CSCE 411: Design and Analysis of Algorithms

Week 7: Graph Algorithms: More DFS

Date: February 24, 2026 — *Nate Veldt, updated by Samson Zhou*

Course Logistics

- Graph algorithms: Chapter 22
- Homework 4 out, due this Friday

1 Depth First Search Algorithm

Recall that a *breadth-first* search explores nodes that are k steps away from node s before exploring any nodes that are $k+1$ steps away.

A *depth-first search* instead explores the *most recently discovered vertex* before backtracking and exploring other previously discovered nodes.

Roughly speaking, this is accomplished by ________________________________.

Recall that unlike in a BFS, a depth-first search (DFS):

- Explores the *most recently discovered vertex* before backtracking and exploring other previously discovered vertices
- All nodes in the graph are explored (rather than just a DFS for a single node s)
- We keep track of a global *time*, and each node is associated with two timestamps for when it is *discovered* and *explored*.

Each node $u \in V$ is associated with the following attributes

Attribute	Explanation	Initialization
u.status	tells us whether a node has been *undiscovered*, *discovered*, and *explored*	u.status $= U$
u.D	timestamp when u is first discovered	NIL
u.F	timestamp when u is finished being explored	NIL
u.parent	predecessor/"discoverer" of u	NIL

```
DFS(G)
  for v ∈ V do
      v.parent = NIL
      v.status = U
  end for
  time = 0
  for u ∈ V do
      if u.status == U then
          DFS-Visit(G, u)
      end if
  end for
```

```
DFS-Visit(G, u)
  time = time + 1
  u.D = time
  u.status = D
  for v ∈ Adj[u] do
      if v.status == U then
          v.parent = u
          DFS-Visit(G, v)
      end if
  end for
  u.status = E
  time = time + 1
  u.F = time
```

1.1 Runtime Analysis

Question 1. *What is the runtime of a depth first search, assuming that we store the graph in an adjacency list, and assuming that* $|E| = \Omega(|V|)$*?*

A $O(|V|)$

B $O(|E|)$

C $O(|V| \times |E|)$

D $O(|V|^2)$

E $O(|E|^2)$

1.2 Properties of DFS

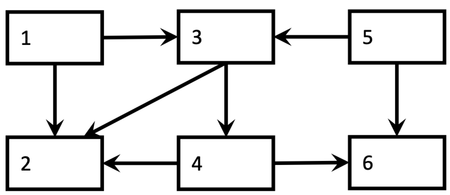
Theorem 1.1. *In any depth-first search of a graph $G = (V, E)$, for any pair of vertices u and v, exactly one of the following conditions holds:*

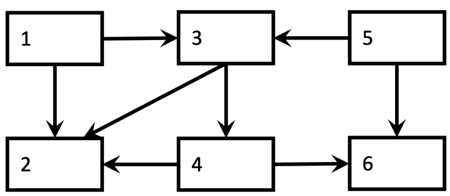
- $[u.D, u.F]$ *and* $[v.D, v.F]$ *are disjoint;* ______________________________

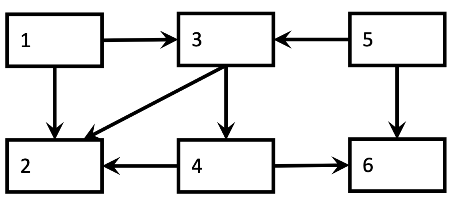
- $[v.D, v.F]$ *contains* $[u.D, u.F]$ *and* ______________________________

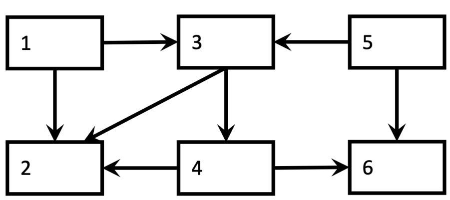
- $[u.D, u.F]$ *contains* $[v.D, v.F]$ *and* ______________________________

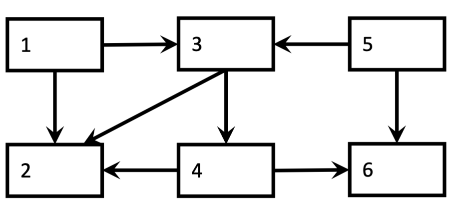
We will not prove this, but we'll give a quick illustration











1	1	2	3	4	5	6	7	8	9	10	11	12
2	1	2	3	4	5	6	7	8	9	10	11	12
3	1	2	3	4	5	6	7	8	9	10	11	12
4	1	2	3	4	5	6	7	8	9	10	11	12
5	1	2	3	4	5	6	7	8	9	10	11	12
6	1	2	3	4	5	6	7	8	9	10	11	12

Corollary 1.2. *v is a descendant of u* $\iff$

1.3 Classification of Edges

Given a graph $G = (V, E)$ performing a DFS on G produces a graph $\hat{G} = (V, \hat{E})$ where

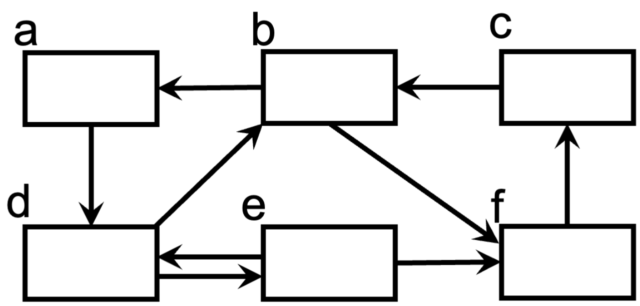
$$\hat{E} = \{(u.\text{parent}, u) \colon v \in V \text{ and } v.\text{parent} \neq NIL\}$$

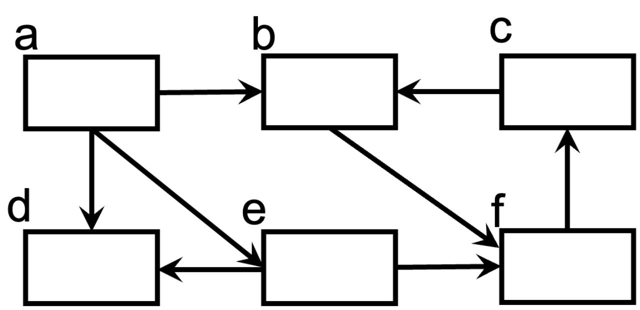
This is called a *depth-first* forest of G.

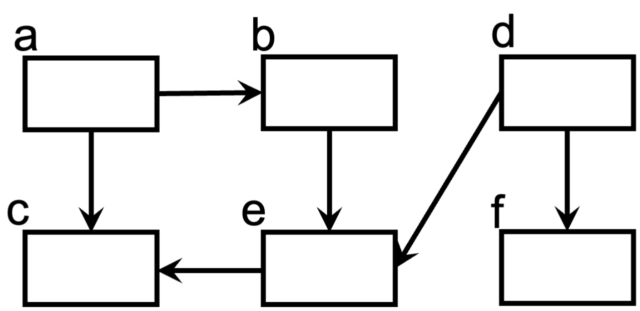
Given any edge $(u, v) \in E$, we can classify it based on the status of node v when we are performing the DFS:

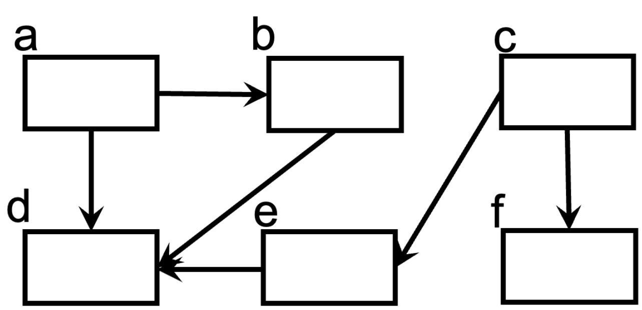
Edge	Explanation	How to tell when exploring (u, v)?
Tree edge	edge in $\hat{E}$	
Back edge	connects u to ancestor v	
Forward edge	connects vertex u to descendant v	*and* $u.D < v.D$
Cross edge	either (a) connects two different trees or (b) crosses between siblings/cousins in same tree	*and* $u.D > v.D$

1.4 Practice









Question 2. *How many of the above graphs were directed acyclic graphs?*

A *1*

B *2*

C *3*

D *4*

E *none of them*

2 Depth First Search: Motivating Problems

Depth first search is used in several applications for analyzing directed graphs. We will now take a closer look at these applications.

Directed graph reminders

2.1 Reachability and Connected Components

Reachability. Given a graph $G = (V, E)$ and node set $S \subseteq V$, node $v \in S$ is *reachable* from node $u \in S$ if ____.

Connected components. For an undirected graph $G = (V, E)$ a connected component is a maximal subgraph in which every node in is ____________________________________

Weakly Connected components If $G = (V, E)$ is directed, a *weakly connected component* is ____

Strongly Connected components If $G = (V, E)$ is directed, a *strongly connected component* is subgraph $S \subseteq V$ in which there is ____

Question 3. *How many weakly connected components and strongly connected components are there in the following graph, respectively?*

A *1 and 3*

B *1 and 2*

C *0 and 1*

D *2 and 3*

2.2 Directed Acyclic Graphs

A *cycle* in a directed graph is a directed path ____

A *Directed acyclic* graph is a directed graph that ______________________.

Examples

2.3 Topological Sorting

A topologically ordering of a directed acyclic graph $G = (V, E)$ is an ordering of nodes so that:

3 Application 1: Checking if G is a DAG

Theorem 3.1. *G is a DAG $\iff$ a DFS yields no back edges. Equivalently:*

Proof First, ($\implies$) we show that if DFS yields a back edge, G is not a DAG.

Next ($\Leftarrow$) we show that if G is not a DAG there will be a back edge.

4 Application 2: Topological Sort

Given a directed acyclic graph $G = (V, E)$, a topological sort of G is an ordering of nodes such that for any $(u, v) \in E$, u comes before v in the ordering.

We can use the following procedure to solve the topological sort problem:

1.

2.

Theorem 4.1. *Ordering nodes in a directed acyclic graph $G = (V, E)$ by reversed finish times will produce a topological sort of G.*

Proof.

1. Let (u, v) be an edge in G

2. Our goal is to show that

   ____

3. When (u, v) is explored, there are three different possibilities for the status of v:

   - **Case 1**: v.status $== U$. This means v becomes a descendant of u.

     Thus, $v.F < u.F$. Reason: _______________________

   - **Case 2**: v.status $== E$, then we also have $v.F < u.F$.

     Reason:

   - **Case 3**: v.status $== D$, this means that v is an ancestor of u, so (u, v) is a back edge.

     But this is impossible. Reason: _______________________

4. In all cases that are possible, ____________________

$\square$

5 The transpose graph and connected component graph

If $G = (V, E)$ is a graph, a *strongly connected component* is maximal subgraph $S \subseteq V$ in which every node is reachable from every other node by following paths in S.

Let $G = (V, E)$ be a graph and assume that $\{C_1, C_2, \ldots, C_k\}$ represent its strongly connected components.
The *connected component graph* $G^{\text{scc}} = (V^{\text{scc}}, E^{\text{scc}})$ is defined as follows:

- There is a node $v_i \in V^{\text{scc}}$ for each component C_i
- There is an edge $(v_i, v_j) \in E^{\text{scc}}$ if and only if there is a directed edge between C_i and C_j

Lemma 5.1. *The connected component graph is* ______________________________

The *transpose graph* of G is $G^T = (V, E^T)$ where

$$E^T = \{(u, v) \colon (v, u) \in E\}$$

Lemma 5.2. *G and G^T have* ______________________________

6 Strongly Connected Components

The following algorithm will compute the strongly connected components of a graph $G = (V, E)$:
Strongly-Connected-Components(G)

1. Find a DFS for G to get finish times $u.F$ for each $u \in V$.
2. Compute the *transpose graph* $G^T = (V, E^T)$
3. Find a DFS for G^T, but in the main loop of DFS, always visit nodes based on the reverse order of finish times from the DFS of G.
4. Output the vertices of each tree in the DFS of G^T.

What is the key to making this work? In the second DFS, we essentially visit all of the nodes in the connected components graph in topologically sorted order.