CSCE 411: Design and Analysis of Algorithms

Lecture : Optimization and Linear Programming

Date: April 22 *Nate Veldt, updated by Samson Zhou*

Course Logistics

- CLRS Chapter 29 and 35
- Last homework due Friday

1 Linear programming

A linear program is a mathematical optimization problem with

- A linear objective function
- Linear constraints

We will use x_i to denote variables—unknowns that we need to find to make the objective function as large as possible, and such that the constraints hold.

objective

Linear Program (LP)

Examples

Maximize $2x_1 + 3x_2$

Subject to: $x_1 + x_2 \leq 5$

$x_1 \geq 1$

$x_2 \geq 0$

Constraints

Minimize $x_4 - x_2 + x_3 - 3x_3$

Subject to: $x_4 + 3x_2 = 10$

$2x_3 - x_1 = 1$

$x_1 \geq 0$

$x_2 \geq 0$

Maximize $x_1^2 + \sin(x_2)$

Subject to $x_2 \in \mathbb{Z}$

$x_1 = \begin{cases} 2 & \text{if } x_2 = 2 \\ 0 & \text{otherwise} \end{cases}$

$x_1^3 + x_2 = 10$

Not an LP

Warm-up problem (from CLRS). As a politician seeking approval ratings, you would like the support of 50 urban voters, 100 suburban voters, 25 rural voters.
For each \$1 spent advertising one of the following policies, the resulting effects are:

Policy	**Urban**	**Suburban**	**Rural**
Zombie apocalypse	-2	+5	-3
Shark with lasers	+8	+2	-5
Flying cars roads	0	0	+10
Dolphins voting	+10	0	-2

Minimize the amount spent to achieve the desired voter support. How can we write this as an algorithmic or mathematic problem?

Create variables:

- Let x_1 be the money spent on ads for preparing for a zombie apocalypse
- Let x_2 be the money spent on ads for sharks with lasers
- Let x_3 be the money spent on ads for roads for flying cars
- Let x_4 be the money spent on ads for allowing dolphins to vote

Then the objective of the problem is:

minimize $x_1 + x_2 + x_3 + x_4$

The constraints of the problem are:

(Urban) $-2x_1 + 8x_2 + 10x_4 \geq 50$

(Suburban) $5x_1 + 2x_2 \geq 100$

(Rural) $-3x_1 - 5x_2 + 10x_3 - 2x_4 \geq 25$

$x_1 \geq 0$ $x_2 \geq 0$ $x_3 \geq 0$ $x_4 \geq 0$

Another example problem. The students of **CSCE 411** are managing their semester project portfolios, which involve three types of projects:

- **Project A:** Basic Algorithms.
- **Project B:** Graph Algorithms.
- **Project C:** Complexity.

Each project type earns a different amount of grade contribution points and consumes a mix of three limited resources: *Self-Study Time*, *Computation Credits*, and *Team Collaboration Hours*.

Project Type	Grade Points	Time (hours)	Credits (units)	Collab Hours
A	10	5	3	2
B	12	6	2	4
C	8	4	4	3

The semester budget for a student is:

- Maximum 60 hours of Self-Study Time.
- Maximum 30 Computation Credits.
- Maximum 36 Collaboration Hours.

Additionally, the professor requires that at least 2 projects from each type be completed for a balanced learning experience. Given the budget and these constraints, devise a project plan to maximize the grade.

Write a linear program for this problem.

Maximize $10x_1 + 12x_2 + 8x_3$

Subject to: $5x_1 + 6x_2 + 4x_3 \leq 60$

$3x_1 + 2x_2 + 4x_3 \leq 30$

$2x_1 + 4x_2 + 3x_3 \leq 36$

$x_1 \geq 2, x_2 \geq 2, x_3 \geq 2$

2 Types of Linear Programs

Linear programs have many variations:

- The objective function can be a maximization or a minimization problem
- The constraints can be equality or inequalities
- Often times, the variables will be greater than or equal to zero

If a particular solution $\overline{x}$ satisfies all of the constraints, we call it a feasible solution. Otherwise, we call it an infeasible solution. The set of solutions that are feasible is called the feasible region

Actually, we can perform different conversions to

- Turn a maximization problem into a minimization problem

$$\max x_1 + x_2 \longrightarrow \min -x_1 - x_2$$

- Turn an equality constraint into inequality constraints

$$x_1 + x_2 = 3 \longrightarrow \begin{array}{l} x_1 + x_2 \geq 3 \\ x_1 + x_2 \leq 3 \end{array}$$

- Turn an inequality constraint into an equality constraint

$$x_1 + x_2 \leq 3 \longrightarrow \begin{array}{l} x_1 + x_2 + s_{12} = 3 \\ s_{12} \geq 0 \end{array}$$

- Turn an unconstrained variable into positive variables

x_1 is arbitrary $\longrightarrow$ $x_1^+ - x_1^- = x_1$

$x_1^+, x_1^- \geq 0$

$x_1 = 3 \Rightarrow$

$x_1^+ = 3 \quad x_1^- = 0$

$x_1 = -3 \Rightarrow x_1^+ = 0, \; x_1^- = 3$

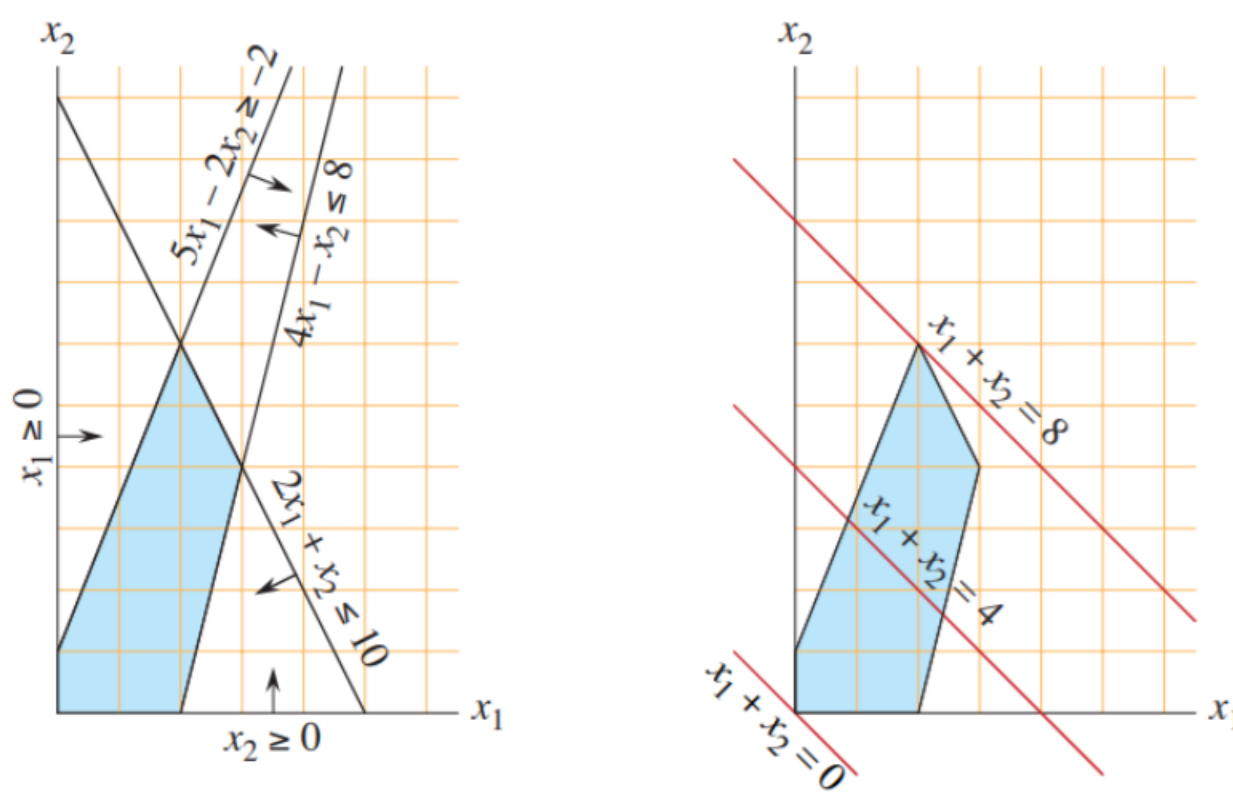
3 Solving a Linear Program

Important Fact: A linear program can be solved in polynomial time, e.g., the ellipsoid algorithm. There are other practical implementations, e.g., the simplex algorithm, that can run in exponential-time in the worst case.

Consider the linear program:

$$
\begin{aligned}
\text{Minimize:} \quad & x_1 + x_2 \\
\text{Subject to:} \quad & 4x_1 - x_2 \leq 8 \\
& 2x_1 + x_2 \leq 10 \\
& 5x_1 - 2x_2 \leq -2 \\
& x_1, x_2 \geq 0 \quad \text{(Non-negativity)}
\end{aligned}
$$

Intuition: The optimal solution must occur at a Vertex of the feasible region.



(a) (b)

4 Duality

Question 1. Consider the linear program:

$$
\begin{aligned}
\text{Maximize:} \quad & x_1 + 2x_2 + 3x_3 + 4x_4 \\
\text{Subject to:} \quad & 10x_1 \leq 10 \\
& x_2 \leq 200 \\
& \boxed{5x_1 + 10x_2 + 15x_3 + 20x_4 \leq 15} \\
& x_1, x_2, x_3, x_4 \geq 0 \quad \text{(Non-negativity)}
\end{aligned}
$$

$x_1 = 0 \quad x_2 = 0$

What can we say about the optimal value?

$x_1 + 2x_2 + 3x_3 + 4x_4 \leq 3$

$x_3 = 1$

$x_4 = 0$

- **A** 1
- **B** 3
- **C** 10
- **D** 15
- **E** 200
- **F** The limit does not exist

Consider the linear program:

$$\begin{aligned} \text{Maximize:} \quad & 3x_1 + x_2 + 4x_3 \\ \text{Subject to:} \quad & x_1 + x_2 + 3x_3 \leq 30 \quad \Leftarrow \quad y_1 \\ & 2x_1 + 2x_2 + 5x_3 \leq 24 \quad \Leftarrow \quad y_2 \\ & 4x_1 + x_2 + x_3 \leq 36 \quad \Leftarrow \quad y_3 \\ & x_1, x_2, x_3 \geq 0 \quad \text{(Non-negativity)} \end{aligned}$$

$2\times$

$4x_1 + 4x_2 + 10x_3 \leq 48$

What can we say about the optimal value?

Coefficient of x_1: $y_1 + 2y_2 + 4y_3 \geq 3$

What if we take the original constraints and add the first two constraints?

$3x_1 + 3x_2 + 8x_3 \leq 54$

The primal solution must be at most 54

48

Can we find a linear combination of the equations that exactly matches the objective?

Create variables:

- Let y_1 be the amount we scale the first inequality
- Let y_2 be the amount we scale the second inequality
- Let y_3 be the third

DUAL

What should be the objective?

min $30y_1 + 24y_2 + 36y_3$

Subject to: $y_1 + 2y_2 + 4y_3 \geq 3$ (coeff of x_1)

$y_1 + 2y_2 + y_3 \geq 1$ (coeff. of x_2)

$3y_1 + 5y_2 + 3y_3 \geq 4$ (coefficient of x_3)

$y_1, y_2, y_3 \geq 0$

The dual program of a linear program is an associated optimization problem where constraints become variables and vice versa, providing bounds on the primal objective

In general, for a primal LP

Maximize $\langle c, x \rangle := c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

subject to:

$$\left.\begin{array}{l} x_1, \\ x_2, \\ \vdots \\ x_n \end{array}\right. \geq 0, \quad m \text{ inequalities} \begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n \leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,n}x_n \leq b_2 \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \dots + a_{m,n}x_n \leq b_m \end{cases}$$

the corresponding dual LP is

Minimize $\langle b, y \rangle = b_1 y_1 + \dots + b_m y_m$

subject to

$$\left.\begin{array}{l} y_1 \\ \vdots \\ y_m \end{array}\right. \geq 0 \quad n \text{ inequalities} \begin{cases} a_{1,1}y_1 + a_{2,1}y_2 + \dots + a_{m,1}y_m \geq c_1 \\ \vdots \\ a_{1,n}y_1 + a_{2,n}y_2 + \dots + a_{m,n}y_m \geq c_n \end{cases}$$

Theorem 4.1 (Weak duality). *Let the primal linear program be a minimization problem and its dual be a maximization problem. If x is a feasible solution to the primal and y is a feasible solution to the dual, then*

$$c^\top x \geq b^\top y.$$

Theorem 4.2 (Strong Duality). *If the primal linear program has an optimal solution over a feasible region $\mathcal{P}$ and satisfies the necessary regularity conditions (e.g., feasibility), then the dual also has an optimal solution over the feasible region $\mathcal{D}$, and the optimal values of the primal and dual objectives are equal. That is,*

$$\min_{x \in \mathcal{P}} c^\top x = \max_{y \in \mathcal{D}} b^\top y.$$

$$\max \langle c, x \rangle$$

subject to $Ax \precsim b$

$x \succsim 0$

$x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

Dual: $\min \langle b, y \rangle$

subject to $A^T y \succsim c$

$y \succsim 0$