On Fine-Grained Distinct Element Estimation

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Distinct Elements

- α servers, server i has a set $S_i \subseteq [1,2,3,\ldots,n]$
- $S = \cup_i S_i$
- Number of distinct elements in dataset is $F_0(S) = \|S\|_0 = |S|$

Communication Model

- Each server can talk to any other server, but only on private channel
- Can designate a specific server as the *coordinator*
- Communication over channel is measured in bits
- Goal: Using minimum total communication, output $(1+\varepsilon)$-multiplicative approximation to $F_0(S)$, i.e., output some number Z such that $Z \le F_0(S) \le (1+\varepsilon)\cdot Z$

Previous Results

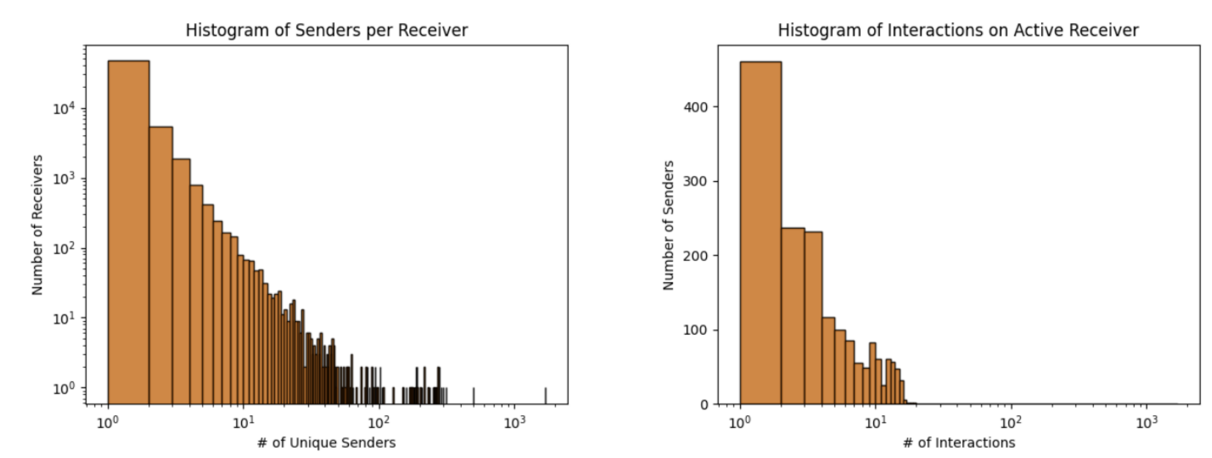
- Theorem [KNW10, Bla20]: There exists a distributed protocol that outputs a $(1+\varepsilon)$-multiplicative approximation to $F_0(S)$, using $O\left(\frac{\alpha}{\varepsilon^2} + \alpha \log n\right)$ bits of communication
- Theorem [WZ14]: Any distributed protocol that outputs a $(1+\varepsilon)$-multiplicative approximation to $F_0(S)$ requires $\Omega\left(\frac{\alpha}{\varepsilon^2} + \alpha \log n\right)$ bits of communication

Bridging Theory and Practice

- There is usually a large number of servers and we want good accuracy, so this means we should require *a lot* of communication!
- Algorithms behave well in practice with little communication
- What's going on?

Our Observations

- In the lower bound instance, all servers have roughly the same number of items in the sets S_i
- In practice, the datasets are "skewed" so a small number of servers have a large number of items, e.g., Zipf's Law, 80-20 Rule, or Pareto's Principle
- This does change the complexity of the problem? If so, how to characterize the complexity?



(a) Histogram of Receivers (b) Histogram of Activity

New Parameterization: Pairwise Collisions

- We define C to be the number of pairwise collisions, i.e., number of triplets (a,i,j) so that $a \in S_i$ and $a \in S_j$ but $i < j$
- Note that if a single server has most of the items, they will not be repeated across the other servers, so C is low and can be constant
- When all servers have similar number of items and there are many intersections, C can be as large as $\alpha^2 \cdot F_0(S)$

	$C = \beta \cdot F_0(S),\ \beta \ge 1$	
	$F_0(S) < \frac{1}{\varepsilon^2}$	$F_0(S) \ge \frac{1}{\varepsilon^2}$
Theorem 1.1	$\mathcal{O}\left(\alpha \log n + \sqrt{\beta}\cdot F_0(S)\cdot \log n\right)$	$\mathcal{O}\left(\alpha \log n + \frac{\sqrt{\beta}}{\varepsilon^2}\log n\right)$
Theorem 1.3	$\Omega(\alpha + \sqrt{\beta}\cdot F_0(S))$	$\Omega\left(\alpha + \frac{\sqrt{\beta}}{\varepsilon^2}\right)$
	$C = \beta \cdot F_0(S),\ \beta < 1,\ C > \varepsilon \cdot F_0(S)$	
	$F_0(S) < \frac{1}{\varepsilon^2}$	$F_0(S) \ge \frac{1}{\varepsilon^2}$
Theorem 1.2	$\mathcal{O}\left(\alpha \log n + \frac{\beta}{\varepsilon^2}\log n\right)$	$\mathcal{O}\left(\alpha \log n + \beta\cdot F_0(S)\cdot \log n\right)$
Theorem 1.4	$\Omega\left(\alpha + \frac{\beta}{\varepsilon^2}\right)$	$\Omega(\alpha + \beta \cdot F_0(S))$

Table 1: A summary of our results for the distributed distinct elements estimation problem on a universe of size n across α servers, parameterized by the number C of collisions across the α servers, and the accuracy parameter $\varepsilon \in (0,1)$.

Results

- Theorem: Suppose $C = \beta \cdot O\left(\min\left(F_0(S), \frac{1}{\varepsilon^2}\right)\right)$. There exists a distributed protocol that outputs a $(1+\varepsilon)$-multiplicative approximation to $F_0(S)$, using $O\left(\min\left(F_0(S), \frac{1}{\varepsilon^2}\right)\cdot \sqrt{\beta}\log n + \alpha \log n\right)$ bits of communication
- Point of comparison: Previously, first term was $\frac{\alpha}{\varepsilon^2}$, which happens if $C = \alpha^2 \cdot F_0(S)$, i.e., many items appear across many servers
- Theorem: Suppose $C = \beta \cdot O\left(\min\left(F_0(S), \frac{1}{\varepsilon^2}\right)\right)$. Any distributed protocol that outputs a $(1+\varepsilon)$-multiplicative approximation to $F_0(S)$ requires $\Omega\left(\min\left(F_0(S), \frac{1}{\varepsilon^2}\right)\cdot \sqrt{\beta} + \alpha \log n\right)$ bits of communication
- Point of comparison: Tight up to a $\log n$ factor in the first term
- Context: Shows that C is a parameter that characterizes the complexity of the problem

Algorithm 2 $(1+\varepsilon)$-approximation to F_0, given an upper bound on the number of collisions

Input: Items given to α players from a universe of size $[n]$, accuracy parameter $\varepsilon \in (0,1)$, upper bound C on the number of pair-wise collisions
Output: $(1+\varepsilon)$-approximation to the number of distinct items

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Let X be a 4-approximation to F_0                                   ▷Lemma 2.4
Let i_0 be the largest integer such that X / 2^{i_0} > 1000 / ε^2
i ← min(0, i_0)
Let S_i be a subset of [n] where each item is subsampled with probability 1/2^i
Assume without loss of generality each player i has a binary vector v^{(i)} ∈ {0,1}^n
Each player sends their total number of items in S_i
Let Z be the sum of these numbers
η ← ε/10, p ← min(1, 100C / (η^2 X^2))
Let T be a subset of S_i where each item is subsampled with probability p
Each player sends their items in T
Let W = Σ_{j∈T} max(0, v_j − 1), where v = Σ_{i∈[α]} v^{(i)} be the excess mass in T
Return Z · 2^i − W · 1/p
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