# CSCE 658: Randomized Algorithms – Spring 2024
# Problem Set 3

Due: Tuesday, February 20, 2024, 5:00 pm CT

**Problem 1.** (30 points total) CountSketch tail bounds.

For any vector $x \in \mathbb{R}^n$ and any integer $k \geq 0$, we define $\textsc{Tail}_k(x)$ to be the vector $x$, but with the $k$ entries of largest magnitude to be set to 0, breaking ties arbitrarily. For example if $x = (-100, 40, 40, 1)$, then $\textsc{Tail}_2(x)$ can be either $(0, 0, 40, 1)$ or $(0, 40, 0, 1)$.

1. (5 points) Show that for any parameter $\alpha \geq 1$ and $k \leq n - 1$, there exists $x \in \mathbb{R}^n$ such that

$$\alpha \cdot \|\textsc{Tail}_k(x)\|_2 < \|x\|_2.$$

   That is, the length of a tail vector of $x$ can be arbitrarily smaller than the length of the vector $x$.

2. (25 points) Show that $\textsc{CountSketch}$ actually provides an $L_2$ tail guarantee. More specifically, for $\varepsilon \in (0, 1)$, suppose we use $\textsc{CountSketch}$ with $\mathcal{O}\left(\frac{1}{\varepsilon^2} \cdot \log n\right)$ buckets to extract estimates $\widehat{x_i}$ for the value of each coordinate $x_i$. Show that with probability $1 - \frac{1}{n^2}$, we simultaneously have that for all $i \in [n]$,

$$|\widehat{x_i} - x_i| \leq \varepsilon \cdot \|\textsc{Tail}_k(x)\|_2,$$

   where $k = \frac{1}{\varepsilon^2}$.

**Problem 2.** (30 points total) AMS Sketch for $F_p$

Let $p \geq 1$. Suppose $f \in \mathbb{R}^n$ is defined by an insertion-only stream of length $m$, where each update increments a coordinate of $f$. Suppose we sample an update $t \in [m]$ in the stream, uniformly at random, and set a counter $c$ to be the number of times the item appears in the stream after time $t$ (including time $t$). After the stream ends, we set $Z = c^p - (c-1)^p$.

For example, suppose the stream consists of the updates $1, 2, 2, 1, 4, 1, 2, 1$, which induces the frequency vector $f = (4, 3, 0, 1)$ and suppose we sample the fourth update of the stream, corresponding to a 1. Then we see a total of three instances of 1, after that time (inclusive), so that $c = 3$ and $Z = 3^p - 2^p$. For $p = 3$ then, we would have $Z = 27 - 8 = 19$.

1. (5 points) Show that $\mathbb{E}[Z] = f_j^p$, *conditioned* on sampling $j \in [n]$.
2. (5 points) Let $F = m \cdot Z$. Show that $\mathbb{E}[F] = \|f\|_p^p$.
3. (10 points) Show that $\mathrm{Var}[F] \leq p \cdot \|f\|_1 \cdot \|f\|_{2p-1}^{2p-1}$.

   HINT: You may use the fact that for all $x \geq 1$ and $p \geq 1$, we have $x^p - (x-1)^p \leq px^{p-1}$.

4. (10 points) Given an algorithm that uses $O\left(\frac{1}{\varepsilon^2} n^{1-1/p}\right) \cdot \log(nm)$ bits of space and with probability at least $\frac{2}{3}$, outputs an estimate $\widehat{F}$ such that

$$(1 - \varepsilon)\|f\|_p^p \leq \widehat{F} \leq (1 + \varepsilon)\|f\|_p^p.$$

HINT: You may use the fact that for all $\|f\|_1 \cdot \|f\|_{2p-1}^{2p-1} \leq n^{1-1/p}\|f\|_p^{2p}$.

**Problem 3.** (30 points total) Oblivious routing

**Problem 4.** (30 points total) Matrix multiplication sketching