CSCE 658: Randomized Algorithms – Spring 2024
Problem Set 4

Due: Tuesday, April 23, 2024, 5:00 pm CT

Problem 1. (30 points total)

1. (10 points) Prove that every graph on m edges with no self-loops has a subgraph on at least $\frac{m}{2}$ edges that is bipartite.

2. (10 points) An independent set of a graph is a subset of vertices that are not connected by an edge in the graph. Prove that for any graph with n vertices, and $m \geq \frac{n}{2}$ edges, there exists an independent set of size at least $\frac{n^2}{4m}$.

3. (10 points) Let n be a sufficiently large parameter. Prove that for every matrix $A \in \{0,1\}^{n \times n}$, there exists a vector $b \in \{-1,+1\}^n$ such that all entries of $Ab \in \mathbb{R}^n$ have magnitude at most $8\sqrt{n \log n}$.

Problem 2. (30 points total)

1. (15 points) Suppose there are p packets that need to be routed over a network of links. Each packet $i \in [p]$ must pick a route from a set R_i of r different possible routes to be sent. Multiple routes can share the same link, but a link only has capacity to support the routing of a single packet. Suppose that for all $i \neq j$ and any route $R \in R_i$, there are at most c other routes $R' \in R_j$ that share a link with R. Prove that if $r \geq 8pc$, then there exists a possible routing of all p packets where no link exceeds its capacity.

2. (15 points) Let G be an undirected graph and suppose each vertex v has a set $C(v)$ of colors and let q be a fixed parameter. A proper list coloring of of the graph assigns each vertex $v \in V$ a color from its set $C(v)$ while ensuring that no edges have two vertices with the same color. Suppose $|C(v)| \geq 10q$ and for all $v \in V$ and $c \in C(v)$, there are most q neighbors u of v that contain c in $C(u)$. Prove that there exists a proper list coloring of the graph.

Problem 3. (30 points total)

1. (10 points) Give an example, with proof, of a primal-dual pair of linear programs, each with at most three variables and three constraints in addition to the non-negativity constraints, such that neither program is feasible.

2. (10 points) Write the linear program for the best fit line with L_1 error, i.e., values (a, b, c) that minimizes
$$\sum_{i=1}^{n} |ax_i + by_i - c|.$$

3. (10 points) Write the linear program for the best fit line with L_∞ error, i.e., values (a, b, c) that minimizes
$$\max_{i \in [n]} |ax_i + by_i - c|.$$

Problem 4. (30 points total)

1. (5 points) Describe the implementation of randomized response for ε-differential privacy. Prove its correctness.

2. (5 points) Use the probability density function of the Laplace distribution to prove that if $X \sim \text{Lap}(b)$, then $\mathbf{Pr}\left[|X| > 4Cb\log n\right] \le \frac{1}{n^C}$ for any constant $C > 0$.

3. (10 points) Suppose that are given a database $x_1, \ldots, x_n$ of counts, so that $x_i \in \mathbb{Z}$ for all $i \in [n]$. To privately release the index of the item with the largest value, i.e., $\text{argmax}_{i\in[n]} x_i$, we first add independent Laplace noise $\text{Lap}\left(\frac{1}{\varepsilon}\right)$ to each value x_i to acquire a value y_i. We then output the index of the noisy item with the largest value, i.e., $\text{argmax}_{i\in[n]} y_i$. Show with proof that the resulting protocol is ε-differentially private. Analyze the correctness/error of the protocol.

4. (10 points) Suppose that are given a database $x_1, \ldots, x_n$ of counts, so that $x_i \in \mathbb{Z}$ for all $i \in [n]$. Suppose that we release $i \in [n]$ with probability proportional to $\exp(s(x_i))$, where $s(x_i) = x_i - \max_{j\in[n]} x_j$. Show with proof that the resulting protocol is ε-differentially private. Analyze the correctness/error of the protocol.