On Socially Fair Low-Rank Approximation and Column Subset Selection

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NEURAL INFORMATION PROCESSING SYSTEMS

Motivation

Unfortunately, real-world machine learning algorithms across a wide variety of domains have recently produced a number of undesirable outcomes from the lens of generalization:

- [BS16] noted that decision-making processes using data collected from smartphone devices reporting poor road quality could potentially underserve poorer communities with less smartphone ownership.
- [KMM15] observed that search queries for CEOs overwhelmingly returned images of white men
- [BG18] observed that facial recognition software exhibited different accuracy rates for white men compared with dark-skinned women.

Biased data or biased algorithms?

- Better training data, fair algorithms

Low-Rank Approximation

- Find rank k matrices $U \in \mathbb{R}^{n\times k}$, $V \in \mathbb{R}^{k\times d}$ that minimizes $\|UV - A\|_F$
- Finding structure among noise
- Matrix completion problem
- Closed form solution to find optimal $U \in \mathbb{R}^{n\times k}$, $V \in \mathbb{R}^{k\times d}$ that minimizes $\|UV - A\|_F$
- V is the top k right singular vectors of A
- Can be computed in polynomial time using singular value decomposition (SVD)

$$n \begin{array}{|ccccccccc|} 1 & 3 & 5 & -2 & 7 & 0 & 11 & 4 & -8 \\ 0 & 0 & -1 & 3 & 13 & 2 & 8 & 6 & 2 \\ 2 & 5 & 6 & 1 & 4 & 0 & -7 & 5 & 3 \\ 8 & 7 & 2 & 1 & -1 & -3 & -2 & -4 & -6 \\ -5 & 3 & -4 & -1 & -2 & -1 & 0 & -3 & -1 \\ 7 & 1 & 3 & 2 & 4 & 1 & 0 & 11 & 1 \end{array}$$
$$d$$

Column Subset Selection

- Find matrices $U \in \mathbb{R}^{n\times k}$, $V \in \mathbb{R}^{k\times d}$ that minimizes $\|UV - A\|_F$, where V is k columns of A
- Finding structure among noise
- Low-rank approximation variant with better interpretability
- NP-hard problem
- Can achieve $O(k)$ approximation by volume sampling or local search
- Approximation algorithms use polynomial time

Social Fairness

- Find rank k matrices $U_1 \in \mathbb{R}^{n_1\times k}$, …, $U_\ell \in \mathbb{R}^{n_\ell\times k}$, $V \in \mathbb{R}^{k\times d}$ that minimizes $\max_{i\in[\ell]} \|U_i V - A_i\|_F$
- Each matrix A_i is the dataset of a protected subpopulation
- Ensure solution is equitable to all subpopulations

$$n_1 \begin{array}{|ccccccccc|} 1 & 3 & 5 & -2 & 7 & 0 & 11 & 4 & -8 \\ 0 & 0 & -1 & 3 & 13 & 2 & 8 & 6 & 2 \end{array} A_1$$
$$d$$
$$n_2 \begin{array}{|ccccccccc|} 2 & 5 & 6 & 1 & 4 & 0 & -7 & 5 & 3 \\ 8 & 7 & 2 & 1 & -1 & -3 & -2 & -4 & -6 \\ -5 & 3 & -4 & -1 & -2 & -1 & 0 & -3 & -1 \end{array} A_2$$
$$\vdots$$
$$n_\ell \begin{array}{|ccccccccc|} 7 & 1 & 3 & 2 & 4 & 1 & 0 & 11 & 1 \end{array} A_\ell$$

Our Results

Lower Bounds

- Fair low-rank approximation is NP-hard to approximation within any constant factor
- Under the exponential time hypothesis (ETH), fair low-rank approximation requires $2^{k^{\Omega(1)}}$ time to approximate within any constant factor
- Recall: Low-rank approximation can be solved in polynomial time

Upper Bounds

- Given accuracy parameter $\varepsilon \in (0,1)$, there exists $(1+\varepsilon)$-approximation algorithm for fair low-rank approximation that uses time $\frac{1}{\varepsilon}\,\text{poly}(n)\cdot(2\ell)^{\text{poly}\left(\ell,k,\frac{1}{\varepsilon}\right)}$
- Given trade-off parameter $c \in (0,1)$, there exists $\ell^c \cdot 2^{\frac{1}{c}} \cdot O\big(k(\log\log k)(\log d)\big)$-approximation algorithm for fair low-rank approximation that uses polynomial time, but with bicriteria rank $O\left(k(\log\log k)\left(\log^2 d\right)\right)$
- There exists $O\big(k(\log\log k)(\log d)\big)$-approximation algorithm for fair column subset selection that uses polynomial time, but with bicriteria rank $O(k\log k)$
- Additional results for fair regression

Algorithm 1 Input to polynomial solver

Input: $\mathbf{A}^{(1)},\ldots,\mathbf{A}^{(\ell)}$, $\mathbf{S}$, α
Output: Feasibility of polynomial system

1: **Polynomial variables**
2: Let $\mathbf{Y} = (\mathbf{VS}) \in \mathbb{R}^{k\times m}$ be mk variables
3: Let $\mathbf{W} = (\mathbf{VS})^\dagger \in \mathbb{R}^{m\times k}$ be mk variables
4: Let $\mathbf{R}^{(i)} \in \mathbb{R}^{k\times k}$ for each $i\in[\ell]$ be ℓk^2 variables
5: **System constraints**
6: $\mathbf{YWY} = \mathbf{Y}$, $\mathbf{WYW} = \mathbf{W}$
7: $\mathbf{A}^{(i)}\mathbf{SWR}^{(i)}$ has orthonormal columns
8: $\alpha \ge \|(\mathbf{A}^{(i)}\mathbf{SWR}^{(i)})(\mathbf{A}^{(i)}\mathbf{SWR}^{(i)})^\dagger\mathbf{A}^{(i)} - \mathbf{A}^{(i)}\|_F^2$
9: **Run polynomial system solver**
10: If feasible, output $\mathbf{V} = (\mathbf{A}^{(1)}\mathbf{SWR}^{(1)})^\dagger\mathbf{A}^{(1)}$. Otherwise, output $\perp$.

Algorithm 2 $(1+\varepsilon)$-approximation for fair low-rank approximation

Input: $\mathbf{A}^{(i)} \in \mathbb{R}^{n_i\times d}$ for all $i\in[\ell]$, rank parameter $k>0$, accuracy parameter $\varepsilon\in(0,1)$
Output: $(1+\varepsilon)$-approximation for fair LRA

1: Let α be an ℓ-approximation for the fair LRA problem
2: Let $\mathbf{S}$ be generated from a random affine embedding distribution
3: **while** Algorithm 1 on input $\mathbf{A}^{(1)},\ldots,\mathbf{A}^{(\ell)}$, $\mathbf{S}$, and α does not return $\perp$ **do**
4: Let $\mathbf{V}$ be the output of Algorithm 1 on input $\mathbf{A}^{(1)},\ldots,\mathbf{A}^{(\ell)}$, $\mathbf{S}$, and α
5: $\alpha \leftarrow \frac{\alpha}{1+\varepsilon}$
6: **end while**
7: Return $\mathbf{V}$

Lower Bounds

- Given vectors $v_1, \ldots v_n \in \mathbb{R}^d$, minimize the distance from these points to a $(n-1)$-dimensional subspace
- Subspace$(n-1,\infty)$ problem is NP-hard to approximate within any constant factor
- Exponential time hypothesis: The 3-SAT problem requires $2^{\Omega(n)}$ runtime.

Upper Bounds

- Theorem [BPR96]: Given a polynomial system in $x_1, \ldots x_n$ over real numbers and m polynomial constraints of degree d with coefficients at most B bits, there exists an algorithm that determines whether there exists a solution to the polynomial system in time $(md)^{O(n)}\cdot\text{poly}(B)$.
- Find "good" approximation α to fair low-rank approximation and then repeatedly decrease α by $(1+\varepsilon)$ and check polynomial system solver using
  - Use dimensionality reduction to improve polynomial solver runtime

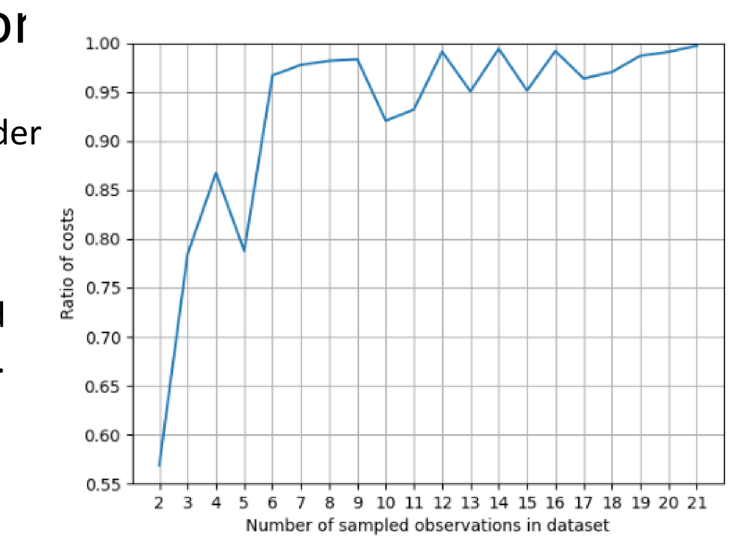
Bicriteria algorithm. $\|x\|_\infty = (1\pm\varepsilon)\|x\|_p$ for large p, so instead minimize over V:

$$\left(\sum_i \left\|A^{(i)}V^\dagger V - A^{(i)}\right\|_F^p\right)^{1/p}$$

- Use Dvoretzky's Theorem to embed into L_p

Experiments

- Credit card dataset with 30,000 observations, 23 features, e.g., previous payment statements and delays, upcoming bill statement, and whether they default. Gender used as the protected attribute
- Baseline: SVD for standard low-rank approximatioı



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https://github.com/samsonzhou/SVWZ24