked around if the non-linearities were included to their full extent - no one even looked!

Think of something you are familiar with, namely a playground swing. If you give the swing a push, it oscillates but quickly damps out, coming back to rest unless it is driven (you keep pushing). It turns out that its motion can (in the long term) be regular (you keep pushing at a regular rate but not too strong) or it could be erratic, never settling down to a steady-state - never exactly repeating any previous motion (you keep pushing at a regular rate but now much stronger pushes).

It turns out that unpredictability was not the reason these systems became interesting. This disorderly behavior of simple systems implies a creative process is taking place. This process generates **complexity** - this corresponds to richly organized patterns - sometimes stable and sometimes unstable - sometime finite and sometime infinite - but always fascinating.

Consider the toy demo of a magnetic chaotic pendulum(+simulation) in class. It consists of a magnet at the end of a pivoted rod (a spherical pendulum) and three magnets on the base (carefully positioned at the corners of an equal sides triangle). We now do a series of repeated experiments.

We start the pendulum off somewhere (arbitrary) and see where it ends up (at one of the three magnets (attractive)) on the floor. Call the three floor magnets (red, blue, green). We then color the starting point with the color the final magnet position. What does the colormap look like? Old style, non-chaos, non-complexity thinking, would answer - three regions- red nearest the red magnet, blue nearest the blue magnet and green nearest the green magnet with sharp boundaries as shown below.

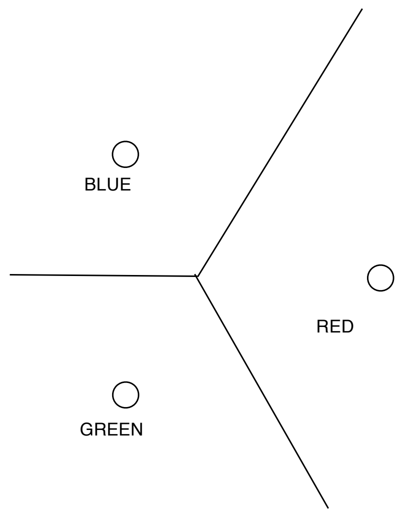


Figure 11: Deterministic Guess

That is not what is observed however. We do get regions of solid color, which means that the pendulum magnet reliably goes to the magnet of that color. But we also get regions where the colors are woven together with infinite complexity, i.e., adjacent to red point, no matter how close you choose to look, no matter how much you magnify the map, there will be green and blue points. This implies, that for all practical purposes (we would need infinitely accurate initial conditions) the pendulum magnet's destiny is impossible to predict! Some illustrations are shown below.

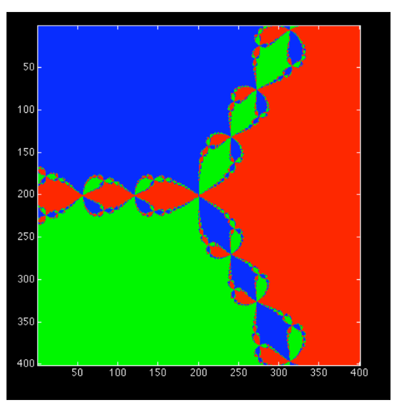


Figure 12: The experimental results

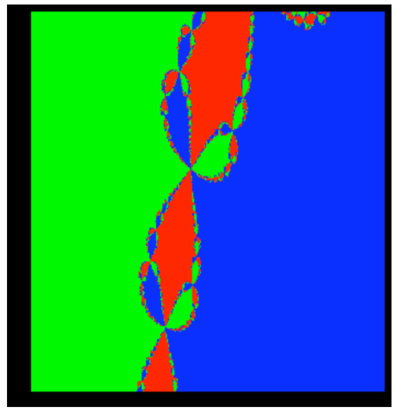


Figure 13: The boundary expanded

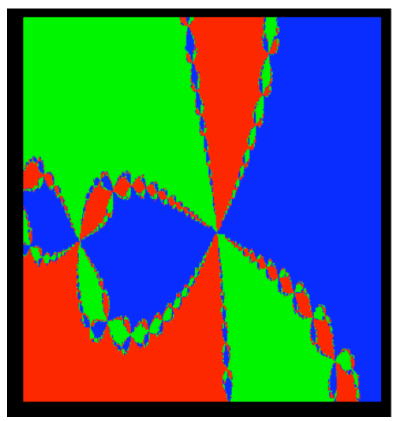


Figure 14: The boundary expanded further

We will discuss the details of these patterns later (they will arise from a completely different and strange source).

The traditional physicist would write down the appropriate equations including angles, friction, driving forces, etc (including all the nonlinearities) and get an unsolvable system. They would then turn to the computer to simulate the system. They repeat the calculation many times generating the color map. Great care must be exercised since computers have a *bit-range* problem and if the simulation is run long enough one obtains only *noise*.

Is there anything hidden in this seeming unpredictability?

Is there any pattern? Is there any structure in the seemingly random behavior?

Physicists seemed to be able to understand the short term motion of the system, but could not extend that understanding to the long term motion when non-linearities were present.

The microscopic (fundamental equations or rules) pieces are clear, but the macroscopic behavior (complex patterns) remained a mystery!

The tradition of looking at systems locally - isolating small scale mechanisms - and then adding the local effects together to figure out the whole system - was beginning to break down.

The standard method of *reductionism* where we understand the smallest building blocks and build the larger system up from the behavior of the smaller blocks was failing.

The whole seemed to be more than a sum of its parts!

Knowledge of the fundamental equations no longer seemed to be the right kind of knowledge at all.

Physicists had to learn again - had to learn to see things that they had learned not to see!

Chaos and instability are not the same at all!

A chaotic system could be stable (means equilibrium) if its type of irregularity persisted in the face of small disturbances (the traditional definition of equilibrium).

The chaos of the pendulum is as stable as the equilibrium of a marble sitting at the bottom of a bowl!

You can add noise, jiggle it, stir it up, interfere with the motion and then when everything has settled down - the transients(due to our messing around) dying away - the system returns to the same peculiar pattern of irregularities as before.

The system is *locally* unpredictable, but *globally* stable.

Real dynamical systems play by a more complicated set of rules they anyone had imagined.

The idea that systems behave in certain quantitative ways that depend not on the detailed physics or model description, but rather only on some general properties of the system is called *universality.*

1.1.3. First Pass on Ideas about Difference Equations or Life's Ups and Downs

Traditionally, physicists study a problem, develop relevant equations with variables that are continuous functions of time (called differential equations) and try to make predictions. They often cannot solve the equations analytically, so they turn to computers. The numerical computer solutions return answers at discrete values of the time variable (not continuous values) and one simply assumes continuity, i.e., nothing strange happens between the closely space discrete time values and we can connect them with continuous curves.

Biologists, on the other hand, use experiment to study systems. They generate equations that give output matching the data from the experiment. Their variables, parameters, driving forces, etc are very complicated and they must idealize the equations. This works in the sense that if the mathematical model had certain types of behavior, then this implied that one could usually guess circumstances where real systems would behave the same way.

Experimental data naturally comes at discrete time intervals (not continuous). Equations relating variables known only at discrete time values are called *difference equations*. In these equations, the process jumps from state to state with no knowledge about "in between".

This is saying that, in these systems, changes at discrete intervals are more important than what is happening at every time point (more important than continuity).

In these equations, a variable *now* depends only on variable(s) *last year(last month,.....)* i.e.,

$$x_{next} = f(x_{now})$$

or

$$x_{n+1} = f(x_n)$$

In such a simple model, we proceed in this way:

1. pick a starting value $x = x_0$
2. x_1(value after Δt)= $f(x_0)$
3. x_2(value after $2\Delta t$)= $f(x_1) = f(f(x_0))$
4. and so on ....

The resulting set of numbers is a discrete *history* or *time-series*. We can observe both stable and unstable behavior in time depending on the nature of the function $f(x)$.

1.1.3.1. Naive Population Model

In this simple model, all factors influencing population growth are contained in one parameter, namely r, and we have the difference equation

$$x_{n+1} = rx_n$$

In this simple model, we have the behavior below as a function of the size of r:

$$r \begin{cases} < 1 & \to \text{exponential decrease (extinction)} \\ = 1 & \to \text{stable} \\ > 1 & \to \text{exponential increase (explosion)} \end{cases}$$

A real population model is needed to do better. We must include (1) competition, (2) supplies, etc. Large supplies and small population implies rapid growth, which leads to competition, which lead to no supplies and thus to rapid decrease, and so on..... Thus, we need extra terms in the equation to restrain growth. The expectation was that we should see population as a function of time behave as shown below.

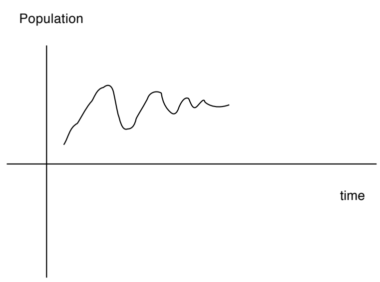


Figure 15: What we expect

i.e., oscillation with decreasing amplitude leading eventually to a steady state.

The simplest equation (we will study this equation in detail later) that limits population growth is

$$x_{n+1} = rx_n(1 - x_n)$$

where r is the rate of growth parameter.

Examples

$$r = 0.5 \quad \to x_{n+1} = 0.5x_n(1 - x_n)$$

Let $x_0 = 0.5$ where $x = 0$ corresponds to extinction and $x = 1$ corresponds to maximum possible value of x. We then have

$$\begin{aligned}
x_0 &= 0.5 \\
x_1 &= 0.5x_0(1 - x_0) = \frac{1}{4}\frac{1}{2} = \frac{1}{8} = 0.125 \\
x_2 &= 0.5x_1(1 - x_1) = \frac{1}{16}\frac{7}{8} = \frac{7}{128} = 0.055 \\
x_3 &= 0.5x_2(1 - x_2) = \frac{7}{256}\frac{121}{128} = 0.026 \\
&\lim_{n\to\infty} x_n = 0 \rightarrow \text{ extinction}
\end{aligned}$$

Now choose

$$r = 2.0 \rightarrow x_{n+1} = 2x_n(1 - x_n)$$

Let $x_0 = 0.8$. We then have

$$\begin{aligned}
x_0 &= 0.8 \\
x_1 &= 0.5x_0(1 - x_0) = 0.32 \\
x_2 &= 0.5x_1(1 - x_1) = 0.44 \\
x_3 &= 0.5x_2(1 - x_2) = 0.49 \\
&\text{and so on .....}
\end{aligned}$$

This looks like our expectation, i.e.,

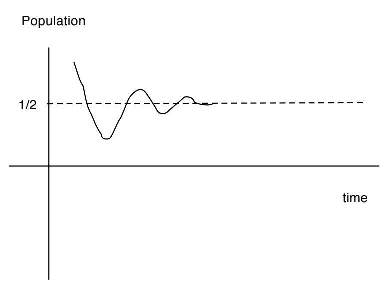


Figure 16: What we get for $r = 2$

For small r we have

lower $r \rightarrow$ steady state at lower value

higher $r \rightarrow$ steady state value increase

This is sensible behavior. But if r gets larger still, the following occurs

1. population does not grow without limit
2. it does not converge to a steady state value

Investigators, however, mistakenly assumed that case #2 was simply oscillating about some equilibrium value and would eventually get to that value. However, this was not the case - it turns out that there is no equilibrium value!

The investigators refused to believe there was no equilibrium value, i.e., if the model disagrees with your own prejudice, then you simply say that you must have left something out of the model!

Paramount was the tradition that *stable solutions are the interesting ones - order is its own reward*. The whole point of simplifying a model was to generate regularity not chaos. Physicists had learned (were trained) not to see chaos even though it seems to be lurking everywhere.

Traditional physics, as I said, means generating differential equations and learning how to solve them. This implies an assumption that the world is a continuum, changing smoothly from place to place and from time to time, not broken into discrete grid points or time steps.

The problem is that differential equations representing real systems cannot be solved exactly (they are nonlinear). We can linearize the equations to make them solvable but then this process eliminates the chaotic behavior!

Traditionally, physicists only looked at nonlinearities as a last resort and usually used approximations which hid any chaos.

The problem is that the real world is chaotic and disorderly so that any approximation which forces regular behavior will not give solutions for real systems.

The message is - do not discard disorder/complexity, but attempt to understand it instead and use its structures/patterns to understand systems properly.

Generally, researchers have looked for periodicity in systems they investigate because periodicity is the most complicated orderly behavior imaginable. If a researcher sees more complicated behavior, they still look for some periodicity wrapped in what they assume to be some kind of random noise.

Robert May(physicist, mathematician and biologist) was studying the equation

$$x_{n+1} = rx_n(1 - x_n)$$

In particular, he was investigating what happens when r gets large?

His investigation showed that larger r values, which meant an increase in nonlinearity, not only changed the steady-state (not only changed the final values), but also determined whether or not any equilibrium was reached at all!

His results were:

1. $0 \le r \le 1$ - end result is always extinction,i.e., $\lim_{n\to\infty} x_n = 0$
2. $1 < r \le 3$ - end result is always a single steady state value(usually oscillates for a while about the final value) - these are called *period-one* states
3. At $r = 3$ we have what is called a *bifurcation*. At that value of r the solutions become *period-two* solutions. The unique steady state breaks apart and the population oscillates between two values - there is no equilibrium - no single steady state solution ever evolves - the steady state is two alternating states!

As r continues to get larger we have further bifurcations into period-four (4 alternating states) and then into period-eight (8 alternating states) and so on. The resulting cyclical behavior (alternating states - never settling down to a steady-state) is stable - all different starting values converge onto same period$-n$ cycle where the value of n depends on r. At each bifurcation, the population goes from a period$-n$ cycle to a period$-2n$ cycle.

This is complex behavior that is also regular.

The continued period-doubling(bifurcations) as r increases implies that the pattern of repetition was breaking down - as r increases, the bifurcations come faster and faster - 4, 8, 16, 32, ... and then they *suddenly* break off.

Beyond a certain point (called a *point of accumulation*) the periodicity gives way to chaos - the fluctuations which never settle down - there are no cycles, no repetitions of any values, all allowed values are passed though with no discernible pattern. The full graph looks like graph below:

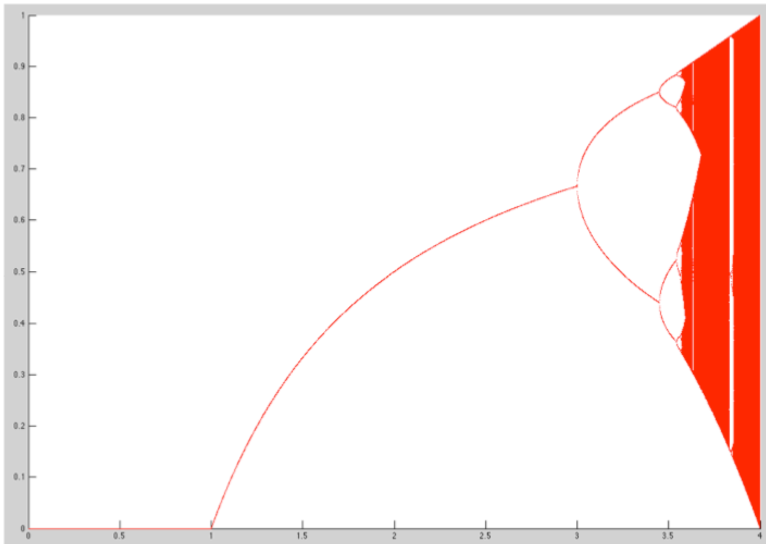


Figure 17: Final states for all values of r

We will discuss this map and these plots in detail later. If this were a real population system, we would think that the changes from period to period$-n$ $\rightarrow$ period$-n+1$ were absolutely random as though blown about by noise.

Yet in the middle of complexity and chaos (as r continues to increase) stable cycles suddenly return - windows suddenly appear with regular period$-n$ cycles, where $n = 3$ *or* 7. The period doubling bifurcations begin all over again at a faster rate rapidly passing through 3,6,12,..... or 7,14,28,..... and then breaking off again into renewed chaos.

We note that in the real world, there is only one particular value of r and thus any observer would see only a single vertical slice (a particular r). Any observer, within a given system (a particular value of r), sees only one kind of behavior - possibly a steady-state or possibly a 7-year cycle or possibly apparent randomness. Any real observer has no way of knowing that this same system, with a slight change in some parameter, could display patterns of a completely different kind.

1.1.4. Some Thoughts

This led to the new science of **simulation physics**. These are theories without full equations or actual parameters that allow us to do experiments that cannot otherwise be done on the actual system - like varying the value of r.

J. Yorke proved that *in any 1-dimensional system, if a regular cycle of period-3* *ever appears, then the same system will also display regular cycles of every other length as well as completely chaotic cycles.*

That is contrary to intuition. One might think that it would be easy to set up a system that would repeat itself with a period-3 oscillation without ever implying chaotic behavior. Yorke showed that this was impossible.

As we will see later, any part of the diagram when magnified turns out to resemble the entire diagram, i.e., the structure of the diagram is infinitely deep - we can continue magnifying forever and the same structure remains - this is called *scale independence*.

The message we are getting here is - chaos is everywhere, it is stable and it is structured.

With powerful computers we can simulate systems and generate many spectacular pictures (as we will see later).

Chaos implies that deterministic systems can produce what looks like random behavior. This behavior has an exquisite fine structure where any piece of it seems indistinguishable from noise.

An example - epidemics - illustrates what we have been talking about. Epidemics come in cycles both regular and irregular - diseases all rise and fall in frequency.

It was realized that the observed oscillations could be reproduced by a set of non-linear equations (the model).

One can now ask - what happens if the system is given a sudden kick - a perturbation - say an inoculation program.

Traditional intuition says that the system will change smoothly in the desired direction.

But actually, huge oscillations can occur even if the long term trend was downward - the path to equilibrium was interrupted by peaks and valleys. Real data corroborates this behavior.

But most health officials, seeing a sharp (short term) rise in disease would assume that the inoculation program was failing!

So physics realized that we were training students very well, but not correctly. We still had to teach traditional topics (they still work in many cases) but we must also teach chaos and its associated methods and we must also teach that one does not always approximate or ignore or simplify because we may be eliminating the important features and that these changes should be expanded into all disciplines because, as we shall see, all fields behave in this way!

1.2. Approach to Chaos

1.2.1. Introduction

There are two major reasons for studying non-linear systems. The first and most basic is that the equations of motion of almost all real systems are non-linear. The second reason is that even a relatively simple system which obeys a non-linear equation of motion can exhibit unusual and surprisingly complex behavior for certain ranges of the system parameters. In addition, in a wide variety of dramatically different non-linear systems, identical features show up.

Much of the existing knowledge of non-linear behavior has been obtained from numerical solutions of the equations. The traditional methods, which lead to analytic (explicit equations) expressions for the motion, fail for most problems in non-linear systems. Numerical integration of the equations of motion is usually necessary.

From the time of Newton until the 20^{th} century, physicists and philosophers viewed the universe as a sort of enormous clock which, once wound up, behaves in a predictable manner.

This idea was dramatically shaken by the discovery of quantum mechanics and the Heisenberg uncertainty principle, but physicists still thought that the motion of classical systems (macroscopic systems) that obey Newton's equations of motion would exhibit predictable or deterministic behavior.

It turns out, however, that even macroscopic systems obeying Newton's equations can exhibit so-called *chaotic* motion or motion that seems very difficult to predict (or is even unpredictable).

The main difference between a chaotic system and a non-chaotic system is the degree of predictability of the motion given the initial conditions to some level of accuracy (note that we are introducing the idea that we might not be able to specify the initial conditions exactly).

Let us look at a particular system, namely, the linear, damped oscillator with sinusoidal driving force. This system satisfies the equation

$$ma = F = D\cos\omega t - bv - cx$$

where

a = acceleration (time rate of change of the velocity)

m = mass

D = amplitude(strength) of driving force

F = total force acting on the mass

b = damping strength (frictional effects)

c = spring constant (governs oscillatory motion)

Examples of such systems are a spring and a pendulum.

Let us do a simulation. In the simulation we plot the motion of two oscillators with starting points(initial conditions) on the same diagram. The plot is in *phase space*, where we plot velocity v (y−axis) versus position x (x−axis). The system at any time is represented by a point in phase space(i.e., a point specifies the position and velocity at an instant of time). Different initial conditions correspond to different starting points in phase space. The path in phase space is called the *trajectory*. In class, we will figure out $x(t)$ and $v(t)$ for each case?

Class demonstration of spring and pendulum

From the class demonstrations we see that the observed motion (for short period of time) corresponds to both the position and the velocity oscillating in time. It corresponds to the values $D = 0.0$ (no driving force), $b = 0.0$ (little or no damping (friction)), $c \neq 0$ and $\omega = \sqrt{c/m} = 2/3$ in the simulation program. The simulation of this case gives (nlo_1.m)

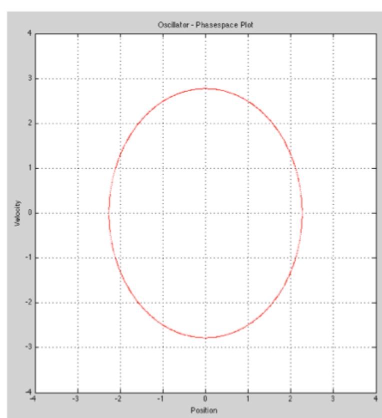


Figure 18: Simple Harmonic Motion - Undamped Oscillations in Phase Space

The frequency $f(f = 2\pi/\omega = 1/T)$ of this motion corresponds to the natural frequency and T corresponds to the period (time for one oscillation). This case is called *simple harmonic motion (SHM).*

In the simulation shown below we use the values $D = 0.0$, $b = 0.5$, $c = 1.5$ and $\omega = 2/3$. In this case we have damping and the motion dies out as it does in the real world(remember the demo after a long period of time).

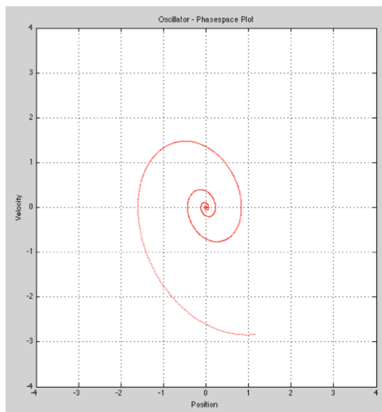


Figure 19: Damped Oscillations in Phase Space

Finally, if we drive the oscillator periodically (add energy to the system by a periodic *kicking* to compensate for friction) we have the simulation $D = 0.9$, $b = 0.5$, $c = 1.0$ and $\omega = 2/3$ as shown below.

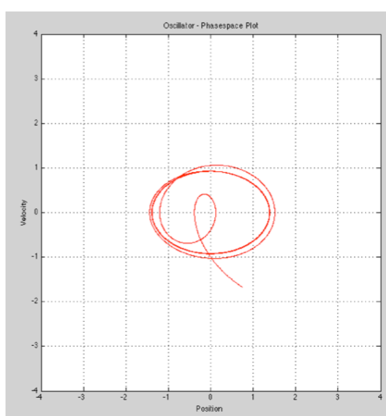


Figure 20: Damped Oscillations in Phase Space

The system clearly starts off with transient motion(interaction between driving force and spring force) and then settles down into a steady state motion (characterized by the driving force). The frequency of this motion corresponds to the driver frequency.

In the following simulation we start the oscillator with two different sets of initial conditions and we observe the following behavior: after a time long enough for the transient motion to die out, the different oscillatory systems will end up with the same motion.

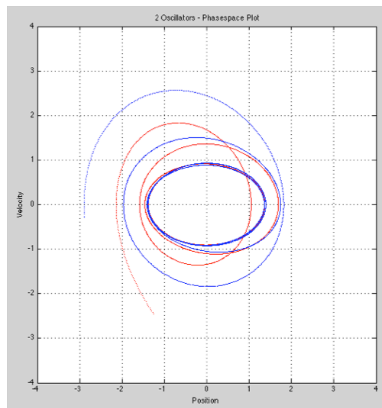


Figure 21: Independence of Starting Point for a Linear System

Thus, the final motion is independent of the initial conditions for a linear system.

If the initial conditions (large amplitude of motion) are such that the motion of the oscillator cannot be approximated as a linear system then this system satisfies the equation

$$ma = F = D\cos\omega t - bv - c\sin x$$

We note that for small (how small?) x, $\sin x \approx x$ and we get the earlier equation. This case corresponds to the pendulum swinging over the pivot point.

For such a nonlinear system, we can illustrate sensitivity to initial conditions as shown in the simulations below where $D = 1.15$, $b = 0.5$, $c = 1.0$ and $\omega = 2/3$:

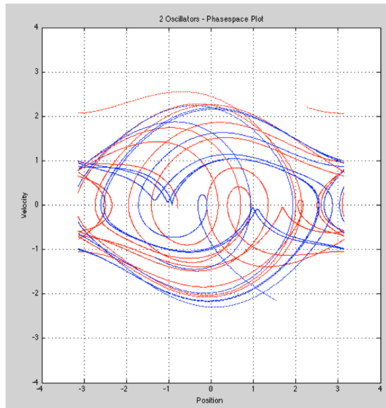


Figure 22: Sensitivity to Starting Point for a Nonlinear System

In this case, we have two very different starting points and we get two very different motions. Thus, the final motion is not independent of the initial conditions for a *nonlinear* system. For anyone that has studied nonlinear systems, this is really not surprising, however.

In the next case, the initial conditions differ by 1 part in 1000 and the simulation shows the motions after 10000 steps. They are beginning to deviate from each other.

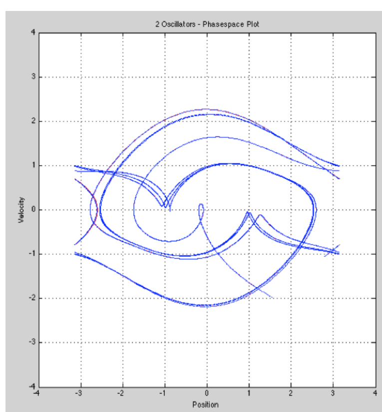


Figure 23: Deviation after a long time

More dramatically, if we let the simulation continue for 100,000 steps and we see that the two motions are totally different.

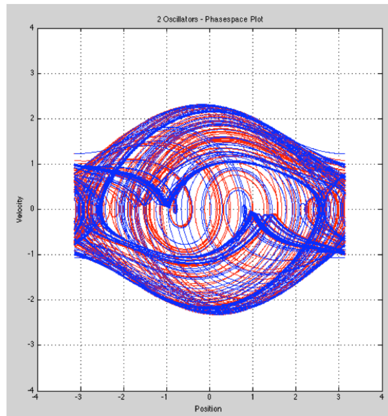


Figure 24: No resemblance at all

In fact, we can watch it in real time in this simulation (nlo_1_mult_nl.m).

We will see in our studies that after as long enough period of time the two motions will not be correlated in any way even though we choose initial conditions that are the same to within the accuracy we are capable of determining in practice.

This sensitivity to initial conditions is the signature of a chaotic system.

We will have many other illustration of the this phenomenon and define chaos shortly.

As we mentioned earlier, one of the best known examples of poor predictability is the weather. At one time it was believed that with a large number of atmospheric measurements and powerful computers to integrate the fluid mechanics equations it would be possible to make long term weather predictions. It is now realized that this was a naive hope and that the weather equations are extremely non-linear and the solutions of the weather equations are exponentially sensitive to the initial conditions; this means that an infinitesimal difference in initial conditions will eventually produce a completely different solution to the equations, that is, no two solutions will have any relation to each other. A simple weather model is the three-dimensional Lorenz equations which generate solutions that fall on a curve in three dimensions called a *strange attractor* (looks like butterfly wings). This led to the famous *butterfly effect* where one could imagine that the perturbation in the weather system due to a butterfly flapping its wings in Africa would grow exponentially into a devastating weather front in North America.

This is illustrated dramatically by the simulation of the Lorenz equations (lorenz0.m and lorenz4.m). Again we observe extreme sensitivity to initial conditions.

Even though the motion of a complex system cannot be precisely predicted, certain features can often still be relied upon. For example, the exact path of a given molecule of water that comes out of a faucet is certainly not predictable. We can say, however, with high probability that the molecule will fall vertically downward within a well-defined cylindrical surface. It is a real challenge to deduce such *robust* features of the solutions of nonlinear equations.

1.3. Introduction to Dynamic Systems Toward an Understanding of Chaos

What is a dynamic system?

A dynamic system is a set of *functions* (rules, equations) that specify how variables change over time. We define

> **Dynamic System**: A set of equations specifying how certain variables change over time. The equations specify how to determine (compute) the new values as a function of their current values and control parameters. The functions, when explicit, are either difference equations or differential equations. Dynamic systems may be stochastic or deterministic. In a stochastic system, new values come from a probability distribution. In a deterministic system, a single new value is associated with any current value.
>
> **Difference Equation**: A function specifying the change in a variable from one discrete point in time to another.
>
> **Differential Equation**: A function that specifies the rate of change in a continuous variable over changes in another variable (time, in these notes).

First example: Alice's height diminishes by half every minute... Clearly, this rule allows us to calculate Alice's height as a function of time

Second example:

$$x_{new} = x_{old} + y_{old}$$
$$y_{new} = x_{old}$$

This second example illustrates a system with two variables, x and y. Variable x is changed by taking its old value and adding the current value of y and y is changed by becoming x's old value. Silly system? Perhaps. We are just showing in this example that a dynamic system is any well-specified set of rules.

Here are some important *distinctions*:

variables (dimensions) vs. **parameters**

discrete vs. **continuous** variables

stochastic vs. **deterministic** dynamic systems

How they differ:

Variables change in time, **parameters** do not.

Discrete variables are restricted to integer values, **continuous** variable are not.

Stochastic systems are one-to-many; **deterministic** systems are one-to-one.

This last distinction will be made clearer as we go along.

Terms

The current *state* of a dynamic system is specified by the current value of its variables, $x, y, z, ...$

> **state**: A point in state space designating the current location (status) of a dynamic system.
>
> **state space (phase space)**: An abstract space used to represent the behavior of a system. Its dimensions are the variables of the system. Thus a point in the phase space defines a potential state of the system.

The process of calculating the new state of a *discrete* system is called *iteration*.

> **iteration**: the repeated application of a function, using its output from one application as its input for the next.

To evaluate how a system behaves, we need the functions, parameter values and *initial conditions* or *starting state*.

> **initial condition**: the starting point of a dynamic system.

In general, non-linear equations are difficult to solve either analytically or numerically. Before investigating in more detail the properties of the non-linear damped, driven oscillator, we will look at a simple system that can be easily solved numerically but still exhibits all the important properties of a chaotic system, that is, there exists a class of elementary model systems that can give insight into the mechanisms leading to chaotic behavior. These are stated in the form of *difference equations*, rather than more complicated differential equations.

A typical difference equation is of the form:

$$x_{n+1} = f(\mu, x_n)$$

where x_n refers to the n^{th} value of x in a sequence of values and x represents a real number on the unit interval $[0, 1]$. μ is a parameter that determines the behavior of the system.

The equation generates the $(n+1)^{th}$ value in the sequence from the n^{th} value in the sequence. The way to think of this system is the following. Think of nT as a time, where T is a basic time interval. Starting from some initial value of x, say x_0, we can generate a sequence of x values as follows:

$$x_1 = f(\mu, x_0) \quad , \quad x_2 = f(\mu, x_1) \quad , \quad x_3 = f(\mu, x_2) \quad , \quad x_4 = f(\mu, x_3) \quad , \ldots\ldots\ldots$$

The function f is called a map of the interval $[0, 1]$ onto itself (all the x's stay inside the same interval).

> **phase space (state space)**: An abstract space used to represent the behavior of a system. Its dimensions are the variables of the system. Thus a point in the phase space defines a potential state of the system. The points actually achieved by a system depend on its iterative function and initial condition (starting point).
>
> **trajectory (orbit)**: A sequence of positions (path) of a system in its phase space. The path from its starting point (initial condition) to and within its attractor.
>
> **orbit (trajectory)**: A sequence of positions (path) of a system in its phase space.
>
> **phase portrait**: the collection of all trajectories from all possible starting points in the phase space of a dynamic system.

The function f can be nonlinear in its argument x_n. Difference equations are readily solved by iteration as we will see, and their numerical solution is much less time consuming than is the case for nonlinear differential equations.

> **linear function**: the equation of a straight line. A linear equation is of the form $y = mx + b$, in which y varies *linearly* with x. In this equation, m determines the slope of the line and b reflects the y−intercept, the value y obtains when x equals zero.
>
> **nonlinear function**: one that's not linear! y would be a nonlinear function of x if x were multiplied by another variable (non-constant) or by itself, that is, raised to some power.
>
> **nonlinear dynamics**: the study of dynamic systems whose functions specifying change are not linear.

To illustrate:

Consider a classic learning theory, the *alpha model*, which specifies how q_n, the probability of making an error on trial n, changes from one trial to the next

$$q_{n+1} = \beta q_n$$

The new error probability is diminished by β $(0 < \beta < 1)$ after each trial.

For example, let the probability of an error on trial 1 be equal to 1, and let $\beta = 0.9$. Now we can calculate the dynamics by iterating the function, and plot the results.

$$q_1 = 1 \quad , \quad q_2 = \beta q_1 = 0.9 \quad , \quad q_3 = \beta q_2 = 0.81$$

and so on. Plotting this result we have

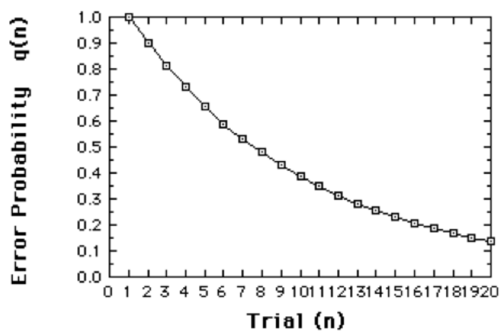


Figure 25: Error Probabilities for the Alpha Model

This *learning curve* is referred to as a *time series*. So far, we have some new ideas, but much is old ......

What is not new

Dynamic Systems – Certainly the idea that systems change in time is not new. Nor is the idea that the changes are probabilistic.

What is s new – *Deterministic nonlinear* dynamic systems.

As we will see, these systems give us:

1. A new meaning to the term *unpredictable*.
2. A different attitude toward the concept of *variability*.
3. Some new *tools* for exploring time series data and for modeling such behavior.

What's a linear function? Well, gee Mikey, it's one that can be written in the form of a straight line. Remember the formula ...

$$y = mx + b$$

where m is the slope and b is the y–intercept.

What's a nonlinear function? Any function that isn't linear! Is the Alpha model a linear model? Yes, because q_{n+1} is a linear function of q_n. But wait!

Its *output*, the plot of its behavior over time (its time series shown earlier) is not a straight line.

Doesn't that make it a nonlinear system? No, what makes a dynamic system *nonlinear* is whether the function specifying the change is nonlinear. Not whether its behavior is nonlinear. $f(x)$ is a nonlinear function of x if it contains x raised to any power other than 1!

Example of the Chaotic Calculator: Consider the *calculator* shown below:

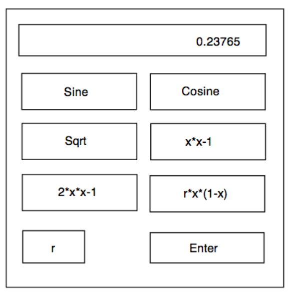


Figure 26: Chaotic Calculator

Let us run the calculator for 50 steps(iterations). We input a number 0.5432 (number does not matter what the initial input is) and hit the appropriate key 50 times. Here are the results:

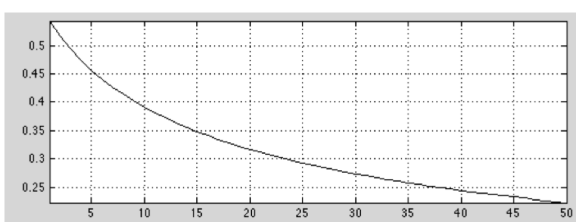


Figure 27: Hitting the Sine key

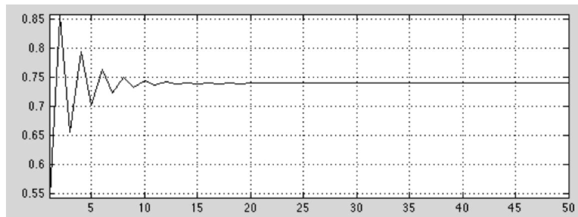


Figure 28: Hitting the Cosine key

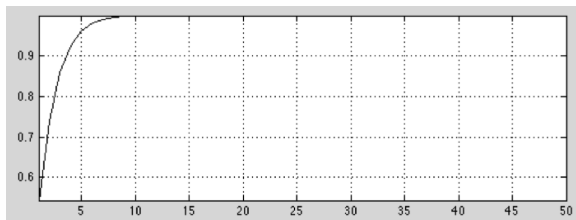


Figure 29: Hitting the Sqrt key

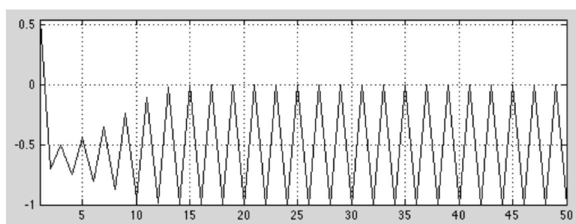


Figure 30: Hitting the $x * x - 1$ key

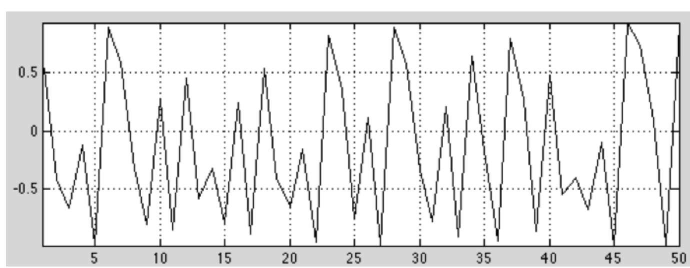


Figure 31: Hitting the $2x * x - 1$ key

The next five examples exhibit properties of the so-called *logistic(or quadratic)* map. This map is

$$x_{n+1} = rx_n(1 - x_n)$$

Note that, if $0 \leq r \leq 4$, then $0 \leq x_n \leq 1$, so that the sequence of x's remains in the interval $[0, 1]$.

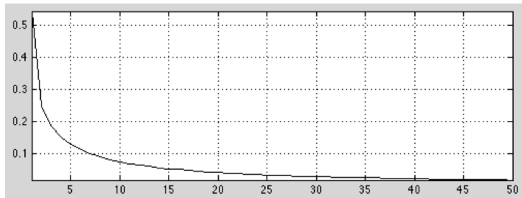


Figure 32: Hitting the $r * x * (1 - x)$ key with $r = 1.0$

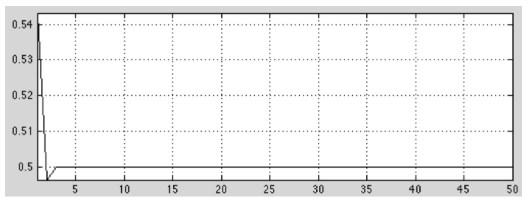


Figure 33: Hitting the $r * x * (1 - x)$ key with $r = 2.0$

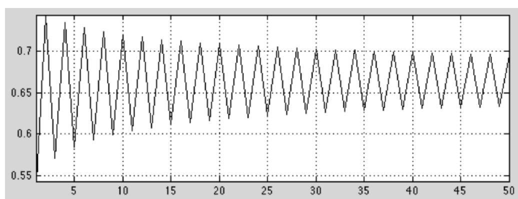


Figure 34: Hitting the $r * x * (1 - x)$ key with $r = 3.0$

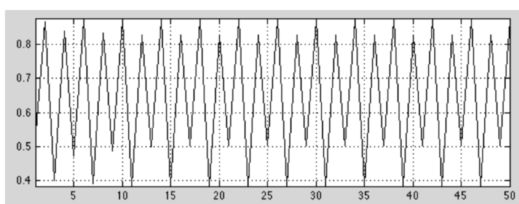


Figure 35: Hitting the $r * x * (1 - x)$ key with $r = 3.5$

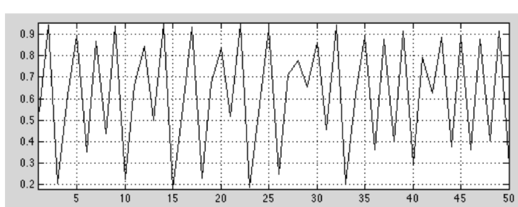


Figure 36: Hitting the $r * x * (1 - x)$ key with $r = 3.8$

First pass ....

Let us develop the logistic equation from first principles (first, change the parameter $r \to \lambda$).

We start, generally, with a model of growth:

$$x_{new} = \lambda x_{old}$$

We prefer to write this in terms of n:

$$x_{n+1} = \lambda x_n$$

This says x changes from one time period, n, to the next, $n+1$, according to λ. If λ is larger than 1, x gets larger with successive iterations. If λ is less than 1, x diminishes.

Let us set λ to be larger than 1 (say $\lambda = 1.5$). We start, year 1 ($n = 1$), with a population of 16 ($x_1 = 16$), and since $\lambda = 1.5$, each year x is increased by 50%. So in years $2, 3, 4, 5, ...$ we have magnitudes $24, 36, 54, ...$

The population is growing *exponentially* larger.

By year 25 the population would be over a quarter million.

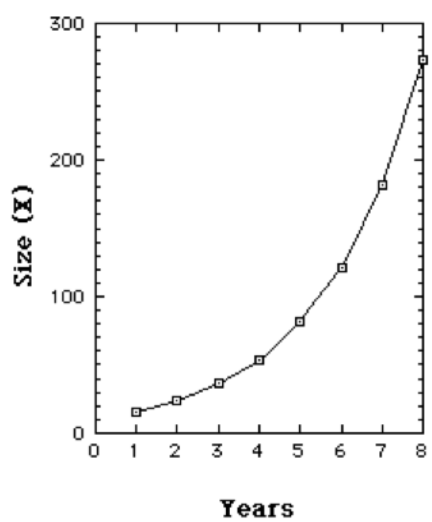


Figure 37: Iterations of the Growth Model with $\lambda = 1.5$

So far, notice, we have a *linear* model that produces unlimited growth.

Limited Growth model - Logistic Map

The Logistic Map prevents unlimited growth by inhibiting growth whenever it achieves a high level. This is achieved with an additional term, $(1 - x_n)$.

The growth measure (x) has also been rescaled so that the maximum value x can achieve is transformed to 1 (so if the maximum size is 25 million, say, x is expressed as a proportion of that maximum).

Our new model is

$$x_{n+1} = \lambda x_n(1 - x_n)$$

with $0 \le \lambda \le 4$.

The $(1 - x_n)$ term serves to inhibit growth because as x approaches 1, $(1 - x_n)$ approaches 0.

Plotting x_{n+1} versus x_n, we see we have a nonlinear relation.

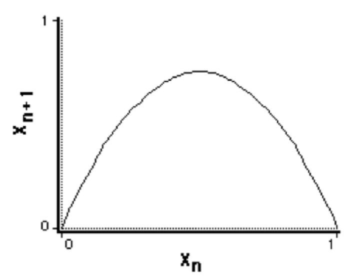


Figure 38: Limited Growth Model: x_{n+1} versus x_n , $\lambda = 3$

We have to iterate this function to see how it will behave. Suppose $\lambda = 3$ and $x_1 = 0.1$. Then we get

$$x_2 = \lambda x_1(1 - x_1) = 0.270$$
$$x_3 = \lambda x_2(1 - x_2) = 0.591$$
$$x_4 = \lambda x_3(1 - x_3) = 0.725$$

as shown below.

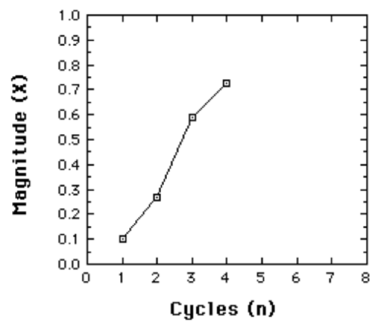


Figure 39: Behavior of the Logistic Map for $\lambda = 3$, $x_1 = 0.1$ iterated to give x_2, x_3 and x_4

It turns out that the logistic map is a very different animal, depending on its control parameter λ.

> **control parameter**: a parameter in the equations of a dynamic system. If control parameters are allowed to change, the dynamic system would also change.

To see this, *we next examine the time series produced at different values of* λ, starting near 0 and ending at $\lambda = 3.99$. Along the way we see very different results, revealing and introducing major features of a chaotic system.

> **time series**: a set of measures of behavior over time.

When λ is less than 1:

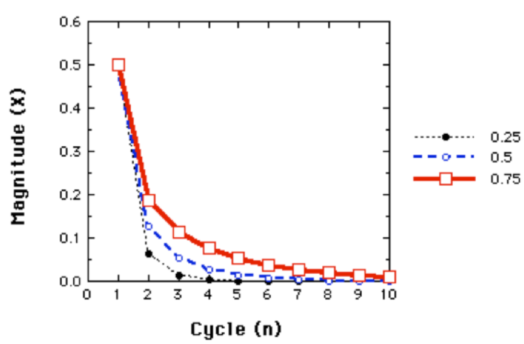


Figure 40: Behavior of the Logistic Map for $\lambda = 0.25, 0.50, \textit{and} 0.75$. In all cases $x_1 = 0.5$

The same fates awaits any starting value. So long as λ is less than 1, x goes toward 0. This illustrates a *one-point attractor*.

attractor: the state that a dynamic system eventually *settles down to*. An attractor is a set of values in the phase space to which a system migrates over time, or iterations. An attractor can be a single fixed point, a collection of points regularly visited, a loop, a complex orbit, or an infinite number of points. It need not be one- or two-dimensional. Attractors can have as many dimensions as the number of variables that influence its system.

When λ is between 1 and 3:

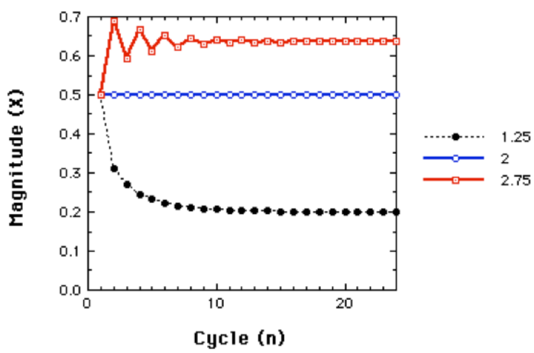


Figure 41: Behavior of the Logistic Map: for $\lambda = 1.25, 2.00, and 2.75$. In all cases $x_1 = 0.5$

Now, regardless, of the starting value, we have non-zero one-point attractors.
When λ is larger than 3:

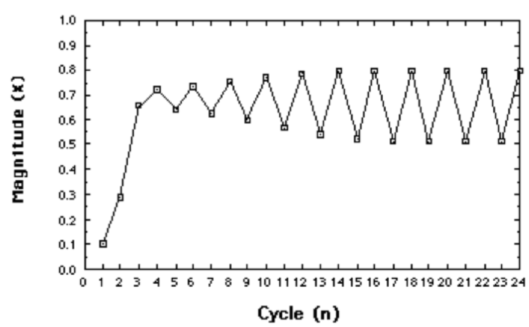


Figure 42: Behavior of the Logistic Map for $\lambda = 3.2$.

Moving beyond $\lambda = 3$, the system settles down to alternating between two points. We have a *two-point attractor*. We have illustrated a *bifurcation*, or *period doubling*.

bifurcation: a qualitative change in the behavior (attractor) of a dynamic system associated with a change in control parameter.

Increasing λ further we have

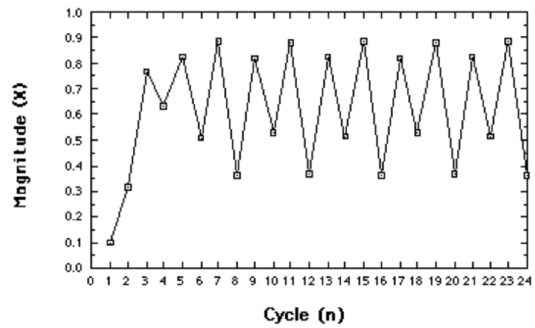


Figure 43: Behavior of the Logistic Map for $\lambda = 3.54$. A four-point attractor

Another bifurcation has occurred so we have a *4-point attractor*. Increasing λ further we can get higher order attractors. Illustrated below is the chaotic behavior of the system (no pattern at all!).

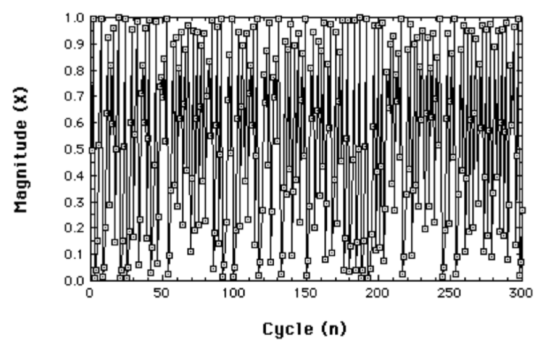


Figure 44: Chaotic behavior of the Logistic Map at $\lambda = 3.99$

So, what is an *attractor*? It is whatever the system *settles down to.*

Here is a very important concept from nonlinear dynamics.

A system eventually *settles down.* But what it settles down to, its *attractor*, need not have *stability*; it can be very *strange.*

> **strange attractor**: an N-point attractor in which N equals infinity. Usually (perhaps always) *self-similar* (discussed shortly) in form.

1.4. More Details

A *fixed point* of a mapping is a point that maps into itself, i.e., $x_{n+1} = x_n$. If there are points which, after more and more iterations of the mapping approach closer and closer to a fixed point, then the fixed point is called an *attractor*.

If $\lambda < 1$, then we have $x_{n+1} \leq x_n$ for all x_n. This implies that the ultimate result of repeated iterations is is inevitably $x = 0$. Thus, when $\lambda < 1$ the mapping has one fixed point, namely $x = 0$, which is an attractor.

We can determine the fixed points for a given value of λ by using the fixed point condition

$$x = \lambda x(1 - x)$$

which has solutions

$$x = \lambda x(1 - x) \rightarrow 1 = \lambda(1 - x) \rightarrow x = 1 - \frac{1}{\lambda}$$

when $x \neq 0$ and $\lambda \geq 1$. For $\lambda < 1$ the solution is $x = 0$ as stated earlier. Let us show that this is true: Suppose $\lambda = 1/2$. We then have

$$x = \frac{1}{2}x(1 - x) \rightarrow x^2 + x + 0 \rightarrow x(x + 1) = 0 \rightarrow x = 0 \ or \ x = -1(not \ allowed)$$

Geometrically, the fixed point is the intersection of the logistic map function(red curve) with the line $x_{n+1} = x_n$ (green curve) using program *fixpt.m* and the fixed point is the circle as shown below for $\lambda = 0.8 < 1$ - the intersection (fixed point) is $x = 0$ as we stated.

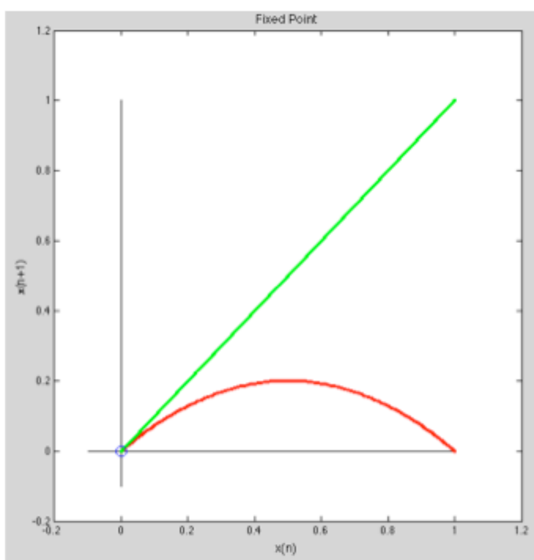


Figure 45: Fixed point for $\lambda = 0.8$

For the case $\lambda = 2.8$ we have a single fixed point at $x = 0.643$ as shown below

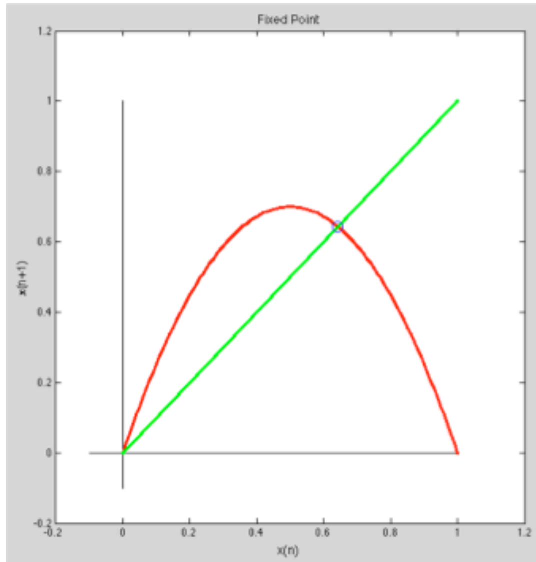


Figure 46: Fixed point for $\lambda = 2.8$

The next question is whether the fixed points are *stable*, that is, whether they are *attractors*. To settle this question, we start with a point near a fixed point and see if the result of repeated mapping converges to the fixed point. We can answer this question using a simple but informative geometrical construction of the iteration process near the fixed point(program *returnmap.m*).

> **limit or fixed points**: these are points in phase space. There are three kinds: attractors, repellers, and saddle points. A system moves away from repellers and towards attractors. A saddle point is both an attractor and a repeller, it attracts a system in certain regions, and repels the system to other regions. (see Appendix - Section 1.7.1 for more details).

This is shown below for the two cases above.

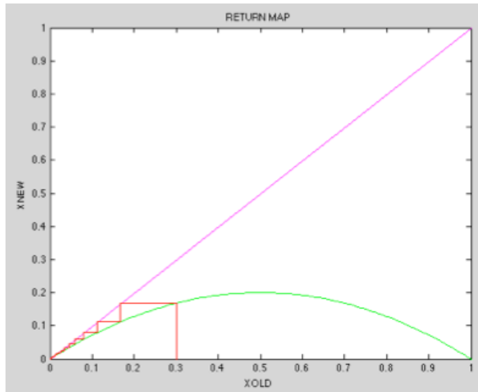


Figure 47: The stable fixed point for $\lambda = 0.8$

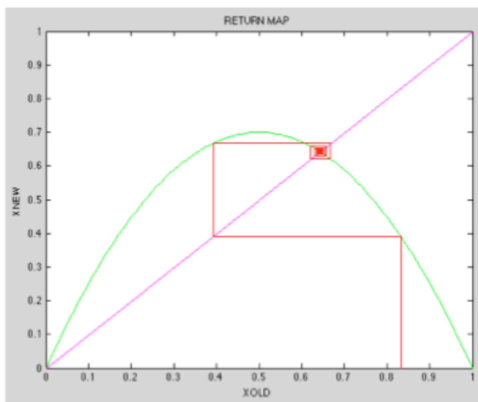


Figure 48: The stable fixed point for $\lambda = 2.8$

The plot is called a return map. For all $0 < \lambda < 1$ the fixed point $x = 0$ is stable. The fixed point $x - 1/\lambda$ is stable for $1 < \lambda < 3$.

Now the criterion for stability of a fixed point is

$$\lambda(1 - x_{fix}) \leq 1$$

or

$$\lambda\left(1 - 2\left(1 - \frac{1}{\lambda}\right)\right) \leq 1 \rightarrow \lambda\left(-1 + \frac{2}{\lambda}\right) \leq 1 \rightarrow -\lambda + 2 \leq 1 \rightarrow 1 < \lambda < 3$$

as expected. This relation also says that there are no stable fixed points for $\lambda > 3$.

As we saw earlier, the sequence does not settle down to a single value (fixed point) in this case, but oscillates between a set of values (sometimes the set will be infinite in number - that as we will see is *chaos*).

For example, look at the return map for $\lambda = 3.3$.

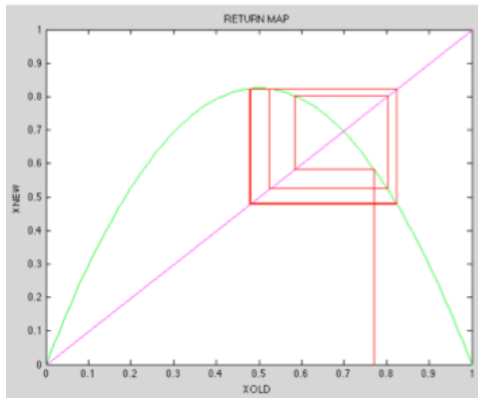


Figure 49: Return Map for $\lambda = 3.3$

Clearly, the iteration is alternating between two distinct points. A plot of the sequence looks like:

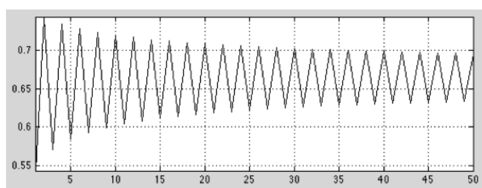


Figure 50: Time Series for $\lambda = 3.3$

In this region we search for points with higher periodicity, that is, points which return to their original value after some number of iterations. For example, period-2 points satisfy $x_{n+2} = x_n$ or

$$x_{n+2} = \lambda x_{n+1}(1 - x_{n+1}) = \lambda^2 x_n(1 - x_n) - \lambda^3 x_n^2(1 - x_n)^2 = F(x_n)$$

This is called the double-map function. If we plot the double map function, the single map function (the original map function) and the line $x_{n+2} = x_n$ for $\lambda = 2.8$ all three curves should intersect at the same point (the fixed point at 0.643) since we already found a period-1 fixed point in this case. This is shown below(*quadplot1.m*):

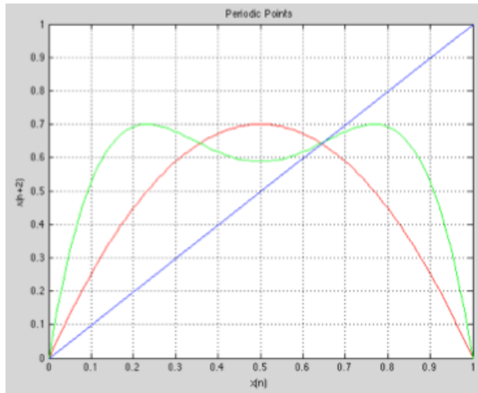


Figure 51: Period-1 fixed point at $\lambda = 2.8$

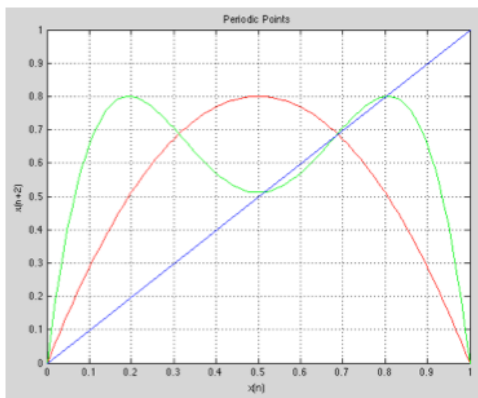


Figure 52: Plot for $\lambda = 3.3$

In this case, there are *three* fixed points in this case. The middle one is the unstable fixed point of the period-1 or single mapping at

$$x = 1 - \frac{1}{3.3} = 0.697$$

The two remaining fixed points of the double mapping are stable in the range $3 < \lambda < 3.449....$ Note that these two points are a single *pair* of period-2 points; calling them x_A and x_B, the mapping takes one into the other, i.e, $x_A = F(x_B)$ and $x_B = F(x_A)$.

This transition, as the value of λ is raised past a critical value (3 in this case), from one stable fixed point to a pair of stable period-2 points, is known as a *bifurcation* or *period doubling*.

> **period-doubling**: the change in dynamics in which a $N-$point attractor is replaced by a $2N-$point attractor.

We can see this in another way by plotting a time series (the sequence of x values) (*logmap1.m*).

The plot below is for $\lambda = 0.8$ and we clearly see the map iterate to the stable fixed point at $x = 0$.

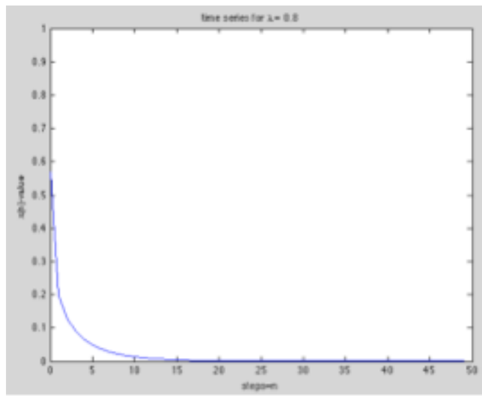


Figure 53: Plot for $\lambda = 0.8$

The next plot below is for $\lambda = 1.8$ and we clearly see the map iterate to the stable fixed point at $x = 0.44$.

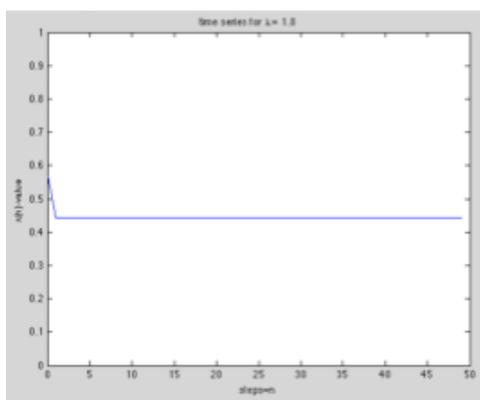


Figure 54: Plot for $\lambda = 1.8$

The next plot below is for $\lambda = 2.8$ and we clearly see the map iterate to the stable fixed point at $x = 0.64$, in agreement with our earlier result.

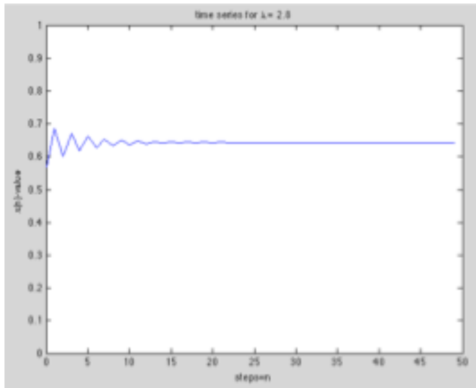


Figure 55: Plot for $\lambda = 2.8$

The next plot below is for $\lambda = 3.3$ and we clearly see the map iterate to two stable fixed points at $x = 0.52$ and $x = 0.80$, in agreement with our earlier result. These are period-2 points.

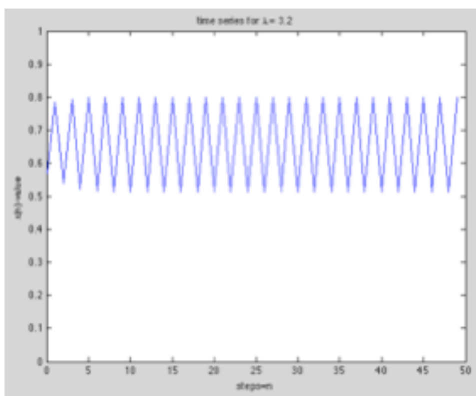


Figure 56: Plot for $\lambda = 3.3$

The next plot below is for $\lambda = 3.5$ and we clearly see the map iterate to four stable fixed points. These are period-4 points.

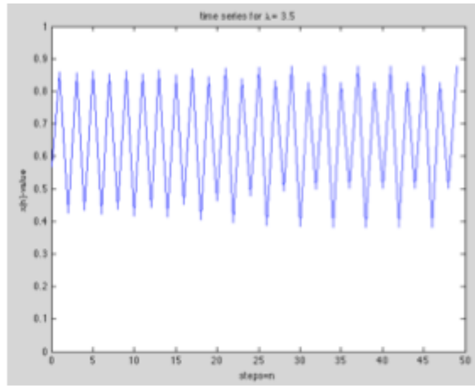


Figure 57: Plot for $\lambda = 3.5$

The last plot below is in a chaotic regime with no periodicity and no fixed points.

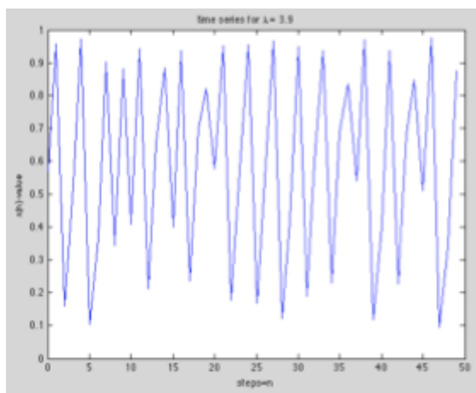


Figure 58: Plot for $\lambda = 3.9$

As λ is raised above 3.499 a second bifurcation occurs (see period-4 points for $\lambda = 3.5$ above), that is, the pair of stable period-2 point turns into a quartet of period-4 points. Such bifurcations occur faster and faster until an infinite number of bifurcations occur at $\lambda = 3.56994....$

We can see the entire structure of the logistic map in the plot below (*logmapall.m*). This is a plot of a large number of x values in a sequence for fixe λ value versus the λ value. Can we guess what it will look like? Using this type of plot we can see all of the other plots.

> **bifurcation diagram**: Visual summary of the succession of period-doubling produced as a control parameter is changed.

Clearly, we can see the period-1, period-2, period-4, etc regions, the bifurcations

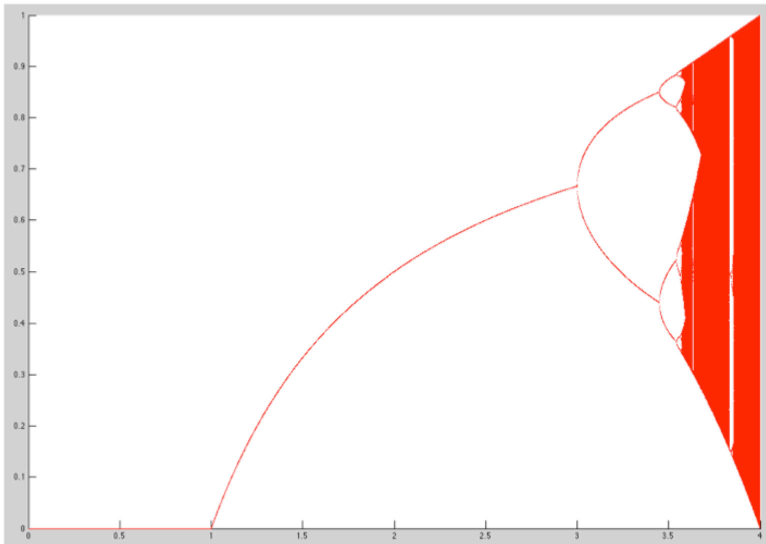


Figure 59: A Bifurcation Diagram

or period doublings, the chaotic regions and many other strange features.

If we blowup the region from 3.545 to 3.575 we have

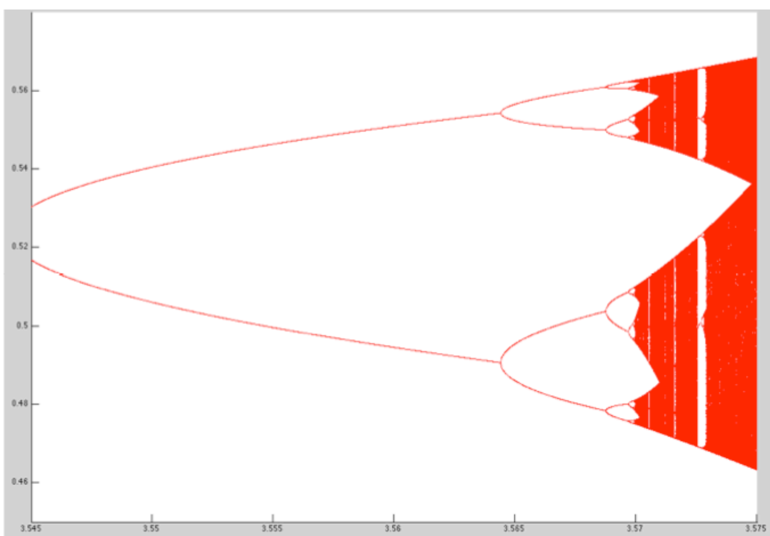


Figure 60: Blowup of region 3.545 to 3.575

Finally if we blowup the region 3.5680 to 3.5710 we have

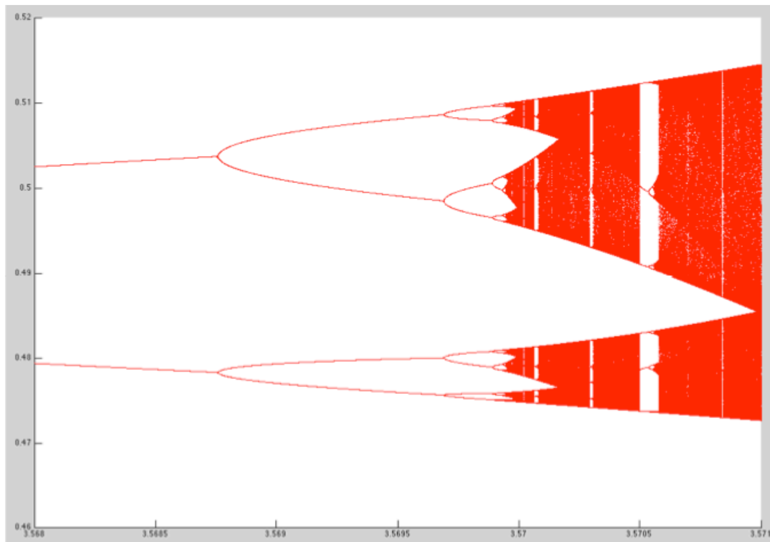


Figure 61: Blowup of region 3.5680 to 3.5710

Denoting by λ_k the critical value of λ at which the bifurcation from a stable period-k set of points to a stable period-(k+1) set occurs, it is found that

$$\lim_{k \to \infty} \frac{\lambda_k - \lambda_{k-1}}{\lambda_{k+1} - \lambda_k} = 4.669201......$$

which known as the *Feigenbaum number*. This ratio turns out to be universal for any map with a quadratic maximum and is seen in a wide range of physical problems. One of the conclusions one can draw from the existence of the Feigenbaum number is that each bifurcation looks similar up to a magnification factor. This *scale invariance* or *self-similarity* plays an important role in the *transition to or onset of chaos* and in the structure of the *strange attractor* that we will discuss shortly.

> **chaos**: Behavior of a dynamic system that has (a) a very large (possibly infinite) number of attractors and (b) is sensitive to initial conditions.

We note from the pictures that above $\lambda_c = 3.56994...$ the attractor set for many (but not all) values of λ shows no periodicity at all. For these values of λ the quadratic map exhibits chaos and is a strange attractor. In the region $\lambda_c < \lambda < 4$ there are *windows* where attractors of small period reappear.

An important property of chaotic motion is extreme sensitivity to initial conditions (as we mentioned earlier). To express the sensitivity quantitatively we introduce the *Lyapunov exponent*.

Lyapunov Number (Liapunov number): The value of an exponent, a coefficient of time, that reflects the rate of departure of dynamic orbits. It is a measure of sensitivity to initial conditions.

Consider two points in phase space separated by distance d_0 at time $t = 0$. If the motion is regular (non-chaotic) these two points will remain relatively close, separating at most according to a power of time. In chaotic motion the two points separate exponentially with time according to

$$d(t) = d_0 e^{\lambda_L t}$$

The parameter λ_L is the Lyapunov exponent. If λ_L is positive the motion is chaotic. A zero or negative coefficient indicates non-chaotic motion. There are as many Lyapunov exponents for a particular system as there are variables. Thus, for the logistic map there is one Lyapunov exponent.

We plot the Lyapunov exponent as a function of λ, the logistic map parameter, below. (See Appendix - Section 1.7.2 for more details)

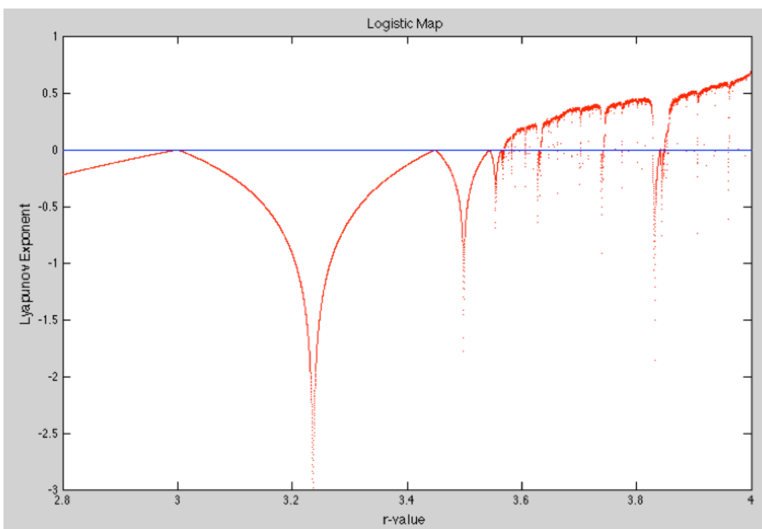
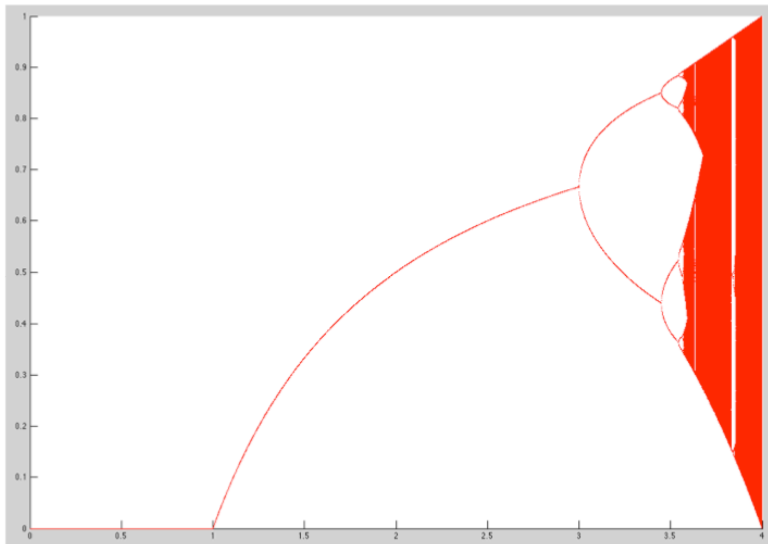
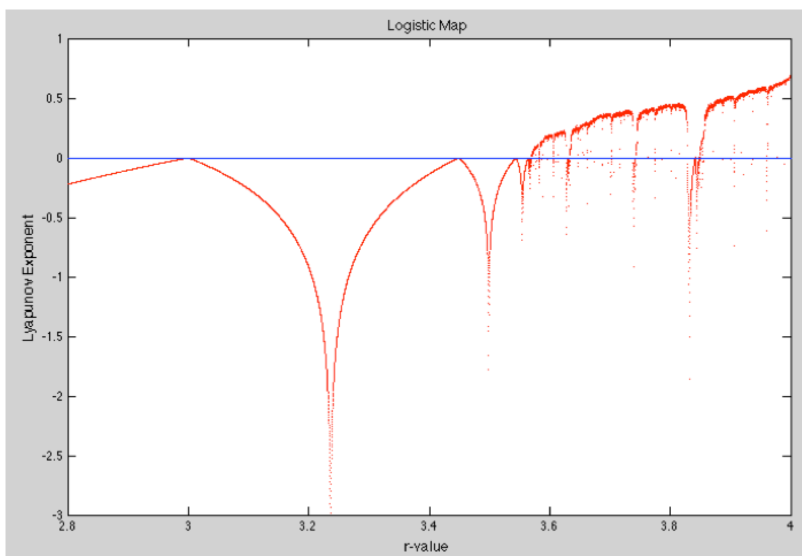


Figure 62: Lyapunov exponent as a function of λ

Let us replot two graphs:





Clearly, the Lyapunov exponent is negative whenever the map is stable and positive whenever the map is chaotic. The value of λ is zero when a bifurcation occurs and the solution becomes unstable. A superstable point occurs where $\lambda = -\infty$. We can see clearly from the plot that when λ goes above zero, there are windows of stability where λ goes negative for a while and period orbits occur amid the chaotic behavior. The relatively wide window just above $\lambda = 3.8$ is apparent.

1.5. Expanding these ideas

Let us repeat many of the ideas we have been discussing for a better understanding.

1.5.1. The Bifurcation Diagram

So, again, what is a *bifurcation*? A bifurcation is a *period-doubling*, a change from an *N-point attractor* to a *2N-point attractor*, which occurs when the control parameter is changed.

A Bifurcation Diagram is a visual summary of the succession of period-doubling produced as λ increases. The next figure shows the bifurcation diagram of the logistic map, λ along the $x-$axis. For each value of λ the system is first allowed to settle down and then successive values of x are plotted for a large number of iterations.

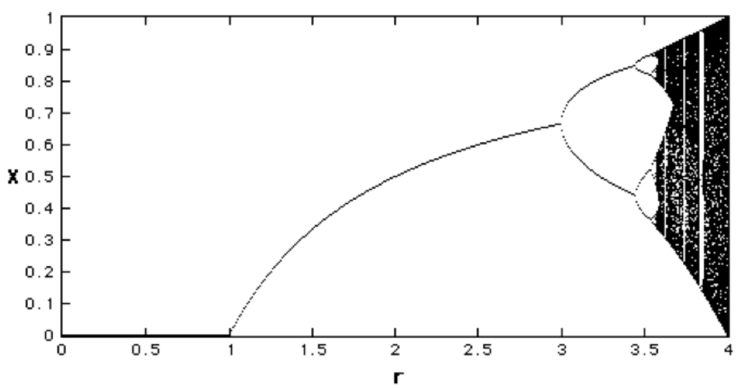


Figure 63: Bifurcation Diagram

We see that for $\lambda < 1$, all points are plotted at 0. 0 is the one-point attractor for λ less than 1. For λ between 1 and 3, we still have one-point attractors, but the attracted value of x increases as λ increases, at least to $\lambda = 3$. Bifurcations occur at $\lambda = 3, 3.45, 3.54, 3.564, 3.569$(*approximately*), etc, until just beyond 3.57, where the system is chaotic. However, the system is not chaotic for all values of λ greater than 3.57.

Let us zoom in a bit.

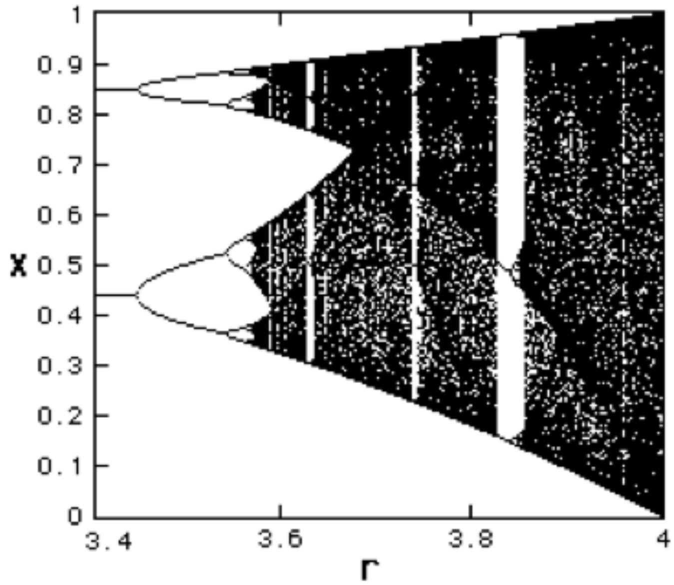


Figure 64: Bifurcation Diagram between 3.4 and 4

Notice again that at several values of λ, greater than 3.57, a small number of $x-$values are visited. These regions produce the *white space* in the diagram. Look closely at $\lambda = 3.83$ and you will see a three-point attractor. In fact, between 3.57 and 4 there is a rich interleaving of chaos and order. A small change in can make a stable system chaotic, and vice versa.

1.5.2. Sensitivity to Initial Conditions

Another important feature emerges in the chaotic region ... To see it, we set $\lambda = 3.99$ and begin at $x_1 = 0.3$. The next graph shows the time series for 48 iterations of the logistic map.

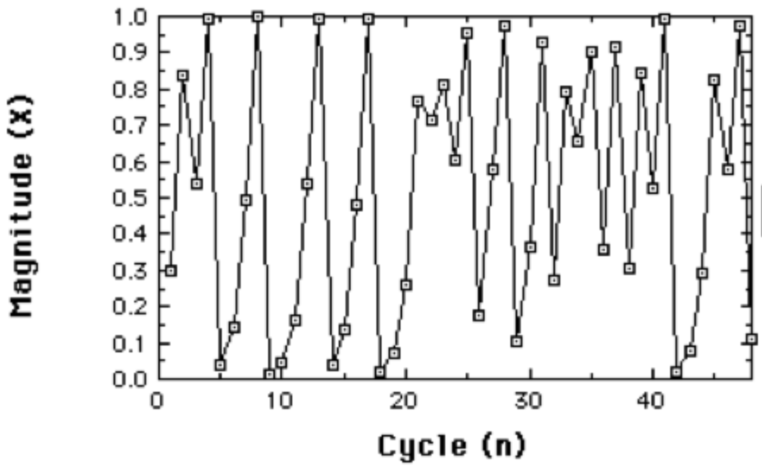


Figure 65: Time series - 48 iterations - $\lambda = 3.99$

Now suppose we alter the starting point a bit. The next figure compares the time series for $x_1 = 0.3$ (open squares) with that for $x_1 = 0.301$ (solid dots).

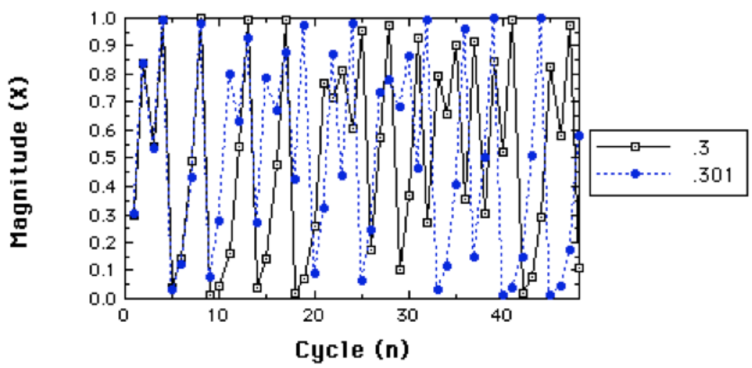


Figure 66: Time series - 48 iterations - $\lambda = 3.99$ - Different starting points

The two time series stay close for about 10 iterations. But after that, they are pretty much on their own - they diverge from each other.

Let us try starting closer together. We next compare starting at 0.3 with starting at 0.3000001....

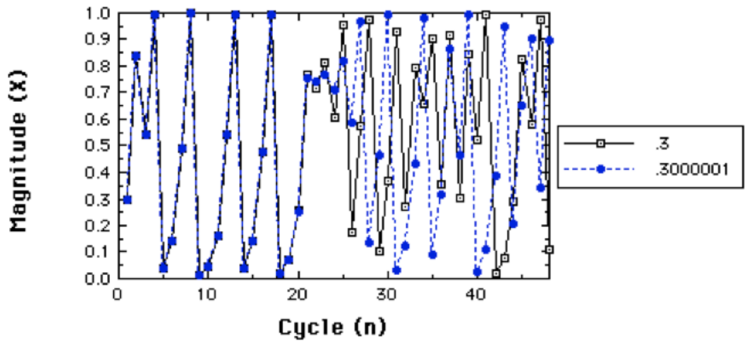


Figure 67: Time series - 48 iterations - λ = 3.99 - Closer starting points

This time they stay close for a longer time, but after 24 iterations they diverge.

To see how independent they become, the next figure provides scatterplots for the two series before and after 24 iterations.

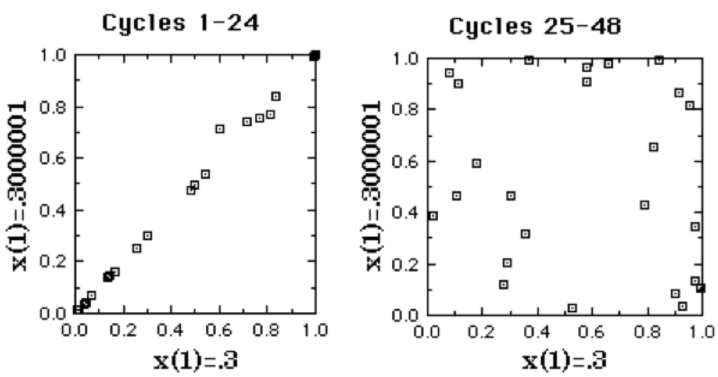


Figure 68: Scatterplots to compare time series

The *correlation* after 24 iterations (right side), is essentially zero. *Unreliability* has replaced *reliability*.

We have illustrated here one of the *symptoms of chaos*.

> **sensitivity to initial conditions**: A property of chaotic systems. A dynamic system has sensitivity to initial conditions when very small differences in starting values result in very different behavior. If the orbits of nearby starting points diverge, the system has sensitivity to initial conditions.

A chaotic system is one for which the *"distance"* between two trajectories or orbits starting from nearby points in its state space diverges over time. The magnitude of the divergence increases *exponentially* in a chaotic system.

So what?

It is unpredictable, *in principle* because in order to predict its behavior into the future we must know its current value precisely.

We have here an example where a slight difference, in the sixth decimal place, resulted in prediction failure after 24 iterations.

We note that six decimal places far exceeds the kind of measuring accuracy we typically achieve with natural biological systems.

1.5.3. Symptoms of Chaos

We are beginning to sharpen our definition of a chaotic system.

First of all, it is a *deterministic* system.

If we observe behavior that we suspect to be the product of a chaotic system, it will be difficult to distinguish from random behavior sensitive to initial conditions.

NOTE: Neither of these symptoms, on their own, are *sufficient* to identify chaos.

Note on technical versus metaphorical uses of terms:

Students of chaotic systems have begun to use the (originally mathematical) terms in a *metaphorical* way.

For example, *bifurcation*, defined here as a period doubling has come to be used to refer to any qualitative change. Even the term *chaos*, has become synonymous, for some with *overwhelming anxiety.*

Metaphors enrich our understanding, and have helped extend nonlinear thinking into new areas. On the other hand, it is important that we are aware of the technical/metaphorical difference.

1.5.4. Two- and Three-Dimensional Systems

First we observe the distinction between variables(dimensions) and parameters.

Consider again the logistic map

$$x_{n+1} = \lambda x_n(1 - x_n)$$

Multiply the right side out

$$x_{n+1} = \lambda x_n - \lambda x_n^2$$

and replace the two λ's with separate parameters, a and b,

$$x_{n+1} = a x_n - b x_n^2$$

Now, the separate *parameters*, a and b, govern growth and suppression, but we still have only one *variable* x.

When we have a system with two or more variables,

1. its current *state* is the current values of its variables
2. it is treated as a *point* in *phase(state) space*
3. we refer to its *trajectory* or *orbit* in time

1.5.5. The Predator-Prey System

This is a 2-dimensional dynamic system in which two variables grow, but one grows at the expense of the other. The number of *predators* is represented by y, the number of *prey* by x

We next plot the phase space of the system, which is a 2-dimensional plot of the possible states of the system.

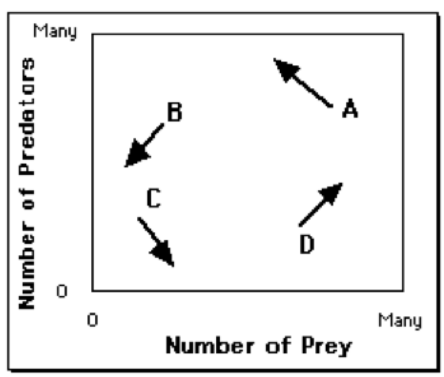


Figure 69: The phase space of the predator-prey system

A = too many predators

B = too few prey

C = few predators and prey; prey can grow

D = Few predators, ample prey

Four states are shown. At *Point A* there are a large number of predators and a large number of prey. Drawn from *Point A* is an arrow or vector showing how the system would change from that point. Many prey would be eaten, to the benefit of the predator. The arrow from *Point A*, therefore, points in the direction of a smaller value of x and a larger value of y.

At *Point B* there are many predators but few prey. The vector shows that both decrease; the predators because there are too few prey, the prey because the number of predators is still to the prey's disadvantage.

At *Point C*, since there are a small number of predators the number of prey can increase, but there are still too few prey to sustain the predator population.

Finally, at *Point D*, having many prey is advantageous to the predators, but the number of prey is still too small to inhibit prey growth, so their numbers increase.

The full *trajectory* (somewhat idealized) is shown next.

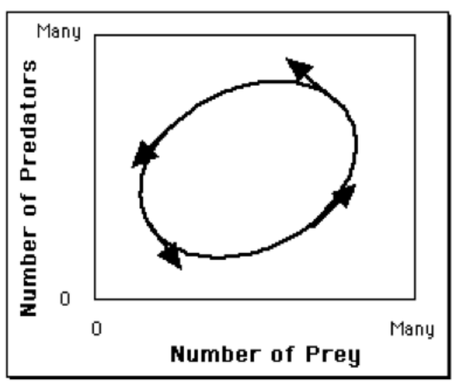


Figure 70: The phase space of the predator-prey system

An attractor that forms a loop like this is called a *limit cycle*.

> **limit cycle**: An attractor that is periodic in time, that is, that cycles periodically through an ordered sequence of states.

However, in this case the system doesn't start outside the loop and move into it as a final attractor. In this system any starting state is already in the final loop. This is shown in the next figure, which shows loops from four different starting states.

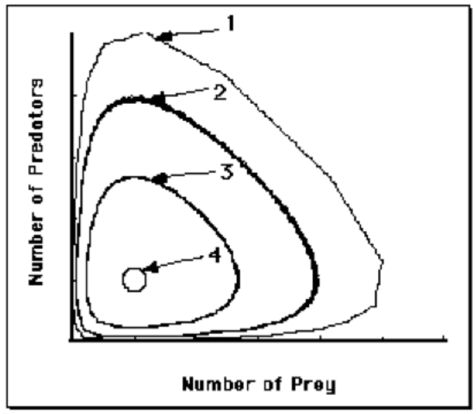


Figure 71: Phase portrait of the predator-prey system showing influence of starting state

Points 1-4 start with about the same number of prey but with different numbers of predators.

Let's look at this system over time, that is, as two time series.

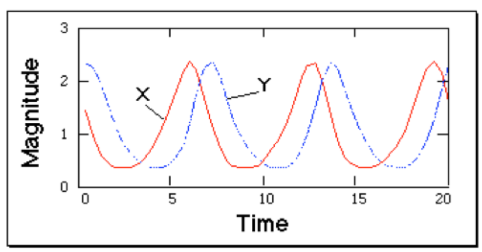


Figure 72: The time series of the predator-prey system

This figures shows how the two variables oscillate, *out of phase*.

1.5.6. Continuous Functions and Differential Equations

Changes in *discrete* variables are expressed with *difference equations*, such as the logistic map.

Changes in *continuous* variables are expressed with *differential equations*

For example, the Predator-prey system is typically presented as a set of two differential equations:

$$\frac{dx}{dt} = (a - by)x \quad , \quad \frac{dy}{dt} = (cx - d)y$$

1.5.7. Types of two-dimensional interactions

Other types of two-dimensional interactions are possible.

mutually supportive: the larger one gets, the faster the other grows

mutually competitive: each negatively affects the other

supportive-competitive: as in Predator-Prey

1.5.8. The buckling column system

The *Buckling Column* system can be used to discuss psychological phenomena that exhibit oscillations (for example, mood swings, states of consciousness, attitude changes).

The model is a single, flexible, column that supports a mass within a horizontally constrained space. If the mass of the object is sufficiently heavy, the column will *give*, or buckle. There are two dimensions, x representing the sideways displacement of the column, and y the velocity of its movement.

Shown next are two situations, differing in the magnitude of the mass.

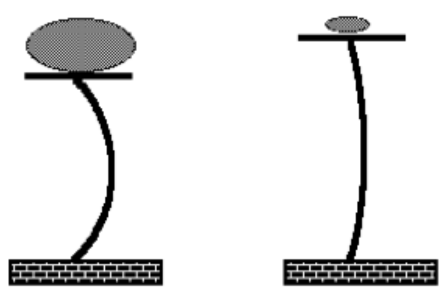


Figure 73: The Buckling Column Model

The mass on the left is larger than the mass on the right. What are the dynamics?

The column is elastic, so an initial give is followed by a springy return and bouncing (oscillations). If there is resistance (friction), the bouncing will diminish and the mass will come to rest. The equations are:

$$\frac{dx}{dt} = y \quad , \quad \frac{dy}{dt} = (1 - m)(ax^3 + b + cy)$$

The parameters m and c represent mass and friction respectively. If there is friction ($c > 0$), and mass is small, the column eventually returns to the upright position ($x = y = 0$), illustrated with the next two trajectories.

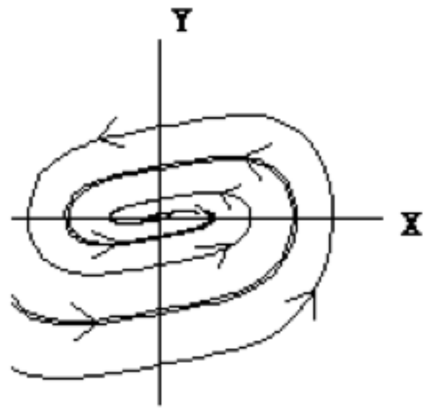


Figure 74: Phase portrait of the buckling column model

For a light mass, the column comes to rest at single point (attractor) for any starting configuration.

With a heavy mass, the column comes to rest in one of two positions (*two-point attractor*), again illustrated with two trajectories.

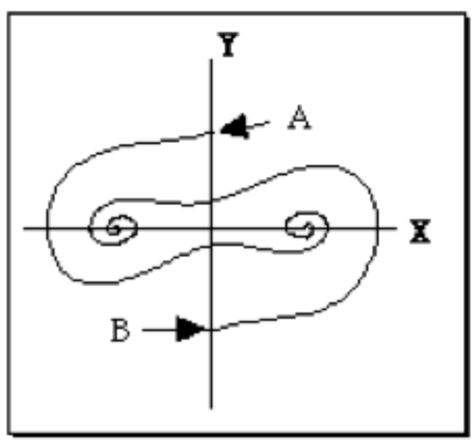


Figure 75: Phase portrait of the buckling column model

Starting at point A, the system comes to rest buckled slightly to the right, starting at B ends up buckled to the left. Now we can introduce another major concept...

1.5.9. Basins of attraction

With sufficient mass, the buckling column can end up in one of two states, buckled to the left or to the right.

What determines which is its fate?

For a given set of parameter values, the fate is determined entirely by where it starts, the initial values of x and y.

In fact, each point in phase space can be classified according to its attractor. The set of points associated with a given attractor is called that attractors' *basin of attraction*.

> **basin of attraction**: A region in phase space associated with a given attractor. The basin of attraction of an attractor is the set of all (initial) points that go to that attractor.

For the two-point attractor illustrated here, there are two basins of attraction. These are shown in the next figure, which has the phase space shaded according to attractor.

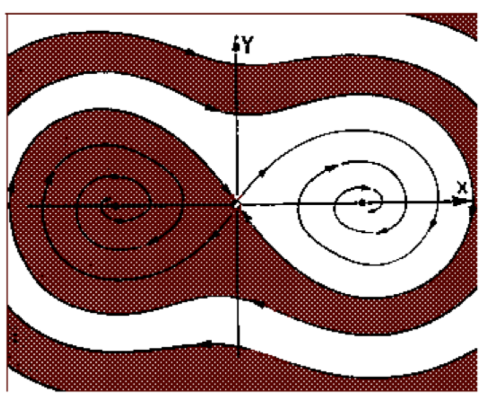


Figure 76: The basins of attraction for the buckling column system

The basin of attraction for one of the attractors is shaded. The basin of attraction for the other attractor is unshaded in the figure.

The term *seperatrix* is used to refer to the boundary between basins of attraction.

1.5.10. Three-dimensional Dynamic Systems - The Lorenz System

Lorenz's model of atmospheric dynamics is a classic in the chaos literature. The model nicely illustrates a three-dimensional system.

$$\frac{dx}{dt} = a(y - x) \quad , \quad \frac{dy}{dt} = x(b - z) - y \quad , \quad \frac{dz}{dt} = xy - cz$$

There are three variables reflecting temperature differences and air movement, but the details are irrelevant to us. We are interested in the trajectories of the system in its phase space for $a = 10$, $b = 28$, $c = 8/3$. Here we plot part of a trajectory starting from $(5, 5, 5)$.

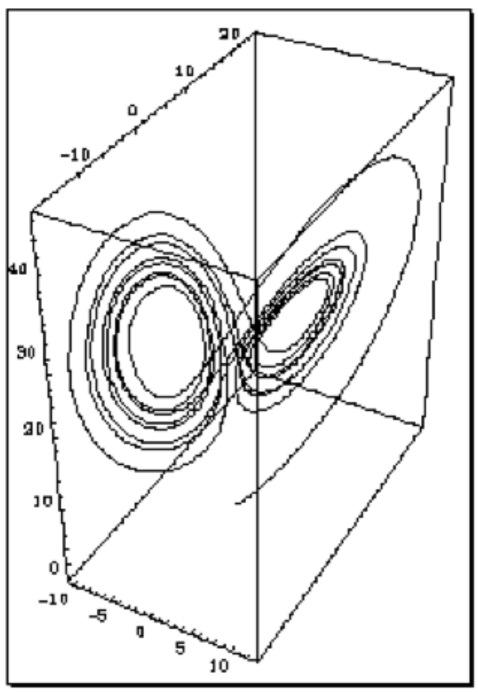


Figure 77: The Lorenz system - only a portion of one trajectory is shown

Although the figure suggests that a trajectory may intersect with earlier passes, in fact it never does.

The Lorenz system shows sensitivity to initial conditions. This is chaos, the first strange attractor, and it has become the icon for chaos.

This is illustrated dramatically by the simulation of the Lorenz equations (lorenz0.m and lorenz4.m). Again we observe extreme sensitivity to initial conditions. In fact, this model was the original source of the idea of sensitivity to initial conditions!

1.5.11. Beasts in Phase space - Limit Points

There are three kinds of *limit points*.

Attractors: where the system settles down to.

Repellers: a point the system moves away from.

Saddle points: and attractor from some regions, repeller to others.

Examples

Attractors: we've seen many

Repellers: the value 0 in the Logistic Map

Saddle points: the point (0,0) in the Buckling Column

Let us now return to a discussion of the non-linear, damped driven oscillator, which is a real physical system.

1.6. The Nonlinear Damped Driven Oscillator

We have a Newton's second law of the form for a pendulum (we have switched variables from x to θ - the angle of a swinging pendulum)

$$m\frac{d^2\theta}{dt^2} + \alpha\frac{d\theta}{dt} + \beta\sin\theta = \gamma\cos(\omega t)$$

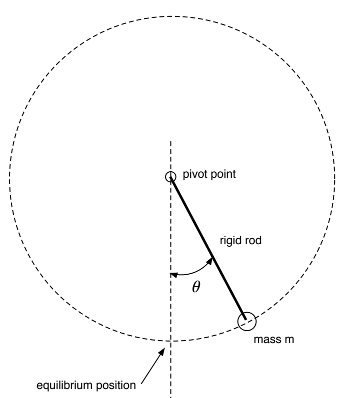


Figure 78: Typical pendulum

First we investigate the motion of this physical system in phase space. We fixed some of the parameters

$$b = 1.0 = \frac{\beta}{m} = \frac{1}{m}\sqrt{\frac{g}{\ell}} \quad , \quad \ell = \text{ pendulum length}$$

$$c = 0.5 = \frac{\alpha}{m} \quad , \quad \alpha = \text{ damping parameter}$$

$$\omega = 0.66666666 = \text{ driving frequency}$$

We will use as a variable (control) parameter (like λ in the logistic map) the constant a where $a = \gamma/m$. In particular, we will look at

$$\begin{aligned}
a = 0.90 &\rightarrow \text{ periodic motion} \\
a = 1.07 &\rightarrow \text{ periodic doubling} \\
a = 1.15 &\rightarrow \text{ chaotic motion} \\
a = 1.35 &\rightarrow \text{ periodic motion} \\
a = 1.45 &\rightarrow \text{ periodic doubling} \\
a = 1.47 &\rightarrow \text{ periodic doubling} \\
a = 1.50 &\rightarrow \text{ chaotic motion}
\end{aligned}$$

We run *phsp.m* to generate the phase space plots shown below:

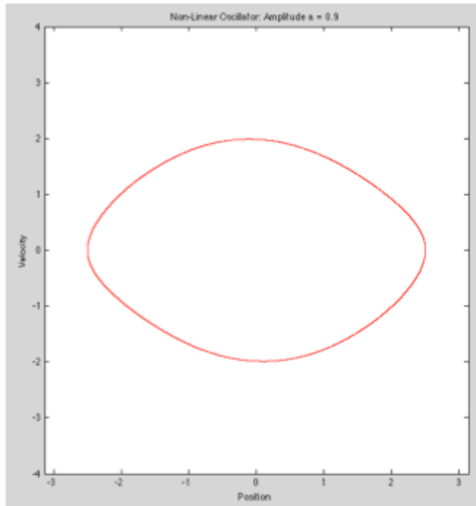


Figure 79: $a = 0.90$ — periodic motion

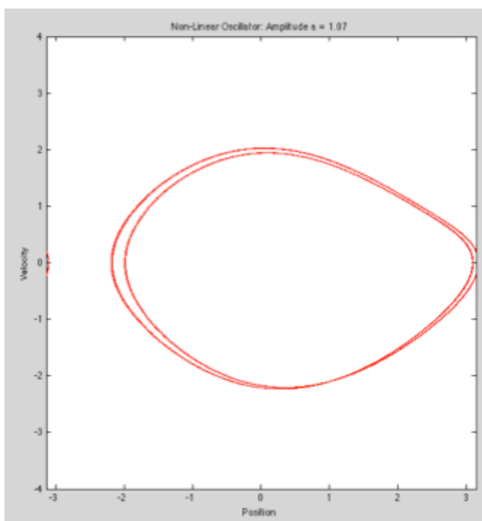


Figure 80: $a = 1.07$ — periodic doubling

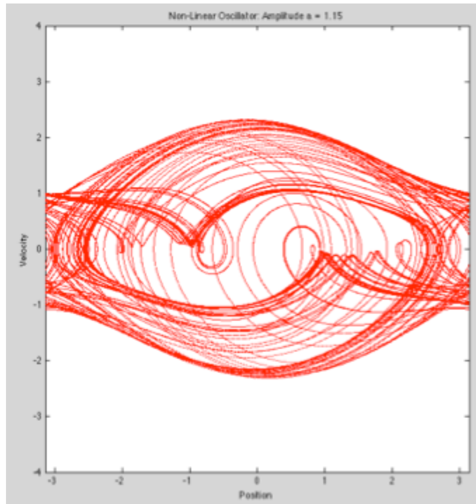


Figure 81: $a = 1.15$ — chaotic motion

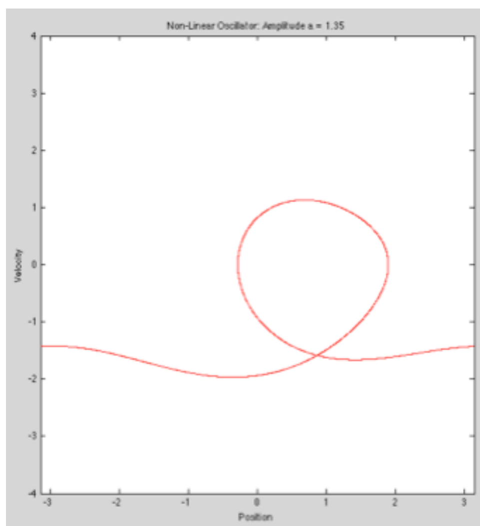


Figure 82: $a = 1.35$ — periodic motion

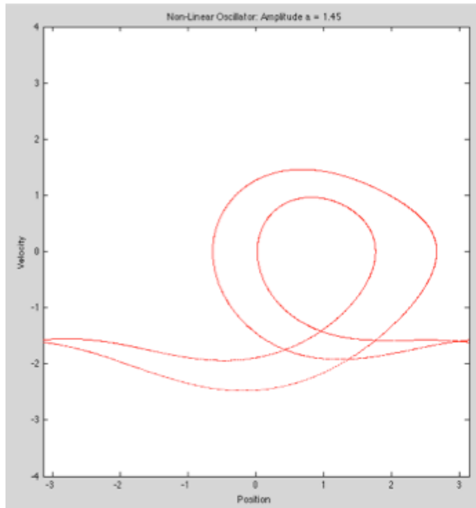


Figure 83: $a = 1.45$ — periodic doubling

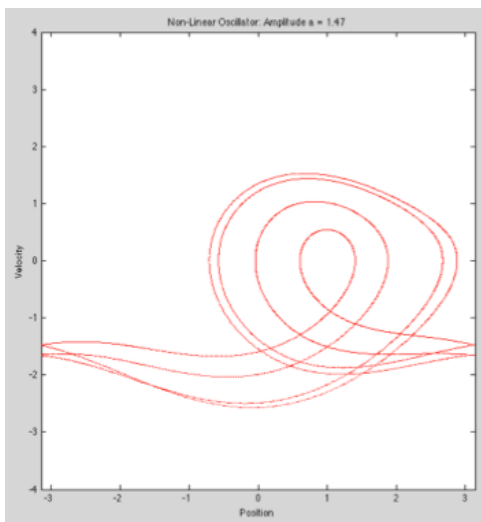


Figure 84: $a = 1.47$ — periodic doubling

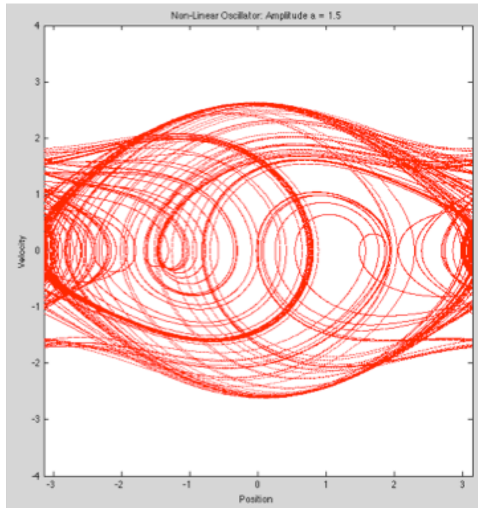


Figure 85: $a = 1.50$ — chaotic motion

The dependence of the system motion on the amplitude is clearly very complex and sensitive to value.

Another way to visualize the behavior of these systems is via *Poincare Plots*. The Poincare plot is the same as using a stroboscope on the motion. In this case we flash the strobe once every cycle of the driving force (frequency = ω). We run *phsppoin.m* to generate the plots shown below:

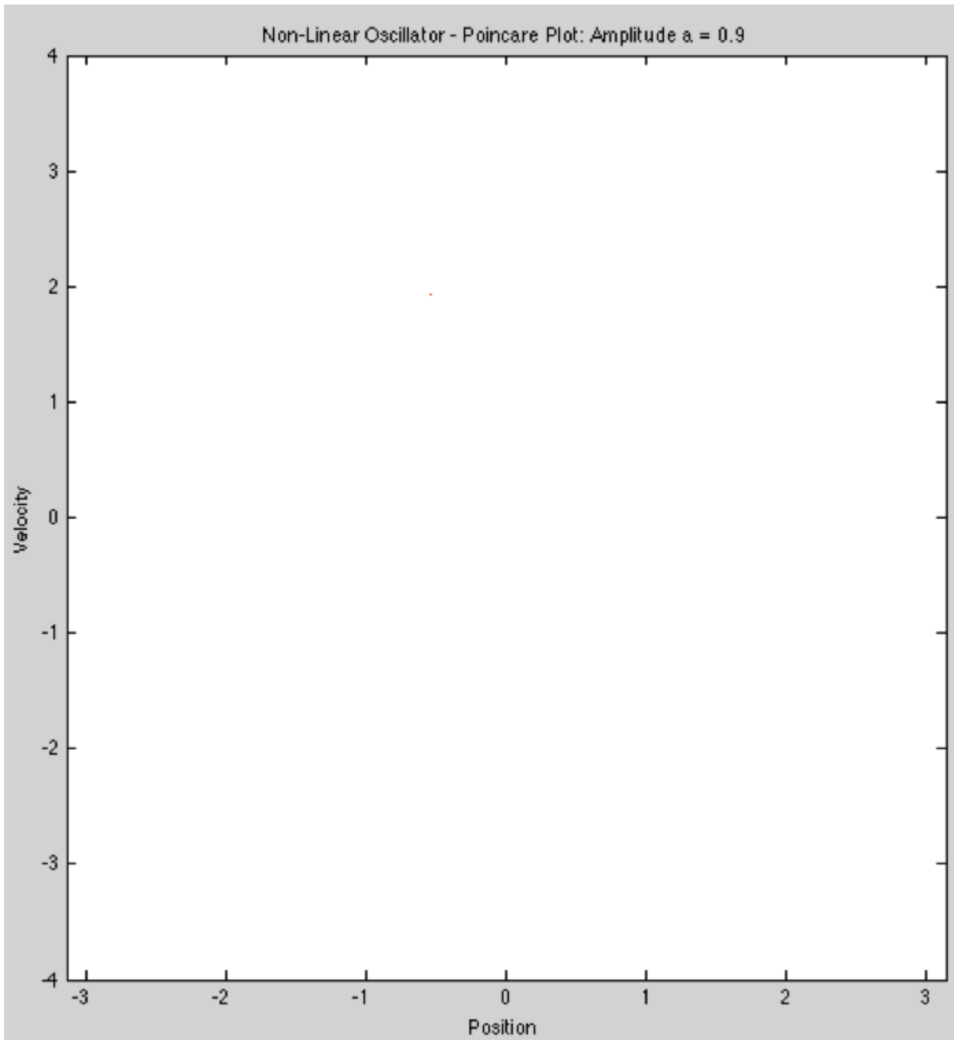


Figure 86: $a = 0.90$ — periodic motion

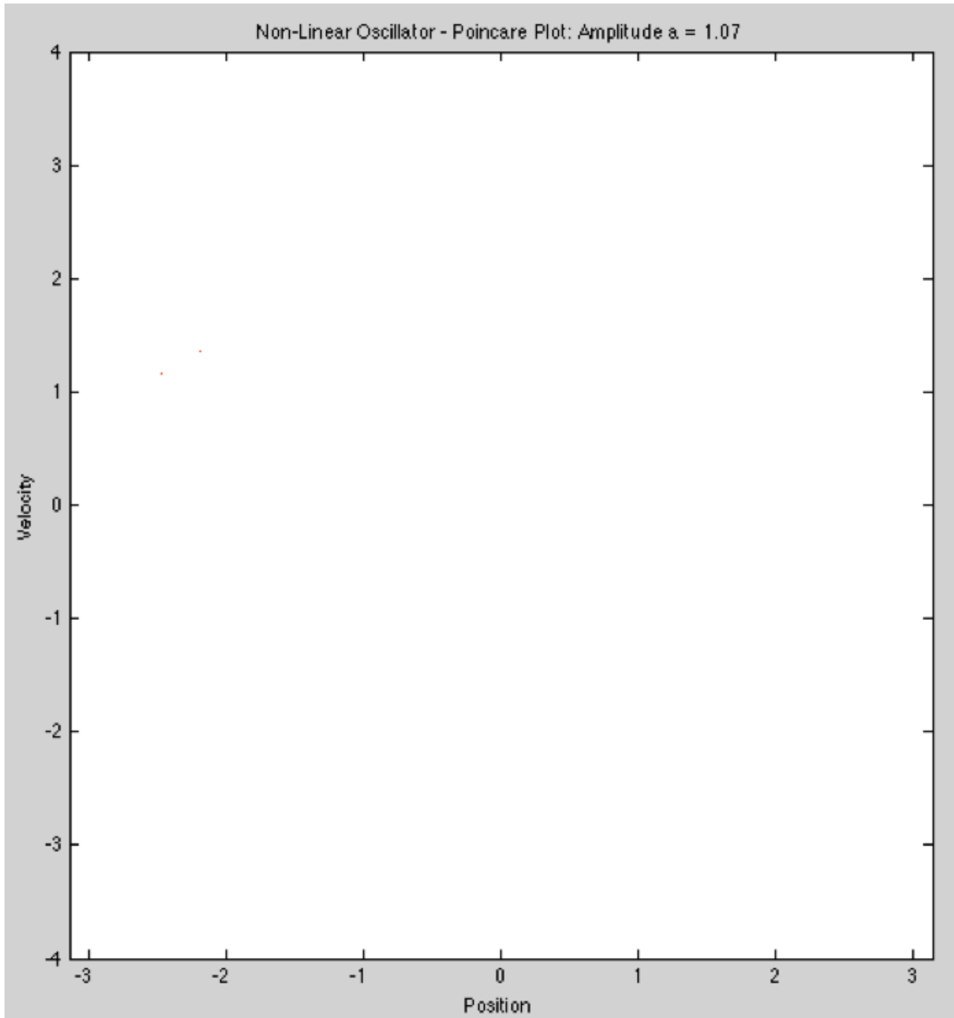


Figure 87: $a = 1.07$ — periodic doubling

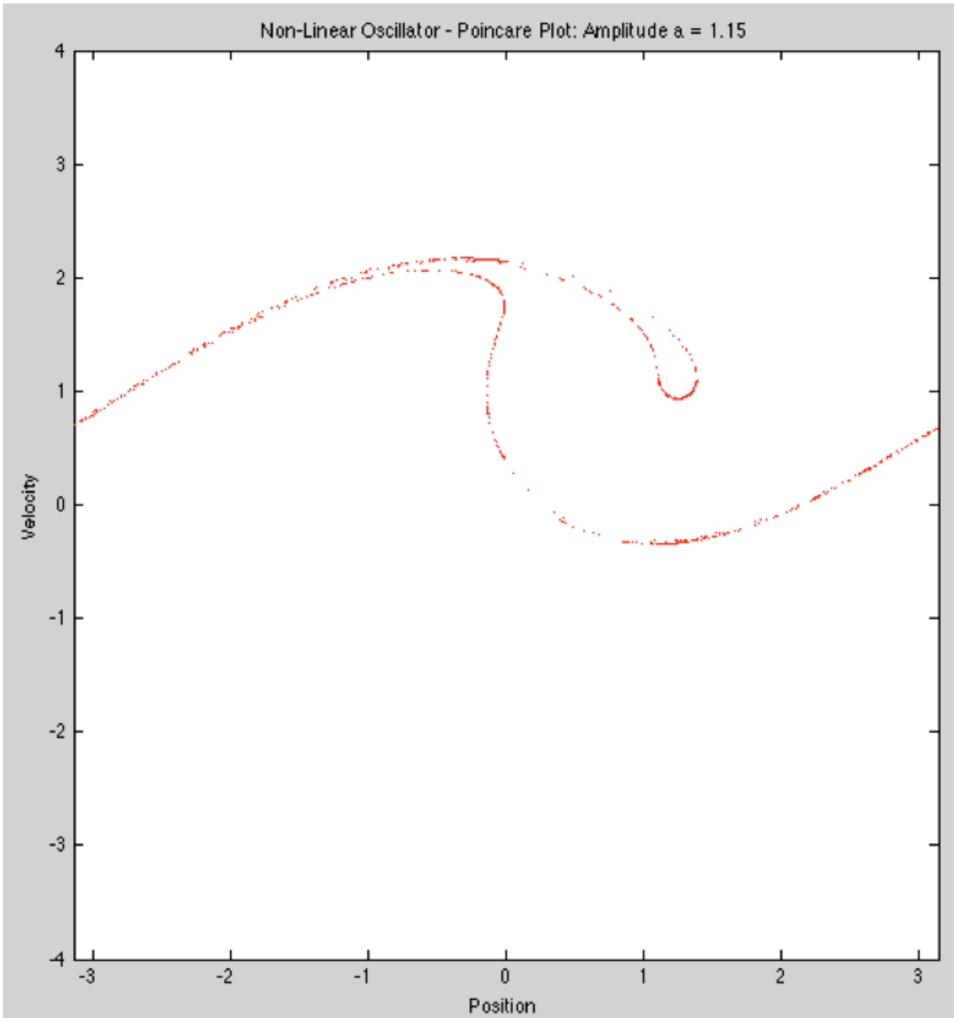


Figure 88: $a = 1.15$ — chaotic motion

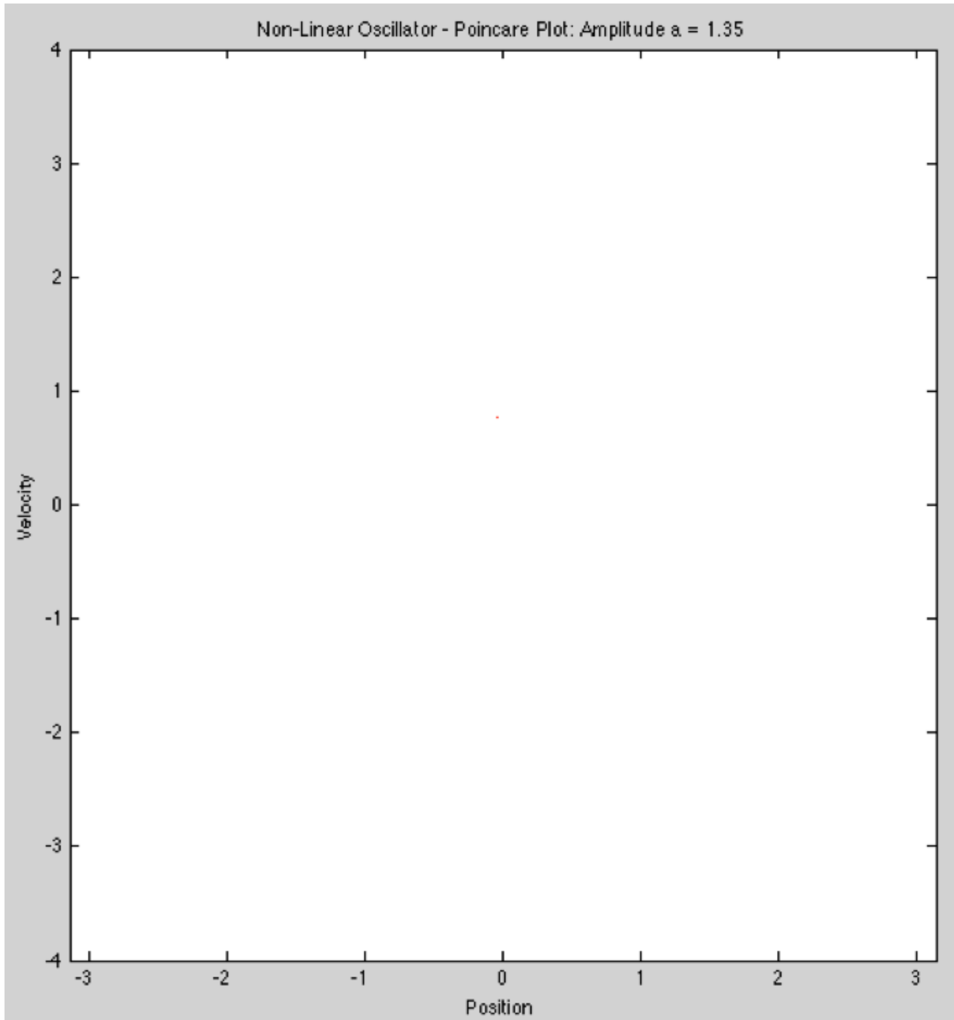


Figure 89: $a = 1.35$ — periodic motion

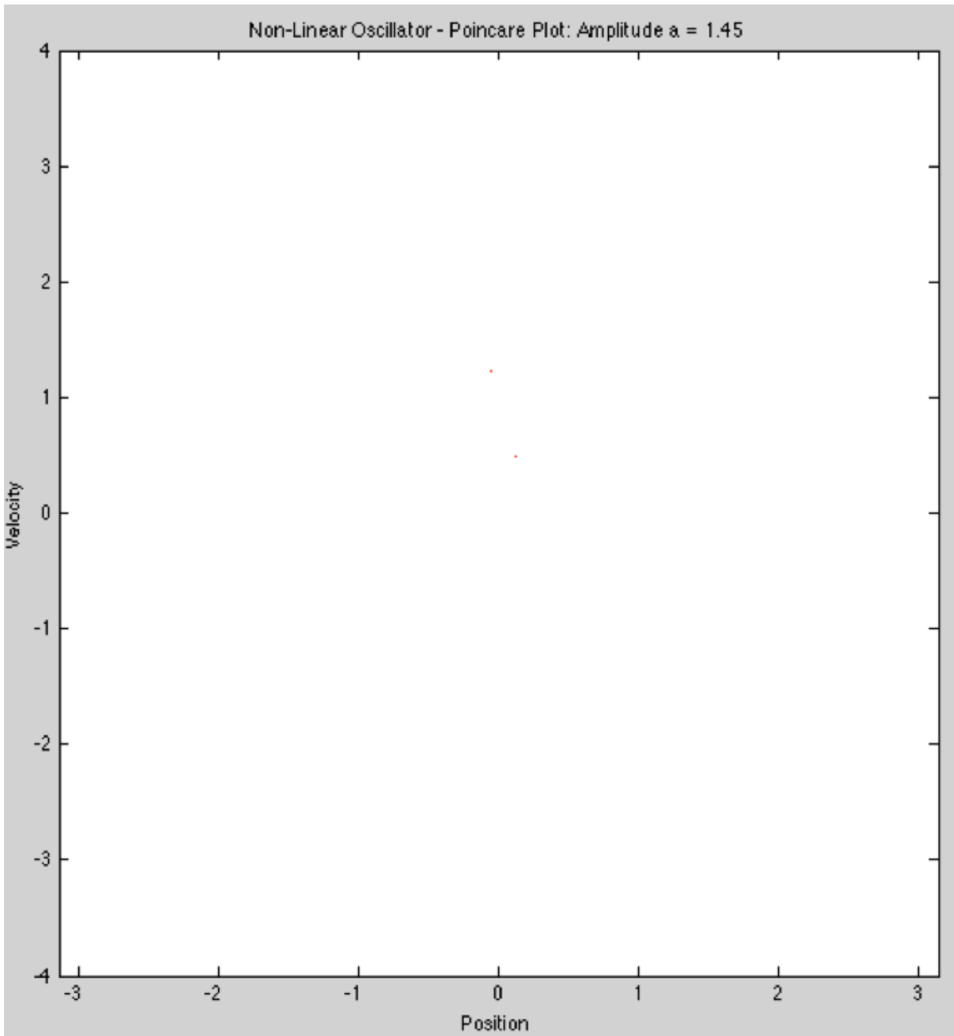


Figure 90: $a = 1.45$ — periodic doubling

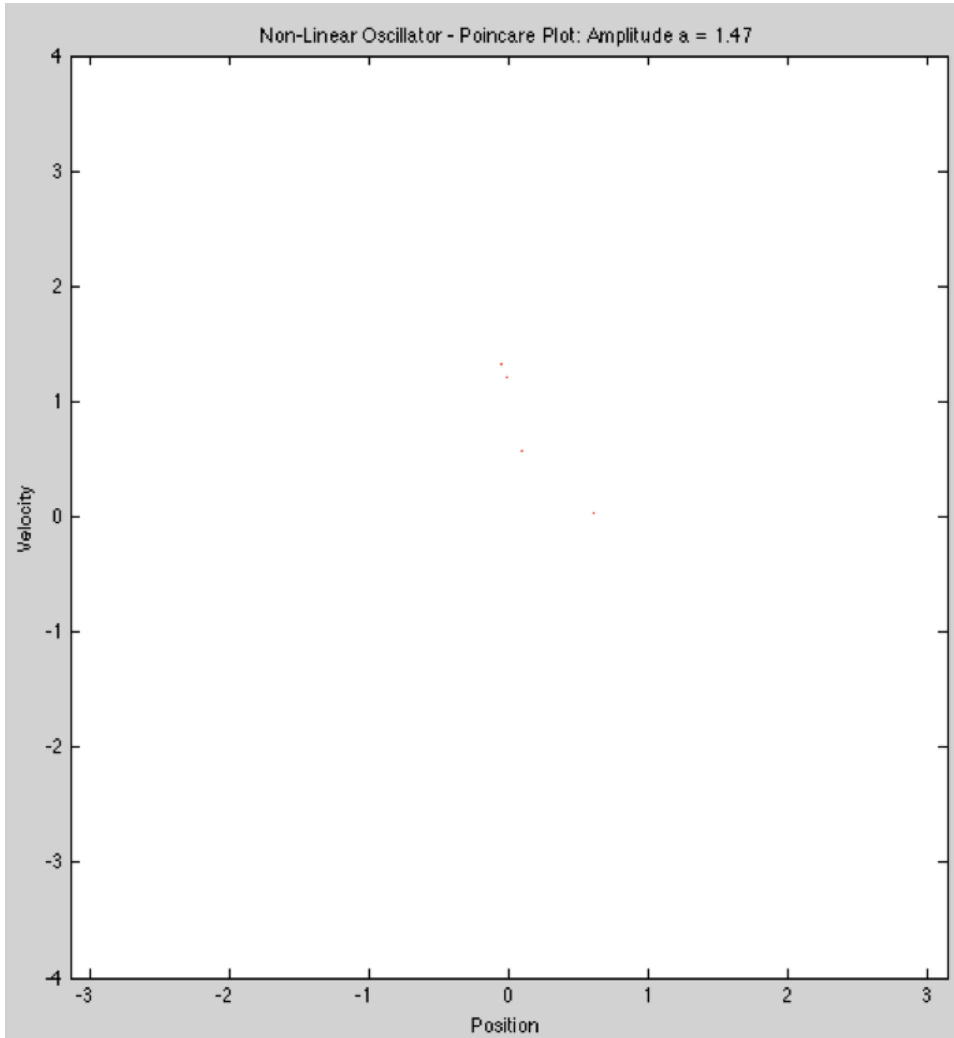


Figure 91: $a = 1.47$ — periodic doubling

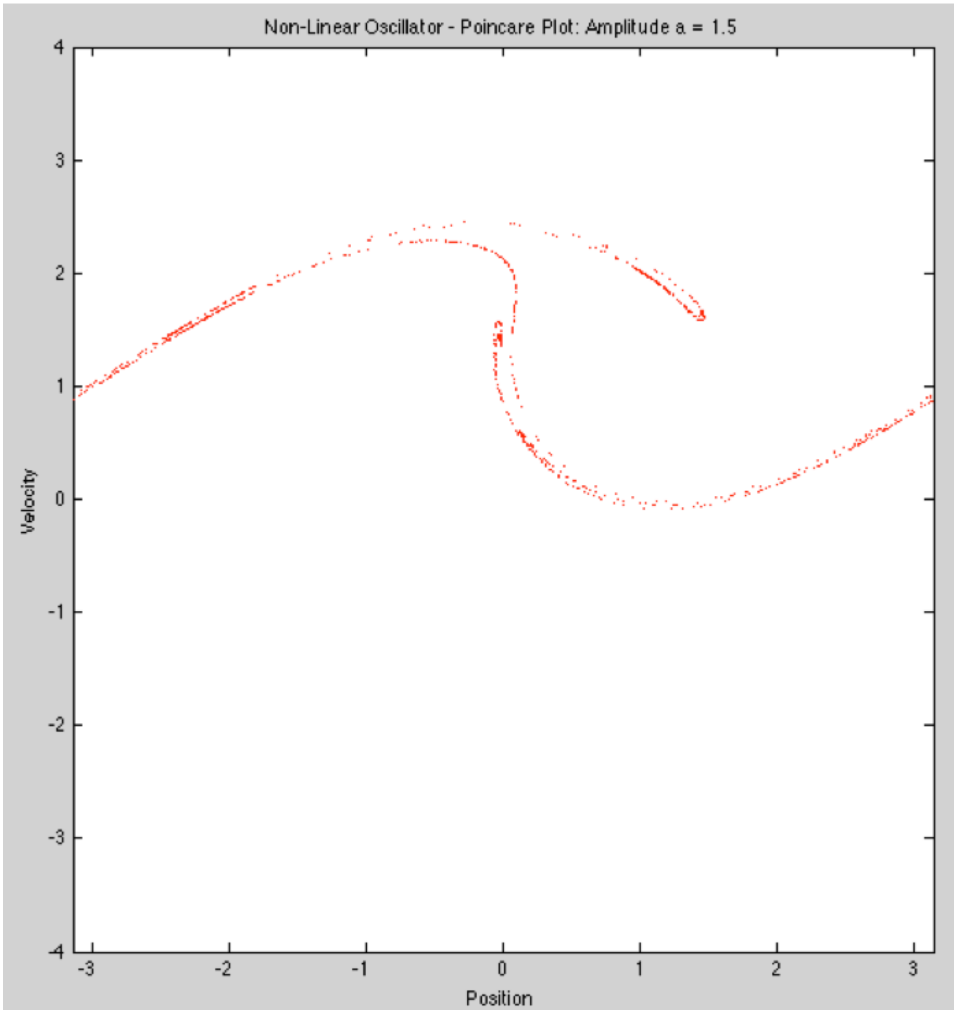


Figure 92: $a = 1.50$ — chaotic motion

The Poincare plot in this case is an *attractor* with an infinite number of points. It is a *fractal curve* (more about this later) with *non-integer dimension*. The steady state motion of the oscillator at this a and ω is not periodic at all; the motion is chaotic. An attractor of this sort is known as a *strange attractor*. Its infinity of points are arranged in a strange *self-similar (fractal)* manner.

Finally we can make a bifurcation plot for the driven oscillator, where we plot the strobe values (from the Poincare plot) versus the driving amplitude.

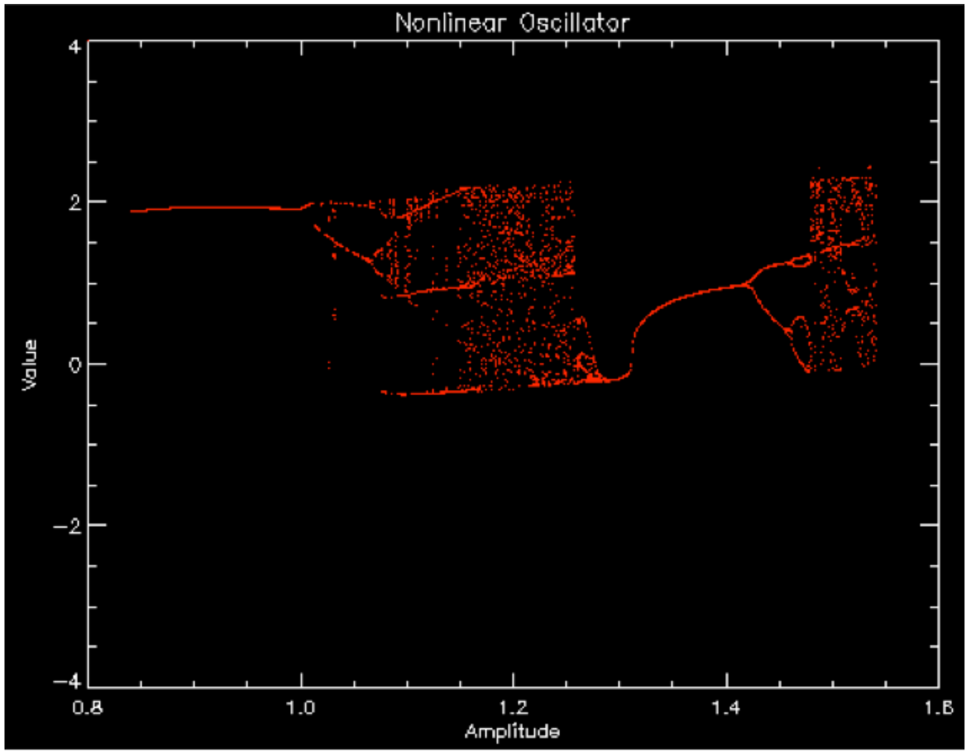


Figure 93: Bifurcation Plot

We see the same structure as in the logistic map bifurcation plot. The various periodic, period-doubling and chaotic regions are clear. The critical points are also clear.

Thus, two systems, which really do not resemble each other in any way except that they are both nonlinear systems, exhibits very similar behaviors.

A movie of the calculation is shown in *oscpoinbif.mpg*.

1.6.1. Zooming in

Let us zoom in on a strange attractor. We consider the Poincare plot for the driven oscillator when $a = 1.50$. It looks like the figures below. It is clear that there is complex structure in a strange attractor or fractal at all levels. The *dimension* of this strange attractor is $D = 1.4954$....(more later).

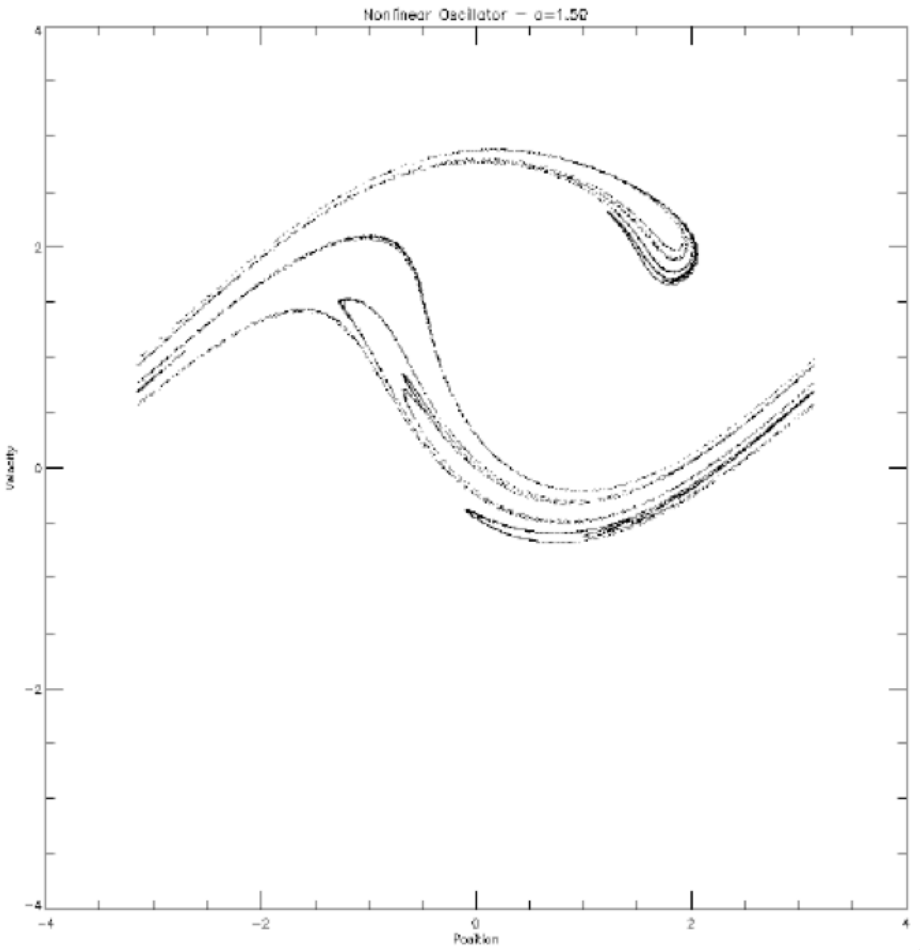


Figure 94: Original Structure

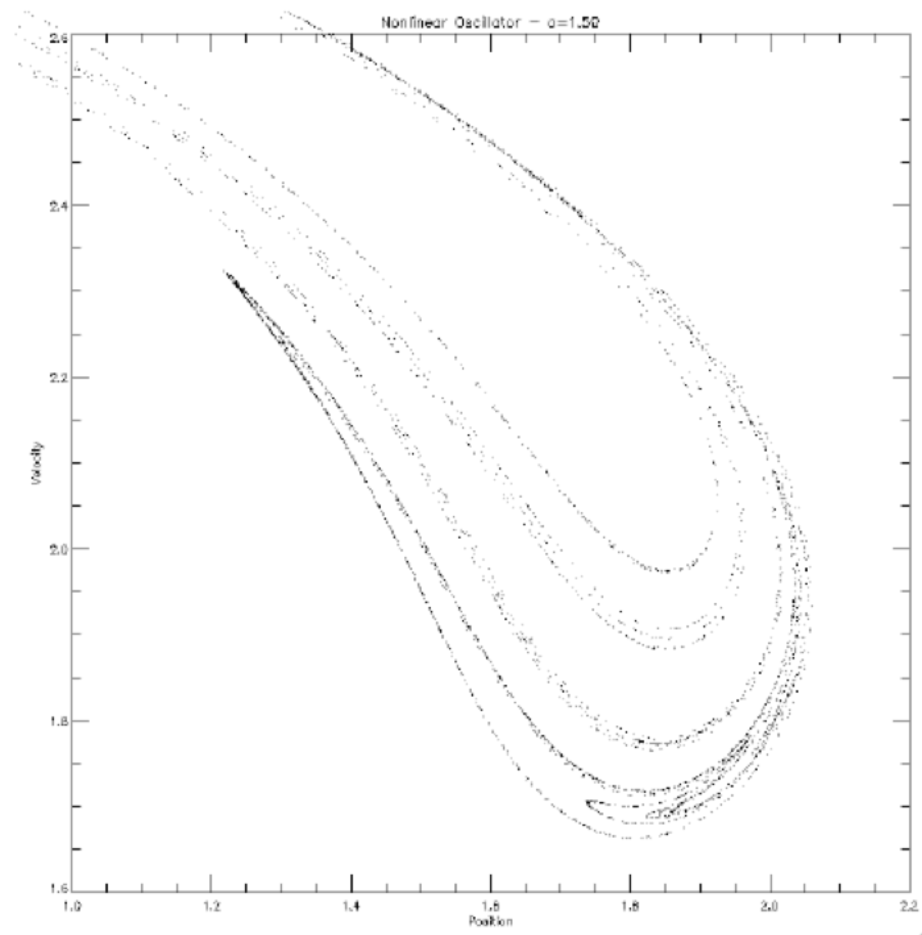


Figure 95: ZOOM #1 below shows the detail present in the attractor.

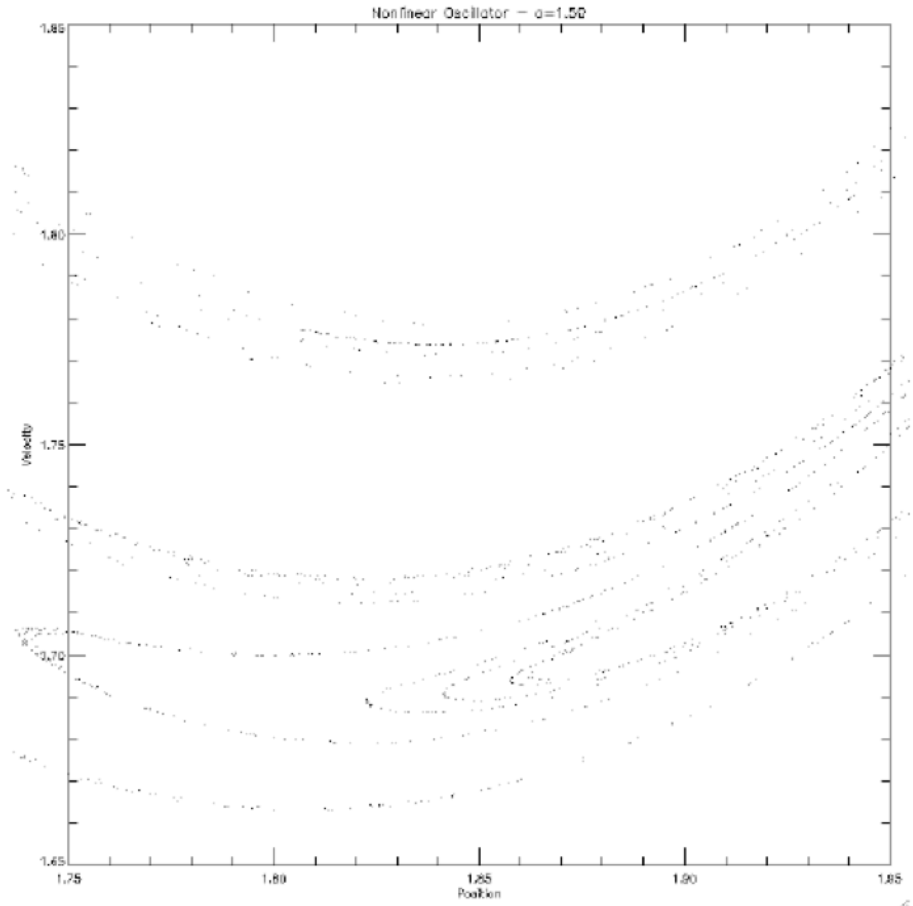


Figure 96: ZOOM #2 below shows further detail present in the attractor.

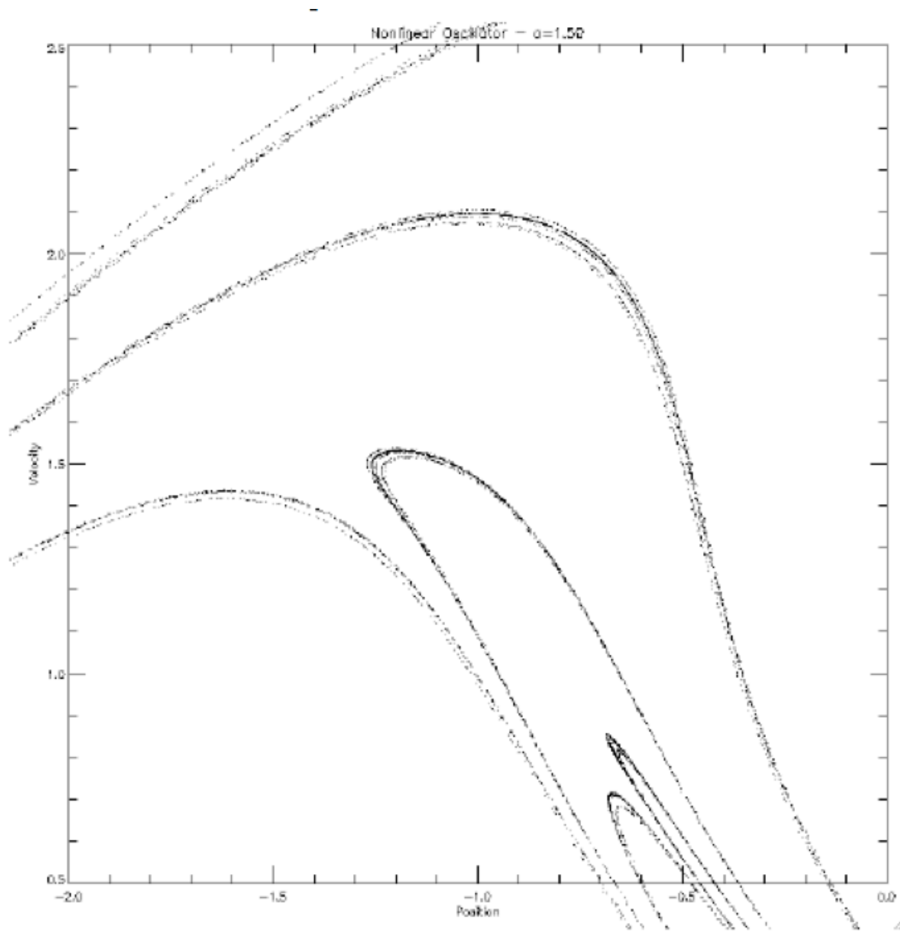


Figure 97: ZOOM #3 Pick another place.......

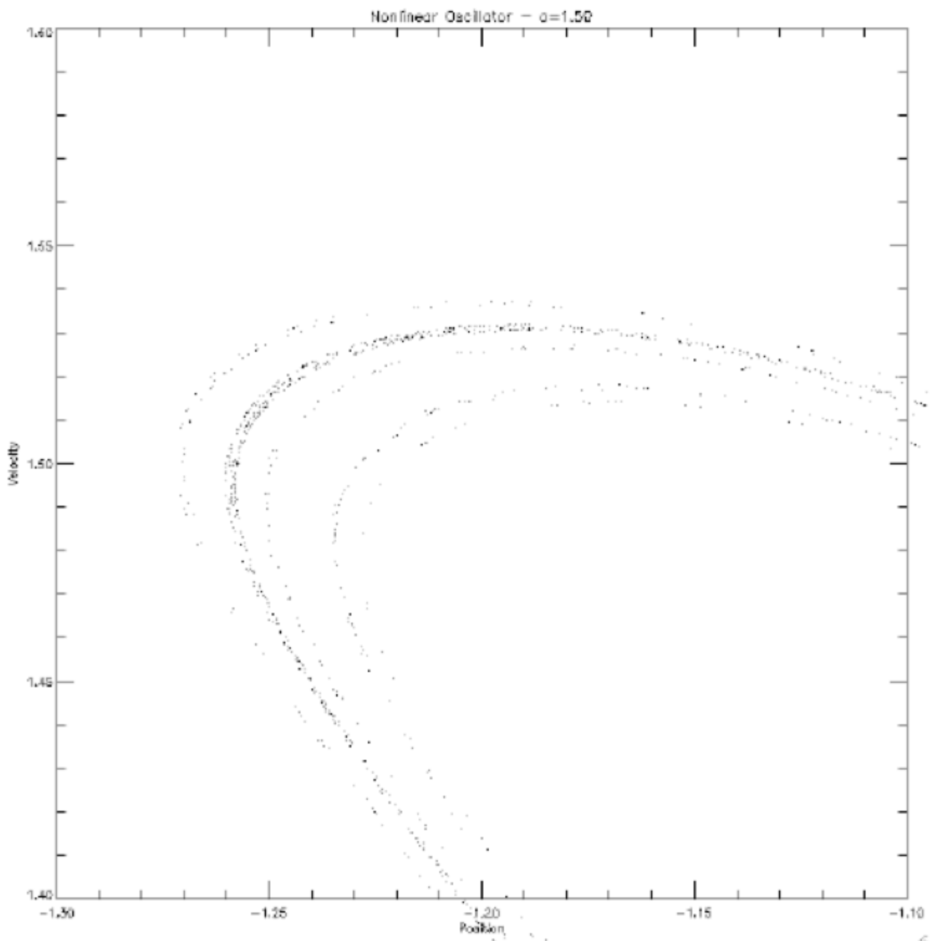


Figure 98: and zoom again .....

Here we are only limited by the resolution of the screen and the accuracy of the calculation.

The simulations of the chaotic attractor and its Poincare section reveal a hierarchical structure that is uncharacteristic of ordinary compact geometrical objects. The chaotic attractor as represented by the Poincare sections will be discussed as fractals - mathematical sets of noninteger dimension(strange attractors) - later in these notes.

These different magnifications clearly reveal the self-similar structure caused by the folding and stretching of the phase volume. the stretching and folding processes lead to a cascade of scales: the attractor consists of an infinite number of layers. the fine structure resembles the gross structure. This property is called *self-similarity*.

Properties

1. The trajectory of a strange attractor cannot repeat.
2. Nearby trajectories diverge exponentially.
3. The attractor is bounded in the phase space.
4. Even though it has an infinite number of different points,the trajectory does not fill the phase space.

A *strange attractor* is a *fractal*(more later), and its *fractal dimension*(more later) is less than the dimension of the phase space.

Self-Similarity

An important (defining) property of a fractal is self-similarity, which refers to an infinite nesting of structure on all size scales. Strict self-similarity refers to a characteristic of a form exhibited when a substructure resembles a superstructure in the same form.

1.7. Appendix

1.7.1. Stability of Fixed Points

An important question is whether a fixed point is stable, that is, they are attractors. To settle this question we start with a point near a fixed point and see if the result of repeated mapping(iteration) converges to the fixed point. We write x_n as

$$x_n = \bar{x} + \delta_n$$

where $\bar{x}$ is a fixed point and δ_n is (at least initially) small in magnitude. Substituting into the map equation and retaining only terms linear in δ_n (since δ_n

is small), we find that

$$\begin{aligned}
&x_{n+1} = \bar{x} + \delta_{n+1} = \lambda(\bar{x} + \delta_n)(1 - (\bar{x} + \delta_n)) \\
&\bar{x} + \delta_{n+1} = \lambda\bar{x}(1 - \bar{x}) + \lambda\delta_n - 2\lambda\bar{x}\delta_n - \lambda(\delta_n)^2 \\
&\delta_{n+1} = \lambda\delta_n - 2\lambda\bar{x}\delta_n \\
&\rightarrow \frac{\delta_{n+1}}{\delta_n} = \lambda(1 - 2\bar{x})
\end{aligned}$$

where we have used the definition of the fixed point $\bar{x} = \lambda\bar{x}(1-\bar{x}) \rightarrow \bar{x} = 1 - 1/\lambda$ and dropped the $\lambda(\delta_n)^2$ term. If $|\delta_{n+1}| < |\delta_n|$, then with repeated mappings the point $x_n = \bar{x} + \delta_n$ moves closer and closer to $\bar{x}$ with increasing n and so this fixed point is called *stable or attracting*. On the other hand, if $|\delta_{n+1}| > |\delta_n|$, then the point moves away from $\bar{x}$ and the fixed point is called *unstable or repelling*.

For the logistic map

$$x_{n+1} = \lambda x_n(1 - x_n) = F(x_n)$$

and we have

$$\left(\frac{dF}{dx}\right)_{x=\bar{x}} = \lambda(1 - 2\bar{x}) = \frac{\delta_{n+1}}{\delta_n}$$

so that the criterion for stability becomes

$$\left(\frac{dF}{dx}\right)_{x=\bar{x}} < 1$$

We note that this result is general for all maps of the form $x_{n+1} = F(x_n)$. Also note that

$$\left(\frac{dF}{dx}\right)_{x=\bar{x}}$$

is the slope of the mapping function in the neighborhood of the fixed point.

1.7.2. Generating the Lyapunov Exponent Curve

Let us look specifically at a one-dimensional map described by $x_{n+1} = f(x_n)$. We choose the initial difference between the two states to be $d_0 = \epsilon$, and after one iteration, the difference d_1 is

$$d_1 = f(x_0 + \epsilon) - f(x_0) \approx \epsilon \left.\frac{df}{dx}\right|_{x_0}$$

where the last result on the right occurs because ϵ is very small. After n iterations, the difference d_n between two initially nearby state is given by

$$d_n = f^n(x_0 + \epsilon) - f^n(x_0) = \epsilon e^{n\gamma}$$

where we have indicated the n^{th} iterate of the map $f(x)$ by the superscript n. If we divide by ϵ and take the logarithm of both sides, we have

$$\ln\left(\frac{f^n(x_0 + \epsilon) - f^n(x_0)}{\epsilon}\right) = \ln\left(e^{n\gamma}\right) = n\gamma$$

Now, because ϵ is very small, we have for γ

$$\gamma = \frac{1}{n} \ln \left(\frac{f^n(x_0 + \epsilon) - f^n(x_0)}{\epsilon} \right) = \frac{1}{n} \ln \left| \frac{df^n(x)}{dx} \right|_{x_0}$$

The value of $f^n(x_0)$ is obtained by iterating the function $f(x_0)$ n times.

$$f^n(x_0) = f(f(.....(f(x_0)).....))$$

We use the derivative chain rule of the n^{th} iterate to obtain

$$\left.\frac{df^n(x)}{dx}\right|_{x_0} = \left.\frac{df(x)}{dx}\right|_{x_{n-1}} \left.\frac{df(x_i)}{dx}\right|_{x_{n-2}} \cdots\cdots\cdots \left.\frac{df(x)}{dx}\right|_{x_0}$$

Taking the limit as $n \to \infty$ we finally obtain

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| \frac{df(x)}{dx} \right|$$

where the sum arises from the logarithm of a product. We plot (*lyapunov.m*)the Lyapunov exponent as a function of λ, the logistic parameter, below.

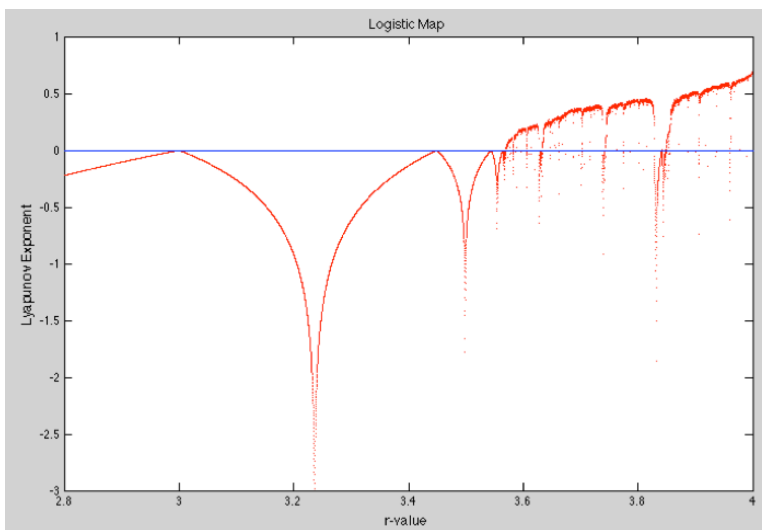


Figure 99: Lyapunov exponent as a function of λ

Clearly, the Lyapunov exponent is negative whenever the map is! stable and positive whenever the map is chaotic. The value of γ is zero when bifurcation occurs because $|df/dx| = 1$ and the solution becomes unstable. A superstable point occurs where $df/dx = 0$ and this implies that $\gamma = -\infty$. We can see clearly from the plot that when γ goes above zero, there are windows of stability where γ goes negative for a while and period orbits occur amid the chaotic behavior. The relatively wide window just above 3.8 is apparent.

Chapter 2

Fractals

2.1. Introduction

> "Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line." (Mandelbrot, 1983)

We now repeat some of the tentative ideas that we mentioned in Chapter 1 and expand them to a better understanding of the term fractal. Benoit Mandelbrot was intrigued by the *geometry of nature*. He was a jack-of-all-trades and was interested in all kinds of phenomena. One of the things he investigated was cotton prices.

Economists said that the cotton prices contained an orderly(regular) part and a random part. This implied that in the long term, prices are driven by *real* forces and that in the short term prices are random.

But the data did not agree!

There were too many large jumps, i.e., the ratio of small price changes to large prices changes was much smaller than expected. In addition there was a large tail (at the high end) in the distribution of price values.

If prices changed at random, then they would be distributed in what is called the *normal distribution* - the *Bell curve*. If this were true than the probability of price change differing from the mean change by three standard deviations would be extremely small - the cotton data did not fit a Bell curve!

Economics rules of the road, articles of faith and prejudices imply that small, transient changes have nothing in common with large, long term changes - that fast fluctuations are random - small scale, short term changes are just noise and thus they are unpredictable. Long term changes, however, are determined by deep macroeconomic forces.

This dichotomy, however, was false. Mandelbrot showed that all changes are bound together - that there is one pattern across all scales. He used computers to show that the cotton prices, which did not obey a normal distribution, produced symmetry from the point of view of scaling - a kind of scale independence. Each particular price change was random and unpredictable, but the sequence of changes was independent of scale.

Mandelbrot also studied transmission of data - in particular errors in transmission.. Mandelbrot found, contrary to intuition, that you can never find a time during during which errors were scattered continuously - within any burst of errors, no matter how short, there would always be periods that were completely error free.

Mandelbrot found geometric relations between bursts of errors and error-free transmission period. The relationship was the same no matter the time scale, namely, the Cantor set, which we will study later. It was a strange scattering of points arranged in clusters - infinitely many but yet infinitely sparse. He showed that transmission errors are a Cantor set arranged in time.

Mandelbrot also investigated the height of the Nile river. He classified variations into

1. Noah effect $\rightarrow$ discontinuous
2. Joseph effect $\rightarrow$ persistence

All other researchers had neglected changes of the "Noah" type. The data showed that persistence applied over centuries as well as over decades - another example of scale independence.

His research showed that trends in nature are real, but that they can vanish as quickly as they arise.

Discontinuities, bursts of noise, Cantor sets - where does all of this fit into the classical geometry of lines, planes and circles. Lines, planes and circles are a powerful abstraction of reality - this kind of abstraction will eliminate complexity and prevent any understanding from being developed. As the quote at the beginning of the chapter states:

clouds $\neq$ spheres

mountains $\neq$ cones

lightning does not travel in straight lines

Mandelbrot proposed that there is a geometry associated with complexity and it implies strange "shapes" that have a meaning in the real world. Euclidean objects are idealizations and cannot represent the essence of the geometry of an object in the real world.

Mandelbrot asked - how long is the coastline of Norway? He said, in a sense, that the coastline is infinitely long or that the answer depends on the size of your ruler. Common sense says that a smaller ruler sees finer detail and measures a longer length - but there must be some limiting value! If the coastline was a Euclidean abstraction, then there would be a limiting value equal to the length.

Thinking about these ideas will lead us to a discussion of *dimension*. We might ask - what is the dimension of a ball of twine?

1. from a great distance it is a point which says its dimension is 0

2. from closer, it fills spherical space which says that its dimension is 3

3. closer still and the twine is now seen and its dimension is now 1, although the dimension is tangled up making use of 3D space

This tells us, in some way, that the *effective* dimension is not always equal to 3.

What about "in between" - non-integer dimension - will nothing be sacred in this class!

As we will see later, non-integer dimensions are associated with coastlines. The coastline will be new kind of curve called a *fractal* and its length will be infinite! We will learn how to determine the dimension of these fractals - the fractal dimension will a way of measuring quantities such as roughness or irregularity in an object - a twisting coastline, despite not be able to measure its length, has a certain degree of roughness. As we will see the fractal dimension will remain constant over different scales.

A simple Euclidean 1D line fills no space at all. We will study a fractal line called the Koch curve that has infinite length, but fits into a finite area, completely filling it. It will be seen to be more than a line, but less than a plane and cannot be represented by Euclidean geometry. Its dimension will be greater than 1 but less than 2 (actually 1.2618).

Smooth curves are 1D objects whose length can be precisely defined between two points. A fractal curve has an infinite variety of detail at each point along the curve implying more and more detail as we zoom in, which implies that the length will be infinite.

Suddenly fractals were everywhere. They were found to be hidden in all kinds of data.

The property that one has infinite detail at each point and the detail looks like the original is called *self-similarity*. Self-similarity is symmetry across scale or recursion or patterns within patterns.

At first sight, the idea of consistency on new scales seems to provide less information. This seems so because it goes against the tradition of reductionism(building blocks). The power of self-similarity begins at much greater levels of complexity. It will be a matter of looking at the *whole, rather than the parts*.

We now turns to a detailed study of fractals. At that point we will have developed all the necessary tools to make and understand a science of self-organization.

2.2. The Nature and Properties of Fractals

Fractals are mathematical sets of point with fractional dimension.

Fractals are not simple curves in Euclidean space. A Euclidean object has integral dimension equal to the dimension of the space the object is being drawn in. In addition, if a Euclidean line connects 2 points in 3-dimensions and also stays within a finite volume, then the length of the line is finite.

A fractal is a line that can stay within a finite volume, but still have an infinite length. This implies that it has a complex structure at all levels of magnification. In comparison, any Euclidean line will eventually look like a straight line at some magnification level.

To describe this property of a fractal, we need to generalize the usual concept of dimension.

The dimensionality D of a space is usually defined as the number of coordinates needed to determine a unique point in that space. When defined in this way, the only allowed values for D are the non-negative integers $0, 1, 2, 3, ...$ There are several ways that the concept of dimension can be redefined so that it still takes on non-negative integer values when considering the systems described above, but can also take on non-negative real number values.

We will adopt a simplified version of the *Hausdorff dimension* called the *box counting* or *capacity* dimension. In the box-counting scheme, the dimension of an object is determined by asking how many *boxes* are needed to cover the object. Here the appropriate *boxes* for coverage are lines, squares, cubes, etc. The size of the boxes is repeatedly decreased and the dimension of the object is determined by how the number of covering boxes scales with the length of the side of the box.

fractal: An irregular shape with self-similarity. It has infinite detail, and cannot be differentiated. Wherever chaos, turbulence, and disorder are found, fractal geometry is at play.

fractal dimension: A measure of a geometric object that can take on fractional values. At first used as a synonym to Hausdorff dimension, fractal dimension is currently used as a more general term for a measure of how fast length, area, or volume increases with decrease in scale.

2.3. The Concept of Dimension

So far we have used the concept of *dimension* in two senses:

1. The *three dimensions* of Euclidean space ($D = 1, 2, 3$)
2. The *number of variables* in a dynamic system

Fractals, which are irregular geometric objects, require a *third meaning.*

In one dimension, we consider a line of length ℓ. We need

1 box(a line) of length ℓ

2 boxes of length $\ell/2$

....

2^m boxes of length $\ell/2^m$

....

If we define δ_m = length of the m^{th} box, then

$$\delta_m = \frac{\ell}{2^m}$$

Thus, the number of boxes $N(\delta_m)$ scales (grows) as

$$N(\delta_m) = \frac{\ell}{\delta_m}$$

In two dimensions, we consider a square of side ℓ. We need

1 box(a square now) of area ℓ^2

4 boxes of length $(\ell/2)^2$

....

2^m boxes of length $(\ell/2^m)^2$

....

Thus, the number of boxes $N(\delta_m)$ scales (grows) as

$$N(\delta_m) = \left(\frac{\ell}{\delta_m}\right)^2$$

Generalizing to D integer dimensions, we have

$$N(\delta_m) = \left(\frac{\ell}{\delta_m}\right)^D$$

Now to do some algebra (the details of the algebra are not important, but the final result is important). We have

$$\begin{aligned}
\log\left(N(\delta_m)\right) &= \log\left(\frac{\ell}{\delta_m}\right)^D = D\log\left(\frac{\ell}{\delta_m}\right) \\
&= D(\log(\ell) - \log\delta_m) \\
&\to D = \frac{\log\left(N(\delta_m)\right)}{\log(\ell) - \log(\delta_m)}
\end{aligned}$$

We then **define** the dimension D by

$$D = \lim_{m\to\infty} \frac{\log\left(N(\delta_m)\right)}{\log(\ell) + \log\left(\frac{1}{\delta_m}\right)}$$

As $m \to \infty$, $\log(\ell)$ becomes negligible compared to other terms and we have

$$D = \lim_{m\to\infty} \frac{\log\left(N(\delta_m)\right)}{\log\left(\frac{1}{\delta_m}\right)} = \lim_{\epsilon\to 0} \frac{\log\left(N(\epsilon)\right)}{\log(1/\epsilon)}$$

where $N(\epsilon)$ = the number p−dimensional cubes of side ϵ needed to completely cover the set.

This is the Hausdorff or fractal dimension D of a set of points in a p−dimensional space.

Wow! That was a very mathematical and tedious derivation. We can make clear the meaning of the result by some examples.

Examples:

(1) **A single point**: only one cube is required. This means that

$$N(\epsilon) = 1 \to \log N(\epsilon) = 0 \to D = 0$$

as expected.

(2) **A line of length ℓ**: The number of cubes of side ϵ required = number of line segments of length ϵ . Therefore

$$N(\epsilon) = \frac{\ell}{\epsilon}$$

and

$$\begin{aligned} D &= \lim_{\epsilon \to 0} \frac{\log\left(\frac{\ell}{\epsilon}\right)}{\log\left(\frac{1}{\epsilon}\right)} \\ &= \lim_{\epsilon \to 0} \frac{\log\left(\frac{1}{\epsilon}\right) + \log(\ell)}{\log\left(\frac{1}{\epsilon}\right)} \\ &= \lim_{\epsilon \to 0} \left(1 + \frac{\log(\ell)}{\log\left(\frac{1}{\epsilon}\right)}\right) = 1 \end{aligned}$$

as expected.

(3) **a square of area ℓ^2**: The number of cubes of side ϵ required = number of squares of side ϵ. Therefore

$$N(\epsilon) = \frac{\ell^2}{\epsilon^2}$$

and

$$\begin{aligned} D &= \lim_{\epsilon \to 0} \frac{\log\left(\frac{\ell^2}{\epsilon^2}\right)}{\log\left(\frac{1}{\epsilon}\right)} \\ &= \lim_{\epsilon \to 0} \frac{2\log\left(\frac{1}{\epsilon}\right) + \log(\ell^2)}{\log\left(\frac{1}{\epsilon}\right)} \\ &= \lim_{\epsilon \to 0} \left(2 + 2\frac{\log(\ell)}{\log\left(\frac{1}{\epsilon}\right)}\right) = 2 \end{aligned}$$

as expected.

So this definition of dimension works for Euclidean objects and clearly makes sense.

An alternative way to think about it. It is always good to look at a new concept in several ways!

2.3.1. The Hausdorff Dimension

If we take an object residing in Euclidean dimension D and reduce its linear size by $1/r$ in each spatial direction, its measure (length, area, or volume) would increase to $N = r^D$ times the original. This is pictured in the figure below.

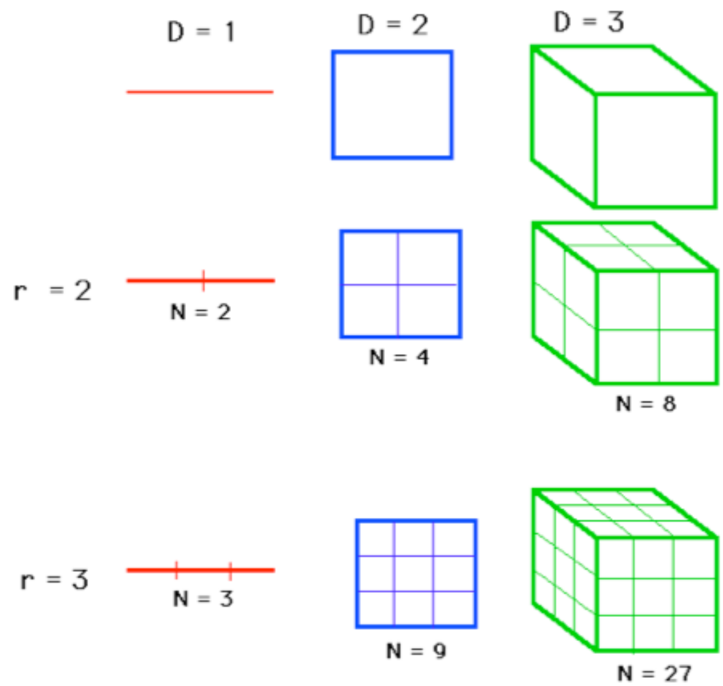


Figure 1: size reductions

We consider $N = r^D$, take the log of both sides and get

$$\log(N) = D \log(r)$$

If we solve for D we get

$$D = \frac{\log(N)}{\log(r)}$$

The point of this exercise: Examined this way, D need not be an integer, as it is in Euclidean geometry. It could be a fraction, as it is in fractal geometry. This generalized treatment of dimension is named after the German mathematician, Felix Hausdorff. It has proved useful for describing natural objects and for evaluating trajectories of dynamic systems.

2.3.2. The length of a coastline

Mandelbrot began his treatise on fractal geometry by considering the question: **How long is the coast of Britain?**

The coastline is irregular, so a measure with a straight ruler, as in the next figure, provides an estimate. The estimated length L, equals the length of the ruler, s, multiplied by the N, the number of such rulers needed to cover the measured object. In the figure below we measure a part of the coastline twice, the ruler(unit of measurement) on the right is half that used on the left.

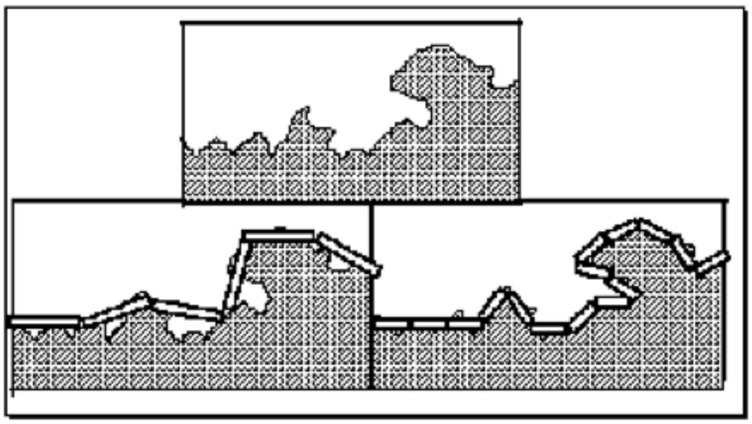


Figure 2: measuring coastline length using rulers of varying lengths

But the estimate on the right is longer. If the scale on the left is one, we have six units, but halving the unit gives us 15 rulers ($L = 7.5$), not 12 ($L = 6$). If we halved the scale again, we would get a similar result, a longer estimate of L. In general, as the ruler gets diminishingly small, the length gets infinitely large.

The concept of *length*, *begins to make little sense*.

2.3.3. The Cantor Set

Consider the figure below.

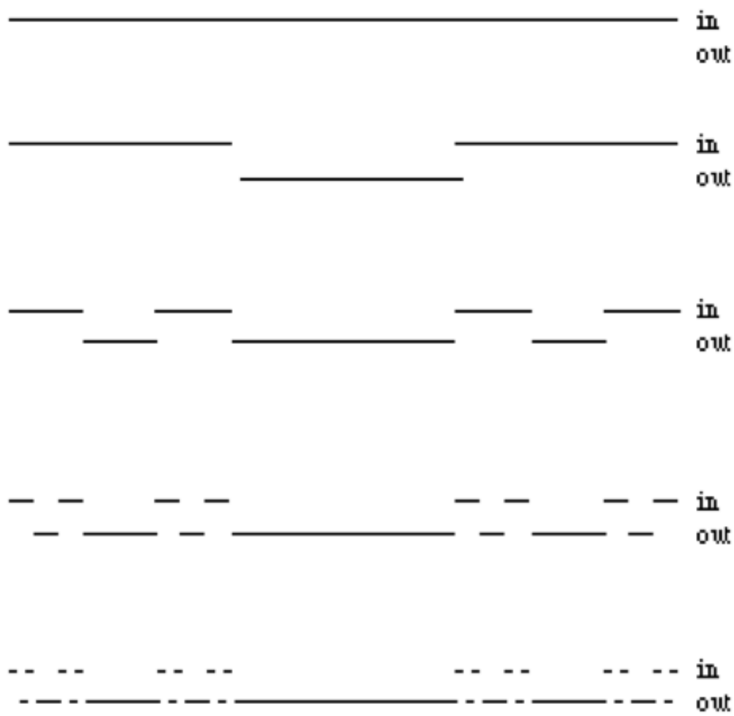


Figure 3: construction of the Cantor set

In the above diagram, points in the interval $(0,1)$ are tracked for 4 iterations. On the 1^{st} iteration, the middle third of the points leave the interval $(0,1)$. The middle thirds of the two *in* segments leave the interval and so on. The bottom row(after infinite iterations) is the Cantor set.

The Cantor set has some stunning properties. For one thing, in the infinite limit it has zero measure (it is the empty line). But the set is far from empty, in fact, it has an uncountable number of points - called "Cantor dust".

To cover the set at step K we need

$$N(K) = 2^K \text{ cubes of side } \epsilon = \left(\frac{1}{3}\right)^K$$

Therefore,

$$D = \lim_{K\to\infty} \frac{\log(2^K)}{\log\left(\left(\frac{1}{3}\right)^K\right)} = \lim_{K\to\infty} \frac{K\log(2)}{K\log(3)} = \frac{\log(2)}{\log(3)} = 0.631$$

The dimension of the Cantor set is *not an integer*!

At first glance one may reasonably wonder if there is anything left. After all, the lengths of the intervals we removed all add up to 1, exactly the length of the segment we started with:

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \cdots = \frac{1}{3}\sum_{n=0}^{\infty}\frac{2^n}{3^n} = \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = \frac{1}{3} \times 3 = 1$$

Yet, remarkably, we can show that there are just as many "points" remaining as there were before we began! This startling fact is only one of the many surprising properties exhibited by the Cantor set.

Before we begin to expose these properties, it is important to be quite precise about this construction. Let us agree that the segments we remove at each stage of the construction are open intervals. That is, in the first step we remove all of the points between 1/3 and 2/3, but leave the end points, and similarly for each successive stage. A little reflection will convince you that these endpoints we leave behind never get removed, since at each stage we are only removing parts that lie strictly between the endpoints left behind at the previous stage. Thus we see that our Cantor set cannot be empty, since it contains 0, 1, 1/3, 2/3, 1/9, 2/9, 7/9, 8/9, 1/27, and so on.

But in fact there is much more that remains. To see this, recall that we may choose any number base to represent real numbers. That is, there is nothing necessary or even special about our common use of base ten; we can just as easily represent our numbers using base two, or base three, or any other base.

$$\frac{1}{3} = 0.33333.... \quad \text{(base 10)}$$

$$\frac{1}{3} = 0.02222.... = 0.1 \quad \text{(base 3)}$$

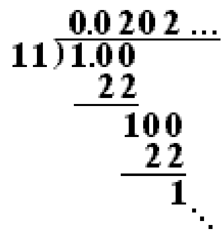
$$\frac{1}{3} = 0.010101010.... \quad \text{(base 2)}$$

When a number is written in base two it is said to be in binary notation, and when it is written in base three it is said to be in ternary notation. Let's focus on the ternary representations of the decimals between 0 and 1. Since, in base three, 1/3 is equal to 0.1, and 2/3 is equal to 0.2, we see that in the first stage of the construction (when we removed the middle third of the unit interval) we actually removed all of the real numbers whose ternary decimal representation have a 1 in the first decimal place, except for 0.1 itself. (Also, 0.1 is the same as 0.0222... in base three, so if we choose this representation we are removing *all* the ternary(composed of 3 parts) decimals with 1 in the first decimal place.) In the same way, the second stage of the construction removes all those ternary decimals that have a 1 in the second decimal place. The third stage removes those with a 1 in the third decimal place, and so on. (Convince yourself that this is so. Begin by noticing that 1/9 is equal to 0.01 and 2/9 is equal to 0.02

in base three.)

Thus, after everything has been removed, the numbers that are left, that is, the numbers making up the Cantor set, are precisely those whose ternary decimal representations consist entirely of 0's and 2's. What numbers does this include, besides the ones already noted above? How many are there?

Lots. Consider $1/4$. This is not one of the endpoints (those all have powers of three in the denominator), but it is not hard to show that $1/4$ is in the Cantor set. Begin by writing 4 in ternary notation (as $11 =$ one "1" plus one "3"), and then use long division to get its ternary decimal representation:



Since the decimal expansion of $1/4$ consists entirely of 0's and 2's, it was never removed during the construction of the Cantor set, so it's still there ... somewhere!

Asking how many numbers are left, as you can easily see, is to ask how many numbers can be represented in ternary notation with no "1" in any decimal place. But this must be as many as there are real numbers in the unit interval - for consider: we may represent all the real numbers between 0 and 1 in binary, and this is just every possible decimal with a 1 or a 0 in each decimal place. And there can be no more and no less of these than there are ternary decimals with a 0 or a 2 in each decimal place. They correspond in an obvious way.

The conclusion is inescapable: once we remove all those intervals, the number of points remaining is no less than the number we started with.

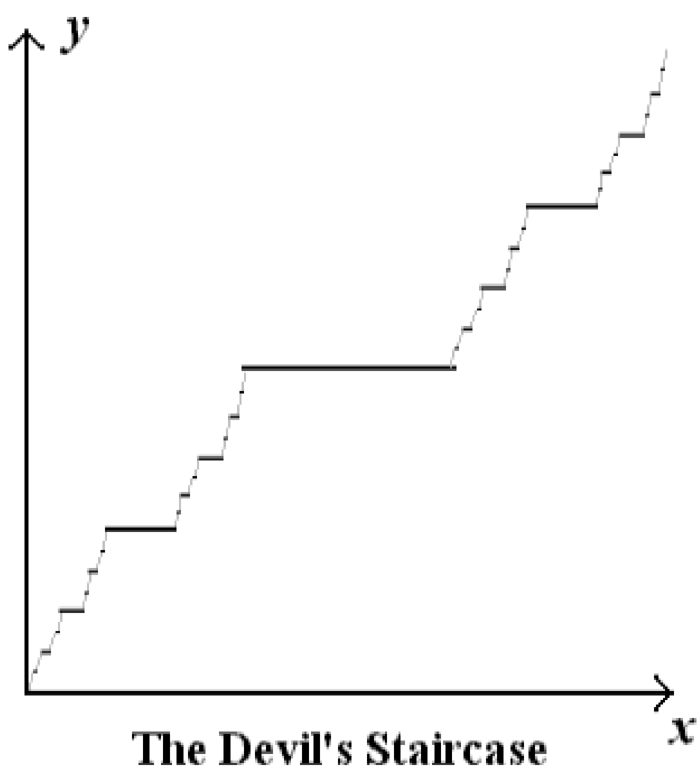
Let us examine that "correspondence" more closely. The idea is evident: for every number whose ternary decimal expansion consists entirely of 0's and 2's, match it with the corresponding number whose binary decimal expansion has 0's in the same place, and 1's wherever the ternary number had 2's. Thus, $1/4$ in ternary gets matched with $1/3$ in binary:

$$\frac{1}{4} = 0.02020202.... \quad \text{(ternary)}$$

$$\frac{1}{3} = 0.010101010.... \quad \text{(binary}$$

This is evidently a function that is surjective (In mathematics, a function f from a set X to a set Y is surjective (or onto) if every element y in Y has a corresponding element x in X such that f(x) = y). Moreover, it is continuous. (Elements that are "close" in the domain are mapped to elements that are "close" in the range.)

We can extend this to a function, called the Cantor function, from the entire unit interval onto itself, by simply agreeing to let its value on the missing intervals be the constant values which equal the values of the original function on the endpoints of those intervals. For example, the Cantor function will map each point in the first middle-third interval (1/3, 2/3) to 1/2, the value of the original function on the points 1/3 and 2/3. (Recall that 1/3 has ternary representation 0.0222... and 2/3 has ternary representation 0.2, which map to 0.0111... and 0.1 respectively, and these both represent the number 1/2 in binary.) If we were to attempt to graph this function, it would look like this:



The flat parts are the images of all of the "middle thirds," and these are all connected by the images of the Cantor set itself. This construction has been called the "Devil's Staircase" since it has infinitely many "steps."

A few more analytical tidbits: Since each interval removed was open, and there were only countably many of them, their union is also open. Thus, the Cantor set (which is the complement of this union) is closed. That is, it contains all of its accumulation points. Moreover, every point of the Cantor set is an accumulation point, since within any neighborhood of a number whose ternary expansion consists entirely of 0's and 2's one may find other such numbers. Consequently, the Cantor set is a perfect set in the topologist's sense. Finally, since any open neighborhood of any point of the Cantor set contains an open set which is disjoint from the Cantor set, we have that the Cantor set is nowhere dense. Altogether a remarkable set. (cantor.m)

The Cantor set is an instructively simple example of a fractal, demonstrating that our geometrical intuitions about space (even such simple spaces as the unit interval) can fail to capture much of the deep structure inherent in those very intuitions.

You might think that such an exotic object cannot have anything to do with "real" systems - but your are wrong as we will see.

2.3.4. The Koch Snowflake

The Koch curve is constructed by recursion as exhibited in the figures below. At each step the middle-third of each segment is replaced with a "V" shaped bulge. This curve turns out to have infinite length, while enclosing (together with its natural base: the original line segment) a finite area. *Demo the Koch Snowflake.*

We begin with a straight line of length 1, called the *initiator*. We then remove the middle third of the line, and replace it with two lines that each have the same length (1/3) as the remaining lines on each side. This new form is called the *generator*, because it specifies a rule that is used to generate a new form.

The rule says to take each line and replace it with four lines, each one-third the length of the original.

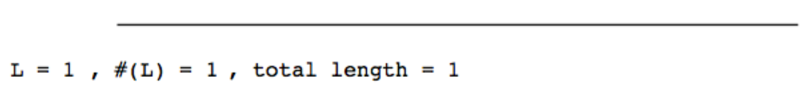


Figure 4: Original Line

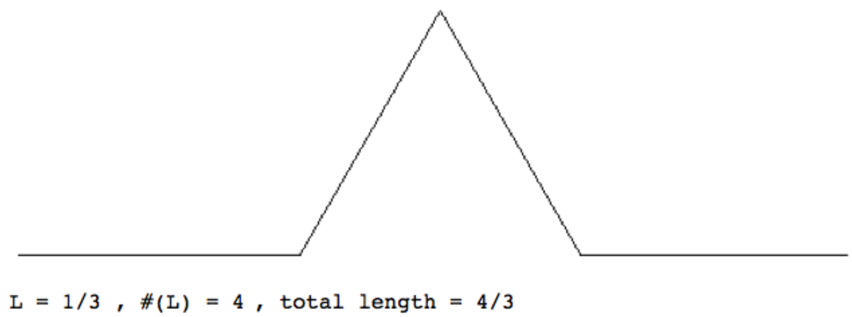


Figure 5: Step #1

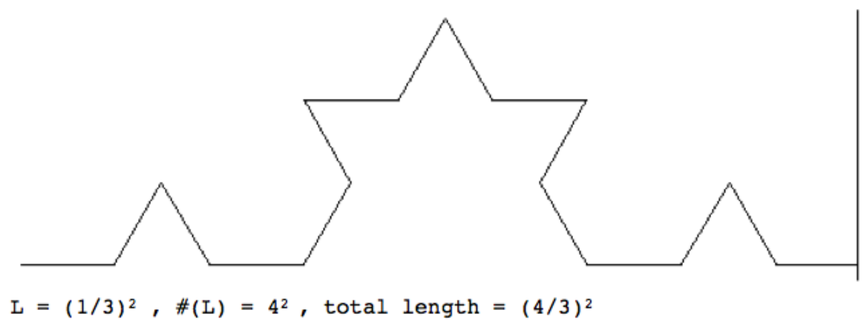


Figure 6: Step #2

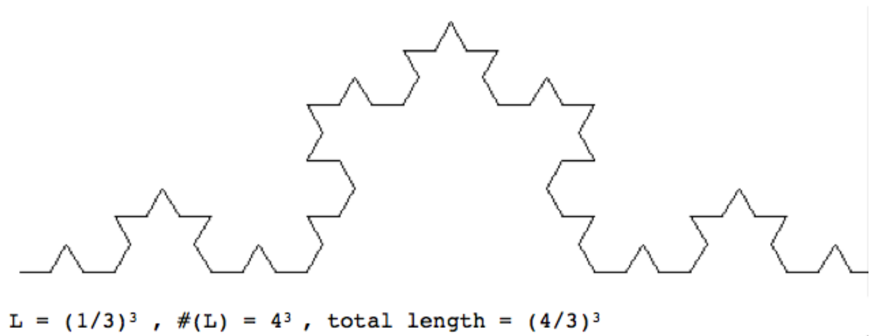


Figure 7: Step #3

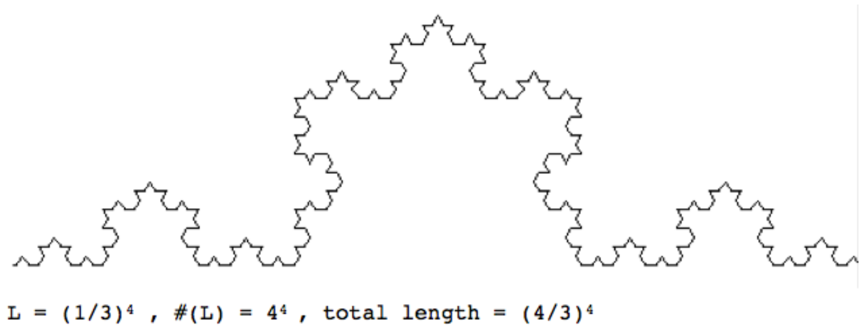


Figure 8: Step #4

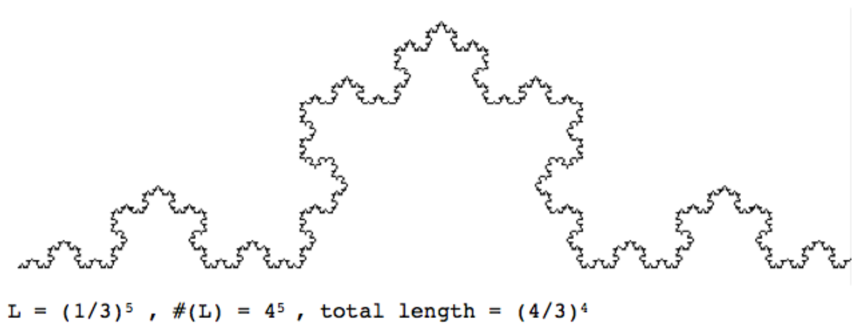


Figure 9: Step #5

The dimension of the Koch snowflake is given by

$$D = \lim_{n\to\infty} \frac{\log(4^n)}{\log\left(\left(\frac{1}{3}\right)^n\right)} = \lim_{n\to\infty} \frac{n\log(4)}{n\log(3)} = \frac{\log(4)}{\log(3)} = 1.26....$$

We get a dimension between 1 and 2. Clearly, the final length of the fractal line

$$\lim_{n\to\infty} \left(\frac{4}{3}\right)^n \to \infty$$

is infinite even though it stays within a finite area of the 2-dimensional Euclidean plane.

2.3.5. Sierpinski Triangle

The figures below illustrate the construction the Sierpinski triangle by iteration:

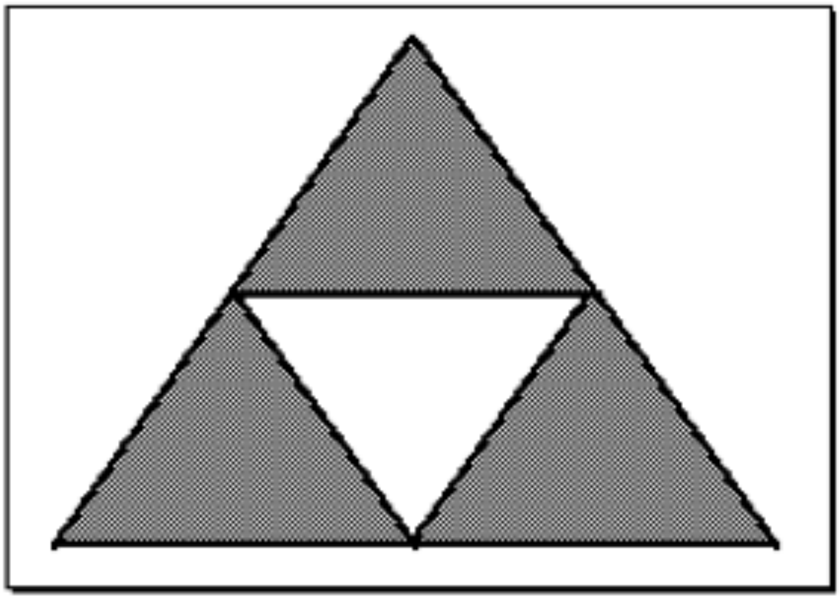


Figure 10: Starting Configuration

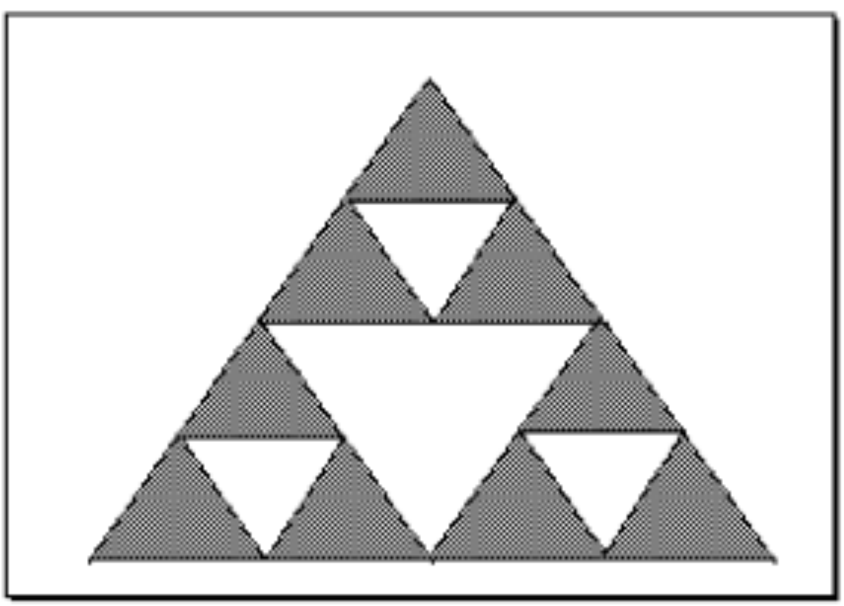


Figure 11: Step #1

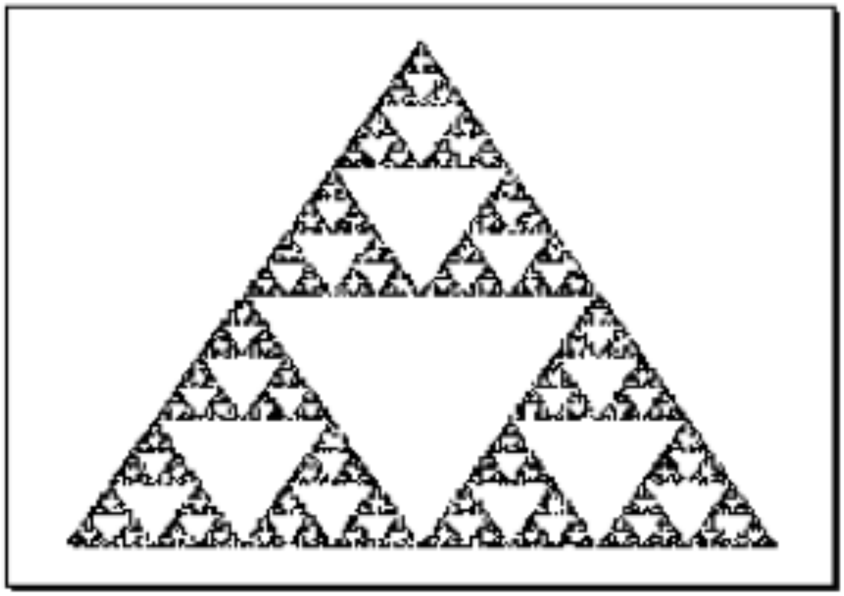


Figure 12: Several steps later

Calculating the dimension.

$$D = \frac{\log(N)}{\log(r)} = \frac{\log(3)}{\log(2)} = 1.585....$$

Again we get a number between 1 and 2.

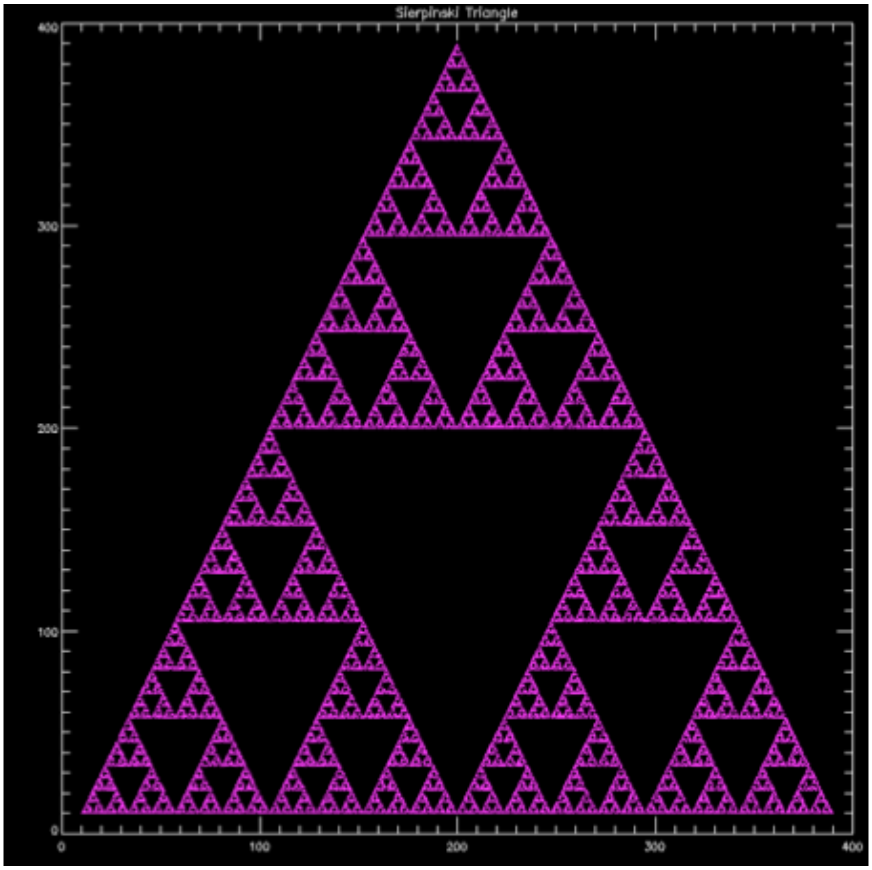


Figure 13: Many iterations

2.3.6. IFS Fractals

One method of generating fractals is to choose dilation(rescaling) and translation transformations at random. Say, for example, that we adopt the iterative dilation/translation map given by:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

$$x_{n+1} = ax_n + by_n + e$$

$$y_{n+1} = cx_n + dy_n + f$$

where the $[a, b, c, d]$ dilation matrix and the $[e, f]$ translation vector are chosen randomly. A fern can be generated using appropriate matrix choices:
run fern.m.

a	b	c	d	e	f	prob.
0.0	0.0	0.0	0.16	0.0	0.0	0.01
0.85	0.04	-0.04	0.85	0.0	1.6	0.85
0.20	-0.26	0.23	0.22	0.0	1.6	0.07
-0.15	0.28	0.26	0.24	0.0	0.44	0.07

Figure 14: $[a, b, c, d, e, f]$ values with probability of choice

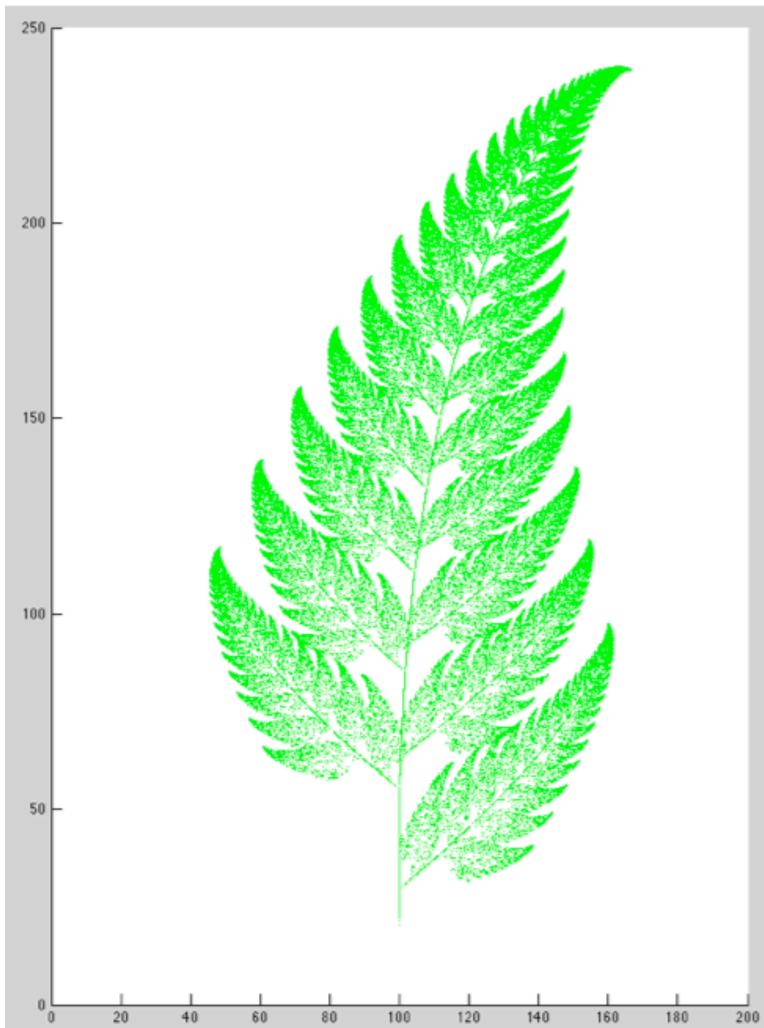


Figure 15: A Fractal Fern

Demo Fern I and Fern II, sierpinski.m, barnsley.m, Sierpinski Arrowhead, Beech-Branch, Bush, ElmBranch, Plant, Tree 1, Tree 2, Tree 3 and Koch Snowflake. L-systems.

2.4. Fractal Dimension Program

We now illustrate a program that carries out the box dimension calculations on complex fractral structures. The ideas of covering a fractal with boxes is shown below.

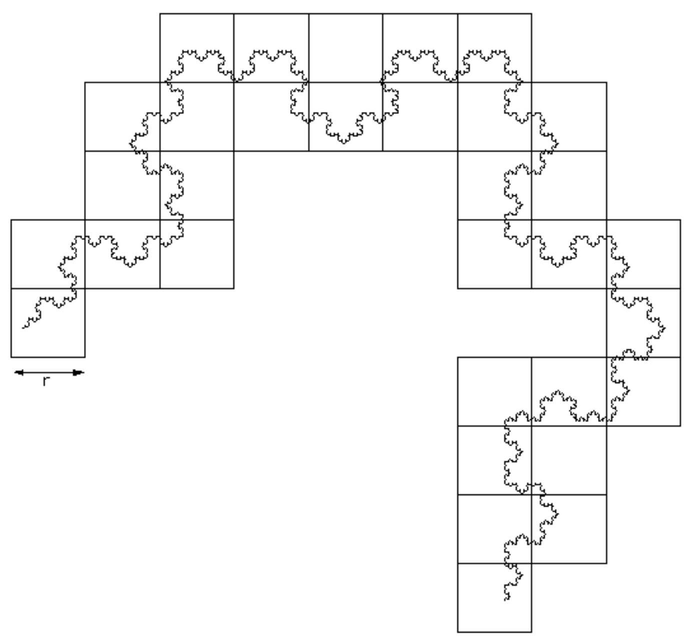


Figure 16: Koch curve covered by boxes of size r

The program simply chooses a "box" size, covers the fractal and counts the number of boxes. It then reduces the box size and repeats until the box is small. At that point it can recognize the limit of plot $\log N(r)$ versus $\log r$ and determine the slope which corresponds to the dimension.

If we apply the program to the Koch snowflake which look like

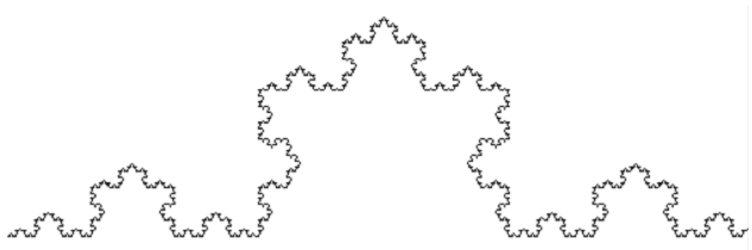


Figure 17: Koch snowflake curve

we get the result

```
reality:fdc_mac boccio$ ./fdc  -i snowflake.tga -b1 200 -b2 8 -n 1 -bf 1.2 -p
FDC - Fractal Dimension Calculator
Initialising GLUT
Creating windows
Creating menus (Right mouse button)
Reading file "snowflake.tga"
   Image dimensions: 1188 x 695 x 24
Creating screen version
   Full screen size: 2560 x 1440
   Display size:     1188 x 695
   Scale factor:     1
Processing starting
   Box size:  200 [  1], Total count = 15
   Box size:  166 [  2], Total count = 19
   Box size:  138 [  3], Total count = 22
   Box size:  114 [  4], Total count = 30
   Box size:   95 [  5], Total count = 31
   Box size:   79 [  6], Total count = 45
   Box size:   65 [  7], Total count = 58
   Box size:   54 [  8], Total count = 70
   Box size:   45 [  9], Total count = 82
   Box size:   37 [ 10], Total count = 109
   Box size:   30 [ 11], Total count = 115
   Box size:   24 [ 12], Total count = 183
   Box size:   20 [ 13], Total count = 225
   Box size:   16 [ 14], Total count = 308
   Box size:   13 [ 15], Total count = 424
   Box size:   10 [ 16], Total count = 547
   Box size:    8 [ 17], Total count = 747
Processing finished
Estimated fractal dimension:  1.211
```

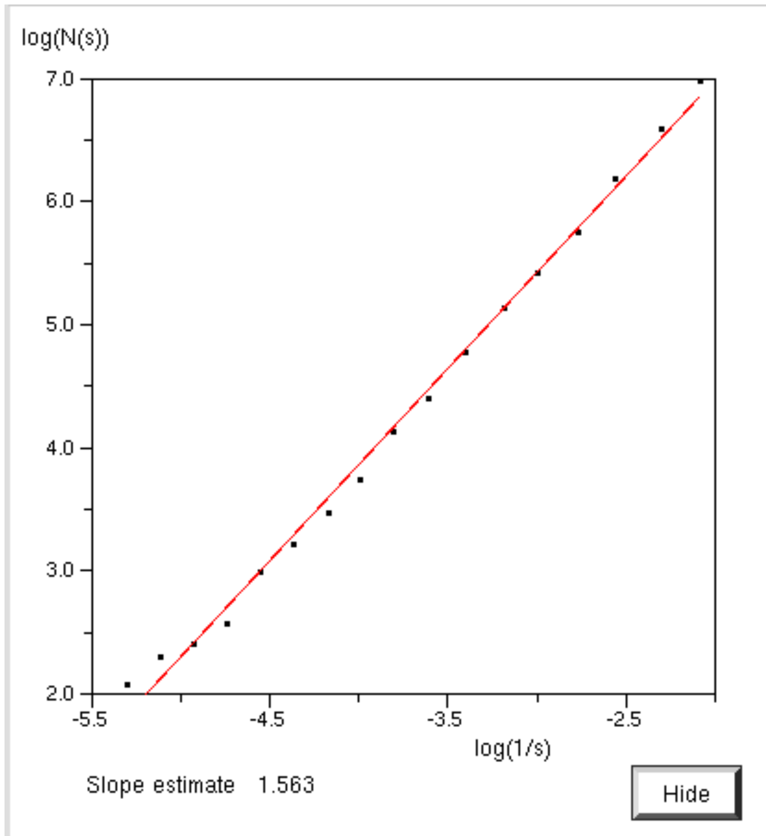


Figure 18: log(N)/log(r) plot

Even more dramatic we apply the program to a fractal tree(created on the computer) as shown below

Figure 19: fractal tree

and get the result

```
reality:fdc_mac boccio$ ./fdc  -i tree.tga -b1 200 -b2 8 -n 1 -bf 1.2 -p
FDC - Fractal Dimension Calculator
Initialising GLUT
Creating windows
Creating menus (Right mouse button)
Reading file "tree.tga"
   Image dimensions: 2048 x 1536 x 24
Creating screen version
   Full screen size: 2560 x 1440
   Display size:     1024 x 768
   Scale factor:     2
Processing starting
   Box size:  200 [  1], Total count = 47
   Box size:  166 [  2], Total count = 63
   Box size:  138 [  3], Total count = 85
   Box size:  114 [  4], Total count = 120
   Box size:   95 [  5], Total count = 170
   Box size:   79 [  6], Total count = 232
   Box size:   65 [  7], Total count = 333
   Box size:   54 [  8], Total count = 467
   Box size:   45 [  9], Total count = 653
   Box size:   37 [ 10], Total count = 928
```

```
   Box size:   30 [ 11], Total count = 1384
   Box size:   24 [ 12], Total count = 2100
   Box size:   20 [ 13], Total count = 2947
   Box size:   16 [ 14], Total count = 4511
   Box size:   13 [ 15], Total count = 6638
   Box size:   10 [ 16], Total count = 10842
   Box size:    8 [ 17], Total count = 16332
Processing finished
Estimated fractal dimension:  1.833
```

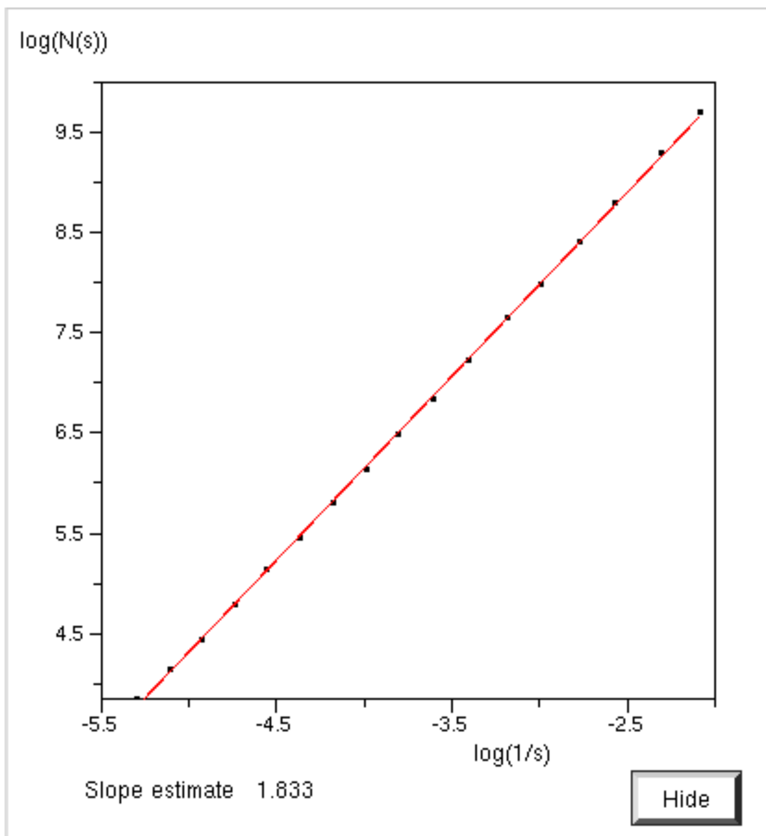


Figure 20: log(N)/log(r) plot

What about a picture of a real trees? I found a picture of a real tree and put it into the program

Figure 21: real tree

and the result is

```
reality:fdc_mac boccio$ ./fdc  -i tree-569.tga -b1 200 -b2 8 -n 1 -bf 1.2 -p
FDC - Fractal Dimension Calculator
Initialising GLUT
Creating windows
Creating menus (Right mouse button)
Reading file "tree-569.tga"
   Image dimensions: 450 x 326 x 24
Creating screen version
   Full screen size: 2560 x 1440
   Display size:     450 x 326
   Scale factor:     1
Processing starting
   Box size:  200 [  1], Total count = 6
   Box size:  166 [  2], Total count = 7
   Box size:  138 [  3], Total count = 11
   Box size:  114 [  4], Total count = 17
   Box size:   95 [  5], Total count = 19
   Box size:   79 [  6], Total count = 23
   Box size:   65 [  7], Total count = 34
   Box size:   54 [  8], Total count = 50
   Box size:   45 [  9], Total count = 57
   Box size:   37 [ 10], Total count = 79
   Box size:   30 [ 11], Total count = 105
```

```
   Box size:   24 [ 12], Total count = 172
   Box size:   20 [ 13], Total count = 227
   Box size:   16 [ 14], Total count = 341
   Box size:   13 [ 15], Total count = 486
   Box size:   10 [ 16], Total count = 761
   Box size:    8 [ 17], Total count = 1073
Processing finished
Estimated fractal dimension:  1.563
```

The log(N)/log(r) graph looks like:

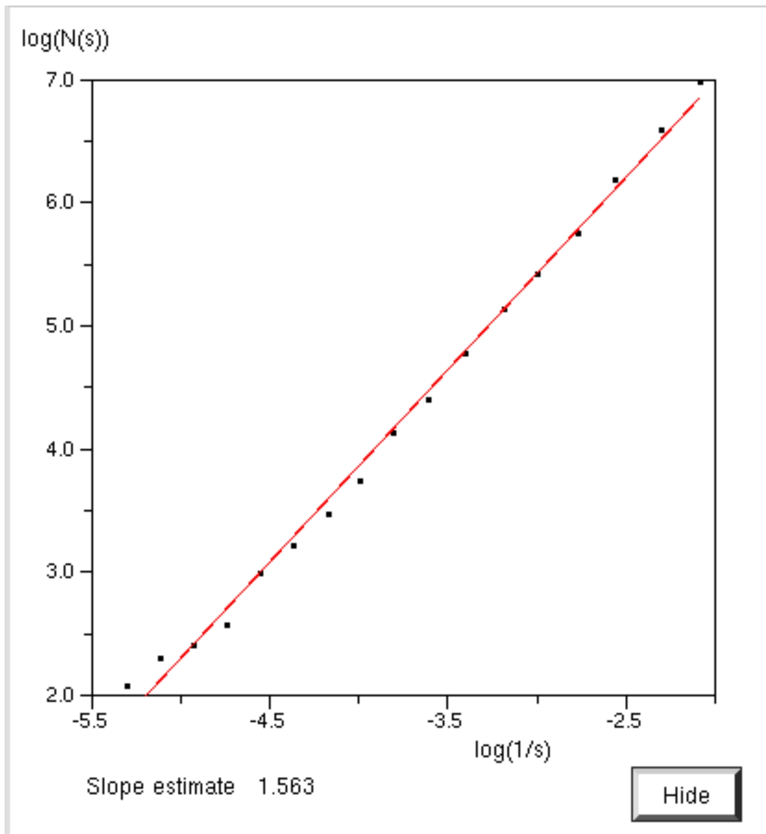


Figure 22: log(N)/log(r) plot

2.5. Complex Maps

To see the self similar nature of fractals even more dramatically we now study 2-dimensional complex maps. In particular we look at the Mandelbrot fractal boundary and the cube roots of 1 via Newton's rule(called the Newton attractor).

2.5.1. Mandelbrot Sets

The key mapping is a quadratic map given by:

$$z_{n+1} = z_n^2 + c$$

where the z_n are complex numbers and c is a complex parameter. It turns out that if $|z_n| > 2$ further iteration of this quadratic map is unbounded, i.e., for different values of c, the trajectories either *stay near the origin* or *escape to infinity*. When calculations are done we stop the iteration when this condition is satisfied.

When investigating these maps, one iterates the equation as follows:

(a) start with a complex value for c inside a rectangular boundary

(b) iterate the equation until either

  (1) $|z_n| > 2$

  or

  (2) a preset number of iterations is exceeded

In case (1) color the starting point (c) in red. In case (2) leave the point (c) black. The *boundary* between the two regions is the *Mandelbrot fractal* as shown below.

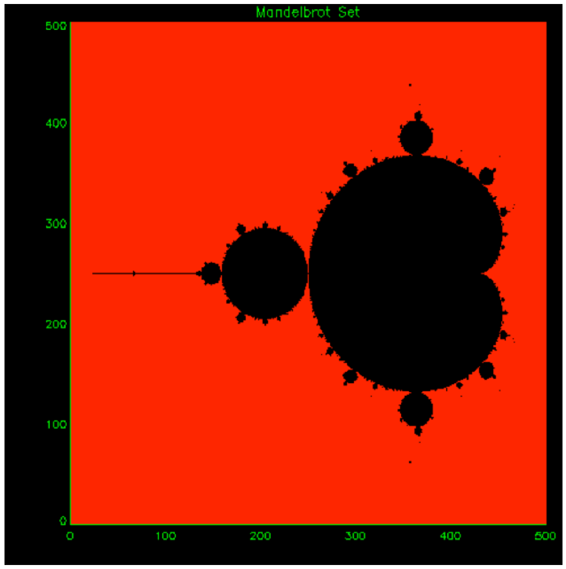


Figure 23: The Mandelbrot Fractal Boundary

Alternately, in case (1) color the starting point (c) by the number of iterations it took to escape. In case (2) leave the point (c) a single color (say dark blue).

The boundary between the two regions is still the Mandelbrot fractal as shown below, but a wide variety of spectacular images can be generated by a clever choice of colormap.

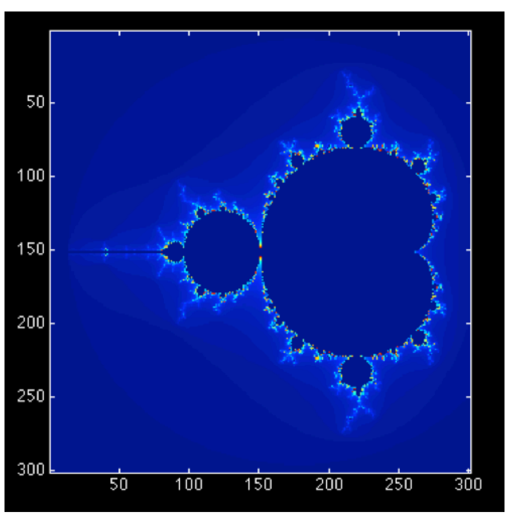


Figure 24: The Mandelbrot Fractal Boundary - Different Coloring Scheme

or choosing a different colormap(no change in information) we get

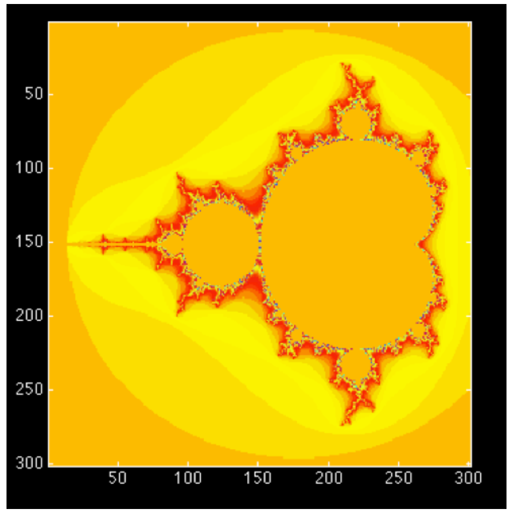


Figure 25: The Mandelbrot Fractal Boundary - Different Coloring Scheme

Run Complex Explorer for colormaps and zooming.

If we zoom far enough we will see the entire repeating itself as all fractals do since they are self-similar.

2.6. So what is a fractal?

It is an irregular geometric object with an infinite nesting of structures at all scales.

2.6.1. Why do we care about fractals?

natural objects are fractals

chaotic trajectories (strange attractors) are fractals

assessing the fractal properties of an observed time series is informative

complex systems that self-organize will have self-similar or fractal behavior at all scales

complexity: While, chaos is the study of how simple systems can generate complicated behavior, complexity is the study of how complicated systems can generate simple behavior. An example of complexity is the synchronization of biological systems ranging from fireflies to neurons.

complex system: Spatially and/or temporally extended nonlinear systems characterized by collective properties associated with the system as a whole–and that are different from the characteristic behaviors of the constituent parts.

2.7. Final Example - The Cube Roots of 1

Another interesting map is called the Newton attractor. In this case one starts with an equation like

$$z^3 - 1 = 0$$

which is to be solved via the classical Newton iteration method for finding zeroes:

$$\begin{aligned} &z^3 - 1 = 0 = f(z) \\ &z_{n+1} = z_n - \frac{f(z_n)}{df(z_n)/dz} = z_n - \frac{z_n^3 - 1}{3z_n^2} = \frac{2}{3}z_n + \frac{1}{z_n^2} \end{aligned}$$

In this case one starts with some initial complex point z_0. One then iterates until the answer stops changing (converges to one of the three complex roots of 1). One might assume that the plane will divide up into three pie shaped wedges such that all starting points in a given wedge are closest to one of the three roots(in its basin) and eventually end up at that closest root (the basin attractor). In fact, this does not happen. The boundaries of the wedges are "nibbled" at by interlaced basins of attraction for all the roots producing a complex fractal boundary as shown below.

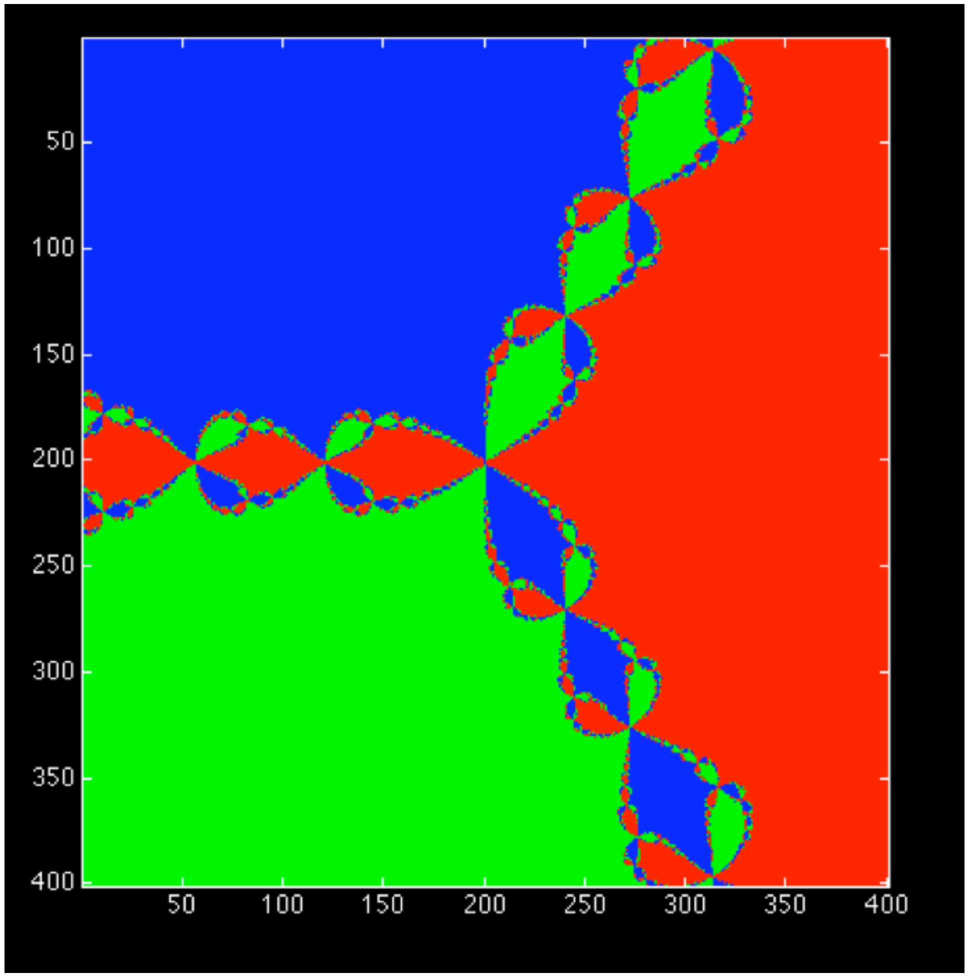


Figure 26: The Newton Attractor Map

If we zoom in on the boundary we see the complex self-similar structure of the fractal strange attractor making up the boundaries between the basins as shown below.

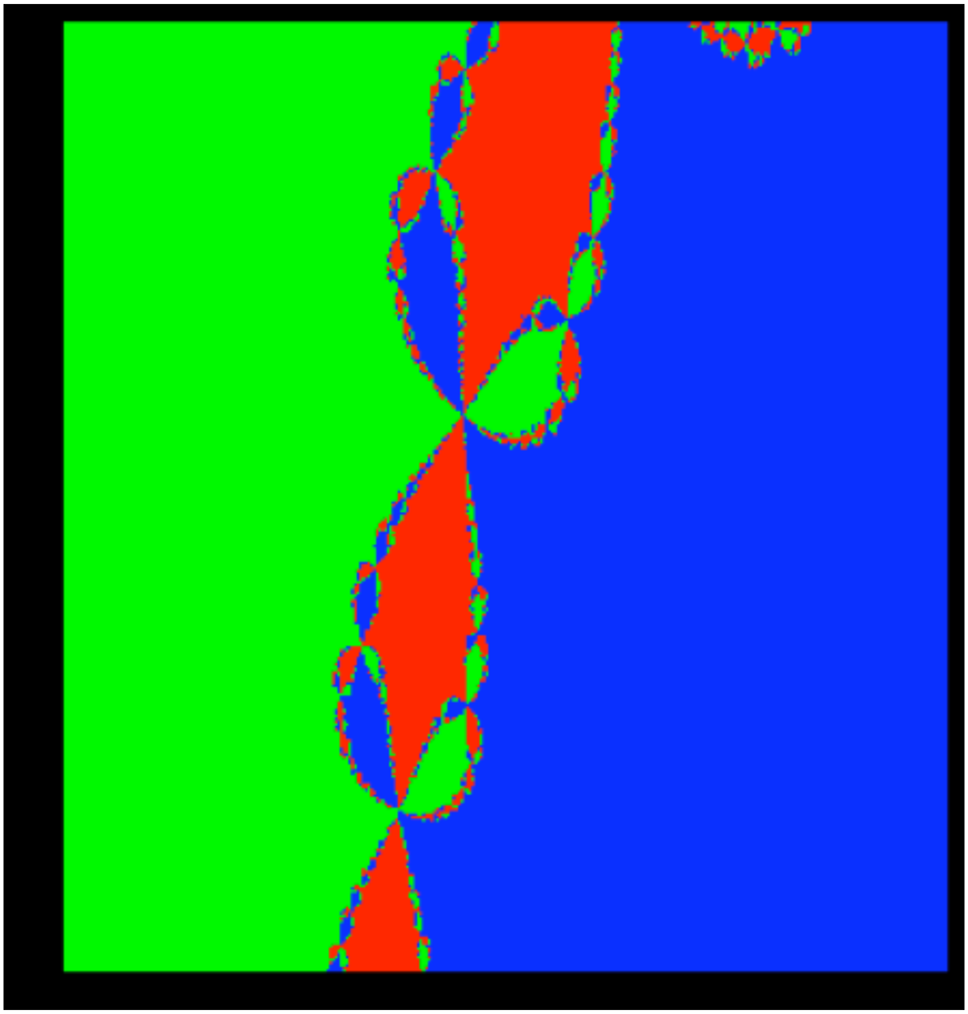


Figure 27: The Newton Attractor Map

Further zooming in produces shows the self-similar structure of the fractal boundary.

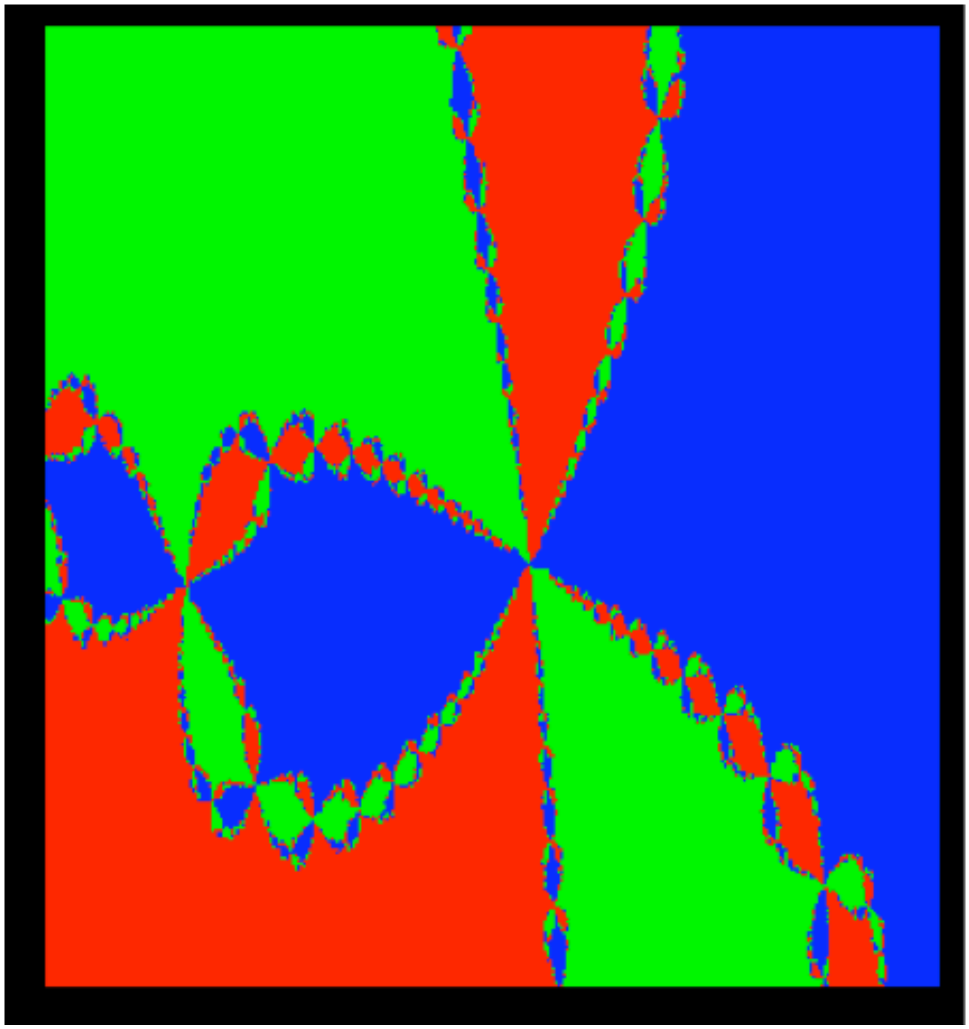


Figure 28: The Newton Attractor Map

2.8. Complexity (first pass about ideas)

While chaos is the study of how simple systems can generate complicated behavior, *complexity* is the study of how complicated systems can generate simple behavior.

Complex systems are spatially and/or temporally extended nonlinear systems characterized by emergent collective properties.

2.8.1. Cellular Automata

What are they?

grid of 0-1 cell values (N cells total)

simple rule (about altering cell status) applied to each cell, simultaneously - iterated in time

rule depends on status of neighbors

2.8.2. Game of Life

$N \times N$ matrix of cell values (that can be real or imaginary values) that change over time according to

(1) the status of *neighbors* (or connected cells, or some external condition)

and

(2) some rule, which can be global (as here) or local. This system evolves over time.

Rules:

Number Neighbors $> 3 \to 0$ (overcrowding)

Number Neighbors $< 2 \to 0$ (loneliness)

Number Neighbors $= 2$ retain current state

Number Neighbors $= 3 \to 1$ (staying alive or birth)

These systems have *attractors*:

static states - *fixed point*

periodic states - *limit cycles*

nonperiodic states (*chaotic or strange attractor*)

2.8.3. Genetic Algorithms(GA)

Approach: GA cell values: not species member, but genotypes.

Goal: system that learns.

2.8.4. Self-organized complexity

Sandpile Model

$N \times M$ cells, each with height h (*grains*)

one cell chosen randomly and h increased by 1

iterated

if cell $h > z$, 1 grain passed to each neighbor (updated)

avalanche has size s (number of cells updated)

continue iteration and count frequency, $D(s)$ of each s

Result: $\log(D(s))$ is *power law* of $\log(s)$

When this occurs it illustrates *self-organized criticality* (more later).

We will study these and many other complex systems next.

Chapter 3

Self-Organized Criticality (SOC), Sandpiles and Evolution

3.1. Introduction

Self-organized criticality is a new way of viewing nature.

The basic picture goes as follows:

> Nature is perpetually out of balance, but organized in a poised state - *the critical state* -where anything (*any size fluctuation*) can happen within well-defined *statistical* laws.

The aim of science in the study of self-organized criticality is to give insight into the fundamental question of why nature is complex, not simple, as the laws of physics seem to imply.

Self-organized criticality explains some ubiquitous patterns existing in nature that we view as complex. Fractal structure and catastrophic events will be seen to be among those regularities.

Applications will range from the study of pulsars and black holes to earthquakes, financial markets, the formation of cities and the evolution of life.

An intriguing consequence of the theory will be that catastrophes can occur for *no reason whatsoever*. Mass extinctions may take place without any external triggering mechanism such as an asteroid impact and so on.

The basic ideas of the theory are simple and mathematical models used in theory are not complicated. We will see that it is easy to set up computer simulations and make predictions.

Unlike many other subjects in physics, the basic ideas are simple enough to be made accessible to non-scientists without being trivialized as we will see in this class.

3.2. Complexity and Criticality

How can the universe start with a few types of elementary particles at the big bang and end up with life, history, economics, and literature? Throughout the history of science, very little scientific effort has been devoted to understanding why nature is complex. This has changed radically in the last two decades.

We will argue in these lectures, that complex behavior in nature reflects the tendency of large systems (having many parts) to evolve into a poised, **critical state**, that is way out of balance and where minor (possibly infinitesimal) disturbances may lead to events, called *avalanches*, or *catastrophes* of all sizes.

We will find that the majority of change in a system will take place through catastrophic events rather than by following a smooth gradual path. Evolution to this very delicate state will be seen to occur without any help from any outside agent.

The state is established solely because of the dynamical interactions among individual elements of system - *the critical state is self-organized.*

This *self-organized criticality* that we will be discussing, is so far the only known general mechanism that is capable of generating complexity.

To make this less abstract, consider the scenario of a child at beach letting sand trickle down to form a pile as shown below
In the beginning, the pile is flat and the individual grains of sand remain close to where they land. Their motion can be understood in terms of simple physical properties. As the process continues, the pile becomes steeper and there will be small sand slides. As time goes on, the sand slides become larger and larger. Eventually, some of sand slides may even span(cover) all or most of pile. At that point, system is far out of balance(balance = equilibrium), and its behavior can no longer be understood in terms of the behavior of individual grains. Avalanches form a dynamic behavior of their own, which can be understood only from a **holistic** description of properties of entire pile rather than from a *reductionist* description of individual grains - *a sandpile has become a complex system.*

Classroom Demo + Computer Simulation

Complex phenomena observed everywhere indicate that, in each such case, nature operates at the self-organized critical state. The behavior of the critical sandpile mimics several phenomena observed across many sciences, which are

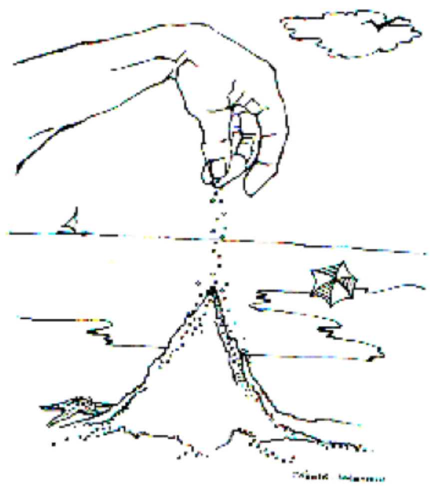


Figure 1: Making a Sand Pile

associated with complexity. But before arguing this is indeed the case, let us try to sharpen the definition of problem.

What is complexity? How have scientists and others addressed problem in the past?

3.2.1. The Laws of Physics Are Simple but Nature Is Complex

Starting from Big Bang, the universe is supposed to have evolved according to laws of physics. By analyzing experiments and other observations, physicists have been very successful in finding those laws. The innermost secrets of matter have been revealed down to very small scales. Matter consists of atoms, which are composed of elementary particles such as electrons, protons, and neutrons, which themselves are made up of quarks, leptons and gluons, and so on. All phenomena in nature, from the largest length scales spanned by universe to smallest, should, according to physicists, be explained by the same laws of physics.

One such law is Newton's second law, $F = ma$, which says that an object (of mass m) subjected to force F responds by accelerating with acceleration $a = F/m$, i.e., proportional to force. This simple law is sufficient to describe how an apple falls to the ground, how planets orbit the sun, and how galaxies are attracted to one another by gravity. Maxwell's equations describe all electromagnetic phenomena, allowing us to understand how an electric motor or an iPod works. Einstein's theory of relativity says that Newton's laws have to be modified for objects moving at high velocities. Quantum mechanics, which tells us that electrons in an atom can only exist in states with specific energies and can jump from one state to another without spending any time in between, explains all microscopic phenomena.

These laws of physics are quite *simple* (on the macroscale). They are expressed in mathematical equations that can all be written down on a couple of notebook pages. However, the mathematics involved in solving these equations, even for simple situations, can be quite complicated. This happens when there are more than two objects to consider. For instance, calculating motion of two planets moving in gravitational field of other planets and the sun is extraordinarily difficult. The problem is insoluble with pen and paper, and can be done only approximately with help of modern computers, but that is usually considered a practical problem rather than a fundamental problem by physics.

The philosophy of physics since its inception has been *reductionist*, i.e., that world around us can be understood in terms of the properties of simple building blocks. Even the Greeks viewed the world as consisting of only a few elements. Once we have broken the world down to its simplest fundamental laws, and the most fundamental entities have been identified, then the job is complete. Some think that once we have accomplished this feat, the role of physics, the *foundation of the sciences*, will be played out, and stage left to the *other* sciences, such as geophysics, chemistry and biology, to sort out the consequences.

In some special cases, physicists have succeeded in explaining the behavior of systems consisting of many atoms, molecules, or electrons. For instance, the behavior of crystals, where very large numbers of atoms neatly occupy the rows and columns of a regular periodic lattice, is relatively well understood from the basic laws of physics. A crystal is a prime example of an *ordered* system, where each atom has its well-defined place on a regular, periodic grid. The crystal is understandable precisely because it looks the same everywhere.

At opposite end of the spectrum from crystals are gases, which also consist of many atoms or molecules. Gases can be understood because their molecules rarely interact (collide with one another). In contrast to crystal, where atoms are ordered on a lattice, atoms in a gas form a random, disordered system. Again, the tractability of system arises from its uniformity. The gas looks same everywhere, although at any given time individual atoms at different locations move with different velocities in different directions. On *average* all atoms behave same way.

However, we do not live in a simple, boring world composed only of planets orbiting stars, regular infinite crystals, and simple gases or liquids. Our everyday situation is not that of falling apples. If we open the window, we see an entirely different picture. The surface of the earth is an intricate conglomerate of mountains, oceans, islands, rivers, volcanoes, glaciers, and earthquake faults, each of which has its own characteristic dynamics. Unlike very ordered or disordered systems, landscapes differ from place to place and from time to time. We will define such systems with large variability as *complex system.* This variability may exist on a wide range of length scales. If we zoom in closer and closer, or look out further and further, we will find variability at each level of magnification, with more and more new details appearing at each stage. In the universe, there is variability on the greatest scale. Just about every week, there is a new report from the Hubble telescope orbiting the earth, or from satellites of various kinds, of some previously undiscovered phenomenon. Many different quantitative general definitions of complexity have been attempted, without much success, so let us think of complexity simply as *variability* for the moment. Crystals and gases and orbiting planets are not complex, but landscapes are.

As if the variability seen in astronomy and geophysics were not enough, complexity has many more layers. Biological life has evolved on earth, with myriad different species, many with billions of individuals, competing and interacting with each other and with the environment. At the end of one tiny branch of biology we find ourselves. We can recognize other humans because we are all different. The human body and brain are formed by an intricate arrangement of interacting cells. The brain may be the most complex system of all because it can form a representation (we can think) of the complex outer world. Our history, with its record of upheavals, wars, religions, and political systems, constitutes yet another level of complexity involving modern human societies with economies composed of consumers, producers, thieves, governments, and economists.

Thus, the world we actually observe is full of all kinds of structure and surprises. How does this variability emerge out of simple laws?

Most phenomena that we observe around us seem rather distant from the basic laws of physics. It is a futile endeavor to try to explain most natural phenomena in detail by starting from particle physics and following the trajectories of all particles. The combined power of all the computers in the world does not even come close to the capacity needed for such an undertaking. The fact that the laws of physics specify everything (that they are deterministic) is irrelevant. The dream arising from the breathtaking progress of physics during the last two centuries combined with the advances of modern high-speed computers - that everything can be understood from *first principles* - has been thoroughly shattered.

About thirty years ago, in the infancy of the computer era, there was a rather extensive effort, known as *limit to growth*, that had the goal of making global predictions(economic). The hope was to be able to forecast, among other things, the growth of the human population and its impact on the supply of natural resources. The project failed miserably because the outcome depended on unpredictable factors that could not be explicitly incorporated into the program.

Perhaps predictions on global warming fall into the same category, since we are dealing with long-term predictions in a complex system, even though we have a reasonable understanding of the physics of weather systems.

The laws of physics can explain how an apple falls but not why Newton, a part of a complex world, was watching the apple or what he thought about the falling apple. Nor does physics have much to say about the origin of the apple.

Ultimately though, we believe that all the complex phenomena, including biological life, do indeed obey physical laws - we are simply unable to make the connection from atoms where we know that the laws are correct, through the chemistry of complicated organic molecules, to the formation of cells, and to the arrangement of those cells into living organisms. There has never been any proof of a metaphysical process not following the laws of physics that would distinguish living matter from any other. One might wonder whether this state of affairs means that we cannot find general *laws of nature* describing why the ordinary things that we actually observe around us are complex rather than simple.

The question of the origin of complexity from simple laws of physics - maybe the biggest puzzle of all - has only emerged as an active science over the last three decades. One reason is that high-speed computers, which are essential in the study, are now generally available. However, even now the science of complexity is shrouded in a good deal of skepticism - it is not clear how any general result can possibly be helpful, because each science seems to work well within its own domain.

Because of our inability to directly calculate how complex phenomena at one level arise from the physical mechanisms working at a deeper level, scientists sometimes throw up their hands and refer to these phenomena as **emergent**. They just *pop out of nowhere*. Geophysics emerges from astrophysics. Chemistry emerges from physics. Biology emerges from chemistry and geophysics, and so on. Each science develops its own jargon, and works with its own objects and concepts. Geophysicists talk about tectonic plate motion and earthquakes without reference to astrophysics, biologists describe the properties and evolution of species without reference to geophysics, economists describe human monetary transactions without reference to biology, and so on. There is nothing wrong with that! Because of the seeming intractability of emergent phenomena, no other modus operandi is possible. If no new phenomena emerged in large systems except that predicted by the dynamics of systems working at a lower level, then we would need no scientists but particle physicists. Quality in some way emerges from quantity. Emergent behavior does not seem to fit with reductionism - but how and why? First let us review a previous approaches to dealing with complex phenomena.

3.2.2. Storytelling Versus Science

The reductionist methods of physics - detailed predictions followed by comparison with reproducible experiments are impossible to use in vast areas of scientific interest. The question of how to deal with complex systems has been clearly formulated by the eminent paleontologist and science writer Stephen Jay Gould in his book **Wonderful Life**:

> *How should scientists operate when they must try to explain the result of history, those inordinately complex events that can occur but once? Many large domains of nature - cosmology, geology, and evolution among them - must be studied with the tools of history. The appropriate methods focus on narrative, not experiment as usually conceived.*

Gould throws up his hands and argues that only *storytelling* can be used in many sciences because particular outcomes are contingent on *single and unpredictable* events. Experiments are irrelevant in evolution or paleontology, because *nothing is reproducible.* History, including that of evolution, is just *one damned thing after another.* We can explain in hindsight what has happened, but we cannot predict what will happen in the future.

The Danish philosopher Sören Kierkegaard expressed the same view in his famous phrase

> *Life is understood backwards, but must be lived forwards.*

Sciences have traditionally been grouped into two categories: hard sciences, in which repeatable events can be predicted from a mathematical formalism expressing the laws of nature, and soft sciences, in which, because of their inherent variability only a narrative account of distinguishable events post mortem is possible. Physics, chemistry and molecular biology belong to the first category; history, biological evolution, and economics belong to the second.

Gould rightfully attributes the variability of things, and therefore their complexity, to *contingency.* Historical events depend on freak accidents, so if the tape of history is replayed many times with slightly different initial conditions, the outcome will differ vastly each time.

Historians explain events in a narrative language where event A leads to event B and C leads to D. Then, because of event D, event B leads to E. However, if the event C had not happened, then D and E would not have happened either. The course of history would have changed into another sequence of events, which would have been equally well explainable, in hindsight, with a different narrative. The discovery of America involved a long series of events, each of crucial historical importance for the actual outcome: Columbus' parents had to meet each other, Columbus had to be born, he had to go to Spain to get funding, the weather had to be reasonable, and so on. History is unpredictable, but not unexplainable. There is nothing wrong with this way of doing science, in which the goal is an accurate narrative account of specific events. It is precisely the overwhelming impact of contingency that makes those sciences interesting. There will always be more surprises in store for us. In contrast, simple predictable systems, such as an apple falling to the ground, become boring after a while.

In the soft sciences, where contingency is pervasive, detailed long-term prediction becomes impossible. The science of evolutionary biology, for example, cannot explain why there are humans and elephants. Life as we see it today is just one very unlikely outcome among myriad other equally unlikely possibilities. For example, life on earth would be totally different if the dinosaurs had not become extinct, perhaps as a consequence of an asteroid hitting the earth instead of continuing in its benign orbit. An unlikely event is likely to happen because there are so many unlikely events that could happen.

But what underlying properties of history and biology make them sensitive to minor accidental events?

In other words, what is the underlying nature of the dynamics that leads to the interdependence of events and thus to complexity?

Why can incidents happen that have dramatic global consequences?

Why is there a dichotomy of the sciences into two quite disparate groups with different methods and styles, since presumably all systems in the final analysis obey the same laws of nature?

Before going into the details of the theory, let us explore, in general terms, what a science of complexity could be.

3.2.3. What Can a Theory of Complexity Explain?

If all that we can do in the soft, complex sciences is to monitor events and make short-term predictions by massive computations, then the soft sciences are no place for physicists to be, and they should gracefully leave the stage for the *experts* who have detailed knowledge about their particular fields. If one cannot predict anything specific, then what is the point?

In a debate in January 1995, John Maynard Smith of the University of Sussex, England, author of **The Theory of Evolution**, exclaimed that he did not find the subject of complexity interesting, precisely because it has not explained any detailed fact in nature.

Indeed, any theory of complexity must necessarily appear insufficient. The variability precludes the possibility that all detailed observations can be condensed into a small number of mathematical equations, similar to the fundamental laws of physics. At most, the theory can explain *why* there is variability, or what typical patterns may emerge, not what the particular outcome of a particular system under particular conditions will be. **The theory will never predict elephants**. Even under the most optimistic circumstances, there will still be room for historians and fiction writers in the future.

A general theory of complex systems must necessarily be abstract. For example, a theory of life, in principle, must be able to describe all possible scenarios for evolution. It should be able to describe the mechanisms of life on Mars, if life were to occur there. This is an extremely important statement. Any general model we might construct cannot have any specific reference to actual species. The model may, perhaps, not even refer to basic chemical processes, or to the DNA molecules that are integral parts of any life form that we know.

We must learn to free ourselves from seeing things the way they are! A radical scientific view indeed! If, following traditional scientific methods, we concentrate on an accurate description of the details, we lose perspective. A theory of life is likely to be a theory of a process, not a detailed account of utterly accidental details of that process, such as the emergence of humans.

The theory must be statistical (involving large numbers) and therefore cannot produce specific details. Much of evolutionary theory as presented for instance in Maynard Smith's book, is formulated in terms of anecdotal evidence for the various mechanisms at work. Anecdotal evidence carries weight only if enough of it can be gathered to form a statistical statement. Collecting anecdotal evidence can only be an intermediate goal. In medicine, it was long ago realized that anecdotal evidence from a single doctor's observation must yield to evidence based on a large, statistically significant set of observations. Confrontation between theories and experiments or observations, essential for any scientific endeavor, takes place by comparing the statistical features of general patterns.

The abstractness and the statistical, probabilistic nature of any such theory might appear revolting to geophysicists, biologists, and economists, expecting to aim for photographic characterization of real phenomena.

Perhaps too much emphasis has been put on detailed prediction, or forecasting, in science in today's materialistic world. In geophysics, the emphasis is on predicting specific earthquakes or other disasters. Funding is provided according to the extent to which the budget agencies and reviewers judge that progress might be achieved. This leads to charlatanism and even fraud, not to mention that good scientists are robbed of their grants. Similarly, the emphasis in economics is on prediction of stock prices and other economic indicators, since accurate predictions allow you to make money. Not much effort has been devoted to describing economic systems in an unbiased, detached way, as one would describe, say, an ant's nest.

Actually physicists are accustomed to dealing with probabilistic theories, in which the specific outcome of an experiment cannot be predicted - instead only certain statistical features can be determined. Three fundamental theories in physics are of a statistical nature. First, statistical mechanics deals with large systems in equilibrium, such as the gas of atoms in the air surrounding us. Statistical mechanics tells us how to calculate average properties of the many atoms forming the gas, such as the temperature and the pressure. The theory does not give us the positions and the velocities of all the individual atoms (and we couldn't care less anyhow). Second, quantum mechanics tells us that we cannot predict both the specific position and velocity of a small particle such as an electron at the same time, but only the probability that an experiment would find the particle at a certain position. Again, we are most often interested only in some average property of many electrons, as for instance the electric current through a wire, which may again be predictable. Third, chaos theory tells us that many simple mechanical systems, for example, pendulums that are pushed periodically may show unpredictable behavior. We don't know exactly where the pendulum will be after a long time, no matter how well we know the equations for its motion and its initial state.

As pointed out by the philosopher Karl Popper, prediction is our best means of distinguishing science from pseudoscience. To predict the statistics of actual phenomena rather than the specific outcome is a quite legitimate and ordinary way of confronting theory with observations.

What makes the situation for biology, economics, or geophysics conceptually different, and what makes it more difficult to accept this state of affairs, is that the outcome of the process is important. As humans, we care about the specific state of the system. We don't just observe the average properties of many small unpredictable events, but only one specific outcome in its full glory. The fact that we may understand the statistical properties of earthquakes, such as the average number of earthquakes per year of a certain size in a certain area, is of little consolation to those who have been affected by large, devastating earthquakes. In biology, it is important that the dinosaurs vanished during a large extinction event and made room for us.

Psychologically, we tend to view our particular situation as unique. It is emotionally unacceptable to view our entire existence as one possible fragile outcome among zillions of others. The problem with understanding our world is that we have nothing to compare it with.

We cannot overcome the problem of unpredictability. Kierkegaard's philosophy represents the fundamental and universal situation of life on earth.

So how can there be a general theory or science of complexity?

If such a theory cannot explain any specific details, what is the theory supposed to explain?

How, precisely can one confront theory with reality?

Without this crucial step, there can be no science.

Fortunately, there are a number of ubiquitous general empirical observations across the individual sciences that cannot be understood within the set of references developed within the specific scientific domains. These phenomena are the occurrence of large catastrophic events, fractals, so-called $1/f$ noise, and Zipf's law.

A litmus test of a theory of complexity is its ability to explain these general observations. Why are they universal, that is, why do they pop up everywhere?

3.2.4. Catastrophes Follow a Simple Pattern

Because of their composite nature, complex systems can exhibit catastrophic behavior, where one part of the system can affect many others in some kind of a domino effect. Cracks in the crust of the earth propagate in this way to produce earthquakes, sometimes with tremendous energies. Scientists studying earthquakes look for specific mechanisms for large events, using a narrative-type individual description for each event in isolation from the others. This occurs even though the number of earthquakes of a given magnitude follows a glaringly simple distribution function known as the Gutenberg-Richter law. It turns out that every time there are about 1,000 earthquakes of say, magnitude 4 on the Richter scale, there are 100 earthquakes of magnitude 5, 10 of magnitude 6, and so on. This law is illustrated in the figure(a) below, which shows how many earthquakes there were of each magnitude in a region of the southeastern United States known as the New Madrid earthquake zone during the period 1974-1983.

Part (b) of the figure shows where those earthquakes took place. The size of the dots represents the magnitudes of the earthquakes. The scale is a logarithmic one, in which the numbers on the vertical axis are 10, 100, 1000 instead of 1, 2, 3.

The Gutenberg-Richter law manifests itself as a straight line in this plot.The horizontal x-axis is also logarithmic, since the magnitude m measures the logarithm of the energy released by the earthquake, rather than the energy itself (a so-called log-log plot). Thus, an earthquake of magnitude 6 is ten times stronger than an earthquake of magnitude 5, and an earthquake of magnitude 4 is ten times stronger than an earthquake of magnitude 3. An earthquake of magnitude 8 is 10 million times more energetic than one of magnitude 1, which corresponds to a large truck passing by. By using worldwide earthquake catalogues, the straight line can be extended to earthquakes of magnitudes 7, 8, and 9. This law is amazing! How can the dynamics of all the elements of a system

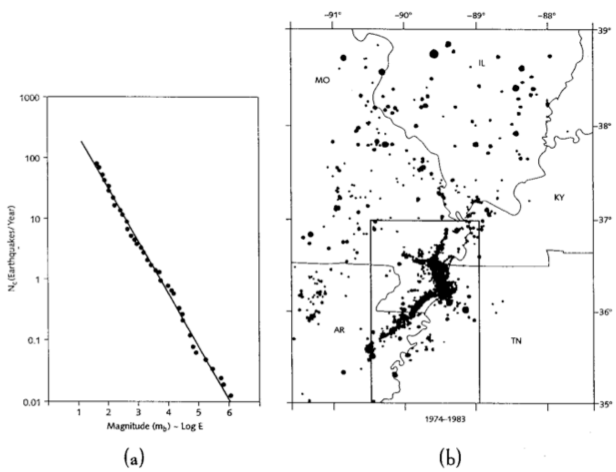


Figure 2: Illustrating Data and Zipf's Law

as complicated as the crust of the earth, with mountains, valleys, lakes, and geological structures of enormous diversity, conspire, *as if by magic*, to produce a law with such extreme simplicity?

The graph shows that large earthquakes do not play a special role; they follow the same law as small earthquakes. Thus, it appears that one should not try to come up with specific explanations for large earthquakes, but rather with a general theory encompassing all earthquakes, large and small.

The importance of the Gutenberg-Richter law cannot be exaggerated. **It is precisely the observation of such simple empirical laws that motivates us to search for a theory of complexity**. Such a theory would complement the efforts of geophysicists who have been occupied with their detailed observations and theorizing on specific large earthquakes and fault zones without concern about the general picture. *One explanation for each earthquake, or for each fault.*

In their fascinating book **Tales of the Earth**, Officer and Page argue that the regularity of numerous catastrophic phenomena on earth, including flooding, earthquakes, and volcanic eruptions, has a message for us on the basic mechanisms driving the earth, which we must unravel in order to deal with those phenomena (or, perhaps, to understand why we cannot deal with them).

In economics, an empirical pattern similar to the Gutenberg-Richter law holds. Benoit Mandelbrot, of IBM'S Watson Center in New York, pointed out in 1966 that the probability of having small and large variations on prices of stocks, cotton, and other commodities follows a very simple pattern, known as a Levy distribution. Mandelbrot had collected data for the variation of cotton prices from month to month over several years. He then counted how often the monthly variation was between 10 and 20 percent, how often the variation was between 5 and 10 percent, and so on, and plotted the result on a logarithmic plot as shown below

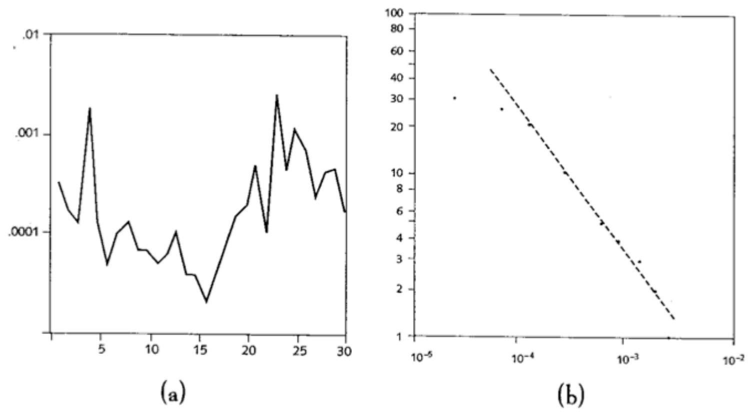


Figure 3: Mandelbrot and Cotton Prices

Just as scientists counted how many earthquakes there were of each size, Mandelbrot counted how many months there were with a given price variation. Note the smooth transition from small variations to large ones. The distribution of price changes follows approximately a straight line, a power law. The price variations are *scale free* with no typical size of the variations, just as earthquakes do not have a typical characteristic size. Mandelbrot studied several different commodities, and found that they all followed a similar pattern, but he did not speculate about the origin of the regular behavior that he observed. Economists have chosen largely to ignore Mandelbrot's work, mostly because it doesn't fit into their generally accepted picture. They would discard large events, since these events can be attributed to specific *abnormal circumstances*, such as program trading for the crash of October 1987, and excessive borrowing for the crash of 1929. Contingency is used as an argument for statistical exclusion. Economists often *cull* or *prune* the data before analysis!

How can there be a general theory of events that occur once?

The fact that large events follow the same law as small events indicates that there is nothing special about those events, despite their possibly devastating consequences.

Similarly in biological evolution, it has been pointed out that the distribution of extinction events follows a smooth distribution where large events, such as the Cretaceous extinction of dinosaurs and many other species, occur with fairly well defined probability and regularity. The data was collected by Jack Sepkoski, who had spent *ten years in the library* researching the fossil records of thousands of marine species. Sepkoski split geological history into 150 consecutive periods of 4 million years. For each period, he estimated what fraction of species had disappeared since the previous period (see figure below)

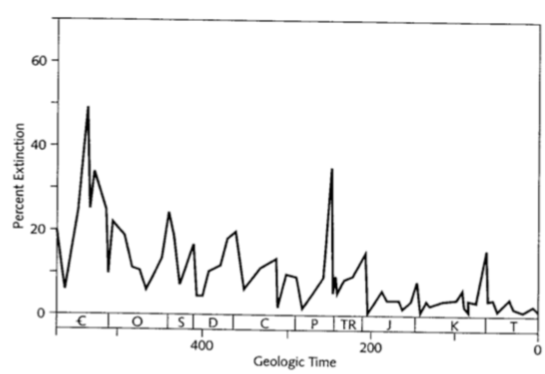


Figure 4: Sepkowski Fossil Data

The estimate is a measure of the extinction rate. Sometimes there were very few extinctions, less than 5 percent, and sometimes there were more than 50 percent extinctions. The famous Cretaceous event in which the dinosaurs became extinct is not even among the most prominent. Scientists simply counted the number of periods in which the relative number of extinctions was less than 10 percent, how many periods the variation was between 10 and 20 percent, and so on, and made a histogram as shown below.

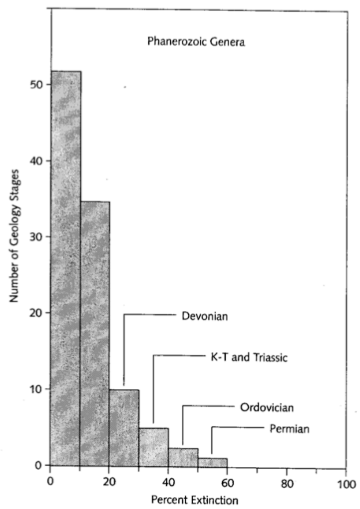


Figure 5: Sepkowski Fossil Data - Histogram

This is the same type of analysis that Mandelbrot made for cotton prices - extinction rates replace price variations, 4-million-year intervals replace monthly ones. The resulting histogram forms a smooth curve, with the number of large events extending smoothly from the much larger number of small events.

Although large events occur with a well-defined probability, this does not mean the phenomenon is periodic, as many thought it was. The fact that an earthquake has not taken place for a long time does not mean that one is due. The situation is similar to that of a gambling roulette. Even if on average black comes out every second time, that does not mean that the outcome alternates between black and red. After seven consecutive reds, the probability that the next event is black is still 1/2. The same goes for earthquakes. That events occur at some average interval does not mean that they are cyclical. For example, the fact that wars happen on average, say, every thirty years, cannot be used to predict the next war. The variations of this interval are large.

Again, specific narratives may explain each large catastrophe, but the regularity, not to be confused with periodicity, suggests that the same mechanisms work on all scales, from the extinctions taking place every day, to the largest one, the Cambrian explosion, causing the extinction of up to 95% of all species, and, fortunately the creation of a sufficiently compensating number of species.

That catastrophes occur at all is quite amazing. They stand in sharp contrast to the theory of uniformitarianism, or gradualism, which was formed in the 19th century by the geophysicist Charles Lyell in his book Principles of Geology. According to his theory, all change is caused by processes that we currently observe, which have worked at the same rate at all times. For instance, Lyell proposed that landscapes are formed by gradual processes, rather than catastrophes like Noah's flood, and the features that we see today were made by slow persistent processes, with time as the *great enabler* that eventually makes large changes.

Lyell's uniformitarian view appears perfectly logical. The laws of physics are generally expressed as smooth, continuous equations. Since these laws should describe everything, it is natural to expect that the phenomena that we observe should also vary in a smooth and gradual manner. An opposing philosophy, catastrophism, claims that changes take place mostly through sudden cataclysmic events. Since catastrophism smacks of creationism, it has been largely rejected by the scientific community, despite the fact that catastrophes actually take place.

3.2.5. Fractal Geometry

Mandelbrot has coined the word **fractal** for geometrical structures with features at all length scales, and was among the first to make the astounding observation that nature is generally fractal. The figure below shows the coast of Norway, which appears as a hierarchical-structure of fjords, and fjords within fjords, and fjords within fjords within fjords.

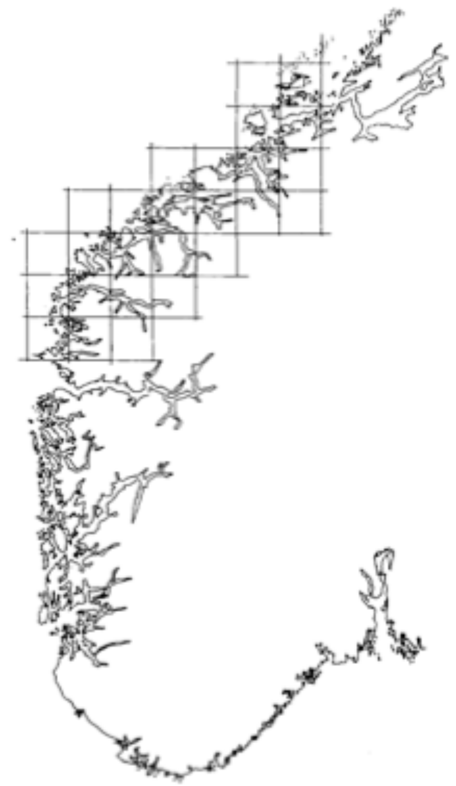


Figure 6: The Coast of Norway

The question **How long is a typical fjord?** has no answer. The phenomenon is *scale free*. If you see a picture of part of the fjord, or part of the coastline, you wouldn't know how large it is if the picture does not also show a ruler. Also, the length measured depends on the resolution of the ruler used for the measurement. A very large ruler that measures features only on the scale of miles will yield a much smaller estimate of the length than if a fine ruler, which can follow details on the scale of meters, is used.

One way of representing this, as we saw earlier, is to measure how many boxes of a certain size δ are needed to cover the coast. We did this earlier in our study of fractals. Obviously the smaller the box, the more boxes are needed to cover the coast. The figure below shows the logarithm of the length L measured with boxes of size δ.

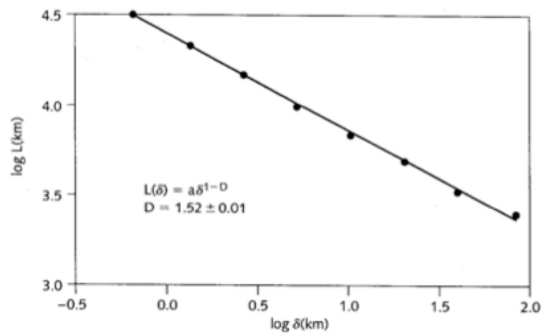


Figure 7: Measuring the Length of the Coast of Norway with Different Rulers

Had the coast been a straight line, of dimension 1(we defined dimension earlier), the number of boxes would be inversely proportional to δ, so that the measured length would be independent of δ, and the curve would be flat. If you measure the length of a line, it doesn't matter what the size of the ruler is. However, the number of boxes needed grows much faster than that since the boxes have to follow the wrinkles of the coastline, so the straight line has a slope. The negative slope of the line gives the *fractal dimension* of the coast, which we defined earlier. Fractals in general have dimensions that are not simple integer numbers. Here, one finds $D = 1.52$, showing that the coast is somewhere between a straight line with dimension 1 and a surface of dimension 2.

A mountain range includes peaks that may range from centimeters to kilometers. No size of mountain is typical. Similarly there are clouds of all sizes, with large clouds looking much like enlarged versions of small clouds. The universe consists of galaxies, and clusters of galaxies, and clusters of clusters of galaxies, and so on. No size of fjord, mountain, or cloud is the *right size.*

A lot of work has been done characterizing the geometrical properties of fractals, but the problem of the dynamical origin of fractals persists - where do they come from? The importance of Mandelbrot's work parallels that of Galileo, who observed that planets orbit the sun. Just as Newton's laws are needed to explain planetary motion, a general theoretical framework is needed to explain the fractal structure of Nature. Nothing exists in previously known general laws that leads to the of emergence of fractals.

3.2.6. *One-Over-f* Noise: Fractals in Time

A phenomenon called $1/f$(one-over-f) *noise* has been observed in systems as diverse as the flow of the river Nile, light from quasars, which are large, very distant objects in the universe, and highway traffic. The figure below shows the light from a quasar measured over a period of eighty years.

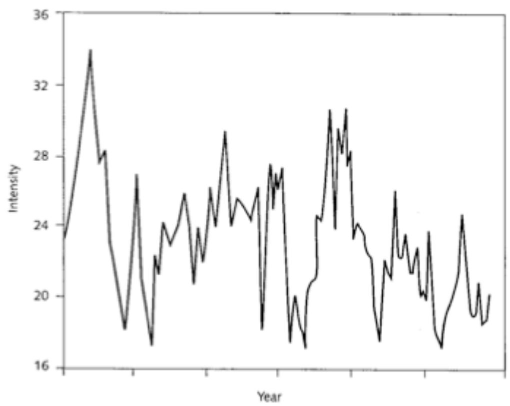


Figure 8: Long Term Measurements of Quasar Light

There are features of all sizes: rapid variations over minutes, and slow variations over years. In fact, there seems to be a gradual decrease over the entire period of eighty years, which might lead to the erroneous identification of a general tendency toward decreasing intensity within a human lifetime, a tendency that needs explanation. The signal can be seen as a superposition of bumps of all sizes; it looks like a mountain landscape in time, rather than space.

The signal can, equivalently be seen as a superposition (addition) of periodic signals of all frequencies. This is another way of stating that there are features at all time scales. Just as Norway has fjords of all sizes, a $1/f$ signal has bumps of all durations. The strength or *power* of its frequency components is larger for the small frequencies - it is inversely proportional to the frequencies. That is why it is called $1/f$ noise, although it might be misleading to call it noise rather than signal. A simple example is the velocity of a car driving along a heavily trafficked highway. There are periods of stop and go of all lengths of time, corresponding to traffic jams of all sizes. A geophysicist spent a lifetime studying the water level of the Nile. Again, the signal is $1/f$, with intervals of high levels extending over short, intermediate, and long periods.

The figure below shows the record of global average temperature variation on earth over the same period.

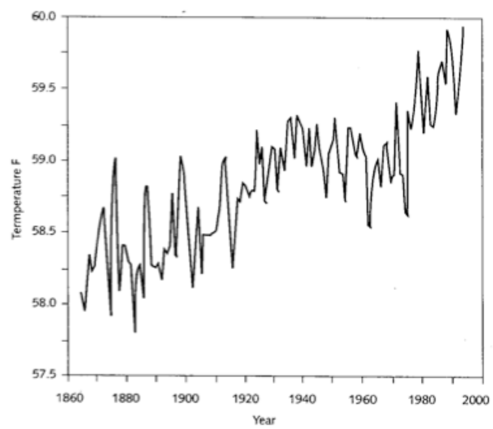


Figure 9: Long Term Measurements of Global Average Temperatures

This record is rising over roughly the same period as the quasar intensity decreases. One could conclude that the changes of quasar intensity and global temperature are correlated, but most reasonable people would not. In fact, the temperature variations can also be interpreted as $1/f$ noise. The apparent increase in temperature might well be a statistical fluctuation rather than an indication of global warming generated by human activity.

One-over-f-noise is different from random *white noise*, in which there are no correlations between the value of the signal from one moment to the next. In the figure below

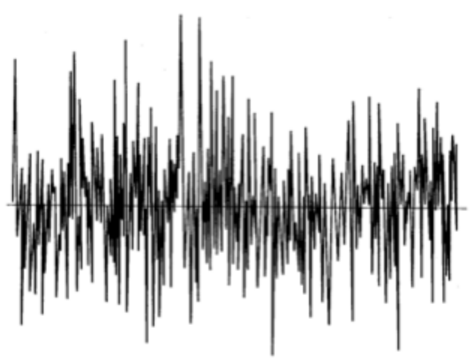


Figure 10: White Noise

the white noise pattern has no slow fluctuations, that is, no large bumps. White noise sounds like the hiss on the radio in between stations rather than music, and includes all frequencies in an equal amount. A simple periodic behavior with just one frequency would be just one tone continuing forever. The $1/f$ noise lies between these two extremes; it is interesting and complex, whereas white noise is simple and boring. Amazingly, despite the fact that $1/f$ noise is ubiquitous, there has been no general understanding of its origin. It has been one of the most stubborn problems in physics.

3.2.7. Zipf's Law

In a remarkable book that came out in 1949, **Human Behavior and the Principle of Least Effort**, Professor George Kingsley Zipf of Harvard University made a number of striking observations of some simple regularities in systems of human origin. The figure below shows how many cities in the world (circa 1920) had more than a given number of inhabitants. There were a couple of cities larger than 8 million, ten larger than 1 million, and 100 larger than 200,000. The curve is roughly a straight line on a logarithmic plot. Note the similarity with the Gutenberg-Richter law, although, of course, the phenomena being described couldn't be more different. Zipf made similar plots for many geographical areas and found the same behavior.

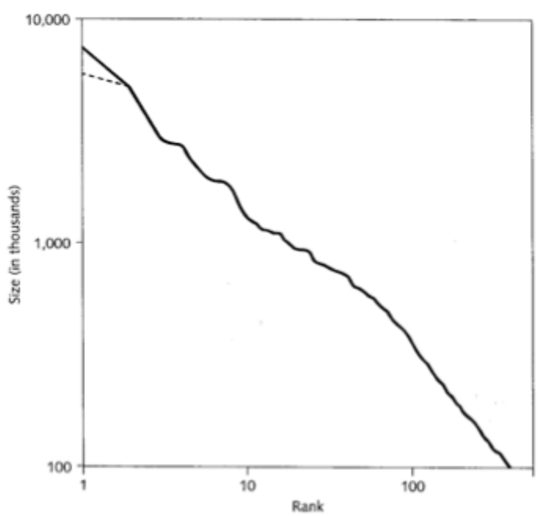


Figure 11: City Population Data

Zipf also counted how often a given word was used in a piece of literature, such as James Joyce's Ulysses or a collection of American newspapers. The tenth most frequently used word (the word off *rank* 10) appeared 2,653 times. The twentieth most used word appeared 1,311 times. The 20,000th most frequent word was used only once. The figure below shows the frequency of words used in the English language versus their ranking. The word of rank 1, *the*, is used with a frequency of 9%. The word of rank 10,*I*, has a frequency 1%, the word of rank 100, *say*, is used with a frequency of 0.1%, and so on. Again, a remarkable straight line emerges.

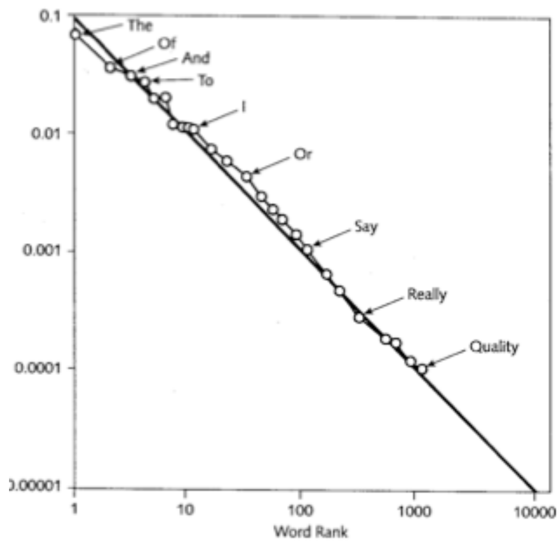


Figure 12: Zipf's Law for the English Language

It does not matter whether the data are taken from newspapers, the Bible, or Ulysses the curve is the same. The regularity expressed by the straight lines in the logarithmic plot of rank versus frequency, with slope near unity is referred to as Zipf's law.

Although Zipf does allude to the source of this regularity being the individual agent trying to minimize his effort, he gave no hints as to how to get from the individual level to the statistical observations. Zipf's law as well as the other three phenomena are emergent in the sense that they are not obvious consequences of the underlying dynamical rules.

Note that all the observations are of statistical nature. The Gutenberg-Richter law is a statement about how many earthquakes there are of each size - not where and when a particular earthquake will or did take place. Zipf's law deals with the number oddities within a given range of populations not with why a particular city has a certain number of inhabitants. The various laws are expressed as distribution functions for measurable quantities. Therefore, a theory explaining those phenomena must also be statistical, as we have already argued.

3.2.8. Power Laws and Criticality

What does it mean that something is a straight line on a double logarithmic plot?

Mathematically, such straight lines are called *power laws*, since they show that some quantity N can be expressed as some power of another quantity s:

$$N(s) = As^{-\tau}$$

Here, s could be the energy released by an earthquake, and $N(s)$ could be the number of earthquakes with that energy. The quantity s could equally well be the length of a fjord, and $N(s)$ could be the number of fjords of that length. It turns out, as we will see, that fractals are characterized by power law distributions. Taking the logarithm on both sides of the equation above we find

$$\log N(s) = \log As^{-\tau} = \log A + \log s^{-\tau} = \log A - \tau \log s$$

This shows that $\log N(s)$ plotted versus $\log s$ is a straight line. The exponent τ is the slope of the straight line. For instance, in Zipf's law the number N of cities with more than s inhabitants was expressed as

$$N(s) = \frac{A}{s} = As^{-1}$$

which is a power law with exponent -1. Essentially all the phenomena to be discussed in this class can be expressed in terms of power laws. The *scale invariance* can be seen from the simple fact that the straight line looks the same everywhere. There are no features at some scale that makes that particular scale stand out. There are no kinks or bumps anywhere. Of course, this must eventually break down at small and large scales. There are no fjords larger than Norway, and no fjords smaller than a molecule of water. But in between these two extremes there are features of all scales. In his beautiful book **Fractals, Chaos, Power Laws: Minutes from an Infinite Paradise**, Manfred Schroeder reviews the abundance and significance of power laws in nature.

Thus, the problem of explaining the observed statistical features of complex systems can be phrased mathematically as the problem of explaining the underlying power laws, and more specifically the values of the exponents.

Let us first, however, consider a couple of approaches that have proven unsuccessful.

3.2.9. Systems in Balance(equilibrium) Are Not Complex

Physicists have had some experience in dealing with large *many body* systems, in particular with systems that are in *balance* or in *stable equilibrium*. A gas of atoms and the sand at a flat beach are large systems in equilibrium; they are *in* *balance*. If an equilibrium system is disturbed slightly for instance by pushing a grain of sand somewhere, not much happens. In general, systems in balance do not exhibit any of the interesting behavior discussed above, such as large catastrophes, $1/f$ noise, and fractal structures.

There is one minor reservation. A closed equilibrium system can show complex behavior characterized by power laws, but only under very special circumstances. There has been spectacular progress in our understanding of systems at a so-called phase transition where the system goes from a disordered state to an ordered state, for instance, when the temperature is varied. Right at the critical point separating these two phases there is complex behavior characterized by scale-free behavior, with ordered domains and fluctuations of all sizes. To reach the critical point, the temperature has to be tuned very accurately in order to have complex behavior. But outside of the laboratory no one is around to tune a parameter to the very special critical point, so this does not provide insight into the widespread occurrence of complexity in nature.

In the past, it has often been more or less tacitly assumed that large systems, such as those we find in biology and economics, are in a stable balance, like the sand at a flat beach. The leading economic theory up to now, the general equilibrium theory assumes that perfect markets, perfect rationality, and so on, bring economic systems into stable Nash equilibria in which no agent can improve their situation by any action. In the equilibrium state, small perturbations or shocks will cause only small disturbances, modifying the equilibrium state only slightly. The system's response is proportional to the size of the perturbation - such equilibrium systems are said to be *linear*. Contingency is irrelevant. Small freak events can never have dramatic consequences. Large fluctuations in equilibrium systems can occur only if many random events accidentally pull in the same direction, which is prohibitively unlikely. Therefore, equilibrium theory does not explain much of what is actually going on, such as why stock prices fluctuate the way they do.

A general equilibrium theory has not been explicitly formulated for biology but a picture of nature as being *in balance* often prevails. Nature is supposed to be something that can, in principle, be conserved - this idea motivates environmentalists and conservationists. No surprise - since in a human lifetime the natural world changes very little, so equilibrium concepts may seem natural or intuitive.

However, if nature is in balance, how did we get here in the first place?

How can there be evolution if things are in balance?

Systems in balance or equilibrium, by definition, do not go anywhere. Does nature as we see it now (or a few years ago before we *started* polluting our environment) have any preferential status from an evolutionary point of view?

Implicitly, the idea of nature being in balance is intimately related to the view that humans are at the center: our natural world is the *right one*.

As pointed out by Gould, the apparent equilibrium is only a period of tranquillity or stasis, between intermittent bursts of activity and volatility in which many species become extinct and new ones emerge. Also, the rate of evolution of individual species, as measured, for instance, by their change in size, takes place episodically in spurts. This phenomenon is called *punctuated equilibrium*. The concept of punctuated equilibrium turns out to be at the heart of the dynamics of complex systems. Large intermittent bursts have no place in equilibrium systems, but are ubiquitous in history, biology and economics.

None of the phenomena described above can be explained within an equilibrium picture. on the other hand, no general theory for large nonequilibrium systems exists. The legendary Hungarian mathematician John von Neumann once referred to the theory of nonequilibrium systems as the *theory of non-elephants*, that is, there can be no unique theory of this vast area of science.

Nevertheless, such a theory of non-elephants will be attempted in this class. The picture that we should keep in mind is that of a steep sandpile, emitting avalanches of all sizes, contrasting with the equilibrium flat sand box.

3.2.10. Chaos is Not Complexity

In the 1980s a revolution occurred in our understanding of simple dynamical systems. It had been realized for some time that systems with a few degrees of freedom could exhibit so-called *chaotic behavior*. Their future behavior is unpredictable no matter how accurately one knows their initial state, even if we had perfect knowledge of the equations that govern their motion, as we have for the swing, or a pendulum, being pushed at regular intervals, as we discussed earlier.

The revolution was triggered by Mitch Feigenbaum of Los Alamos National Laboratory. He had constructed a simple and elegant theory for the transition to chaos for a simple model predator-prey system. The model was actually invented several years earlier by the British biologist Robert May. The number of individuals, x_n, who are alive one year can be related to the number of species that are alive the following year, x_{n+1} by a simple *map*(we studied this map earlier):

$$x_{n+1} = \lambda x_n(1 - x_n)$$

Feigenbaum studied this map using a simple pocket calculator. Starting with a random value of x_n, the map was used repeatedly to generate the populations at subsequent years. For small values of the parameter λ, the procedure would eventually approach a *fixed point* at which the population remains constant ever after. For larger values of λ the map goes into a cycle in which every second year the population returns to the same value. For even larger values of λ, the map first goes into a four-cycle, then an eight-cycle, until at some point (the Feigenbaum point) it goes into a completely chaotic state. In the chaotic phase, a small uncertainty in the initial value of the population is amplified as time passes, precluding predictability. Feigenbaum constructed a beautiful mathematical theory of this scenario. This was the first theory of the transition from regular periodic behavior to chaos. Chaos theory shows how simple systems can have unpredictable behavior. Chaos signals have a white noise spectrum, not $1/f$. One could say that chaotic systems are nothing but sophisticated random noise generators. If the value of x (or the position of the regularly pushed swing) is plotted versus time, the signal looks much like the white noise shown shown earlier. It is random and boring. Chaotic systems have no memory of the past and cannot evolve.

However, precisely at the *critical* point where the transition to chaos occurs, there is complex behavior, with a $1/f$ like signal. The complex state is at the border between predictable periodic behavior and unpredictable chaos. Complexity occurs only at one very special point, and not for the general values of λ where there is real chaos. The complexity is not robust! Since all the empirical phenomena we have discussed - fractals, $1/f$ noise, catastrophes, and Zipf's law occur ubiquitously, they cannot depend on some delicate selection of temperature, pressure, or whatever, as represented by the parameter λ.

Borrowing a metaphor from the English theologian William Paley, *"nature is operated by a blind watchmaker"*, who is unable to make continuous fine adjustments.

Also, simple chaotic systems cannot produce a spatial fractal structure like the coast of Norway. In the popular literature, one finds the subjects of chaos and fractal geometry linked together again and again, despite the fact that they have little to do with each other. The confusion arises from the fact that chaotic motion can be described in terms of mathematical objects known as strange attractors embedded in an abstract phase space. These strange attractors have fractal properties, but they do not represent geometrical fractals in real space like those we see in nature.

In short, chaos theory cannot explain complexity.

3.2.11. Self-Organized Criticality

The four phenomena discussed here - regularity of catastrophic events, fractals, $1/f$ noise, and Zipf's law are so similar, in that they can all be expressed as straight lines on a double logarithmic plot, that they make us wonder if they are all manifestations of a single principle. Can there be something like Newton's law, $F = ma$, of complex behavior? Maybe self-organized criticality is that single underlying principle.

Self-organized critical systems evolve to the complex critical state without interference from any outside agent. The process of self-organization takes place over a very long transient period. Complex behavior, whether in geophysics or biology, is always created by a long process of evolution. It cannot be understood by studying the systems within a time frame that is short compared with this evolutionary process. The phrase *"you cannot understand the present without understanding history"* takes on a deeper and more precise meaning. The laws for earthquakes cannot be understood just by studying earthquakes occurring in a human lifetime, but must take into account geophysical processes that occurred over hundreds of millions of years and set the stage for the phenomena that we are observing. Biological evolution cannot be understood by studying in the laboratory how a couple of generations of rats or bacteria evolve. The canonical example of SOC is a pile of sand. A sandpile exhibits punctuated equilibrium behavior, where periods of stasis are interrupted by intermittent sand slides. The sand slides, or avalanches, are caused by a domino effect, in which a single grain of sand pushes one or more other grains and causes them to topple. In turn, those grains of sand may interact with other grains in a chain reaction. Large avalanches, not gradual change, make the link between quantitative and qualitative behavior, and form the basis for emergent phenomena.

If this picture is correct for the real world, then we must accept instability and catastrophes as inevitable in biology, history and economics. Because the outcome is contingent upon specific minor events in the past, we must also abandon any idea of detailed long-term determinism or predictability. In economics, the best we can do, from a selfish point of view, is to shift disasters to our neighbors. Large catastrophic events occur as a consequence of the same dynamics that produces small ordinary everyday events. This observation runs counter to the usual way of thinking about large events, which, as we have seen, looks for specific reasons, for instance, a falling asteroid causing the extinction of dinosaurs, to explain large cataclysmic events. Even though there are many more small events than large ones, most of the changes of the system are associated with the large, catastrophic events. Self-organized criticality can be thought of as the theoretical justification for catastrophism.

3.3. The Discovery of Self-Organized Criticality

In 1987 Chao Tang, Kurt Wiesenfeld, and Per Bak constructed the simple, prototypical model of self-organized criticality, namely, the sandpile model. Calculations on the model showed how a system that obeys simple, benign local rules can organize itself into a poised state that evolves in terms of flashing, intermittent bursts rather than following a smooth path. They did not set out with the intention of studying sandpiles. As with many other discoveries in science, the discovery of sandpile dynamics was accidental. We now describe the events leading to the discovery. In hindsight, things could have been much simpler, but it is interesting to SEE how the actual process involved quite convoluted paths.

3.3.1. Science at Brookhaven

They were working at Brookhaven National Laboratory, an excellent intellectual environment.

Their research in self-organized criticality involved a combination of the physics of equilibrium-critical phenomena involving very many particles and chaos theory for simple dynamical systems which they had been studying. Much of the best physics is at carried out by small groups of imaginative scientists left alone to do whatever they wish to do. Good science is not necessarily expensive science.

3.3.2. Where Does $1/f$ "Noise" Come From?

They became obsessed with the origin of the mysterious phenomenon of $1/f$ "noise" or more appropriately, the $1/f$ "signal" that is emitted by numerous sources on earth and elsewhere in the universe. They had endless discussions in the physics coffee room, the intellectual center of Brookhaven.

Most attempts to explain $1/f$ noise were ad hoc theories for a single system, with no general applicability, which was unsatisfactory. Since the phenomenon appears everywhere, there must be a general, robust explanation. Systems with few degrees of freedom, like the angle and velocity of a single pendulum or other equilibrium systems cannot generally show $1/f$ noise or any other complex behavior, since fine-tuning is always necessary. Thus, one comes to the conclusion that $1/f$ noise would have to be a cooperative phenomenon where the different elements of large systems act together in some concerted way. Indeed, all the sources of $1/f$ noise were such large systems with many parts. For instance, the fluctuations of the water level of the Nile must be related to the landscape and weather pattern of Africa, which can certainly not be reduced to a simple dynamical system.

One thought was that $1/f$ noise could be related to the spatial structure of matter. Systems in space have many degrees of freedom; one or more degrees of freedom is associated with each point in space. The systems had to be *open*, that is, energy had to be supplied from outside, since closed systems in which energy would not be supplied would approach an ordered or disordered equilibrium state without complex behavior. However, at that time there were no known general principles for open systems with many degrees of freedom.

3.3.3. Susan Coppersmith's Dog Model

This was the situation when Susan Coppersmith, a scientist from Bell Laboratories in New Jersey, visited in late 1986. She had called a few days before.

She had some new ideas that she was dying to discuss with someone and asked to give a talk at Brookhaven. She said there was nobody at Bell Labs to talk to. In a collaboration at Bell Labs, she was working on charge density waves in solid systems. Charge density waves (CDWs) can be thought of as a periodic arrangement of electronic charges, interacting with the regular lattice of atoms in a crystal. She had discovered a simple but remarkable effect.

We can think about CDWs in terms of a simple metaphor. The situation is (very)roughly equivalent to a reluctant dog being pulled along a hilly surface with an elastic leash as shown in the figure below.

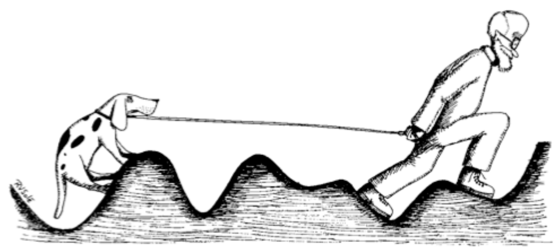


Figure 13: Reluctant Dog Model

At some point the dog will slip, and jump from one bump to the next bump. Because there will still be tension in the string after the jump, the dog will end up at a position near the top of a bump, rather than sliding to the equilibrium position at the bottom of a valley. The dog sits near the top for a while until the tension has built up again to overcome the dog's friction, and the dog will jump again. This can be seen as a trivial example of *punctuated equilibria*, although with no large events or catastrophes.

This is a simple nonequilibrium open system where energy is supplied from the outside by means of the leash. Actually, a CDW can be thought of as a string of particles (dogs), connected with springs, which is pulled across a washboard by means of an external electric field acting as a constant force. They studied the situation where the chain would be pulled for some time, and then allowed to relax, and then pulled again. The upshot of the analysis was that after many pulses, most of the particles, just like the dog, would stay near the top of the potential(hill) between the pulses. Obviously particles sitting near the top are much more unstable than particles at the bottom. It would take only a very small push to upset the balance. They called the resulting state *minimally stable*. The result of the theory could not possibly be more different from the behavior of equilibrium systems, where all the particles would end up near the bottoms of the valleys in the washboard potential.

Indeed, it appeared that it was possible to say something general about open nonequilibrium systems that would distinguish them completely from equilibrium systems. Of course, the resulting configuration has no components of complexity whatsoever, no hints of fractals or $1/f$ noise. But it was the first systematic analysis of large dynamical systems out of equilibrium, once and for all demonstrating the futility of thinking about them in equilibrium terms. New thinking was necessary.

3.3.4. On Coupled Pendulums

The study of such *coupled* systems, where many parts interact with one another, continued. Specifically, they looked at a network of coupled torsion pendulums. The figure below shows a one-dimensional version where the pendulums are connected along a line.

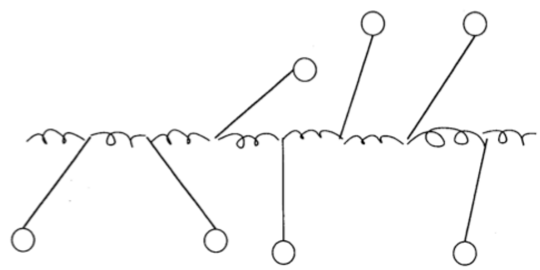


Figure 14: Coupled Pendulum Model

Torsion pendulums can make full rotations around their point of support, not just oscillate around their equilibrium point like a clock pendulum. In contrast to previous studies of chaotic behavior in single pendulums, they studied the limit where there were many coupled pendulums. On the computer, they put many pendulums on a regular two-dimensional grid. Neighbor pendulums were connected with springs like those you find in a clock. Energy was pumped into the system by selecting one pendulum randomly and pushing it so that it would make one revolution. Because of the network of connected pendulums, this push would put pressure on the neighbor pendulums by winding up the spring, perhaps forcing one or more of those pendulums also to rotate. The springs were chosen to be sloppy; it would take several rotations of one pendulum before the force on the neighboring pendulums would be strong enough to cause a rotation. Their system was also *dissipative*. If pushed once and left alone, a pendulum would make only a single revolution and then stop because of friction. One might think of the pendulums as rotating in syrup.

To simplify the calculations they used a representation where they would keep track only of the number of revolutions, called the *winding numbers*, the pendulums would perform; they wouldn't bother with the exact patterns of rotation. The tension of the springs would depend on the difference in the full number of rotations between neighbor springs. Because of the connecting springs, the winding numbers of neighbor torsion springs cannot differ too much. The dynamics involved only integer numbers, not continuous real numbers; this simplification greatly speeded up the computer calculations.

3.3.5. The Philosophy of Using Simple Models: On Spherical Cows

Why would we simulate a simple system of oversimplified pendulums instead of a realistic model of something going on in nature?

Why don't we do calculations on the real thing?

The answer is simple: there is no such thing as doing calculations on the real thing. One cannot put a frog into the computer and simulate it in order to study biology. Whether we are calculating the orbit of Mercury circling the sun, the quantum mechanics of some molecule, the weather, or whatever, the computer is only making calculations on some mathematical abstraction originating in the head of the scientist. We make pictures of the world. Some pictures are more realistic than others. Sometimes we feel that our modeling of the world is so good that we are seduced into believing that our computer contains a copy of the real world, so that real experiments or observations are unnecessary. Obviously, if we want our calculation to produce accurate quantitative results, such as on the weather, or accurate predictions, such as of the rate of global warming, the demands are much more stringent than when only qualitative behavior is asked for. This is true notably for computer modeling but also for pen-and-paper analytical calculations like those performed by the geneticists in the 1930s. The absence of computers put even more severe limitations on the type of calculations that could be done. When scientists in the past made theories of evolution, for example, they made theories of simple models of evolution. Instead of calculating the probabilities of reproduction and survival in the real world, all of this information might be condensed into a single abstract number called *fitness*, which would enter the calculation. We are always dealing with a model of the system.

The large dynamical systems that we are interested in, like the crust of the earth, are so complicated that we cannot hope to make accurate enough calculations to predict what will happen next. We would have to construct a full-sized model of California in order to predict where and when the next large earthquake would take place. This is clearly a losing strategy!

The physicist's agenda is to understand the fundamental principles of the phenomenon under investigation. She tries to avoid the specific details, such as the next earthquake in California. Before asking how much we have to add to our description in order to make it reproduce known facts accurately we ask how much we can throw out without losing the essential qualitative features. Our strategy is to strip the problem of all the flesh until we are left with the naked backbone and no further reduction is possible. We try to discard variables that we deem irrelevant. In this process, we are guided by intuition. In the final analysis, the quality of the model relies on its ability to reproduce the behavior of what it is modeling!

Thus, how would we physicists make a suitable model of, say, biological evolution? The biologist might argue that since there is sexual reproduction in nature, a theory of evolution must necessarily include sex. The physicist would argue that there was biology before there was sex, so we don't have to deal with that. The biologist might point out that there are organisms with many cells, so we must explain how multicellular organisms developed. The physicist argues that there are also single-cell organisms, so we can throw multicellular organisms out! The biologist argues that most life is based on DNA, so that should be understood. The physicist emphasizes that there is simpler life based on RNA, so we don't have to deal with DNA. She might even argue that there must have been a simpler reproductive chemistry before RNA, so that we don't have to deal with that either, and so on. The trick is to stop the process before we throw out the baby with the bathwater. Once we have identified the basic mechanisms from the simple models, we leave it to others to put more meat on the skeleton, to add more and more specific details, if one so wishes, to check whether or not more details modify the results.

In this particular study, the underlying philosophy is that general features, such as the appearance of large catastrophes and fractal structure, cannot be sensitive to the particular details. This is the principle of *universality.* We hope that important features of large-scale phenomena are shared between seemingly disparate kinds of systems, such as a network of interacting economics agents, or the interactions between various parts of the crust of the earth. This hope is nourished by the observation of the ubiquitous empirical patterns in nature – fractals, $1/f$ noise, and scaling of large events among them - discussed earlier. Since these phenomena appear everywhere, they cannot depend on any specific detail whatsoever.

Universality is the theorist's dream come true. If the physics of a large class of problems is the same, this gives her the option of selecting the simplest possible system belonging to that class for detailed study. One hopes that a system is so simple that it can be studied effectively on a computer, or maybe laws of nature can be derived by mathematical analysis, with pen and paper, from that stripped-down description or model. Simple models also serve to strengthen our intuition of what goes on in the real world by providing simple metaphoric pictures.

The concept of universality has served us well in the past. It has scored a couple of spectacular successes in recent years. Wilson's theory of phase transitions for which he was awarded the Nobel Prize proved its universality by demonstrating that the basic properties of a system near a phase transition had nothing to do with the microscopic details of the problem. It doesn't matter whether we are dealing with a liquid-gas transition, a structural transition where a crystal deforms, or a magnetic transition where the little magnets or spins start pointing in the same direction. Wilson's calculations were based on the Ising model, the simplest possible model of a phase transition, and they agreed with experiments on much more complicated real systems (we will look at the Ising model later).

Similarly Feigenbaum's studies of the transition to chaos was based on a *map* that can only be seen as a caricature of a real predator-prey ecological system. I don't think that either Feigenbaum or May ever claimed that the map describes anything in real biology. Feigenbaum argued that near the transition to chaos the dynamics had to be the same for all systems undergoing a transition to chaos through an infinite sequence of *bifurcations* at each of which the periodicity would be doubled. The contrast between the simplicity of the model, and the depth of the resulting behavior is astonishing. Although Feigenbaum's theory was based on a grossly oversimplified model, experiments on many kinds of complicated systems have beautifully confirmed it. The phenomenon is quite universal.

Thus, the scientific process is as follows: We describe a class of phenomenon in nature by a simple mathematical model, such as the Feigenbaum map. We analyze the model either by analytical means, with pen and paper, or by numerical simulations. There is no fundamental difference between these two approaches; they both serve to elucidate the consequences (predictions) of the simple model. Often, however, simulations are easier than mathematical analysis and serve to give us a quick look at the consequences of our model before starting analytical considerations. The beauty of the model can be measured as the range between its own simplicity and the complexity of the phenomena that it describes, that is, by the degree to which it has allowed us to condense our description of the real world.

Without the concept of universality we would be in bad shape. There would be no fundamental *emergent* laws of nature to discover, only a big mess. Of course, we have to demonstrate that our models are robust, or insensitive to changes, in order to justify our original intuition. If unfortunately, it turns out that they are not, we are back to the messy situation where detailed models of the highly complex phenomena is the only possible approach - the so-called weatherman's approach.

The obsession among physicists to construct simplified models is well illustrated by the story about the theoretical physicist asked to help Cornell Agricultural School raise cows that would produce more milk. For a long time, nobody heard from her, but eventually she emerged from hiding, in a very excited state. *"I now have figured it all out"*, she says, and proceeds to the blackboard with a piece of chalk and draws a circle. *"Consider a spherical cow (simple model): .*

. . ." Here, unfortunately it appears that universality does not apply. We have to deal with the real cow!

3.3.6. The Pendulums Become Critical

This is why one does computer simulations on networks of coupled pendulums and not realistic models of earthquakes or whatever. If you have difficulty visualizing the system of coupled pendulums, so much the better - it will only serve to illustrate the value of having good metaphors. The pendulums are not good enough metaphors. It is too difficult to grasp what is going on; things are still too messy!

If the pendulums were pushed in random directions, one at a time, nothing interesting would happen. Most of the pendulums would be near the equilibrium position. However, they realized that if they always pushed the pendulums in the same direction, say clockwise, there would be an increased tendency for the pendulums to affect each other. The springs connecting the pendulums would slowly be wound up and store energy. As the process of pushing a single pendulum at a time continued, more and more pendulums would stay near the upward position rather than the downward position. Because of the increased instability of the pendulums, there would be chain reactions caused by a domino effect. Pushing a single pendulum might cause others to rotate. How far would this domino process continue? Obviously if we started from the position where all the springs were relaxed, there would be no way that pushing a single pendulum once would cause other pendulums to rotate. But suppose the process of pumping up the pendulums went on for a very long time. What would set the limit of the chain reaction? What would be the natural scale of the disturbances? How many pendulums could be turned by a single push?

The idea popped up that maybe there was no limit whatsoever! It appeared that there was essentially nothing in the system that could possibly define a limit! Maybe, even if the system was dissipating with lots of friction, the constant energy supply from pushing the pendulums might eventually drive the system to a state where once a single pendulum started rotating somewhere, there would be enough stored energy to allow a chain reaction to go on forever, limited only by the large total number of pendulums?

This was programmed into a computer. The model was a small system with pendulums on a grid of size 50 by 50, a total of 2,500 pendulums. Each pendulum was connected with its four neighbors, in the up, down, left, and right directions. Starting from having all pendulums in the equilibrium configuration, one arbitrary pendulum would be wound up by one revolution. This would put more pressure on the neighbors. Then another pendulum would be chosen, and so on. For a while there were only single rotations, but at some point one spring would be wound up enough to trigger another pendulum to rotate. Continuing further, at some point there would be enough energy stored in the springs that there would be large chain reactions, where one pendulum would trigger the next by a domino effect. This process is called an avalanche. The avalanches would become bigger and bigger. Eventually after thousands of events, they would grow no further. As the simulation continued, there would be a stream of avalanches, some small, some intermediate, and a few big ones.

If one measures how many avalanches there were of each size, just like the earthquake scientists had measured how many earthquakes there were of each magnitude. The size of an avalanche was measured as the total number of rotations following a single kick. There were many more small ones than large ones. The figure below shows the results.

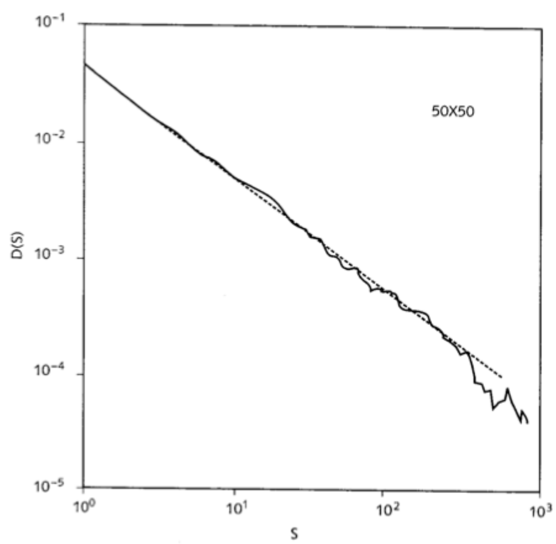


Figure 15: Coupled Pendulum Data Plot

The x–axis shows the size of the avalanches. The y–axis shows how many avalanches there were of that size. It is a log-log plot as earlier plots.

Their data fall approximately on a straight line, which indicates that the number of avalanches of size s is given by the simple power law

$$N(s) = As^{-\tau}$$

where the exponent τ defined as the slope of the curve, is approximately equal to 1.

The pendulums obeyed the same Gutenberg-Richter power law as for earthquakes! At the lower end, the straight line is limited by the fact that no avalanche can be smaller than one pendulum rotation. At the upper end, there is a cutoff because no avalanche can be bigger than one with all the pendulums rotating. The scattering of points around the straight line are statistical fluctuations, just like in real experiments. Some points are above the line, some below. If we let the simulation continue longer and longer, these fluctuations become smaller and smaller, just like the ratio of sixes you get when you throw a dice will converge toward 1/6 as the number of throws increases.

The system had become *critical*! There were avalanches of all sizes just as there were clusters of all sizes at the *critical* point for equilibrium phase transitions. But no tuning was involved. We had just blindly pushed the pendulums. There is no temperature to regulate, no λ parameter to change. The simple behavior of the individual elements following their own simple local rules had conspired to create a unique, delicately balanced, poised, global situation in which the motion of any given element might affect any other element in the system. The local rule was simply a specification of the total number, n, of revolutions the four neighbors should perform, to induce a single revolution of a given pendulum. The system had self-organized into the critical point without any external organizing force. Self-organized criticality (SOC) had been discovered. It was as if some *invisible hand* had regulated the collection of pendulums precisely to the point where avalanches of all sizes could occur. The pendulums could communicate throughout the system.

Once the poised state has been reached, the *criticality* is similar to that of a nuclear chain reaction. Suppose you have a collection of radioactive atoms emitting neutrons. Some of those neutrons might become absorbed by other atoms, causing them to emit neutrons of their own. A single neutron leads to an avalanche. If the concentration of fissionable atoms is low, the chain reaction will die out very soon. If the concentration is high, there will be a nuclear explosion similar to that in an atomic bomb. At a unique critical concentration there will be avalanches of all sizes, all of which will eventually stop. Again, one has to *tune* the nuclear chain reaction by choosing precisely the correct amounter radioactive material to make it critical. In nuclear reactors this tuning is very important and is carried out by inserting neutron-absorbing graphite rods. In general the reactor is not critical. There is absolutely no self-organization involved in a nuclear chain reaction, so in this all-important aspect the situation is entirely different.

We therefore find that criticality and therefore complexity can and will emerge *for free* without any watchmaker tuning the world.

3.4. The Sandpile Paradigm

The discovery of the coupled-pendulums case of self-organized criticality was very important. An open dissipating system had naturally organized itself into a *critical* scale-free state with avalanches of all sizes and all durations. The statistics of the avalanches follow the Gutenberg-Richter power law. There were small events and large events following the same laws. A simple model for complexity in nature had been discovered.

The variability that we observe around us might reflect parts of a universe operating at the self-organized critical state. While there had been indications for some time that complexity was associated with criticality, no robust mechanism for achieving the critical state had been proposed, nor had one been demonstrated by actual calculation on a real mathematical model. Of course, this was only the beginning. For instance, one still had to show that the activity has an $1/f$ like signal, and that the resulting organization had a fractal geometrical structure. This was only the beginning!

Maybe the ultimate understanding of scientific topics is measured in terms of our ability to generate metaphoric pictures of what is going on. The physics of the messy system of pendulums is far from transparent. Their intuition about the system was poor. It turns out that there was a simpler picture that could be applied to their self-organized critical dynamics. By a change of language the rotating pendulums could be describing as toppling grains of sand in a pile of sand. Instead of counting revolutions of pendulums, they count toppling grains in a sandpile. Although the mathematical formulation was exactly the same for the sand model as for the pendulum model, the sand picture led to a vastly improved intuitive understanding of the phenomenon. Sandpiles are part of our everyday experience, as any child who has been playing on the beach knows. Rotating coupled pendulums are not.

Before discussing the mathematical formulation of their model, let us recall the sandpile experiment mentioned earlier. Consider a flat table, onto which sand is added slowly one grain at a time. The grains are added at random positions. The flat state represents the general equilibrium state; this state has the lowest energy, since obviously we would have to add energy to rearrange the sand to form heaps of any shape. Because the grains tend to get stuck due to static friction, the landscape formed by the sand will not automatically revert to the ground state when we stop adding sand(as would be the case for a liquid).

Initially the grains of sand will stay more or less where they land. As we continue to add more sand, small sand slides or avalanches occur. The grain may land on top of other grains and topple to a lower level. This may in turn cause other grains to topple. The addition of a single grain of sand can cause a local disturbance, but nothing dramatic happens to the pile. In particular, events in one part of the pile do not affect sand grains in more distant parts of the pile. There is no global communication within the pile at this stage, just many individual grains of sand.

As the average height increases, an avalanche from a single grain is more likely to cause other grains to topple. Eventually the average height reaches a certain value and cannot increase any further, because the amount of sand added is balanced on average by the amount of sand leaving the pile by falling off the edges. This is called a stationary state, since the average amount of sand and the average height are constant in time. It is clear that to have this average balance between the sand added to the pile, and the sand leaving along the edges, there must be communication throughout the entire system. There will occasionally be avalanches that span the whole pile. This is the self-organized critical(SOC) state.

The addition of grains of sand has transformed the system from a state in which the individual grains follow their own local dynamics to a critical state where the emergent dynamics are global. In the stationary SOC state, there is one complex system, the sandpile, with its own emergent dynamics. The emergence of the sandpile could not have been anticipated from the properties of the individual grains.

The sandpile is an open dynamical system, since sand is added from outside. it has many degrees of freedom, or grains of sand. A grain of sand landing on the pile represents potential energy measured as the height of the grain above the table. When the grain topples, this energy is transformed into kinetic energy. When the toppling grain comes to rest, the kinetic energy is dissipated, that is, transformed into heat in the pile. There is an energy flow through the system. The critical state can be maintained only because of energy in the form of new sand being supplied from the outside.

The critical state must be robust with respect to modifications. This is of crucial importance for the concept of self-organized criticality to have any chance of describing the real world; in fact, this is the whole idea. Suppose that after the same system has reached its critical stationary state we suddenly start dropping wet sand instead of dry sand. Wet sand has greater friction than dry sand. Therefore, for a while the avalanches would be smaller and local. Less material will leave the system since the small avalanches cannot reach the edge of the table. The average height of the pile becomes larger. This, in turn, will cause the avalanches to grow, on average. Eventually we will be back to the critical state with system-wide avalanches. The average height at this state will be higher than the original ones. Similarly, if we dry the sand, the pile will sink to a more shallow shape by temporarily shedding larger avalanches. If we try to prevent avalanches by putting local barriers, *snow* screens, here and there, this would have a similar effect: for a while the avalanches will be smaller, but eventually the slope will become steep enough to overcome the barriers, by forcing more sand to flow somewhere else. The physical appearance of the pile changes, but the dynamics remain critical. The pile bounces back to a critical state when we try to force it away from the critical state.

3.4.1. The Sandpile Model

We have defined the physics, but so far everything is simply a product of imagination, mixed with some intuition from actual experience. How do we go from here to make a representation, a model, that reproduces these features? The sandpile model that Kurt, Chao, and Bak studied is easy to define and simulate on the computer.

The table where the sand is dropped is represented by a two-dimensional grid. At each square of the grid, with coordinates (x, y), we assign a number $Z(x, y)$, which represents the number of grains present at that square. For a table of size $L = 100$, the coordinates x and y are between 1 and 100. The total number of sites is $L \times L$. We are using *theoretical physicist's sand*, with ideal grains that are regular cubes of size 1, which can be stacked neatly on top of one another, not the irregular complicated ones that you find on the beach.

The addition of a grain of sand to a square of the grid is carried out by choosing one site randomly and increasing the that site by 1: height Z at that site by 1:

$$Z(x, y) \to Z(x, y) + 1$$

This process is repeated again and again. To have some interesting dynamics, we apply a rule that allows a grain of sand to shift from one square to another, a *toppling rule*. Whenever the height Z exceeds a critical value Z_{cr} that may arbitrarily be set, say to 3, one grain of sand is sent to each of the four neighbors. Thus, when Z reaches 4, the height at that site decreases by 4 units,

$$Z(x, y) \to Z(x, y) - 4$$

for $Z(x, y) > X_{cr}$, and the heights Z at the four neighbor sites go up by 1 unit,

$$\begin{aligned} Z(x \pm 1, y) &\to Z(x \pm 1, y) + 1 \\ Z(x, y \pm 1) &\to Z(x, y \pm 1) + 1 \end{aligned}$$

The toppling process is illustrated in the figure below:

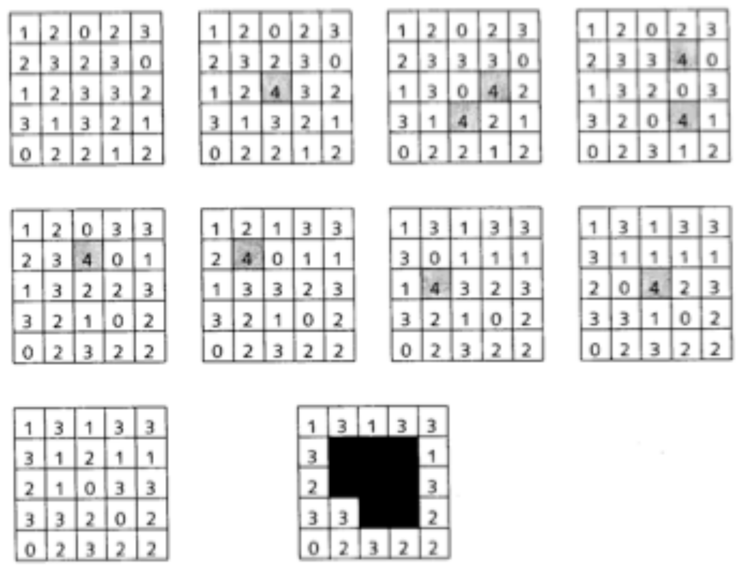


Figure 16: Toppling Process

If the unstable site happens to be at the boundary where x or y, is 1 or 100, the grains of sand simply leave the system; they fall off the edge of the table and we are not concerned with them any longer.

These few simple equations completely define the model. No mathematics more complicated than adding and subtracting numbers between 1 and 4 is needed. Nevertheless, the consequences of these rules are horrifyingly complicated, and can certainly not be deduced from a simple inspection of the equations, which represent the local dynamics of each of the sand grains. We follow the general procedure outlined earlier, and start studying the model by direct computer simulations.

Class Demo - MATLAB sand.m and Avalanche Model

This physicists' sandpile is a gross oversimplification of what really happens. First, real grains have different sizes and shapes. The instabilities in a real sandpile occur not only at the surface but also through the formation of cracks in the bulk. The toppling depends on how the individual grains lock together. Once a grain is falling, its motion is determined by the gravity field, which accelerates the grain, and the interaction with other grains, which tends to decelerate the motion. Stopping the motion depends on many factors, such as the shape of the grains it bumps into and its velocity at that point, and not just the height or slope of the pile at the neighbor points. One could go on and on with objections like this. One quickly realizes that it is a losing strategy to make a realistic model of the sandpile, which at first glance might have seemed a reasonably simple object. So why is the model acceptable at all? Its validity is based on the intuition that the model contains the essential physics, namely that grains interact and may cause each other to topple. That this is indeed correct can be justified (or falsified) only a posteriori by comparing with experiments.

Second, we are not particularly interested in sand. We hope that the sand dynamics that we observe are general enough that they can be applied to a much larger class of phenomena. Peter Grassberger, a computational physicist, came up with an amusing representation of the model. He asks us to think about a large office where bureaucrats sit at tables organized in rows as shown in the figure below. Every now and then a piece of paper from the outside enters the desk of a random bureaucrat. He does not deal with it until he finds too many pieces of paper on his desk. He then sends one piece of paper to each of his four neighbors.

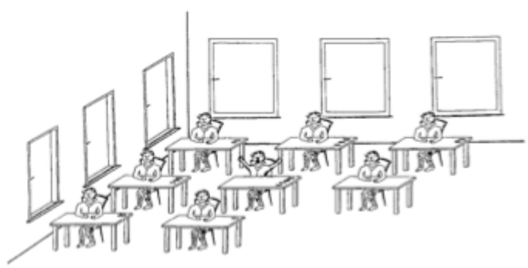


Figure 17: Bureaucrat Model Process

Everybody follows this rule, except those who are placed along the walls, who simply throw the paper out the window. Jumping forward a little bit, we shall see that a single piece of paper entering the office can lead to a bureaucratic catastrophe where millions of transfers of paper take place (if the office is large enough!). Each bureaucrat may perform many transactions within such an avalanche.

In the beginning of the process, where all the heights are low, there are no unstable sites. All sites have Z less than 3, so the sand stays precisely where it happens to land. After many steps of adding a single grain to a square of the grid, the height somewhere must necessarily exceed 3, and we have the first toppling event. It is unlikely that the height at any of the four neighbor squares exceeds 3 this soon, so there will be no further activity of toppling grains. As the process continues, it becomes more likely that at least one of neighbors will reach its critical height, so the first event induces a second event. One toppling event leads next, like falling dominos. As more sand is added, there will be bigger and bigger landslides, or avalanches, although there will still also be small ones.

The above figure shows a sequence of toppling events in a very small system. The numbers in the squares represent the heights. A grain of sand lands on a site with height 3, causing that site to topple. Two of the neighbor sites had $Z = 3$, so those two sites topple next, at the second time step, sending a total of eight grains to their neighbors, including two grains to the original site. Eventually the system comes to rest. We notice that there were precisely nine couplings, so that avalanche had size $s = 9$. We also monitor the total duration, that is, the number of update steps, $t = 7$, of that avalanche.

Eventually the entire sandpile enters into a stationary state where the average height of all sites does not increase further. The average height is somewhere between 2 and 3. The pile can never reach the highest possible stable state, where all the heights are 3, since long before that simple state is reached the pile has broken down due to large avalanches. We can monitor this by counting the total number of grains in the pile at all times. In the stationary state, most avalanches are small and do not reach the edge, so they cause the pile to grow. This is precisely compensated by fewer, and generally larger, avalanches that reach the edge and cause many grains of sand to leave the pile.

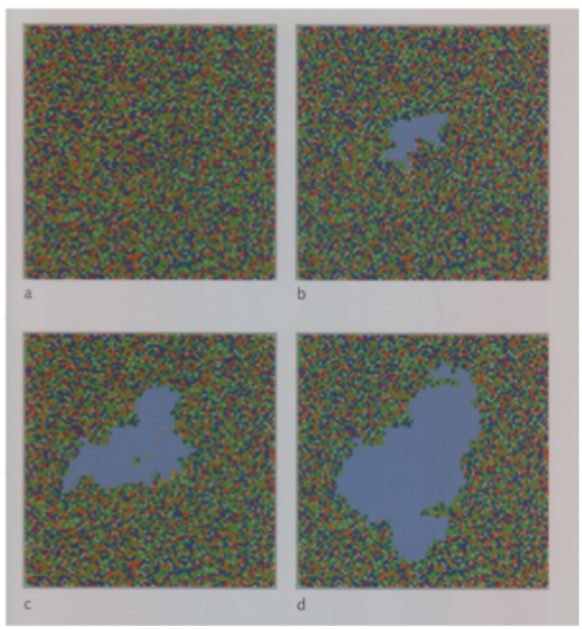


Figure 18: Computer Simulation

Photo (a) above shows a configuration in the stationary state, just after the completion of an avalanche for a very large pile. Here, instead of the numbers, a color code is used. Red is $Z = 3$, blue is $Z = 2$, green is $Z = 1$, and gray is $Z = 0$. The picture looks like a big mess, with no organized structure whatsoever. But nothing can be further from the truth. The pile has organized itself into a highly orchestrated, susceptible state through the process of repeatedly adding sand and having avalanches travel through the pile again and again.

We can realize the intricate properties of the configuration of sand, not by directly inspecting the colors but by dropping one more grain of sand.

If a *red* site is hit, this triggers an avalanche. Photo (b) shows what has happened after a few time steps. The light blue area represents all the grains that have fallen. The yellow and white spots represent active sites chat are about to topple, where $Z > 3$. The next picture shows the situation a little later, where the avalanche has covered a larger area. Eventually the avalanche comes to a stop after approximately half the sites in the pile have toppled at least once. Most sites have actually toppled several times. The particular configuration at the end of the avalanche is very different than the one we started out with.

This was a very big avalanche. More often than not the avalanches are smaller. We now follow the same procedures as the geophysicists when making statistics of earthquakes. By successively adding sand after each avalanche has stopped we generate a large series of avalanches, say, 1 million avalanches.

We then make a *synthetic* earthquake catalog by counting how many avalanches there are of each size. The *magnitude* of avalanches is the logarithm of the size of the avalanche. As usual, we plot the logarithm of the number of avalanches of a given magnitude versus that magnitude. The number of avalanches of each size for a system of linear size 50 is plotted in the figure below, which shows data from their first sandpile simulation. The straight line indicates that the avalanches follow the Gutenberg-Richter power law, just like the real earthquakes, although the slopes are different.

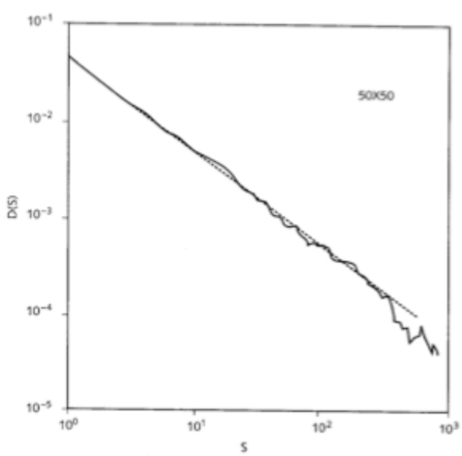


Figure 19: Simulation Data Plot

We do not have to wait millions of years to generate many earthquakes, so our statistical fluctuations are smaller than those for earthquakes, where we must deal with the much smaller number that nature has generated for us. The exponent τ of the power law, that is, the slope of the curve in the figure, was measured to be approximately 1.1. The power law indicates that the stationary state is critical. We conclude that the pile has self-organized into a critical state.

One can show, by analyzing the geometry of the sandpile, that the profile of the sandpile is a fractal, like Norway's coast. The avalanches have carved out fractal structures in the pile. The power law also indicates that the distribution of avalanches follows Zipf 's law. Instead of plotting how many avalanches there are of each size, we could equally well plot how large the biggest avalanche was (the avalanche of *rank* 1), how large the second biggest avalanche, of rank 2 was, how large the tenth biggest avalanche was, and so on, precisely the same way that Zipf plotted the ranking of cities. This is just another way of representing the information from the original power law. The straight line shows that the sandpile dynamics obey Zipf's law.

Our simple model cannot by any stretch of the imagination represent the formation of real cities in a human society or the process by which James Joyce wrote Ulysses, where we are dealing with humans, not sand grains. Nevertheless, one might speculate that Zipf's law indicates that the world population has organized itself into a critical state, where cities are formed by avalanches of human migrations.

They had to check that the criticality is robust with respect to modifications of the model. The power law should prevail no matter how they modify the sandpile. They tried a long sequence of different versions. Instead of having the same critical height equal to 3, a version where the critical height varies from site to site was tried. Snow screens were simulated by preventing sand from falling between certain neighbor sites, selected randomly, by having the sand arranged on a triangular grid instead of the square grid. They also tried adding grains of different sizes, that is, they increased Z not by unity when grains are falling but by some random number between 0 and 1. They massaged the model so that a random amount of sand topples when the site becomes unstable. They selected the sites to which the sand would topple in a random way, and not to the nearest neighbors. In all cases, the pile organized itself into a critical state with avalanches of all sizes. The criticality was unavoidable.

One might speculate that the criticality is caused by the randomness of the way that the system is driven - we add new grains at random positions. In fact this is not important at all. We can drive the system in a deterministic way with no randomness whatsoever. The phenomenon of SOC is essentially a deterministic phenomenon, just like the chaos studied by Feigenbaum. The fact that the randomness of adding sand does not affect the power law indicates that the randomness is irrelevant for the complex behavior we are observing.

This fact is important to realize when studying much more complicated systems. Economics deals with the more or less random behavior of many agents, whose minds were certainly not made up at the beginning of history. Nevertheless, this randomness does not preclude the system's evolving to the delicate critical state, with well-defined statistical properties. This is a fascinating point that is difficult to grasp.

How can a system evolve to an organized state despite all the obvious randomness in the real world?

How can the particular configuration be contingent on minor details, but the criticality totally robust?

3.4.2. Life in the Sandpile World

The dynamics of the nonequilibrium critical state could hardly be more different than the quiet dynamics of a flat beach. How would a local observer experience the situation? During the transient stage, when the sandpile was relatively shallow, his experience would be monotonous. Every now and then there would be a small disturbance passing by, when a few grains topple in the neighborhood. If we drop a single grain of sand at one place instead of another, this causes only a small local change in the configuration. There is no means by which the disturbance can spread system-wide. The response to small perturbations is small. In a noncritical world nothing dramatic ever happens. It is easy to be a weather (sand) forecaster in the flatland of a non-critical system. Not only can he predict what will happen, but he can also understand it. The action at some place does not depend on events happening long before at faraway places. *Contingency* is irrelevant.

Once the pile has reached the stationary critical state, though, the situation is entirely different. A single grain of sand might cause an avalanche involving the entire pile. A small change in the configuration might cause what would otherwise be an insignificant event to become a catastrophe. The sand forecaster can still make short time predictions by carefully identifying the rules and monitoring his local environment. If he sees an avalanche coming, he can predict when it will hit with some degree of accuracy. However, he cannot predict when a large event will occur, since this is contingent on very minor details of the configuration of the entire sandpile. The massive contingency in the real world could be understood as a consequence of self-organized criticality.

The sand forecaster's situation is similar to that of the weatherman in our complex world: by experience and data collection he can make *weather* forecasts of local grain activity, but this gives him little insight into the *climate*, represented by the statistical properties of many sand slides, such as their size and frequency.

Most of the time things are completely calm around him, and it might appear to him that he is actually living in a stable equilibrium world, where nature is in balance. However, every now and then his quiet life is interrupted by a punctuation - a burst of activity where grains of sand keep tumbling around him. There will be bursts of all sizes. He might be tempted to believe that he is dealing with a local phenomenon since he can relate the activity that he observes to the dynamical rules of the sand toppling around him. But he is not; the local punctuation that he observes is an integrated part of a global cooperative phenomenon.

Parts of the critical system cannot be understood in isolation. The dynamics observed locally reflect the fact that it is part of an entire sandpile. If you were sitting on a flat beach instead of a sandpile, the rules that govern the sand are precisely the same, following the same laws of physics, but history has changed things. The sand is the same but the dynamics are different. The ability of the sand to evolve slowly is associated with its capability of recording history. Sand may contain memory; one can write letters in the sand that can be read a long time later. This cannot happen in an equilibrium system such as a dish of water.

In the critical state, the sandpile is the functional unit, not the single grains of sand. No reductionist approach makes sense. The local units exist in their actual form, characterized for instance by the local slope, only because they are a part of a whole. Studying the individual grains under the microscope doesn't give a clue as to what is going on in the whole sandpile. Nothing in the individual grain of sand suggests the emergent properties of the pile.

The sandpile goes from one configuration to another, not gradually but by means of catastrophic avalanches. Because of the power law statistics, most of the topplings are associated with the large avalanches. The much more frequent small avalanches do not add up to much. Evolution of the sandpile takes place in terms of revolutions, as in Karl Marx's view of history. Things happen by revolutions, not gradually, precisely because dynamical systems are poised at the critical state. Self-organized criticality is nature's way of making enormous transformations over short time scales.

In hindsight one can trace the history of a specific large avalanche that occurred. Sand slides can be described in a narrative language, using the methods of history rather than those of physics. The story that the sand forecaster would tell us goes something like this:

> Yesterday morning at 7 A.M., a grain of sand landed on site A, with coordinates (5,13). This caused a toppling to site B at coordinates (5,12). Since the grain of sand resting at B was already near the limit of stability this caused further topplings to sites C, D, and E. We have carefully monitored all subsequent topplings, which can easily be explained and

> understood from the known laws of sand dynamics, as expressed in the simple equations. Clearly, we could have prevented this massive catastrophe by removing a grain of sand at the initial triggering site. Everything is understood.

However, this is a flawed line of thinking for two reasons. First, the fact that this particular event led to a catastrophe depended on the very details of the structure of the pile at that particular time. To predict the event, one would have to measure everything everywhere with absolute accuracy, which is impossible. Then one would have to perform an accurate computation based on this information, which is equally impossible. For earthquakes, we would have to know the detailed fault structure and the forces that were acting on those faults everywhere in a very large region, like California. Second, even if we were able to identify and remove the triggering grain, there would sooner or later be another catastrophe, originating somewhere else, perhaps with equally devastating consequences.

But most importantly, the *historical account does not provide much insight into what is going on, despite the fact that each step follows logically from the previous step.* The general patterns that are observed even locally, including the existence of catastrophic events, reflect the fact that the pile had evolved into a critical state during its entire evolutionary history which took place on a much longer time scale than the period of observation. The forecaster does not understand why the arrangement of grains happened to be precisely such that it could accommodate a large avalanche. Why couldn't all avalanches be small?

There is not much that an individual can do to protect himself from these disasters. Even if he is able to modify his neighborhood by flattening the pile around him, he might nevertheless be swept away by avalanches from far away through no fault of his own. Fate plays a decisive role for the sandpile inhabitant. In contrast, the observer on the flat noncritical pile can prevent the small disasters by simple local measures, since he needs information only about his neighborhood in order to make predictions, assuming that he has information on the arrival of grains to the pile. It is the criticality that makes life complicated for him.

The sandpile metaphor has reached well beyond the world of physicists thinking about complex phenomena; it contains everything - cooperative behavior of many parts, punctuated equilibrium, contingency, unpredictability, fate. It is a new way of viewing the world.

3.4.3. Can We Calculate the Power Laws with Pen and Paper?

The sandpile model is utterly simple to describe. It takes only a couple of lines of text to define the model completely. Why do we have to go through the computer simulation? The computer calculation does not prove anything in the mathematical sense. Can't we make a simple pen-and-paper calculation that will tell us what will happen without the simulation? For instance, can we calculate the exponent τ for the distribution of avalanches? The model is so simple and transparent that one would expect to be able to calculate everything. For other complicated phenomena, like the transition to chaos, or phase transitions in equilibrium systems, scientists like Feigenbaum and Wilson were eventually able to create beautiful analytical theories providing deep insight into the origins.

Surprisingly we cannot! Some of the best brains in mathematical physics have been working on the problem, including Mitch Feigenbaum and Leo Kadanoff of the University of Chicago, and Itamar Procaccia of the Weizmann Institute in israel. Together with a couple of very bright graduate students they considered a model that is even simpler than this one: the grains were arranged in a one-dimensional pile where sand was stacked on a line, not a two-dimensional plane. The model self-organizes to the critical point, but no analytical results could be derived. For instance, they were unable to prove that the avalanches follow a power law.

The mathematics is prohibitively difficult. But how can it be otherwise? We are dealing with the most complex phenomena in nature, involving a slow buildup of information through a long history; why should we necessarily expect a simple mathematical formula to describe this state?

The model is simple, but nevertheless too difficult for theoretical physicists and mathematicians to analyze efficiently. At least so far no one has been able to deal with it satisfactorily.

Later we shall see that for some other models we can achieve a good deal of analytical insight. We shall also see that there are other models that describe surface growth, traffic, and biological evolution, where pen-and-paper theories, or analytical theories as we call them, can be formulated.

3.5. Real Sandpiles and Landscape Formation

Experiments on sandpiles can be viewed as the first test of self-organized criticality. If the theory that large dynamic systems organize themselves to a critical state cannot even explain sandpiles, then what can it explain? The abstract model grossly oversimplifies real sand, but one can still hope that experiments live up to predictions. However, nature has no obligation to obey our ideas; our intuition could be entirely wrong. Theory has to be confronted eventually with real-world observations, so we study sandpiles and we ask, Do they or don't they self-organize to the critical state?

3.5.1. Norwegian Rice Piles

The most careful experiment was performed by a group at the University of Oslo, Norway. They created the ultimate sandpile experiment. They studied grains of rice, not sand. In principle, it should not matter very much what kind of material is used. The details should not be important. Grains of rice have a convenient size that allows for a visual study of the motion of individual grains. The Norwegian group first went to the local supermarket to buy long grain rice, which have more friction than sand and do not keep rolling. They are more likely to get stuck again once they start sliding. The experiment was designed to be similar to the computer models exhibiting self-organized criticality so it was important to monitor the bulk avalanches and not just the rice falling off the edges. The rice pile was confined to the space between two glass plates, through which the dynamics of the pile could be observed, either directly or with a video recorder. Rice was slowly fed into the gap at the upper corner at a slow rate of twenty grains per minute. Experiments were performed at various spacings between the plates and at various slow feeding rates. Long runs were important in order to get good statistics, particularly for the very few large avalanches, and the large system size was important in order to have a wide range of avalanche sizes.

Motion of grains was monitored with a CCD video camera. Frames were taken every fifteen seconds, and digitized signal was sent to a computer, identifying the positions of all the rice grains. The pile grew until it reached a stationary state. Once the stationary state was reached, the camera and the computer started monitoring the motion of rice grains. The figure below shows a propagating avalanche in the stationary state. The profiles of the rice pile at two consecutive measurements are shown below.

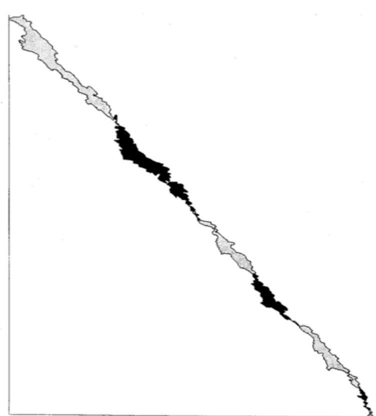


Figure 20: Experimental Rice Profiles

The gray area shows the rice that was present at the first measurement, and not at the second, i.e., the amount of rice that had fallen. Conversely, the black areas show where the rice went. Those areas were not filled at the first measurement, only after the second. Thus, an avalanche had occurred in the fifteen-second interval between the two measurements.

The size of an avalanche was defined as the total amount of downward motion of grains between two successive frames, that is the number of grains falling weighted by the distance they fell. The size of the avalanche measured this way is equal to the energy lost, or dissipated into heat.

In the stationary state, the rice grains get stuck in intricate arrangements, where they lock into each other, allowing for steep slopes, even with overhangs as shown in the photo below:

Figure 21: Actual Rice Pile

An analysis of the surface profile shows that it is a fractal structure just like the coast of Norway, with bumps and other features of all sizes. The figure below shows a sequence of avalanches that occurred in a period of 350 minutes during one run.

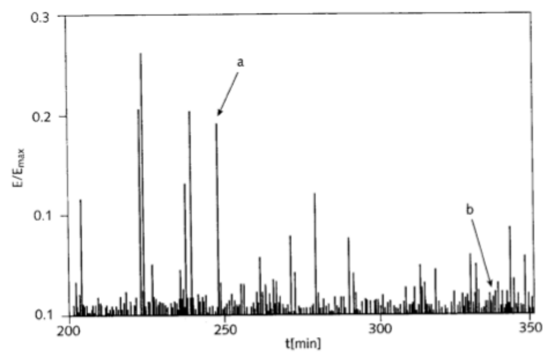


Figure 22: Raw Rice Pile Avalanche Data

On the basis of such measurements, one can count the number of avalanches of each size. For the long grain rice the distribution of avalanches is a power law, indicative of SOC behavior. The distribution was measured for different sizes of piles (see figure below)):

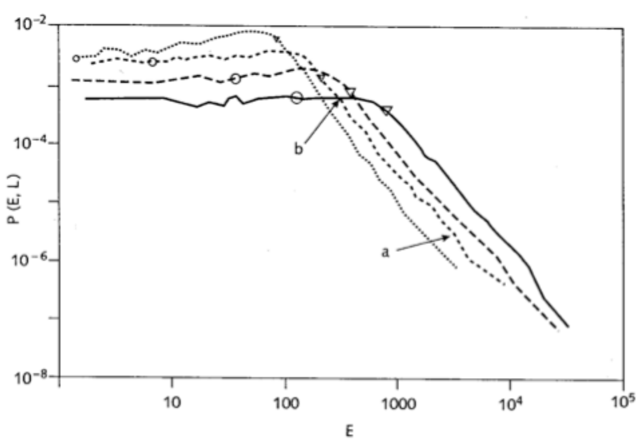


Figure 23: Rice Pile Data - Histogram

The larger the pile, the larger the avalanche. The same scaling behavior was observed for avalanches ranging in size from a few grains to several thousand grains. The scientists showed that the curves for different sizes of systems followed a systematic behavior, known as *finite size scaling* (see figure below) unique to critical systems. Thus, SOC can indeed be observed in the laboratory sandpiles, if one has persistence and patience.

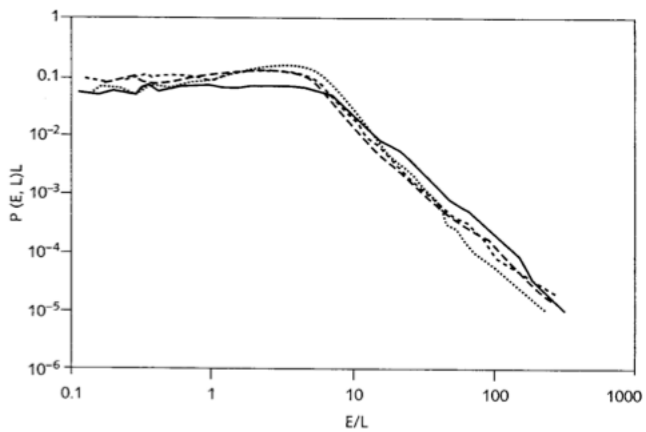


Figure 24: Scaling Behavior

By coloring a few grains, the experimenters were able to trace the motion of the individual grains. This turned out to be surprisingly complicated. The sliding grains were not confined to the surface; the grains made complicated excursions of long duration through the pile. No grains would stay forever in the pile. They all eventually would leave, but some grains remained in the pile for an extremely long time. This behavior is not understood at all, but it does not affect the SOC behavior, as evidenced by the measured power law. It would be interesting if the duration of grains conformed to another power law. Experimentalists might wish to have avalanches spanning an even larger range of magnitude in experiments of longer duration. However, no laboratory experimentalist has the infinite patience nature has, and no laboratory has the space nature has, so there are limits on the systems that can be studied. Observations of real phenomena, such as the distribution of earthquake magnitudes, might show scaling behavior, i.e. power laws, over much wider ranges than the short-term experiments in the lab. After all, it took billions of years for the morphology of the earth to reach its present state. On the other hand, laboratory experiments allow study under systematically varying conditions, whereas nature represents only a single experiment. This is the problem that one generally encounters when studying emergent phenomena such as large avalanches: the experiment must contain everything from the shortest length scale of the microscopic entities to the largest where the emergent phenomena occur. In contrast, the *reductionist* scientist sees a need to study things at only the smallest scale.

Nevertheless, the Norwegian rice experiments show conclusively that SOC occurs in piles of granular material within the limits defined by the laboratory conditions.

The two images below are of your professor setting up a rice pile avalanche during a class at Swarthmore in 2012.

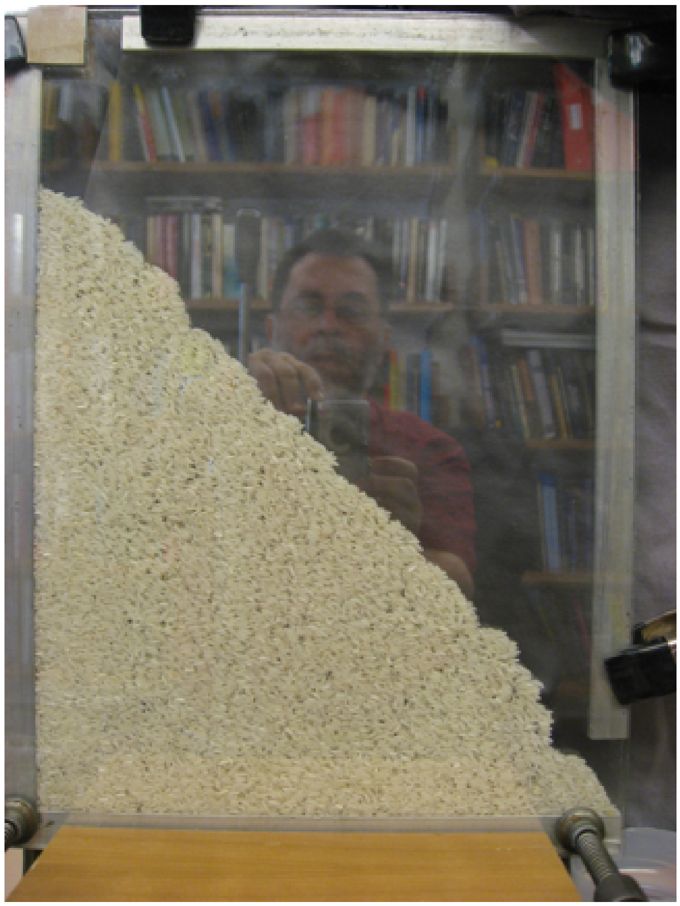


Figure 25: Before avalanche

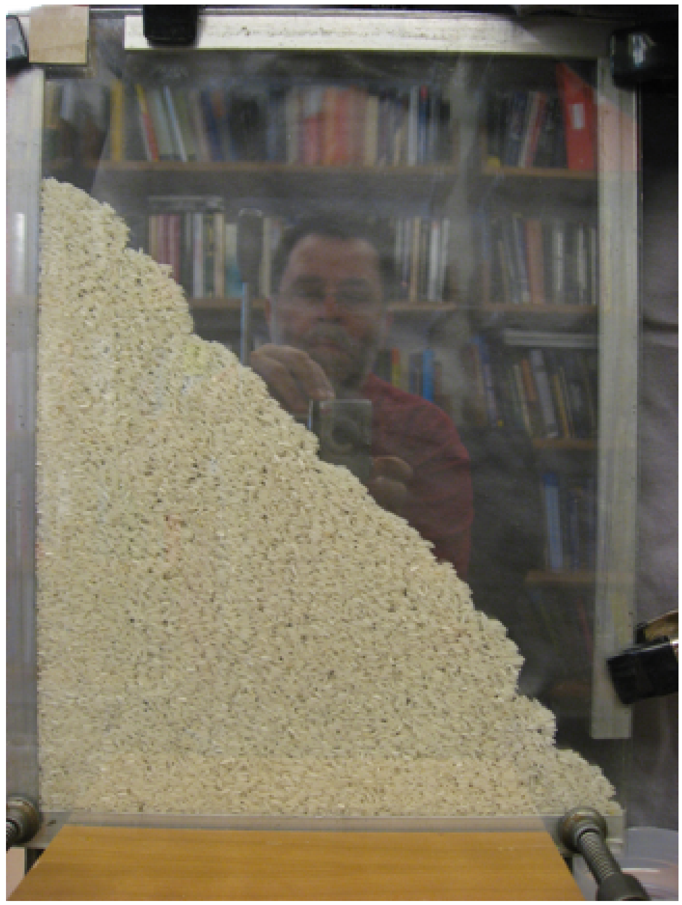


Figure 26: After avalanche

3.5.2. Viesek's Landslide Experiment: The Origin of Fractals

Viesek, is a Hungarian physicist who devoted most of his career to studying fractal phenomena. He developed a general formalism for describing the growth of surfaces by random deposition of material. Viecsek also constructed a fascinating model for self-organization of a flock of birds. He showed that it was possible for the birds to fly in formation in the same direction without a leader - the individual birds would simply follow their neighbors. The flock migration is a collective effect, as is SOC. (Class Demos)

Viesek did an experiment that not only confirmed the evolution of a sandpile to the critical state, but also threw light on the mechanisms for landscape formation in nature. He asked: Why do landscapes look the way they do? They decided to build their own mini-landscape, subjected to erosion by water. This type of laboratory experiment may be an interesting contribution to geomorphology, the science of how real geological structures are formed.

A granular pile was erected by slowly pouring a mixture of silica and potting soil onto a table. The initial *landscape* had the shape of a ridge. The ridge was watered by commercial sprayers modified to suit the experiment. As the water penetrated the granular pile, parts of the pile became saturated, and these wet parts slid down the surface, like avalanches or mud slides.

The purpose of the experiment was to gather information on the distribution of the sizes of the landslides in this micro-model of landscape formation by water erosion. This was done by video recording the changes in the profile of the ridge, just as for the rice pile. The information was fed to a computer for analysis. Since each experiment eventually caused a complete breakdown of the pile, the experiment had to be repeated many times to get a sufficiently large number of avalanches. In all, Viesek performed nine independent erosion experiments with between ten and thirty mudslides in each experiment. All the data were combined to form a single histogram of landslide sizes, which exhibited a power law shape with an exponent near 1, indicating self-organized criticality.

The experimenters measured many other properties of the landscapes formed by the erosion process. The distribution of velocities of the landslides is another power law. Most importantly, they measured the geometrical properties of the resulting contours of the landscape. They found that it is a fractal with similar features at all length scales! Thus, Viesek's group had demonstrated in a real experiment that fractals can be generated by a self-organized critical process, precisely as predicted from the sandpile simulations and as found also by the Norwegian group.

Mandelbrot, who coined the term fractal, rarely addressed the all-important question of the dynamical origin of fractals in nature, but restricted himself to the geometrical characterization of fractal phenomena. The Hungarian experiment showed directly that fractals can emerge as the result of intermittent penetrations, or avalanches, carving out features of all length scales.

Thus, it is a very tempting suggestion that fractals can be viewed as snapshots of SOC dynamical processes! In real life, where time scales are much longer than in the laboratory, landscapes may appear static, so it may not be clear that we are dealing with an evolving dynamical process. In the past, geophysicists have fallen into this trap when dealing, for instance, with earthquakes as a phenomenon occurring in a preexisting fault structure. The chicken (geometric fractal structure of the network of faults, or the morphology of landscapes) and the egg (earthquakes, landslides) were treated as two entirely different phenomena. The geophysicists did not realize that the earthquakes and the fault structures could be two sides of the same coin, different manifestations of one unique underlying critical dynamical process.

3.5.3. Himalayan Sandpiles

Do sand slides in nature obey the power laws indicative of SOC that were observed in the laboratory under controlled circumstances? To shed some light on this, scientists investigated sand slides in the Himalayas. They examined data from two road-engineering projects. On two mountain roads in Nepal, the six-kilometer Mussoori-Tehrie road and the two-kilometer stretch on the recently completed Mussoori bypass, avalanches were cleared off the road. The smallest landslides had a volume of 1/1000 cubic meters, which is about a shovelful. The largest avalanches were $10,000,000$ cubic meters, so the landslide volumes spanned a colossal range of eleven orders of magnitude, compared with the two or three orders of magnitude covered by the laboratory experiments.

Thus, in contrast to the early sandpile experiments, there were events of all sizes. The distribution of avalanches follows a power law over about six orders of magnitude. The power law was not obeyed for avalanches smaller than one cubic meter. This is simply because not all avalanches involving a few shovelfuls were recorded, just as not all small earthquakes are. Also, the small sand slides may have been removed by cars and yaks traveling along the roads. In any case there was scaling extending over an enormous range. It was noted that the avalanches originate from a steep *supercritical* state that erodes and produces avalanches. They point out that one obvious laboratory setup *would be systematically drying or vibrating an overly steep pile of wet sand.*

3.5.4. Sediment Deposition

Rocks formed by sediment deposition form a layered structure. One process for the formation of the layers works as follows. First, by various transport processes, sediment is deposited at the edge of the continental shelf and along the continental slopes. The slope eventually becomes unstable, causing avalanche-like events known as slumps. The slump creates a region of mud, which flows along the sea bottom. Eventually the mud current slows down when it reaches the relatively flat basin plain, at which point the sediment it has carried finally settles down. Deposits produced this way are called turbidites. Turbidite events occur on time scales ranging from minutes to days, whereas the time between deposition events in any location is thought to be on the order of years to thousands of years. We are dealing with an intermittent, punctuated equilibrium phenomenon. By studying the thickness of layers, ranging from centimeters to several meters, one can estimate the distribution of avalanches causing the sedimentation.

Scientists at MIT carried out a detailed study of turbidity deposits. Turbidites can be observed at the Kingston Peak formation along the Amargosa River near the southern end of Death Valley California. See the figures below.

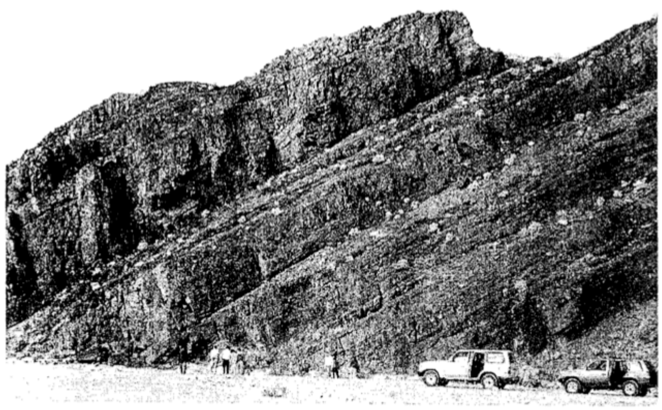


Figure 27: Turbidites in Death Valley

The turbidites were formed approximately 100 million years ago. The sample that MIT studied was obtained by drilling a hole several hundred meters deep and recovering the sediments from that hole.

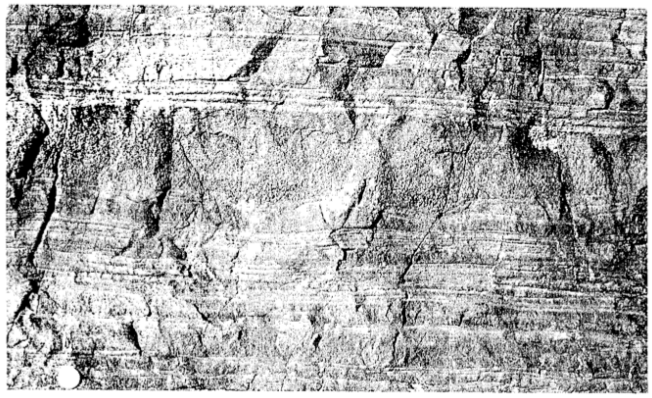


Figure 28: MIT Sample

See the penny at the bottom left to get a sense of the scale.

They counted how many layers exceeded a certain thickness, and made the usual log-log histogram as in the figure below.

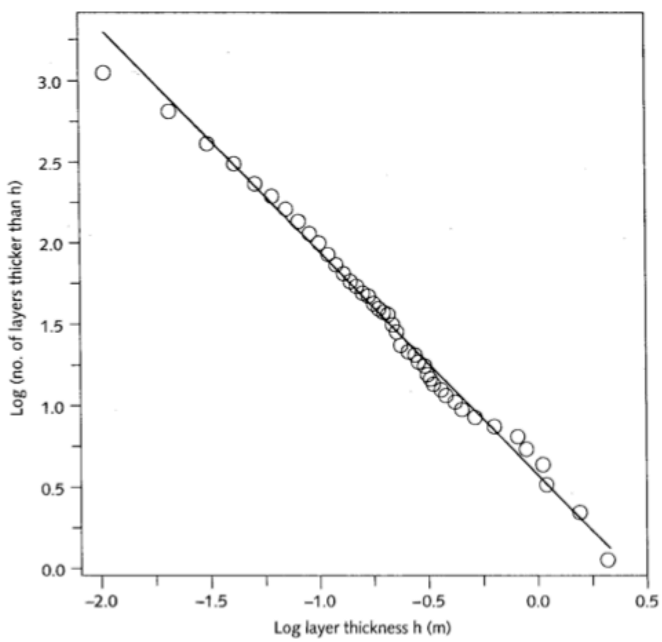


Figure 29: Log-Log Plot

Indeed, there is a power law distribution of layer thicknesses, as the theory of

SOC predicts.

3.5.5. Geomorphology: Landscapes Out of Balance

Landscapes are prime examples of complex systems. Simple systems do not vary much from one place to another. Landscapes are different. We can look around and orient ourselves by studying the landscape precisely because every place is different from every other place. Complexity involves surprises. Every time we turn a corner, we see something new. What are the general principles governing the formation of landscapes? So far, there has been no general framework for discussing and describing landscape formation.

It puzzles me that geophysicists often show little interest in the underlying principles of their science. Perhaps they take it for granted that the earth is so complicated and messy that no general principles apply, and that no general theory (in the physicist's sense) can exist. There are outstanding exceptions, however. Turcotte of Cornell University has been involved in discovering the general mechanisms. In particular, he has performed extended analysis of many fractal phenomena and constructed simple mathematical models reproducing some general features in geology and geophysics.

Another exception is Rinaldo of the University of Padova. His university may be considered the cradle of modern science. In the fifteenth century the idea of studying the human body by observing and describing, rather than by unsubstantiated philosophical arguments, originated at Padova.

Rinaldo is a hydrologist. He studies the flow of water on earth. He has been particularly interested in the complicated dynamics of the flow of water from the Adriatic Sea back and forth into the lagoons of Venice. In the best tradition of the University of Padova, Rinaldo wanted to identify some general principles for the formation of landscapes. Together with his colleagues he initiated a theoretical study of the formation of river networks and the effects of the rivers on landscapes. Small rivers, starting essentially everywhere, join each other to form larger rivers,which merge to form even larger rivers, and so on until the largest rivers run into the oceans.

It is known that the branching structure of rivers follows a simple power law known as Horton's law. Horton defined the order of river segments as the number of links to other segments that has to be passed before the river reaches the ocean. Horton's law states that the number of segments of each order increases as a power law in the order. This hierarchical structure indicates that river networks are fractal, just as the hierarchical structure of fjords along Norway's coast indicates that the coast is fractal. Another empirical law says that the length L of a river scales with the area A that is drained by that river as

$$L = 1.4A^{0.6}$$

Could it be that these and other power laws for river networks are indicative of SOC?

In sandpile models, the criticality comes about from a combination of two processes: energy is supplied by adding sand, and energy is dissipated by toppling of the grains of sand. Rinaldo's group speculated that landscape formation occurs by a similar process, in which energy is supplied by an uplifting process (by plate tectonic or some other geological process) and dissipated through erosion by wind and water.

In Rinaldo's model, erosion takes place if the stress on a riverbank from the water flow exceeds a critical value. The stress at a given point depends on the flow of water through that point, and the slope s of the landscape. The flow of the water is proportional to the area A that is drained by the river branch, assuming that the rain falls at the same rate everywhere. The formula for the stress was taken to be

$$stress = \sqrt{As^3}$$

(although the exact expression is not important).

The simulation is quite simple: Starting from a given landscape with a river network, the stress is calculated everywhere using the formula above. The sites where the stress exceeds the critical value are identified. Erosion is simulated by removing one unit of material at each of those sites. After the erosion takes place, a new landscape, with a new network of rivers, has emerged, and the process is repeated. The river pattern is constructed from the resulting contours of the landscape by having the water always running in the direction of steepest descent from any point. The erosion is combined with a general uplifting that uniformly increases the slope s of the landscape everywhere.

The landscape settles into a stationary state, with a fractal network of rivers traversing a frontal landscape. The figure below shows a snapshot of the river network.

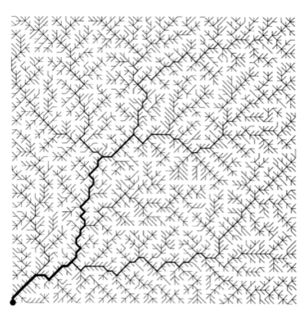


Figure 30: Snapshot of a River Network

Many aspects of the computed river network are in agreement with empirical observations, such as Horton's law and the law for the drainage area for a river of a given length. The power laws show that the stationary state is critical. The photo below shows the corresponding landscape that was generated by the process.

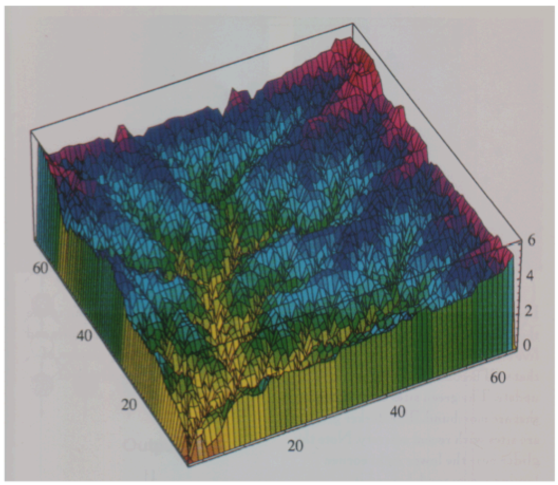


Figure 31: Simulation Result

Rinaldo's computer simulations of landscape formation represent a new and refreshing way of looking at geophysics. Instead of simply describing all geophysical features by a simple cataloging process, or *stamp collecting*, the simulations reveal the general mechanisms. Observing details may be entertaining and fascinating, but we learn from the generalities.

Rinaldo concludes that the fractal structure of river networks on the surface of the earth is a manifestation that the crust of the earth has self-organized into a critical state, forming landscapes *out of balance*. No other dynamic mechanism for the formation of fractals in geophysics has been proposed. The variability of landscapes can be viewed as an SOC phenomenon.

Landscapes are snapshots of a dynamic critical process.

Figure 32: Satellite Image

Figure 33: Satellite Image

3.6. The Crust of the Earth is Critical

Earthquakes may be the cleanest and most direct example of a self-organized critical phenomenon in nature. Most of the time the crust of the earth is at rest, in periods of stasis. Every now and then the apparent tranquillity is interrupted by bursts of intermittent, sometimes violent, activity. There are a few very large earthquakes and many more smaller earthquakes. The small earthquakes do not affect us at all, so scientific efforts have been directed toward trying to predict the few large catastrophic ones. Scientists have taken a very direct approach, formulating individual theories, or explanations, for individual earthquakes or earthquake zones; there has not been much effort directed toward a general understanding of the earthquake phenomenon. The geophysics community is very conservative. For instance, the theory of plate tectonics as a general explanation for the shifting of crustal plates that creates earthquakes was put forward in **The Origin of Continents and Oceans** by the German meteorologist Alfred Wegener in 1912, but not even found worthy of discussion until the late 1960s. Among its obvious appealing features, it explains the similar shape and geological composition of the west coast of Africa and the east coast of south America.

Don't get me wrong. I have the deepest respect for the type of science where you put on your rubber boots and walk out into the field to collect data about specific events. Such science provides the bread and butter for all scientific enterprise. I just wish there was a more open-minded attitude toward attempts to view things in a larger context.

This issue was once raised among a group, not of geophysicists, but of physicists who study quantum foundations at a conference. *"Why is it that you guys are so conservative in your views, in the face of the almost complete lack of understanding of what is going on in your field?"*. The answer was as simple as it was surprising. *"If we don't accept some common picture of the universe, however unsupported by the facts, there would be nothing to bind us together as a scientific community. Since it is unlikely that any picture that we use will be falsified in our lifetime, one theory is as good as any other"*.

The explanation was social, nonscientific.

Explanations for earthquakes typically relate the earthquakes to specific ruptures of specific faults or fault segments. This might be reasonable, but then, of course, one has to explain the fault pattern independently. Analogously, our sand theorist may correctly conclude that the origin of sand slides is toppling sand, but that does not provide any insight into the properties of large slides. The fact that earthquakes are caused by ruptures at or near faults does not in itself explain the remarkable Gutenberg-Richter law.

Scientists are poor at making earthquake predictions, and not for lack of effort. All kinds of phenomena in nature have been viewed as precursors of large earthquakes, such as the behavior of animals, the variations in the ground water level, and the occurrence of minor earthquakes. The latter approach, trying to recognize earthquake patterns preceding major quakes, seems, at least in principle, plausible. However, there has been no success. In particular, there have been claims that earthquakes are periodic at some locations, but the statistics were never based on more than two to four intervals. Notably, it appeared that in the Park Field earthquake region in California there was a periodicity of approximately 20 years. Some years ago a major and expensive project was set up to study the next earthquake. The last event in that area took place in the 1950s, and the scientists are still waiting. The earthquake predictors have had much less success than their meteorologist colleagues. Richter (of the Gutenberg-Richter law, and the Richter scale for earthquake magnitudes) once said.

> "Only fools, charlatans, and liars predict earthquakes"

The phenomenon is surrounded by much folklore. Because of the poor statistics of the very few large quakes, one can say just about anything about earthquakes without being subjected to possible falsification. The predictions will not be challenged within the lifetime of the person making the prediction!

Indeed, after an earthquake one can report what happened in some detail. One can identify the fault that was responsible and pinpoint the epicenter. This information might convince scientists working on earthquakes that one should be able to predict large events. "With a little more funding" one might become successful. However, our experience with sandpile modeling tells us that things do not generally work out that way. Because we can explain with utmost precision what *has* happened does not mean that we are able to predict what *will* happen.

It seems reasonable to take some time to acquire a general understanding of earthquakes before jumping into predicting specific events. Let us look at the extensive work that has been performed during the last few years, supporting the view that earthquakes are a SOC phenomenon. The Gutenberg-Richter law, discovered long before anybody thought about landscape self-organization, epitomizes what SOC is all about. The distribution of earthquake magnitudes is a power law, ranging from the smallest measurable earthquake, whose size is like a truck passing by to the largest devastating quakes killing hundreds of thousands of people. I cannot imagine a theory of earthquakes that does not explain the Gutenberg-Richter law.

The Gutenberg-Richter law is a statistical scaling law - it states how many earthquakes there are of one size compared with how many there are of some other size. It does not say anything about a specific earthquake. The law is an empirical law - it stems from direct measurements.

One might think that there is something special about the largest events on the curve - of magnitude 9 or so for a worldwide catalogue. It appears that there must be some particular physics on the scale that prevents larger quakes from taking place. This is probably an illusion. The largest events merely represent the largest magnitude that we typically can expect in a human lifetime. Even if the Gutenberg-Richter law extends beyond earthquakes of magnitude 10, we may not have had the opportunity to observe even a single one. A superhuman living for a million years might well have observed a few earthquakes of magnitude 10, involving, for example, most of the earthquake zone ranging from Alaska to the southern tip of South America. To this superhuman, earthquakes of magnitude 9 might appear uninteresting. Similarly, a mouse living only for a year or so, might find an earthquake of magnitude 6 terribly interesting, since this is the largest it can expect to experience in its lifetime. Unfortunately it is not yet possible to check by geological observations whether or not there have been earthquakes of magnitude, say 10 in the last 10,000 years.

The scaling law says that there can be nothing special about earthquakes of magnitude 8 or 9 because there is nothing special about a human lifetime of 100 years or so (the average time interval for such events) in a geophysical context, in which the time scale for tectonic plate motion is hundreds of millions of years. That is not necessarily a bad situation, since the physics is the same on all scales, one might acquire insight into earthquakes of magnitude 8 or 9 by studying the much more abundant quakes of magnitude 5 or 6, the statistics of which are more available. It is pointless to hang around for dozens of years to get better data on large earthquakes.

3.6.1. Self-Organization of Earthquakes

The Gutenberg-Richter law first appeared in 1988 during a conference on fractals; the conference was on fractal structures in nature. One of the speakers of this conference was Yakov Kagan of UCLA, who addressed the importance of scale-free behavior of earthquakes and earthquake zones. He pointed out that faults form fractal patterns, and presented worldwide earthquake data showing power law behavior of earthquake magnitudes over seven decades. Kagan gave a sharp rebuttal to much of the folklore surrounding the earthquake business, such as *characteristic earthquake* sizes.

Are earthquakes like the sand slides in our sandpile model?

Tectonic plate motion would provide the energy for the earthquakes. The ruptures would correspond to toppling grains. Slowly increasing pressure from the tectonic plates grinding into one another eventually must cause rupture somewhere. Just as toppling grains can affect one another in a domino process, one rupture can lead to another by the transfer of force, and sometimes lead to a large chain reaction representing a large earthquake. In a larger perspective, one might think of the plate motion as the source of *landscape upheaval* and the earthquake as the *erosion*, whose combined effects organize the crust of the earth to the critical state.

The next experiment developed was a block-spring picture of earthquake generation as shown below, in which the fault is represented by a two-dimensional array of blocks in contact with a rough surface.

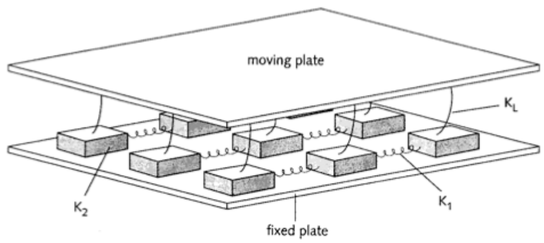


Figure 34: Block-Spring Earthquake Simulator

In the real world one cannot localize the earthquakes to single preexisting faults. The Gutenberg-Richter law concerns the statistics of earthquakes over an extended region like California. Of course, we cannot construct a realistic computer model of California and follow its evolution through hundreds of millions of years, as we would like to do. In the block-spring model, the blocks are connected to a constantly moving plate by leaf springs. The leaf springs representing pressure on the material near the fault due to the tectonic plate motion. The blocks are also connected with each other by coil springs.

Each element sticks to the surface when the sum of the spring forces is less than a threshold. The leaf springs exert a constantly increasing force on all the blocks. When the force on a particular block becomes larger than the threshold, the block slips instantaneously in the direction of the moving plate. Because of the coil springs, this increases the force on the four neighboring springs, and might cause the force on one or more of those blocks to exceed the critical value, so that they too, would slip. This could lead to the chain reaction representing the earthquake.

The arithmetic of the block-spring model was very much the same as for the coupled pendulums. Pulling the blocks by the leaf spring was like slowly winding up all the pendulums simultaneously, until one of them would make a rotation, initializing an avalanche. The slip of a block corresponds to the rotation of a pendulum. In turn, we knew that the rotation of a pendulum is equivalent to the toppling of a grain of sand in the sand model. Thus, the three models are mathematically identical; if you have studied one, you have studied them all! Thus, the Gutenberg-Richter law is the fingerprint that the crust of the earth has self-organized to the critical state.

A paper about the earthquake ideas was sent to the world's most prestigious journals, first to Nature and then to Science. Their article was rejected by both journals - by geophysicists who did not understand what it was all about. The idea of having a general theory of the phenomena of earthquakes was unacceptable. However, the referees should be given credit for revealing their identity, which is not required in the normally anonymous refereeing process. To appreciate the pain and annoyance that one might feel because of such a decision, it should be pointed out that essentially anything can be published, no matter how insignificant even in Nature. Most published material sinks like a rock and never surfaces again. It is precisely, when you have something potentially new and interesting that you get into trouble. Ironically, dozens of articles applying their ideas to various natural phenomena have since appeared with great regularity in those same journals.

Eventually, the article was published in the Journal of Geophysical Research by its editor, Albert Tarantola, who took the matter in his own hands and published the article despite its rejection by his referees. Shortly thereafter, there were more than 100 articles in the literature supporting the view of earthquakes as an SOC phenomenon.

Their model was immensely oversimplified and wrong in one respect. Their original sand model was conservative, that is, all the sand that topples ends up at the neighboring sites. There is no sand lost in the process. That is quite reasonable for sandpiles. For earthquakes, on the other hand, a careful analysis of the block-spring model shows that there is no reason for conservation of forces. The amount of force that is transmitted to the neighbors may be less than the release of force on the sliding block. As soon as the condition of conservation was relaxed in the sand model, by letting not one grain of sand arrive at the neighbor sites, but, say only 0.9 grains, the Gutenberg-Richter law would be obeyed only up to a cutoff-magnitude that would depend on the degree of conservation. There would be only small earthquakes. The block- spring model would not be critical!

3.6.2. A Misprint Leads to Progress

The solution to this problem was found by accident. An extended version of their earthquake article was written for a book, **Fractals in the Earth Sciences**. Extensive calculations were made on the continuous version of the sandpile, where all the heights are raised uniformly until there is an instability somewhere. The authors of the book had for some time been excited about the appearance of fractals everywhere in geophysics. They saw the possibility of SOC being the underlying dynamic mechanism for a variety of geophysical phenomena. There was a minor misprint in the preprint of that article that was circulated to colleagues.

Let us recall from the discussion of the sand model that when the height, representing the force f acting on a particular part of the crust of the earth, reaches $f = 4$, it relaxes to $f - 4$, while transmitting one force unit to each of its four neighbor blocks. Instead, we wrote that f goes to 0. For the first toppling in an avalanche this is no problem, since f of the toppling site is exactly 4. However, for some of the subsequent toppling events f is greater than 4, so the relaxation is greater than 4, and only 4 units of force are transmitted. Thus, there is a net loss of force in the process if f is reset to 0.

In Oslo, scientists decided to test the SOC earthquake theories by pulling a sheet of sandpaper across a carpet. The motion was not smooth, but jerky. They measured a power-law distribution of the sizes of the slip events. They simulated earthquakes using the instructions in the preprint. Indeed, they reproduced the Gutenberg-Richter law, but with other exponents than the ones predicted. The misprint was discovered. Inadvertently, they had studied a model that had no conservation of force, but nevertheless exhibited SOC. This was of great importance, since at that time there was a growing suspicion among scientists working on dynamic phase transitions, that SOC occurred only if the system was *tuned* to be conservative, indicating that one would not, in general, observe criticality in nature!

It was very difficult to rebut those claims at that time.

They first showed that the analysis of $1/f$ noise in the original sandpile article was not fully correct. Fortunately, it was then showed that for a large class of models, $1/f$ noise does indeed emerge in the SOC state.

In another development, they started with the block-spring picture and transformed it into a mathematical *sandpile*-like algorithm: Each block is subjected, as usual, to a constantly increasing force from the moving rod, and a force from the neighbor blocks. Whenever the force on any block exceeds the critical value $f = 4$, the force on that block is reduced to 0, while a fraction α of that force is transferred to each of its four neighbors. In the special case that the fraction α is $1/4$, the model reduces to the deterministic conservative version of the original sand model. When α is less than $1/4$, the model is nonconservative. It cannot be emphasized enough that the setup in the figure does not really represent how earthquakes work. **It is our spherical cow**. The earthquake cannot be localized to individual, preexisting faults; it is a three-dimensional distributed phenomenon. The Gutenberg-Richter law is not a property of a fault, but a property of the entire crust, or at the very least a large geographical area. Ideally, we would like to have the fractal systems of faults be created by the earthquake dynamics itself in the model, simulating the entire geological process that formed the crust and eventually carried it to the critical state. The model is merely intended to show that such behavior is indeed within the realm of the possible.

The model gives earthquakes of all sizes following the Gutenberg-Richter law!

See figure below:

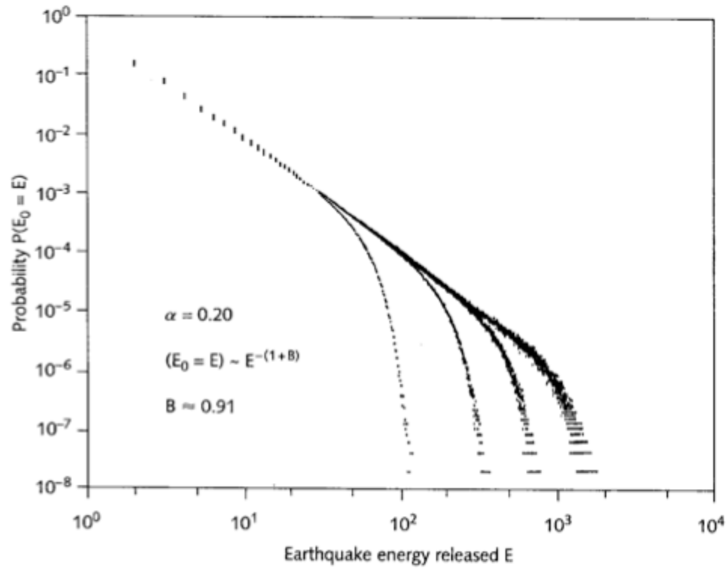


Figure 35: Earthquake Model Results

What was particularly interesting about this result was

(1) the model was derived from a careful analysis of the original block-spring model, which was already well known and accepted in the community (they did not have to pull some new "ad hoc" physics out of the hat)

(2) the model required no tuning in order to be critical.

The power law was valid for a wide range of values of the parameters. They could even include various types of randomness in their model without destroying the criticality.

The various curves in the figure correspond to various numbers of blocks. When the number of blocks in the system increases, the power law extends to larger events in a systematic way known as finite size scaling, which only critical systems obey. Conversely if the system is not critical, the cutoff will not be affected by system size.

The model was still very simplified. When there is a rupture in any solid material, not only the nearest neighborhood is affected. In reality, the elastic forces extend to very large distances. Taking this into account, they constructed a much more elaborate model of fracture formation. Starting with a nonfractured solid, a fractal pattern of fault zones emerges, together with a power-law distribution of fracture events. This simulation showed that a fractal fault pattern and the Gutenberg-Richter law could both be derived within a single mathematical model. The results are much more in tune with real earthquakes, where the seismic activity is distributed over a large area and not confined to individual fault. Some earthquakes involve interactions between faults, where the rupture along one fault puts pressure on another fault,which then ruptures during the same earthquake.

3.6.3. Rumbling Around Stromboli

Volcanic activity, like that of earthquakes, is also intermittent, with events of all sizes. A team from the University of Perugia, Italy has measured bursts of acoustic emission, that is rumbling sounds, in the area around Stromboli in Italy. They placed piezoelectric sensors coupled to the free ends of steel rods tightly cemented into holes drilled into the rocks. One sensor was placed at a large distance from the volcano and another was placed nearer to it. The sensors measured the distribution of the strengths of the burst of activity. The figure below shows the distribution for the two signals.

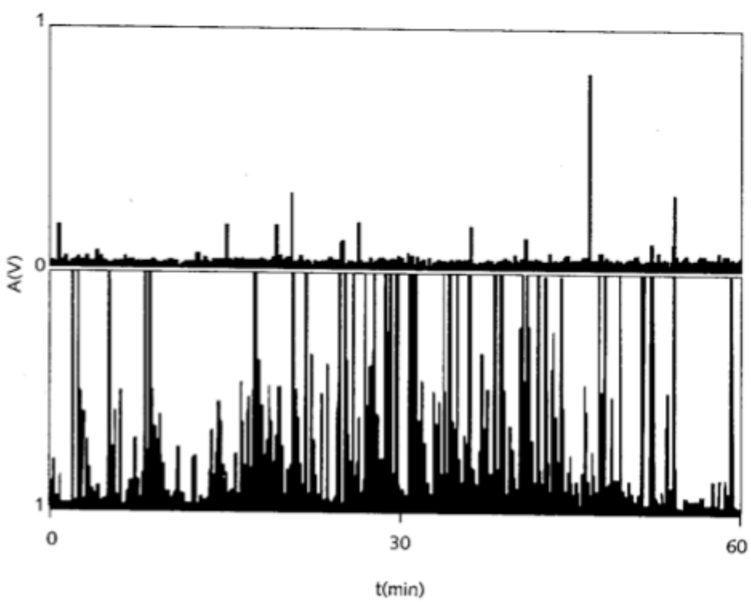


Figure 36: Volcanic Activity Data

Although one signal was weaker than the other, the straight lines on the logarithmic plot below have the same slope, with an exponent approximately equal to 2. They claimed that this indicates that volcanic activity is an SOC phenomenon.

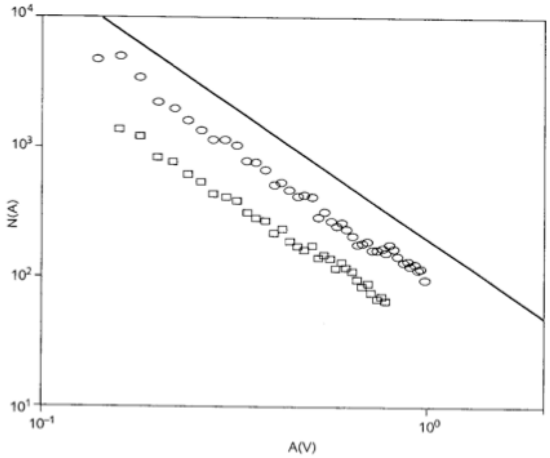


Figure 37: Log-Log Plots

It seems that the human brain has not developed a language to deal with complex phenomena. We see patterns where there are none, like the Man in the Moon, and the inkblots shown in Rorschach psychological tests. The human mind cannot directly read the boring straight line in logarithmic plots from observation of geophysical phenomena. First we tend to experience phenomena as periodic even if they are not, for example, at gambling casinos and in earthquakes and volcanos. When there is an obvious deviation from periodicity, like the absence of an event for a long time, we say that the volcano *has become dormant* or the earthquake fault is *no longer active*. We try to compensate for our lack of ability to perceive the pattern properly by using words, but we use them poorly. Nothing really, happens in an earthquake fault zone in a human lifetime - the phenomenon is a stationary process over millions of years.

3.6.4. The Crust of the Earth, Is Critical

The picture that emerges is amazing and simple. The crust of the Earth, working through hundreds of millions of years, has organized itself into a critical state through plate tectonics, earthquake dynamics, and volcanic activity The crust has been set up in a highly organized pattern of rocks, faults, rivers, lakes, etc., in which the next earthquake can be anything from a simple rumble to a cataclysmic catastrophe. The observations summarized by the Gutenberg-Richter law are the indications that this organizational process has indeed taken place.

So far, we have been viewing earthquakes, volcanic eruptions, river network formation, and avalanches causing turbidity deposition as separate phenomena, but they are all linked together. Earthquakes cause rivers to change their pattern. In Armenia after the 1988 earthquakes near Spitak, a small river had suddenly found a new path through the rocky landscape, and was displaced hundreds of meters from the original riverbed. The shift was not caused by erosion, as is usually the case. Also, it has been proposed that rare external events occurring over a large region, for example, earthquakes or storms, are the dominant source of the turbidity deposits, i.e., the aggregation of material at the continental shelf is not caused by a smooth transport process. The distribution of turbidity deposits simply mirrors the statistics of earthquakes.

In the final analysis, the crust of the Earth can probably be thought of as one single critical system, in which the criticality manifests itself in many different ways. The sandpile theory explains only one level in a hierarchy. The sand must come from somewhere else - maybe another critical system - and it must go somewhere else - perhaps driving yet another critical system. The sandpile describes only one single step in the hierarchical process of forming complex phenomena. Similarly, the crustal plates are fractal structures themselves, indicating that they originate from another critical process, possibly associated with the convective motion of the material in the earth's interior.

3.7. The "Game of Life" : Complexity is Critical

So far we have discussed many phenomena on Earth. However, one geophysical phenomenon was left out, the most complex of all, namely biological life. In the early days of self-organized criticality, one did not think about biology at all; one had only inert dead matter in mind. However, this situation has radically changed. Let us now discuss this.

We can construct some simple mathematical models for evolution of an ecology of interacting species. However, to appreciate the content of the theory that came out at the end, a historical account of the activities seems most suitable.

We start with the **Game of Life**, a toy model of the formation of organized, complex, societies. We will show that the game operates at, or at least very near, a critical state.

The Game of Life is a *cellular automaton*; a simple device that could be used as a laboratory for studying complex phenomena. Cellular automata are much simpler than the continuous partial differential equations usually used to describe complex, turbulent phenomena, but their behavior is similar. Cellular automata are defined on a grid similar to the one on which our sand model is defined. Cellular automata can be defined in any dimension. On each point of the chosen grid, there is a number that can be either 0 or 1. At each time step, all the numbers in all the squares are updated simultaneously according to a simple rule. In one dimension, the rule specifies what the content of each cell at the next time step should be, given the state of that particular cell, and its neighbors to the left and to the right, at the present time. The rule could be, for instance, that the cell should assume 1 if two or more of those three cells are 1 otherwise 0.

One can show that in one dimension there are $2^8 = 256$ such rules. Starting from, say, a random configuration of 0s and 1s, some rules lead to a boring state, in which the numbers freeze into a static configuration after some time. Sometimes the rules lead to a *chaotic* state, in which the numbers will go on changing in a noisy way without any pattern, like a television channel where there is no signal. Sometimes, the rule leads to regular geometrical patterns. It was speculated that there was a fourth class that unfortunately was never defined (and therefore not found), in which the automaton would produce new complex patterns forever.

It has now been demonstrated by computer simulations that none of the one-dimensional automata show truly complex behavior; they can all be classified into the first three classes.

No one has produced any theory of cellular automata. The automata are *computationally irreducible*, or *undecidable*, which means that the only way to find the outcome of a specific rule with a specific initial condition is to simulate the automaton on a computer. However, while this view might seem like the end of the story for a mathematician, this does not prevent the physicist from a statistical, probabilistic description of the phenomenon. Many problems that physicists deal with, such as dynamic models of phase transitions, might well be undecidable. The problem of computational irreducibility doesn't keep the physicists awake at night, since there are approximate methods available that give eminently good insight into the problem.

In two dimensions, there is an even richer world than in one dimension. Often, the neighborhood that is considered when updating a site is restricted to eight neighbors - those at the left, right, up, and down positions, and those at the four corners at the upper and lower left and right positions - and to the site itself There are a total of 2^{512} possible rules specifying how to update a cell, that is a number written as 1 followed by more than 150 zeros. It is obviously impossible to investigate them all, even with a computer.

Many years before, the mathematician John Conway of Princeton University had studied one of these zillions of two-dimensional rules of the Game of Life. Presumably he was trying to create a model of the origin of complicated structures in societies of living individuals. Although the Game of Life has never been taken seriously in a biological context, it has nevertheless served to illustrate that complex phenomena can be generated from simple local rules. The Game of Life is described very well on a Wikipedia page and in a number of classic articles by Martin Gardner in Scientific American in the beginning of the 1970s. Gardner involved his readers in an exciting hunt for amazingly complicated and fascinating phenomena in this simple game. The game is played on a two-dimensional grid as follows. On each square, there may or may not live an individual. A live individual is represented by a 1, or a blue square in the photos below.

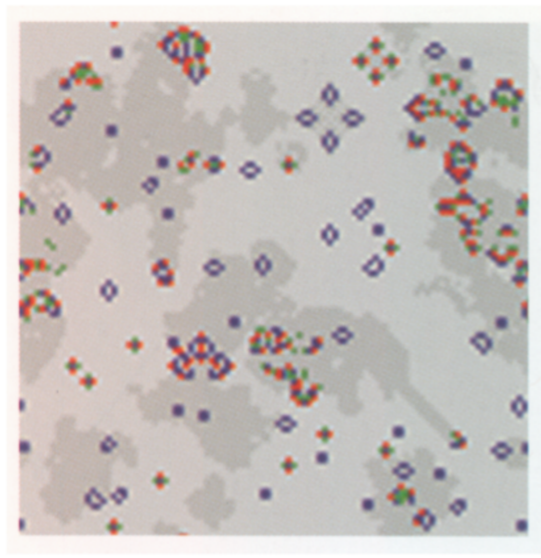


Figure 38: Game of Life - Example 1

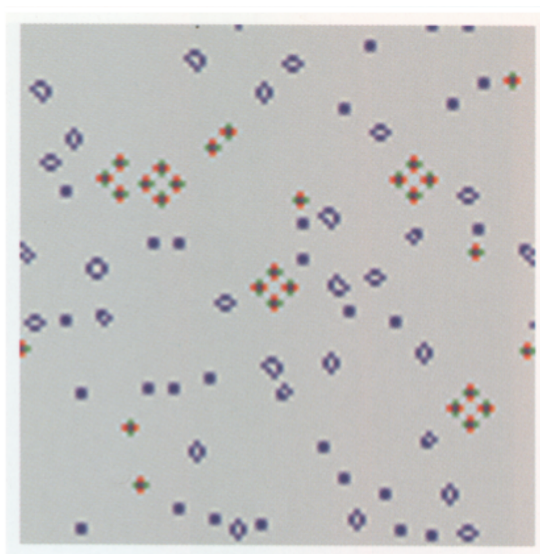


Figure 39: Game of Life - Example 2

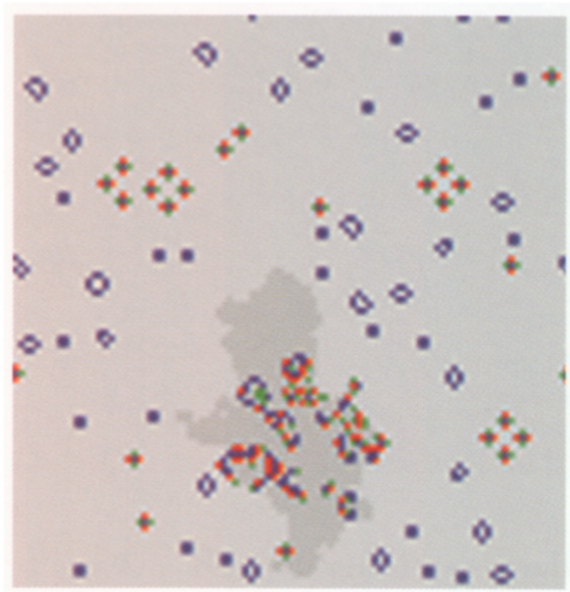


Figure 40: Game of Life - Example 3

The absence of an individual is represented by a 0, or a light gray square. At each time step, the total number of live individuals in the nine-cell neighborhood of a given cell is counted. If that number is greater than 3, an individual at that cell dies, presumably of overcrowding. If the number is 1 or 0, he will die of loneliness. A new individual is born on an empty square only if there are precisely three live neighbors. The red sites are empty sites where a new individual will be born at the next time step. In the figures, individuals who are going to die at the next update are shown as green squares, and empty sites where an individual will be born are shown as red squares. Notice that each red site indeed is surrounded by precisely three blue cells among its eight neighbors.

A myriad of complicated structures can be constructed from these rules. The photos show some stable blue clusters of live individuals. Note that the number of live neighbors in the neighborhood of each live site is either 2 or 3.

There are also configurations that propagate through the lattice. The simplest is the glider shown in the first photo near the lower right corner. In a small number of time steps, the glider configuration reproduces itself at a position that is shifted in a diagonal direction of the grid. It keeps moving until it hits something. The gray areas show where there has been recent activity, so the path of the glider is shown as a gray trail behind it. *Blinkers* shift back and forth between two states, one with three individuals on a horizontal line, the other with three individuals on a vertical line. The blinking comes about by the death of two green sites and the simultaneous birth at two red sites. There are more complicated formations involving four blinkers, as shown in the first photo. There are incredibly ingenious configurations, such as glider guns, which produce gliders at a regular rate and send them off in the diagonal direction. There are even structures that bounce gliders back and forth. The number and variety of long-lived structures in the Game of Life is evidence of its emergent complexity. Conway's interest in the game was in its ability to create this fascinating zoo of organisms (DEMO).

What makes the Game of Life tick? What is special about the particular rule that Conway had chosen? If one starts the game from a totally random configuration of live individuals, the system will come to rest after a long time in a configuration in which there are only stable static structures and simple blinkers. All moving objects, such as the gliders, will have died out. It appears that the Game of Life might operate at a critical state. To test this hypothesis, a computer simulation can be done.

One starts from a random configuration and lets it come to rest in a static configuration. Such a static configuration, with stable clusters and blinkers only, is shown in the second photo above. We then make a single "mutation" in the system, by adding one more individual, or removing one at a random position. This is analogous to the addition of a single grain to the sandpile model at a random position. The addition of a single individual may cause a live site to die because the number of live individuals in its neighborhood becomes too large. It may also give rise to the birth of a new live site by increasing the number of live neighbors of dead sites from 2 to 3. This creates some activity of births and deaths for a while, where new clusters of living objects are coming and going, and gliders are moving back and forth. Eventually the system again comes to rest at another configuration with static objects, or simple periodic blinkers, only. Then we would make another mutation, and wait for the resulting disturbance to die out. Sometimes the Game of Life comes to rest after a small number of extinction and creation events, sometimes after a large number of events. The process is repeated again and again. The process that starts when a new individual is added or removed and stops when a static configuration is reached is called an avalanche. The size s of the avalanche is the total number of births and deaths occurring before the avalanche stops. The duration t of the avalanche is the total number of time steps involved. The size s is greater than the duration t because at each time step there are usually many births and deaths taking place simultaneously. The last photo above shows a snapshot of an avalanche in progress. The gray area indicates sites where at least one individual was born or died during the avalanche. Because of the magnitudes of the largest avalanches, which involved up to 100 million births and deaths, the computations are very time-consuming. The distribution was the usual power law, shown in the figure below.

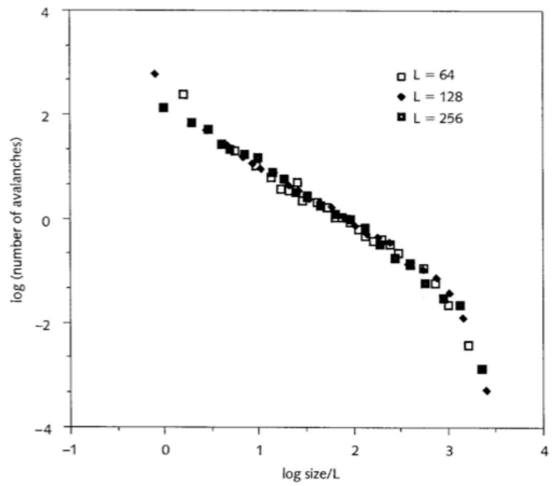


Figure 41: Data from Game of Life

The exponent, measured in the usual way as the slope of the curve, is $\tau = 1.3$. This shows that the Game of Life is critical! Surprisingly, one can make a theory for this value of τ based on a connection of the Game of Life to sophisticated theories of particle physics, as we shall later. The number of time steps follows another power law,with an exponent that can be calculated from the same theory. This mind-boggling connection was found by Maya Paczuski at Brookhaven.

Many other computer simulations have been performed and confirm these results. It might be an incredible accident that the Game of Life is critical; the Game of Life does not exhibit robust criticality. If you change the rules, you destroy the criticality.

Self-organized critical systems must be precisely critical without any tuning. If the criticality in the Game of Life is not self-organized, then it is accidental. John Conway must have tuned it to be extremely near to criticality. Conway is the watchmaker in the Game of Life! We don't know how much Conway experimented before he arrived at the Game of Life, unique among millions of millions of other rules. He was interested in the endogenous complexity of his creatures(Endogenous processes include the self-sustained circadian rhythms of plants and animals). But calculations show that at the same time that he had succeeded in constructing something that exhibited a vast amount of complexity, he had (inadvertently) tuned the system to be critical! At the time of Conway's work, little was known about the concept of critical phenomena even in equilibrium systems, so Conway cannot have known anything about criticality. Among those many possible rules, he had arrived precisely at the one that is critical. One can wonder what in the world made him hit upon this absurdly unlikely model, in view of the fact that the world's largest computers have not yet been able to come up with another complex one.

Only the critical state allows the system to *experiment* with many different objects before a stable complex one is generated. Supercritical, chaotic rules will wash out any complex phenomenon that might arise. Subcritical rules will freeze into boring simple structures. The message is strikingly clear. The phenomena, like the formation of the *living* structures in the Game of Life, that we intuitively identify as complex originate from a global critical dynamics. Complexity like that of human beings, which can be observed locally in the system is the local manifestation of a globally critical process. None of the noncritical rules produce complexity.

Complexity is a consequence of criticality.

3.8. Is Life a Self-Organized Critical Phenomenon?

The step from describing inert matter to describing biological life seems enormous, but maybe it isn't. Perhaps the same principles that govern the organization of complexity in geophysics also govern the evolution of life on earth. Then nature would not suddenly have to invent a new organizational principle to allow live matter to emerge. It might well be that an observer who was around when life originated would see nothing noteworthy - only a continuous transition (which could be an *avalanche*) from simple chemical reactions to more and more complicated interactions with no sharp transition point indicating the exact moment when life began. Life cannot have started with a chemical substance as complicated as DNA, composed of four different, complicated molecules called nucleotides, connected into a string, and wound up in a double helix. DNA must itself represent a very advanced state of evolution, formed by massively contingent events, in a process usually referred to as prebiotic(inorganic or organic chemistry in the natural environment before the advent of life on Earth) evolution. Perhaps the processes in that early period were based on the same principles as biology uses today so the splitting into biotic and prebiotic stages represents just another arbitrary division in a hierarchical chain of processes.

Maybe a thread can be woven all the way from astrophysics and geophysics to biology through a continuous, self-organized critical process. At this time, all the intermediate stages of evolution progressing from chemistry to life are distant history, so we see geophysics and biology as two separate sciences.

Biology involves interactions among millions of species, each with numerous individuals. One can speculate that the dynamics could be similar to that of sandpiles with millions of interacting grains of sand. However, the realization of this idea in terms of a proper mathematical description is a long and tedious process. Much of the thinking along these lines was like walking around in circles without being able to make a suitable model of evolution, but eventually these efforts paid off in a rather surprising turn of events.

Complexity deals with common phenomena in different sciences, so the study of complexity benefits from an interdisciplinary approach. However, because of the sociology of science, it takes someone at the top to change the course of science. Most scientists in the rank and file do not venture into new areas that have not been approved from above. There is good reason for this since young scientists are dead in the water if they step out of traditional disciplines or don't adhere to the orthodoxy A very bad state of affairs!!.

Traditionally, cross-disciplinary research has not been very successful. The fundamental entities dealt with in the various sciences are completely different atoms, quarks, and strings in physics; DNA, RNA, and proteins in biology, and buyers and sellers in economics. Attempts to find common ground have often been contrived and artificial. At universities, the different sciences are historically confined to specialized departments with little interaction. This has left vast areas of science unexplored. However, a new view is emerging that there could be common principles governing all of those sciences, not directly reflected in the microscopic mechanisms at work in the different areas. Maybe similarities arise due to the way the various building blocks interact, rather than to the way they are composed.

Traditional university and government laboratory environments have a tendency to freeze into permanent patterns as their scientists become older. We need meetings where scientists from various fields come together at seminars and conferences. These meetings would force us to place science in a greater perspective. In our everyday research, we tend to view our own field as the center of the world. This view is strengthened by our peer groups, which are, because of the compartmentalization of science, working along the same line. No mechanism for changing directions exists, so more and more efforts go into more and more esoteric aspects of well-studied areas that once paid off, such as high-temperature superconductivity, surface structures, and electronic band structures, without any restoring force. Nobody has an incentive to step back and ask himself *"Why am I doing this?"* In fact, many scientists are put off if you ask this question.

3.8.1. Sandpiles and Punctuated Equilibrium

Many scientists working in many different disciplines over a period of years took the ideas of sandpiles further then anyone could have imagined.

In particular, Stephen Jay Gould had set forth the ideas of *punctuated equilibria* in evolution. Punctuated equilibrium is the idea that evolution occurs in spurts instead of following the slow, but steady path that Darwin suggested. Long periods of stasis with little activity in terms of extinctions or emergence of new species are interrupted by intermittent bursts of activity The most spectacular events are the Cambrian explosion 500 million years ago, with a proliferation of new species, families, and phyla(the sequence is Life - Domain - Kingdom - Phylum - Class - Order - Family - Genus - Species)

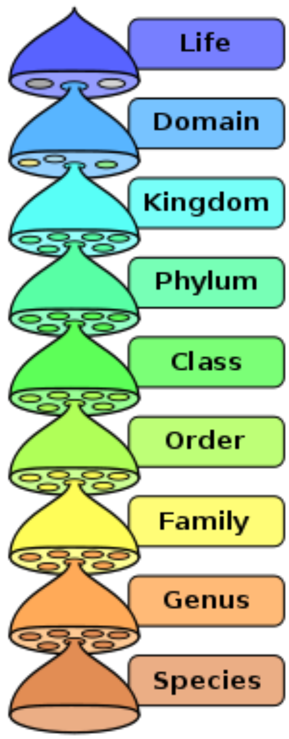


Figure 42: The Hierarchy

and the extinction of the dinosaurs about sixty million years ago. The evolution of single species follows the same pattern. For long periods of time, the physical properties, like the size of a horse or the length of the trunk of an elephant, do not change much; these quiet periods are interrupted by much shorter periods, or punctuations, where their attributes change dramatically. Darwin, however, argued and believed strongly that evolution proceeds at a constant rate.

Indeed, sandpiles exhibit their own punctuated equilibria. For long periods of time there is little or no activity This quiescent state is interrupted by rapid bursts, namely the sand slides, roaming through the sandpile, changing everything along their way. The similarity between the avalanches in the sandpile and the punctuations in evolution is astounding. Punctuations, or avalanches are the hallmark of self-organized criticality. Plotting the original Sepkoski data for extinction events in the evolutionary history of life on earth the same way as for the sandpile

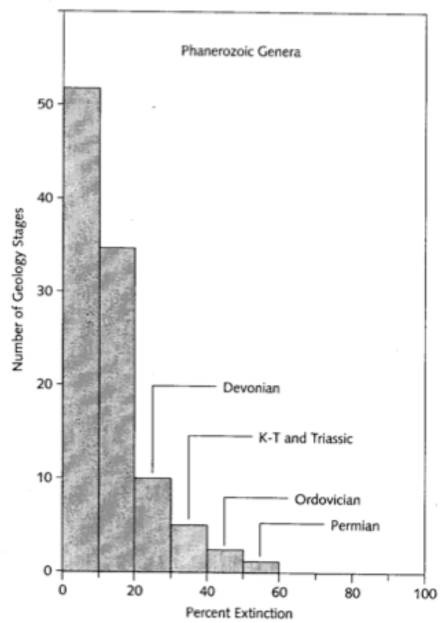


Figure 43: Sepkowski Extinction data

we find that the data are consistent with a power law, with the large extinction events occurring at the tail of the distribution. Could it be that biological evolution operates at the self-organized critical state? The idea has enormous implications for the view of life on earth. Life would be a global, collective, cooperative phenomenon, where the complex structures of individual creatures would be manifestations of the dynamics of this critical state, just like the organisms in Conway's Game of Life. But how could one express the idea in a theoretical framework, in view of the inherent difficulties that were encountered when modeling a system as straightforward as a sandpile?

3.8.2. Interacting Dancing Fitness Landscapes

Before going further we need to take a look at the important concept of *fitness landscapes*, described by Sewall Wright in a very remarkable article, **The Shifting Balance-Theory** in 1952. The physical properties of biological individuals, and thus their ability to survive and reproduce, depend on the *traits* of that individual. The *genotype* is the genetic makeup of a cell, an organism, or an individual (i.e. the specific allele makeup of the individual) usually with reference to a specific character under consideration. An *allele* is one of two or more forms of a gene. Sometimes, different alleles can result in different observable phenotypic traits, such as different pigmentation. A *trait* is a distinct variant of a phenotypic character of an organism that may be inherited, environmentally determined or be a combination of the two. For example, eye color is a character or abstraction of an attribute, while blue, brown and hazel are traits. A phenotypic trait is an obvious and observable trait; it is the expression of genes in an observable way. An example of a phenotypic trait is hair color, there are underlying genes that control the hair color, which make up the genotype, but the actual hair color, the part we see, is the phenotype. The phenotype is the physical characteristics of the organism. The phenotype is controlled by the genetic make-up of the organism and the environmental pressures the organism is subject to. This ability to survive and reproduce is referred to as *fitness.* A trait could be the size of the individual, the color or the thickness of the skin, the ability of the cell to synthesize certain chemicals, and so on. The traits express the underlying genetic code. If there is a change of the genetic code, that is, a change in the genotype, there may or may not be a change of one or more of these traits, that is, a change in the physical appearance or phenotype and therefore a change in fitness.

Wright suggested that fitness, when viewed as a function in the many-dimensional trait-space with each dimension representing a trait, forms a rough landscape, as illustrated in the simple 1-dimensional figure below.

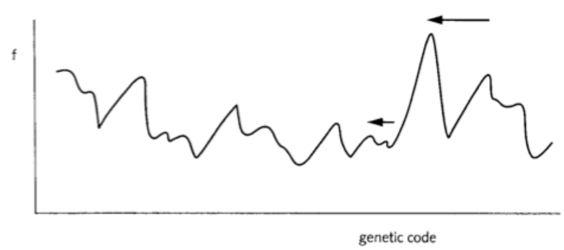


Figure 44: Fitness Landscape

Since the traits reflect the underlying genes, one might think of the fitness as being a function of the genetic code. Some genetic combinations correspond to particularly fit individuals and are shown as peaks in the diagram. Some other combinations give rise to individuals with little viability, and are represented by valleys. As the genetic code is varied over all possible combinations, the fitness curve traces out a landscape (easily visualized as an ordinary surface for two traits). There are numerous peaks and valleys corresponding to the many very different possibilities of having fit(and unfit) genes. A mutation corresponds to taking a step in some direction in the fitness space. Sometimes that will be a step down, to a state with lower fitness, and sometimes that would be a step up, to a state with higher fitness.

A species can be thought of as a group of individuals localized around a point in the fitness landscape. In the following discussion, we will represent an entire species population in terms of a single point, which is referred to as the *fitness* *of the species.* Each individual member of a species undergoes random mutations. The fitter variants, by definition, will have larger survival rates, and will proliferate and take over the whole population. Downhill steps will be rejected, uphill steps accepted. Hence by random mutation and selection of the fitter variants, the whole species will climb uphill. At this level there is not much difference between Darwin's selection of of fitter variants among random mutation and Lamarque's picture of evolution as being directed toward higher fitness - it is only a matter of time scales. Both lead to hill climbing. Darwin's theory provides a mechanism for Lamarque's directed evolution. In other words, even if Lamarque was wrong and Darwin was right, this may not have any fundamental consequences for the general structure of macroevolution.

Many early theories of evolution, can be understood simply as a detailed description of this uphill climbing process in a situation where the mountains have a constant slope, and are infinitely high. The fitnesses increase forever, implicitly representing the view that evolution is progress. They didn't even touch the complexity and diversity of evolution - everything was neat and predictable.

Unfortunately there is a pervasive view among biologists that evolution is now understood, based on these early theories, so that there is no need for further theoretical work. This view is explicitly stated even in Dawkins' book, **The Blind Watchmakers**. Nothing prevents further progress more than the belief that everything is already understood, a belief that has repeatedly been expressed in science for hundreds of years. In all fairness, all that Dawkins is saying is that Darwin's mechanism is sufficient to understand everything about evolution, but how do we know this is true in the absence of a theory that relates his mechanism at the level of individuals to the macroevolutionary level of the global ecology of interacting species?

In Sewall Wright's picture, however, the uphill climb must necessarily stop when the fitness *reaches a peak*. When you sit on top of a mountain, no matter which direction you go, you will go downhill. If we take a snapshot of biology, we can think of the various species as sitting near peaks in their landscapes. To get from one peak to a better one, the species would have to undergo several simultaneous, orchestrated mutations. For instance, a grounded species would have to spontaneously develop wings to be able to fly. This is prohibitively unlikely. Therefore, in Wright's picture evolution would come to a end when all species reach a local maximum. There may be maxima somewhere else, but there is no way to get there. Evolution will get to a *frozen* state with no further dynamics.

What is missing in Sewall Wright's fitness landscape? The important omission was the interaction between species. Species affect each other's fitnesses. When a carnivorous animal develops sharper teeth, that reduces the fitness of its prey; vice versa, if the prey develops thicker skin, or if the animal becomes quicker, or if it becomes extinct, that affects the livelihood of its prospective predators. An excellent example is: if a frog develops a sticky tongue in order to catch a fly the fly can react by developing slippery feet. Diagrammatically, the interactive ecology can be illustrated as in the figure below.

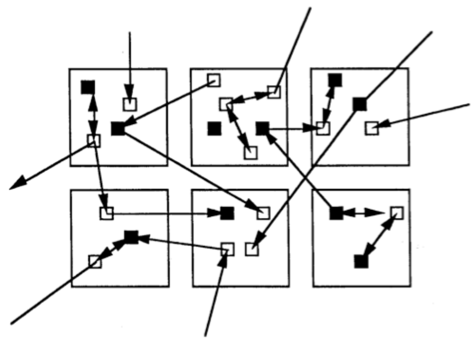


Figure 45: Illustrating Interactive Ecology

The squares represent species. An arrow from one species to another indicates that the latter species depends on the physical properties of the first. Sometimes, the arrows point only in one direction. For instance, our body contains numerous viruses and bacteria that benefit from us, but don't affect us. Often the arrows point in both directions when the two species have symbiotic relations to each other, or when a parasite benefits from, but harms, its host.

Biology might be thought of as the dynamics of a collection of interactive species living in a global ecology. The fitness landscapes of the various species are *deformable rubber landscapes* that interact with one another. The landscapes may change. When a species mutates and changes its own properties, it changes the shapes of the landscapes seen by other species. A species living happily on top of one of the hills of its own fitness landscape may suddenly find itself way down the slope of the mountain. But then the species can respond by climbing to a new top, by random mutation and selection of the fitter variants. Using Stuart's metaphoric example: A frog may improve its ability to catch flies by developing a sticky tongue; the fly can respond by developing slippery feet. The fly has to evolve just to stay where it was before. It never actually improves its fitness; it must evolve in order to simply survive as a species.

This is called the *Red Queen effect* (DEMO - NLOGO), after a character in Lewis Carroll's **Through the Looking-Glass**. *"Well, in our country"* said Alice, still panting a little, *"you'd generally get to somewhere else - if you ran very fast for a long time as we've been doing"*. *"A slow sort of country"* said the Queen. *'Now, here, you see, it takes all the running you can do, to keep in the same place"*.

We are living in *the fast place* where you have to run in order to go nowhere, not the slow place with a static landscape. In the absence of interactions between species, evolution would come to an abrupt halt, or never get started in the first place. Of course, the fitness landscapes could change because of external effects, such as a change of climate that would change the landscapes of all species.

The solution is to consider coevolution of interacting species, rather than evolution of individual species in isolation that comes to a grinding halt. Coevolution of many species can be described conceptually in terms of fitness landscapes that affect one another - *interacting dancing fitness landscapes*. This picture is a grossly simplified representation of the highly complicated population dynamics of real species coevolving in the real world, but nevertheless it represents a monumental computational challenge to find the ramifications of this view. It could provide a valuable metaphor. The competition between two species is quite well understood in terms of predator-prey models, but what are the consequences for a global ecology with millions of interacting species?

Scientists implemented fitness landscapes in terms of interacting models, called *NKC models*. They represented each species by a string of N 0's and 1's, (100000...111100), representing the states of N genes or traits. In the simplest version of the model, they would associate a random number to each of the $2N$ configurations, representing the fitness of that configuration. A little black square might represent a 1, a white square a 0. The randomness represents our lack of knowledge of the couplings. So far, the model represents a single landscape. If one tries to flip a single bit, from 1 to 0 or from 0 to 1, one finds either a lower fitness or a higher fitness. Selecting the higher value represents a single step uphill in the fitness landscape.

Thus, a very complicated process, namely the mutation of a single individual and the subsequent selection of that fitter state for the whole species population, was boiled down to a change of a single number. A single flip corresponds to a *mutation* of the entire population of a given species, or, equivalently extinction of one species followed by the replacement of another with different properties! Here and in the following discussion this process is referred to as *mutation of the species*. Many evolutionary biologists, such as John Maynard Smith, the author of the bible on traditional evolutionary thinking, **The Theory of Evolution**, insist on locating the mechanisms of evolution in the individual, and find concepts like species mutation revolting. Of course, the basic mechanisms are operating at the individual level; we are simply using a more coarse-grained description to handle the entire macroevolution. Each step involves many generations. Stephen Jay Gould uses the same terminology in some of his books, precisely to be able to discuss evolution on a larger scale than is usually done by geneticists. Not even the gradualists would question that differential selection of the fitter variant leads to the drift of entire species. It is precisely this drift of species that is eventually described by earlier theories. The coarse graining does not in itself produce "punctuated equilibria" since we envision this single step to take place in a smooth, gradual way, just as a single falling grain, containing many individual atoms, does not constitute a punctuation or avalanche in the sandpile. In the final analysis, if using a fine-enough time scale, everything, even an earthquake, is continuous. Punctuated equilibria refers to the fact that there is a vast difference in time scales for the periods of stasis, and the intermittent punctuations. The periods of stasis may be 100 million years, while the duration of the punctuations may be much less than a million years.

Eventually, when the process of selecting the fitter variant is continued for some time, the species will eventually reach a local peak from which it cannot improve further from single mutations. Of course, by making many coordinated mutations the species can transform into something even more fit, but this is very unlikely. Each species is coupled to a number C of other species, or, more precisely, to one particular trait (which could be decided by one gene) in each of C other species, where C is a small integer number. This situation is described in the last figure where the small black and white squares could represent genes that are 1 and 0, respectively. The two genes that are coupled could represent, for instance, the slippery foot of the fly, and the stickiness of the surface of the tongue of the frog. If that particular gene in one of the species flips, the viability of the other interacting species is affected. The fitness of the frog depends not only on its own genetic code, but also on the genetic code of the fly. In the model, this coupling is represented by assigning a new random number to a species if the gene to which it is coupled mutates. The interacting species could either be neighbors on a two-dimensional grid, or they could be chosen randomly among the $N - 1$ other species.

A mathematical biologist should in principle be able to study this type of system by using the much more cumbersome methods of coupled differential equations for population dynamics, called Lotka-Volterra equations, or replicator equations. In those equations, the increase or decrease of the population of a species is expressed in terms of the populations of other species. But the computational costs are so tremendous that it limits the systems that can be studied to include very few interacting species, say two or three. Indeed, the dynamics of coevolution of a small number of species have been previously studied, for instance in the context of predator-prey, or parasite-host coevolution. This is *insufficient* for our purposes, where the conjecture is that the complexity comes from the limit of many interacting species. The limit by which the number of species is very large, in practice infinity, had never before been investigated. The spirit is the same as for our sandpile or slider block models for earthquakes. Instead of following the details of the dynamics, a coarsened representation in terms of integer numbers is chosen. A species is either there or not. We do not keep track of the population of the species, just as we did not keep track of the rotation angle in our pendulum models.

Because of their simpler, though still enormously complicated structure, one can study the situation in which there was a large number of species, each interacting with C other species. One starts from an arbitrary configuration in which each of, say, 100 species are assigned a random sequence of numbers 1 and 0.

At each time step, one makes a random mutation for each species. If this would improve the fitness of the species, the mutation is accepted, that is a single 1 was replaced by a 0, or vice versa. If the fitness was lowered, the mutation was rejected, and the original configuration was kept.

If the value of C is low, the collective dynamics of the ecology would run only for a short time. The first mutation might knock another species out of a fitness maximum. That species will mutate to improve its fitness. This might affect other species. Eventually, the domino process will stop at a *frozen* configuration where all the species are at the top of a fitness peak, with no possibility of going to fitter states through single mutations. All attempts to create fitter species by flipping a single gene would be rejected at that point. This is similar to the situation with no coupling between species. In theoretical biology such a state is called an *evolutionary stable state* (ESS), and has been studied in great detail by mathematical biologists, in particular by John Maynard Smith. Economists call such states, in which no one can improve their situation by choosing a different strategy *Nash equilibria*. There is a rather complete mathematical theory of those equilibria derived within the mathematical discipline known as game theory. However, game theory does not deal with the important dynamical problem of how to get to that state, and where you go once the state ceases to be stable.

If, on the other hand, each species interacts with many other species, that is, C is large, the system enters into a "chaotic" mode in which species are unable to reach any peak in their fitness landscape, before the environment, represented by the state of other species, has changed the landscape. This can be thought of as a collective *Red Queen* state, in which nobody is able to get anywhere. Evolution of the single species to adjust to the ever-changing environment is a futile effort.

Both these extremes are poor for the collective well-being of the system. In one case, species would freeze into a low-lying peak in the fitness landscape with nowhere else to go. We might say that *"Everybody is trapped in the foothills"*. In the second case, evolution is useless because of the rapidly varying environment. As soon as you have adjusted to a given landscape, the landscape has changed. There is no real evolution in either of those two cases. This leaves but one choice: the ecology has to be situated precisely at the critical state separating those extremes, that is, at the phase transition between those extremes. Here, the species could benefit from a changing environment, allowing them to evolve to better and better fitness by using the slowly changing environment as stepping stones, without having that progress eliminated by a too rapidly changing environment. The critical state is a good place to be. We are there, because that's where, on average, we all do best.

This shows a kind of free-market view of evolution. If left to itself the system will do what's best for all of us. Unfortunately evolution (and the free market) is more heartless than this. First one would wait for the system to relax to a frozen state. Then one would make one arbitrary additional mutation, and let the system relax again to a new stationary state. Each mutation would generate an avalanche. It turns out that the system will never organize itself to the critical point. The result was always the same. The model would converge either to the frozen phase or to the chaotic phase, and only if the parameter C was tuned very carefully would we get the interesting complex, critical behavior. There was no self-organized criticality. Models that are made critical by tuning a parameter, although plentiful, are of little interest in our context.

This was the first crude attempt to model a complete biology. It was a quite frustrating state. On the one hand we had a picture of self-organized criticality that empirically seemed to fit observations of punctuated equilibria and other phenomena. On the other hand, we were totally unable to implement that idea in a suitable mathematical framework, despite frantically working on the problem. One is even able to prove by rigorous mathematics that the models could never self-organize to the critical point.

However, apart from the question as to what type of dynamics may lead to a critical state, the idea of a poised state operating between a frozen and a disordered, chaotic state makes an appealing picture for evolution. A frozen state cannot evolve. A chaotic state cannot remember the past. This leaves the critical state as the only alternative.

Unfortunately, life is not all happiness. In all of the work so far, we had selected a random species for mutation in order to start avalanches. It turned out that all we had to do was to choose the least fit species, which would have the smallest valley in the landscape to jump in order to improve its fitness. After years of hard work and little progress, persistence finally paid off.

3.9. Mass Extinctions and Punctuated Equilibria in a Simple Model of Evolution

Darwin's theory is a concise formulation of some general observations for the evolution of life on earth. In contrast to the laws of physics, which are expressed as mathematical equations relating physical observable quantities, there are no Darwin's equations describing biological evolution in the language of rigorous mathematics. Therefore, it is a very important matter to determine if Darwin's theory gives an essentially complete description of life on earth, or if some other principles have to be included. Darwin's theory concerns evolution at the smallest scale, microevolution. We do not know the consequences of his theory for evolution on the largest scale, macroevolution, so it is difficult to confront, and possibly falsify, the theory by observations on the fossil record.

It was at the time of Darwin that Charles Lyell formulated the philosophy of uniformitarianism, or gradualism. It was Lyell's view that everything should be explainable in terms of the processes that we observe around us, working at the same rate at all times. For instance, geological landscape formations are supposed to be formed by smooth processes, and the full scale of events, even those of the greatest extent and effect, must be explained as smooth extrapolations from processes now operating, at their current observable rates and intensities. In other words, the small scale behavior may be extended and smoothly accumulated to produce all scales of events. No new principles need be established for the great and the lengthy processes; all causality resides in the smallness of the observable present, and all magnitudes may be explained by extrapolation.

Darwin accepted Lyell's uniformitarian vision in all its uncompromising intensity. Darwin believed that his mechanism, random mutation followed by selection and proliferation of the fitter variants, would necessarily lead to a smooth gradual evolution. Darwin went so far as to deny the existence of mass extinctions. Since biology is driven by slow and small mutations operating at all times and all places, how can the outcome be anything but smooth? Uniformitarianism underlies many views and opinions in Darwin's **The Origin of Species**, including his hostility to mass extinction. Darwin saw evolution as a slow, gradual process. Darwin claims, *"We see nothing of these slow changes in progress until the hand of time has marked the long lapse of ages"*. This is gradualism in a nutshell.

This view about external influences is often shared, without further ado, by many evolutionary biologists. Nills Eldridge, the co-promoter of the phenomenon of punctuated equilibria, belongs to that group and concludes that Darwin's theory is incomplete because, as Eldridge believes, it cannot explain the catastrophic extinctions. The external cause could be a change in weather pattern, a volcanic eruption, or an extraterrestrial object hitting the earth. Recently, it has been suggested that cosmic neutrinos from collapse of nearby supernovas, hitting the earth at regular intervals, are responsible. It seems to be a widespread assumption that some cataclysmic impact must be responsible for mass extinction, so the debate has been about which external force was responsible.

To a large degree, Lyell's uniformitarian view is a healthy one. Indeed the microscopic mechanisms are solely responsible for the behavior at all scales. Nothing new has to be introduced at any scale.

However, the uniformitarian theory fails to realize that a *simple extrapolation* does not necessarily take us from the smallest to the largest scale. A physicist might represent Lyell's philosophy simply as a statement that we live in a linear world. The assumption that a large effect must come from a large impact also represents a linear way of thinking. However, we may be dealing with highly nonlinear systems in which there is no simple way (or no way at all) to predict emergent behavior. We have already seen in different contexts that microscopic mechanisms working everywhere in a uniform way lead to intermittent, and sometimes catastrophic, behavior. In self-organized critical systems most of the changes often concentrate within the largest events, so self-organized criticality can actually be thought of as the theoretical underpinning for catastrophism, the opposite philosophy to gradualism.

Thus, the science of genetics, which might be thought of as the atomic theory of evolution, does not provide an answer to the question of the consequences of Darwin's theory, precisely because we cannot extrapolate directly from the microscopic scale to the macroscopic scale. G. L. Simpson, in his famous book **Tempo and Mode in Evolution** states this observation very explicitly in his introduction:

> (Geneticists) may reveal what happens to a hundred rats in the course of ten years under fixed and simple conditions, but not what happened to a billion rats in the course of ten million years under the fluctuating conditions of earth history Obviously, the latter problem is much more important.

Stephen Jay Gould uses this argument to justify that only a historical, narrative approach to studying evolution is possible, underlining the importance of his own science, paleontology, which deals with the study of the fossil record. Indeed, such studies are indispensable for providing insight into the mechanisms for evolution on a grander scale.

The physicist's approach, however, is to explore, by suitable mathematical modeling, the consequences of Darwin's theory. Perhaps then we can judge if some other principles are needed. If the theory of self-organized criticality is applicable, then the dynamics of avalanches represent the link between Darwin's view of continuous evolution and the punctuations representing sudden quantitative and qualitative changes. Sandpiles are driven by small changes but they nevertheless exhibit large catastrophic events.

The early mathematical models studied were absurdly simplified models of evolution, and failed to capture the essential behavior. There was no self-organized critical state and no punctuated equilibrium. It turns out that the successful strategy was to make an even simpler model, rather than one that is more complicated. Insight seldom arises from complicated messy modeling, but more often from gross oversimplifications. Once the essential mechanism has been identified, it is easy to check for robustness by tagging on more and more details. It is usually easy to start at the simple and proceed to the complicated by adding more and more ingredients. On the other hand, it is an art to start at the complicated and messy and proceed to the simple and beautiful. The goal is not the reductionist one of identifying the *correct* underlying equations for evolution in all its details, but to set up some simple equations with the goal of illustrating robust processes.

3.9.1. Can We Model Darwin?

We now look at a simple mathematical model for interfaces moving in a random medium due to Sneppen. While superficially this might not seem useful, it turns out it is. Think, for instance, of coffee being absorbed by a paper napkin. The boundary between the wet paper and the dry paper forms an interface. The paper has some *pinning* sites where it is difficult for the interface to pass, as, for instance narrow pores in the napkin. See figure below.

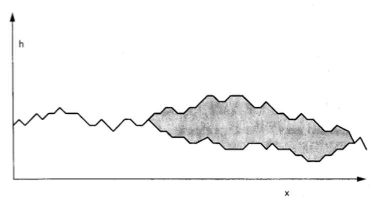


Figure 46: Illustrating a Pinning Site

In his model, growth takes place at each time step at the site with the smallest value of the pinning force. The interface shifts upward by one length unit and is assigned a new random pinning force. This type of dynamics, where activity occurs at the place with the smallest or the largest value of some force, is called *extremal dynamics*. Because of the elasticity of the interface, the growth at one site reduces the pinning force on the neighbor sites, making them likely candidates for growth at the next instance. Sneppen showed that the surface organizes itself to a critical state, with avalanches of all sizes. In other words, interface growth is a self-organized critical phenomenon.

These new ideas could be incorporated into the models. *Extremal* turned out to be the magic word. Sneppen's model worked because the site with the *least* pinning force was selected for action. In fact, in the continuous deterministic sandpile models, which describe a bowl of sugar that is gradually tilted, avalanches start at the point with the largest slope. In earthquakes, the rupture starts at the location where the force first exceeds the threshold for breakage. Maybe extremal dynamics is the universal key to self-organized criticality. Could the principle be applied to models of evolution and thereby produce punctuated equilibria?

In the old computer simulations, new coevolutionary avalanches were initiated by making a random mutation of a random species, that is by changing an arbitrary 1 to a 0 or vice versa somewhere in the NKC model. Now we choose the species positioned at the lowest foothill in the Sewall Wright's fitness landscape for elimination, and replace it with a new species.

Didn't Darwin invoke survival of the fittest, or, equivalently, elimination of the least fit?

One might think of this fundamental step either as a mutation of the least fit species, or the substitution of the species with another species in its ecological niche, which is defined by its coupling to the other species with which it interacts. Such an event is called a pseudo-extinction event. This is in line with Gould's picture of speciation: it takes place because of the *differential success of essentially static taxa*(A taxonomic(Taxonomy is the practice and science of classification) category or group, such as a phylum, order, family, genus, or species). It is a matter of definition as to how many steps are needed to conclude that a species has become extinct and a new one has emerged, i.e., when a real extinction event has taken place. According to Sepkoski, *"A species is what a reputable taxonomist defines as such"*. In our model, the number of species is conserved - only the fitter of the original species and its mutated version is conserved.

The basic idea is that the species with the lowest fitness is the most likely to disappear or mutate at the next time step. These species (by definition) are most sensitive to random fluctuations of the climate and other external forces. Also, by inspection of the fitness landscape, it is obvious that in general the species sitting at the lowest fitness peak has the smallest valley to overcome in order to jump to a fitter peak. That is, the smallest number of coordinated mutations are needed to move to a better state. In fact, laboratory experiments on colonies of bacteria show that bacteria start mutating at a faster rate when their environment changes for the worse, for instance when their diet changes from sugar to starch.

However, we want a simpler representation of the fitness landscapes than the cumbersome NKC landscapes. In the NKC models, a specific fitness was assigned to each combination of 1's and 0's in the genetic code. For a species with a twenty-bit code, interacting with four other species, we would have to store 2 to the 24^{th} power random numbers, that is more than ten million numbers for each species. If there are $1,000$ species, we would have a total of more than 10 billion numbers. In our model, we would not keep track of the underlying genetic code, but represent each species by a single fitness value, and update that value with every mutation of the species. One does not know the explicit connection between the configuration of the genetic code and the fitness anyhow, so why not represent the fitness with a random number, chosen every time there is a mutation? One then has to keep track of only $1,000$ fitness values.

In the model one chooses the species to be situated on the rim of a large circle. Each species is interacting with its two neighbors on the circle. This could represent something like a food chain, where each species has a predator on its left and a prey on its right. In principle, it could also have a symbiotic relationship with either. In the beginning of the simulation, one assigns a random number between 0 and 1 to each species. This number represents the overall fitness of the species, which can be thought of as positioned on a fitness peak with that random value of fitness. Then, the species with the lowest fitness is eliminated and replaced by another species. What would the fitness of the new species be? Several possibilities are tried that worked equally well. The fitness of the new species after a mutation is unlikely to be much improved. One would not expect a jump from a very low peak to a very high peak. Thus, first one replaces the least fit species with a species with a fitness between 0 percent and 10% higher than the original one. One might also also try a version in which the new fitness is restricted to be between its old value and 1. However, for mathematical simplicity one might try to use a species with a completely random fitness. That means one assigns a new random number between 0 and 1 to that site. Of course, this does not represent real life. The important point is that the outcome of the simulation be robust with respect to these variations, so with a little bit of luck it might be broad enough to include real evolution.

The crucial step that drives evolution is the adaptation of the individual species to its present environment through mutation and selection of a fitter variant. Other interacting species form part of the environment. One could in principle choose to model evolution on a less coarse-grained scale by considering mutations at the individual level rather than on the level of species, but that would make the computations prohibitively difficult.

The idea that adaptation at the individual (or the species) level, is the source of complexity is not new. Zipf's observation that organization stems from the individuals' pursuit to *minimize their efforts* can be put in that category. In his book **Hidden Order**, John Holland, a computer scientist at the Santa Fe Institute and the University of Michigan, also locates the source of complexity to the adaptation of individuals. His observation is correct, but perhaps not particularly deep. Where else could complexity come from? Holland is best known for inventing *genetic algorithms* for problem solving. In these algorithms, the possible solutions to a given problem are represented by a genetic string of 1's and 0's, and the solutions evolve by random mutations and selection of the most fit variants, which is the variant that best solves the problem. The crucial issue is, again, *how* to go from the microscopic individual level to the higher level of many individuals where complexity occurs. We shall see that this happens because myriad successive individual adaptation events eventually drive the system of individuals into a global critical state.

How should one represent the interactions with other species? The reason for placing the species on a circle was to have a convenient way of representing who is interacting with whom. A given species would interact with its two neighbors, one to the left and one to the right. If the species that changes is the frog, the two neighbors could be the fly and the stork. One wants to simulate the process by which the neighbors are pushed down from their peak and adjust by climbing the nearest peak available in the new landscape. One possibility was to choose the resulting fitness as some fixed amount, say 50% lower than the original peak. One must try this and many different algorithms for choosing the new fitness of the neighbor. The programs are so simple that the programming for each version takes no more than a few minutes, and the computer run takes only a few seconds to arrive at some rough results. Again, the interactions can be chosen arbitrarily, which is crucial since without this type of robustness the model could not possibly have anything to do with real evolution. In the end one settles on a version where the fitnesses of the neighbors would simply change to new random numbers between 0 and 1.

In summary, the model was probably simpler than any model that anybody had ever written for anything: random numbers are arranged in a circle. At each time step, the lowest number, and the numbers at its two neighbors, are each replaced by new random numbers. That's all! This step is repeated again and again. What could be simpler than replacing some random numbers with some other random numbers? Who says that complexity cannot be simple? This simple scheme leads to rich behavior beyond what one could imagine. The complexity of its behavior sharply contrasts with its simple definition.

In a business context, the process would correspond to a manager firing the least efficient worker and his two coworkers, and then replacing them with three new workers coming in from the street. The abilities that the two coworkers had learned by working with their poor performing colleague would be useless. Of course, the manager's rule is not fair, but neither are the laws of nature.

At the start of the computer simulation, the fitnesses on average grow since the least fit species are always being eliminated(Dante Matlab programs + NL-OGO). The figure below shows the fitness of the least fit species versus time.

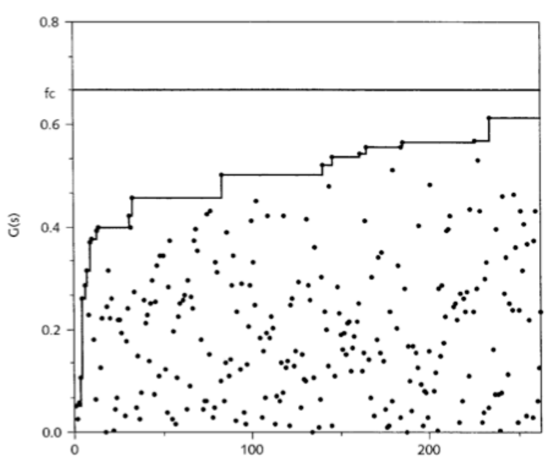


Figure 47: Fitness of Least Fit Species versus Time

Although there are fluctuations up and down, there is a general tendency of the average fitness to increase. Eventually the fitnesses do not grow any further on average. All species have fitnesses above some threshold. The threshold appears to be very close to 2/3. No species with fitness higher than this threshold will ever be selected for spontaneous mutation; they will never have the lowest fitness. However, their fate may change if their weak neighbors mutate.

Let us consider a point in time when all species are over the threshold. At the next step the least fit species, which would be right at that threshold, would be selected starting an *avalanche* or *cascade* or *punctuation* of mutation (or extinction) events that would be causally connected with this triggering event. There is a domino effect in the ecology. After a while, the avalanche would stop, when all the species are in the state of "stasis" where all the fitnesses again will be over that threshold.

The figure below shows a snapshot of all the fitnesses of all the species in the midst of an avalanche in an ecology consisting of 300 species.

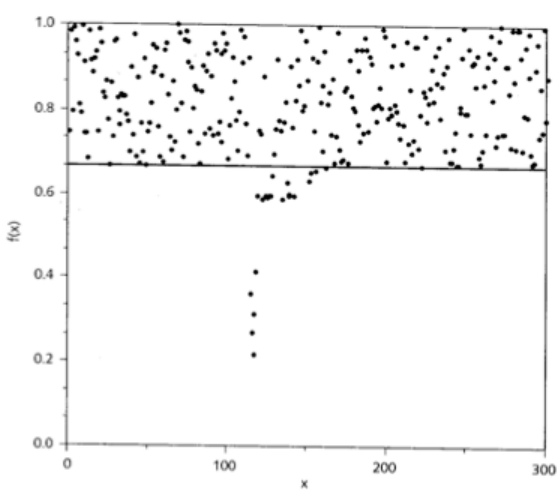


Figure 48: Avalanche Occurring

Note that most species are above the threshold but there is a localized burst of very active species with fitnesses below the threshold. Those species will be selected for mutation again and again, as the avalanche moves back and forth in the ecology. The species with high fitnesses are having a happy life, until the avalanche comes nearby and destroys their pleasant existence. In some sense, nature is experimenting with all kinds of mutations, until it arrives at a stable complex network of interacting species, where everybody is stable, with fitnesses above the threshold. One can think of this as a learning process in which nature creates a network of functionally integrated species, by self-organization rather than by design. The *blind watchmaker* is at work. The Cambrian explosion 500 million years ago, and the Permian extinction 250 million years ago in which 96% of all species became extinct, were the biggest avalanches that have occurred so far. At the Cambrian explosion, nature experimented with many different designs, most of which were discarded soon after, but out of the Cambrian explosion came a sustainable network of species.

A similar behavior was observed in the Game of Life. An avalanche of unstable, low-fit individuals with short life propagates until the seemingly accidental emergence of a stable network of organisms. If one monitors the duration of the avalanche, that is the total number of mutation events in each avalanche, and made a histogram of how many avalanches of each size were observed, one finds the all-important power law. There were indeed avalanches of all sizes

$$N(s) = As^{-\tau}$$

with τ being approximately equal to 1. Small avalanches and large avalanches are caused by the same mechanism. It does not make sense to distinguish between background extinctions happening all the time, and major ecological catastrophes.

The result appears to be universal. The system self-organizes to the critical state. It was no longer a fundamental mystery to us how an interacting ecology could evolve to a "punctuated equilibrium" state with ecological catastrophes of all sizes. Of course one might want to put some more meat on the skeleton of the model that we had constructed, but one is confident that the fundamental conclusion would survive. Darwin's mechanism of selecting the fitter variant in an ecology of species leads not to a gradually changing ecology but to an ecology in which changes take place in terms of co-evolutionary avalanches, or punctuations. The numerical simulations had demonstrated that there is no contradiction between Darwin's theory and punctuated equilibria. The model is in the spirit of Darwin's theory, but nevertheless exhibits punctuated equilibria.

Let us briefly return to what went wrong in the previous attempts to model punctuated equilibria. First of all, the idea that the critical point represents a particularly *fit* or good state was misguided. When we see ourselves and other species as *fit* this means that we are in a period of stasis in which we form a stable, integrated part of a complex ecological network. Let it be cooperation or competition. The key point is that the network is self-consistent, just like Conway's creatures in the Game of Life.

We are *fit* only as long as the network exists in its current form. We tend to see fitness as something absolute, perhaps because we view the present period of stasis as permanent, with a preferential status. However, when the period of stasis is over, it is a new ball game and our high fitness might be destroyed. Actually in a greater perspective, our present period might not even be a major period of stasis, but a part of an avalanche. Life is unstable and volatile. Dead, inert material is stable and in this sense fit. Ironically, evolution cannot be seen as a drive toward more and more fit species, despite the fact that each of the steps that constitutes evolution may improve the fitness. What one species (humans) may see as its superior fitness may better be characterized as a self-consistent integration into a complex system. Seen in isolation, the emergence of organisms as complicated as us is a complete mystery. Biology constructed the solution to the fitness problem together with the problem itself by a process of self-organization involving billions of species.

It is a much simpler task to construct a complicated crossword puzzle by a coevolutionary process than to solve it by trial and error for each word in isolation. Evolution is a collective Red Queen phenomenon where we all keep running without getting anywhere.

The simple model barely constitutes a skeleton on which to construct a theory of macroevolution. It is not the last word on the matter. It is a simple toy model that demonstrates how, in principle, complexity in an interacting biology can arise. It is the beginning of a new way of thinking, not the end. It ignores an embarrassing range of real phenomena in evolution. There is no process by which the number of species can change. Why are there species in the first place? Also, the fitness landscape is introduced ad hoc. In a more realistic theory, the landscape itself should be self-organized in the evolutionary process. However, the model is a useful starting point for these considerations. Scientists have augmented the model to make it more complete. They have included speciation. The mutating species gives rise to two or three new species, each with its own fitness, which enter the ecology in competition with all the other species. Some start their simulation with a single species. This results in pathogenetic tree structures, with a hierarchical organization similar to the taxonomic classification of species into phyla, genera, and families. The models still organize to the critical state. The exponents of the power law are different however.

Why is it that the concept of punctuated equilibrium is so important for our understanding of nature? Maybe the phenomenon illustrates better than anything else the criticality of a complex system. Systems with punctuated equilibria combine features of frozen, ordered systems, with those of chaotic, disordered systems. Systems can remember the past because of the long periods of stasis allowing them to preserve what they have learned through history, mimicking the behavior of frozen systems; they can evolve because of the intermittent bursts of activity.

Digression to study critical points via *percolation*.

3.9.2. Life at a Cold Place

In real life there is no Grim Reaper looking for the least stable species, asking it to put up (mutate) or shut up (go extinct). Things must happen in parallel everywhere. A real-time scale for the mutations has to be introduced. Species with low fitness, at the low peaks in the fitness landscape, have a short time scale for jumping to better maxima; species with high fitness are less inclined to mutate because a larger valley has to be traversed to find a more fit peak.

The barrier that has to be traversed can be thought of as the number of coordinated mutations of the DNA that have to occur to take the species from one maximum to a better one. The number of random mutations that have to be tried out increases exponentially with this barrier. Thus, the time scale for crossing the barrier is roughly exponential in the fitness. One can think of the probability of a single mutation as given by an effective temperature T. For high temperatures, there is a high mutation rate everywhere, and the dynamics are very different from the punctuated equilibrium behavior discussed here. There cannot be large periods of stasis in systems that are disturbed at a high rate. If the sandpile is shaken vigorously all the time it cannot evolve to the complex, critical state. it will be flat instead. For low temperatures, or for low mutation activity, the dynamics studied here are recovered without explicitly searching for the species with the lowest fitness. We arrive at the conclusion that complex life can only emerge at a cold place in the universe, with little chemical activity - not a hot sizzling primordial soup with a lot of activity.

3.9.3. More Details of the Model

The *fitness landscape* we have been describing represents the ability of species to survive as a function of their genetic code. In some theoretical models the landscape might be defined in terms of a spin-glass model, where the fitness (negative energy) depends on the configuration of spins. Single spin flips represent adaptive moves of *mutations*. Accepting only a new configuration if it increases the fitness, the species evolve to a local fitness maximum. This adaptive motion is fast. Further evolution takes place only if non-beneficial moves are accepted with some low probability, so the species are almost always at local fitness maxima (critical state). This is agreement with the observation that the fossil record tends to lack intermediate stages between recorded species).

The stability of each species is characterized by a barrier height separating its local fitness maximum from other better maxima. The barrier height is a measure of the number of bits, or amount of genetic code, which has to be changed. Single bit mutation occurs often, but complicated modifications, such as developing wings to allow a creature to fly, are prohibitively unlikely to occur since they involve large coordinated evolutionary moves. The time scale for mutation is exponential in the barrier height. When the fitness is high, it is difficult to find better maxima nearby, so those states are relatively stable. When the fitness is low it is more likely to find nearby better states, so the barriers are low.

For each species, i, we therefore consider only the smallest barrier, B_i. The barriers are a measure of stability. The jump across a barrier can be thought of as either a mutation of the species or the substitution of one species by a better one in an ecological niche. Since the smallest barriers generally are related to the lowest fitness and the highest barriers correspond to the highest fitness, the barriers are also a measure of fitness. Since the small barriers are unstable, a collection of non-interacting species would converge towards a deeply frozen *dead* state with the highest barriers, or fitness. Nothing is more fit than stable, inert material.

However, the fundamental driving mechanism for biology is that species interact with each other, for geographical or other reasons. For instance, the interaction could represent the fact that two species are consecutive links of a food chain. When a species makes an adaptive move, it changes the fitness landscape of its neighbors. A species with a high barrier and unable to mutate on its own might eventually be affected by a mutating neighbor, causing a reduction of the barrier which facilitates its mutation.

The model, intended to represent the main feature of all of this, is defined and simulated as follows:

1. N species are arranged on a 1-dimensional line with periodic boundary conditions ($N \leftarrow 1$ and $n \rightarrow 1$)

2. A random barrier, B_i, equally distributed between 0 and 1, is assigned to each species.

3. At each time step, the ecology is updated by locating the site with the lowest barrier and mutating it by assigning a new random number to that site

4. At each time step, the two nearest neighbors to the right and left, respectively, have their landscapes changed by assigning new random numbers to the sites also.

We note that instead of representing the species explicitly in terms of their genetic code, for instance by a spin-glass type model, we are working directly with the resulting fitness, assuming that any adaptive move simply leads to a new random fitness. The selection of the smallest barrier is justified by the exponential separation of time scales. In the beginning, subsequent events are quite uncorrelated in space, but as the barrier heights increase, it becomes more and more likely that near neighbors of events become correlated. After an extensive transient period, the distribution becomes stationary. At this point, the distribution $C(x)$ of the distance x between subsequent mutations looks like a straight line on a log-log plot indicating a power law distribution - hence the system is critical. This result does not depend on the initial conditions, so the critical state is a global attractor for the dynamics - hence it is self-organized.

All mutations turn out to take place through barriers which are less than a self-organized critical value $B_C = 0.67 \pm 0.01$. The threshold defines the maximal waiting time between successive mutations. Evolution takes place at a fast pace through small barriers and there is no time (and no need) to make large individual mutations. In contrast, the need for large coordinated mutations makes the traditional non-cooperative Darwinian evolution particularly slow. In non-interactive biology, the species would reach a state with all fitnesses equal to 1, but extremely slowly. Even if at each step the least fit species are extinct in the model, the resulting fitness in the ecology is far out of equilibrium - that is, far from the optimal one with unit barriers only. This can be thought of a a collective *Red Queen* effect, where adaptive moves do not bring you any further benefits. Life is synonymous with volatility and evolution rather than stability and fitness.

3.9.4. Comparison with Real Data

To arrive at an overview of evolution in the model, one can make a space time plot of the evolutionary activity as shown in the figure below. The x−axis is the species axis, and the y−axis is time. The plot starts at an arbitrary time after the self-organized critical state has been reached. A black dot indicates a time that a given species undergoes a mutation. The resulting graph is a fractal.

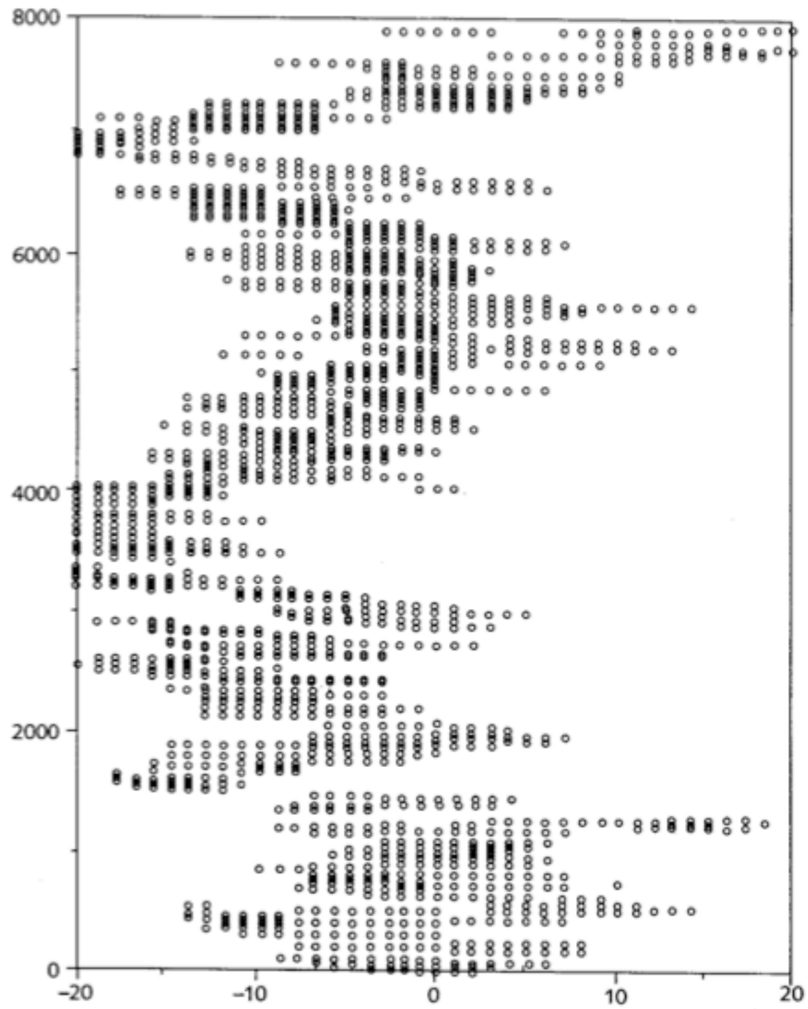


Figure 49: Space Time Plot of Evolutionary Activity

Starting from a single mutating species, the number S of species that will on average be affected after a large number R of updates will be a power law

$$S = R^D$$

where the exponent D is called the *fractal dimension* of the graph.

To monitor the fate of individual species, let us focus on a single species, for instance the one situated on the origin of the species axis, as we move along the vertical time direction. Obviously there are long periods with no black dots when not much is going on. These are the periods of stasis. Also, there are some points in time when there is a lot of activity. Let us count the number of mutation events as we move along the time direction. The figure below shows the accumulated number of mutations of the selected species as a function of the time.

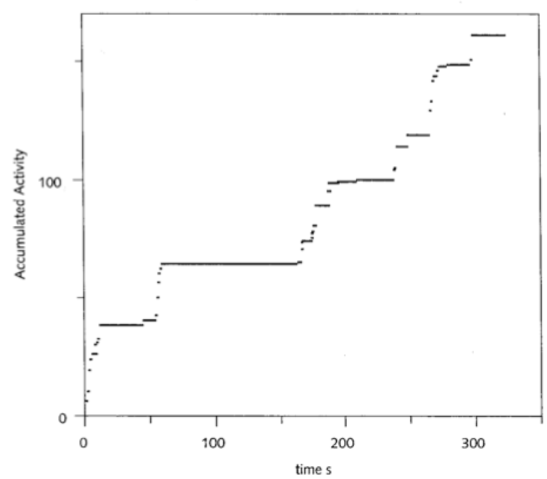


Figure 50: Accumulated Mutations as a Function of Time

One can think of this number as representing the amount of physical change, such as the size of a horse, versus time. The "punctuated equilibrium" nature of the curve is obvious.There are long periods of stasis where there is no activity, separated by bursts of activity. Such a curve is called a Devil's staircase because of its many steps, some very large, but most very small. Between any two steps, there are infinitely more steps. The Devil's staircase was invented by the German mathematician Georg Cantor (1845-1918) in the nineteenth century, and for a long time it was thought that no physical system could possibly show such intricate behavior.

One can measure the distribution of the durations of the periods of stasis, or the return times between mutations. There are no real jumps in the curve, only periods with a large number of very rapid small increases. In the fossil record, one might not be able to resolve these small, rapid increases, so the resulting variation appears as a jump. For comparison, the figure below shows how the thoracic width of the radiolarian Pseudocubus vema has evolved during the last five million years.

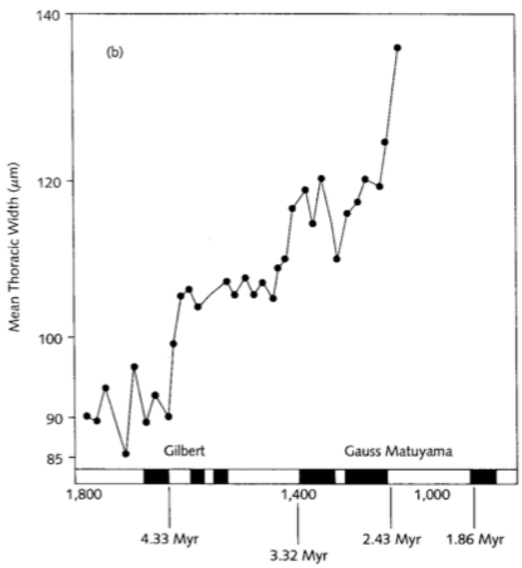


Figure 51: Thoracic Width over 5 Million Years

This curve has a quite similar structure to the previous figure. Note that there are no large jumps in the curve. The punctuations are simply periods where there is a large amount of evolutionary activity. The evolution of the size of the horse follows a similar pattern.

In our crude model, the single step can be thought of as representing either an extinction event, in which the niche of the species that became extinct is filled by another species, or a pseudo-extinction event, in which a species mutates into a different species. In either case, the original species does not exist after the event. In real evolution the same question may arise. Species may become extinct,or they may mutate through several steps into something quite different. The statistical properties of avalanches in our model should be similar to the statistical properties of extinction events in biological history. Therefore, it makes sense to compare the results of simulations with the record of extinction events in the fossil record.

By running the computer long enough, one can accumulate enough data to make the statistics of the model very accurate. In one run, more than $400,000,000,000$ pseudoextinctions were made. That is more than eighty mutations for each person in the world. One can also make several runs on the computer, whereas there is only one evolution of history on earth. It is impossible for even very meticulous paleontologists like John Sepkoski to compete with this, making it difficult to compare our predictions with reality. Sepkoski looked at *only* $19,000$ real extinctions of species.

To make comparisons with data, one simulates the evolution model in real time units as discussed above. One samples the rate of extinctions (or pseudo-existence) taking place in temporal windows of a few hundred time steps, to be compared with Sepkoski's binning of data in intervals of four million years. In this way one is able to generate a synthetic record of extinctions as shown below.

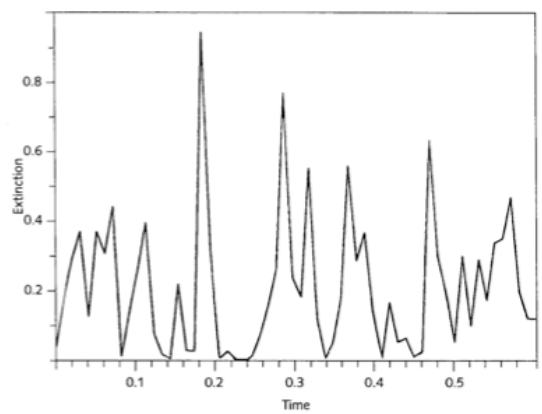


Figure 52: Synthetic Record of Extinctions

Note the similarity with Sepkoski's data.

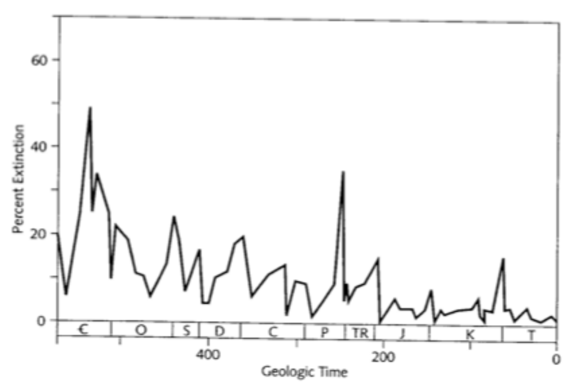


Figure 53: Sepkowski's Data

Raup's histogram of Sepkoski's data

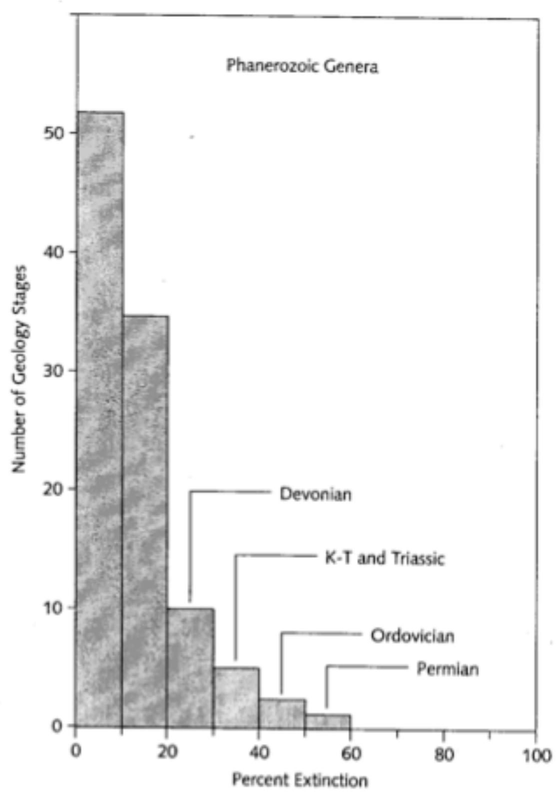


Figure 54: Sepkowski's Data - Histogram

can be reasonably well fitted to a power law with exponent between 1 and 3. The figure below shows, for comparison, the distribution of extinction events from the model.

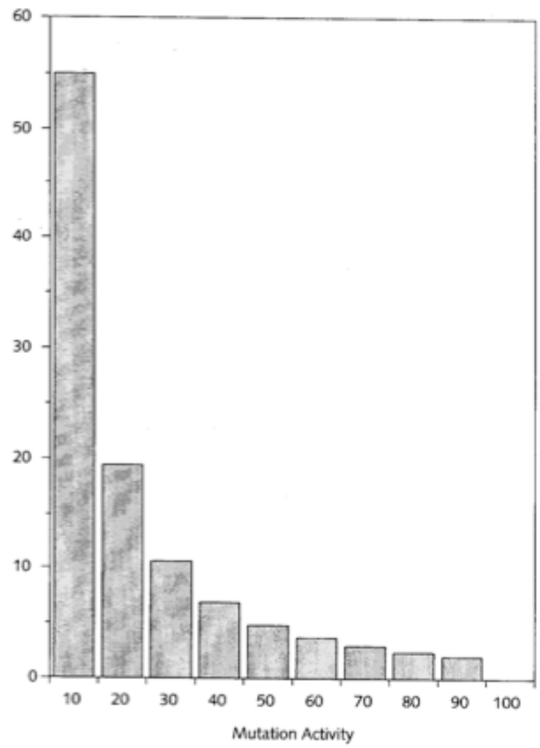


Figure 55: Distribution of extinctions from the Model

The important point is that the histogram is a smooth curve with no off-scale peaks for large extinction events. It would certainly be nice to have a finer resolution on the data, with extinctions measured, say, every one million years.

Sepkoski also noted that extinctions within individual families were correlated with extinctions in other families across the various taxa. One may say that the evolutions of different species *march to the same drummer.* This is exactly what to expect from our simulation, in which extinction events, including mass extinctions, can be thought of as the radiation of adaptive changes of individual species.

The figure below shows the accumulated mutations of a single species, the Devil's staircase, together with a plot of the global activity of extinctions.

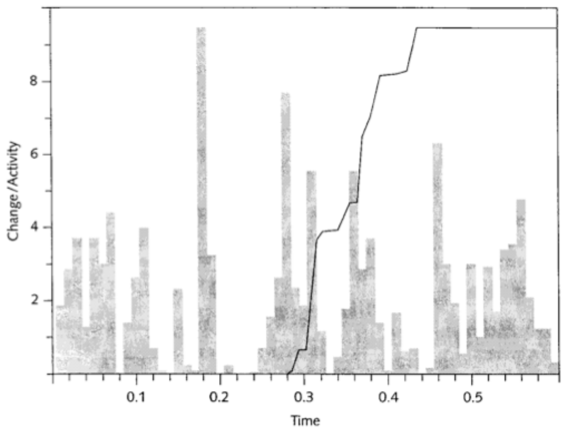


Figure 56: Accumulated Mutations; Devils Staircase; Global Activity of Extinctions

A real time scale in which the mutations rate was represented by a low temperature was used. The individual species change during periods when there is a large general activity, as observed by Sepkoski, although not all avalanches affect the species that we are monitoring. No outside *drummer* is necessary, however. The synchronized extinctions are a consequence of the criticality of the global ecology, linking the fates of the various species together, like the sand grains of the sandpile model.

Although large events occur with a well-defined frequency, they are not periodic, neither in real evolution nor in our simulation. For real evolution, the actual statistical properties of the extinction record supports the view that biological evolution is a self-organized critical phenomenon.

3.9.5. On Dinosaurs and Asteroids

Implicit in all proposed causes of mass extinction so far, including the theory involving an asteroid impact, is a presumed equality between cause and effect. According to this philosophy mass extinction must be caused by a cataclysmic external event, and the only way to understand the extinction event is to identify that event. Alvarez's theory of an asteroid hitting the earth sixty five million years ago, and thereby causing the extinctions of dinosaurs is widely accepted. The asteroid has been identified as one falling near the Yucatan peninsula in Mexico. The remnants of a large crater and a layer of iridium spreading worldwide at about the same time are taken as evidence of the theory.

One reason may be that no *alternative* theory has emerged, in the sense that no other cataclysmic impact has been suggested. The impact theory has been accepted despite two major shortcomings. First, many dinosaurs appear to have died out at least a couple of million of years before the asteroid hit. At the very least, the dominance of the dinosaurs was already greatly reduced at that time. It defies logic to claim that an asteroid hitting when the dinosaurs were on the way out was responsible for their demise. There would be no obvious need for the asteroid. The real question would be why the dinosaurs were going downhill in the first place. Second, no causal relationship between the asteroid and the resulting extinction has been established. What actually killed the dinosaurs? All we have are loose, unsubstantiated, speculations about climate changes caused by the asteroid. And why were the dinosaurs affected and not certain other species?

The fact that extinctions are synchronized is taken as further evidence of an external force working across families. Indeed, in an equilibrium linear world there would be no other possibility. A massive extinction event requires a massive external impact. This is not the case in our self-organized critical world.

Model calculations demonstrate that it is at least conceivable that the intermittent behavior of evolution, with large mass extinctions, can be due to the internal dynamics of biology. It can be argued that extinction is caused by bad luck due to external effects, rather than by intrinsically bad genes. We argue that even in the absence of external events, good genes during periods of stasis are no guarantee of survival. Extinctions may take place also due to bad luck from freak evolutionary events endogenous(proceeding from within) to the ecology. This cannot rule out that extinction events were directly caused by some external object hitting the earth. Of course, in the greater picture, nothing is external, so in the final analysis catastrophes must be explainable endogenously(from within). However, the fact that the histogram of extinction events is a smooth curve indicates that the same mechanism is responsible for small and large extinction events, because otherwise the size and frequencies of large events would have no correlations with the smaller extinction events. Certainly the small extinctions taking place all the time have nothing to do with extraterrestrial impacts.

In fact, it is quite simple and natural to reconcile the two viewpoints. In these models the avalanches were initiated by events that we thought of as mutations of a single species. One might also think of the initiating event as having an external cause. Think of the sandpile model in which the avalanches are initiated by dumping a grain of sand from the outside. Within this latter interpretation, the asteroid hitting the earth merely represents a triggering event, which initially would affect only a single or a few species. Maybe it destroyed some vegetation because of lack of sunshine. The demise of these species would destroy the livelihood of other species, and so on. The resulting mass extinction would be a domino process *caused* by this initial event. The mass extinction could only take place because the stage had been set by the previous evolutionary history, preparing the global ecology in the critical state. The model has been extended to include the effect of external perturbations as sketched here.

One still finds self-organized criticality (SOC) with a power law distribution of avalanches, although the value of the exponent, $\tau = 2$, is different from our model and possibly in better agreement with other observations.

3.9.6. Dante Chialvo's Evolutionary Game

This is a pedagogical version of the model, which we will do in class.

Dante arranged twenty students in a circle and gave them 20-sided dice. The students represented the species, and the number on her die is her fitness. Our random number generator is replaced by a throw of the dice. At each step the student with the lowest number, that is the species with the least fitness is selected. She throws her die, and so do her two neighbors. The new random numbers represents their new fitnesses. In case two of them share the lowest number, the one to go extinct would be decided by a roll of a tie-breaker die. The student who now would hold the lowest number is then selected for extinction and so on. A twenty-first student would do the bookkeeping at the blackboard. He would monitor and plot the running smallest number of all the dice. That would trace out a curve looking like the figure below.

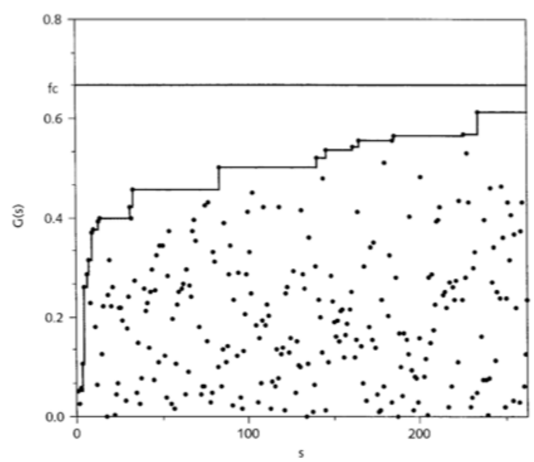


Figure 57: Dante Chialvo's Evolutionary Game Results

After several rolls, most of the students would be looking at numbers exceeding a critical fitness threshold of 13, that is near the fraction 0.667 found in our model. The bookkeeper then starts measuring the avalanche distribution. An avalanche starts when the lowest number among all the students exceeds 13, and it stops when the lowest number exceeds 13 again. The whole dynamics can be followed in detail. Because of the small number of students and their limited patience, the resulting statistics are lousy compared with what can be obtained from the high-speed digital computer. Punctuated equilibrium behavior can be detected by plotting the accumulated activity of a single, selected student. If we count how many times he has thrown his die up to a time t, the resulting curve will look somewhat like the figure below.

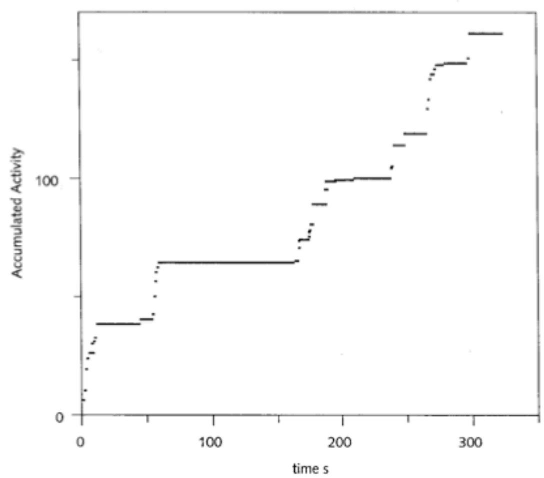


Figure 58: Dante Chialvo's Evolutionary Game - Accumulated Activity

For long periods of time, the periods of stasis, he does not throw the die at all, while other students are busy, but this inactivity is interrupted by relatively short periods where he and his neighbors get busy.

3.9.7. Self-Organized Crlticality and Gaia

In a seminal work, Jim Lovelock, an English scientist, came up with the fascinating idea that all life on earth can be viewed as a single organism. This idea has struck many scientists as preposterous since it flies in the face of the usual reductionist approach and smacks of New-Age philosophy. Lovelock's idea is that the environment, including the air that we breathe, should not be viewed as an external effect independent of biology, but that it is endogenous to biology. The oxygen represents one way for species to interact. Lovelock noted that the composition of oxygen has increased dramatically since life originated. The oxygen content is far out of equilibrium. The layer of ozone, an oxygen molecule, that protects life on earth did not just happen to be there, but was formed by the oxygen created by life itself. Therefore, it does not make sense to view the environment, exemplified by the amount of oxygen, as separate from biological life. One should think of the earth as one single system.

What does it mean to say that the earth is one living organism? One might ask in general: What does it mean that anything, such as a human, is one organism? An organism may be defined as a collection of cells or other entities that are coupled to each other, so that they may exist, and cease to exist, at the same time, that is, they share one another's fate. The definition of what represents an organism depends on the time scale that we set. In a time scale of 100 million years, all humans represent one organism. At short time scales, an ant's nest is an organism. There is no fundamental difference between genetically identical ants carrying material back and forth to build and operate their nest, and genetically identical human cells organizing themselves in structures and sending blood around in the system to build and operate a human body. Thus a single organism is a structure in which the various parts are interconnected, or *functionally integrated* so that the failure of one part may cause the rest of the structure to die, too. The sandpile is an organism because sand grains toppling anywhere may cause toppling of grains anywhere in the pile.

One might think of self-organized criticality as the general, underlying theory for the Gaia hypothesis. In the critical state the collection of species represents a single coherent organism following its own evolutionary dynamics. A single triggering event can cause an arbitrarily large fraction of the ecological network to collapse, and eventually be replaced by a nonstable ecological network. This would be a *mutated* global organism. At the critical point all species influence each other. In this state they act collectively as a sing1e meta-organism, many sharing the same fate. This is highlighted by the very existence of-large-scale extinctions. An asteroid might have directly impacted a small part of the organism, but a large fraction of the organism eventually died as a result.

Within the SOC picture, the entire ecology has evolved into the critical state. It makes no sense to view the evolution of individual species independently. Atmospheric oxygen might be thought of as the bloodstream connecting the various parts of our Gaia organism, but one can envision organisms that interact in different ways. The vigorous opposition to the Gaia hypothesis, which represents a genuine holistic view of life, represents the frustration of a science seeking to maintain its reductionist view of biological evolution.

3.9.8. Replaying the Tape of Evolution

In real life we cannot *rewind the tape of evolution*, but in our simple model we can! History and biological evolution are massively contingent on spurious incidents. The question of what if this or that did not happen has been the source of endless speculations by historians. In real life, we never know what would have happened. We cannot extrapolate from our present situation into the future (or from the past into the present). Where will the stock market be in a year from now? Or tomorrow?

One could argue that it is actually the sensitivity of real life to minor spurious events that makes fiction possible, or believable. One could not think of literature, apart from the most boring, describing life in a noncritical universe where everything is ordered and predictable. That world could not be subjected to realistic and believable manipulations by the fiction writer. Nor can one have a literature in a world where everything is totally random and chaotic, because then what happens tomorrow has nothing to do with what happens today.

The importance of contingency in economics must be stressed. As an example, we can argue that the victory of the VHS system over Betamax for video recording, or combustion engines over steam engines, was dependent on spurious historical events rather than on the technical superiority of the winning project. In traditional equilibrium economics, however, the best product always wins.

Stephen Jay Gould has emphasized the role of contingency in determining the history of life on earth. The importance of contingency could be understood as a consequence of self-organized criticality. What if we were actually able to replay history under slightly different circumstances? In real life, everything occurs only once in its full glory so we can't do that. But in our simple model of evolution we can perform the computer simulation again,with only a tiny modification somewhere.

How could we make this idea concrete? We decided to *rewind the tape of evolution.* At first we run the evolution model as usual (fat curve) and monitor the accumulated number of mutations at one site as shown below recovering the usual punctuated equilibrium Devil's staircase. We then identify the event that initiated one of the larger avalanches involving that particular site. Of course, that could be done only in hindsight.

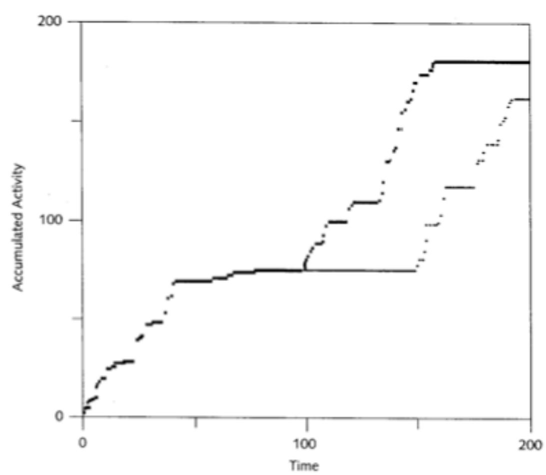


Figure 59: Replaying the Tape of Evolution

This event happened to be at a distance from the particular species that we monitored. We eliminated that event by replacing the fitness with a higher value and thus preventing extinction there. This interruption could correspond to changing the path of an asteroid, or preventing the frog from developing its slippery tongue. We then ran the simulation again. The random numbers that were chosen were the same as before for species not affected by the small change that we had made. New random fitnesses were chosen whenever needed for species that were affected by the change, and any future event that was affected. At the point where the minor perturbation was made, history changed. The accumulated number of mutation events in the replay of evolution was monitored as in the original history.

The new result is shown as the thin curve in the figure. The large punctuation is gone. However, that did not prevent disasters at all. Other punctuations happened at later points. Thus large fluctuations cannot be prevented by local manipulation in an attempt to remove the source of the catastrophe. If the dinosaurs had not been eradicated by an asteroid (if they indeed were), some other large group of species would be eliminated by some other triggering event.

Because of the large sensitivity of the critical state, a small perturbation will eventually affect the behavior everywhere. Chaos scientists call this the butterfly effect. A butterfly, moving ins wings in South America will affect the weather in the United States. What they have in mind is a simple system like the Feigenbaum map, or a pushed pendulum, or small number of coupled differential equations. If one gives the pendulum a microscopic extra push, the position of the pendulum at later times will greatly differ from the original trajectory in an unpredictable way. Of course, the global weather is not a simple chaotic system, so these considerations appear irrelevant. Our evolution model illustrates the butterfly effect for a complex system. Any small change of any event will sooner or later affect everything in the system. If the initial event caused a large avalanche, the effect will take place sooner rather than later. We believe that the effect that we have described is the real butterfly effect, in contrast to the one found in simple chaotic systems that have no relevance to evolution or any other complex system.

To illustrate the connection between criticality and punctuated equilibria, we also ran a simulation for a noncritical system. We stopped evolution before it had evolved to the critical point, and did the same two computer runs, with and without eliminating an extinction event. The noncritical evolution is gradual, with no large intermittent bursts. Changing or eliminating one roll of a die does not have any dramatic outcome whatsoever. In particular, species that are distant from the event that was eliminated were not affected at all in the simulation.

3.10. The Theory of the Punctuated Equilibrium Model

Let us now take a brief look into the mathematical analytical theory of the punctuated equilibrium model. The main reason for dealing with grossly over-simplified toy models is that we can study them not only with computer simulations but also with mathematical methods. This puts our results on a firmer ground, so that we are not confined to general grandiose, philosophical claims.

As a fringe benefit, the insight achieved from the study of the simple evolution model can be applied to the Game of Life. providing a spectacular, totally unexpected link between theory on the most microscopic level - particle theory - and the complex behavior of Conway's Game of Life.

3.10.1. What Is a Theory?

Curiously, the concept of what constitutes a theory appears to be different in biology and physics. In biology, Darwin's thoughts about evolution are always referred to as a theory even though it is only a verbal characterization of some general observations. There is nothing wrong with that. According to one of the most fundamental principles of science, a theory is a statement about some phenomenon in nature that in principle can be confronted with reality and possibly falsified. The description can be either verbal or mathematical. In physics, we use the language of mathematics to express our theories. To confront the theory with reality we solve equations and compare with experiments. The result of the theory is a number that is compared with a number measured by some apparatus. If there is disagreement, we return to the drawing board. When theories are expressed verbally in terms of much less precise languages, the confrontation with facts is much more cumbersome and leaves space for endless discussions among expert as to what constitutes the better description. Sometimes the experimental observation itself without any condensation into more general principles, is viewed as a theory.

The science of paleontology is an empirical observational science like astronomy and experimental particle physics. However, there seems to be a belief based on some misguided inferiority complex acknowledged and discussed at great length in the paleontologist Stephen Jay Gould's **Wonderful Life**, that the science becomes more respectable if the word theory can be attached to it.

This ambiguity about what counts as a theory is interesting. If Stephen Jay Gould were asked *"Wouldn't it be nice if there were a theory of punctuated equilibria?"* I think he would respond *"Punctuated equilibria is a theory"*.

3.10.2. The Random Neighbor Version of the Evolution Model

How does one go about constructing a theory in the physicist's sense? The construction of a simple model in conjunction with computer simulations does not in itself constitute a full-fledged theory. Although the numerical results do provide predictions to compare with observations, they give only a limited amount of insight into the physical process of self-organized criticality. The main advantage of having simple models of complex phenomena is that one might eventually be able to deal with them with powerful mathematical methods. For that reason, the evolution model was stripped down as far as possible. The computer simulations act as a guideline for the analytical approach. They help one focus ideas. The model and the numerical simulation serve as a bridge between nature and a mathematical theory. The main theoretical issues to be addressed are the process by which the model organizes itself to the critical state, and the characterization of the critical state, expressed for instance in terms of the critical exponents for the power laws characterizing the critical state, which eventually should be compared with observations.

After constructing our model, and doing the first preliminary computer studies, one then attempts an analytic approach. One wants a version that would yield to rigorous analysis and also gives a rigorous way of properly defining the avalanches in terms of the activity below the critical threshold. Instead of placing the species in a circle, we let each species interact with two randomly selected species in the system. At each time step one would select the species with the lowest fitness, and two other random species, and provide all three with new random fitnesses. In Dante Chialvo's game version, that would correspond to a situation in which the student with the lowest value on the die and two other random students in the class would roll their dice attach time step.

One can calculate the fitness threshold above which all species would find themselves after a transient time. The threshold is at 1/3, to be compared with 0.667 for the chain model. This number in itself is of no importance. We also calculated the exponent of the power law for the avalanche distribution, $\tau = 3/2$. There would be slightly fewer catastrophic events than in the original model in which τ was 1.07. This exponent seems to be in better agreement with the distribution of extinction events. The resulting mathematics turned out to be highly complicated despite the simple nature of the model. The avalanche process in the random neighbor model can be thought of as a *random walk*. At a given stage of the propagating avalanche, there will be a number of active species with fitnesses below the threshold. At the next time step, the number of active species will take a random step: the number will either increase or decrease by 1. The process continues until there are no more active species, and the avalanche is over.

There is another solvable model with a good deal more complexity. A particle physicist became interested in the world of self-organized criticality and came up with a model in which each species is explicitly characterized by many traits, each of which gives a contribution to the fitness of the species. At each time step, the single trait with the lowest fitness among all the species is *mutated*, that is, the corresponding fitness is replaced by a random number between 0 and 1. This trait interacts with one trait of the species to the right in a food-chain geometry and one trait to the left. Those traits are also assigned random new fitnesses. When there is exactly one trait for each species, the model reduces to the original punctuated equilibrium model. Surprisingly in the limit where there are many traits, the model can be solved exactly by very sophisticated mathematical methods. The punctuated equilibrium evolution for a single species is shown in the figure below.

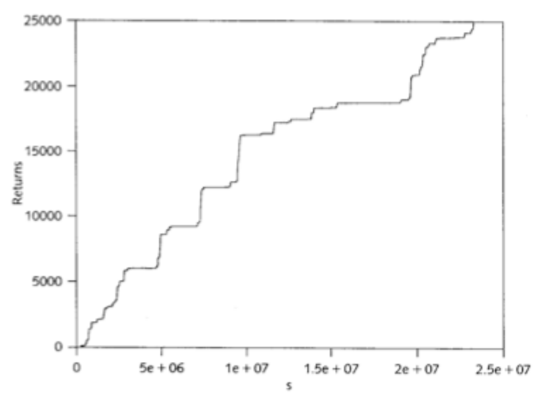


Figure 60: Punctuated Equilibrium Evolution

The distribution of avalanche sizes is a power law with exponent 3/2.

3.10.3. The Self-Organization Process

The general process of self-organization in the punctuated equilibrium model has been studied. In contrast to sand models and earthquake models, it is possible to construct a mathematical theory for the slow process in which the ecology organizes itself to the critical state.

Then a breakthrough!

Ever since the inception of SOC, there has been a lack of analytical (pen-and-paper) progress on SOC. There were a few examples of exact results and beautiful approximate schemes for calculating the exponents, but there was essentially no progress on the important question of how the system becomes attracted to the critical state. However, this situation changed for the better.

The approach to the critical point follows a characteristic pattern as show below.

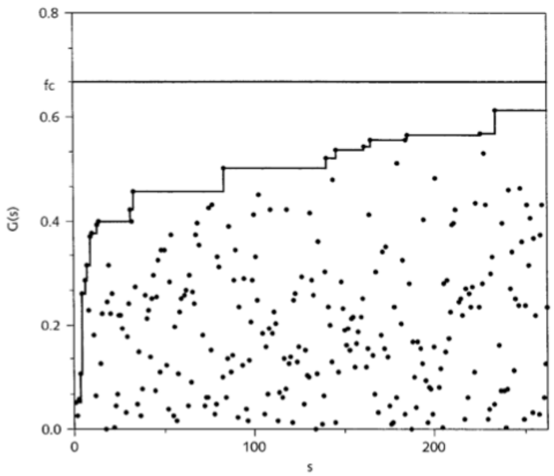


Figure 61: Approach to the Critical Point

The value of the largest fitness belonging to any species that has mutated up to a given time follows the stepwise curve shown in the figure. The steps of that curve show the points in time when that fitness grows. For a while after the step, there are lower fitnesses in the system, but eventually these low fitnesses are erased, and the curve has another small up-step. We call this curve the *gap* curve (and the equation that describes it the *gap* equation) since there are no species with fitnesses below the curve at the points in time when there is a step. The mutation activity between the steps are called *avalanches*. The avalanches represent cascades of extinction events. One can show that the mutations during the avalanches are connected in a tree-like structure to the first mutation in the avalanche, that is, they are generated by a domino effect. After the completion of the avalanche where the curve makes a step, the activity jumps to somewhere else in the ecology, generally not connected with any species that mutated in the previous avalanche.

As the plateaus of the fitness curve reach higher and higher values, the avalanches, on average, become bigger and bigger. Eventually the size of avalanches reaches infinity, limited only by the total number of species in the system, and the stepwise envelope curve ceases to increase. It gets stuck at the value $f_c = 0.667$. At that time the system has become critical and stationary. During the avalanches the fitnesses of some species are, by definition of the avalanches, less than the critical value, but at the end of an avalanche all fitnesses are again above the critical value. Thus, the self-organization can be described by an inescapable divergence of the size of avalanches. This divergence is described by a power law with an exponent γ(gamma),where $\gamma = 2.7$ in the model in which the interacting species were arranged on a circle.

The asymptotic approach of the gap f to the critical value as a function of time is yet another power law:

$$f(t) = f_c - A\left(\frac{t}{N}\right)^{-1/(\gamma-1)}$$

Here, t is the total number of update steps, N is the number of species, and A is a constant factor. This equation is the fundamental equation for the process of self-organization. It shows that as t becomes larger and larger the gap f gets closer and closer to the critical value f_c. The envelope in last figure follows that formula. The critical state with the unique value of the gap is an attractor for the dynamics, in contrast to non-self-organized critical systems where tuning is necessary. We call this equation the *gap equation.*

A similar process is responsible for the criticality in sandpile models, although the insight here is mostly numerical. As the pile becomes steeper and steeper, the sand slides become larger and larger, until they reach the critical slope where they diverge and cover the entire system; this prevents further growth.

3.10.4. The Critical State

Once the system reaches the critical state, the evolutionary dynamics are described in terms of the spatiotemporal fractal shown in the figure below.

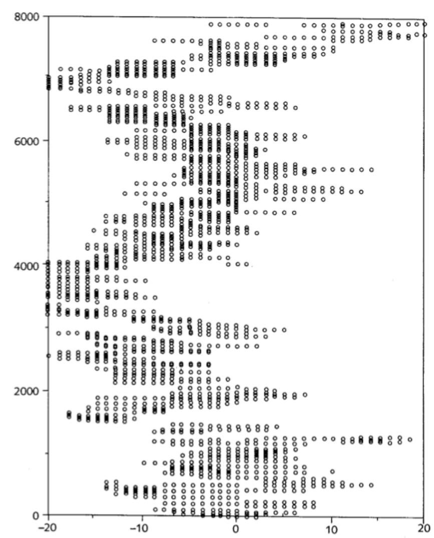


Figure 62: Evolutionary Dynamics as a Spatiotemporal Fractal

We have already defined the fractal dimension D of this fractal, and we have also defined the exponent τ for the avalanches. Interestingly, all other quantities that one might think of measuring can be expressed in terms of those two exponents. For instance, the exponent

$$\rho = \frac{1}{\gamma - 1}$$

in the gap equation for the relaxation of the critical state is a simple algebraic expression

$$\rho = \frac{1 + \frac{1}{D - \tau}}{1 - \frac{1}{D}}$$

Another formula that we have derived allows us to determine the threshold very accurately. It turns out to be $f_c = 0.66700$, and not $2/3$ as we believed for a long time; it just happens to be very close.

Another quantity that we have ignored for some time is the power spectrum, i.e., the quantity that is supposed to show $1/f$ type noise. Again,we consider the mutation activity of a single species as time progresses. The punctuated equilibrium behavior, with periods of stasis of all durations separating bursts of activity gives rise to a power spectrum

$$S(f) = \frac{1}{f\alpha}$$

where the exponent

$$\alpha = 1 - \frac{1}{D}$$

For our model, the exponent is 0.58; for the theoretical model the exponential be found to be exactly 3/4.

Thus, everything is quite well understood for the punctuated equilibrium model. The existence of the self-organized critical state has been proven. The resulting dynamics can in terms of an underlying spatiotemporal fractal. spectrum is $1/f$ like; there are avalanches of all sizes. It provides insight into the origin of all results discussed earlier. Of course, our models are necessarily quite abstract, but they are robust. One can change features of the models without changing the criticality. This feature makes us believe that the models may be general enough to span the real world.

3.10.5. Revisiting Earthquakes

Earthquakes are a self-organized critical phenomenon; the punctuated equilibrium model can roughly be thought of as an earthquake model, simply by a change in terminology. The fitness landscape in the evolution model is equivalent to the heterogeneous barrier distribution over a fault plane that generates earthquakes. We have the two-dimensional version in mind, in which each species affects its four nearest neighbors. Mutation corresponds to rupture.

In seismology, a nonuniform distribution of strengths over a fault plane is described in terms of *barriers* or *ruggedness*, which are considered to cause the complex rupture process of earthquakes. The fitnesses in the evolution model can be thought of as the ruggedness in a fault model. During an earthquake, a rupture starts from the weakest site in the crust with the minimum barrier strength. When the site breaks, the stress in the neighborhood changes. This can be modeled by assigning new random numbers to the new barriers at all those sites. Rupture propagates as long as the new barriers are weaker than the threshold for rupture. The earthquake stops when the minimum barrier becomes stronger than the threshold. Another earthquake starts from the site with the minimum barrier after some time when the tectonic stress is increased again. All these phenomena follow the punctuated equilibrium model.

To summarize, we view the entire dynamics of the fault zone as the dynamics of the evolution model depicted in the last figure. We are dealing with one single dynamic process, not one process for each earthquake. Also the dynamics cannot be understood as a phenomenon associated with faults created by some independent process. The fault structure and the earthquakes are both generated by one process. There is only one spatiotemporal fractal structure. The spatial and temporal structures are two sides of the same coin. The temporal behavior at a specific site is given as a vertical cut in this fractal, and the spatial behavior is given as a horizontal cut.

How does this correspond to reality? We must consider the time intervals it would take for earthquakes in California to return to the same small area, that is, we look at the distribution of periods of stasis between earthquakes at a given location. We measure the distribution of these return times for 8,000 earthquakes. The result is shown in the figure below.

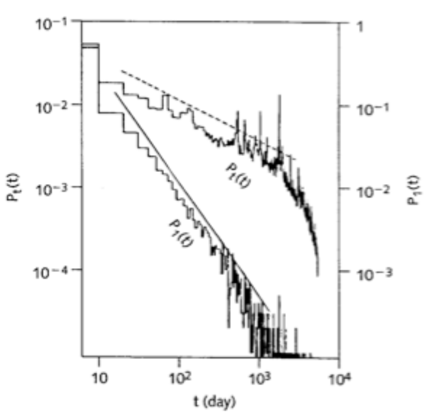


Figure 63: 8000 Earthquakes Data

Strikingly, it is a power law,with an exponent of 1.4, very similar to our exponent of 1.58. We also consider the distribution of times from a given earthquake to any subsequent earthquakes in the same region, not just the first earthquake. That is another power law, with exponent 0.5, compared with our exponent 0.42. Finally we measure the distribution of spatial distances from one earthquake to the next consecutive one. That is another power law with exponent 1.7. The fact that there are power laws in both space and time suggests that there is one underlying space time fractal for the activity pattern of earthquakes in California, and that it is very possible that this fractal is generated by a dynamic process following rules similar to our evolution model.

The empirical result demonstrates that earthquakes are a self-organized critical phenomenon, with all of its hallmarks. The empirical power law for the return time, i.e., the periods of stasis, is interesting because it demonstrates that earthquakes are not periodic. There is a tendency, even among scientists, to view events that occur with some degree of regularity as periodic, as we have already seen in connection with extinction data. The power law indicates that the longer you have waited since a large earthquake at a given location, the longer you can expect still to have to wait, contrary to common folklore. Earthquakes are clustered in time, not periodic. The same goes for evolution. The longer a species has been in existence, the longer we can expect it to be around in the future. Cockroaches are likely to outlast humans. I have often been asked what the realization that nature operates at a self-organized critical state is *good for*. How can that help us predict or prevent earthquakes? How can I use it to make money on the stock market? If I am so smart, why am I not rich? Usually I don't like to answer these questions, not because I don't believe that the basic insight into how things work will not pay off at some time, but because I believe that acquiring insight is in itself a worthwhile effort. There is one business that is entirely based on the statistical properties of events: the insurance business. I should be able to make a profit selling earthquake insurance! I would approach residents in earthquake areas where there has not been a major earthquake for a long time. The sales pitch would point out the *obvious* fact that an earthquake *is due*; nevertheless I would sell earthquake insurance at a price that is lower than that of my competitors. On the other hand, I would stay away from areas where there has recently been a major earthquake.

3.11. Self-Organization to Criticality Further Thoughts

The idea that living systems tend to self-organize has been around for a quarter of a century. The concept was introduced to understand the apparent chicken-and-the-egg problem of what came first: proteins or nucleic acids. The main tenet is that biological systems are organized by the information present in them, and that the information in turn originates in the self-organized state by means of selection. As a consequence, one witnesses the establishment of structure or order in such a way that the entropy of the system is not maximal, i.e., the system is not in equilibrium.

3.11.1. Digression on Information and Entropy

There is little doubt now that the complexity of living systems is just a reflection of the information stored in the genome.

Artificial living systems can help tremendously in understanding the mechanism that allows information to be transferred from the environment into the genome.

3.11.1.1. Complexity and Information

Stochastic, from the Greek "stochos" or "aim, guess", means of, relating to, or characterized by *conjecture* and *randomness*. A stochastic process is one whose behavior is *non-deterministic* in that a state does not fully determine its next state. We now investigate the basic process that allows the stochastic transfer of information. In the end, we will find that the main agent in this transfer is Darwin's principle of survival of the fittest.

We start by abstracting the mechanism down to its simplest form, operating on self-replicating binary(0's and 1's = bits) strings.

Imagine such a string self-replicating (contains context (environment) dependent instructions needed for self-replication) in an environment consisting of other strings and instructions for self-replication (physics and chemistry). Also imagine their is an agent which induces errors (bit flips) or noise. Two possible ways exist, namely, random bit-flips like cosmic ray mutation and incorrect copying (transcription) of instructions during self-replication.

We don't answer question of where self-replication instructions came from - this is the origin-of-life question.

Let us follow the evolution of such a string in an environment full of potential information - the environment is complex.

1^{st} **Observation**: Noise-based mutation probability is non-uniform across bits of the string. Bit positions essential to self-replication resist mutation; this is result of a simplest mechanism - if flipped, string cannot self-replicate and messed up code dies out. Quickly replaced by normal string (reverses lethal mutation). Non-essential bits that are mutated are not corrected and eventually sample all possible values - these mutations do not affect the fitness of string. Nonuniform substitution rates are a reflection of importance of bits (hot (variable) and cold (conserved)).

Imagine a random mutation that (by chance) increases the rate of replication. For example, the mutation of hot bit coupled with the simultaneous mutation of another bit that together make better use of environment. Such changes will be passed on and amplified (better replication). In a finite world all strings will eventually carry this allele (particular bit value) - bit has reverted from hot to cold - it has frozen. Success = better exploitation of environment. This requires information about environment, i.e., when string gets better adapted to environment, environment information must have been written into the genome. Information slowly trickles into genome as population takes advantage of properties of the environment - genomes become correlated with the environment. This is how life evolves. More and more information acquired will correspond to greater complexity.

Entropy and Information

Entropy is a measure of the disorder in a system, or a measure of our lack of knowledge about the system. If no knowledge is specified, the entropy or disorder of a system is always a maximum. A system is a combination of all possible states with equal probability.

What happens at the instant information enters the genome? We consider this event a measurement performed by the genome on the environment. In a measurement, the correlation between the measurement device and the measured system increases, the amount of information (I) increases (the entropy $\propto -\log I$ decreases). These measurements are performed spontaneously and randomly. Once information acquired in this spontaneous manner, it is not released spontaneously (would lower entropy).

Information that can be accessed by measurement is bounded by the entropy of system. **entropy = potential information**. If a system is isolated (closed) its entropy or information content is constant (it redistributes within the system during a measurement).

In general, if a system S is measured by a device M that is not part of S, then the result is a correlation of some of M's variables with S. Thus, information is a quantity that needs the specification of two ensembles - what is being measured and by what. Thus, information is never context-free - information is always information about something - it is conditional information. The entropy is also conditional entropy.

For example, the entropy of molecules stashed in the corner of a box is going to increase until it reaches a maximum value at equilibrium where all the the molecules are uniformly distributed in the box.

The inverse(nonequilibrium) process that confines all the molecules in a corner of the box is a measurement, thereby creating information and reducing the (conditional) entropy.

Entropy and information are entirely statistical concepts (requiring ensembles for their definition). We cannot determine the entropy of one binary string. Nevertheless, it appears intuitive that not all bit-strings are alike; some appear more regular than others.

Continuing ......

Many people believe that self-organization is one of the hallmarks of living systems. The concept of self-organization will allow us to take a very different look at the evolution of self-replicating systems, from the point of view of statistical theories that admit critical points, i.e., a branch of physics usually concerned with phase transitions in condensed matter systems. The idea is to abstract the interaction between the self-replicating elements in the system to such a degree that they can be described by simple theoretical models. It is then the task of the theorist to isolate those characteristics of the models that carry over to living systems from those that are just an artifact of the abstraction.

3.11.2. Self-Organization and Sandpiles - repeat of earlier stuff

The paradigm for the self-organized critical state is the sandpile that Bak, Tang, and Wiesenfeld(BTW) introduced, and that we will describe again below. Its usefulness lies partly in the fact that it is so simple, yet displays some of the uncanny traits of natural systems such as power law behavior and self-organization. Power law behavior is seen in many physical systems, such as thermal noise in electronic devices (shot noise), the flashing of fireflies, turbulent fluid flow, activity patterns in neural networks, the distribution of earthquake sizes (the Gutenberg-Richter law), the distribution of solar flares and sunspots, the intensity fluctuations of quasars, and the size distribution of initial masses of stars (the Salpeter law), to name a few. Specifically, it was the frequency distribution of noise in many physical systems known as $1/f$ noise that prompted the idea of SOC. In general we distinguish between three main types of power laws in physical systems. First, we have the power spectral density distribution (such as $1/f$ noise), where

$$P(f) \sim \frac{1}{f^{\alpha}}$$

where f is the frequency and $P(f)$ is the power at that frequency. In general, this function describes which frequency is the most dominant in the temporal behavior of the system under consideration. The exponent does not necessarily have to be $\alpha = 1$ such as in $1/f$ noise, but it is in general a small real number. Another kind of power law appears in size definitions

$$N(s) \sim \frac{1}{s^{\beta}}$$

which reflects a distribution of frequency of events $N(s)$ as a function of the size s of events. This is the kind of distribution observed in the Gutenberg-Richter law. Finally, we distinguish a power law in the temporal distribution of events, where τ is either the duration of an event (as in sandpile experiments), or better, the time between events (also known as inter-event-interval distribution):

$$N(\tau) \sim \frac{1}{\tau^{\gamma}}$$

Without trying to define SOC, let us immediately describe the sandpile. In the simplest, one-dimensional model, imagine a linear lattice of L sites, on which we can distribute grains of sand one at a time. Let the number of grains deposited on site j be denoted by $h(x_j)$ (the height of the pile at site x_j, as shown in the figure below:

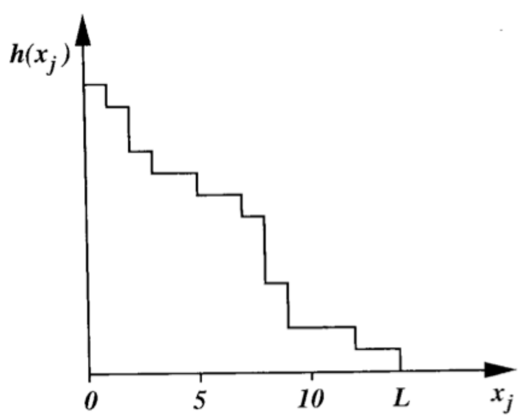


Figure 64: Sandpile on a linear lattice with L sites

The rules of the game are now such that grains of sand are free to accumulate as long as the height difference between adjacent sites does not exceed two (as for example is the case between the eighth and ninth site in the above figure). Such a situation is unstable, and a grain has to tumble from site j to site $j+1$. Should site $j+1$ become unstable due to this process, the tumbling continues up until there are no more sites with a height difference between them exceeding two. This is called the minimally stable state. Indeed, dropping a grain randomly anywhere on the pile may result in either no transport of grains, or in a cascade involving any number of sites. Let us write down the update rules for the sandpile. They can be implemented in a straightforward manner as a simple computer program updating the **slopes** $z_j = h(x_j) - h(x_{j+1})$. Here, we write rules in a general manner, where 1 grain topples if the critical slope $z_c (= 2)$ is exceeded (a supercritical site):

Rule (i) - "adding of sand":

$$z_j \to z_j + 1$$
$$z_{j-1} \to z_{j-1} - 1$$

Rule (ii) - "tumbling of grains" (see figure below):

$$\text{For } z_j > z_c : z_j \to z_j - 2$$
$$z_{j\pm1} \to z_{j\pm1} + 1$$

The lattice is furthermore subject to the boundary condition $h(x_L) = 0$, which means that sand slides off the pile at that end.

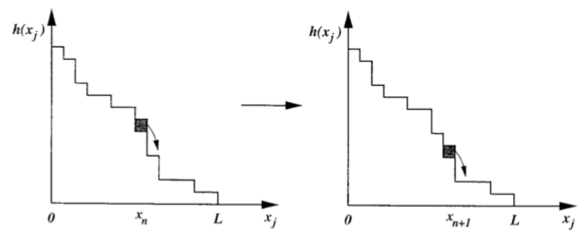


Figure 65: Tumbling of supercritical site x_n

We can easily imagine what happens if we run such a sandpile for an extended amount of time, dropping a grain of sand on a random site, updating the pile such that no more sliding occurs, and repeating. Surely we will witness many events where only one grain topples, a good number of events where a few grains are involved in an avalanche, and a few rare events where all the sites are involved in the avalanche. We may then ask what is the distribution of sizes of events. While this distribution can be obtained quite easily by simply performing the experiment, it is surprisingly difficult to arrive at the law from first principles, i.e., treating the model that we outlined above analytically. In the figure below we show the abundance distribution of avalanches of size s for a one-dimensional sandpile of linear dimensions $L = 32, 64, 128$. We plotted the function on a logarithmic scale for both the size s and the abundance $N(s)$, to highlight the power law behavior. Indeed, if the functional form is

$$N(s) = Cs^{-\tau}$$

plotting the logarithm of $N(s)$ against the logarithm of s will result in the functional dependence:

$$\log(N(s)) = \log(C) - \tau \log(s)$$

i.e., a linear law with slope $-\tau$. For the one-dimensional sandpile, we can fit the function with an exponent of $\tau = 1.0 \pm 0.1$ at small sizes. Note the strong dependence on the system size, however as shown below.

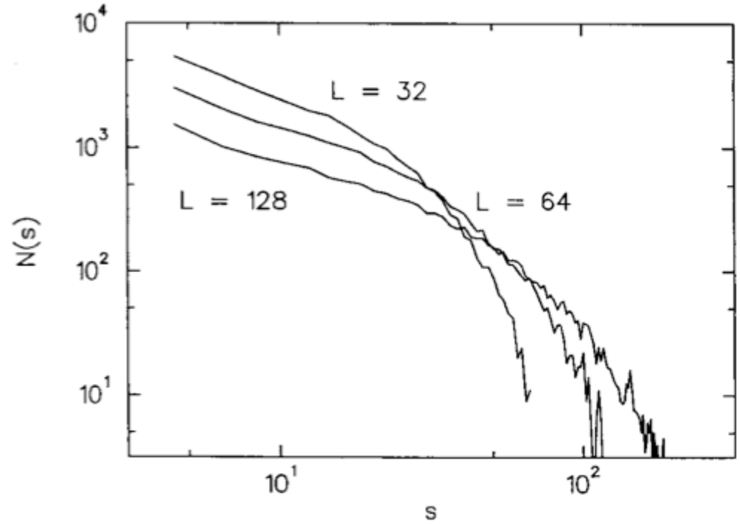


Figure 66: Distribution $N(s)$ of avalanches(size s) for 1−dimensional sandpile

The experiment that yielded these results was performed in such a way as to first fill the empty lattice with sand up to the point where adding a single grain will almost always produce a sliding event. This means, of course, that the left side of the lattice (x_0) is filled to roughly its maximal height given $h(x_L) = 0$, and almost all of the slopes are at the critical point $z_j = 2$. This is called the critical or minimally stable, state. Now, dropping a grain may result in an avalanche that disturbs the entire pile. Note, however, that while this critical state is disturbed by the avalanche, after the event the pile will return to the critical state due to the continuing dropping of the grains. Thus, the average slope of the pile represents a fixed point of the system: the pile will always return to it, but the state itself is highly unstable, and any small perturbation will result in another runaway event. This gives us a first indication of why sandpiles may be a good place to look for an analogy to the dynamics of living systems. There is a certain robustness to the pile, and it responds to perturbational events of all sizes that may even be catastrophic, but still will always return to an equilibrium state which, in turn, is singularly prone to another perturbation. While the analogy is clearly very rough at this point, we will continue to refine the model to make it closer to natural systems, and find that the pile, as abstract as it is, may embody some of the same principles that govern simple systems of self-replicating entities, albeit in an entirely different medium.

Before examining the dynamics of the sandpile in a more rigorous manner, let us first generalize it to 2 dimensions. Rather than first constructing a lattice with a height distribution $h(x, y)$ and then calculating the slopes, let us immediately work with the slopes $z(x, y)$, defined at the coordinates x, y of a 2−dimensional lattice. The rules for updating the lattice are again very simple. Adding a grain at site (x, y) results in

$$
\begin{aligned}
&z(x-1, y) \to z(x-1, y) - 1 \\
&z(x, y-1) \to z(x, y-1) - 1 \\
&z(x, y) \to z(x, y) + 2
\end{aligned}
$$

while a toppling event takes place if

$$
\begin{aligned}
&z(x, y) > z_c : z(x, y) \to z(x, y) - 4 \\
&z(x, y \pm 1) \to z(x, y \pm 1) + 1 \\
&z(x \pm 1, y) \to z(x \pm 1, y) + 1
\end{aligned}
$$

i.e., if the slope at one site is supercritical, it is distributed evenly among its four neighbors to the north, east, south, and west. In the examples we show here we choose $z_c = 4$, but this is not necessary. If the lattice is filled with sand again like in the one-dimensional case (for example by dumping an excessive amount of sand on the table and updating until no toppling takes place anymore), we can witness avalanches of all sizes by randomly dropping grains on the lattice at an arbitrary site and updating until no more site is affected. An example of an avalanche(25×25 lattice) is shown in the figure below.

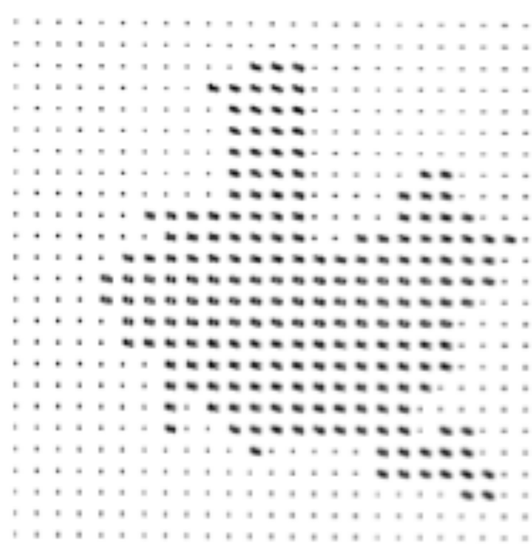


Figure 67: 2-dimensional lattice(25x25)-avalanche of 207 sites

We can repeat this experiment many times and ask again about the distribution of avalanche sizes(AVALANCHE code). Again, we will find power law behavior. However, this time the exponent of the power law is slightly different. In the figure below, we show the(binned) abundance distribution of 20,000 avalanches on a 50×50 lattice, fitted with an exponent $\tau = 1.12 \pm 0.05$.

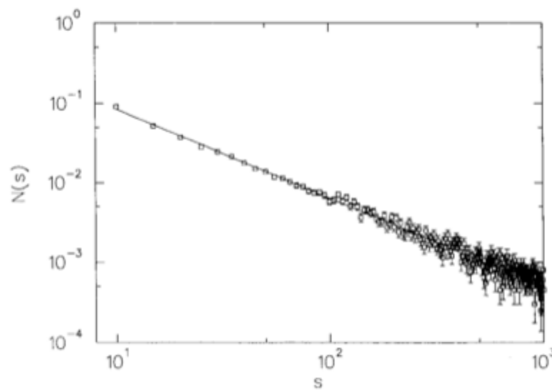


Figure 68: Size distribution(20000 avalanches) on 50x50 lattice

Clearly, the power of decay of the size distribution depends on the geometry of the system. For a three-dimensional lattice, BTW found an exponent $\tau_3 \approx 1.37$. The important point is that the law of decay does not involve any scale, as would be the case for an exponential law (such as radioactive decay, where the scale is the half-life) or a logarithmic law, which also requires a scale. Indeed, the power laws occur precisely in the case where there is no scale to set an average. More precisely, there is no scale of the same order as the range where we observe power law behavior. This is an important point that needs to be stressed, as for any system that we investigate experimentally or numerically, there is one fundamental scale: the size of the system under investigation, as we have seen in the one-dimensional case. Of course for the sandpiles introduced here, this is the size of the lattice. Clearly, we cannot find avalanches that involve more sites than are in the lattice, so we cannot expect the power law to be valid close to this limit. In general, therefore, we must be sure to study finite size effects by fitting the power law in such a way that the finite size is taken into account. In the figure above, this was unnecessary, as we stopped the fit at an avalanche size 10^3, whereas the ultimate cutoff is of course 2500, the maximal number of sites.

3.11.2.1. Summary

Let us try to summarize the main requirements that must be met in order to observe a self-organized critical system. Note that this list does not specify if any of the conditions are necessary or sufficient. Because as of yet, there is no universal theory of SOC, the list looks somewhat like a shopping list, but gives a good idea about where to look to find SOC: It appears that we want

1. A dissipating dynamical system with (locally) interacting degrees of freedom.

2. Propagation of fluctuations described by a dissipating transport equation.

3. Noise that can propagate through the entire system.

4. An infinitesimal driving rate.

Let us go through these one by one. Dissipation plays an important role in self-organizing systems for the simple reason that it would be almost impossible to transmit signals in a system that is both noisy and nondissipative. There is a very general theorem of thermodynamics that states that fluctuations, which as we shall see essentially represent the signal, are always accompanied by dissipation. We can intuitively understand that it would be hard to encode and decode signals into the fluctuations if there were no way to damp them. Local interactions are obviously an important ingredient in self-organization.

The second condition is almost a consequence of the first: only in very awkward systems that are dissipating and have locally interacting degrees of freedom is the transport of fluctuations not described by a dissipating transport equation (such as a reaction-diffusion equation). However, it is important to require that the entire system is accessible to the fluctuations and therefore to noise (third condition). Indeed, we cannot imagine self-organization to occur in a system that has parts that are not connected to the rest. It is, after all, the fluctuations that provide for the communication between all parts of the system, which results in self-organization.

Finally, and most importantly, the system has to be driven at an infinitesimal rate. This most important condition will reappear in many different guises. First, any system that is to self-organize has to be *driven* i.e., there must be a force that is responsible for providing the fluctuations that may or may not result in catastrophically big events. For the sandpile, this is of course the dropping of the grains, and in the protocol outlined earlier we made sure to specify that one waits until the avalanche is over before dropping another grain. This is not, strictly speaking, a physical situation. Since the system is driven by the dropping of the grains, we ought to define a constant driving rate: the number of grains dropped per unit time. However, even if this is only specified on average, we encounter the possibility, for every finite driving rate, that a grain of sand is dropped before the last avalanche is completed. We see immediately that if we increase the driving rate in such a manner that we do not wait for the avalanche to finish most of the time, we reach a situation of *steady flow* of sand. Clearly such a state is neither self- organized nor critical. Indeed, SOC only results in the limit where the driving rate is infinitesimally small. In this limit, the system will always return to its critical state, which is so prone to disruption. It is this self-tuning feature that has attracted the most attention, as all standard statistical systems that possess a critical point (such as the freezing transition from water to ice, or the freezing transition in magnetic Ising systems) sport a parameter (the temperature in the latter examples) that has to be fine-tuned to obtain this state. In SOC, the system apparently self-tunes to this state. Before attempting to understand this feature, we first describe another model that displays SOC and then show how SOC may be an important ingredient in the behavior of populations of self-replicating entities, specifically artificial ones.

3.11.3. SOC in Forest Fires

Here we investigate another simple model that displays SOC but that is easier to analyze in a systematic manner. The analytic treatment yields some important insights into the limits of self-organized behavior and points to possible generalizations. The Forest Fire model was first introduced by Bak, Chen and Tang. We will henceforth refer to their model as Model I. It was subsequently improved by Drossel and Schwabl(Model II). Let us first consider Model I.

Imagine a 2-dimensional lattice where each site can be in any one of the following states:

T [tree, susceptible to being burned]

B [a burning tree]

A [ashes: a tree that has burned down]

The dynamics of the model are determinedly the following update rules. In one time-step, any $B \rightarrow A$, i.e., a burning tree ceases to burn after being reduced to ashes. At the same time, any $BT \rightarrow AB$, i.e., a burning tree will ignite an adjacent one, while leaving ashes only at the next update. Also, new trees can grow from ashes; $A \rightarrow AT$ with a small probability $p < 1$. As we shall see, Model I turns out not to be critical. Rather, the dynamics are more similar to disease-spreading dynamics. The reason for this is that the system is not driven, so rather than returning to a critical state in a dissipating manner, we are witnessing waves of live and dead trees in the system. The crests of these waves are separated by a fixed distance that provides a scale in the system. We thus notice that the absence of a dissipating element renders the dynamics periodic rather than critical.

The required driving was added in the form of a small probability for trees to start burning spontaneously, i.e., a probability for lightning strike. Thus, the added rule is that a tree will start burning: $T \rightarrow B$ with a small probability $f << 1$. Implementing this algorithm reveals that the dynamics of the forest is such that, after a transition period, the forest settles into a steady state with a constant mean forest (nonburning trees) density $\bar{\rho}$. Let us estimate the dynamics of the forest in this steady state.

We calculate the time between two lightning strikes hitting a tree. This can be done in a general manner for forests of all dimensions, even though it becomes a stretch to imagine any forest with dimension larger than $d = 2$. If the forest is d–dimensional with linear dimension L (such that the total volume of forest is L^d), we obtain this time scale-the time between two strikes - by multiplying the probability for a tree to be hit, f , by the mean tree density $\bar{\rho}$ and the volume of trees, to obtain the rate at which trees are hit:

$$R_f = f\bar{\rho}L^d$$

The time between hits is $1/R_f$. We can then calculate the average number of trees growing between two lightning strikes, which is just the time between such strikes times the density of ashes $(1 - \bar{\rho})$ multiplied by the probability of tree growth p and again the volume of forest L^d. Thus, the average number of growing trees $\bar{s}$ is

$$\bar{s} = \frac{p(1-\bar{\rho})L^d}{f\bar{\rho}L^d} = \frac{p}{f}\frac{1-\bar{\rho}}{\bar{\rho}}$$

Since we are in a steady state situation, this is also the average number of trees *burning*. Due to the nearest-neighbor interactions in this model, fires form clusters of burning trees surrounded by ashes. As a consequence, the average number of burning trees is the same as the average cluster size in this model. As we shall see, the average cluster size in a model in which there is no scale other than the size of the system diverges as the system size is made arbitrarily large. This is a key element in SOC systems, so let us study this in more detail. Clearly, the distribution we are interested in is the size distribution of burning clusters, $N(s)$. Armed with this distribution, we can write down the total number of burning trees as

$$N_b = \sum_{1}^{s_{max}} sN(s)$$

and thus the probability for any site to be burning is

$$P(s) = \frac{sN(s)}{N_b}$$

where s_{max} is the maximum size of a cluster. Then, the *mean* number of burning trees is

$$\bar{s} = \sum_{1}^{s_{max}} sP(s) = \frac{\sum_1^{s_{max}} s^2 N(s)}{\sum_1^{s_{max}} sN(s)}$$

Let us determine what happens if we assume a power law form for the distribution of clusters $N(s)$:

$$N(s) \propto \frac{1}{s^\tau}$$

with a critical exponent τ. We find, not surprisingly, that the average size of the cluster is determined entirely by the only scale in the problem, s_{max}, as

$$\bar{s} \propto \begin{cases} s_{max}^{3-\tau} & \text{if } 2 < \tau < 3 \\ \frac{s_{max}}{\log(s_{max})} & \text{if } \tau = 2 \end{cases}$$

consequently, as s_{max} diverges when the system size tends to infinity $L \to infty$), the average size of clusters also must diverge. Let us go back to the estimate we derived previously

$$\bar{s} = \frac{p}{f}\frac{1-\bar{\rho}}{\bar{\rho}}$$

It becomes clear then that the average cluster size diverges only in the limit $f/p \to 0$, which we recognize as the condition for an infinitesimally small driving rate! Thus, the system approaches the critical point in this limit. Note that this condition more precisely is a condition for the separation of time scales. According to this, we should only observe SOC behavior in the limit where the time scale for growing trees is much smaller than the time scale to ignite them.

3.11.4. SOC in the Living State

Let us now make the concepts introduced in these abstract models more palpable as far as living systems are concerned. If self-organization to criticality is a feature of living systems, what distribution is showing power laws? What is the agent of self-organization? How are fluctuations transported throughout the system? What is the critical threshold parameter? We shall attempt to answer all of these questions for a simple system of self-replicating strings. First, we shall try to argue for the existence of a self-organized critical state in populations of self-replicating strings and then carry out experiments on the computer to support the claim.

There is no shortage of distributions in natural living systems that show power law behavior. We must be careful, however, not to attempt to find an SOC explanation for all of them. Most intriguing is perhaps the distribution of extinction events throughout the fossil record, as we saw earlier, depicted in the figure below

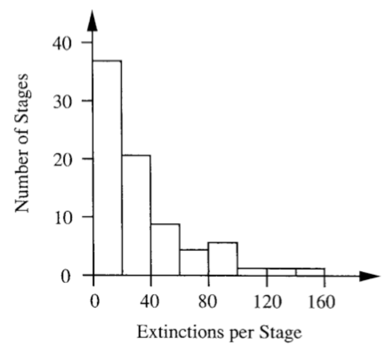


Figure 69: Distribution of extinction intensities based on recorded times of extinction of 2316 marine animal families

and the episodic nature of evolution associated with it: the punctuated equilibrium scenario advocated by Gould and Eldredge. The general idea is that some sort of ecosystem forms between all the competing species that makes them intimately connected. A small perturbation of this equilibrium would then result in extinction events of all sizes, which will leave a system that slowly creeps back to the self-organized state. Here, we shall investigate a simpler model that does not assume such a strong interconnection between the species.

Rather, the self-organizing agent we consider is information. We shall assume a population of strings of code whose only distinguishing mark is its genotype (describes the genetic constitution of an individual, that is the specific allelic makeup of an individual, usually with reference to a specific character under consideration), reflecting (in the cold, i.e., non-volatile, spots) the information it has stored about its environment.

Definitions of terminology are at http://en.wikipedia.org/wiki/Genotype.

The idea will be that if the population is sufficiently equilibrated, almost all of the strings (in a certain confined region) exhibit the same information. Consequently, they are susceptible to avalanches of all sizes instigated by rare mutation events that introduce new, powerful information into the population.

Let us set up our model (theoretically at first) such that we can identify the critical threshold parameter (the analogue of the sandpile's critical slope). Imagine a population of N strings self-replicating in an environment with limited resources (so that the number of strings is constant). Each string is composed of ℓ instructions taken from an alphabet of size D. The total number of different strings is the maximum number of genomes

$$N_{max} = D^{\ell}$$

usually a very large number. In the current population at time t, let there be N_g different such genotypes in existence. Note that the maximum number of genotypes represents the total volume of state space available to the population, while N_g is the volume currently occupied by the population. The population of N strings falls into subpopulations of n_i strings each of genotype i currently in existence, with each genotype i furthermore characterized by its replication rate ϵ_i. We then know that

$$N = \sum_i^{N_g} n_i$$

and we can define the average replication rate:

$$\langle \epsilon \rangle = \sum_i^{N_g} \frac{n_i}{N} \epsilon_i$$

Note that the average replication rate is intrinsically time-dependent and we have the additional constraint

$$\frac{dN}{dt} = 0 \rightarrow N = \text{ constant}$$

We can write down a simplified equation that determines the time dependence of the genotype occupation-number n_i, if we introduce the mutation rate R as the probability for mutating a site per unit time, and the corresponding probability for a single string of length ℓ to be hit by such a mutation, $p \approx R\ell$. Then, in an approximation where the average number of strings removed per unit time is proportional to the average replication rate, we can show that

$$\frac{dn_i(t)}{dt} \approx (\epsilon_i - \langle \epsilon \rangle) n_i(t)$$

This is a typical cosmic-ray mutation rate that affects each string in the population in the same manner. Several assumptions go into the writing this last equation, which here serves only to identify a critical parameter. On the one hand, we assumed that mutations are Poisson-random events that simply remove the genotype under consideration. In more realistic systems we need to include errors arising from an erroneous copy operation in the replication process. Such mutations behave differently than the external mutations we are considering here. Also, we ignore the fact that a mutation might produce a genotype of the type i by a mutation hitting some genotype j. In the limit where $N_g << N_{max}$ this is a safe assumption, but it does imply that last equation violates the number conservation equation $dN/dt = 0$. If we ignore all these problems for the moment and introduce the growth factor

$$\gamma_i = \epsilon_i - \langle \epsilon \rangle - R\ell$$

one can immediately solve the equation in the static limit, i.e., in the limit where the average replication rate is approximately independent of time, as

$$n_i(t) = N_0^i e^{\gamma_i t}$$

where N_0^i represents the population of genotypes of type i at time $t = 0$.

This equation implies that each genotype must either grow exponentially if its growth factor is larger than zero, or shrink in the same manner if it has a replication rate that renders the growth factor negative. Such a behavior is, however, incompatible with our assumption of equilibrium that guaranteed that the average replication rate is approximately time-independent. Thus, growth factors unequal to zero, while entirely possible, cannot possibly be a fixed point of the population, i.e., a point to which the population always returns. Rather, this point must be characterized by $\gamma_i \approx 0$ for *almost all* genotypes. This must then be the critical threshold parameter that organizes the entire population. Let us imagine a population poised at this state, where almost all γ_i vanish (except for those few negative γ_i that are results of mutations and are quickly purged from the population). Since no genotype dominates another, the situation is quasi-stable, even though genotypes are still being created and go extinct at a small rate that is determined by the number of hot instructions of the main genotype. Indeed, it is in general fair to assume that the population is dominated by a quasi-species, which we can imagine as the genotype specified by its cold instructions only. While there can in principal be several quasi-species in the population, they must all be *degenerate* in the replication rate (must all have the same replication rate), as otherwise one would drive the others to extinction. If enough time has elapsed since the establishment of the multi-species, we realize that the information that characterizes it has spread throughout the population. The population then is very much connected, held together by the invisible thread that is their common cold sites: their complexity. Every mutation that changes the replication rate of a string to one higher than the previous ϵ_{best}, the replication rate of the quasi-species, constitutes a ripple: a fluctuation in γ_i. Because the population has organized itself to a state where $\gamma_i \approx 0$ almost everywhere, such a fluctuation can indeed travel throughout the entire population, a criterion that we remember as being on our shopping list for self-organized criticality.

Thus, we find that in this simple model information indeed organizes the population, and since the information is part of the population, we can safely say that the population self-organizes. But is it a critical state, and which parameter determines the infinitesimal driving rate? The answer to both of these questions lies in the magnitude of the mutation probability per string. First, we easily recognize the small driving of the sandpile (initiated by the dropping of the grains) in the driving which results from mutations creating ever-fitter genotypes. In the same manner we see that a condition for the emergence of a self-organized critical state is that avalanches due to new inventions (i.e., events that created a substantially better replication rate) must be rare, so rare, in fact, that most avalanches are long over (i.e., have eradicated and driven to extinction each and every inferior genotype) before a new invention sweeps through the population. Otherwise, we would witness overlapping avalanches just as in the continuously driven sandpile, and no critical behavior could emerge. On the other hand, the probability of a mutation leading to a fitter genotype must not be too small, as otherwise we would not find any avalanches on the time scale of our observation, and again no SOC would result. Thus, SOC in living systems is predicated upon a mutation rate just right so that the critical state is formed. While this sounds much like the tuning of parameters in standard statistical phase transitions, we shall argue that this critical mutation rate is actually achieved for most reasonable parameters of the living state, and furthermore there is evidence, both experimental and theoretical, that living systems actually arrange for this mutation probability to be just in the critical range.

Let us now turn to observable consequences of self-organized criticality in such simple systems of self-replicating strings. First, we expect to see a punctuated picture of evolution, where fitness does not change gradually, but in bursts that are initiated by a genotype that found a better way to exploit the environment. The time between such events must be large compared to the time it takes to communicate this new information to the rest of the population. This latter time is the relaxation time of the system. Let us look at the figure below, a typical time-evolution of fitness (i.e., replication rate) of a population of self-replicating strings; results from computer code of the **tierra** world - a UNIX simulation code.

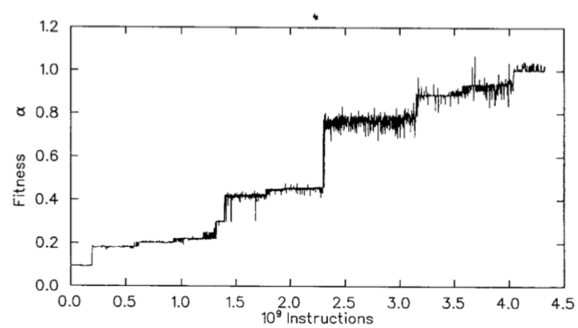


Figure 70: Normalized replication of best of population measured every million executed instructions for a population of tierran strings, with $D = 32$, $\ell \approx 100$ and $N \approx 1000$

The staircase structure of the fitness curve is very evident, underlining that evolution in that system indeed proceeds in such a way that periods if long stasis (the plateaus in the figure) are interrupted by very brief periods of invention that move the population to a new plateau rather quickly, compared to the length of the plateaus. Time is measured in total number of instructions executed by the population, which is a convenient measure even though it does not scale well with population size.

Let us first look for any signs of periodicity in the dynamics. This would be revealed by a preferred frequency in the power spectrum of a time-series such as in the above figure. This spectrum is shown in the figure below, and reveals no bumps of any kind: the spectrum is, except for the finite size effects at small and large frequencies, a pure power law $P(f) \sim 1/f^2$:

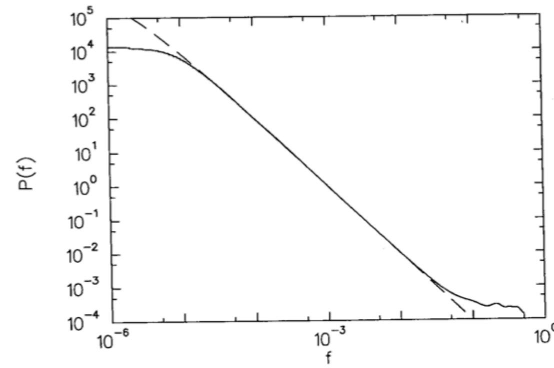


Figure 71: Square of Fourier transform (power spectrum) of the time-series shown in the last figure. The dashed line is a fit to $P(f) \sim 1/f^2$

Note that power law behavior for the spectrum is a necessary, but not a sufficient, condition for self-organized criticality. It reveals that there are no periodic dynamics in the population, but does not rule out certain random processes that have a power law frequency spectrum but show no signs of self-organization or critical behavior. To extricate the latter, we must look at temporal correlations and thus ask about the distribution of plateau lengths (in units *millions of executed instructions*). This seems to be a natural question, since the beginning of a plateau certainly signals the birth of a new multi-species, while the end of a plateau heralds its demise. Thus, the length of a plateau reflects the time of domination of a multi-species. Since the total number of representatives of that multi-species that ever lived must be proportional to the length of time it dominated the population, the length of the plateaus may also tell us something about the absolute size (total number that ever lived) of a species. Clearly, however, a single run such as depicted above will not be able to reveal this size distribution to us.

Let us couch the question about the distribution into a more technical language. We are here interested in the inter-event interval distribution, where an event is defined as the emergence of a genotype noticeably fitter than those present before that event. The restriction to noticeably larger fitness improvements is a rather practical one, enabling us to pick out visually the beginning of a new epoch. In each run, we witness between five and ten such events on average, but sometimes less and sometimes more. In the particular set of runs that are reported here, the strings were adapting to a specific landscape created by the user: one in which the strings were guided to learn how to add integer numbers. The specific kind of landscape is not our concern here, even though this is a subject of fundamental importance shortly. We shall only demand that the landscape contains enough information (to be discovered by the strings) that no population will ever exhaust it during the time that we spend in observing the population. This, of course, is just the requirement that there always be a very small driving rate for the population: if the landscape is exhausted, or in other words, if there can be no more fitness improvement, we expect to lose the SOC behavior.

For such effectively infinite landscapes, we observe the adaptive behavior of the population at a fixed mutation rate and count the number and lengths of epochs. The distribution of number of epochs as a function of the length of the epoch is our inter-event-interval distribution. Let us look at the results, obtained in 50 runs under identical conditions, at a moderate mutation probability of the order 10^{-6}. The sizes were obtained by measuring (painstakingly) the lengths of every epoch by hand (visual inspection first, then extracting the beginning and end of the epoch through the identification of the jumps in fitness), so as not to bias the analysis. As the data obtained with *tierra* are somewhat noisy, a program designed to find the beginning and end of an epoch might easily be fooled. Still, the 512 epoch sizes that were extracted from these runs need careful statistical interpretation. For example, let us try first to bin the data into bins of length 100 (millions of executed instructions). For each data point that we obtain (the first one covering epochs of lengths between 1 and 100, centered at 50), we estimate the error in that number to be due entirely to statistics, i.e., we assume a $\sqrt{N}$ error, which is certainly appropriate for those bins with very few entries (at large epoch sizes), but could underestimate the error for small epochs, which may be affected by systematic trends. The result of such an analysis is shown in the figure below.

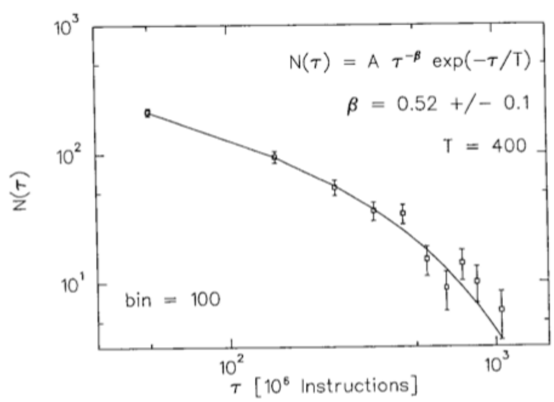


Figure 72: Fit of the binned abundance distribution with bin size 100

We attempt to fit the data with the smallest number of parameters that seems reasonable to us, in the form of a power law

$$N(\tau) = A\frac{1}{\tau^{\beta}}e^{-t/T}$$

where T is our cutoff parameter. The cutoff simply reflects the fact that our measurements are finite in time: most runs were terminated after between 500 and 2000 million instructions had passed (while a few were allowed to exceed the 4000 million mark, such as the one in earlier figure. Nevertheless, we will certainly undersupply lifetimes larger than 500 million. This is taken into account here by an exponential decay. There is a certain amount of freedom in the choice of a cutoff function, which can affect the result of the fit. Consequently, care must be taken in that choice also. In general, a full finite-size scaling analysis, which would involve repeating the experiment and cutting off the experiment at different times, should substitute for choosing a cutoff function.

As the resulting fit with $\beta = 0.52$ and $T = 400$ has a $\chi^2 = 1.1$ per point, we can at least be confident that we are not over-fitting the data. Yet we should not be overconfident, as we find that β depends strongly on the size of the bins chosen. Choosing a bin size of 200 results in data with smaller error bars, but the fit changes accordingly, as we can see in the figure below.

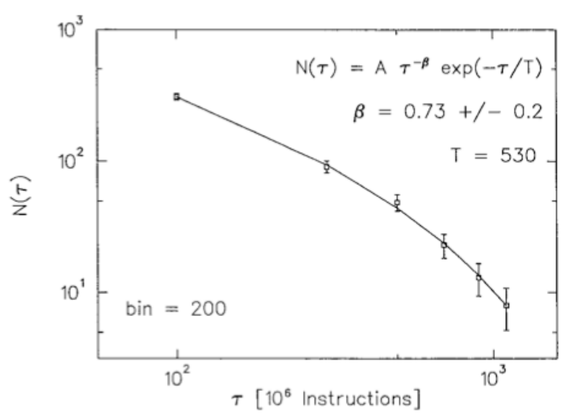


Figure 73: Fit of the binned abundance distribution with bin size 200

At the same time, our measure of confidence, the χ^2 coefficient drops to $\chi^2 = 0.44$, suggesting that we overfit the data.

In the face of such adversity, we can choose another path to extract meaningful results. Consider a different distribution $M(\tau)$, obtained from our distribution $N(\tau)$, by asking about the distribution of events with a size larger than τ. As this distribution, which has the same power law behavior, can be obtained from the first:

$$M(\tau) = \frac{1}{\tau}\int_\tau^\infty N(\tau)d\tau$$

but has much better statistics (due to the effect of summing the distribution at each τ), we may have better luck fitting that distribution without the need of large bins. Fortunately, the functional form for the fit of $M(\tau)$ is dictated to us, as the integral above can be done analytically, resulting in an incomplete Γ–function:

$$M(\tau) = A'\frac{1}{\tau}\Gamma(1-\beta,\tau/T)$$

Note that fitting $M(\tau)$ should yield the same result as fitting $N(\tau)$ (as we use the same parameters and the same functional form for N), except for the statistics and a much smaller bin size. The result is shown in figure below, and suggests $\beta = 0.57$.

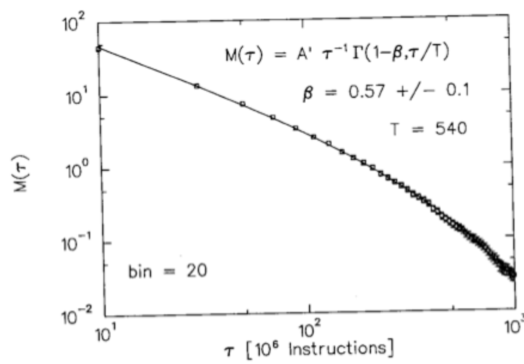


Figure 74: Fit of the integrated abundance distribution

The analysis presented here leaves us fairly confident that in the populations we studied, the time between avalanches is indeed distributed as a power law. Still, we cannot be very confident in the numbers that resulted from the fits, and we must wait for a much more extensive investigation with at least an order of magnitude more epochs, and an analysis of the influence of system size and mutation rate on the critical parameter β, to be convinced that such systems self-organize to a critical state. Fortunately, with the *avida* system, such an investigation is within the realm of possibility.

Even though the results we have in hand now are not fully conclusive, let us speculate about their significance. It appears that there is no temporal scale of the order of the observational time in the system that would determine an average plateau length $\bar{\tau}$. In other words, a numerical estimate of the average plateau length $\bar{\tau}$ will always reflect the length of time the system is observed, instead:

$$\bar{\tau} \propto \tau_{max}$$

This implies that $\bar{\tau}$ is, in self-organized critical systems, a meaningless quantity that is never independent of the way we observe the system. Therefore, it can also not be used to make any *predictions* about the temporal behavior of same. In other words, it is impossible, armed just with statistics gathered from the past, to infer how much longer, on average, the current epoch will last. This is a somewhat satisfying side effect of criticality and the absence of scales.

If the time between events has a scale-free distribution (or, more precisely, is governed only by the size of the system), what about the distribution of *sizes* of events? The latter can be defined in two ways. If we consider as the size of an event the total number of members of the new dominating species that will be produced (or equivalently the total number of subspecies it will spawn) before it is driven to extinction by a new species, we have very good evidence (both from natural and artificial systems) that this distribution is also scale free as we will see shortly.

As far as the distribution of sizes of vertical jumps is concerned, the situation is somewhat more tricky, as this may depend on the kind of landscape on which the population is evolving. The landscape used in the analysis above is probably too poor, and the runs not long enough, to take a stab at estimating this distribution. Still, the results are compatible with the assumption that there is also no scale in the distribution of fitness increases, at least in an infinite landscape. This would indicate that any time the population climbs a new peak in the landscape, the number of possible increases in fitness, and the size of such possible increases, is *unchanged*. This is in stark contrast to finite landscapes, where each improvement reduces the number of possible future improvements. We will discuss fitness landscapes in much more detail shortly.

In the meantime, let us speculate about the nature of fitness curves in landscapes where neither the time nor the size of a fitness jump is dictated by any scale. Such curves are fractals, and look similar at any scale of observation. So, if, time series (plateau diagram) were a true fractal, then what appears as a plateau to the naked eye would, under the magnifying glass, be resolved into many very small jumps (as is almost discernible at the beginning of the fitness history in the figure). Because of the finiteness of the actual landscape in the experiment, and the noisiness of the data, this aspect is difficult to discern for the later part of the history. Fractals of the kind we have in mind are termed *Devil's Staircases* in the literature. If evolutionary histories are generally Devil's staircases, a number of interesting consequences can be obtained for evolutionary systems in general. First, it would imply that no fitness jump is too big to be explained by common mutational events such as the ones that drive the populations in *tierra*. As the evolutionary system self-organizes to the critical state, it poises itself for such grandiose events to occur, albeit in an unpredictable manner. Thus, there is no need for more than one theory to explain all sizes of evolutionary advances. Second, it would imply that, because the fitness histories are self-similar, certain global aspects of evolution occurring on very large time scales may already be present in the microscopic histories, and can be inferred from them.

Before treading on arguably more solid ground, let us issue a few *caveats* about the preceding discussion. The arguments presented applied to finite populations of self-replicating strings, in which the only means of competition was acquiring a higher replication rate. While this may have been a scenario present on the very early earth, such populations are unrealistic in the present world. specifically, we must keep in mind that if speciation occurs on a higher level of taxonomy (such as to a genus), we must remember that the population may then segment into parts that do not compete directly anymore. Also, the evolution of cells, or more generally hosts that carry the information present in the genome, may affect the dynamics of populations. While as a consequence of these limitations the lessons learned from simple systems of self-replicating strings must not strictly be applicable to all living systems, it is conceivable that the central trait, self-organization to criticality, is a universal characteristic of all evolving systems.

Chapter 4

Systems Exhibiting SOC

4.1. The Brain

The human brain is able to form images of the complex world surrounding us, so it might seem obvious that the brain itself has to be a complex object. However, it is not necessarily so. We have seen that complex behavior can arise from models with a simple architecture through a process of self-organization. Perhaps the brain is also a fairly simple organ.

Starting from a native state with little structure, the information about the surrounding environment is coded into the brain by a process in which the brain self-organizes into a critical state. In analogy with the sandpile, a "thought" may be viewed as a punctuation, i.e., a small or large avalanche triggered by some minor input in the form of an observation, or by another thought.

The brain contains trillions of neurons. Each neuron may be connected to thousands of other neurons. The wiring mechanisms of individual neurons are fairly well understood, but how do trillions of neurons work together to form the emergent process we call thinking? Comparing with the way computers work, the function of the computer is not apparent from the properties of the individual transistors making up the computer. Those who construct computers do not even have to know how transistors work. The function of the computer comes from the way these interconnected transistors work together.

There is at least one major conceptual difference between the computer and the brain. The computer was built by design. An engineer put together all the circuits and made it work. No engineer - no computer. However, there is no engineer around to connect all the synapses of the brain. Even more to the point, there is no engineer available to make adjustments every time the outer world poses the brain with a new problem. One might imagine that the brain is ready and hard-wired from birth, with its connections formed through biological evolution, with all possible scenarios coded into the DNA. This does not make any sense. Evolution is efficient, but not that efficient. Indeed, the amount of information contained in the DNA is sufficient to determine general rules for neural connections but vastly insufficient to specify the whole neural circuitry. While there is some hard wiring - a lobster brain is different from a human brain - the functionality has to evolve during the lifetime of an individual. This means that the structure has to be set/organized rather than designed. Brain function is essentially created by the problems the brain has to solve. Thus, to understand how the brain functions it is important to understand the process of self-organization. It is not enough to take the brain apart at some given instant and map out all existing connections, just as we don't understand the sandpile by just making a map of all the grains at some given point in time. Essentially all modeling of brain function from studying models of neural networks has ignored the self-organized aspects of the process, but has concentrated on designing a working brain model by engineering all the connections of inputs and outputs. This is good enough if the system is going to be used for some engineering purpose, such as pattern recognition, but it is basically misguided when it comes to understanding brain function.

4.1.1. Why Should the Brain Be Critical?

One may argue at least two different ways that the brain must be critical. First, consider a brain that is exposed to some external signal, representing for instance a visual image. The input signal must be able to access everything that is stored in the brain, so the system cannot be subccritical, in which case there would be access to only a small, limited part of the information. Grains dropped on a subcritical sandpile can only communicate locally by means of avalanches. The brain cannot be supercritical either, because then any input would cause an explosive branching process within the brain, and connect the input with essentially everything that is stored in the brain. This can be seen in a different way. Consider a neuron somewhere in the brain, and an output neuron at a distance from that neuron. By changing the properties of the neuron, for instance by increasing or decreasing the strength of its connection with a neighbor neuron, it should be possible to affect the output neurons in the brain; otherwise that neuron would not have any meaningful function. If the brain is in the frozen subcritical state, there will be only a local effect of that change. If the brain is in a chaotic disordered state with neurons firing everywhere, it is not possible to communicate with the output neuron, and affect its signal properly, through all the noise. Hence, the brain must operate at the critical state where the information is just barely able to propagate. At the critical state the system has a very high sensitivity to small shocks. A single grain of sand can lead to a very large avalanche. We say that a critical system has a large susceptibility. Of course, the avalanches at the critical state in the sandpile do not perform any meaningful function, so our problem is to teach the avalanches to connect inputs with the correct outputs.

How does the brain organize itself to the critical state? In the sand model, the criticality was ensured by adding grains of sand at a very slow rate, one grain at the time.

Work has been done on neural network models of steering processes, such as tracking a flying target. The network was kept at a critical state by a feedback mechanism that would keep the output, rather than the input low.

We then construct a toy brain model using these ideas.

4.1.2. The Monkey Problem

One of the problems in describing brain function is the uncertainty in determining what exactly is the problem that the brain is actually *solving*. What, precisely is the function of the brain? It isn't enough to simply say that it is *thinking*. A good deal of brain research traces the location of the activity of the brain when a person is subjected to various stimuli, but gives next to no insight on general principles. Before constructing a model it important to define a specific problem that the brain has to solve.

A hungry monkey is confronted with the following problem. To get food, it must press one of two levers. At the same time it is shown a signal that can either be red or green. If the red signal(#1)l is on, it has to press the left lever; if the green signal(#2) is on, it has to press the right lever. The signal is all the information the monkey has in order to figure out which is the right button at every instant. The signal switches back and forth between the red one and the green one. If the correct lever is pressed, the monkey will get a couple of peanuts. The monkey learns the correct reaction after a "learning" period of trial and error. If the outside world changes, i.e., the "correct" buttons are switched, the monkey should be able to modify its behavior.

A block diagram (entire network had 256 neurons) of the situation is shown below.
The outer world sends a signal to some of the neurons in the brain, through the eyes of the monkey. The resulting action of the monkey is fed back to the outer world, which in turn provides feedback to the monkey and its brain by either giving or denying food. After a number of wrong tries the monkey learns to perform properly.

The fact that the function of the brain has to be self-organized puts severe constraints on any brain model. In the model, neurons were arranged on a two-dimensional grid. The neurons in each row are connected with three neurons at the row below that neuron, as indicated by the arrows in the block diagram. One also needs to study a network where the connections were completely random. This network functions almost as well, but is more difficult to illustrate graphically. The firing signal from the environment is represented by pulses

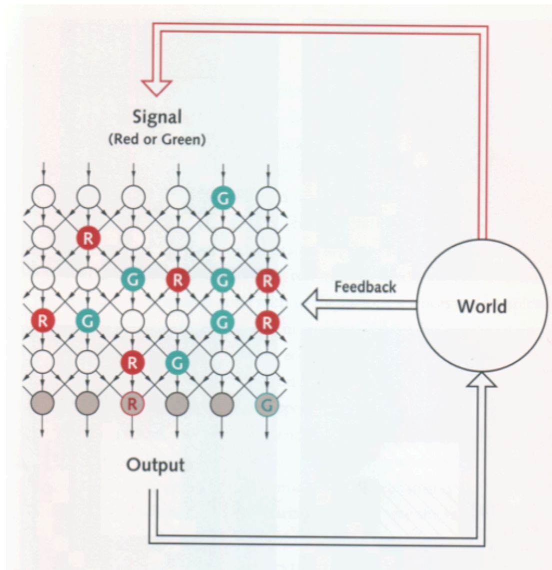


Figure 1: Block Diagram

that are fed into a number of random neurons, the red ones if the signal is red, the green ones if the signal is green. It is a simple matter to define the initial network. It took only a couple of sentences to specify the geometry. Not much more information is needed to specify a larger network. The brain model *at birth* is a simple structure.

At each step, each of the neurons are either in a *firing* state or a *non-firing* state, according to whether their input voltage, or potential, exceeds a threshold. Firing neurons send electric signals to other neurons, driving their potentials closer to the threshold. This is very similar to the sand model, in which a toppling occurs if the height exceeds a threshold. The signal from a firing neuron is sent to the inputs of the three neurons in the layer below. The input of each neuron depends on the strengths of the connections between that neuron and the one that fired. In addition, a small amount of noise was added to all the inputs.

The output is given by the neurons in the bottom row.

The input signals, each with a fixed number of random input neurons, were chosen. For each input, a pair of output neurons was defined in the bottom row. The input signals were switched each 200 time steps (or when complete success, meaning that the selected output neurons were active while all other neurons were not for 50 time steps). The feedback from the environment was sent to all the neuron connections in a totally democratic way. This could represent some hormone, or some other chemicals fed into the brain. In this sense, our model is fundamentally different from most other neural network models in which an amount of outside computation, not performed by the network itself, has to be carried out to update in detail the strength of the connections.

If there is a positive feedback signal, that is, the proper output neurons fired, all the connections between simultaneously firing neurons are strengthened whether or not they were responsible for the favorable result. If there is a negative signal, all these-firing connections are weakened slightly. All other non-firing connections are left alone.

This type of scheme has been tried before without much success, precisely because of the weak communication between inputs and outputs, which makes learning prohibitively slow. Also, whenever the red signal is on, the pattern that is favorable for the green light is forgotten, and vice versa. An extra ingredient is needed. If there are too many output cells firing, all thresholds are raised. The function of this mechanism is to keep the activity as low as possible, and it results in setting up the brain in a critical state. To think clearly you have to keep cool! If the activity becomes too low, for instance if the brain is asleep with no output, the thresholds of all neurons are lowered and more neurons fire. The monkey becomes hungry and activates the brain. Note that all of these processes are biologically reasonable; they can be performed by chemicals being sent around in the brain without a specific address.

The figure below shows the performance, measured as the relative number of output neurons that are firing correctly.

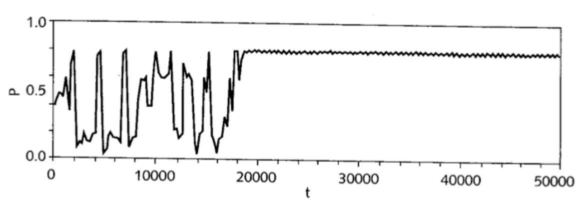


Figure 2: Brain Performance

After an initial period of very erratic response, which we call the learning period, the toy brain eventually learns to switch quickly back and forth between the correct responses. The transition is very sharp.

4.1.3. The Brain and River Networks

What happens inside the network during the learning process? Through a complicated organizational process, the system creates internal contacts or connections between selected parts of the input signal and the correct output cells. The process can be thought of as the formation of a river network with dams (thresholds). When the output is incorrect the riverbeds are raised (the firing connections are weakened) and the dams are lowered (the thresholds are lowered), which causes water to flow elsewhere; this is the process of *thinking*. During that period there is an increased activity in the brain.

If at some point the response happens to include the correct cells, the riverbeds are deepened but all the dams are raised, which prevents the signal from going elsewhere. The system tries to lower the activity as much as possible while still connecting with the correct output. At some point the thresholds will become too high and the output becomes too low (the monkey loses concentration), but the system immediately responds by lowering the threshold. The small amount of random noise prevents the network from locking into wrong patterns, with too deep riverbeds, from which it cannot escape. It allows the network to explore new possibilities.

The process is somewhat similar to the mechanism for the evolution model. During periods of low fitness (improper connections with output) there is a relatively great activity where the system tries many different connections until it finds a state with correct connections (high fitness), in which most of the neurons are passive, just as the species in the evolution model have fitnesses above threshold in the periods of stasis, in which they have *learned* to connect properly with the environment.

The ability of fast switching is related to the system operating at the critical point. The signal is barely able to propagate through the system. The flow pattern is very similar to critical river networks, and does not look like a flooded system with large lakes. A traditional neural network model corresponds to a hooded system with roughly half the neurons firing at all times, resulting in poor communication. At the critical point, the system can easily switch from a state in which one system of rivers is flowing to a state in which another system of rivers is flowing to a different output. We exploit the high susceptibility of the critical state.

The network is robust to damage. After $150,000$ steps, a block of thirty neurons was removed from the network. After a transient period the network had relearned the correct response by carving new rivers in the network. The performance is shown in below.

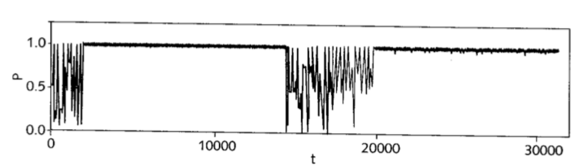


Figure 3: Network Performance

In other words, instead of using some features of the input signal the system learns to use other features. Think of this as replacing *vision* with smell.

Our toy brain model is not a realistic model of the brain. Its sole purpose is to demonstrate that aspects of brain function can be understood from a system starting with a minimum amount of structure. The ability of the brain to function is intimately related to its dynamics, which organize knowledge about the outer world into critical pathways of firing networks in an otherwise quiet medium. The criticality allows for fast switching between different complicated patterns without interference. The memory is encoded as a network of riverbeds waiting to be filled up under the relevant external stimulus.

The feedback between reality and the individual through perception of the physical world determines the interconnected structure of the brain.

4.2. On Economics

So far we have proceeded from astrophysics to geophysics, and from geophysics to biology and the brain. We now take yet another step in the hierarchy of complete phenomena, into the boundary between the natural world and the social sciences - a dangerous area for me to venture! Humans interact with one another. Is it possible that the dynamics systems of human societies are self-organized critical systems? After all, human behavior is a branch of biology, so why should different laws and mechanisms be introduced at this point? Here two specific human activities will be considered, namely economics and traffic. Perhaps these phenomena are simpler than other human activities. At least, the activities can be quantified and measured, in terms of prices, volumes, and velocities. That might be the reason that economics exists as a discipline independent of other social sciences.

4.2.1. Equilibrium Economics Is Like Water

Traditional equilibrium interpretations of economics resemble the description of water flowing between reservoirs. Goods and services flow easily from agent to agent in amounts such that no further flow or trade can be advantageous to any trading partner. A small change in the economy, such as a change in the interest rate, causes small flows that adjust the imbalance.

Specifically, consider two agents each having a number of apples and oranges. One has many oranges and few apples, the other has many apples and few oranges. Since having too many apples or oranges may not be desirable, they trade some of their apples and oranges. Before trading, oranges are worth more for the agent with too few oranges than for the other agent. They trade a precise amount such that oranges are worth exactly the same number of apples for the two agents, which removes any incentive for further trading. At that point it is not to anybody's advantage to trade further. The agents are **perfectly rational**, so they both know how many apples to buy and sell, and what the exchange rate should be. They are perfectly predictable. In our water flow analogy, the water in two connected glasses of water will flow from one glass to the other until the equilibrium point where the levels in the two glasses are the same.

In equilibrium systems, everything adds up nicely and linearly. It is trivial to generalize to many agents; this simply corresponds to connecting more glasses of water. The effect on the water level from adding several drops of water is proportional to the number of drops. One does not have to think about the individual drops. In physics, we refer to this kind of treatment where only a global macrovariable, such as the water level, is considered as a *mean field approximation*. Traditional economics theories are mean field theories in that they deal with macrovariables, such as the the gross national product (GNP), the unemployment rate, and the interest rate. Economists develop mathematical equations that are supposed to connect these variables. The differences in individual behavior average out in this kind of treatment. No historical accident can change the equilibrium state, since the behavior of rational agents is unique and completely defined. Mean field treatments work quite well in physics for systems that are either very ordered or very disordered. However, they completely fail for systems that are at or near a critical state. Unfortunately, for economists, there are many indications that economics systems are in fact critical.

Traditional economics does not describe much of what is actually going on in the real world. There are no stock market crashes, nor are there large fluctuations from day to day. Contingency plays no role in perfectly rational systems in which everything is predictable. Equilibrium economics does not even work for the simple example of - the agents trading apples and oranges. Neither one knows how much oranges and apples are worth for the other agent. When offering apples for sale, they may sell too cheaply, or ask too high a price, so that the proper equilibrium will never be reached. They may end up with more apples than they want. Agents in reality are not perfectly rational. In discussions with traditional economists, one might argue that their economics theory has to include me, and that I certainly am not perfectly rational, as they themselves argue so convincingly. The obsession with the simple equilibrium picture probably stems from the fact that economists long ago believed that their field had to be as *scientific* as physics, meaning that everything had to be predictable. What irony! In physics detailed predictability has long ago been devalued and abandoned as a largely irrelevant concept. Economists were imitating a science whose nature they did not understand!

Perfect rationality makes things nice and predictable. Without this concept, how can you characterize the degree of ignorance among agents, and how can you then predict anything? There exists a stubbornness to give up the ideas of perfect rationality. The absurdity of the *perfect rationality concept* in a world of real people is clear. But many still prefer the *perfect rationality concept.*

4.2.2. Real Economics is Like Sand

But economics is like sand, not like water. Decisions are discrete, like the grains of sand, not continuous, like the level of water. There is friction in real economics, just like in sand. We don't bother to advertise and take our apples to the market when the expected payoff of exchanging a few apples and oranges is too small. We sell and buy stocks only when some threshold price is reached, and remain passive in between, just as the crust of the earth is stable until the force on some asperity exceeds a threshold. We don't continually adjust our holdings to fluctuations in the market. In computer trading, this threshold dynamics has been explicitly programmed into our decision pattern. Our decisions are sticky. This friction prevents equilibrium from being reached, just like the friction of sand prevents the pile from collapsing to the flat state. This totally changes the nature and magnitude of fluctuations in economics.

Economists close their eyes and throw up their hands when it comes to discussing market fluctuations, since there cannot be any large fluctuations in equilibrium theory. *"Explanations for why the stock market goes up or down belong on the funny pages"*, says Claudia Goldin, an economist at Harvard. If this is so, one might wonder, what do economists explain?

The various economic agents follow their own, seemingly random, idiosyncratic behavior. Despite this randomness, simple statistical patterns do exist in the behavior of markets and prices. Already in the 1960s, a few years before his observations of fractal patterns in nature, Benoit Mandelbrot analyzed data for fluctuations of the prices of cotton and steel stocks and other commodities. Mandelbrot plotted a histogram of the monthly variation of cotton prices. He counted how many months the variation would be 0.1% (or −0.1%), how many months it would be 1%, how many months it would be 10%, etc as shown below.

He found a *Levy distribution* for the price fluctuations, as shown below.

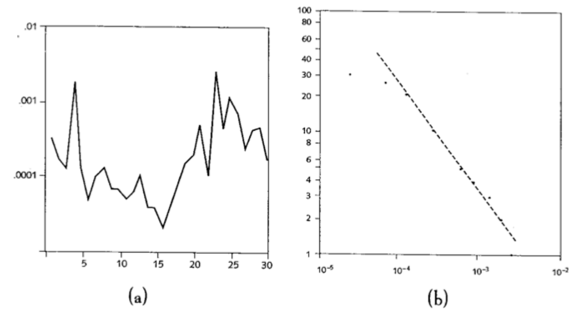


Figure 4: Mandelbrot Cotton Price Data

The important feature of the Levy distribution is that it has a power law tail for large events, just like the Gutenberg-Richter law for earthquakes. His findings have been largely ignored by economists, probably because they don't have the faintest clue as to what is going on.

Traditionally, economists would disregard the large fluctuations, treating them as *atypical* and thus not belonging in a general theory of economics. Each event received its own historical account and was then removed from the data set. One crash would be assigned to the introduction of program trading, another to excessive borrowing of money to buy stocks. Also, they would *detrend* or *cull* the data, removing some long-term increase or decrease in the market. Eventually they would end up with a sample showing only small fluctuations, but also totally devoid of interest. The large fluctuations were surgically removed from the sample, which amounts to throwing the baby out with the bathwater. However, the fact that the large events follow the same behavior as the small events indicates that one common mechanism works for all scales just as for earthquakes and biology.

How should a generic model of an economy look? Maybe very much like the punctuated equilibrium model for biological evolution described earlier. A number of agents (consumers, producers, governments, thieves, and economists, among others) interact with each other. Each agent has a limited set of options available. He exploits his options in an attempt to increase his happiness (or *utility function* as the economists call it to sound more scientific), just as biological species improve their fitness by mutating. This affects other agents in the economy who now adjust their behavior to the new situation. The weakest agents in the economy are weeded out and replaced with other agents, or they modify their strategy, for instance by copying more successful agents.

This general picture has not been developed yet. However, there is a simplified toy model that offers a glimpse of how a truly interactive, holistic theory of economics might work.

4.2.3. Simple Toy Model of a Critical Economy

We consider a sandpile-type model of economics. The idea was to construct a simplified network of consumers and producers. This model was a very productive, though rather painful, collaboration, reflecting the very different modes of operating in physics and economics.

Theoretical economists like to deal only with models that can be solved analytically with pen and paper mathematics. Physics is a much simpler science than economics, but nevertheless very rarely are we able to *solve* problems in the mathematical sense. Even the world's most sophisticated mathematics is insufficient to deal rigorously with many problems in physics. Sometimes we use numerical simulations; sometimes we use approximate theories. Surely, some of these approximations must look horrifying to a pure mathematician. However, although sometimes based on sheer intuition, they work well and provide a good deal of insight into the relevant physics. The physicist performs one dirty mathematical trick after another. Invariably, however, there is a mathematician running after him, who will eventually almost catch up with him and yell, *"It was all right what you did"*.

It appears to me that economics, because of the complexity of the systems involved, does not call for exact mathematical solutions. Indeed, the model that was developed, despite its simplicity, could not be solved mathematically. Numerical simulations were done on the model. Indeed, the model was critical, with avalanches of all sizes. Continued work on the problem eventually led to a model that could be solved mathematically without sacrificing the scientific content, to everybody's satisfaction.

The model is illustrated in in the figure below.

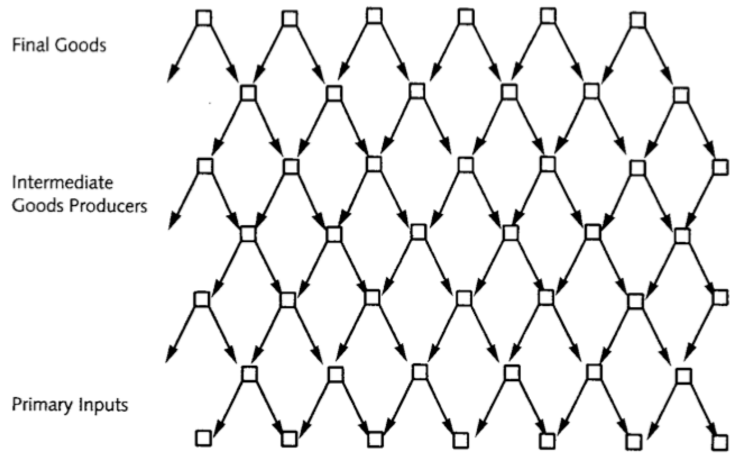


Figure 5: Economics Model

It is a network of producers, who each buy goods from two vendors, produce goods of their own, and sell them to two customers. The producers may start the process by having random amounts of goods stored, or they may start with nothing. It makes no difference. At the beginning of each period, say each week, the producers receive orders of one or zero units from each customer. If they have a sufficient amount of goods in stock, they transfer it to the customers; if not, they send orders to their two vendors, receive one unit from each, and produce two units to fill the order. If they have one unit left after these transactions, they store it until next week. Each producer thus plays the dual role of selling to his customers, and buying from his vendors. The process starts at the upper row in the network, which represent consumers, and ends at the bottom row, which represents the producers of raw materials.

First,we considered the situation where each week there would be a single *shock* triggering the economy with only one consumer demanding goods. This initial demand leads to a *trickle-down* effect in the network. The figure below shows a typical state of the network, with each producer marked by the number of goods he has in stock after completing the previous week's trades. An empty circle indicates zero units, a full circle one unit. The first vendor has nothing in stock. He receives two units from his vendors, sells one unit to the consumer, and keeps one unit in stock for the next week. His vendors actually did not have the demanded products in stock, and had to send orders further down in the network. After a number of events, the avalanche stops. The figure shows the extent of the avalanche, and the amount the vendors have in stock at the end of the week.

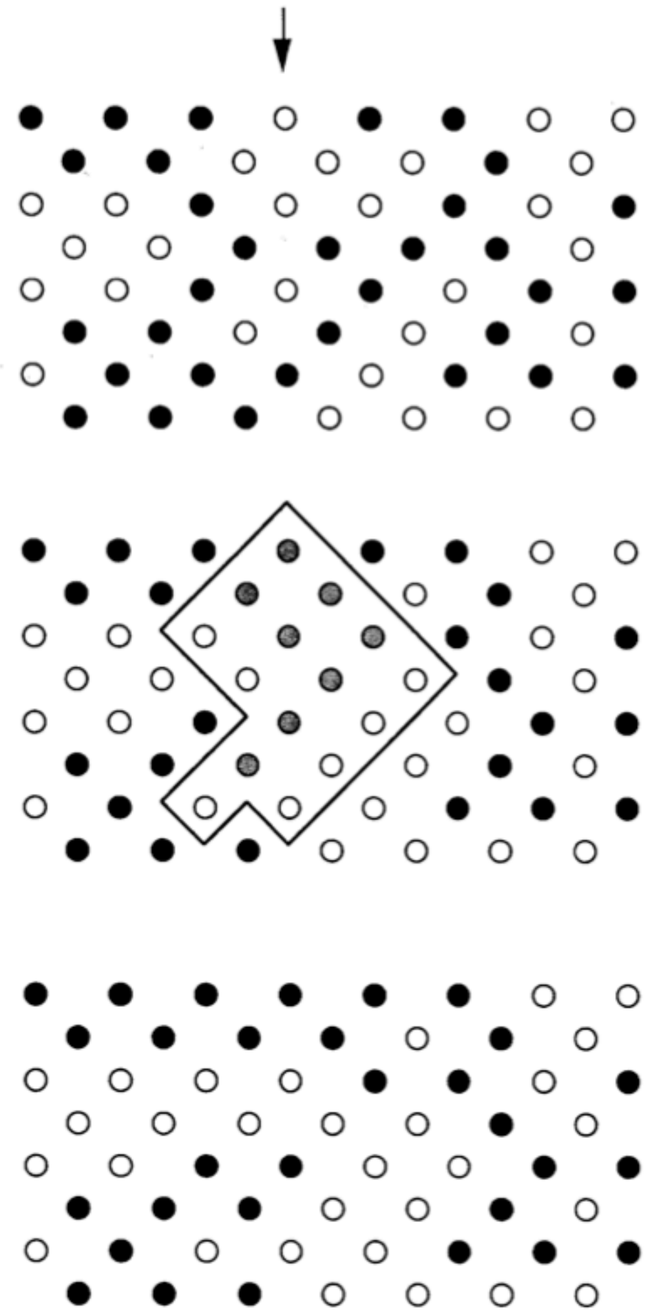


Figure 6: Typical State of the Network

Thus, small shocks can lead to large avalanches. The contribution of the event to the GNP is the area of the avalanche, that is the total amount of goods produced during the avalanche.

We could solve the model because we could relate it to another model that had previously been solved by Deepak Dhar at the Tata Institute in Bombay, in the context of sandpiles. The model is directional, in the sense that information is only transmitted down in the network, not up. Dhar showed that the distribution of avalanches is a power law

$$N(s) = As^{-\tau}$$

with $\tau = 4/3$.

To go from the power law to the Levy law observed by Mandelbrot, all that one has to do is to consider the situation in which each week there are several customers, and not just one, each demanding final goods. Each demand leads to an avalanche, so each day there are many avalanches of different sizes. One can show rigorously that for very many customers the result of the distribution of the total activity is the Levy function. They were able to demonstrate this by means of a simple mathematical calculation that would satisfy any physicist's demand for rigor. Nevertheless, the methods did not satisfy the very demanding economists, who didn't yield before they found the formula for how to add power law distributed random variables to arrive at the Levy distribution in a mathematics textbook.

4.2.4. Fluctuations and Catastrophes Are Unavoidable

The conclusion is that the large fluctuations observed in economics indicate an economy operating at the self-organized critical state, in which minor shocks can lead to avalanches of all sizes, just like earthquakes. The fluctuations are unavoidable. There is no way that one can stabilize the economy and get rid of the fluctuations through regulations of interest rates or other measures. Eventually something different and quite unexpected will upset any carefully architectures balance, and there will be a major avalanche somewhere else in the system.

In contrast to our critical economy, an equilibrium economy driven by many independent minor shocks would show much smaller fluctuations. Those fluctuations are given by a Gaussian curve, better known as the *bell curve* which has negligible tails. There is no possibility of having large fluctuations or catastrophes in an equilibrium economy.

Although economists do not understand the large fluctuations in economics, the fluctuations are certainly there. Karl Marx saw these fluctuations in employment, prices, and production as a symbol of the defunct capitalistic society. To him, the capitalistic society goes from crisis to crisis. A centralized economy would eliminate the fluctuations to the benefit of everybody or at least the working class. Marx argued that a large avalanche, namely a revolution, is the only way of achieving qualitative changes.

Alan Greenspan, former chairman of the Federal Reserve, manipulated the interest rate in order to avoid inflationary bursts even with the prospect of slowing down the economy. Common to Greenspan's and Marx's view is the notion that fluctuations are bad and should be avoided in a healthy economy.

If economics is indeed organizing itself to a critical state, it is not even in principle possible to suppress fluctuations. Of course, if absolutely everything is decided centrally, fluctuations could be suppressed. In the sandpile model, one can carefully build the sandpile to the point where all the heights are at their maximum value, $Z = 3$. However, the amount of computations and decisions that have to be done would be astronomical and impossible to implement. And, more important, if one indeed succeeded in building this maximally steep pile, then any tiny impact anywhere would cause an enormous collapse. The Soviet empire eventually collapsed in a mega-avalanche (not predicted by Marx). But maybe, as we shall argue next in a different context, the most efficient state of the economy is one with fluctuations of all sizes.

4.2.5. The Two Faces of Emergence in Economics

In Economics, there is not one definition of Emergence that is universally agreed upon, nor for progress to be made in the field does there need to be one. For the purposes here, however, I will use a stripped down definition which if not common to all emergent processes, does at least summarize what is at the core of most examples. These core characteristics include:

(1) At least two levels of organization,

(2) A multitude of individual agents at the lower level of organization who operate by following simple rules, and

(3) An aggregate outcome at the higher level that results from the interaction of these individual agents, but which is not easily derivable from the rules that the individual agents follow. Many times, therefore, this aggregate outcome comes as a surprise to the observer because nothing in the rules at the lower level seem to predetermine the aggregate outcome.

If we take these three characteristics to be a canonical representation of Emergence, then Economics was the first discipline to have emergent processes at its core. In 1776, Adam Smith wrote in *The Wealth of Nations*:

> It is not from the benevolence of the butcher, the brewer, or the baker that we expect our dinner, but from their regard to their own interest. We address ourselves, not to their humanity but to their self-love, and never talk to them of our own necessities but of their advantages (Book I, Chapter 2) ... every individual...neither intends

> to promote the public interest, nor knows how much he is promoting it...he intends only his own security; and by directing that industry in such a manner as its produce may be of the greatest value, he intends only his own gain, and he is in this, as in many other cases, led by an invisible hand[1] to promote an end which was no part of his intention. Nor is it always the worse for the society that it was no part of it. By pursuing his own interest he frequently promotes that of the society more effectually than when he really intends to promote it. (Book IV, Chapter 2)

The above quotations from the *The Wealth of Nations* are certainly the most famous in all of economics, and they remain the central dogma of the discipline to this day. What distinguishes an economist from other social scientists (and other people in general) is a faith that self interest at the lower level, when channeled through competitive markets, will result in a beneficial outcome at the aggregate level. Modern economics has discovered many exceptions to this rule, but they remain the exception and Adam Smith's insight remains the rule. With the exception of evolutionary biology, there is no modern academic discipline that has a concept of emergence so at its core.

This discussion will be organized as follows:

(1) Providing the technical knowledge of economics that is necessary to follow the arguments.

(2) Introduce two explicit emergent models that reach conclusions which are at variance with the sunny vision of Adam Smith.

(3) Finally, discuss Macroeconomics where emergent modeling has not proceeded very far but where it is very much needed.

Economic Theory and Practice

The Gold Standard in economics is economic efficiency (or Pareto efficiency). An allocation of resources is efficient if it is impossible to make any one person better off without making someone else worse off. At an efficient allocation, all waste has been squeezed out of the system in the sense that the only way to improve the well being of one person is by taking resources away/harming someone else. At an efficient allocation, all possibilities for mutually improving the well being of individuals have been exhausted. Economics is, in essence, the study of how to know when these conditions are met, when they are violated, and what to do when they are violated.

[1] As an 18th century man, Adam Smith was referring to Providence, or God, when he used the phrase "invisible hand". So Smith's views cannot be considered fully modern because emergence, as now understood, does not consider the phenomena that emerge at the higher level to be designed by anyone. But, when modern economists refer to the "invisible hand", they mean the impersonal forces of supply and demand which is consistent with the meaning of Emergence.

Adam Smith never used the phrase "economic efficiency" and did not know the formal conditions under which it could be achieved. However, when economics was formalized in the 19th century, it became clear that under suitable assumptions, competitive markets, of the kind that Adam Smith championed, would achieve economic efficiency. This insight is so central to modern economics that it is known as the First Fundamental Theorem of Welfare Economics. It also became clear (not formalized until the 20th century) that many of the institutions that Adam Smith condemned, such as monopolies, resulted in economic inefficiency. The corpus of knowledge developed after Adam Smith, which in so many ways confirmed his intuitions, is called Neoclassical or Walrasian economics.

A curious thing about Neoclassical economics is that the economy is not modeled as an emergent process, in fact quite the opposite. The Neoclassical tradition is so far removed from Emergence that many of its central propositions can be derived from, and illustrated by, an economy with just one individual. That one individual is called "the representative agent" or with a little more literary flair, Robinson Crusoe. Economists use a Robinson Crusoe economy as a pedagogic tool to derive the conditions for economic efficiency. If you only have one person, economic efficiency is synonymous with Robinson behaving sensibly and not wasting any of his resources. The Robinson Crusoe economy enables economists to turn what is a hard problem of market analysis into what is, in essence, an engineering problem as Robinson seeks to maximize his lifetime utility.

Modern economists have become so accustomed to using Robinson Crusoe as an explanatory tool, that they may not realize how much of Adam Smith's original insight is lost. For Smith, what was surprising was that individuals motivated by self-interest could nevertheless promote the interest of society. In a Robinson Crusoe economy, there is no society, and it is completely unsurprising that Robinson Crusoe promotes his own self-interest. While not all propositions in Neoclassical economics can be understood by studying an economy with one individual, it is surprising how many can. Still, this surprise is diametrically opposed to Adam Smith's surprise. What made Smith's insight so remarkable was that there was a disconnect between the two levels of analysis: the rule at the level of the individual was self-interest, but what emerged at the societal level was what we now call economic efficiency. In the Robinson Crusoe correspondence, the rule at the level of the individual is optimization and the outcome at the societal level is what we now call a Pareto efficient/optimal allocation. There is nothing surprising about optimality flowing from the behavior of an individual to an entire economy when the economy contains only one individual.

It should be emphasized that the First Fundamental Theorem of Welfare Economics is not a form of misplaced anthropomorphism. The Welfare Theorem is a rigorously proved proposition that does not conceive of the economy as one large individual. The problem comes when economists start making positive statements about the real economy on the basis of a representative agent. The conditions under which the behavior of an entire economy can be predicted from the behavior of one individual are very severe. They basically amount to assuming that everyone in the economy is identical in terms of tastes and income. This, of course, is never true. So to take the simplest possible example, if it is the case that when the fish are running, Robinson spends more time fishing because the price of fish in terms of foregone leisure has declined; we cannot conclude that for an entire economy, the demand for fish will go up as the price falls. The First Fundamental Theorem of Welfare Economics does not guarantee this correspondence. What the Theorem guarantees is that if the economy is competitive and certain other assumptions hold, then when the price of fish falls, whatever outcome emerges will be efficient.

The Dark Side of Emergence

So, the first face of Emergence in economics, which comes down to us from Adam Smith, is a very positive one: the road to heaven may be paved with bad intentions. Agents acting selfishly can, nevertheless, create an aggregate outcome such that it is impossible to make someone better off without making someone else worse off. This is an amazingly strong statement. As the Nobel Laureate Kenneth Arrow wrote in *General Competitive Analysis*, a textbook that codified Neoclassical economics for a generation of economists:

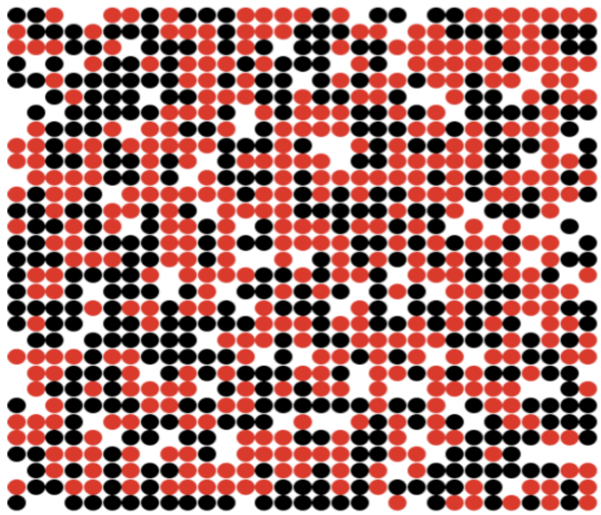
> the notion that a social system moved by independent actions in pursuit of different values is consistent with a final coherent state of balance, and one in which the outcomes may be quite different from those intended by the agents, is surely the most important intellectual contribution that economic thought has made to the general understanding of social processes

The Schelling Segregation Model:

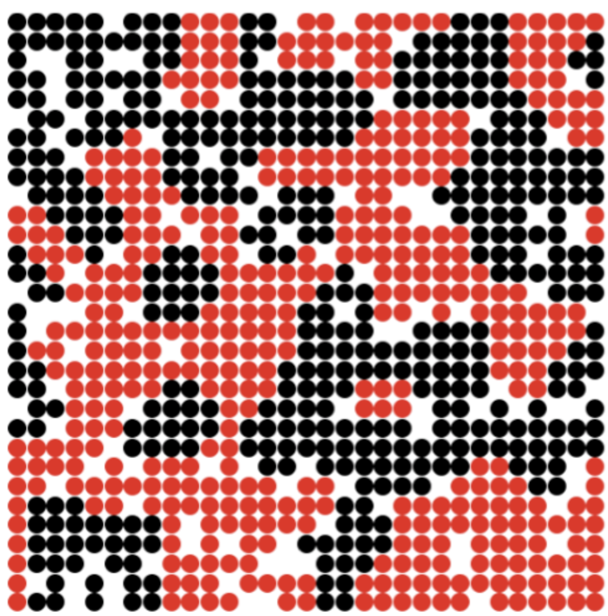
The first economist to create an emergent model whose outcomes were not socially desirable was Thomas Schelling in *Micromotives and Macrobehavior (1978).* Schelling analyzed how neighborhoods would emerge given that people had some preference to live near people like themselves.

Figure[2] below illustrates a "society" where people (the red and black dots) are distributed randomly throughout the space.

[2]NetLogo Model created by Uri Wilensky 1998



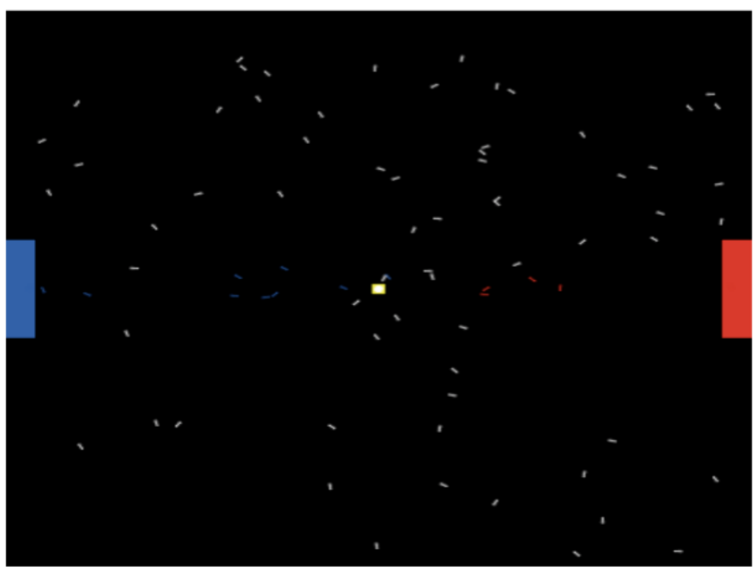
Each individual has 8 neighbors, and we assume that they will move unless 3/8th (37.5%) of their neighbors are of the same color as themselves. This is not a strong preference for segregation, and as a result, in the figure, 72.1% of the people are happy - meaning that they have at least 3 neighbors of their same color. Nevertheless, when you move people around until no one is unhappy, figure below emerges which has a substantial degree of segregation.



In first figure, approximately 50% of ones neighbors shared the same color, but in second figure the number is over 80%. The surprise is that a relatively mild preference for living with people of the same color results in a substantial degree of segregation. So the rule at the lower level, "move if less than 3/8ths of your neighbors are of the same color" results in an aggregate outcome where more than 80% are of the same color.

Antz:

The Schelling Model does not relate directly to economics. While the outcome is bad given a social preference for integration, one cannot say that the outcome is inefficient. In fact, since in the final equilibrium everyone is satisfied with their neighborhood, one could say the outcome is efficient. Of more relevance to economics is the model in the next figure . Ostensibly, it is a model of ants who have a nest in the middle of the graph and forage for food from two equidistant food sources (red and blue) at the edges of the graph. There are three kinds of ants: ants that have no source affiliation, blue ants who forage at the blue source, and red ants who forage at the red source. Initially all ants have no affiliation, but when they discover one of the sources they become that kind of ant and bring food back to the nest and then go out again to that source. If a blue ant encounters an unaffiliated ant (one that has not yet discovered a source), then that ant is recruited to become a blue ant (similarly for red ants). The final effect that makes the model interesting is that an affiliated ant that is not carrying food can be converted to the other color, with some probability, if it encounters an ant of the other color.



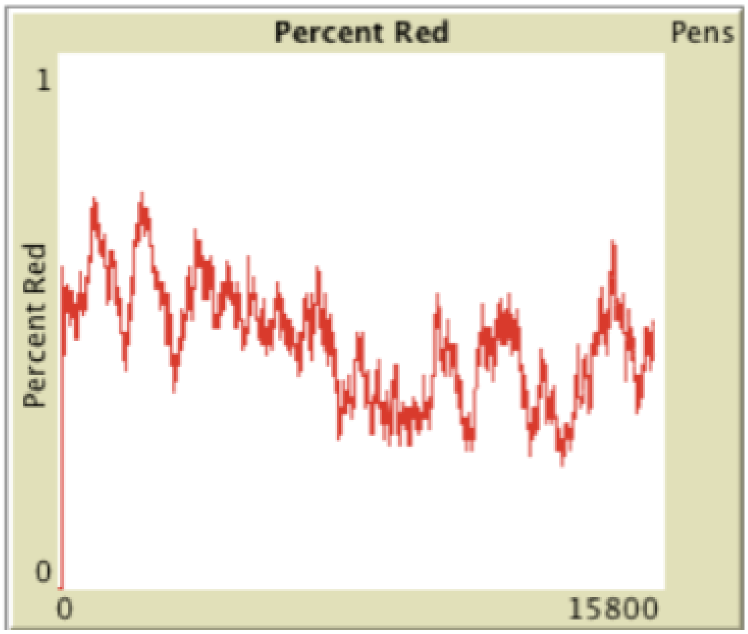
Possible economic applications for such a model are choosing to adopt one of two alternative technologies, choosing to do business with one of two alternative firms, etc. The model then neatly illustrates two opposing views of how this competition will evolve:

(1) Since the food sources are equidistant from the nest, equally plentiful, and the ants initially move randomly, one might think that 50% of the ants will be red and 50% will be blue. In the context of this model, this would be the "competitive" outcome, and it is what would be predicted by what economists call the Hotelling model - an equilibrium model of spacial competition.

(2) Since ants can recruit and convert other ants that they meet, one might think that if one source develops a lead in ant affiliation, it will build on that lead and ultimately all the ants will be of that color. This is sometimes called the "first stake in the ground" theory .

Which of these two outcomes emerges is not only of academic interest. One of the major driving forces behind the stock market bubble of the late 1990's was the belief that if a firm developed a lead in internet customers, it would lock in that customer base and have very high profits in the future even if it was currently suffering severe loses. What the model shows is that, as expected, if there is no recruitment or conversion of ants, the Hotelling result emerges: approximately 50% of the ants are red and 50% are blue. Surprisingly, this result is essentially unchanged if there is recruitment but no conversion. With recruitment and no conversion, the ability of ants affiliated with a given source to recruit other ants does not tip the scales irreversibly to the first source found. The fact that ants are randomly searching for a source at the beginning insures a nearly equal split between sources. In either case without the possibility of conversion, the model is in equilibrium when all ants are affiliated with some source, recruitment simply speeds this process up.

The next figure illustrates the case of recruitment with a conversion rate of 75% and plots the proportion of red ants. As can be seen, even after 15,000 periods the model does not settle into an equilibrium, the percentage of red ants fluctuates widely. Why is this? The reason is the complex interplay between positive and negative feedback that is at work in the model. Positive feedback results from the fact that when there are more ants of a particular color, it is more likely that an unaffiliated ant will meet an ant of that color and be recruited plus the fact that there are more "missionary" ants of that color to convert ants of the opposing color. If these were the only mechanisms in operation, eventually all ants would be of one color. Negative feedback results from the fact that when there are more ants of a particular color, there are necessarily more ants who are not carrying food of that color. For ants of the other color, therefore, there are many potential converts. For the ants of the minority color, the graph is a target rich environment. If the conversion rate is set high enough, these two forces are continually at war with one another and neither of the two intuitions discussed above is correct.



The last figure illustrates the danger in telling top-level stories or finding patterns in top-level phenomena when the underlying process is emergent. Looking at the time series in last figure, macroeconomists might analyze the tops and bottoms of the percent red ants as peaks and troughs of business cycles and seek macroeconomic explanations for their occurrence. Technical stock market analysts might look at the pattern of percent red ants and claim to be able to predict future movements. There is a lot of confusion in the Emergence literature as to whether emergent phenomena are necessarily random and/or unpredictable. The Ant model is completely deterministic and in this sense completely predictable. Where technical analysts go wrong is in assuming that one can predict the movements in the graph based on past movements in the graph. In order to predict the movements of the red ants, one needs to know where every ant is and how it is interacting with every other ant. The nonlinear deterministic processes in Chaos Theory are also predictable in the above sense because they contain no random elements, but the computing power necessary to make these predictions may be so large that the distinction between "unpredictable in principle" and "unpredictable in practice" may be mute. But we know from how this model was constructed that telling aggregate stories of movements during particular time periods is nonsense because all of the observed phenomena were caused by interactions at the local level.

A New Kind of Economics

Just as there are multiple definitions of Emergence, there are multiple descriptions of how an economics based on emergent principles differs from traditional neoclassical/Walrasian economics. As with my definition of canonical emergence, I will state the minimum set of characteristics that distinguish what has come to be know as "agent based computational economics" from traditional economics.

A fortiori, agent based computational economics is populated by heterogeneous agents. I say *a fortiori* because it is in the very nature of Emergence that agents interacting at a local level cannot be identical. So, for example, in the segregation model, agents differ by their initial position in the grid and therefore by who their immediate neighbors are. Even if agents were initially programmed to be identical, their local interaction with one another would be different and they would soon cease to be identical. This necessary lack of a representative agent means that one cannot in emergence adopt the Robinson Crusoe methodology where economic efficiency flows to the whole economy from the maximizing behavior of one individual. Still, heterogeneity in no way negates the First Theorem of Welfare Economics. A strength of the First Theorem and of Walrasian economics is that both are perfectly capable of dealing with any degree of heterogeneity. So, while some economists see heterogeneity as a hallmark of agent based computational economics, its real role is to eliminate the possibility of using the intellectually suspect Robinson Crusoe methodology.

What fundamentally differentiates agent based computational economics from traditional Walrasian economics is that economic activity occurs outside of equilibrium. Equilibrium is a state of rest for any system: it is the "state of balance" referred to in the quotation from Kenneth Arrow. Agent based models can certainly have an equilibrium. In the segregation model, for example, equilibrium occurs when everyone is content with the color distribution of their neighbors. This equilibrium is not unique, if by unique we mean an identical final pattern of dots; it depends critically on the initial placement of the dots and also on the order in which people get to move. It is generally the case that once out of equilibrium trades or economic activity are allowed, the ultimate equilibrium, if there is one, will not be unique.

While the existence of an equilibrium is important for the analysis, uniqueness of the equilibrium is not a central concern. The essential question is whether the equilibrium will be efficient. Under standard assumptions, what assures efficiency in traditional Walrasian economics is a fictitious character called the Walrasian auctioneer who aggregates all supply and demand information and allows trading only at market clearing prices. In other words, equilibrium prices are first established by the auctioneer and then trading takes place. It is never the case that someone wants to supply or demand something at current prices and cannot find a willing buyer or seller. The Walrasian auctioneer is the economics version of a top-down coordinator, and it is a hallmark of emergent processes that there is no such coordinator. Without the auctioneer, one must generate the final equilibrium from the local interaction of the individual agents. Under what circumstances such an equilibrium will be efficient is an open question.

Macroeconomics

Macroeconomics is the study of the economic activity of the economy taken as a whole. To carry out this study, macroeconomists create economy-wide aggregates of individual real world variables. Some of these aggregates are sums of individual variables, such as gross domestic product, which is the sum total of the economy's production of goods and services for a given time period; other aggregates are averages of individual variables, such as the price level or the inflation rate, which average individual prices and their percentage changes. The goal of macroeconomics is to understand the movements of and the relationships between these various aggregates. If we conceive of the economy as an emergent system, then from this description, it should be obvious that macroeconomics is inherently studying its top-level behavior.

Modern macroeconomics began with the publication in 1936 of *The General Theory of Employment, Interest, and Money* by John Maynard Keynes. Virtually since its inception, there has been a research agenda to provide micro-foundations for the relationships between the macroeconomic aggregates. For the most part, this research program has used traditional neoclassical/Walrasian economics to provide the micro-foundations. Such an approach contained within itself an internal contradiction which only became fully obvious in the 1970?s with the advent of what is known as New Classical economics. We have seen that traditional Walrasian economics shares with Adam Smith an optimistic view of the workings of the economy. The central message of The General Theory, however, was that the performance of the economy would many times be sub-optimal. Because the central tendencies of Walrasian and Keynesian economics are diametrically opposed, the effort to provide micro-foundations for Keynesian macroeconomics has yet to produce a model that is convincing to most economists.

What I wish to argue here is that the reason for this failure may be that we are using the wrong microeconomic paradigm. Instead of using traditional neoclassical/Walrasian analysis perhaps we should be thinking in terms of emergent processes. This has both positive and normative implications. In terms of positive analysis, we can see that the ants model exhibits internally generated apparent cyclical behavior. I say apparent because there unquestionably is not a mechanism generating a fixed periodicity to these cycles. The cyclical behavior emerges from the local interaction of the ants. This is in stark contrast with the standard macroeconomic explanation for apparent cyclical behavior that is found in both Keynesian and New Classical macroeconomics. According to the standard view, apparent cyclical behavior is generated when the economy is hit by an aggregate exogenous disturbance. The internal mechanisms of the economy then augment and ultimately dampen down this disturbance, and the only reason that there appears to be business cycles is that the economy is hit later on by another disturbance. There is a large literature dealing with endogenous business cycles, but this is not the majority view among macroeconomists.

A key premise behind the standard view is that macroeconomic events must have macroeconomic causes: changes in the macroeconomic aggregates must be the result of macroeconomic disturbances. This is precisely what an emergent perspective calls into question (remember the extinction of the dinosaurs). What the standard view calls a macroeconomic disturbance can be, as in the ants model, the bubbling up to the macroeconomic surface of small events at the local level. Some events at the local level are nullified at the local level: so, for example, an ant not carrying food converts to the opposing color, but then meets an ant of its original color and converts back. We never see these events at the top-level and are completely unaware of their existence. But sometimes, a local interaction, or the random occurrence of many local interactions of the same type, is propagated by positive feedback into a bigger and bigger event until it emerges at the top-level as a macroeconomic event.

In the give and take between micro and macroeconomists, a standard line by microeconomists is, "there is no such thing as macroeconomics", by which they mean that all that really exists is individual behavior and its aggregation into markets. Emergent processes, of course, have precisely this quality. The case can be made that all of what is observed at the top-level is epiphenomenal and that the only reality is the local interactions. I would argue, however, that this does not imply that one has to give up on aggregate relationships or on the possibility of finding higher level laws. The paradigm should be Boyle?s Law where an aggregate equation describes the top-level behavior of a gas with no reference to the interactions of the individual gas molecules. The aggregate relationship must, of course, be consistent with what is happening at the micro level; but in an emergent system, there is no presumption that the aggregate outcome will have a simple correspondence to the micro rules.

With respect to normative analysis, while both Keynesian and New Classical economics share a common view that macroeconomic outcomes have macroeconomic causes, they come to diametrically opposite normative conclusions. New Classical economics follows the Walrasian tradition and relies very heavily, almost exclusively, on representative agent modeling. It is not surprising, therefore, that it comes to the conclusion that the economy is operating efficiently. The hope that Emergence holds out for Keynesian economics is that by banishing the representative agent, it also eliminates the all too easy correspondence between optimality at the individual level and efficiency at the top level. This opens up the rich possibility that individuals behaving optimally will interact with one another in such a way that the top-level outcome will not be efficient.

It is important not to claim too much for emergent modeling and too little for the traditional Walrasian approach. Traditional economics has analyzed a full set of conditions under which markets do not result in efficiency. The most empirically important are:

(1) externalities which result in excessive pollution and

(2) information asymmetries which result in malfunctions of insurance and financial markets.

Likewise, it is not the case that all emergent models will result in inefficiency, sometimes the emergent outcome may mirror the Walrasian outcome. The point is that the central tendency of these modelling strategies is different.. But, whether such an approach will bear fruit, remains to be seen.

4.3. Traffic Jams

Taking a broader view, economics deals with the way humans interact, by exchanging goods and services. In the real world, each agent has limited choices, and a limited capability of processing the information available; he has *bounded rationality*. In some sense, the situation of the individual agent is like a car driver in traffic on a congested highway. His maximum speed is limited by the cars in front of him (and perhaps by the police); his distance to the car in front of him is limited by his ability to brake. He is exposed to random shocks from the mechanical properties of his car and from bumps in the road.

A simple cellular automaton model was constructed for one-lane highway traffic. Cars can move with velocity 0, 1, 2, 3, 4, or 5. This velocity defines how many *car lengths* each car will move at the next time step. If a car is moving too fast, it must slow down to avoid a crash. A car that has been slowed down by a car in front will accelerate again when given an opportunity. The ability to accelerate is less than the ability to break, that is, it takes more time steps to go from 0 to 5, than to brake from 5 to 0. Depending on the total number of cars on the road, there are two possible situations. If there are few cars there is a free flow of cars with only small traffic jams. If the density is high there is massive congestion.

They considered the traffic emerging from one large traffic jam. Think of the Long Island Expressway which runs along Long Island, starting at the Queens Midtown Tunnel leading into Manhattan. They came up with a theory that could describe the traffic coming out of the tunnel in the rush hour, where the largest possible number of cars would be pumped into the expressway. Everybody living on Long Island is familiar with the resulting huge traffic jams that can occur on the expressway which has been called *the world's largest parking lot*. The figure below shows the computer-simulated traffic jams. The horizontal axis is the highway, the vertical axis is time. Time is increasing in the downward

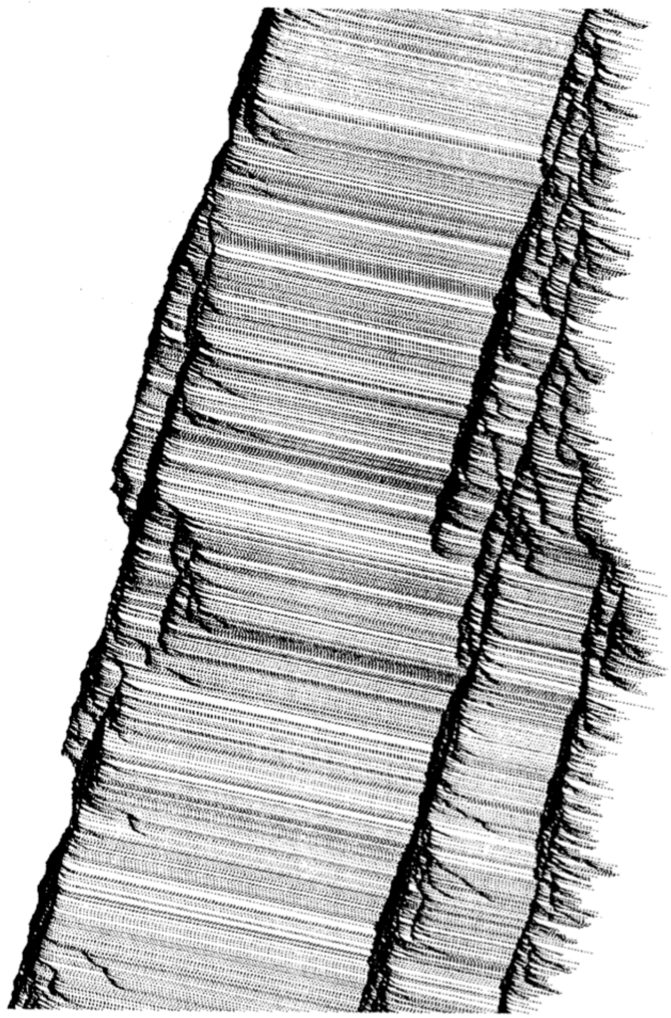


Figure 7: Computer Simulated Traffic Jams

direction. The cars are shown as black dots.
The cars originate from a huge jam to the left, which is not seen, and all move to the right. The diagram allows us to follow the pattern in time and space of the traffic. At each time step, the position of each car is shifted to the right by an amount equal to the velocity of that car. Traffic jams are shown as dense dark areas, where the distance between the cars is small. Also, the positions of cars between two successive horizontal lines are only shifted slightly since their velocities are low.

Traffic jams may emerge for no reason whatsoever! They are *phantom* traffic jams. A small random velocity reduction from 5 to 4 of a single car is enough to initiate huge jams. We have met the same situation before: for earthquakes, for biological evolution, for river formation, and for stock market crashes. A cataclysmic triggering event (like a traffic accident) is not needed. Our natural intuition that large events come from large shocks has been violated. It does not make any sense to look for specific reasons for the jams.

The jams are fractal, with small subjams inside big jams ad infinitum. This represents the irritating stop-and-go driving pattern that we are all familiar with in congested traffic. On the diagram, it is possible to trace the individual cars and observe the stop-and-go behavior.

Traffic jams move backward, not forward, as can be seen in the figure. For comparison, a similar diagram for the traffic on a real highway in Germany is also showed in the figure below.

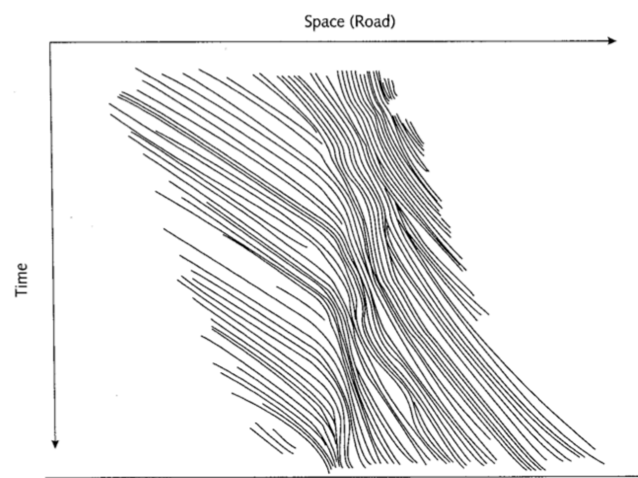


Figure 8: Real Traffic Jam

The picture is based on photos of the highway taken at regular intervals. Note that the general features are the same as for the computer simulations, including the backward motion of tile jams. Eventually the jams dissolve. From extensive computer simulations, the number of traffic jams of each size was calculated. Of course (you guessed it) they found a power-law distribution. The exponent for the power law appeared to be close to 3/2. This suggested an elegant but simple theory of the phenomenon, a *random walk theory*.

Each jam starts at a random nucleation point, at the top of the figure. At each time step, the size of the jam can either increase, with a certain probability or decrease, with the same probability. Because of this 50 – 50 situation, the process is critical. This process can be solved mathematically and gives a power law distribution,with an exponent that is exactly 3/2 as suggested by the simulations. We have met the random walk picture of self-organized critical systems before, in the context of the random neighbor model of evolution.

Highway traffic is a classic example of $1/f$ noise. Over 30 years ago, the traffic was measured on the Kanai Expressway in Japan as a function of time, by standing on a bridge over the highway and measuring the times that cars passed under the bridge. They observed a curve similar to that of light from a quasar. When measuring the power spectrum, they found components of all frequencies, with the famous $1/f$ distribution. The same measurement was done on the computer-simulated traffic data. Standing on a bridge and monitoring the traffic corresponds to measuring the patterns of black dots along a vertical line. The signal is a Devil's stair-case, just as for the evolution model. They also found a $1/f^{\alpha}$ noise in the computer simulations as shown in the figure below.

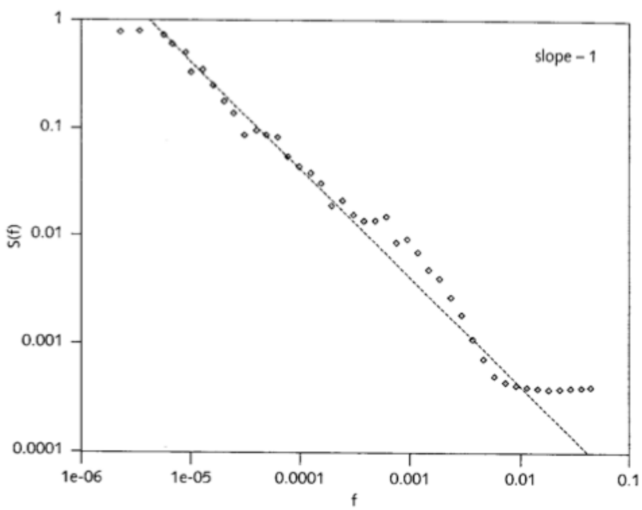


Figure 9: Real Data

Moreover, they were able to prove mathematically that $\alpha = 1$ from a cascade mechanism, where subjams at each time step can grow or die or branch into more jams. For once, we have an accurate and complete understanding of the elusive $1/f$ noise in a model system that actually describes reality. As for the other phenomena that we have studied, $1/f$ noise is due to scale-free avalanches in a self-organized critical system. In the case of traffic, the $1/f$ noise is the mathematical description of the irritating, unpredictable stop-and-go behavior in traffic jams.

They then studied the situation in which there were only very rare random fluctuations initiating traffic jams. Interestingly they point out that technological advancements such as cruise control or radar-based driving support would tend to reduce the fluctuations around maximum speed, and thus increase the range of validity of the results. One unintended consequence of these flow control technologies is that, if they work, they would in fact push the traffic system closer to its underlying critical point, thereby making prediction, planning, and control more difficult, in sharp contrast to the original intentions. Note the analogy with attempts to regulate economy (or sandpiles). Self-organized criticality is a law of nature for which there is no dispensation.

They made one final observation. Traffic jams are a nuisance, amplified by our lack of ability to predict them. Sometimes we are slowed down by a large jam, sometimes we are not. One might suspect that there would be a more efficient way of dealing with the traffic. In fact there might not be. *The critical state, with jams of all sizes, is the most efficient state.* The system has self-organized to the critical state with the highest throughput of cars. If the density were slightly less, the highway would be underutilized, if the density were slightly higher, there would be one big permanent, huge jam, absorbing a large fraction of the cars. In both cases, the throughput would be less.

More precisely *the critical state is the most efficient state that can actually be reached dynamically.* A carefully engineered state where all the cars were moving at velocity 5 would have higher throughput, but it would be catastrophically unstable. This very efficient state would collapse long before all the cars became organized.

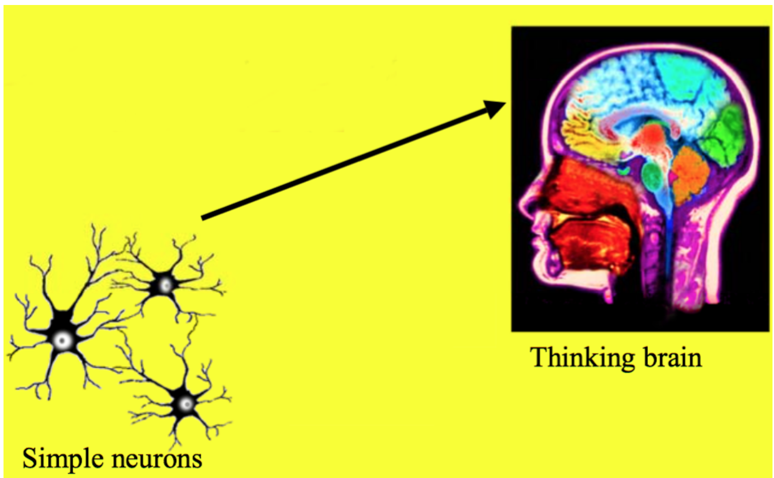
This gives some food for thought when applied to economics in general. Maybe Greenspan and Marx are wrong. The most robust state for an economy could be the decentralized self-organized critical state of capitalistic economics, with fluctuations of all sizes and durations. The fluctuations of prices and economic activity may be a nuisance (in particular if it hits you), but that is the best we can do!

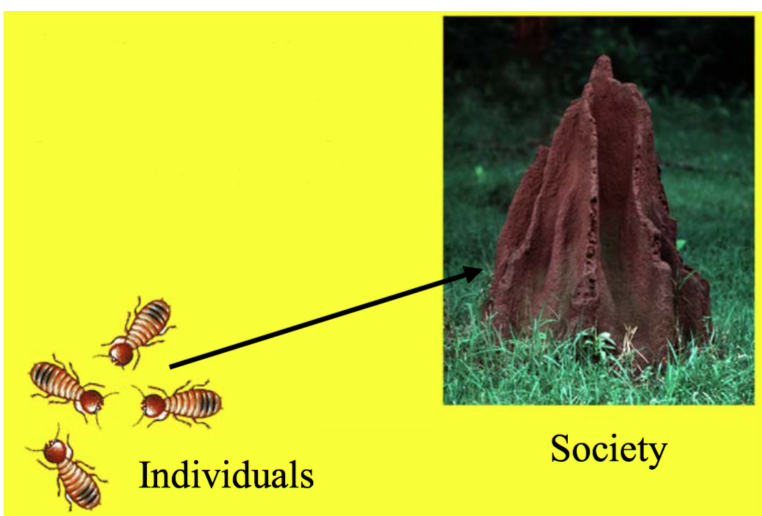
The self-organized critical state with all its fluctuations is not the best possible state, but it is the best state that is dynamically achievable.

4.4. Self-organization in biological systems

Let us continue our discussions with a "biological magical mystery tour".

What are the mechanisms for integrating biological subunits into a coherently functioning entity?





Remember, self-organization is a process in which pattern at the global level of a system emerges solely from numerous interactions among the lower-level components of the system. Moreover, the rules specifying interactions among the system's components are executed using only local information, without reference to the global pattern. In other words, the pattern is an emergent property of the system, rather than a property imposed on the system by an external influence.

4.4.1. Self-Organized Patterns in Nature - Non-living Systems

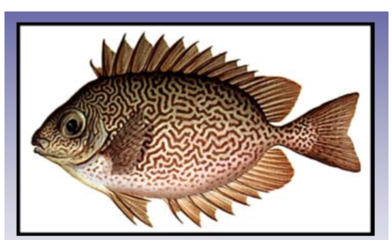
Sand dune ripples

Paint wrinkles

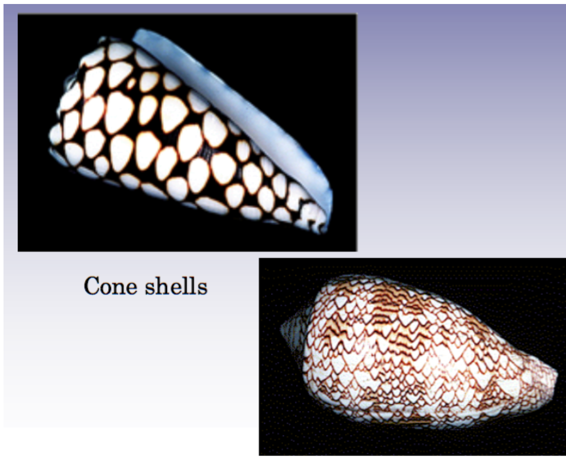
Mud cracks

4.4.2. Self-Organized Patterns in Nature - Living Systems

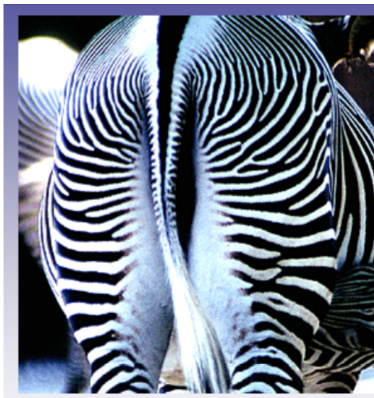
Giraffe coat



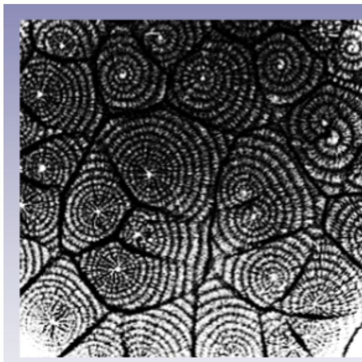
Vermiculated rabbitfish



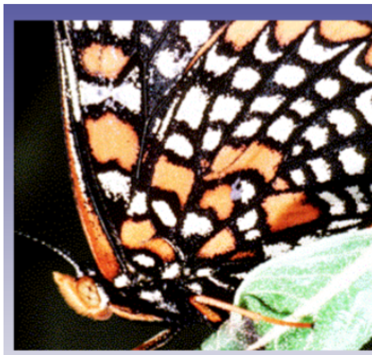
Cone shells



Zebra

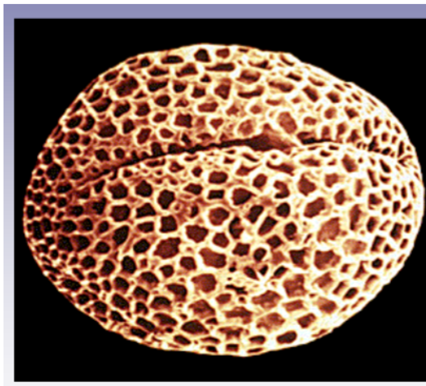


Slime mold



Checkerspot butterfly

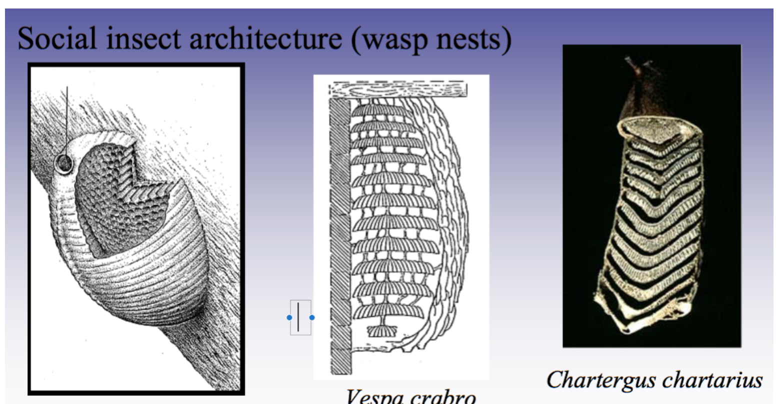
Morel mushroom



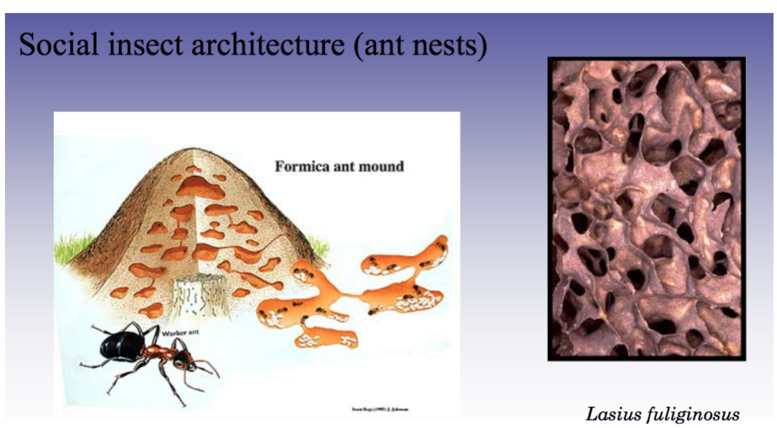
Forsythia pollen grain

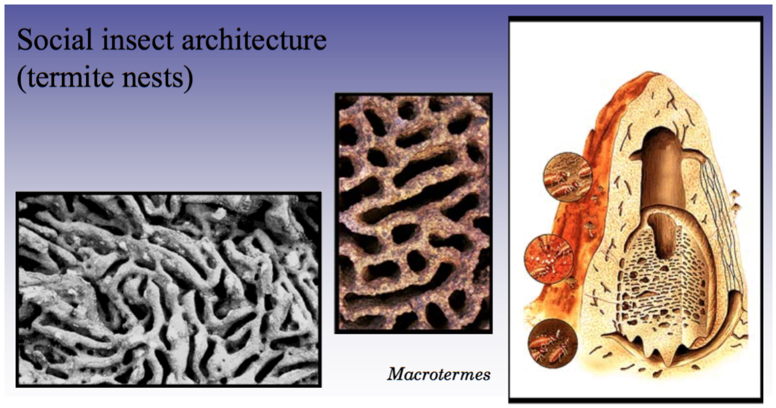
Cabbage

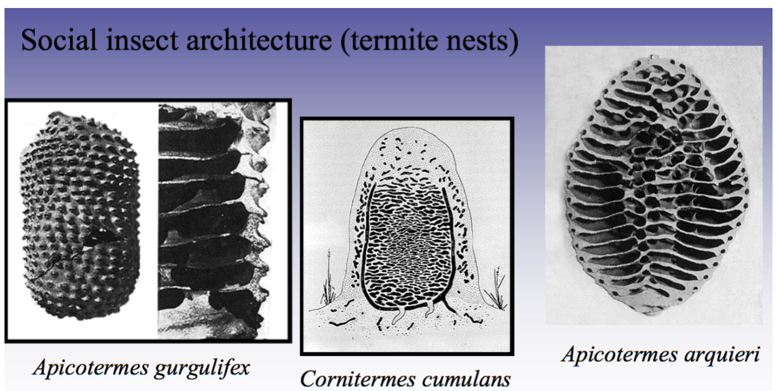
4.4.3. Self-Organized Patterns in Nature



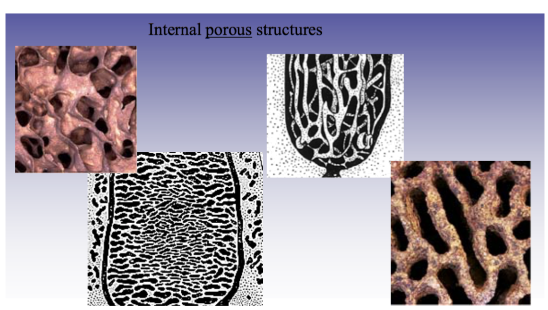
Synoeca surinama



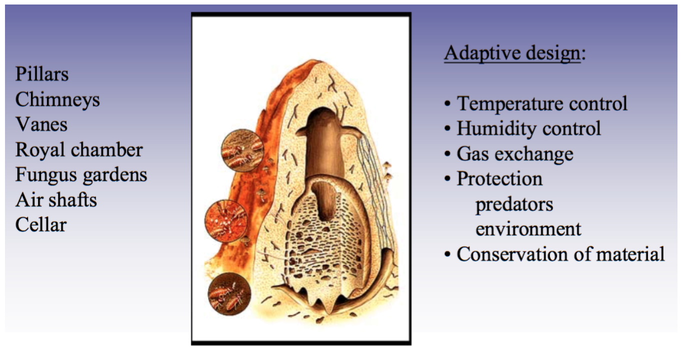




4.4.4. Architectural Features Shared by Social Insect Nests



Among the termites, we reach a pinnacle of nest complexity



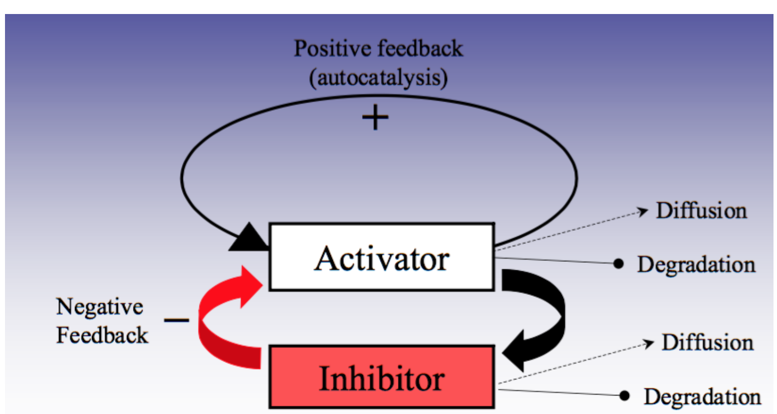
Common architectural themes:

- Porous structures
- Mottled, spongy patterns of spaces among building material
- Surface structures - Ripples, cracks, pillars, evaginations
- "Positive" and "negative" space (substance and voids)
- "Competition" for building material or space

4.4.5. How do insects build these structures?

Proposition: Social insects have evolved simple behavioral rules for generating these complex architectures.

One such set of simple rules is based upon an *activation-inhibition mechanism.*



What happens? The activator autocatalyzes its own production, and also activates the inhibitor. The inhibitor disrupts the autocatalytic process. Meanwhile, the two substances diffuse through the system at different rates, with the inhibitor migrating faster. The result: local activation and long-range inhibition.

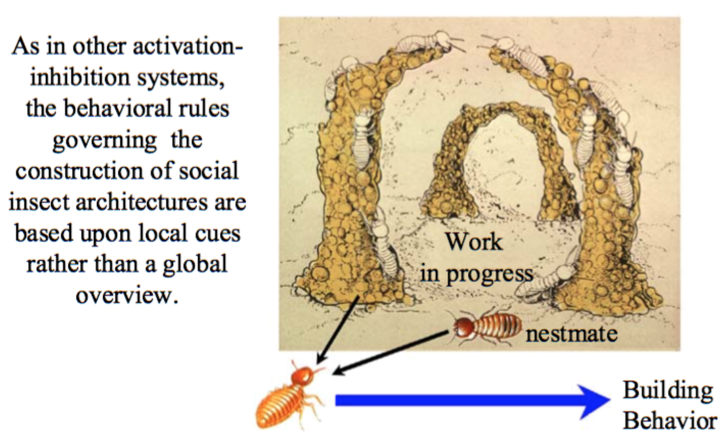
What is the relationship between activation-inhibition mechanisms and self-organization? They share a common mechanism:

> Starting point: a homogeneous substrate (lacking pattern)
>
> Positive feedback (local activation or attraction)
>
> Negative feedback (long-range inhibition, depletion, decay)

4.4.6. What can self-organization achieve?

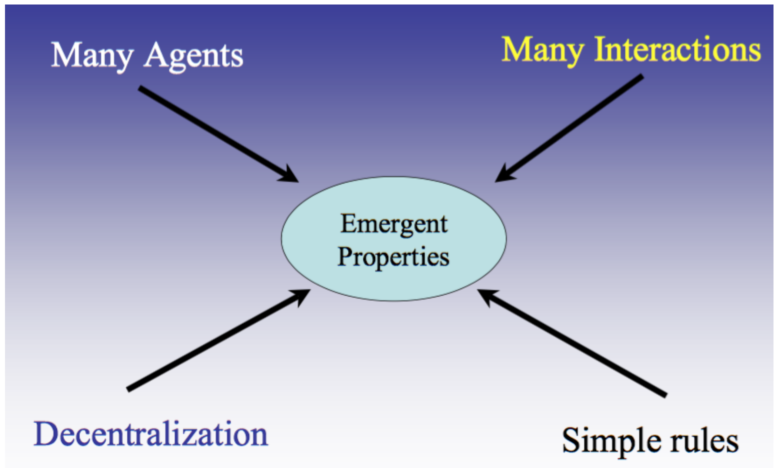
In the case of the termite mound, I suggest the following type of scenario:

Starting with a homogeneous, flat landscape, the random movements of the termites, and their dropping and picking up behavior leads to tiny surface irregularities which become the site of rising pillars. Once a pillar has emerged, this structure acts as a source of heterogeneity that modifies the actions of individual builders. The activity, in turn, create new stimuli that trigger new building actions. Complexity unfolds progressively; increasingly diverse stimuli result from previous building activities, and facilitate the construction of ever more complex structures.



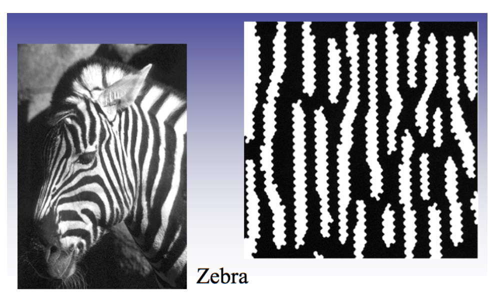
This PLUS a lot of handwaving might give you a termite mound!

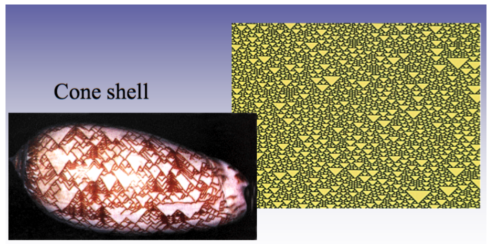
4.4.7. Summary: Distinguishing Features of Complex, Self-Organizing Systems

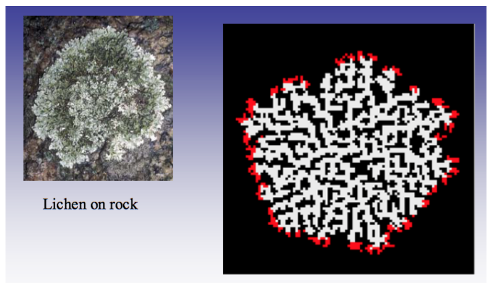


- Large numbers of units (agents)
- Large numbers of interactions
- Simple rules of interaction
- Decentralized organization
- Emergent properties

As we have seen many times the computer modeling is relatively easy, but unraveling the actual biological mechanisms is extremely difficult. Some simulations are shown:







4.4.8. Adaptive advantages of self-organized systems

Robustness

Error tolerance

Self-repair

Ease of implementation

Simple agents

Why is all of this important?

Many biological systems have evolved decentralized solutions to their vital challenges. Through self-organization, evolution has stumbled upon a wide range of extremely efficient, relatively simple solutions for solving very complex problems

Why has evolution "chosen" these types of solutions?

Biological Constraints - One of the mysteries of biology is how the enormous amount of morphogenic, physiological and behavioral complexity of living organisms can be achieved with the limited amount of genetic information available within the genome.

Self-organization is one solution to this problem!