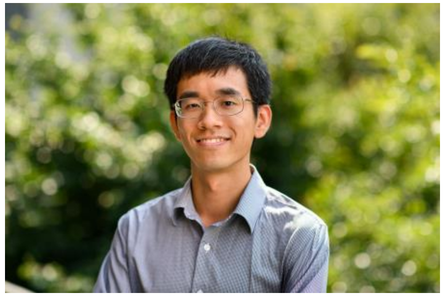
A Strong Separation for Adversarially Robust ℓ_0 Estimation for Linear Sketches



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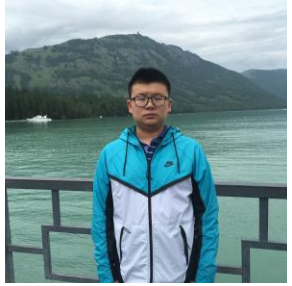

Streaming Model

- Input: Elements of an underlying data set S, which arrives sequentially
- Output: Evaluation (or approximation) of a given function
- Goal: Use space *sublinear* in the size m of the input S

Lots of problems…

- Graph problems: Matchings, MST, MAX-CUT
- Geometric problems: Clustering, facility location
- Statistical problems: Heavy-hitters, norm/moment estimation, quantile estimation
- Algebraic problems: Subspace embeddings, regression, low-rank approximation
- String problems: pattern matching, periodicity
- Others: CSPs, coding theory, submodular optimization, etc

Distinct Elements

- Given a set S of m elements from $[n]$, let f_i be the frequency of element i. (How often it appears)
- Let F_0 be the frequency moment of the vector:

$$F_0 = |\{i : f_i \neq 0\}|$$

- Goal: Given a set S of m elements from $[n]$ and an accuracy parameter ε, output a $(1+\varepsilon)$-approximation to F_0
- Motivation: Traffic monitoring

Insertion-Only Streams

- Each update of the stream can only increase a coordinate of the frequency vector $x \in \mathbb{R}^n$

$$1\ 4\ 2\ 1\ 3\ 4\ 4\ 1 \rightarrow [3, 1, 1, 3, 0] := x$$

4 Distinct Elements

Streaming Algorithms for ℓ_0 Estimation

$(1+\varepsilon)$-multiplicative approximation streaming algorithms for distinct elements estimation using space:

- $O(\log n)$, assuming constant ε and random oracle [FlajoletMartin85]
- $O(\log n)$, assuming constant ε [AlonMatiasSzegedy99]
- $O\left(\frac{1}{\varepsilon^2}\log n\right)$ [Bar-YoseffJayramKumarSivakumar02]
- $O\left(\frac{1}{\varepsilon^2}\log\log n + \log n\right)$ assumes random oracle, additive error, i.e., HyperLogLog [FlajoletFusyGandouetMeunier07]
- $O\left(\frac{1}{\varepsilon^2}+\log n\right)$ [KaneNelsonWoodruff10], [Blasiok20]

Streaming Algorithms for ℓ_0 Estimation

- Sample the elements of the universe $[n]$ at rate $\frac{1}{2^i}$ into set S_i for $i = 0,1,\ldots,O(\log n)$
- Pick set S_i with roughly $\frac{1}{\varepsilon^2}\log n$ items in the stream
- Output $|S_i| \cdot 2^i$ as constant-factor approximation to the number of distinct elements

Adversarially Robust Streaming

- Input: Elements of an underlying data set S, which arrives sequentially and *adversarially*
- Output: Evaluation (or approximation) of a given function
- Goal: Use space *sublinear* in the size m of the input S

1

1

Adversarially Robust Streaming

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1 4

2

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1 4 2

3

Adversarially Robust Streaming

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1 4 2 1

4

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1 4 2 1

4

Adversarially Robust Streaming

- Input: Elements of an underlying data set S, which arrives sequentially and *adversarially*
- Output: Evaluation (or approximation) of a given function
- Goal: Use space *sublinear* in the size m of the input S

- Adversarially Robust: “Future queries may depend on previous queries”
- Motivation: Database queries, adversarial ML

Robust Algorithms for ℓ_0 Estimation

$(1+\varepsilon)$-multiplicative approximation adversarially robust streaming algorithms for distinct elements estimation using space:

- $\tilde{O}\left(\frac{1}{\varepsilon^3}\right) \cdot \text{polylog}(n)$, via sketch switching [Ben-EliezerJayaramWoodruffYogev20]
- $\tilde{O}\left(\frac{1}{\varepsilon^{2.5}}\right) \cdot \text{polylog}(n)$, via differential privacy [HassidimKaplanMansourMatiasStemmer20]
- $\tilde{O}\left(\frac{1}{\varepsilon^2}\right) \cdot \text{polylog}(n)$, via difference estimators [WoodruffZhou21]

Robust Algorithms for ℓ_0 Estimation

$(1+\varepsilon)$-multiplicative appr ... streaming algorithms fo

- $\tilde{O}\left(\frac{1}{\varepsilon^3}\right) \cdot$
  Elie
- $\tilde{O}\left(\frac{1}{\varepsilon^2}\right.$
  [Has
- $\tilde{O}\left(\frac{1}{\varepsilon^2}\right) \cdot \text{poly}$ ... ors ... ruffZhou21]

Nearly tight results for adversarial robustness on insertion-only streams

Insertion-Deletion Streams

- Each update $u_t = (a_t, \Delta_t)$ can increase or decrease a coordinate $a_t \in [n]$ of the underlying frequency vector $x \in \mathbb{R}^n$ by $\Delta_t \in \mathbb{Z}$
- For simplicity, we assume $\Delta_t \in \{-1, +1\}$
- In the robust setting, each update u_t can be chosen adversarially

Insertion-Deletion Streams

- $\tilde{O}(m^{1/3})$ space algorithm for distinct element estimation, where m is the length of the stream [Ben-EliezerEdenOnak22]

- Nothing known for constant-factor approximation in space polynomial in n

Linear Sketch

- Algorithm maintains Ax for a matrix A throughout the stream
- The algorithm then outputs $f(Ax)$ for some post-processing function f

- All insertion-deletion streaming algorithms on a sufficiently long stream might as well be linear sketches [LiNguyenWoodruff14, AiHuLiWoodruff16]

Reconstruction Attack on Linear Sketches

- Linear sketches are "not robust" to adversarial attacks, must use $\Omega(n)$ space [HardtWoodruff13]
- Approximately learn sketch matrix A, query something in the kernel of A
- Iterative process, start with $V_1, \ldots, V_r$
- Correlation finding: Find vectors weakly correlated with A orthogonal to V_{i-1}
- Boosting: Use these vectors to find strongly correlated vector v
- Progress: Set $V_i = \operatorname{span}(V_{i-1}, v)$

Reconstruction Attack on Linear Sketches

- Attack randomly generates Gaussian vectors

- Analysis uses rotational invariance of Gaussians to observe

- Attack ONLY works on *real-valued inputs (main difficulty!)* and ONLY against ℓ_p norm estimation for $p > 0$

Our Contribution

- There is a constant $\varepsilon = \Omega(1)$ so that any linear sketch giving a $(1+\varepsilon)$-approximation to ℓ_0 on an adversarial insertion-deletion stream using $r < n^c$ rows, for a constant $c > 0$, can be broken in $\tilde{O}(r^8)$ queries

Upcoming

- Attack intuition

Questions?

Gap ℓ_0 Norm Problem

- Let α and β be fixed constants
- Distinguish between the case where $\|x\|_0 < \alpha n$ or $\|x\|_0 > \beta n$
- Algorithm allowed to arbitrarily output when neither case holds
- Any multiplicative $(1 + \varepsilon)$-approximation algorithm to ℓ_0 can solve the gap problem, for sufficiently small ε

A Motivating Question

- Some coordinates of the input vector are significant, i.e., learned well by the sketching matrix A, but most of them are not

- What does significant mean?

A Motivating Question

- Intuitively, a sketch matrix A may preserve a "large" amount of information about some coordinates and a "small" amount of information about other coordinates
  - There can be a row of A that is nonzero in only a single column
  - A can be sampled such that a random set of $O(1)$ coordinates has large information
  - There can be coordinates that only appears in columns with a large number of nonzero entries

$$Ax := \overset{A}{\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}} \underset{x}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$x := [0,1,0,1,0,0,0]$$

$$A_1 := [0,0,0,1,0,0,0] \rightarrow \langle A_1, x \rangle = 1$$

$$A_2 := [1,-1,1,1,0,1,1] \rightarrow \langle A_2, x \rangle = 0$$

Attack Outline

- Adversary wants to gradually learn the sketching matrix

- Strategy:

    1. Iteratively identify the significant coordinates and set them to zero in all future queries

    2. After we have learned all significant coordinates, the query algorithm must rely on the other coordinates, for which the sketch Ax only has "small" information

Attack Outline

- Consider an extreme example where the sketch Ax is a subset S of r coordinates of x, unknown to the adversary

- Attack:
  1. Identify S
  2. Place zeros in S and nonzeros elsewhere

Interactive Fingerprinting Code Problem

- An algorithm $\mathcal{P}$ selects a set $S \subset [n]$ of coordinates unknown to the fingerprinting code $\mathcal{F}$

- $\mathcal{F}$ must identify S by making adaptive queries $c^t \in \{0,1\}^n$
- $\mathcal{P}$ must answer consistently with some coordinate in c^t, i.e., $a^t = c_i^t$ for some $i \in [n]$
- BUT $\mathcal{P}$ can only observe c_i^t for $i \in S$ → needs to distinguish between inputs that are all zeros and all ones restricted to S

- Used for watermarking, traitor-tracing schemes [BonehShaw98]

Interactive Fingerprinting Codes

- There exists an interactive fingerprinting code with length $\tilde{O}(n^2)$ [SteinkeUllman15]

- Gap ℓ_0 norm problem needs to distinguish between $\|x\|_0 < \alpha n$ or $\|x\|_0 > \beta n$

- Stronger requirement than fingerprinting code (which just needs to distinguish between all zeros and all ones)

Significant Coordinates (I)

- How to quantify significant coordinates?
- i is significant if there exists:
  - an elementary vector e_i that is a row of A

$$\begin{array}{cc} & A \\ Ax := & \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 999 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{cc} & \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ x & \end{array} = \begin{bmatrix} 1 \\ 1000 \end{bmatrix}$$

$$x := [0,1,0,1,0,0,0]$$

$$A_1 := [0,0,0,1,0,0,0] \rightarrow \langle A_1, x \rangle = 1$$

$$A_2 := [1,999,1,1,0,1,1] \rightarrow \langle A_2, x \rangle = 1000$$

Significant Coordinates (II)

- Since the algorithm has Ax, it can recover $y^\top Ax$ for any vector $y \in \mathbb{R}^r$
- If there exists $y \in \mathbb{R}^r$ such that $(y^\top A)_i^2 \geq \frac{1}{s}\|y^\top A\|_2^2$, then i is significant ("leverage score" of column i is large)

Significant Coordinates (II)

- How to quantify significant coordinates?
- i is significant if there exists:
  - an elementary vector e_i that is a row of A
  - $y \in \mathbb{R}^r$ such that $(y^\top A)_i^2 \geq \frac{1}{s} \|y^\top A\|_2^2$

$$A_1 \coloneqq [10, 10, 10, 10, 10, 10, 3] \rightarrow \langle A_1, x \rangle = 103$$

$$x \coloneqq [2,3,5,0,0,0,1]$$

Reveals information about x_n modulo 10

$$A_1 := \left[1, 1, 1, 1, 1, 1, \frac{3}{10}\right] \rightarrow \langle A_1, x\rangle = 10.3$$

$$x := [2,3,5,0,0,0,1]$$

Fractional part of $(y^\top A)_n$ is large, for y selecting the first row of A

Significant Coordinates (III)

- i is significant if there exists $y \in \mathbb{R}^r$ such that $(\mathrm{FRAC}(y^\top Ax)_i)^2 \geq \frac{1}{s} \sum_i (\mathrm{FRAC}(y^\top Ax)_i)^2$

Significant Coordinates

- How to quantify significant coordinates?
- i is significant if there exists:
  - an elementary vector e_i that is a row of A
  - $y \in \mathbb{R}^r$ such that $(y^\top A)_i^2 \geq \frac{1}{s}\|y^\top A\|_2^2$
  - $y \in \mathbb{R}^r$ such that $(\mathrm{FRAC}(y^\top A)_i)^2 \geq \frac{1}{s}\sum\left(\mathrm{FRAC}(y^\top A)_j\right)^2$

- If i satisfies condition 1 or 2, then i satisfies condition 3

Pre-processing the Sketch Matrix

- The algorithm has access to linear sketch Ax
- Pre-process the matrix A into a larger matrix A' that separates the significant coordinates
- Only gives the algorithm "more" information

$$\underset{A}{\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 999 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}} \longrightarrow \underset{A'}{\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

Pre-processing the Sketch Matrix

- Resulting matrix A' is a combination of a sparse part S and a dense part D

$$A' = \begin{bmatrix} S \\ D \end{bmatrix}$$

$$A' = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Pre-processing the Sketch Matrix

- Sparse part S has at most one nonzero entry per column
- Dense part D has no significant columns

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

- Show only $O(rs \log n)$ rows added to A

- Note that if there were no dense part, we can use fingerprinting code to attack S

Pre-processing the Sketch Matrix

- Sparse part S has at most one nonzero entry per column
- Dense part D has no significant columns

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

- Show only $O(rs \log n)$ rows added to A

If we subsample coordinates of a random input $x \in \{-1,1\}^n$ at rate $\frac{1}{s}$ and repeat independently $s \log n$ times, obtaining sketches $Ax^1, \ldots, Ax^{s \log n}$, we can recover any significant coordinate by looking at fractional parts of L_2 norm. But $O(rs \log n)$ bits of information in $Ax^1, \ldots, Ax^{s \log n}$

Overall Attack

1. Pre-process the matrix A into a matrix A' that is a combination of a sparse part S and a dense part D
2. Attack sparse part S using fingerprinting code
3. Attack dense part D

Upcoming

- Attack on dense part

Questions?

Attacking the Dense Part

- Design a family of distributions $\mathcal{D}$ over $[-R, \dots, -1,0,1, \dots, R]$ with $R = \mathrm{poly}(r\, s \log n)$ such that:
  - For $D_p \in \mathcal{D}$ with $p \in [\alpha, \beta]$, we have $\Pr_{X \sim D_p}[X = 0] = p$
  - For any $q, p \in [\alpha, \beta]$, the total variation distance between Dx_p and Dx_q is small, i.e., $\frac{1}{\mathrm{poly}(n)}$

- We show these are the only distributions needed for the fingerprinting code – dense part does not help for these!

Attacking the Dense Part

- Design a family of distributions $\mathcal{D}$ over $[-R, \dots, -1,0,1, \dots, R]$ with $R = \text{poly}(r\, s \log n)$ such that:
  - For $D_p \in \mathcal{D}$ with $p \in [\alpha, \beta]$, we have $\Pr_{X \sim D_p}[X = 0] = p$
  - For any $q, p \in [\alpha, \beta]$, the total variation distance between Dx_p and Dx_q is small, i.e., $\frac{1}{\text{poly}(n)}$

- If $x \sim D_p^n$, then $\text{Ex}[\|x\|_0] = pn$ and if $x \sim D_q^n$, then $\text{Ex}[\|x\|_0] = qn$, *but the marginal distribution of* Dx *is nearly identical for* $x \sim D_p^n$ *and* $x \sim D_q^n$, *so the algorithm must use* Sx

Overall Attack

1. Pre-process the matrix A into a matrix A' that is a combination of a sparse part S and a dense part D
2. Attack sparse part S using fingerprinting code
3. Attack dense part D using the family of distributions $\mathcal{D}$

Bounding the Total Variation Distance

- Let P be the probability distribution corresponding to Dx_p and Q be the probability distribution corresponding to Dx_q
- To bound the total variation distance between P and Q, we multiply the support size ($n^{O(rs \log n)}$) of P and Q by a pointwise bound:

$$|P(x) - Q(x)| = \left| \frac{1}{(2\pi)^r} \int_{u \in [-\pi,\pi]^r} e^{i\langle u,x \rangle} \left(\hat{P}(u) - \hat{Q}(u) \right) \mathrm{d}u \right|$$

$$\leq \frac{1}{(2\pi)^r} \int_{u \in [-\pi,\pi]^r} \left| \hat{P}(u) - \hat{Q}(u) \right| \mathrm{d}u$$

Bounding the Total Variation Distance

- For a symmetric product distribution, we can write

$$\hat{P}(u) = \mathrm{E}_{z\sim P}\left[e^{-i\langle u,z\rangle}\right]$$
$$= \prod_{j\in[n]} \sum_{k\geq 0} \left(M_p(2k)\right) \cdot f(D^j, u, k)$$

where $M_p(2k) = \left(\sum_{m\geq 0} P_m m^{2k}\right)$ is the $2k$-th moment of the distribution and f is a rapidly decaying function independent of P

Expand $e^{-i\langle u,z\rangle}$ into cosines and this is where fractional parts arise – use preprocessing to bound these fractional parts!

Bounding the Total Variation Distance

- To analyze the total variation distance, we have

$$\left|\hat{P}(u) - \hat{Q}(u)\right| = \prod_{j \in [n]} \sum_{k \geq 0} \left(M_p(2k) - M_q(2k)\right) \cdot f(D^j, u, k)$$

so if the moments of the distributions of P and Q match, up to a sufficiently large k, this helps TVD be small!

Constructing Hard Distributions

- Design a family of distributions $\mathcal{D}$ over $[-R, \dots, -1,0,1, \dots, R]$ with $R = \mathrm{poly}(r\, s \log n\,)$ such that:
  - For $D_p \in \mathcal{D}$ with $p \in [a, b]$, we have $\Pr_{X \sim D_p}[X = 0] = p$
  - For any $q, p \in [a, b]$, the total variation distance between Dx_p and Dx_q is small, i.e., $\frac{1}{\mathrm{poly}(n)}$
  - Moments of the distributions of D_p and D_q match

Moment Matching

- Want $E_{X\sim D_p}[X^k] = E_{X\sim D_q}[X^k]$ for all $k \leq K = O(r \log n)$
- [LarsenWeinsteinYu20, Sherstov] There is a degree R polynomial Q such that for all $t < R^{1/2}$ and constant ε:

$$|Q(0)| > \varepsilon$$

Allows creating distributions with large probability at 0

$$\sum_{i=0}^{R} (-1)^i \binom{R}{i} \cdot Q(i) \cdot i^t = 0$$

Allows creating distributions with first t matching moments

Overall Attack

1. Pre-process the matrix A into a matrix A' that is a combination of a sparse part S and a dense part D
2. Attack sparse part S using fingerprinting code
3. Attack dense part D using the family of distributions $\mathcal{D}$

Main Result

- There is a constant $\varepsilon = \Omega(1)$ so that any linear sketch giving a $(1+\varepsilon)$-approximation to ℓ_0 on an adversarial insertion-deletion stream using $r < n^c$ rows, for a constant $c > 0$, can be broken in $\tilde{O}(r^8)$ queries

Additional Results

- Any linear skech that produces 1.1-approximation to ℓ_0 on an adversarial insertion-deletion stream using $r \ll n$ rows can be broken in $\tilde{O}(r^3)$ queries, if the calculations are performed on finite fields $\mathbb{F}_p$

- There exists an attack on any real-valued linear sketch with "bounded subdeterminants" that produces $O(1)$-approximation to ℓ_0 on an adversarial insertion-deletion stream with $r \ll n$ rows, using $\mathrm{poly}(r)$ queries

Future Directions

Attacks on linear-sketches for ℓ_0 estimation on adversarial insertion-deletion streams

Attacks on streaming algorithms for ℓ_0 estimation on adversarial insertion-deletion streams

Attacks on linear-sketches for ℓ_p estimation on adversarial insertion-deletion streams

Attacks on streaming algorithms for ℓ_p estimation on adversarial insertion-deletion streams