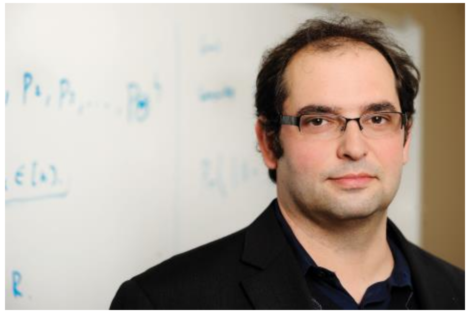
Private Data Stream Analysis for Universal Symmetric Norm Estimation



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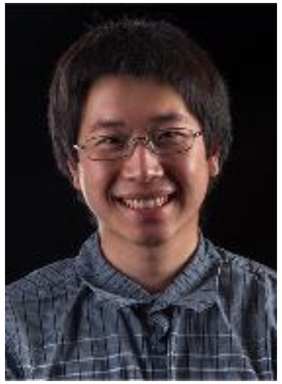


Symmetric Norms

❖ A norm is symmetric if it is invariant under permutations and sign flips on an input frequency vector

$$a = [1,3,-2,0,0,5,-2,4]$$
$$b = [1,3,2,0,0,5,2,4]$$
$$c = [0,0,1,2,2,3,4,5]$$
$$\|a\| = \|b\| = \|c\|$$
$$L(a) = L(b) = L(c)$$

L_p Norms

- Let F_p be the frequency moment of the vector $f \in R^n$:

$$F_p = f_1^p + f_2^p + \cdots + f_n^p$$

- Then the L_p norm of the frequency vector f is:

$$L_p(f) = \left(F_p(f)\right)^{1/p}$$

- Goal: Given an accuracy parameter α, output a $(1+\alpha)$-approximation to L_p
- Motivation: Entropy estimation, linear regression

Differential Privacy

- [DworkMcSherryNissimSmith06] Given $\varepsilon > 0$ and $\delta \in (0,1)$, a randomized algorithm $A: U^* \to Y$ is (ε, δ)-differentially private if, for every neighboring frequency vectors f and f' and for all $E \subseteq Y$,

$$\Pr[A(f) \in E] \leq e^{\varepsilon} \Pr[A(f') \in E] + \delta$$

Multiple Privately Queries

- Privately query $f \in R^n$ multiple times?

- Add noise to each query with scale parameter depending on the number Q of queries
- Accuracy degrades as the number Q of queries increases

Can we answer multiple queries without sacrificing accuracy?

"Beating the union bound"
"Avoid privacy analysis per algorithm"

Streaming Model

- Input: Elements of an underlying data set S, which arrives sequentially
- Output: Evaluation (or approximation) of a given function
- Goal: Use space *sublinear* in the size m of the input S

1 0 1 1 1 0 0 1

Symmetric Norms in the Streaming Model

- Given a stream S of m elements from $[n]$, let f_i be the frequency of element i. (How often it appears)

- Goal: Given a stream S of length m that defines a frequency vector $f \in R^n$ and an accuracy parameter α, output a $(1+\alpha)$-approximation to $||f||$, using space sublinear in n and m

Our Result

There exists an (ε, δ)-differentially private algorithm such that:

- Input: on a stream S of length m that defines a frequency vector $f \in R^n$ that
- Output: a set C, from which the $(1+\alpha)$-approximation to any symmetric norm with maximum modulus of concentration M can be computed with probability $1-\delta$.
- The algorithm uses $M^2 \cdot \mathrm{poly}\left(\frac{1}{\alpha}, \frac{1}{\varepsilon}, \log(n, m), \log\frac{1}{\delta}\right)$ space

Applications

- For L_p norms, $M(\ell) = O(\log m)$ for $p \in [1,2]$ and $M(\ell) = O\left(n^{1/2-1/p}\right)$ for $p > 2$ [MilmanSchectman86, KlartagVershynin07]
- Our algorithm achieves space $\text{poly}\log(m)$ for $p \in [1,2]$ and $\tilde{O}\left(n^{1-2/p}\right)$ for $p > 2$ in the constant α and $\delta = \frac{1}{\text{poly}(m)}$ regime
- Matches known lower bounds up to log factors [Bar-YossefJayramKumarSivakumar04]
- For top k norms, $M(\ell) = \tilde{O}\left(\sqrt{\frac{n}{k}}\right)$ [BlasiokBravermanChestnutKrauthgamerYang17]

Maximum Modulus of Concentration

- Maximum modulus of concentration [MilmanSchectman86] of a norm measures the worst-case ratio of the maximum value to the median value of a norm on the L_2-unit sphere for any restriction of the coordinates

- Intuitively, quantifies the “difficulty” of computing a norm

Modulus of Concentration

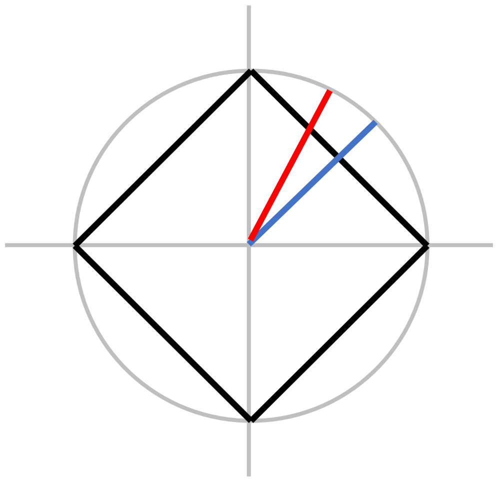
- Let $f \in R^n$ be a random vector drawn from the uniform distribution on the L_2-unit sphere S^{n-1}

- Let b_L denote the maximum value of $L(f)$ over S^{n-1} and let M_L denote the median of $L(f)$, i.e., the unique value such that $\Pr[L(f) \geq M_L] \geq \frac{1}{2}$ and $\Pr[L(f) \leq M_L] \geq \frac{1}{2}$

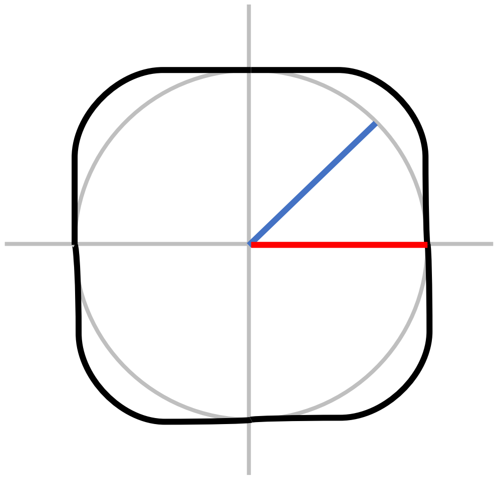
- The ratio $\mathrm{mc}(L) = \frac{b_L}{M_L}$ is the modulus of concentration of L

Modulus of Concentration

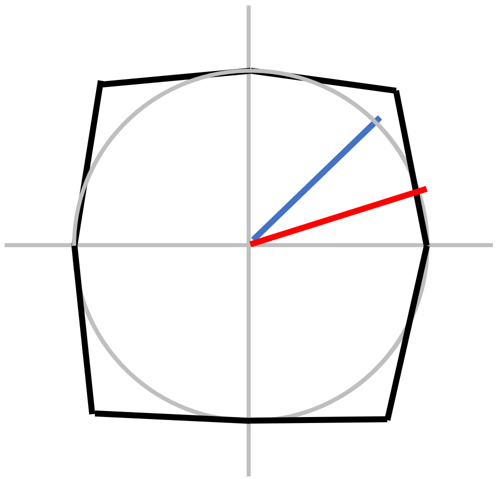
b_L is the maximum value of $L(f)$ over S^{n-1}
M_L is the median of $L(f)$



$b_L = \sqrt{n}$
$M_L \approx \sqrt{n}$



$b_L = 1$
$M_L \approx n^{-1/6}$



Maximum Modulus of Concentration

- Maximum modulus of concentration of a norm is the maximum of the modulus of concentration of the norm restricted to sub-coordinates of R^n

- Definition is robust to "average" norms that "hide" challenging behavior embedded in lower-dimensional space

- $L(x) = \max\left(\frac{L_1(x)}{\sqrt{n}}, L_\infty(x)\right)$

Symmetric Norms

- A norm is symmetric if it is invariant under permutations and sign flips on an input frequency vector

$$a = [1,3,-2,0,0,5,-2,4]$$
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$$\|a\| = \|b\| = \|c\|$$

Approximating Symmetric Norms

- Only care about number of coordinates in each range $[\xi^i, \xi^{i+1})$ for some $\xi > 1$ a function of the desired accuracy parameter α

$\xi = 2$

$v = [0,0,1,2,2,3,4,5]$

#coordinates in [1,2): 1

#coordinates in [2,4): 3

#coordinates in [4,8): 2

Level Sets

- Level set i is the set of coordinates with magnitude in range $[\xi^i, \xi^{i+1})$

$$v = [0,0,\underbrace{1}_{\text{Level set 0}},\underbrace{2,2,3}_{\text{Level set 1}},\underbrace{4,5}_{\text{Level set 2}}]$$

Contribution of Level Sets

❖ The *contribution* of the level set is the "amount" the level set contributes to the norm of the entire frequency vector

$$v = [0,0,1,2,2,3,4,5]$$
$$v' = [0,0,0,2,2,3,0,0]$$

Level set 1

Important Level Sets

- A level set is *important* if its contribution is an $\frac{\alpha}{O(\log m)}$ fraction of the norm of the entire frequency vector

- It suffices to estimate the contribution of the important level sets within $\left(1+\frac{\alpha}{O(\log m)}\right)$-approximation [BlasiokBravermanChestnutKrauthgamerYang17]

Important Level Sets

❖ Intuition: Important level sets must either have large magnitude coordinates or a large number of coordinates

$$v = [1,1,1, \dots, 1,1,10000]$$

$$[1,1,1, \dots, 1,1,0]$$
$$[0,0,0, \dots, 0,0,10000]$$

❖ How to privately release important level sets?

Important Level Sets

- Definition: Define thresholds T_1 and T_2. A level set i is "high" if $\xi^i \geq T_1$. A level set i is "medium" if $\xi^{i+1} \leq T_1$ and $\xi^i \geq T_2$. A level set i is "low" if $\xi^{i+1} \leq T_2$

- Intuition: Important high level sets have large coordinates, important low level sets have a large number of coordinates, important medium level sets have a combination of the two

Heavy-Hitters

- Given a set S of m elements from $[n]$, let f_i be the frequency of element i. (How often it appears)
- Let L_2 be the norm of the frequency vector:

$$L_2 = \sqrt{f_1^2 + f_2^2 + \cdots + f_n^2}$$

- Goal: Given a set S of m elements from $[n]$ and a threshold ε, output the elements i such that $f_i > \varepsilon L_2$...and no elements j such that $f_j < \frac{\varepsilon}{16} L_2$
- Motivation: DDoS prevention, iceberg queries

CountSketch

- Given a threshold/accuracy parameter α, there exists a one-pass streaming algorithm COUNTSKETCH that outputs an estimated frequency for each element, with additive error $\alpha \cdot L_2(f)$
- The algorithm uses $O\left(\frac{1}{\alpha^2}\log^2 m\right)$ space

CountSketch

- COUNTSKETCH with threshold/accuracy parameter $O\left(\frac{\text{poly}(\alpha,\varepsilon)}{M \text{ poly} \log m}\right)$ will find the important high level sets because their magnitude is so large, but it will miss the others

$$c = [1,1,1,\ldots,1,1,10000]$$

$$[1,1,1,\ldots,1,1,0]$$
$$[0,0,0,\ldots,0,0,10000]$$

Subsampling the Universe

- Sample coordinates of the universe with probability $\frac{1}{2^j}$ for $j = 0,1,\ldots,O(\log n)$ [IndykWoodruff05]

$$c = [1,1,1,1,1,1,1,\ldots,1,1,1,1,1,10000]$$
$$[1,0,1,0,0,1,0,\ldots,1,0,0,1,1,10000]$$
$$[1,0,0,0,0,1,0,\ldots,0,0,0,1,0,0]$$

- The important medium and low level sets will be heavy-hitters in the subsampled streams!

Subsampling the Universe

- ❖ Sample coordinates of the universe with probability $\frac{1}{2^j}$ for $j = 0,1,\ldots,O(\log n)$ [IndykWoodruff05]

$$c = [1,1,1,1,1,1,1,\ldots,1,1,1,1,1,10000]$$
$$[1,0,1,0,0,1,0,\ldots,1,0,0,1,1,10000]$$
$$[1,0,0,0,0,1,0,\ldots,0,0,0,1,0,0]$$

- ❖ Will find the important medium and low level sets

Towards Privacy

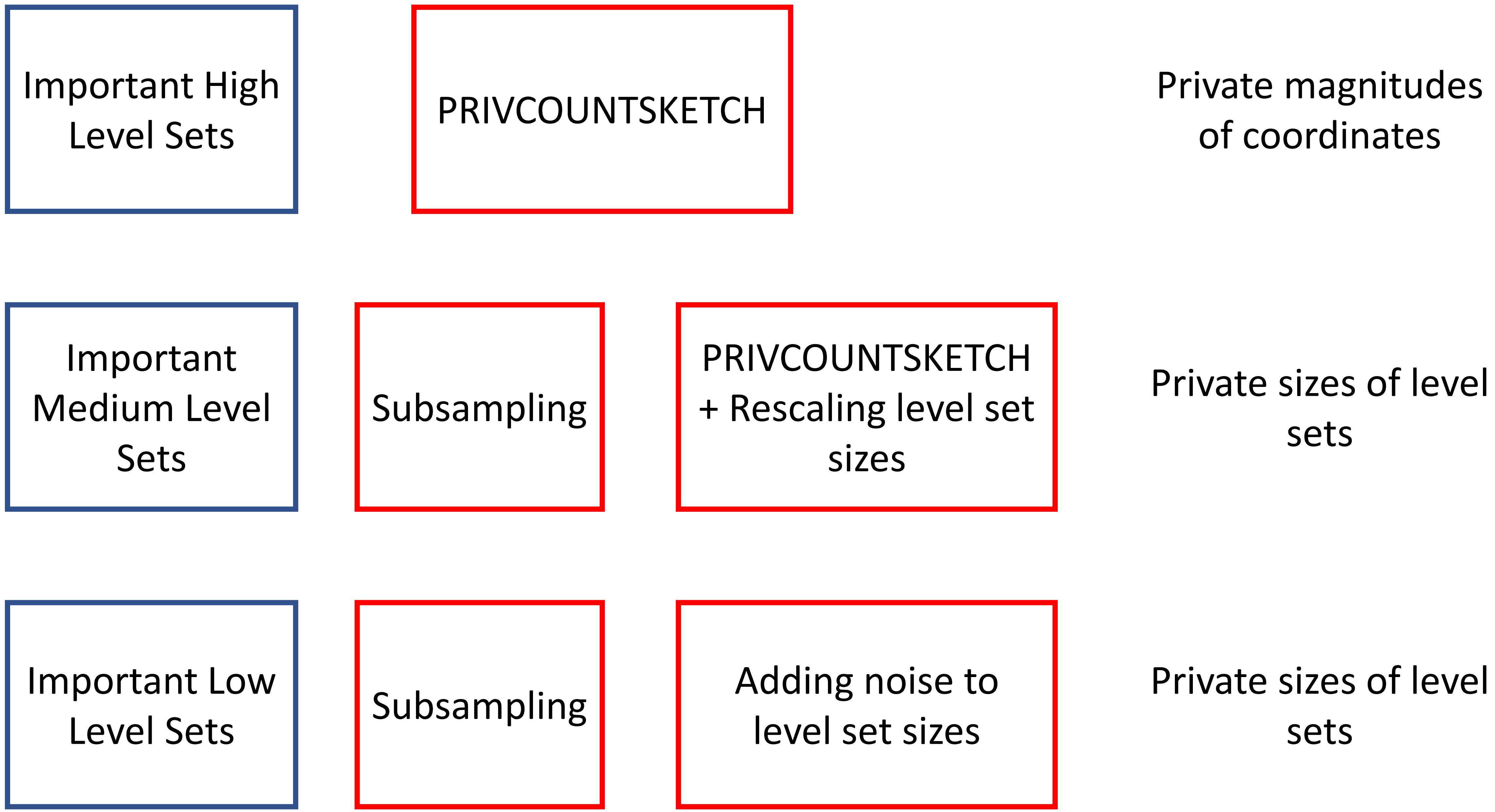
- PRIVCOUNTSKETCH, private release of heavy-hitters, by adding Laplacian noise to each coordinate

- Even though PRIVCOUNTSKETCH estimates n frequencies, only $O\left(\frac{1}{\alpha^2}\right)$ frequencies are released, so only need to add Laplacian noise with scale $O\left(\frac{1}{\alpha^2}\right)$

Towards Privacy

- Even Laplacian noise with scale $O\left(\frac{1}{\alpha^2}\right)$ is too much noise for important low level sets

- Instead add Laplacian noise to the *size* of each important low level set


Important High Level Sets
PRIVCOUNTSKETCH
Private magnitudes of coordinates
Important Medium Level Sets
Subsampling
PRIVCOUNTSKETCH + Rescaling level set sizes
Private sizes of level sets
Important Low Level Sets
Subsampling
Adding noise to level set sizes
Private sizes of level sets

Additional Challenges

❖ Privately identify coordinates for each level set → Two instances of PRIVCOUNTSKETCH

❖ Classification error for each level set from privacy noise → Thresholds are robust for high, medium, and low important levels

❖ Classification error for each level set from frequency estimation → Randomly choose boundaries of each level set

Summary

There exists an (ε, δ)-differentially private algorithm such that:

❖ Input: on a stream S of length m that defines a frequency vector $f \in R^n$ that

❖ Output: a set C, from which the $(1+\alpha)$-approximation to any symmetric norm with maximum modulus of concentration M can be computed with probability $1-\delta$.

❖ The algorithm uses $M^2 \cdot \text{poly}\left(\frac{1}{\alpha}, \frac{1}{\varepsilon}, \log(n,m), \log\frac{1}{\delta}\right)$ space

❖ Algorithm splits important level sets into high, medium, and low coordinates and separately releases private statistics for each