CSCE 411: Design and Analysis of Algorithms

Lecture : Optimization and Linear Programming

Date: April 22 *Nate Veldt, updated by Samson Zhou*

Course Logistics

- CLRS Chapter 29 and 35
- Last homework due Friday

1 Linear programming

A linear program is a mathematical optimization problem with

- A linear objective function
- Linear constraints

We will use x_i to denote variables—unknowns that we need to find to make the objective function as large as possible, and such that the constraints hold.

Examples

Warm-up problem (from CLRS). As a politician seeking approval ratings, you would like the support of 50 urban voters, 100 suburban voters, 25 rural voters.
For each \$1 spent advertising one of the following policies, the resulting effects are:

Policy	**Urban**	**Suburban**	**Rural**
Zombie apocalypse	-2	+5	-3
Shark with lasers	+8	+2	-5
Flying cars roads	0	0	+10
Dolphins voting	+10	0	-2

Minimize the amount spent to achieve the desired voter support. How can we write this as an algorithmic or mathematic problem?

Create variables:

- Let _____ be the money spent on ads for preparing for a zombie apocalypse
- Let _____ be the money spent on ads for sharks with lasers
- Let _____ be the money spent on ads for roads for flying cars
- Let _____ be the money spent on ads for allowing dolphins to vote

Then the objective of the problem is:

The constraints of the problem are:

Another example problem. The students of **CSCE 411** are managing their semester project portfolios, which involve three types of projects:

- **Project A:** Basic Algorithms.
- **Project B:** Graph Algorithms.
- **Project C:** Complexity.

Each project type earns a different amount of grade contribution points and consumes a mix of three limited resources: *Self-Study Time*, *Computation Credits*, and *Team Collaboration Hours*.

Project Type	**Grade Points**	**Time (hours)**	**Credits (units)**	**Collab Hours**
A	10	5	3	2
B	12	6	2	4
C	8	4	4	3

The semester budget for a student is:

- Maximum 60 hours of Self-Study Time.
- Maximum 30 Computation Credits.
- Maximum 36 Collaboration Hours.

Additionally, the professor requires that at least 2 projects from each type be completed for a balanced learning experience. Given the budget and these constraints, devise a project plan to maximize the grade.

Write a linear program for this problem.

2 Types of Linear Programs

Linear programs have many variations:

- The objective function can be a maximization or a minimization problem
- The constraints can be equality or inequalities
- Often times, the variables will be greater than or equal to zero

If a particular solution $\overline{x}$ satisfies all of the constraints, we call it a ______________________________. Otherwise, we call it an ________________________________. The set of solutions that are feasible is called the _________________________.
Actually, we can perform different conversions to

- Turn a maximization problem into a minimization problem

- Turn an equality constraint into inequality constraints

- Turn an inequality constraint into an equality constraint

- Turn an unconstrained variable into positive variables

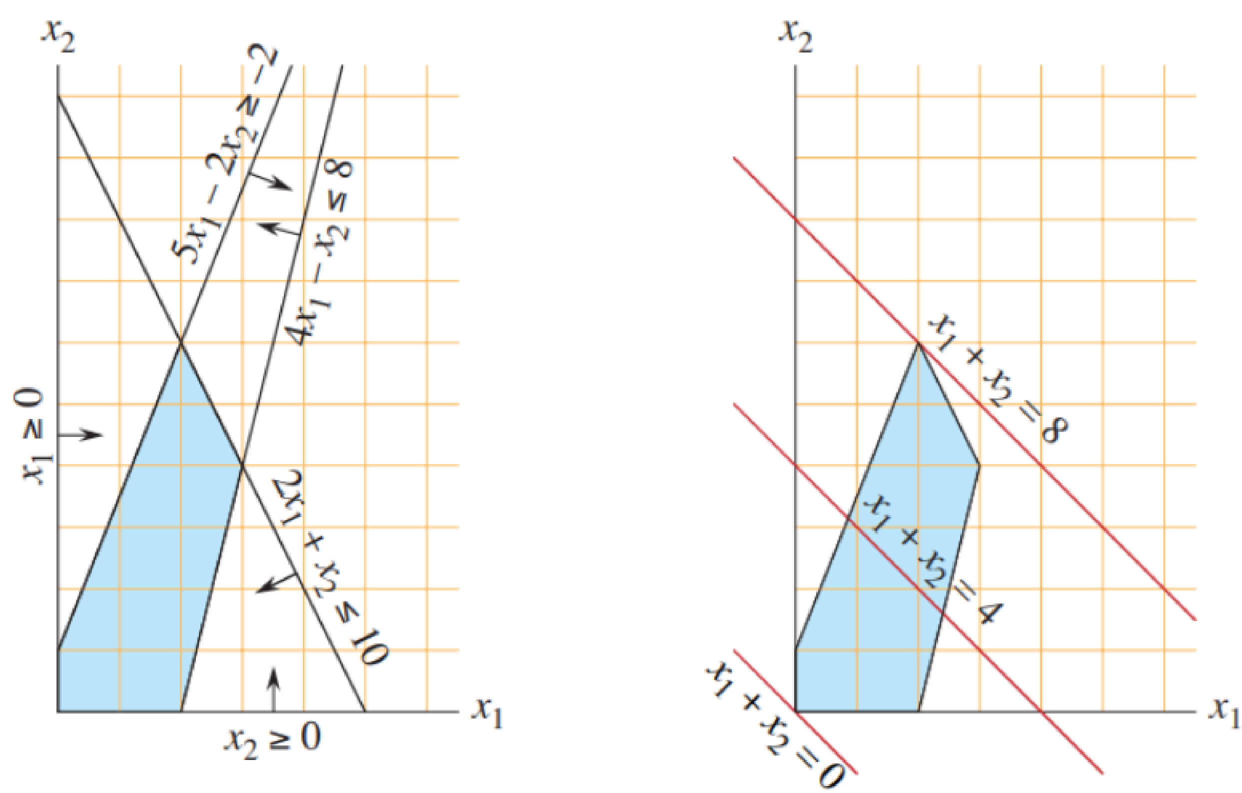
3 Solving a Linear Program

Important Fact: A linear program can be solved in polynomial time, e.g., the ellipsoid algorithm. There are other practical implementations, e.g., the simplex algorithm, that can run in exponential-time in the worst case.

Consider the linear program:

$$
\begin{aligned}
\text{Minimize:} \quad & x_1 + x_2 \\
\text{Subject to:} \quad & 4x_1 - x_2 \le 8 \\
& 2x_1 + x_2 \le 10 \\
& 5x_1 - 2x_2 \le -2 \\
& x_1, x_2 \ge 0 \quad \text{(Non-negativity)}
\end{aligned}
$$

Intuition: The optimal solution must occur at a ___________ of the feasible region.



(a) (b)

4 Duality

Question 1. Consider the linear program:

$$\begin{aligned} \text{Maximize:} \quad & x_1 + 2x_2 + 3x_3 + 4x_4 \\ \text{Subject to:} \quad & 10x_1 \leq 10 \\ & x_2 \leq 200 \\ & 5x_1 + 10x_2 + 15x_3 + 20x_4 \leq 15 \\ & x_1, x_2, x_3, x_4 \geq 0 \quad \text{(Non-negativity)} \end{aligned}$$

What can we say about the optimal value?

A 1

B 3

C 10

D 15

E 200

F The limit does not exist

Consider the linear program:

$$\begin{aligned} \text{Maximize:} \quad & 3x_1 + x_2 + 4x_3 \\ \text{Subject to:} \quad & x_1 + x_2 + 3x_3 \leq 30 \\ & 2x_1 + 2x_2 + 5x_3 \leq 24 \\ & 4x_1 + x_2 + x_3 \leq 36 \\ & x_1, x_2, x_3 \geq 0 \quad \text{(Non-negativity)} \end{aligned}$$

What can we say about the optimal value?

What if we take the original constraints and add the first two constraints?

The primal solution must be at most _____

Can we find a linear combination of the equations that exactly matches the objective?
Create variables:

- Let _____ be the ____
- Let _____ be the ____
- Let _____ be the ____

What should be the objective?

The ________________________ of a linear program is an associated optimization problem where ____ and _________________, providing bounds on the _____________________________

In general, for a primal LP

the corresponding dual LP is

Theorem 4.1 (Weak duality)**.** *Let the primal linear program be a minimization problem and its dual be a maximization problem. If x is a feasible solution to the primal and y is a feasible solution to the dual, then*

$$c^\top x \geq b^\top y.$$

Theorem 4.2 (Strong Duality)**.** *If the primal linear program has an optimal solution over a feasible region $\mathcal{P}$ and satisfies the necessary regularity conditions (e.g., feasibility), then the dual also has an optimal solution over the feasible region $\mathcal{D}$, and the optimal values of the primal and dual objectives are equal. That is,*

$$\min_{x \in \mathcal{P}} c^\top x = \max_{y \in \mathcal{D}} b^\top y.$$