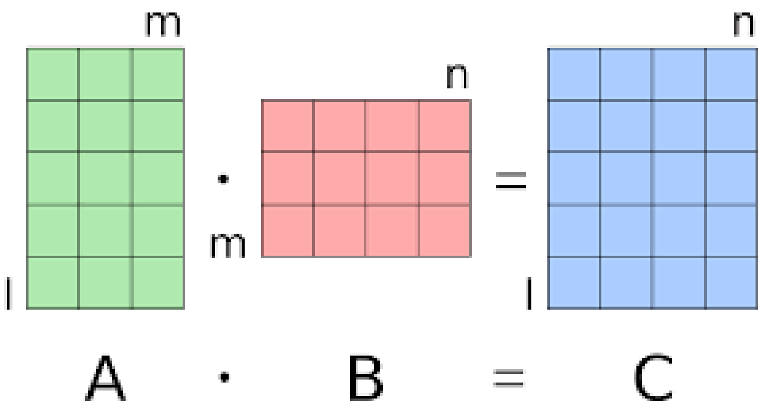
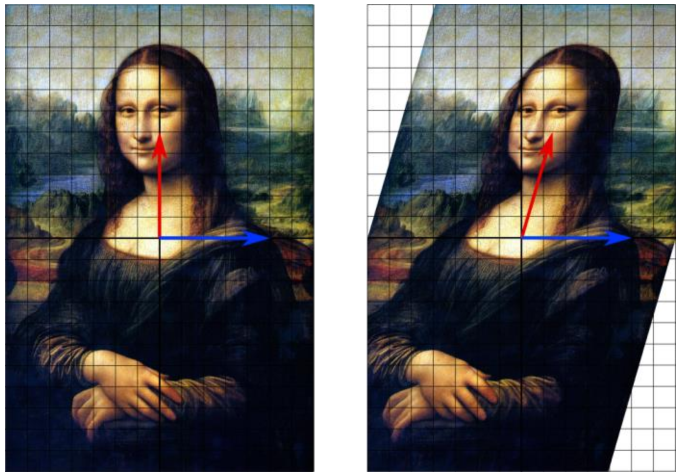
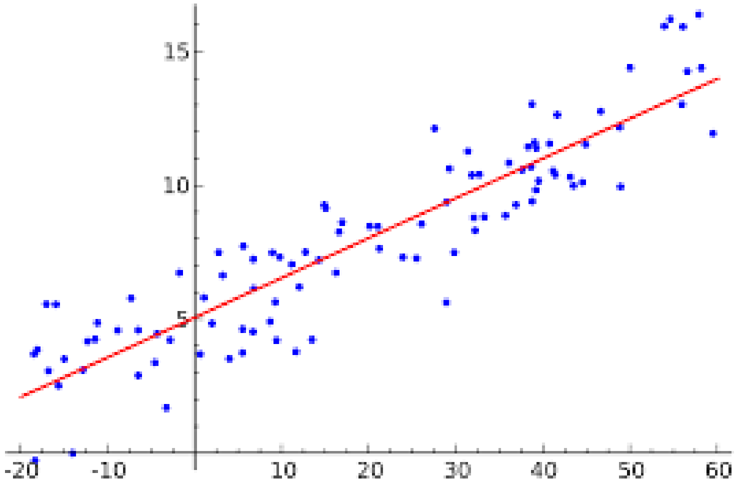
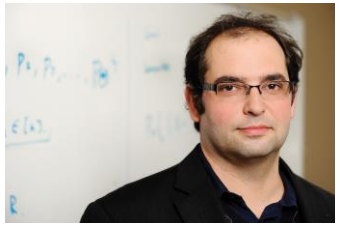
1 0 1 1 1 0 0 1 1 0 1

Near Optimal Linear Algebra in the Online and Sliding Window Models









Vladimir Braverman

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UMass Amherst

Christopher Musco

NYU

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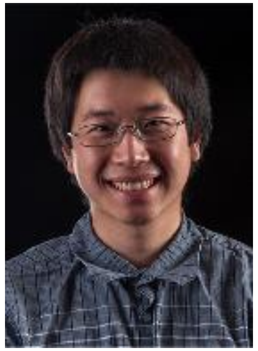

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Model #1: Streaming / Sliding Window Model

- Input: Elements of an underlying data set S, which arrives sequentially
- Output: Evaluation (or approximation) of a given function
- Goal: Use space *sublinear* in the size of the input S

- Sliding Window: "Only the n most recent updates form the underlying data set S"

1 0 1 1 1 0 0 1

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1 0 1 1 1 0 0 1 1

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$$1\ 0\ \boxed{1\ 1\ 1\ 0\ 0\ 1\ 1\ 0}$$

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- **Input**: Elements of an underlying data set S, which arrives sequentially
- **Output**: Evaluation (or approximation) of a given function
- **Goal**: Use space *sublinear* in the size of the input S

- **Sliding Window**: “Only the n most recent updates form the underlying data set S”
  - Emphasizes recent interactions, appropriate for time sensitive settings

$$1\ 0\ 1\ \boxed{1\ 1\ 0\ 0\ 1\ 1\ 0\ 1}$$

Randomized Numerical Linear Algebra (randNLA) on Sliding Windows

$$
\begin{array}{cc}
n & \begin{array}{ccccccccc}
1 & 3 & 5 & -2 & 7 & 0 & 11 & 4 & -8 \\
0 & 0 & -1 & 3 & 13 & 2 & 8 & 6 & 2 \\
2 & 5 & 6 & 1 & 4 & 0 & -7 & 5 & 3 \\
8 & 7 & 2 & 1 & -1 & -3 & -2 & -4 & -6 \\
-5 & 3 & -4 & -1 & -2 & -1 & 0 & -3 & -1 \\
7 & 1 & 3 & 2 & 4 & 1 & 0 & 11 & 1
\end{array} \\
 & d
\end{array}
$$

❖ Rows arrive one-by-one in the data stream

Randomized Numerical Linear Algebra (randNLA) on Sliding Windows

1	3	5	-2	7	0	11	4	-8
0	0	-1	3	13	2	8	6	2
2	5	6	1	4	0	-7	5	3
8	7	2	1	-1	-3	-2	-4	-6
-5	3	-4	-1	-2	-1	0	-3	-1
7	1	3	2	4	1	0	11	1
1	2	5	-5	4	1	23	4	-3

n

d

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Randomized Numerical Linear Algebra (randNLA) on Sliding Windows

1	3	5	-2	7	0	11	4	-8
0	0	-1	3	13	2	8	6	2
2	5	6	1	4	0	-7	5	3
8	7	2	1	-1	-3	-2	-4	-6
-5	3	-4	-1	-2	-1	0	-3	-1
7	1	3	2	4	1	0	11	1
1	2	5	-5	4	1	23	4	-3
0	5	0	0	7	0	1	31	6

n

d

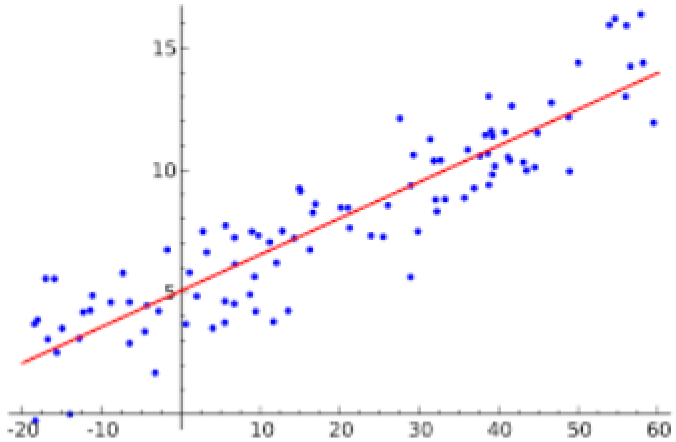
- ❖ Rows arrive one-by-one in the data stream

Why randNLA on Sliding Windows?

- Sliding window emphasizes *efficiency* and *recency*, good for massive data sources and time-sensitive information
- We use linear algebra in optimization strategies and prediction of future patterns based on past data, don't want outdated information
- Principal Component Analysis (PCA), Low-Rank Approximation (LRA), Regression

		-1		
			1	
1	1	-1	1	-1
1				-1
		-1		

1	1	-1	1	-1
1	1	-1	1	-1
1	1	-1	1	-1
1	1	-1	1	-1
1	1	-1	1	-1



Results: Sliding Window Model

- $O(d^2)$ space randomized sliding window algorithm for spectral sparsification (space optimal, up to lower order terms)
- $O(dk)$ space randomized sliding window algorithm for low-rank approximation (space optimal, up to lower order terms)
- $O(d^3)$ space randomized sliding window algorithm for ℓ_1 subspace embedding

Model #2: Online Model

- Input: Elements of an underlying data set S, which arrives sequentially
- Output: Evaluation (or approximation) of a given function
- Goal: Use space *sublinear* in the size n of the input S

- Online Model: "Each time must make irrevocable decision to the output"
  - Send decisions to some further applications downstream

Results: Online Model

- Online algorithm for low-rank approximation that samples $O(k)$ rows (space optimal, up to lower order terms)
- Online algorithm for row subset selection that samples $O(k)$ rows (space optimal, up to lower order terms)
- Online algorithm for principal component analysis that embeds into a matrix with dimension $O(k)$ (space optimal, up to lower order terms)
- Online algorithm for ℓ_1 subspace embedding that samples $O(d^2)$ rows

Challenges

P

$$A = \begin{bmatrix} -2 & 7 & 0 & 11 & 4 & -8 \\ 3 & 13 & 2 & 8 & 6 & 2 \\ 1 & 4 & 0 & -7 & 5 & 3 \\ 1 & -1 & -3 & -2 & -4 & -6 \\ -1 & -2 & -1 & 0 & -3 & -1 \\ 2 & 4 & 1 & 0 & 11 & 1 \end{bmatrix}$$

- Approach 1: Use smooth histogram [BO07] technique for sliding windows
- …Functions are not smooth
- Approach 2: Use sketching techniques for subspace embeddings
- …Cannot undue expirations from sliding window

Challenges

-2	7	0	11	4	-8

P

3	13	2	8	6	2
1	4	0	-7	5	3
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5	0	4	1	0	-11

A

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P

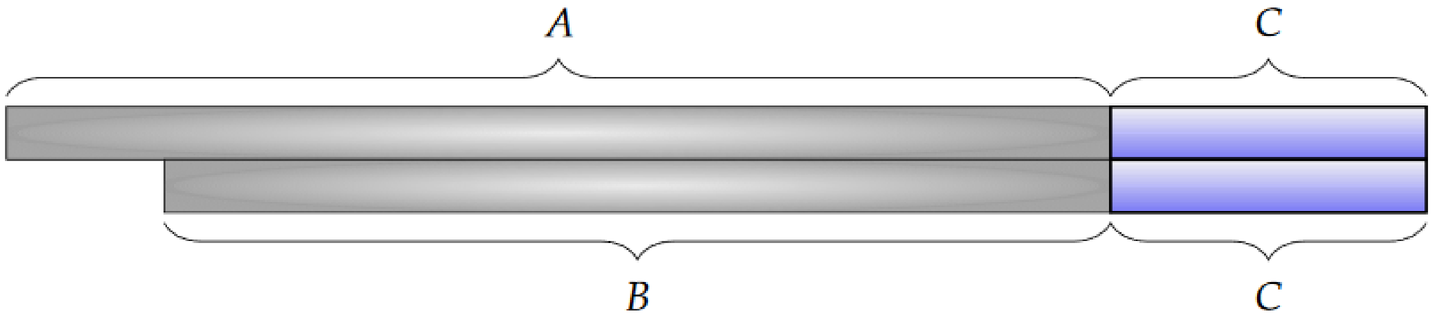
$$\begin{array}{c} \begin{matrix} -2 & 7 & 0 & 11 & 4 & -8 \end{matrix} \\ \begin{bmatrix} 3 & 13 & 2 & 8 & 6 & 2 \\ 1 & 4 & 0 & -7 & 5 & 3 \\ 1 & -1 & -3 & -2 & -4 & -6 \\ -1 & -2 & -1 & 0 & -3 & -1 \\ 2 & 4 & 1 & 0 & 11 & 1 \\ 5 & 0 & 4 & 1 & 0 & -11 \end{bmatrix} \end{array}$$

A

- Approach 1: Use smooth histogram [BO07] technique for sliding windows
- …Functions are not smooth
- Approach 2: Use sketching techniques for subspace embeddings
- …Cannot undue expirations from sliding window
- Approach 3: Use sampling techniques for data streams, e.g., sample rows that are unique
- …Recent rows seem to be more important

Sliding Window Algorithms

- Suppose we are trying to approximate some given function
  1. Suppose we have a streaming algorithm for this function
  2. Suppose this function is "smooth": If $f(B)$ is a "good" approximation to $f(A)$, then $f(B \cup C)$ will always be a "good" approximation to $f(A \cup C)$.



- Smooth histogram framework [BO07] gives a sliding window algorithm for this function

Smooth Histogram

- Suppose we are trying to approximate some given function
- Smooth histogram framework [BO07] gives a sliding window algorithm for this function
- Start a new instance of the streaming algorithm (along with existing instances) each time a new element arrives
- Each time there are three instances that report “close” values, delete the middle one
- Use different checkpoints to “sandwich” the sliding window

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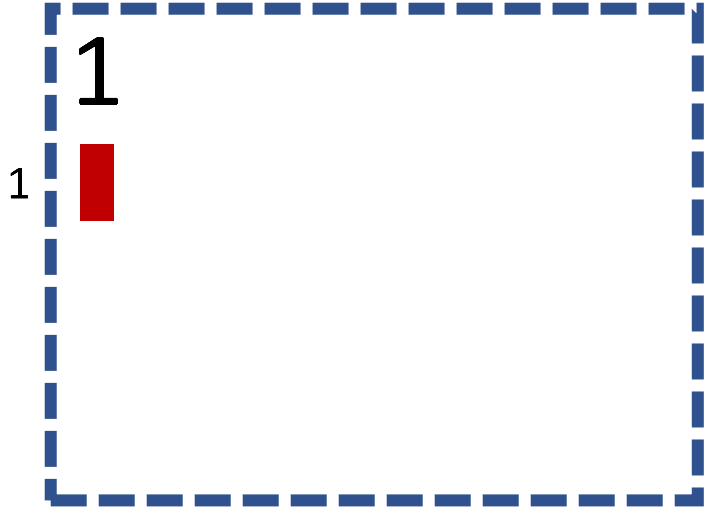

- Example: Number of ones in sliding window (2-approximation)

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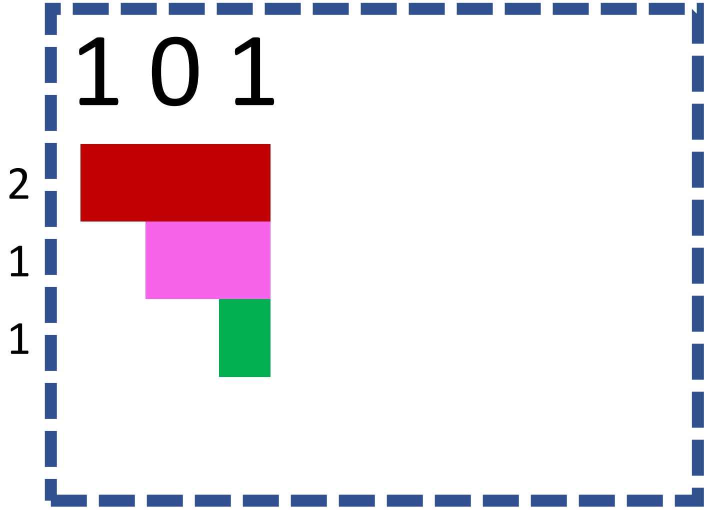

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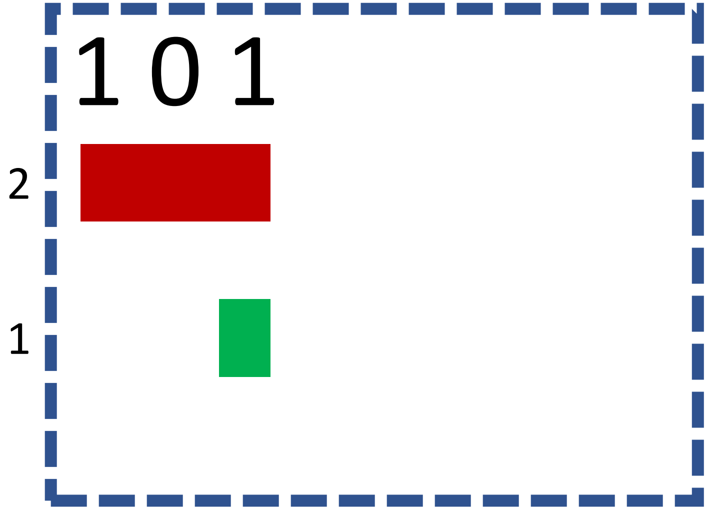

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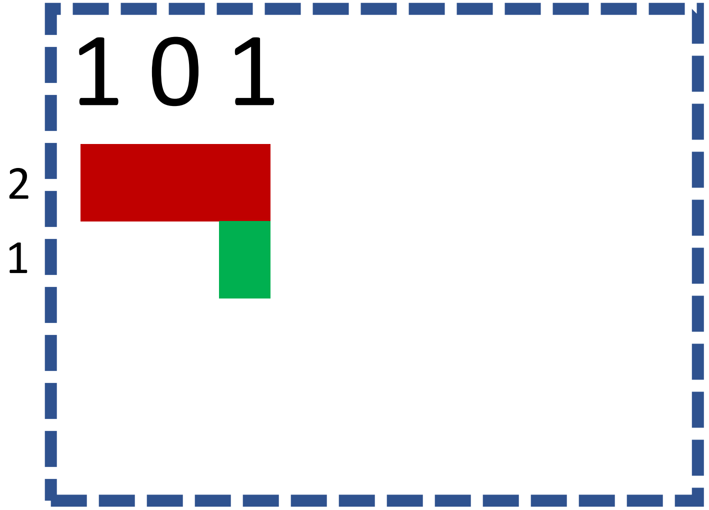

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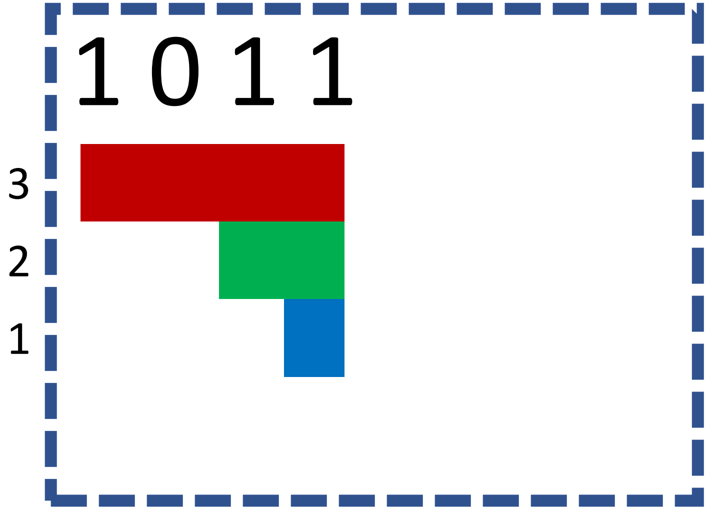

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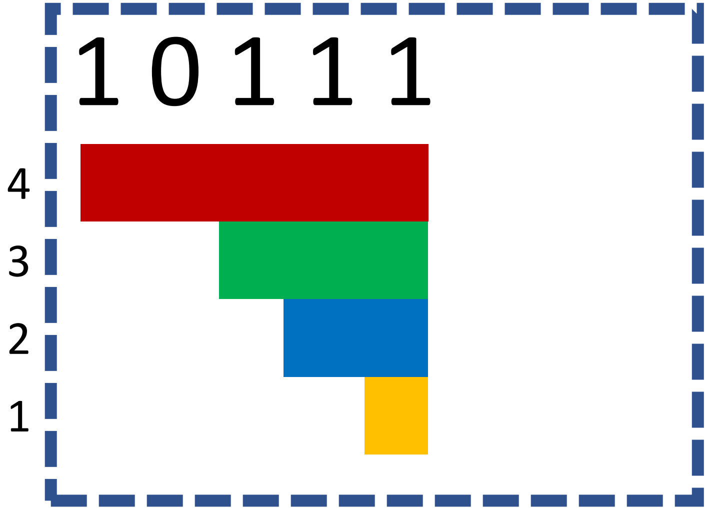

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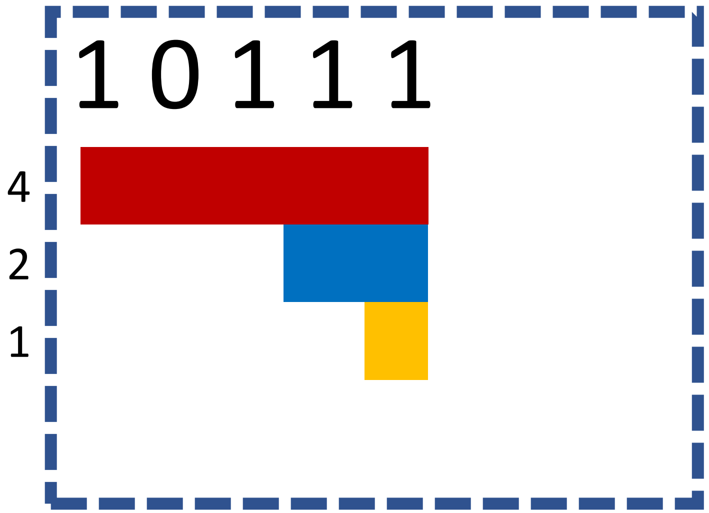

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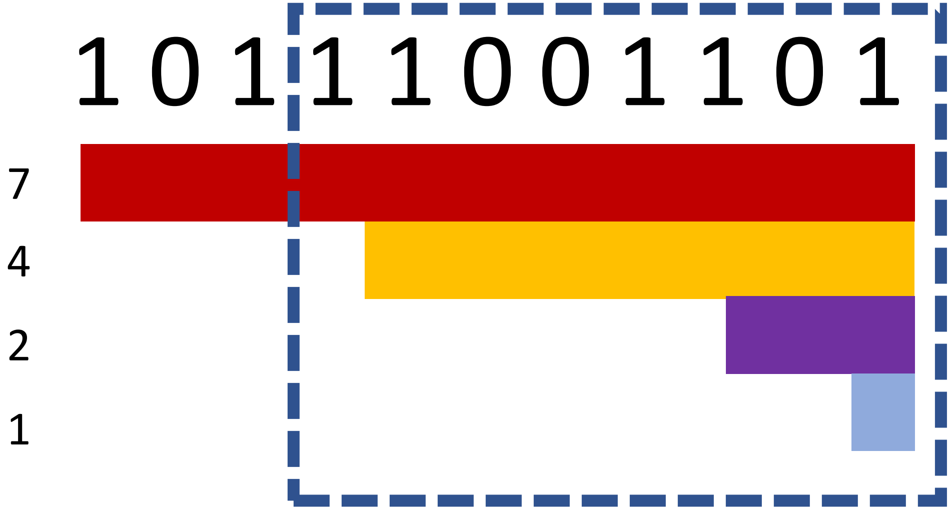

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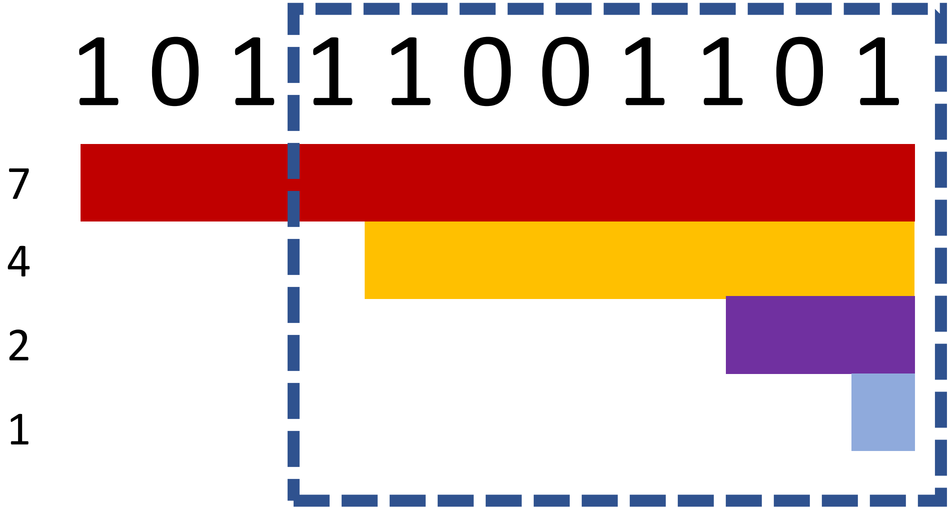

- ❖ Example: Number of ones in sliding window (2-approximation)
- ❖ Number of ones in sliding window is at least 4 and at most 7
- ❖ 4 is a good approximation

Streaming Model Algorithms

- Quantiles, heavy-hitters, norm estimation, distinct elements, sampling
- Matchings, triangle counting, spanners, sparsifiers, densest subgraph,…
- Minimum enclosing ball, clustering, facility location, volume maximization,…
- Numerical linear algebra (matrix multiplication, spectral approximation,…)
- Submodular optimization
- Strings (pattern matching, periodicity, distance, palindromic detection,…)
- Codeword testing

Smooth Functions

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Smooth Functions

[BGO13, BGLWZ18] [BOZ09]

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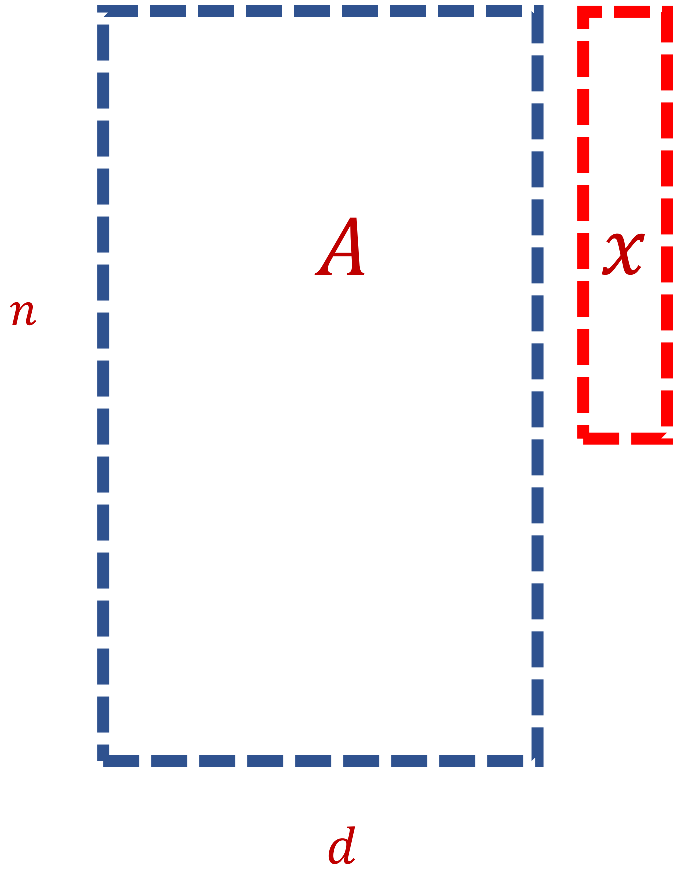
Spectral Approximation

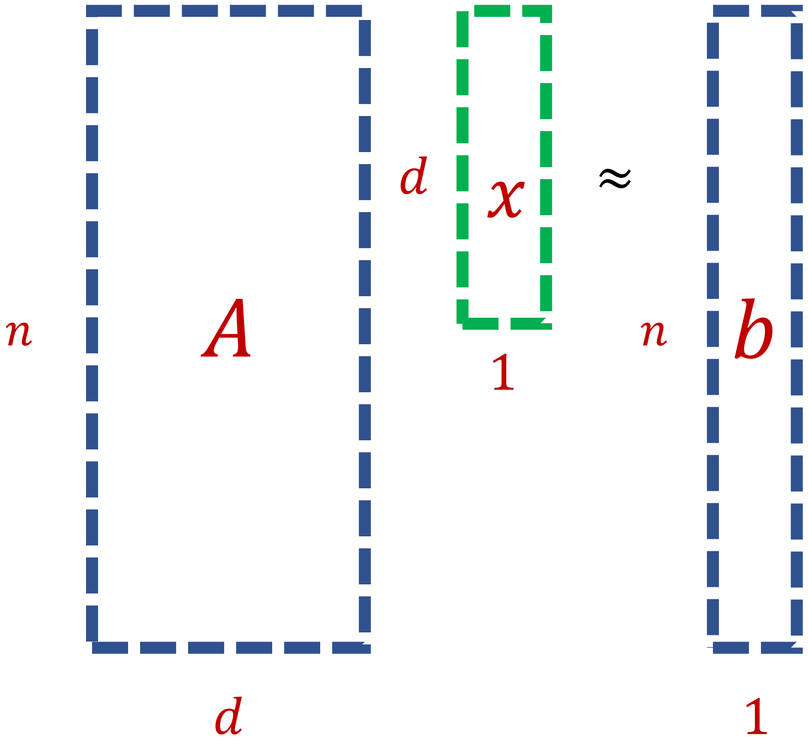


- Spectral approximation: Given $\epsilon > 0$ and $A \in R^{n\times d}$, find matrix $M \in R^{m\times d}$ with $m \ll n$, such that for *every* $x \in R^d$,

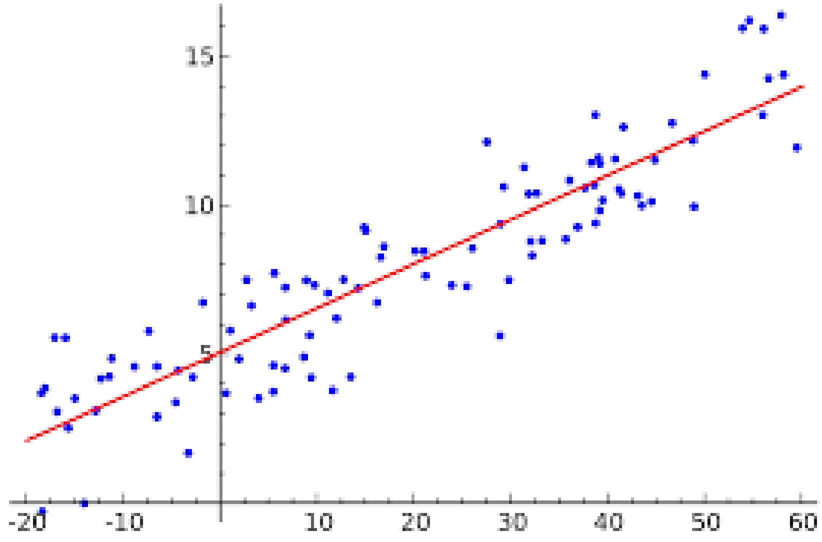
$$(1-\epsilon)\|Ax\|_2 \leq \|Mx\|_2 \leq (1+\epsilon)\|Ax\|_2$$

- Equivalent to $(1-\epsilon)A^\top A \preccurlyeq M^\top M \preccurlyeq (1+\epsilon)A^\top A$

Linear Regression



- Find the vector x that minimizes $\|Ax - b\|_2$
- "Least squares" optimization



Schatten Norms

- Spectral approximation: Given $\epsilon > 0$ and $A \in R^{n\times d}$, find matrix $M \in R^{m\times d}$ with $m \ll n$, such that for *every* $x \in R^d$,

$$(1-\epsilon)\|Ax\|_2 \le \|Mx\|_2 \le (1+\epsilon)\|Ax\|_2$$

- Equivalent to $(1-\epsilon)A^\top A \preccurlyeq M^\top M \preccurlyeq (1+\epsilon)A^\top A$
- Singular value: square root of eigenvalue of $A^T A$
  - $\sigma_1(A) \ge \sigma_2(A) \ge \cdots \ge \sigma_n(A)$
- Schatten p norm: $\|A\|_p = \left(\sigma_1^p + \sigma_2^p + \cdots \sigma_n^p\right)^{\frac{1}{p}}$

Linear Algebra Background

- A symmetric matrix $M \in R^{d \times d}$ is positive semi-definite (PSD) if $x^\top M x \geq 0$ for all column vectors $x \in R^d$
- All eigenvalues of PSD matrix M are non-negative

- If $A - B$ is PSD, we write $B \preccurlyeq A$

- For any row vector $v \in R^d$, $v^\top v$ is a PSD matrix
- Sum of two PSD matrices is a PSD matrix

Initial Approach (Smooth PSD Histogram)

- If $(1-\epsilon)B^\top B \preccurlyeq A^\top A \preccurlyeq (1+\epsilon)B^\top B$, then for any matrix C,

$$(1-\epsilon)(B^\top B + C^\top C) \preccurlyeq A^\top A + C^\top C \preccurlyeq (1+\epsilon)(B^\top B + C^\top C)$$

- The singular values of the matrices behave “smoothly”
- Maintain histogram based on the singular values

- Each substream represents a matrix A
- Keep $A^\top A$ and merge whenever there are three matrices within $(1+\epsilon)$ in Loewner ordering

Initial Approach (Smooth PSD Histogram)

- Space? Each instance stores a matrix $A^\top A$
- A has at most n rows but $A^\top A \in R^{d\times d}$
- How many instances? d singular values, each of them polynomially bounded
- $O\left(\frac{1}{\epsilon} d \log n\right)$ instances

- Total space: $O\left(\frac{1}{\epsilon} d^3 \log n\right)$ (in words)

Summary

- **Algorithm**: Mimic smooth histogram by keeping checkpoints whenever singular values "jump" and storing $A^\top A$
- **Correctness**: smoothness of Loewner ordering
- **Space**: $O\left(\frac{1}{\epsilon} d^3 \log d\right)$

Questions?

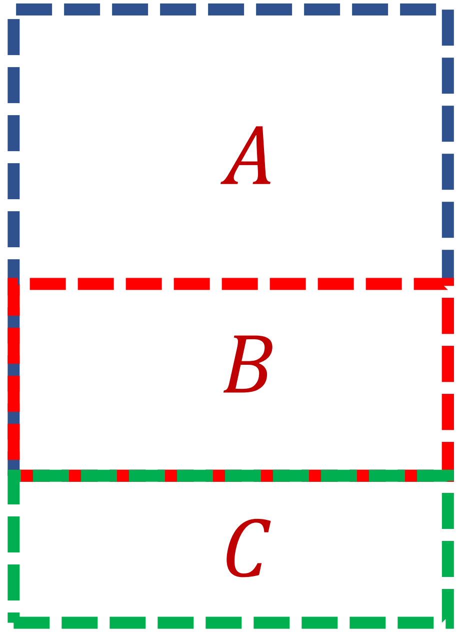
Initial Approach (Smooth PSD Histogram)

- Deterministic algorithm: $O\left(\frac{1}{\epsilon}d^3\log n\right)$ space (in words)

- Outputs spectral approximation of $A^\top A$ rather than A (does not generalize to other norms)
- Update time can be $O\left(\frac{1}{\epsilon}d^4\log n\right)$
- Can be done in $\tilde{O}\left(\frac{1}{\epsilon^2}d^2\right)$ space in streaming

Intuition

- To decrease the space, we first observe there is a lot of similar structure between instances A, B, C: most rows are shared!
- Try subsampling approach?
- Uniform sampling of rows is generally poor
- Importance sampling: Matrix multiplication uses squared row norm, but it doesn't work here…



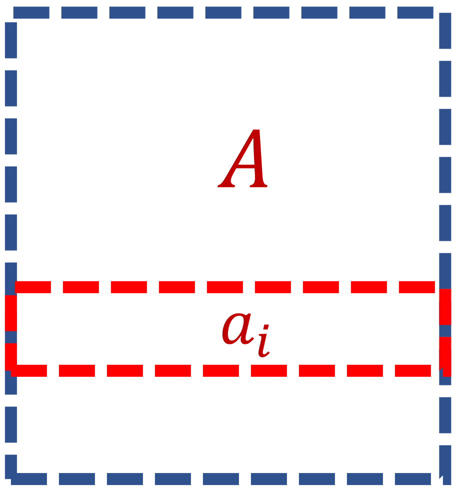
Leverage Scores

- Intuition: how unique a row is (importance sampling)
- $\ell_i = \max \frac{\langle a_i, x\rangle^2}{\|Ax\|_2^2}$ are the *leverage scores* of A (in this case of row a_i)

$$\begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix}$$

- Take $x = (1 \quad -1)$ to see that $\ell_1 = 1$
- Take $x = (0 \quad 1)$ to see that $\ell_2 = 1$

- $\ell_i = a_i(A^\top A)^{-1} a_i^\top, \ \sum \ell_i = d$



Challenges

- ❖ r_3 does not look important, so we do not sample r_3

$$
\begin{matrix}
1 & 3 & 5 & -2 & 7 & 0 & 11 & 4 & -8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 100 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{matrix}
$$

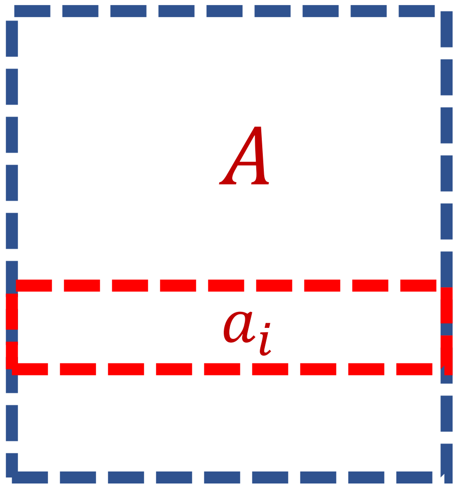
Challenges

- r_3 does not look important, so we do not sample r_3
- Stream proceeds: All rows before r_3 expire, new rows are all zeros
- Cannot possibly get approximation of A without storing r_3
- This implies we should *always* store the most recent row!
- Need a new sense of importance accounting for both uniqueness AND recency of a row

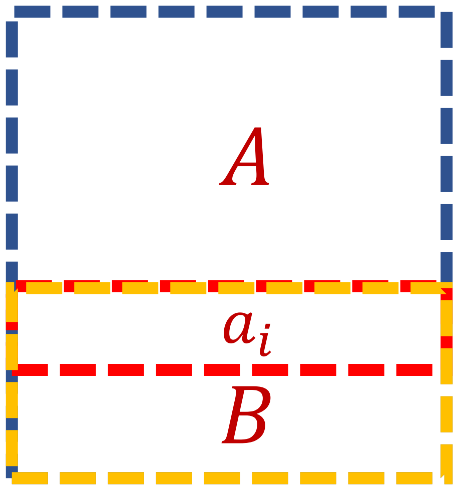
$$
\begin{array}{ccccccccc}
1 & 3 & 5 & -2 & 7 & 0 & 11 & 4 & -8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 100 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

Reverse Online Leverage Scores

- Leverage score of row a_i is $\ell_i = a_i (A^\top A)^{-1} a_i^\top$
- Rows before a_i might be deleted so they shouldn't count towards the importance of a_i

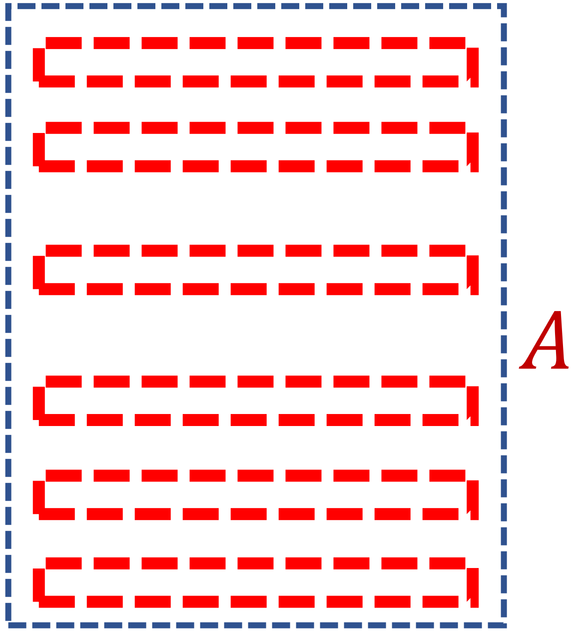


- Reverse online leverage score of row a_i is $\tau_i = a_i (B^\top B)^{-1} a_i^\top$ where B are the rows after a_i



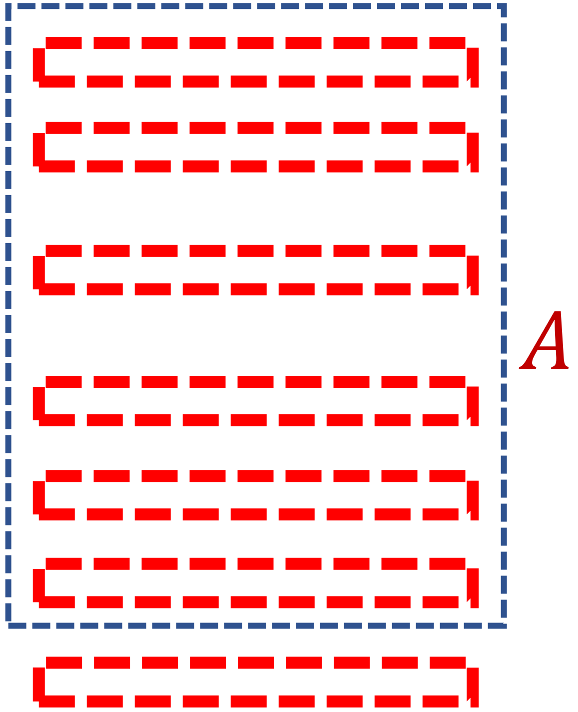
Algorithm

- Algorithm: sample (and rescale) a number of rows



Algorithm

- Algorithm: sample (and rescale) a number of rows
- New row arrives – store it



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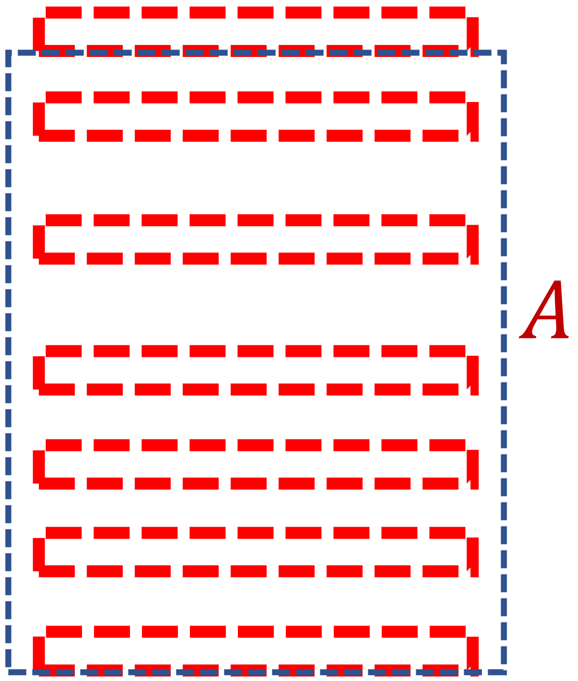

Algorithm

- Algorithm: sample (and rescale) a number of rows
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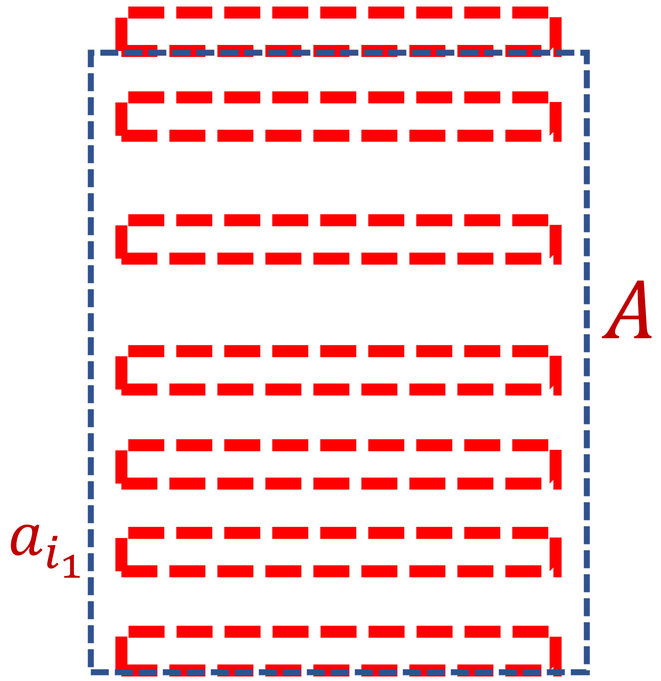
- For each sampled (and rescaled) row a_i, sample the row with probability $\propto \tau_i \leftrightarrow$ downsampling



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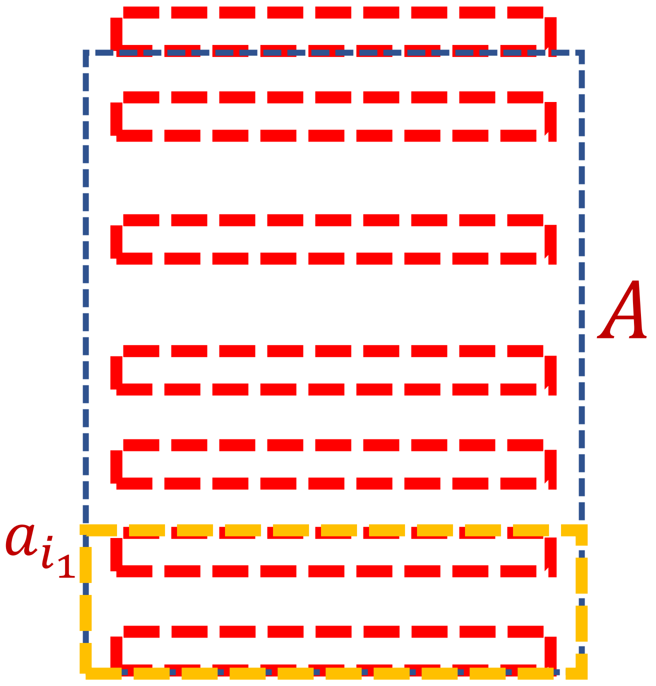

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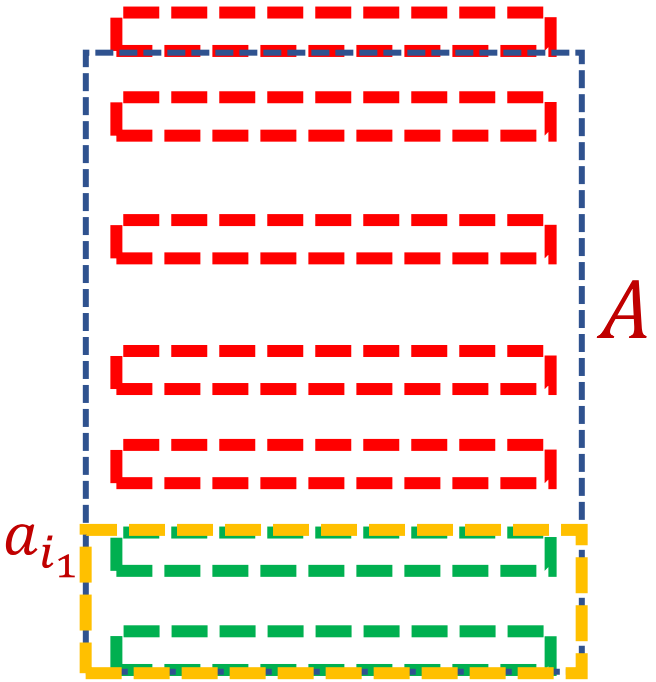

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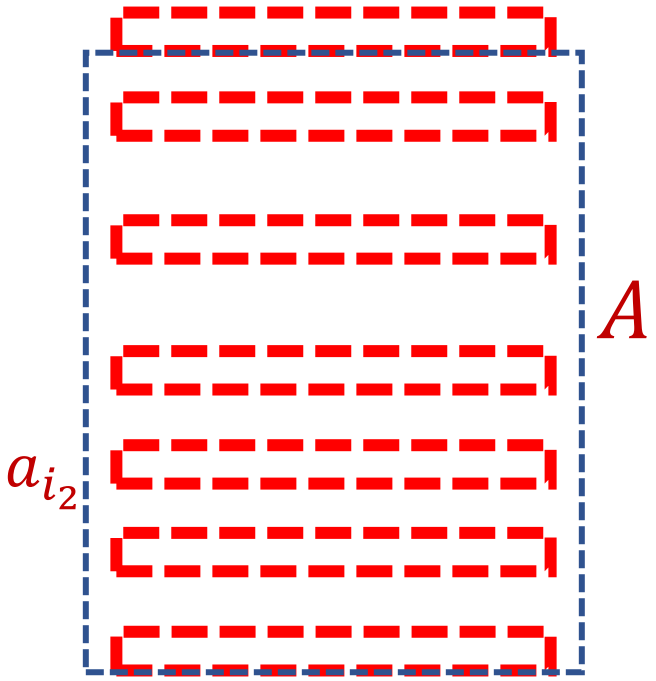

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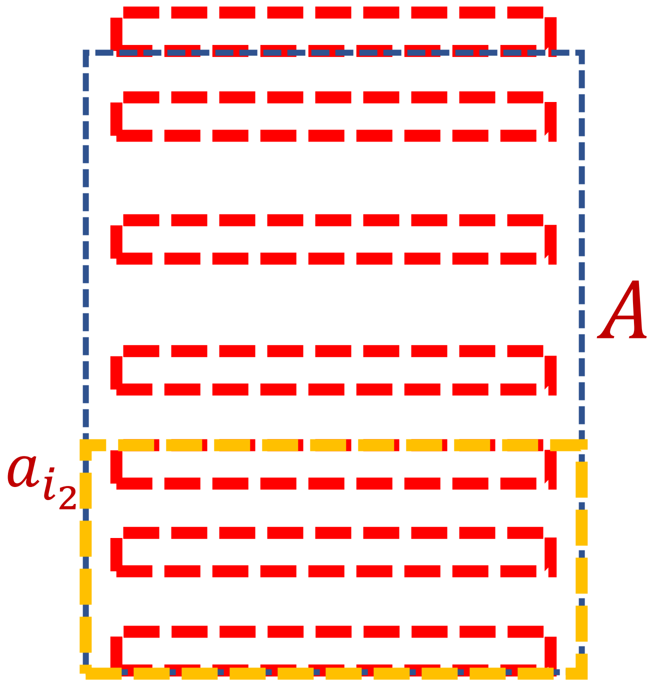

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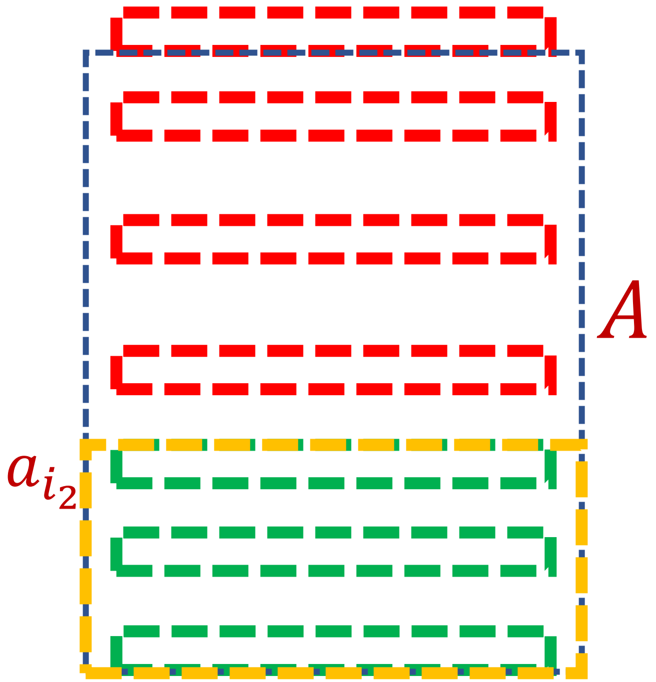

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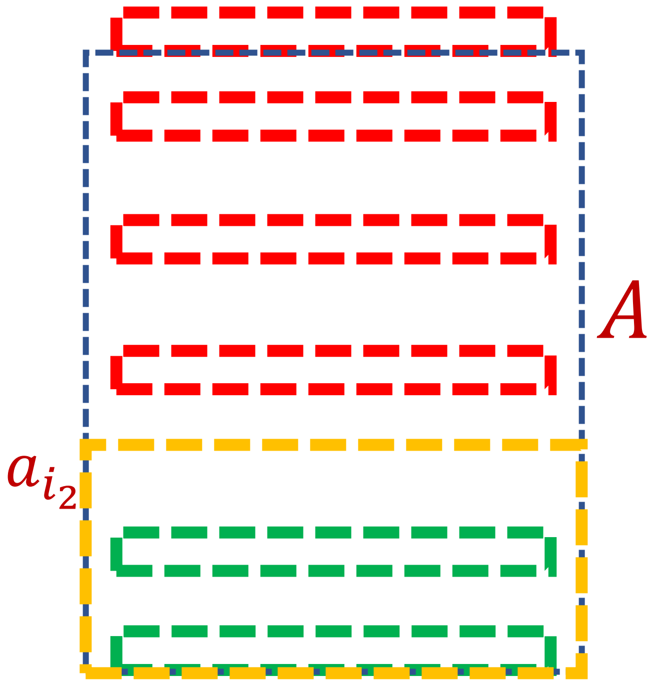

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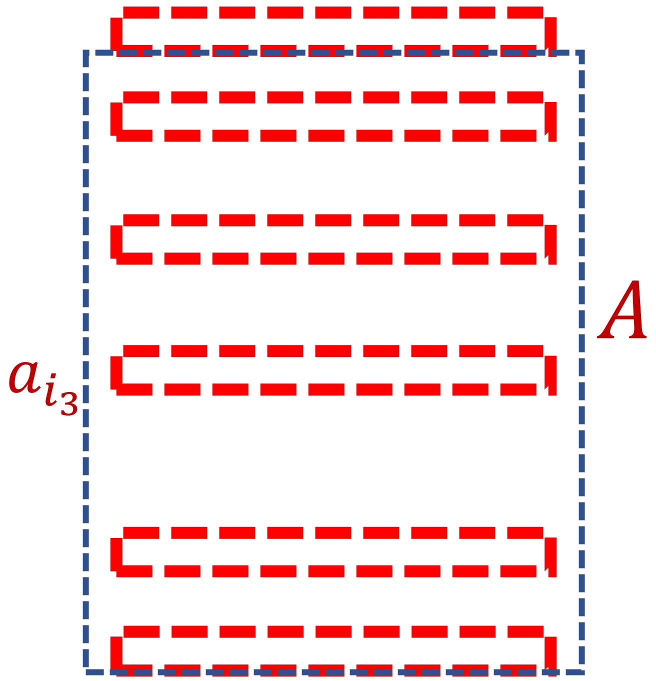

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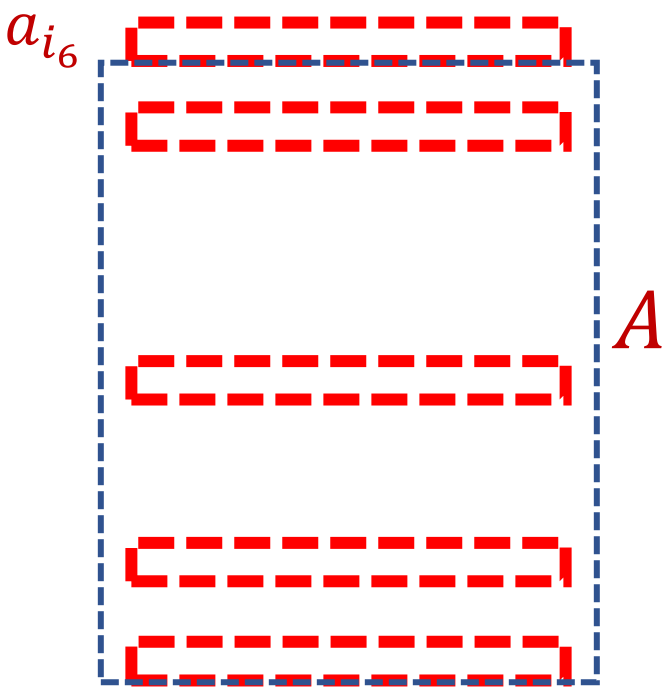
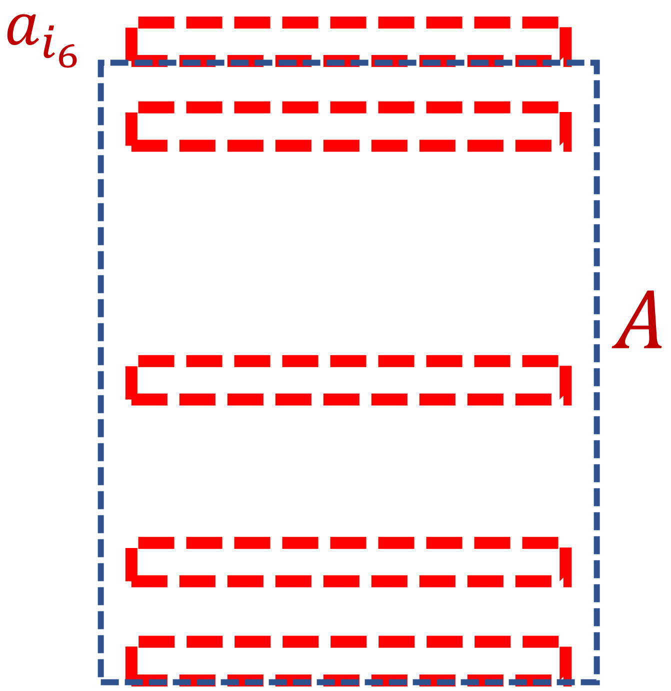
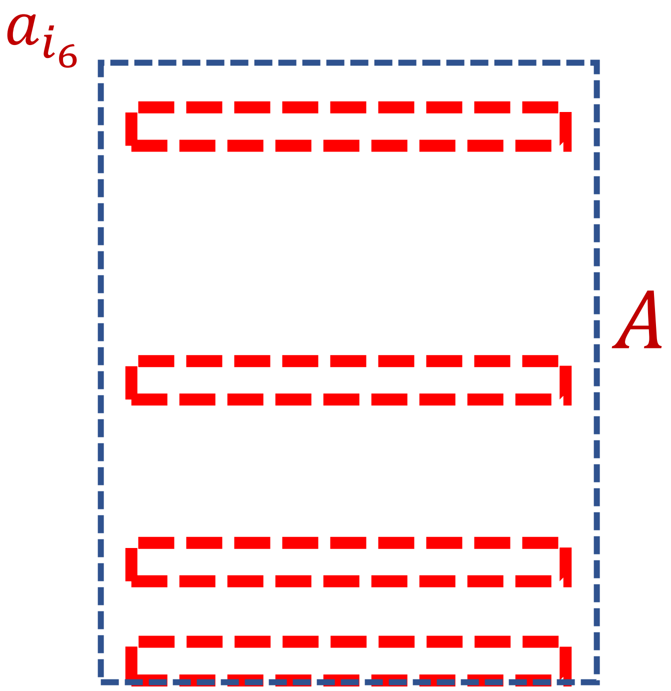


Algorithm

- Algorithm: sample (and rescale) a number of rows
- New row arrives – store it
- For each sampled (and rescaled) row a_i, sample the row with probability $\propto \tau_i \leftrightarrow$ downsampling
- Delete expired rows



Algorithm



- Algorithm: sample (and rescale) a number of rows
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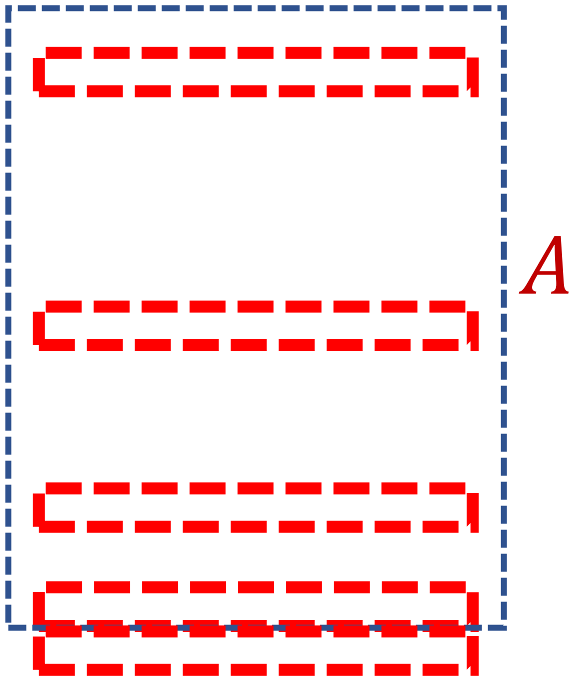
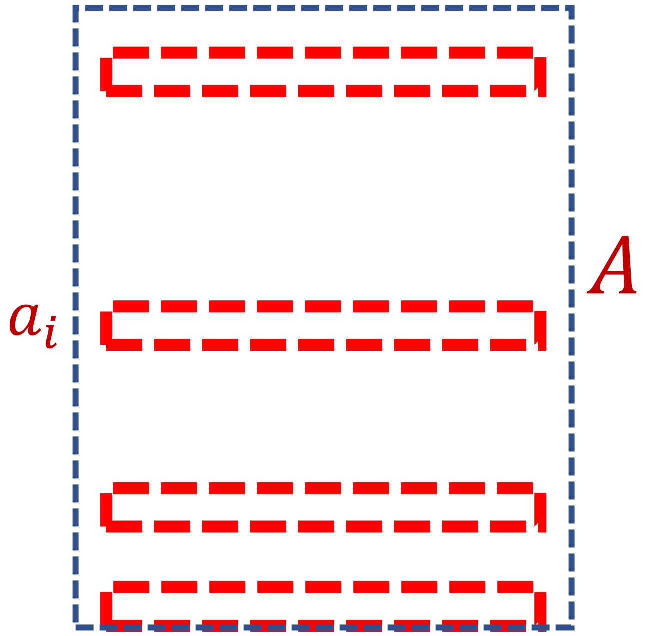
Algorithm

- Algorithm: sample (and rescale) a number of rows
- New row arrives – store it
- For each sampled (and rescaled) row a_i, sample the row with probability $\propto \tau_i \leftrightarrow$ downsampling
- Delete expired rows
- New row arrives – repeat



Correctness

- Correctness: Show an invariant that each sampled matrix is good approximation to row a_i is sampled with probability $\propto$ *final* reverse online leverage score
- Shown by martingale argument and monotonicity

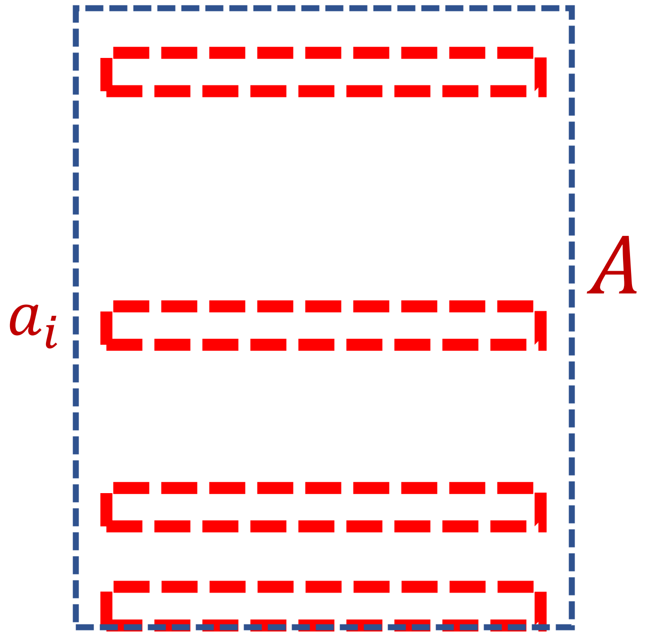


Correctness

- Want to show: $(1-\epsilon)\|Ax\|_2 \le \|Mx\|_2 \le (1+\epsilon)\|Ax\|_2$ for all x
- Approach 1: Fix $x \in R^d$ and observe that row a_i contributes $\frac{1}{p_i}\langle a_i, x\rangle^2$ to $\|Mx\|_2^2$ with probability p_i and 0 otherwise and consider $\left|\|Mx\|_2^2 - \|Ax\|_2^2\right|$ (Bernstein-type martingale)
- Show $(1-\epsilon)\|Ax\|_2 \le \|Mx\|_2 \le (1+\epsilon)\|Ax\|_2$ for all x in an ϵ-net of a unit ball
- Approach 2: Observe row a_i contributes $a_i^\top a_i$ to $M^\top M$ with probability p_i and 0 otherwise and consider $\|M^\top M - A^\top A\|_2$ (matrix Freedman martingale)

Spectral Approximation (Summary)

- Correctness: Show an invariant that each sampled matrix is good approximation to row a_i is sampled with probability $\propto$ *final* reverse online leverage score
- Shown by martingale argument and monotonicity
- Outputs a matrix $M \in R^{m \times d}$ such that $(1-\epsilon)\|Ax\|_2 \leq \|Mx\|_2 \leq (1+\epsilon)\|Ax\|_2$ for all $x \in R^n$ [DMM06, SS08, CMP16]

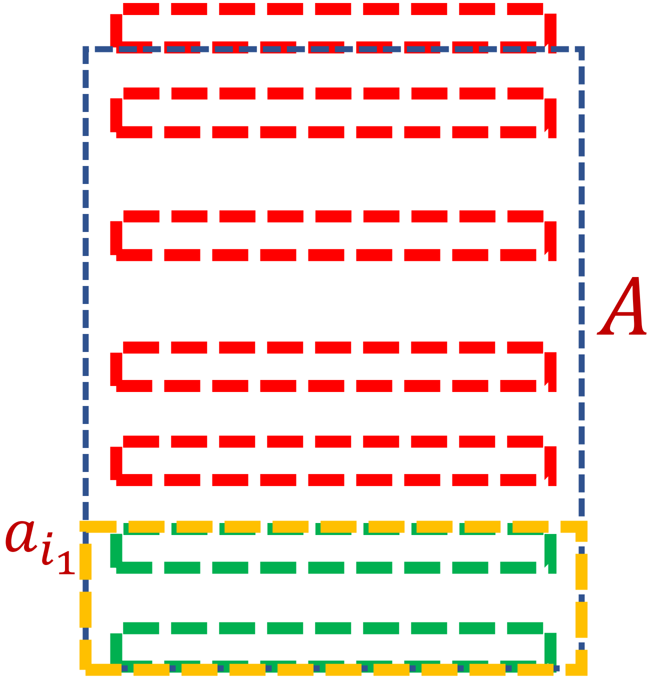


- Space? Must bound $\sum \tau_i = \tilde{O}\left(\frac{1}{\epsilon^2} d\right)$ [CMP16]

Summary

- **Algorithm**: Start from most recent row and downsample through (reverse) online leverage score
- **Correctness**: expectation is correct, bound the variance and use matrix martingale
- **Space**: $\widetilde{O}\left(\frac{1}{\epsilon^2} d^2\right)$

Questions?



Low-Rank Approximation

$$
\begin{array}{cc}
n & \boxed{\begin{matrix}
1 & 3 & 5 & -2 & 7 & 0 & 11 & 4 & -8 \\
0 & 0 & -1 & 3 & 13 & 2 & 8 & 6 & 2 \\
2 & 5 & 6 & 1 & 4 & 0 & -7 & 5 & 3 \\
8 & 7 & 2 & 1 & -1 & -3 & -2 & -4 & -6 \\
-5 & 3 & -4 & -1 & -2 & -1 & 0 & -3 & -1 \\
7 & 1 & 3 & 2 & 4 & 1 & 0 & 11 & 1
\end{matrix}} \\
 & d
\end{array}
$$

- Find rank k matrix A_k that minimizes $\|A_k - A\|_F$
- Finding structure among noise
- Matrix completion problem

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Low-Rank Approximation

- Low-rank approximation: Given $\epsilon > 0$ and $A \in R^{n \times d}$, find matrix $M \in R^{m \times d}$ with $m \ll n$ such that

$$(1-\epsilon)\|A - A_k\|_F \leq \|M - M_k\|_F \leq (1+\epsilon)\|A - A_k\|_F$$

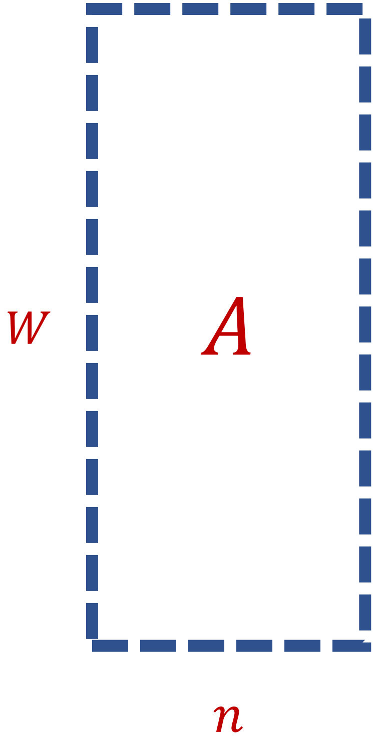
- Spectral approximation algorithm solves low-rank approximation: $\tilde{O}\left(\frac{1}{\epsilon^2} d^2\right)$ space

- Can be done in $\tilde{O}\left(\frac{1}{\epsilon^2} kd\right)$ space in streaming

Low-Rank Approximation

- Low-rank approximation: Given $\epsilon > 0$ and $A \in R^{n \times d}$, find matrix $M \in R^{m \times d}$ with $m \ll n$ such that

$$(1-\epsilon)\|A - A_k\|_F^2 \leq \|M - M_k\|_F^2 \leq (1+\epsilon)\|A - A_k\|_F^2$$

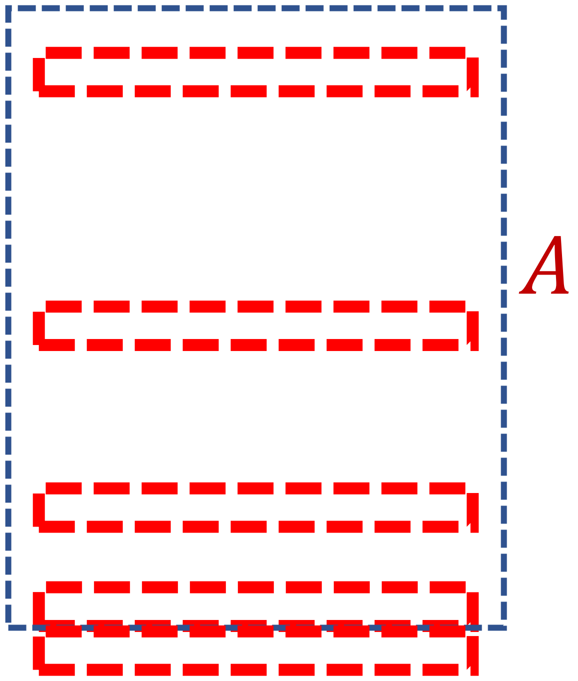


- Leverage score: $a_i (A^\top A)^{-1} a_i^\top$

- Ridge leverage score: $\ell_i = a_i (A^\top A + \lambda I_n)^{-1} a_i^\top$, where $\lambda = \frac{\|A - A_k\|_F^2}{k}$

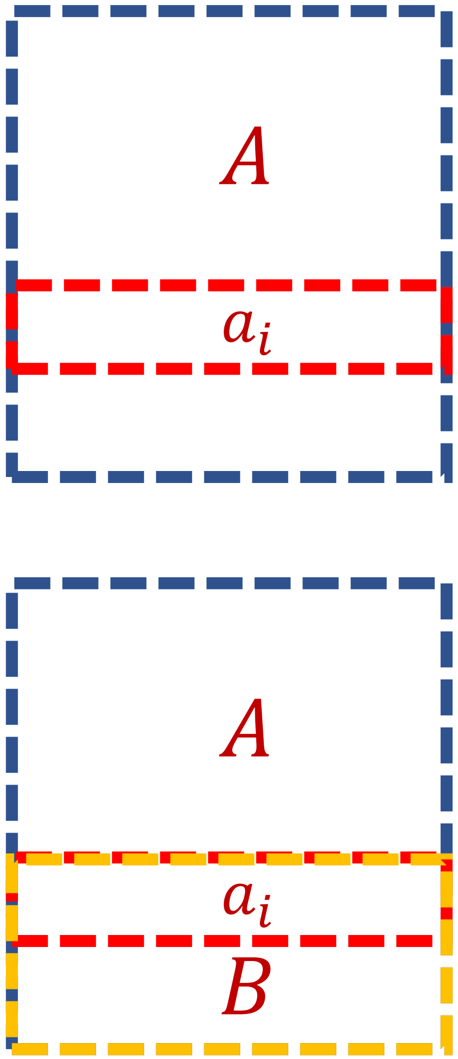
Algorithmic Template (Sliding Windows)

- Algorithm: at each time, repeatedly downsample each stored row a_i based on a probability distribution $\propto \tau_i$ "reverse online score" that accounts for both importance AND recency of a row with respect to the desired function
- Correctness: Show an invariant that each sampled matrix is a "good approximation" and row a_i is sampled with probability $\propto$ *final* reverse online score
- Space? Must bound $\sum \tau_i$



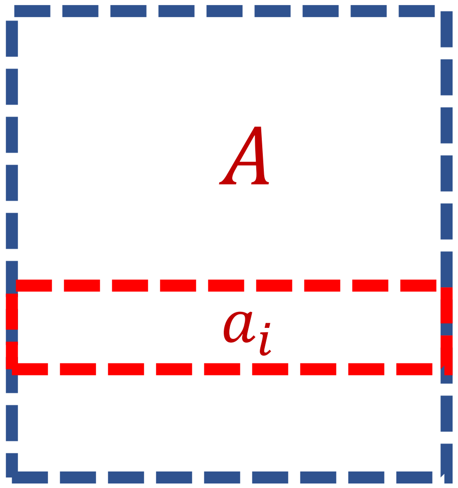
Template

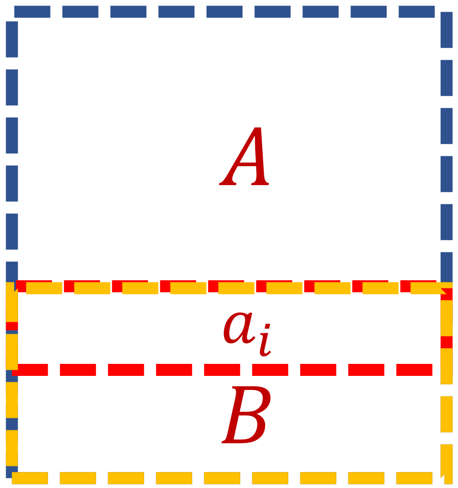
- Suppose we know $\lambda = \frac{\|A - A_k\|_F^2}{k}$
- Reverse online leverage score: Sample each row a_i with probability $p_i \propto \tau_i = a_i(B^\top B + \lambda I_n)^{-1} a_i^\top$
- Monotonicity of ridge leverage score
- Outputs a matrix $M \in R^{m \times d}$ such that $(1 - \epsilon)\|A - A_k\|_F \leq \|M - M_k\|_F \leq (1 + \epsilon)\|A - A_k\|_F$ [CMM15]



Template

- Issue #1: Compute $\lambda = \frac{\|A - A_k\|_F^2}{k}$
- Issue #2: Bound $\sum \tau_i$





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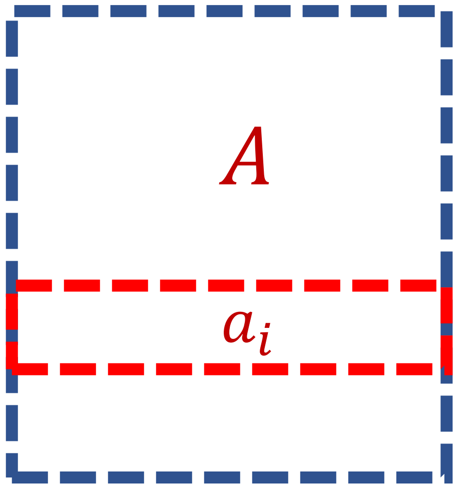
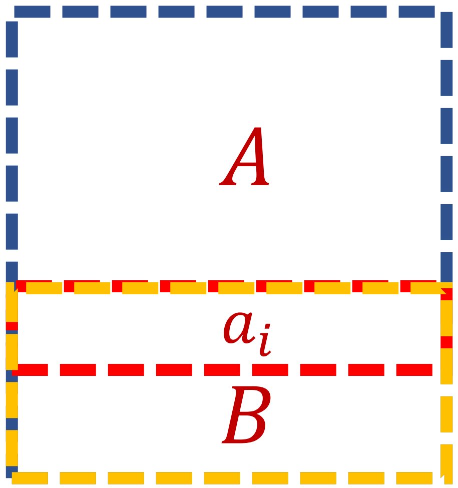


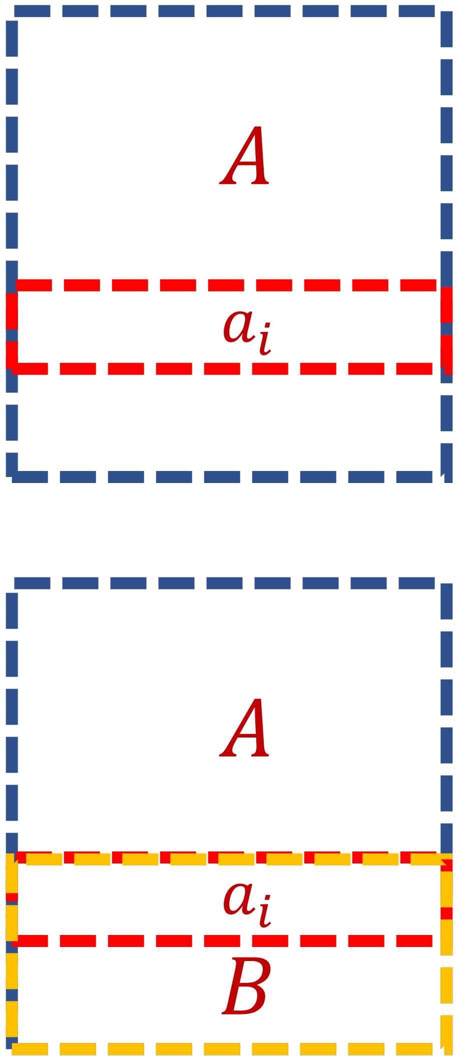
- Observation: it suffices to have a constant factor approximation of $\lambda = \frac{\|A - A_k\|_F^2}{k}$
- Use projection-cost preserving sketch [CEMMP15] to reduce the dimension of each row
- Feed reduced rows into spectral approximation algorithm



Template

✓ Issue #1: Compute $\lambda = \frac{\|A - A_k\|_F^2}{k}$

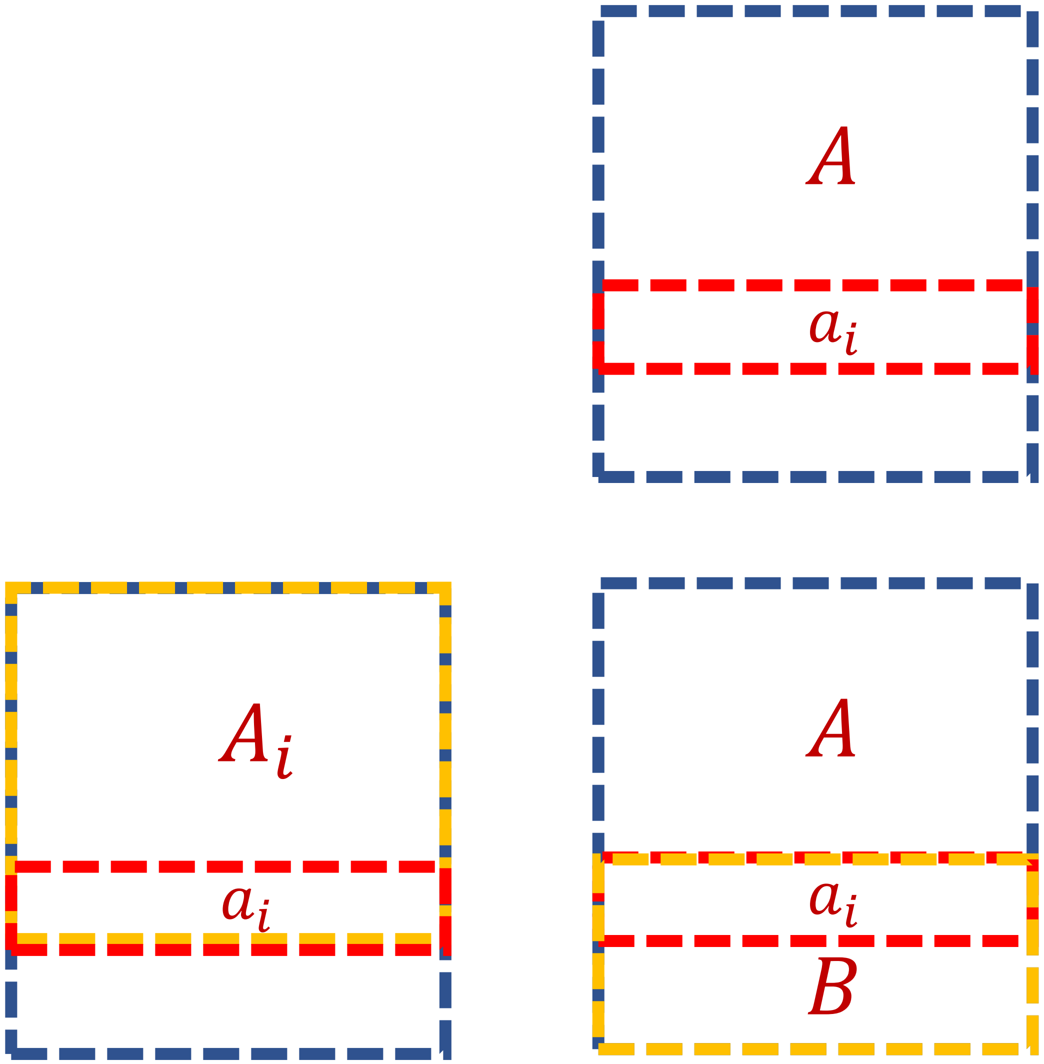
❖ Issue #2: Bound $\sum \tau_i$



Template

- ✓ Issue #1: Compute $\lambda = \frac{\|A - A_k\|_F^2}{k}$
- ❖ Issue #2: Bound $\sum \tau_i$

- ❖ A_i be the first i rows of A
- ❖ $\tau_i = a_i \left(A_i^\top A_i + \lambda I_n \right)^{-1} a_i^\top$
- ❖ Bound $\sum \tau_i$



Number of Total Rows

- $\det(A^\top A + \lambda I_d) \geq \lambda^d \, e^{\sum \tau_i / 2}$ [CMP16]
- $\frac{\sum \tau_i}{2} + d \log \lambda \leq \log \det(A^\top A + \lambda I_d)$

- $\log \det(A^\top A + \lambda I_n) = O(k \log n) + d \log \lambda$

- $\sum \tau_i = O(k \log n)$

Using Matrix Determinant Lemma [CMP16]

- $\det(A + v^\top v) = \det(A)(1 + vA^{-1}v^\top)$
- $\det(A^\top A + \lambda I_d + v^\top v) = \det(A^\top A + \lambda I_d)\,(1 + v(A^\top A + \lambda I_d)^{-1}v^\top)$
- $\tau_i = a_i\left(A_i^\top A_i + \lambda I_d\right)^{-1} a_i^\top$

$$\begin{aligned}\det(A^\top A + \lambda I_d) &= \det(A_{n-1}^\top A_{n-1} + \lambda I_d)\,(1 + a_n(A_{n-1}^\top A_{n-1} + \lambda I_d)^{-1}a_n^\top) \\ &= \det(A_{n-1}^\top A_{n-1} + \lambda I_d)\,(1 + \tau_{\mathrm{n}}) \\ &\geq \det(A_{n-1}^\top A_{n-1} + \lambda I_d)\left(1 + e^{\tau_n/2}\right)\end{aligned}$$

$$\det(A^\top A + \lambda I_d) \geq \lambda^d\, e^{\sum \tau_i/2}$$

Number of Total Rows

- $\det(A^\top A + \lambda I_d) \geq \lambda^d \, e^{\sum \tau_i/2}$ [CMP16]
- $\frac{\sum \tau_i}{2} + d \log \lambda \leq \log \det(A^\top A + \lambda I_d)$
- $\boxed{\log \det(A^\top A + \lambda I_d) = O(k \log n) + d \log \lambda}$
- $\sum \tau_i = O(k \log n)$

Number of Total Rows

- $\det(A^\top A + \lambda I_d) = \prod \sigma_i (A^\top A + \lambda I_d)$
- Small singular values: $\sigma_{k+1} + \ \ldots + \sigma_d = \|A - A_k\|_F^2 + \lambda(d-k)$
- By AM-GM,

$$\prod_{i=k+1}^{i=d} \sigma_i \le \left(\frac{\|A - A_k\|_F^2 + \lambda(d-k)}{d-k} \right)^{d-k}$$

- Large singular values: $\sigma_i \le \|A\|_2^2 + \lambda$ for $1 \le i \le k$

Number of Total Rows

$$\prod_{i=1}^{i=d} \sigma_i \leq \left(\frac{\|A - A_k\|_F^2 + \lambda(d-k)}{d-k} \right)^{d-k} \left(\|A\|_2^2 + \lambda \right)^k$$

small singular values

large singular values

For $\lambda = \frac{\|A - A_k\|_F^2}{k}$, $\det(A^\top A + \lambda I_d) \leq \lambda^{d-k} e^k \left(\|A\|_2^2 + \lambda \right)^k$

$$\log \det(A^\top A + \lambda I_d) = O(k \log n) + d \log \lambda$$

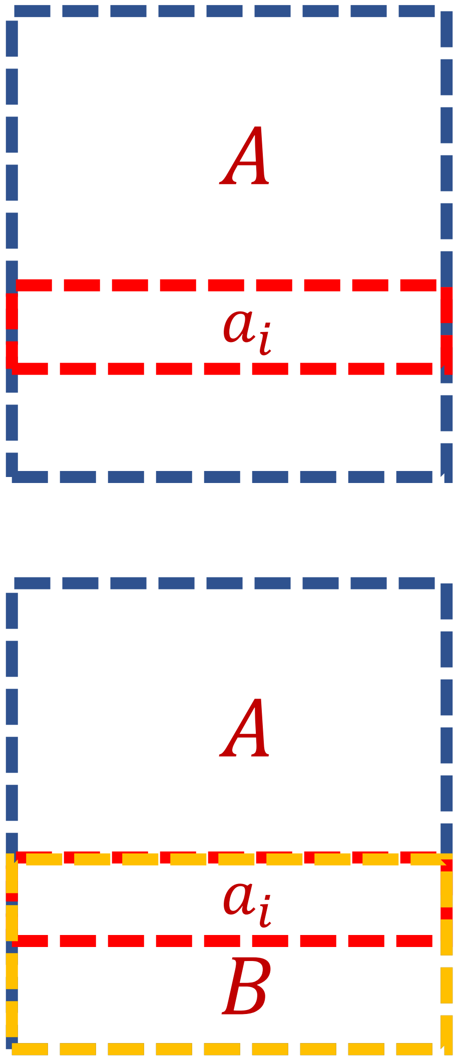
Number of Total Rows

- $\det(A^\top A + \lambda I_d) \geq \lambda^d \, e^{\sum \tau_i / 2}$ [CMP16]
- $\frac{\sum \tau_i}{2} + d \log \lambda \leq \log \det(A^\top A + \lambda I_d)$

- $\boxed{\log \det(A^\top A + \lambda I_d) = O(k \log n) + d \log \lambda}$

Template

- ✓ Issue #1: Compute $\lambda = \frac{\|A-A_k\|_F^2}{k}$
- ✓ Issue #2: Bound $\sum \tau_i$



Low-Rank Approximation (Summary)

- Correctness from a similar argument
- $\sum \tau_i = O(k \log n)$
- Both sampling algorithms can run in *input-sparsity time*

- Sum of reverse online scores = sum of online scores
- Gives many space efficient *online* algorithms essentially for free!
- Improvements on online principal component analysis, column subset selection in [BLVZ19]

Structural Results for Reverse Online Scores

- Reverse online leverage scores (for spectral approximation): $\sum \tau_i = O(d \log \kappa)$ [CMP16]
- Reverse online ridge leverage scores (for low-rank approximation): $\sum \tau_i = O(k \log \kappa)$
- Reverse online ℓ_1 sensitivities, online Lewis weights (for ℓ_1 subspace embedding): $\sum \tau_i = O(d \log n \log \kappa)$

- Sum of reverse online scores = Sum of online scores!
- Our structural results essentially give our online results for free!

Results: Sliding Window Model

- $O(d^2)$ space randomized sliding window algorithm for spectral sparsification (space optimal, up to lower order terms)
- $O(dk)$ space randomized sliding window algorithm for low-rank approximation (space optimal, up to lower order terms)
- $O(d^3)$ space randomized sliding window algorithm for ℓ_1 subspace embedding

Results: Online Model

- Online algorithm for low-rank approximation that samples $O(k)$ rows (space optimal, up to lower order terms)
- Online algorithm for row subset selection that samples $O(k)$ rows (space optimal, up to lower order terms)
- Online algorithm for principal component analysis that embeds into a matrix with dimension $O(k)$ (space optimal, up to lower order terms)
- Online algorithm for ℓ_1 subspace embedding that samples $O(d^2)$ rows

Results: Connections

- Online coreset: algorithm whose output can be used to approximate all prefixes of a stream

- Online coresets give deterministic sliding window algorithms!
- $O(d^2)$ space deterministic sliding window algorithm for spectral sparsification (space optimal, up to lower order terms)
- $O(dk)$ space deterministic sliding window algorithm for low-rank approximation (space optimal, up to lower order terms)
- $O(d^2)$ space deterministic sliding window algorithm for ℓ_1 subspace embedding (space optimal, up to lower order terms)

Future Work?

- ❖ Space complexity for ℓ_p spectral approximation?
- ❖ Time decay models instead of sliding window
  - ❖ Polynomial decay, exponential decay,…
- ❖ Faster deterministic algorithms?
- ❖ Other update models for sliding windows (ex. entrywise)?
- ❖ Better bounds with bit complexity assumptions?

Summary

- $O(n^3)$ space deterministic sliding window algorithm for spectral sparsification
- $O(n^2)$ space randomized sliding window algorithm for spectral sparsification
- $O(nk)$ space randomized sliding window algorithm for low rank approximation
- Results for ℓ_1 subspace embedding (with bit complexity assumptions), matrix multiplication on sliding windows
- Results for online principal component analysis, low rank approximation, column subset selection

Reverse Online Leverage Scores

- $\det(A^\top A + \lambda I_n) \geq \lambda^n \, e^{\sum \tau_i/2}$
- $\log \det(A^\top A + \lambda I_n) = O(k \log n)$

- $\sum \tau_i = O(k \log n)$

- Also gives a space efficient *online* algorithm for low-rank approximation!
- Can use slightly different estimator for $\lambda = \frac{\|A - A_k\|_F^2}{k}$ [AN13]

Reverse Online Leverage Scores

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- Also gives a space efficient *online* algorithm for low-rank approximation!
- Can use slightly different estimator for $\lambda = \frac{\|A - A_k\|_F^2}{k}$ [AN13]

ℓ_1 Spectral Approximation

- Given $\epsilon > 0$ and $A \in R^{W \times n}$, find matrix $M \in R^{m \times n}$ with $m \ll W$, such that for *every* $x \in R^n$

$$(1-\epsilon)\|Ax\|_1 \leq \|Mx\|_1 \leq (1+\epsilon)\|Ax\|_1$$

- Robust to outliers, but unstable solution and possibly multiple solutions

ℓ_1 Leverage Scores

- Previous ℓ_2 leverage scores: $\ell_i = \max \frac{\langle a_i, x\rangle^2}{\|Ax\|_2^2}$
- ℓ_1 leverage scores: $\ell_i = \max \frac{|\langle a_i, x\rangle|}{\|Ax\|}$
- Sample each row a_i with probability $p_i \propto \ell_i$ gives ℓ_1 spectral approximation [DDHKM07]
- Bound the sum of the reverse online ℓ_1 leverage scores

ℓ_1 Leverage Scores

- Make nice assumptions: the entries of A and x are bounded integers
- Can show that if $||Ax||_1$ increases by $(1+\epsilon)$, $||Ax||_2^2$ must increase by $\left(1+\frac{\epsilon}{\mathrm{poly}(n)}\right)$
- Can use deterministic algorithm to find these breakpoints

- Use separate instances of streaming ℓ_1 spectral approximation algorithm starting at each of these breakpoints [DDHKM07, CP15]

Matrix Multiplication Lower Bounds

- Distributional INDEX: $\{0,1\}^n \times [n]$
- Alice has string $S \in \{0,1\}^n$ chosen uniformly at random
- Bob has index $i \in [n]$ chosen uniformly at random and must output $S[i]$ with probability $\frac{2}{3}$
- Requires $\Omega(n)$ bits of communication from Alice to Bob [MNSW98]

Matrix Multiplication Lower Bounds

- Alice has $S \in \{0,1\}^{n/c^2\epsilon^2}$
- Creates matrix $M \in \{-c, c\}^{\frac{1}{c^2\epsilon^2} \times n}$ in the natural way (stuffs string into matrix by sign)
- Creates matrix $A = [M \;\; E]$, where E is $c^2\epsilon^2 n$ instances of e_1, …, $c^2\epsilon^2 n$ instances of e_{c^2/ϵ^2}
- If $\|A^\top A - B^\top B\|_F \leq \epsilon \|A^\top A\|_F$, then Bob can recover the signs of most entries in M and hence can recover most of the symbols of S

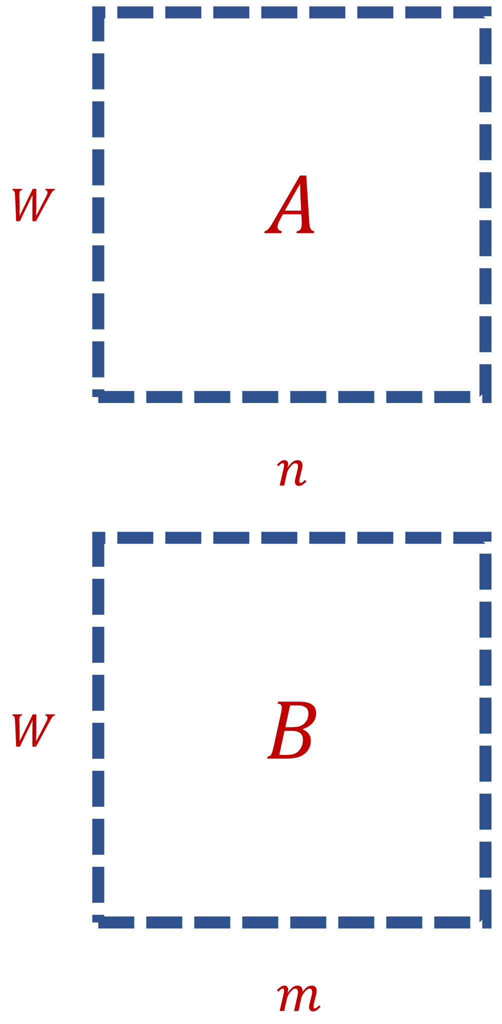
Linear Algebra Background

- Vectors $u, v \in R^n$
- Inner product: $\langle u, v \rangle = \sum u_i v_i \in R$
- Outer product: $u \otimes v = uv^T \in R^{n \times n}$

$$\begin{matrix} u_1 v_1 & u_1 v_2 & \cdots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \cdots & u_2 v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_n v_1 & u_n v_2 & \cdots & u_n v_n \end{matrix}$$

Linear Algebra Background

- Matrices: $A \in R^{W \times n}, B \in R^{W \times m}$
- $(A^\top B)_{i,j} = \langle a_i b_j \rangle$, where a_i is the i^{th} column of A and b_j is the j^{th} column of B.
- $A^T B = \sum_{i=1}^{W} a_i \otimes b_i = \sum_{i=1}^{W} a_i^\top b_i$



Approximate Matrix Multiplication

- Vector norm: $\|x\|_p = \left(x_1^p + x_2^p + \cdots x_n^p\right)^{\frac{1}{p}}$
- Frobenius norm: $\|A\|_F = \sqrt{\sum A_{i,j}^2}$
- Matrices: $A \in R^{W\times n}, B \in R^{W\times m}, W \gg m, n$
- Output $A^T B$

- Can we find $C \in R^{d\times n}, D \in R^{d\times m}, d \ll W$ such that $C^\top D \approx A^\top B$?
- What does $\approx$ mean?

Approximate Matrix Multiplication

- Given $\epsilon > 0$, find $C \in R^{d \times n}, D \in R^{d \times m}$ such that

$$\|A^\top B - C^\top D\|_F \le \epsilon \|A^\top B\|_F$$

- Information Retrieval: $A \in R^{W \times n}$ rows represent documents, columns represent occurrence of each word
- High entries of $AA^\top$ correspond to "similar" documents

Approximate Matrix Multiplication (Offline) [DK01]

- Given $\epsilon > 0$ and $A \in R^{W \times n}, W \gg n$, find $B \in R^{d \times n}$ such that

$$\|A^\top A - B^\top B\|_F \leq \epsilon \|A^\top A\|_F$$

- Intuition: Large entries in $A^\top A$ come from large entries in A
- Importance sampling: Sample row a_k of A proportional to its squared row norm
- Sample row a_k of A with probability $p_k \propto \frac{\|a_k\|_2^2}{\|A\|_F^2}$
- Rescale each sampled row by $\frac{1}{\sqrt{p_k}}$

Approximate Matrix Multiplication (Offline)

- Analyze $\mathrm{E}[\|A^\top A - B^\top B\|_F^2]$

- Step 1: Show that $B^\top B$ is an unbiased estimator:
- $\mathrm{E}[B^\top B] = \sum p_k \left(\frac{1}{\sqrt{p_k}} a_k^\top \frac{1}{\sqrt{p_k}} a_k\right) = A^\top A$
- Step 2: Bound the variance of $(B^\top B)_{i,j}$:
- $\mathrm{Var}\left[(B^\top B)_{i,j}\right] \leq \sum \frac{1}{p_k} \left(a_k^\top a_k\right)_{i,j}^2$

Approximate Matrix Multiplication (Offline)

- $\mathrm{E}[\|A^\top A - B^\top B\|_F^2] = \mathrm{E}\left[\sum_{i,j}(A^\top A - B^\top B)_{i,j}^2\right]$
- $\mathrm{E}[\|A^\top A - B^\top B\|_F^2] = \sum_{i,j} \mathrm{Var}\left[(A^\top A - B^\top B)_{i,j}\right]$
- $\mathrm{E}[\|A^\top A - B^\top B\|_F^2] \leq \sum_{i,j,k} \frac{1}{p_k}\left(a_k^\top a_k\right)_{i,j}^2 = \sum_k \frac{1}{p_k}\|a_k\|_2^4$

- For $p_k = \frac{c\|a_k\|_2^2}{\|A\|_F^2}$, $\mathrm{E}[\|A^\top A - B^\top B\|_F^2] \leq \frac{1}{c}\|A\|_F^4$.
- $\sum p_k = c := \frac{1}{\epsilon^2}$, so total number of sampled rows is $O\left(\frac{1}{\epsilon^2}\log n\right)$ whp

Approximate Matrix Multiplication

- Goal: Sample row a_i of A with probability $p_{\mathrm{i}} \propto \frac{\|a_i\|_2^2}{\|A\|_F^2}$

- See a_i in the sliding window model, can compute $\|a_i\|_2^2$
- Cannot compute $\|A\|_F^2$ without seeing all the rows

- How would we do matrix multiplication in the streaming model?

Approximate Matrix Multiplication

- How would we do matrix multiplication in the streaming model?
- Track $\|A\|_F^2$
- Suppose we have sampled row a_i of A with probability $p_\mathrm{i} \propto \frac{\|a_i\|_2^2}{\|A\|_F^2}$
- New row arrives a_t: $\|A\|_F^2$ increases by $\|a_t\|_2^2$
- What do we do with a_i?
- Downsample: keep a_i with probability $\frac{\|A\|_F^2}{\|A\|_F^2+\|a_t\|_2^2}$
- Sampled a_i with probability $p_i \propto \frac{\|a_i\|_2^2}{\|A\|_F^2+\|a_t\|_2^2}$

Approximate Matrix Multiplication

- Note it suffices to have $\widehat{A}$ a 2-approximation of $\|A\|_F^2$
- Why? Sample row a_i of A with probability $p_{\mathrm{i}} \propto \frac{2\|a_i\|_2^2}{\widehat{A}}$
- Frobenius norm is *smooth*
- Use smooth histogram to maintain $\widehat{A}$

- Smooth histogram for Frobenius norm
- Separate instance of matrix multiplication streaming algorithm for each instance tracking the Frobenius norm

Approximate Matrix Multiplication

- Total space: $O\left(\frac{1}{\epsilon^2}\log n\right)$ rows → $O\left(\frac{n}{\epsilon^2}\log^2 n\right)$ bits of space
- Can decrease to $O\left(\frac{n}{\epsilon^2}\log n\left(\log\log n + \log\frac{1}{\epsilon}\right)\right)$ with bit representation tricks

- Also give $\Omega\left(\frac{n}{\epsilon^2}\log n\right)$ space lower bound

Questions?

Spectral Approximation (Offline)

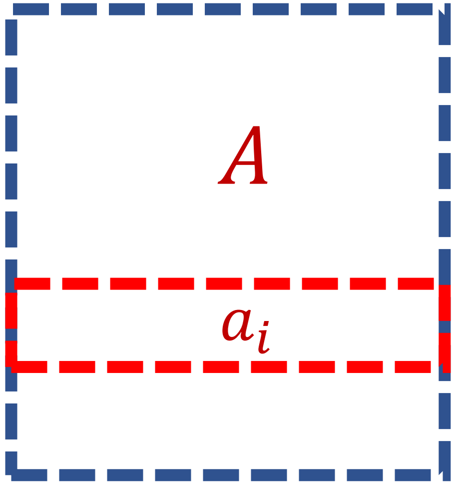
- Spectral approximation: Given $\epsilon > 0$ and $A \in R^{W \times n}$, find matrix $M \in R^{m \times n}$ with $m \ll W$, such that for *every* $x \in R^n$,

$$(1-\epsilon)\|Ax\|_2 \leq \|Mx\|_2 \leq (1+\epsilon)\|Ax\|_2$$

- How would we do this offline? Hint: fundamental tool of dimensionality reduction
- Although Johnson-Lindenstrauss reduces the number of rows and sparse JL can preserve sparsity, we want to focus on sampling

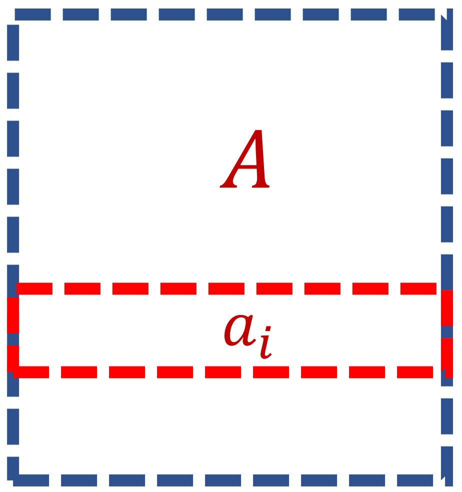
Linear Algebra Background

- Singular Value Decomposition (SVD): $A = U\Sigma V^T \in R^{W\times n}$
  - $U \in R^{W\times W}$ is an orthonormal matrix (rows, columns orthonormal)
  - $\Sigma \in R^{W\times n}$ is a rectangular diagonal matrix with non-negative entries
  - $V \in R^{n\times n}$ is an orthonormal matrix (rows, columns orthonormal)
- $\|u_i\|_2^2$ are the *leverage scores* of A (in this case of row a_i)
- Intuition: how "unique" a row is (recall importance sampling)
- $\ell_i = \max \frac{\langle a_i, x\rangle^2}{\|Ax\|_2^2} = \|u_i\|_2^2 = a_i(A^\top A)^{-1}a_i^\top$
- $\sum \ell_i = n$



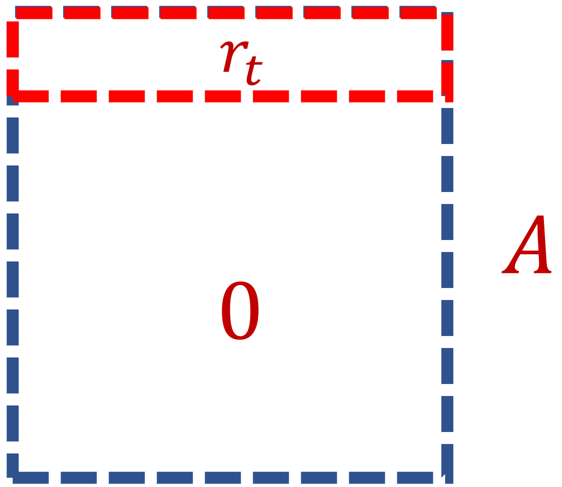
Spectral Approximation (Offline)

- Sample each row a_i with probability $p_i \propto \ell_i = a_i(A^\top A)^{-1} a_i^\top$
- Outputs a matrix $M \in R^{m \times n}$ such that $(1-\epsilon)\|Ax\|_2 \leq \|Mx\|_2 \leq (1+\epsilon)\|Ax\|_2$ for all $x \in R^n$ [DMM06, SS08]
- $\sum \ell_i = n$, so the total number of rows sampled is $\propto \tilde{O}(n)$
- Leverage scores are monotonic with more rows, so the offline approach can be adapted to streaming through downsampling



Spectral Approximation (Sliding Window)

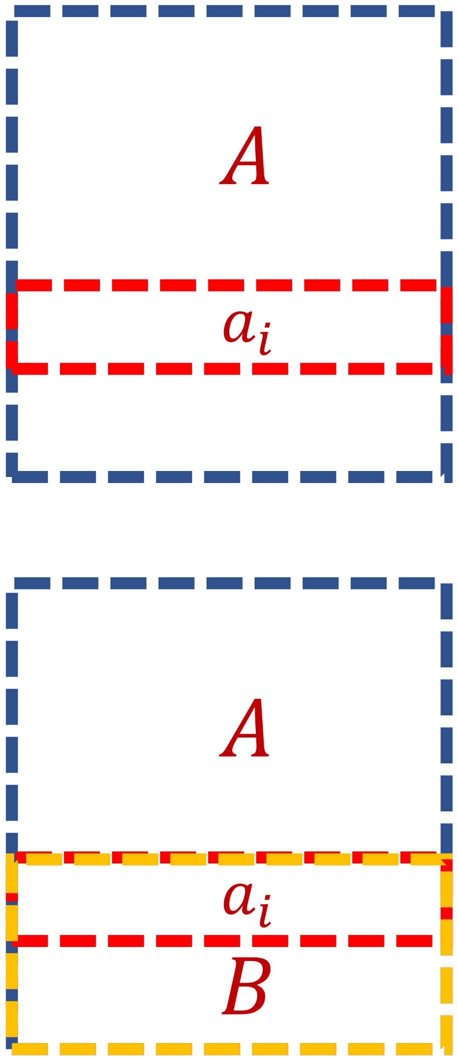
- Consider the sliding window model: we see rows $r_1, r_2, \ldots$
- Leverage score of r_t tells us r_t is not important, so we do not sample r_t
- Stream proceeds: All rows before r_t expire, new rows are all zeros
- Cannot possibly get approximation of A without storing r_t
- This implies we should *always* store the most recent row!
- Need a new sense of importance accounting for both uniqueness AND recency of a row



Reverse Online Leverage Scores

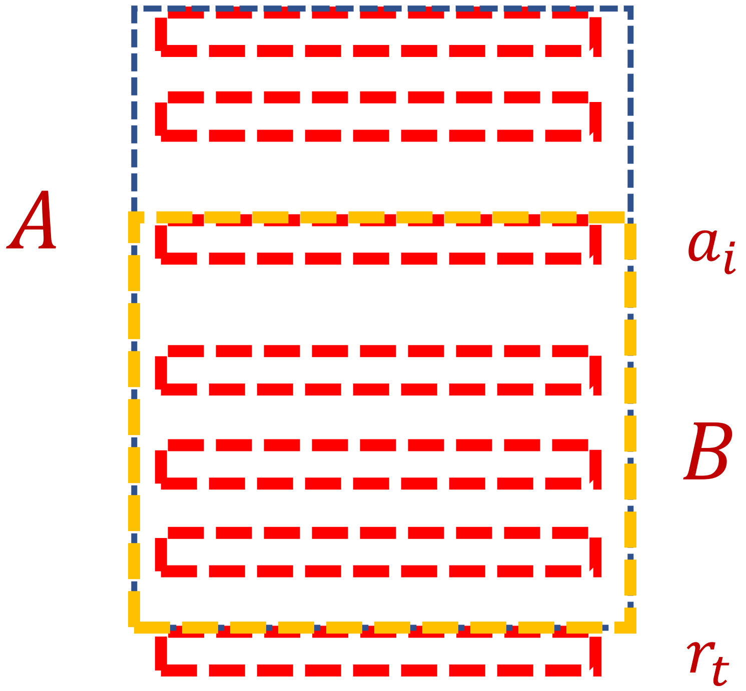
- Leverage score of row a_i is $\ell_i = a_i(A^\top A)^{-1}a_i^\top$
- Rows before a_i might be deleted so they shouldn't count towards the importance of a_i

- Reverse online leverage score of row a_i is $\tau_i = a_i(B^\top B)^{-1}a_i^\top$ where B are the rows after a_i



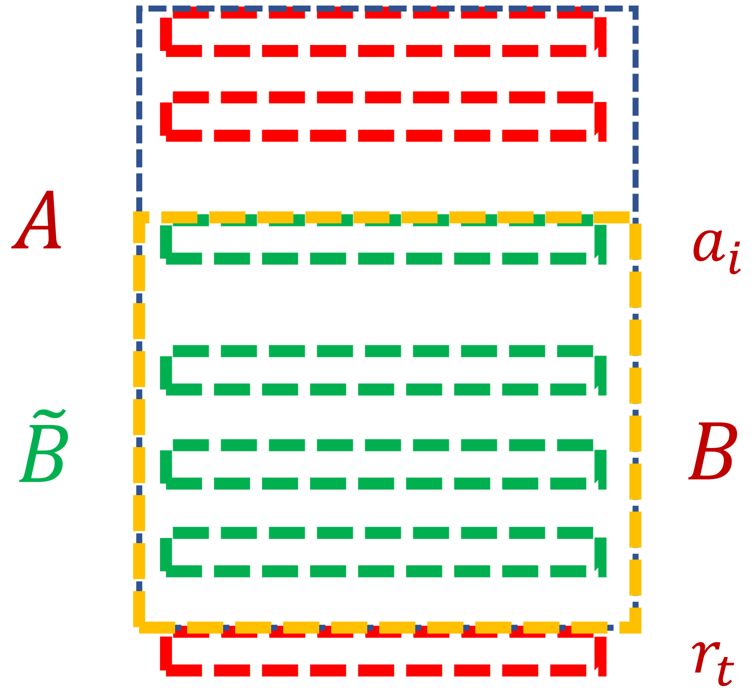
Correctness

- Correctness: Show an invariant that each row a_i is sampled with probability $\propto$ *final* reverse online leverage score
- Let B be the rows after a_i before row r_t
- Suppose before the arrival of row r_t, row a_i has been sampled with probability p_i, where $c_1\tau_B(a_i) \leq p_i \leq c_2\tau_B(a_i)$



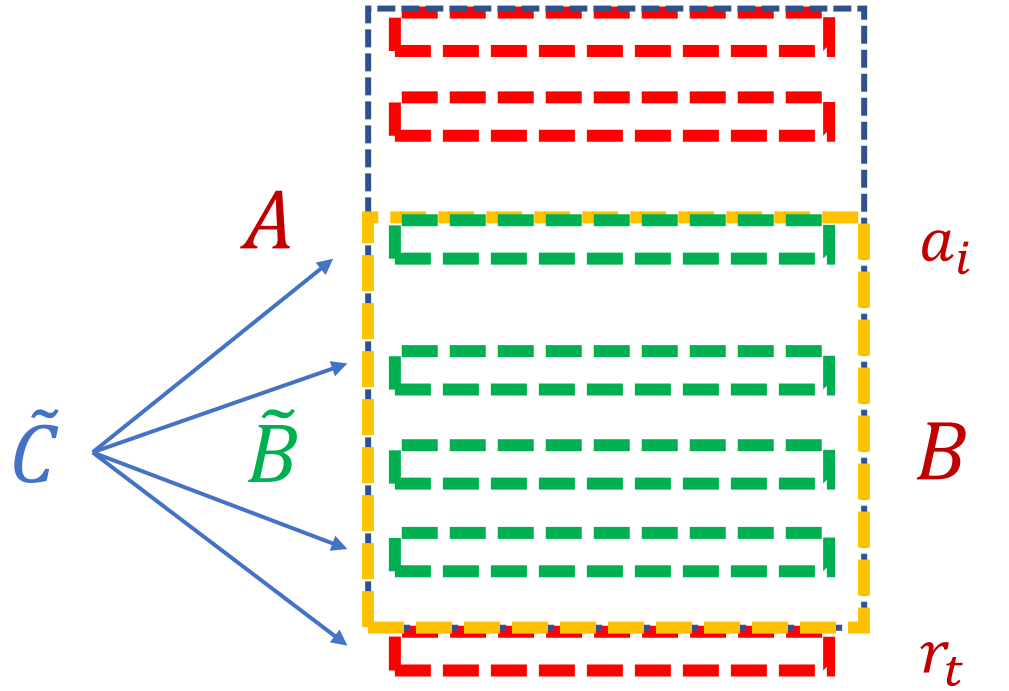
Correctness

- Suppose before the arrival of row r_t, row a_i has been sampled with probability p_i, where $c_1 \tau_B(a_i) \leq p_i \leq c_2 \tau_B(a_i)$
- $(1-\epsilon)B^\top B \preccurlyeq \tilde{B}^\top \tilde{B} \preccurlyeq (1+\epsilon)B^\top B$ [DMM06, SS08]



Correctness

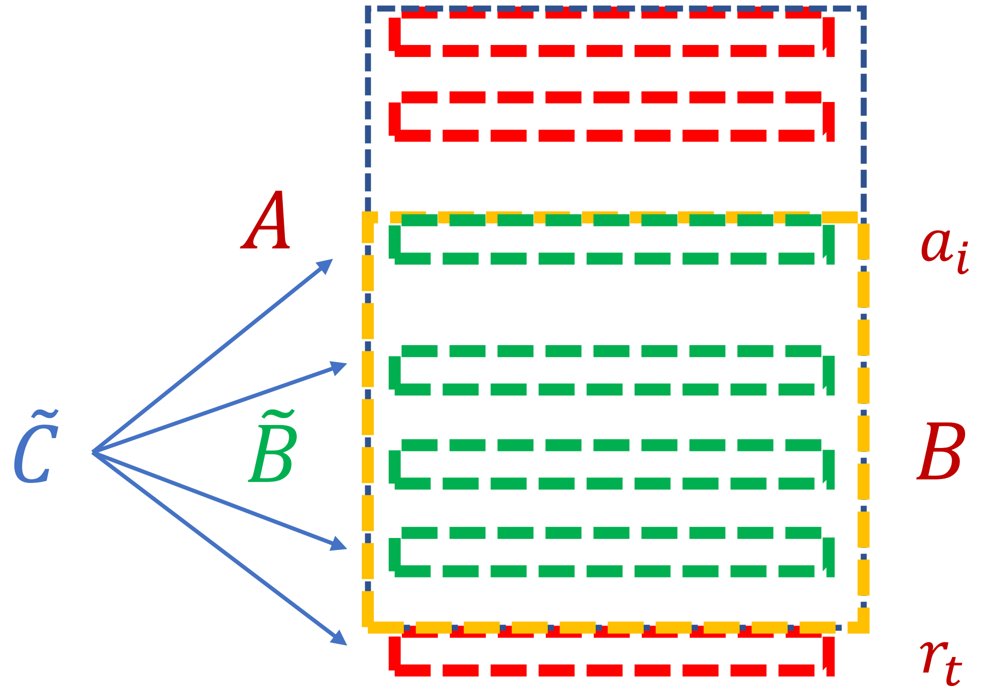
- Suppose before the arrival of row r_t, row a_i has been sampled with probability p_i, where $c_1\tau_B(a_i) \le p_i \le c_2\tau_B(a_i)$
- $(1-\epsilon)B^\top B \preccurlyeq \tilde{B}^\top\tilde{B} \preccurlyeq (1+\epsilon)B^\top B$ [DMM06, SS08]
- Let $\tilde{C}$ be $\tilde{B}$ appended by r_t
- a_i remains with probability $\propto \tau_{\tilde{C}}\left(\frac{a_i}{\sqrt{p_i}}\right)$



Correctness

- a_i remains with probability $\propto \tau_{\tilde{C}}\left(\frac{a_i}{\sqrt{p_i}}\right)$
- Reverse online leverage score:

$$\left(\frac{a_i}{\sqrt{p_i}}\right)(\tilde{C}^\top\tilde{C})^{-1}\left(\frac{a_i}{\sqrt{p_i}}\right)^\top = \left(\frac{a_i}{\sqrt{p_i}}\right)(\tilde{B}^\top\tilde{B}+r_t^\top r_t)^{-1}\left(\frac{a_i}{\sqrt{p_i}}\right)^\top$$

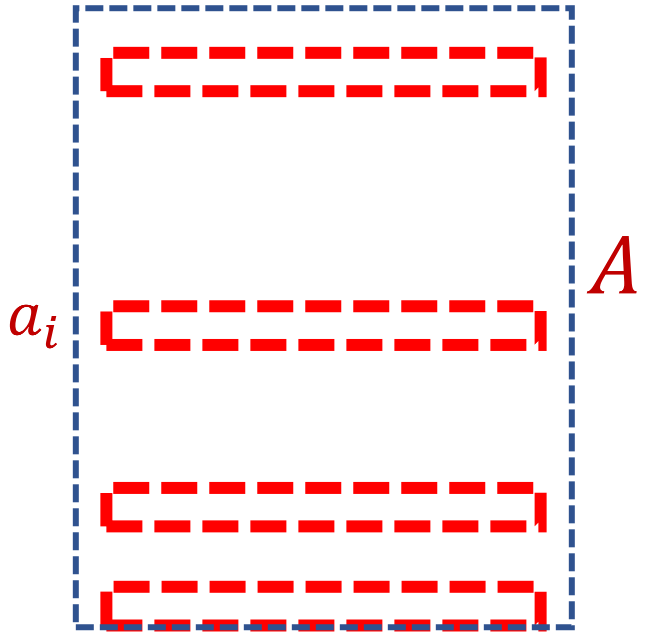


- Recall $(1-\epsilon)B^\top B \preccurlyeq \tilde{B}^\top\tilde{B} \preccurlyeq (1+\epsilon)B^\top B$, so $(1-\epsilon)C^\top C \preccurlyeq \tilde{C}^\top\tilde{C} \preccurlyeq (1+\epsilon)C^\top C$
- a_i survives w.p. $c_1\tau_C(a_i) \le p_i \le c_2\tau_C(a_i)$

Algorithm

- Correctness: Show an invariant that each row a_i is sampled with probability $\propto$ *final* reverse online leverage score
- By monotonicity, a_i is sampled with probability $\propto$ leverage score
- Outputs a matrix $M \in R^{m\times n}$ such that $(1-\epsilon)\|Ax\|_2 \leq \|Mx\|_2 \leq (1+\epsilon)\|Ax\|_2$ for all $x \in R^n$ [DMM06, SS08]

- Space? Must bound $\sum \tau_i$



Reverse Online Leverage Scores

- Online algorithm: see rows sequentially and irrevocably store or discard row, output spectral approximation at the end
- Sum of reverse online leverage scores = sum of online leverage scores [CMP16]
- $\sum \tau_i = \tilde{O}\left(\frac{1}{\epsilon^2} n\right) \rightarrow \tilde{O}\left(\frac{1}{\epsilon^2} n^2\right)$ space algorithm

Results

- Smooth histogram does not work for: vector induced p norms, generalized regression, low-rank approximation
- (Vector induced p norm: $||A||_p = \max||Ax||_p$ for $||x||_p = 1$)

Problem	Space	Reference
Deterministic Spectral Approximation	$\tilde{\mathcal{O}}\left(\frac{n^3}{\varepsilon}\right)$	Theorem 1.1
Spectral Approximation	$\tilde{\Theta}\left(\frac{n^2}{\varepsilon^2}\right)$	Theorem 4.5
Rank k Approximation	$\tilde{\Theta}\left(\frac{nk}{\varepsilon^2}\right)$	Theorem 5.8
Online Rank k Approximation	$\tilde{\Theta}\left(\frac{nk}{\varepsilon^2}\right)$	Theorem 6.4
Covariance Matrix Approximation	$\tilde{\Theta}\left(\frac{n}{\varepsilon^2}\right)$	Theorem 6.20, Theorem 6.24

Results

- If the entries of A and x are bounded integers, $O\left(\mathrm{poly}\left(n, \frac{1}{\epsilon}\right)\right)$ space algorithm for ℓ_1 spectral approximation:

$$(1-\epsilon)\|Ax\|_1 \leq \|Mx\|_1 \leq (1+\epsilon)\|Ax\|_1$$

- Algorithms can be slightly modified to run in *input sparsity time*
  - Only require constant factor approximation to reverse online leverage score
  - Use sparse JL for subspace embedding

Low-Rank Approximation (Offline)

- SVD: $A = U\Sigma V^T$ with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$
- Let Σ_k be the matrix with diagonal entries $\sigma_1, \ldots, \sigma_k$
- $M = U\Sigma_k V^T$ is *optimal* solution

Streaming Model

- Input: Elements of an underlying data set S, which arrives sequentially
- Output: Evaluation (or approximation) of a given function in space *sublinear* in the size of the input
  - Quantiles, heavy-hitters, norm estimation, distinct elements, sampling
  - Matchings, triangle counting, spanners, sparsifiers, densest subgraph,…
  - Minimum enclosing ball, clustering, facility location, volume maximization,…
  - Numerical linear algebra (matrix multiplication, spectral approximation,…)
  - Submodular optimization
  - Strings (pattern matching, periodicity, distance, palindromic detection,…)
  - Codeword testing

Initial Approach (Smooth PSD Histogram)

- Recall smooth function: If $f(A)$ is a "good" approximation to $f(B)$, then $f(A \cup C)$ will always be a "good" approximation to $f(B \cup C)$.
- Monotonic, polynomially bounded, all properties of real values…

- Use partial (Loewner) ordering on PSD matrices
  - If matrix $A \in R^{r \times n}$ is a submatrix of $B \in R^{s \times n}$, then $A^\top A \preccurlyeq B^\top B$
- The singular values of $A^\top A$ are respectively at most those of $B^\top B$

- Have "monotonicity", what about smoothness?