# Reyleigh Monotonicity in Electric Networks From Nonreversible Markov Chains

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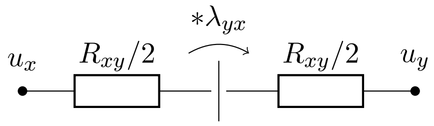
## 1 Introduction

In this report, we further the exploration into monotonicity in the electric networks introduced in Balazs-Folly, consisting of units as in the schematic below.




As in Balazs-Folly, we write $z \sim x$ for neighbouring vertices $z$ and $x$ in the graph, and for later use we define

$$\gamma_{xy} := \sqrt{\lambda_{xy}} = \frac{1}{\gamma_{yx}}, \quad D_{xy} := \frac{2\gamma_{xy}}{1+\lambda_{xy}} C_{xy}, \quad D_x := \sum_{z \sim x} D_{xz}\gamma_{zx}.$$

where $C_{xy} = 1/R_{xy}$ is the Ohmic conductance.

We will consider networks that satisfy the 'Markovian property', which says every vertex $x$ must satisfy

$$\sum_{z \sim x} D_{xz}\gamma_{xz} = \sum_{z \sim x} D_{xz}\gamma_{zx}.$$

This is equivalent to Kirchoff's current law (the current must be divergence free) when the voltage at every vertex is fixed at 1V, and is the necessary condition for the network to correspond to a Markov chain.

We will explore the notion of bistochasticity, that is, when the rows *and* columns of the probability transition matrix associated with the network sum to 1. We see that this property can be written as

$$D_x = \sum_{z \sim x} D_{xz}\gamma_{zx} = \text{constant}$$

at every vertex $x$. Our aim is to determine whether bistochasticity is a sufficient condition for the effective resistance across a network to be monotone in the resistances of the comprising units.

## 2 A Turblent Random Walk

We consider a nonreversible simple random walk on $\mathbb{Z}^2$. Denote the probability measure conditioned on initial position $\mathbf{X}_0 = (x, y)$ by $\mathbb{P}_{(x,y)}$. Then define the transition probablilties for the random walk as follows.

If $x + y$ is odd, then

- $\mathbb{P}_{(x,y)}(\text{moving up}) = \mathbb{P}_{(x,y)}(\text{moving down}) = \frac{1}{3}$ and
- $\mathbb{P}_{(x,y)}(\text{moving left}) = \mathbb{P}_{(x,y)}(\text{moving right}) = \frac{1}{6}$.

If $x + y$ is even, then

- $\mathbb{P}_{(x,y)}(\text{moving up}) = \mathbb{P}_{(x,y)}(\text{moving down}) = \frac{1}{6}$ and
- $\mathbb{P}_{(x,y)}(\text{moving left}) = \mathbb{P}_{(x,y)}(\text{moving right}) = \frac{1}{3}$.

We observe that the walk has a tendancy to move around the individual squares of the integer lattice in a specific direction, creating a 'turbulence' in each square that is in the opposite direction to that of its neighbours. Figure 1 shows the direction of each transition with probability $\frac{1}{3}$.

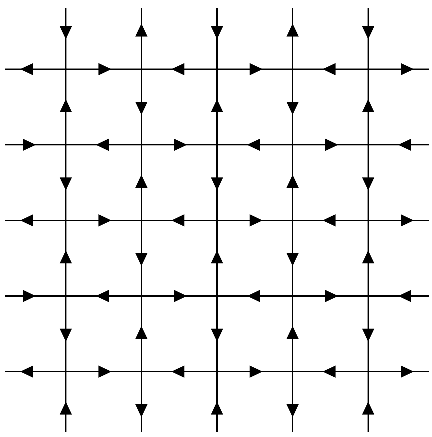


Figure 1

It is clear from the asymmetry of the 'turbulances' that the chain is nonreversible. This is easily verifiable with Kolmogorov's criterion: for a chain to be reversible, the probability of traversing through any loop must be equal in both directions. Choosing a loop around any individual square, we have probability $(1/3)^4$ in one direction, and probability $(1/6)^4$ in the other.

We also observe that the chain is bistochastic: for any given vertex, the sum of the transition probabilities entering that vertex from its neighbouring vertices is 1. This means that its stationary distribution must be uniform across $\mathbb{Z}^2$. From this we can obtain an electric network that fits the Markov chain.

### 2.1 Choosing the electric network

When identifying the electric network for a finite chain with transition matrix $P$ and stationary distribution $\mu$, we made the choices

$$D_{xy} := \sqrt{\mu_x \cdot P_{xy} \cdot \mu_y \cdot P_{yx}};$$

$$\gamma_{xy} := \sqrt{\frac{\mu_x \cdot P_{xy}}{\mu_y \cdot P_{yx}}}.$$

However in the case where the stationary distribution is uniform over an infinite number of vertices there is no way to normalise the stationary distribution, so we choose $\mu_x$ to be some constant for every $x \in \mathbb{Z}^2$. This gives us the freedom to pick any constant for $D_{xy}$ – for the sake of simplicity, choose it such that the conductances $C_{xy} = 1$ for each edge. We also obtain $\lambda_{xy} = 2$ when $x \to y$ with probability $\frac{1}{6}$ and $\lambda_{xy} = 1/2$ when $x \to y$ with probability $\frac{1}{3}$.

This gives us an infinite grid of unit resistors, each with a voltage amplifier of amplification 2 in the opposite direction to the probability turbulance. We can now ask what is the effective resistance between the origin and infinity?

### 2.2 Recurrence/transcience of the turbulent random walk: a semi-probabilistic approach

We can determine whether the random walk is recurrent by analysing the two-step random walk, $(\mathbf{X}_{2n})_{n\in\mathbb{Z}}$. The trasition probabilities of the two-step Markov chain are identical at every vertex, and they are symmetrical both vertically and horizontally (see Figure 2). So the chain must be reversible and bistochastic. This means we can use classical electrical network methods to determine whether the two-step Markov chain is recurrent.

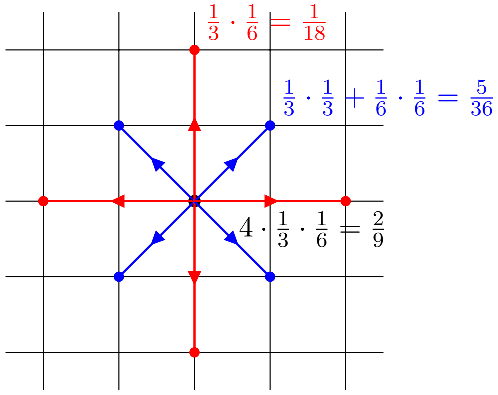



Figure 2: The two-step transition probabilities

We first acknowledge that the two-step chain has a laziness – i.e. it has a positive probability of not changing states – at every state, which cannot be easily translated to a classical resistor network. We can fix this by rescaling the transition probabilities, only looking at the steps where the chain jumps to a different state. This gives the transition probabilities shown in figure 3.

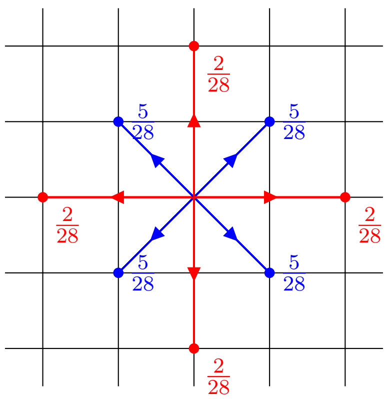



Figure 3: The probabilities rescaled to avoid laziness

Since this two-step chain is reversible, we can now turn to an infinite network of classical resistors and determine whether the effective resistance to infinity is infinite; if it is, our two-step chain is recurrent. We can calculate (as in Doyle and Snell) that the horizontal and vertical edges (red) should each correspond to $\frac{1}{2}$ Ohm resistors and the diagonals (blue) should be $\frac{1}{5}$ Ohm resistors.

Next we short circuit concentric squares of wires and use Rayleigh's Monotonicity Law: short circuiting nodes together decreases the resistance between the nodes (to 0) and hence decreases the effective resistance across the whole network. Figure 4 shows a way of shorting along diagonals, from which we can calculate the effective resistance between these sets of shorted nodes.

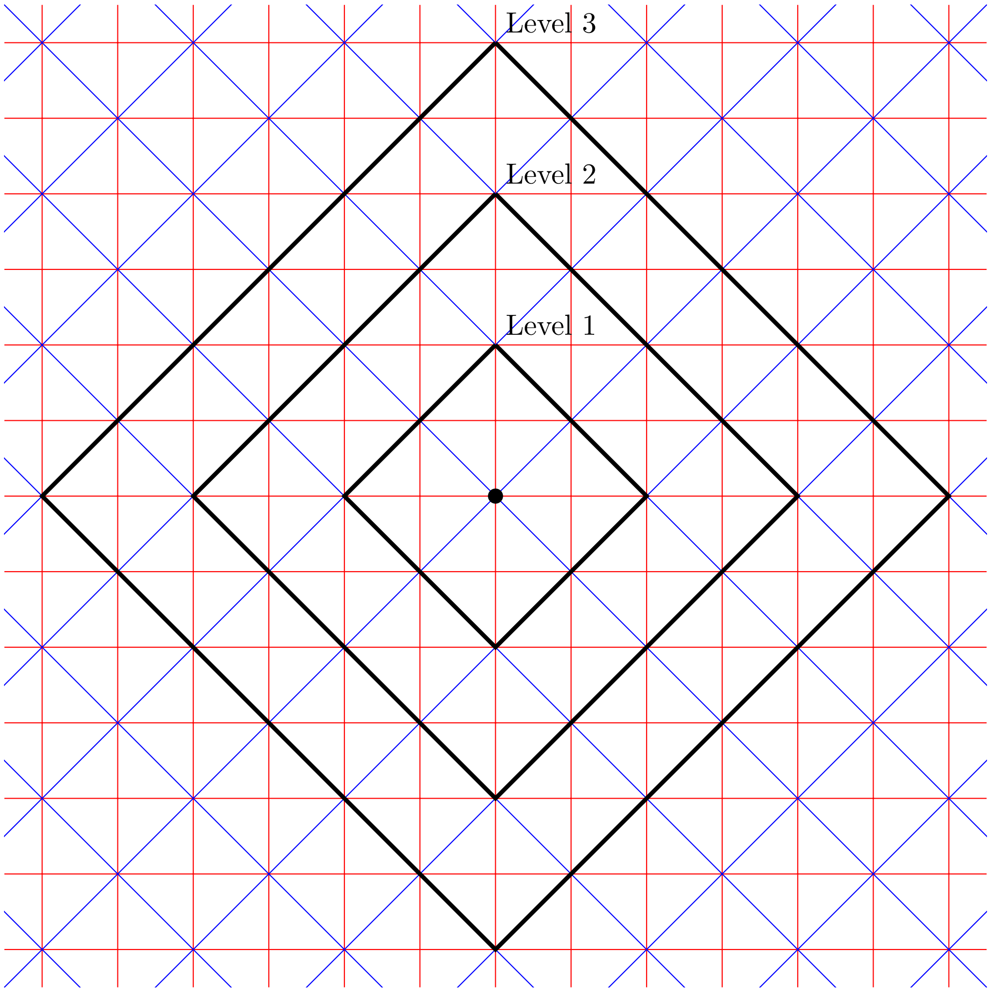



Figure 4: The black lines show where we have short circuited. The red lines represent $\frac{1}{2}$ Ohm resistors and the blue lines represent $\frac{1}{5}$ Ohm resistors.

We can now see that between the origin and level 1, there are 4 red and 4 blue resistors in parallel (discount the red resistor at each corner that connects level 1 to itself); between levels 1 and 2 there are 20 reds and 12 blues; between levels 2 and 3 there are 36 reds and 20 blues, etc. Thus we can transform this circuit into the one shown in Figure 5, and use the formula for resistors in parallel to work out the effective resistance between each level.

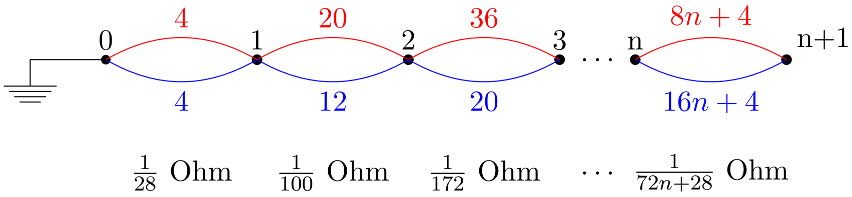



Figure 5

Since these resistances are now in series, we can sum them up to find that the effective resistance from the origin to the $(n+1)$th level is

$$\sum_{k=0}^{n} \frac{1}{72n+4},$$

which explodes as $n \to \infty$. Hence the effective resistance to infinity is infinite, and so the two-step Markov chain with laziness removed is recurrent. It isn't hard to reason from this that the original turbulent Markov chain must be recurrent.

### 2.3 Using MATLAB to calculate the effective resistance across a network

Using the well-known voltage divider equation, it can be easily shown that for every vertex $x$ not in the starting or ending sets

$$u_x = \frac{\sum_y \lambda_{xy} u_y C_{xy}^{\text{se}}}{\sum_z C_{xz}^{\text{se}}} = \frac{\sum_y \lambda_{xy} u_y C_{xy}/(1+\lambda_{xy})}{\sum_z C_{xz}/(1+\lambda_{xz})}.$$

Using these equations and MATLAB's symbolic equation solving, we can solve for the voltages at every point in the circuit, and hence for the currents and effective resistance.

Let's use this to solve for the voltages in a $5 \times 5$ section of our grid with the outer edges all short circuited together.

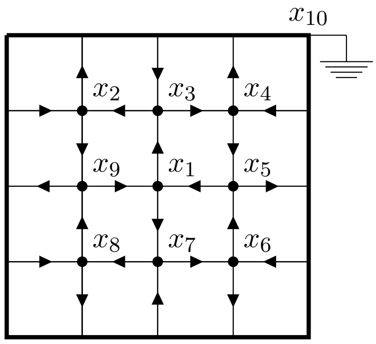



Figure 6: $x_1$ is held at 1V and the boundary $x_{10}$ is grounded. The arrows represent unit resistors with voltage amplifiers with parameter 2 in the direction of the arrow.

Inputting amplifiers into a log-antisymmetric matrix *lambda* (if two nodes $x$ and $y$ aren't connected, put $\lambda_{xy} = 1$) and the conductances into the symmetric matrix $C$. (Here non-existent edges have zero conductance, and note the corner vertices have two connections in parallel to the boundary, hence the 0.5 Ohms. Also the opposing direction of amplifiers cancel out.)

```
% Solve for the voltages in an electric network for nonreversible Markov chains

syms R S C l lambda k real;

% Input voltage amplifiers as log-antisymmetric matrix:

lambda = [  [1  1  .5  1  2  1  .5  1  2  1]
            [1  1  2   1  1  1  1   1  .5 1]
            [2  .5 1  .5  1  1  1   1  1  2]
            [1  1  2   1  .5 1  1   1  1  1]
            [.5 1  1   2  1  2  1   1  1  .5]
            [1  1  1   1  .5 1  2   1  1  1]
            [2  1  1   1  1  .5 1  .5  1  2]
            [1  1  1   1  1  1  2   1  .5 1]
            [.5 2  1   1  1  1  1   2  1  .5]
            [1  1  .5  1  2  1  .5  1  2  1]];

% Input conductances as symmetric matrix:

C = [   [0  0  1  0  1  0  1  0  1  0]
        [0  0  1  0  0  0  0  0  1  .5]
        [1  1  0  1  0  0  0  0  0  1]
        [0  0  1  0  1  0  0  0  0  .5]
        [1  0  0  1  0  1  0  0  0  1]
        [0  0  0  0  1  0  1  0  0  .5]
        [1  0  0  0  0  1  0  1  0  1]
        [0  0  0  0  0  0  1  0  1  .5]
        [1  1  0  0  0  0  0  1  0  1]
        [0  .5 1  .5 1  .5 1  .5 1  0]];

n = length(C); u = sym('u',[1,n],'real'); U_div = sym('U',[1,n],'real');
```

```

% Set start and end voltages
u(1) = 1; u(n) = 0;


for k=2:n-1
    % set up expressions for voltage dividers (Proposition 3):
    U_div(k)=simplify(dot(u,C(k,:).*lambda(k,:)./(1+lambda(k,:)))/dot(C(k,:),1./(1+lambda(k,:))));
end

% solve voltage divider simultaneous equations
usol = struct2cell(solve( u(2:n-1) == U_div(2:n-1), u(2:n-1)));
for k = 1:n-2
    u(k+1)=usol{k};
end

% Currents
currents = sym('i',[n,n]);
for x = 1:n
    for y = 1:n
        currents(x,y) = simplify(2*C(x,y)/(1+lambda(y,x))*(lambda(y,x)*u(x)-u(y)));
    end
end

% Effective resistance
Reff = simplify((u(1)-u(n))/sum(currents(1,:)));
```

The code calculates the voltages at the points $(x_1, x_2, \ldots, x_{10})$ to be (1, 10/29, 13/29, 10/29, 23/58, 10/29, 13/29, 10/29, 23/58, 0), and effective resistance between $x_1$ and $x_{10}$ to be 29/68 Ohms.

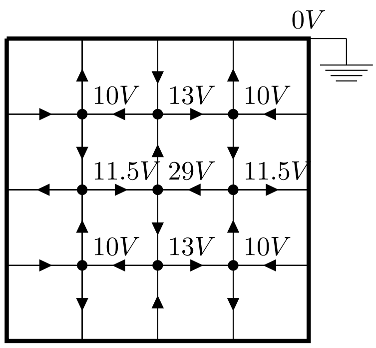



Figure 7: Scaling up the voltages by a factor of 29 makes it easier to read. We see that the voltages from the origin up and down are greater than left and right, which coincides with the direction of the amplifiers.

### 2.4 Where are the currents flowing?

The MATLAB code calculates the current passing through each node and stores them in an antisymmetric matrix. For this network, the current matrix is

$$\begin{bmatrix} 0 & 0 & 30/29 & 0 & 4/29 & 0 & 30/29 & 0 & 4/29 & 0 \\ 0 & 0 & -32/87 & 0 & 0 & 0 & 0 & 0 & 17/87 & 5/29 \\ -30/29 & 32/87 & 0 & 32/87 & 0 & 0 & 0 & 0 & 0 & 26/87 \\ 0 & 0 & -32/87 & 0 & 17/87 & 0 & 0 & 0 & 0 & 5/29 \\ -4/29 & 0 & 0 & -17/87 & 0 & -17/87 & 0 & 0 & 0 & 46/87 \\ 0 & 0 & 0 & 0 & 17/87 & 0 & -32/87 & 0 & 0 & 5/29 \\ -30/29 & 0 & 0 & 0 & 0 & 32/87 & 0 & 32/87 & 0 & 26/87 \\ 0 & 0 & 0 & 0 & 0 & 0 & -32/87 & 0 & 17/87 & 5/29 \\ -4/29 & -17/87 & 0 & 0 & 0 & 0 & 0 & -17/87 & 0 & 46/87 \\ 0 & -5/29 & -26/87 & -5/29 & -46/87 & -5/29 & -26/87 & -5/29 & -46/87 & 0 \end{bmatrix}.$$

The next step is to calculate the currents moving along the two edges that connect each corner point $x_2$, $x_4$, $x_6$ and $x_8$ to the boundary $x_{10}$, as the code only works out the total current passing between two connected nodes, regardless of how many edges connect them. Since the boundary is grounded at 0V, the calculation is very simple. The currents scaled up by a factor of 87 are shown in the figure below:

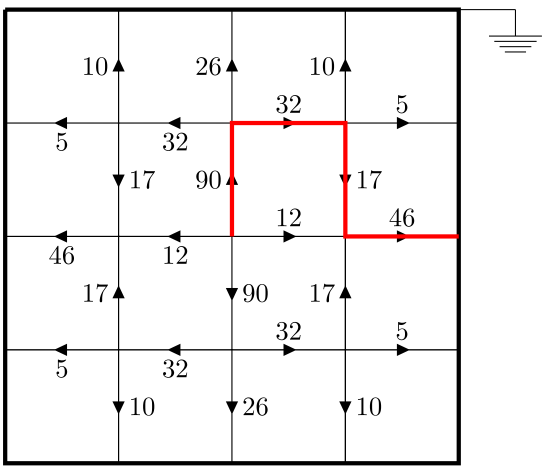



Figure 8: The most 'popular' path – i.e. the path with the most current – is up, right, down, right (alongside the three other identical paths symmetrically opposite).

## 2.5 What happens when we short circuit?

Now let's short circuit along the square of radius 1 from the origin, and see what happens to the currents, voltages and effective resistance.

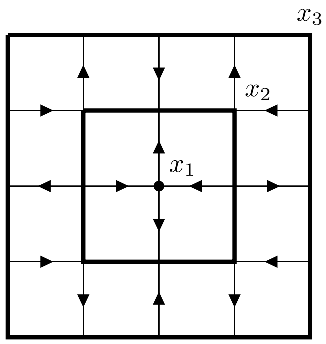
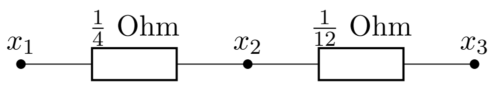



Figure 9: Short circuiting combines eight nodes into one, leaving us with three

Intriguingly – though perhaps not surprisingly – the amplifiers of two identical units of resistance $R$ and amplification $\lambda$ connected in parallel, but oriented in opposite directions, cancel out. So we can compact pairs of opposing parallel units into resistors of resistance 1/2. This circuit then boils down to




so our effective resistance is $\frac{1}{3} < \frac{29}{68}$. This doesn't break the monotonicity law.

We can again calculate the currents passing through each edge.

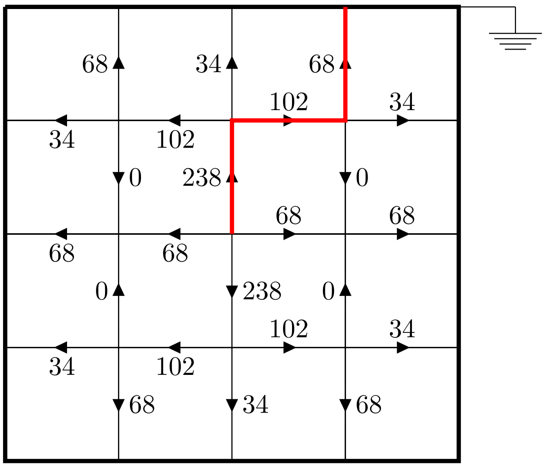



Figure 10: This time most of the current goes up, right, up, (and along the three other identical paths symmetrically opposite).

### 2.6 Short circuiting another way

We see from picture above that the current doesn't flow at all along four of the edges, which previously comprised the paths with the most current flowing before we short circuited. However our intuition tells us that the amplifiers should make it easier for the current to flow in this direction. Let's see what happens if we short circuit the horizontal edges of the inner square but leave the vertical edges un-short circuited.

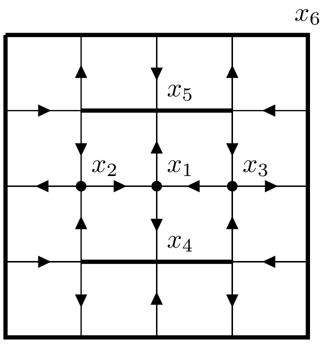



Figure 11: This time we are left with six nodes

Matlab calculates the voltages at each vertex to be $(u_1, \ldots, u_6) = (1, \frac{83}{148}, \frac{83}{148}, \frac{175}{296}, \frac{175}{296}, 0)$, and the effective resistance is $74/127 \approx 0.58$ which is much larger than the value for the original network. However, short-circuiting like this loses bistochasticity, so we do not have a counterexample.

# 3 A Network With Four Vertices

In this section we consider a network with four vertices and with components as seen below.

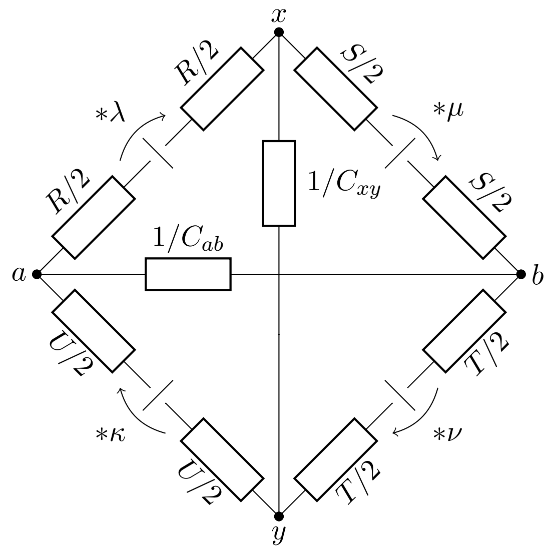



Figure 12

We will use this network to see whether bistochasticity is sufficient for monotonicity.

## 3.1 When is this network Markovian?

For this network to be Markovian, we require a constant current around the circuit when all the vertices are held at 1V. In that case,

$$\left(1 - \frac{R_{xa}}{2} \cdot i\right) \cdot \lambda_{xa} - \frac{R_{xa}}{2} \cdot i = 1,$$
$$\implies i = \frac{2}{R_{xa}} \frac{\lambda_{xa} - 1}{\lambda_{xa} + 1}$$

and the same for the other three edges, (replace $xa$ with $ay$, $yb$, and $bx$). Rearranging and substituting in our variable names, the network is Markovian if and only if

$$\frac{\lambda + 1}{\lambda - 1} R = \frac{\mu + 1}{\mu - 1} S = \frac{\nu + 1}{\nu - 1} T = \frac{\kappa + 1}{\kappa - 1} U,$$

which gives us $S$, $T$ and $U$ in terms of $R$. We also see that for each resistance to be positive, either every $\lambda_{xy} > 1$ or every $\lambda_{xy} < 1$.

### 3.2 When is it bistochastic?

Bistochasticity requires $D_a = D_x = D_b = D_y$. We calculate

$$\begin{aligned} D_a &= \frac{2\lambda}{(1+\lambda)R} + \frac{2}{(1+\kappa)U} + C_{ab} \\ &= \frac{2}{(1+\lambda)R}\left(\lambda + \frac{\lambda-1}{\kappa-1}\right) + C_{ab} \\ &= \frac{2}{(1+\lambda)R} \cdot \frac{\lambda\kappa - 1}{\kappa - 1} + C_{ab}. \end{aligned}$$

Similarly,

$$\begin{aligned} D_x &= \frac{2}{(1+\lambda)R} \cdot \frac{\lambda\mu - 1}{\mu - 1} + C_{xy} \\ D_b &= \frac{2}{(1+\lambda)R} \cdot \frac{(\lambda-1)(\nu\mu-1)}{(\mu-1)(\nu-1)} + C_{ab} \\ D_y &= \frac{2}{(1+\lambda)R} \cdot \frac{(\lambda-1)(\kappa\nu-1)}{(\kappa-1)(\nu-1)} + C_{xy}. \end{aligned}$$

Solving $D_a = D_x$ we see that

$$\begin{aligned} & D_a - D_x = 0 \\ \iff & \frac{2}{(1+\lambda)R} \cdot \left(\frac{\lambda\kappa-1}{\kappa-1} - \frac{\lambda\mu-1}{\mu-1}\right) = C_{xy} - C_{ab} \\ \iff & \frac{\lambda\mu - \lambda\kappa + \mu - \kappa}{(\kappa-1)(\mu-1)} = \frac{(1+\lambda)R \cdot (C_{xy} - C_{ab})}{2} \\ \iff & \frac{\mu - \kappa}{(\kappa-1)(\mu-1)} = \frac{R(C_{xy} - C_{ab})}{2}, \end{aligned}$$

and $D_b = D_y$ becomes

$$\begin{aligned} & D_b - D_y = 0 \\ \iff & \frac{2(\lambda-1)}{(1+\lambda)(\nu-1)R} \cdot \left(\frac{\nu\mu-1}{\mu-1} - \frac{\kappa\nu-1}{\kappa-1}\right) = C_{xy} - C_{ab} \\ \iff & \frac{1}{(\nu-1)} \cdot \frac{\nu\mu - \kappa\nu + \kappa - \mu}{(\kappa-1)(\mu-1)} = \frac{R(C_{xy} - C_{ab})(1+\lambda)}{2(\lambda-1)} \\ \iff & \frac{\mu - \kappa}{(\kappa-1)(\mu-1)} = \frac{R(C_{xy} - C_{ab})(1+\lambda)}{2(\lambda-1)}. \end{aligned}$$

We see that the left hand sides are both the same, so we conclude that $C_{xy} = C_{ab} := 1/Q$ since $(1+\lambda)/(\lambda-1) = 1$ has no solutions, and thus $\mu = \kappa$. Then

$$D_a = D_x = \frac{2(\lambda\mu - 1)}{R(1+\lambda)(\mu-1)} + 1/Q$$

and

$$D_b = D_y = \frac{2(\lambda-1)(\mu\nu-1)}{R(1+\lambda)(\mu-1)(\nu-1)} + 1/Q$$

Solving the equality of these two expressions gives

$$\frac{(\lambda-1)(\mu\nu-1)}{\nu-1} = \lambda\nu - 1$$
$$\iff \lambda = \nu$$

We've now found the general solution for the parameters that allow the network to be bistochastic. We can vary $Q$, $R$, $\lambda$ and $\mu$ as we please in the following circuit.

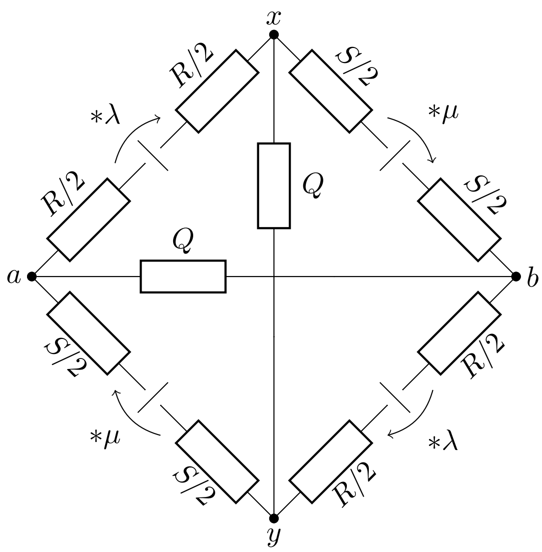



Figure 13: Here $S = R \cdot \frac{(\lambda+1)(\mu-1)}{(\lambda-1)(\mu+1)}$.

Applying 1V at vertex $a$ and grounding vertex $b$, we can use voltage divider equations at $x$ and $y$ to obtain expressions for the voltages $u_x$ and $u_y$:

$$u_x = \frac{(\mu-1)(\lambda+1)R + 2\lambda(\mu-1)Q}{2(\mu-1)(\lambda+1)R + 2(\mu\lambda-1)Q}$$
$$u_y = \frac{(\mu-1)(\lambda+1)R + 2(\mu-1)Q}{2(\mu-1)(\lambda+1)R + 2(\mu\lambda-1)Q}.$$

We can now find the currents along each vertex and hence the total current entering and leaving the system:

$$i_b = \frac{(\lambda+1)(\lambda\mu-1)R + 2(\lambda-1)(\lambda+1)Q}{R(\lambda+1)\left(2(\mu-1)(\lambda+1)R + 2(\mu\lambda-1)Q\right)} + \frac{1}{Q} \tag{1}$$

$$= \frac{(\lambda+1)^2(\mu-1)R^2 + 2(\lambda+1)(\lambda\mu-1)RQ + 2(\lambda-1)(\lambda+1)Q^2}{RQ(\lambda+1)\left(2(\mu-1)(\lambda+1)R + 2(\mu\lambda-1)Q\right)} \tag{2}$$

### 3.3 Monotonicity in $R$

We will differentiate $i_b$ with respect to $R$. First, rewrite (1) as

$$i_b = \frac{aR+b}{cR^2+dR} + \frac{1}{Q}$$

where coefficients $a$, $b$, $c$ and $d$ are

$$a = (\lambda+1)(\lambda\mu-1), \quad b = 2(\lambda-1)(\lambda+1)Q, \quad c = (\lambda+1)^2(\mu-1), \quad d = 2(\mu\lambda-1)Q.$$

Then $i_b$ is decreasing if and only if

$$\begin{aligned}
\frac{\partial i_b}{\partial R} &= \frac{a(cR^2+dR) - (aR+b)(2cR+d)}{(cR^2dR)^2} < 0 \\
\iff & a(cR^2+dR) < (aR+b)(2cR+d) \\
\iff & acR^2 + 2bcR + bd > 0.
\end{aligned}$$

But we see that

$$\begin{aligned}
ac &= (\lambda+1)^3 + (\lambda\mu-1)(\mu-1) > 0, \\
bc &= 2(\lambda-1)(\lambda+1)^2(\lambda\mu+1)(\mu-1)Q > 0, \text{ and} \\
bd &= 2(\lambda-1)(\lambda\mu-1)(\lambda\mu+1)Q^2 > 0
\end{aligned}$$

and so the above inequality is definitely true for any $R > 0$. Hence $i_b$ is decreasing in $R$.

### 3.4 Monotonicity in $Q$

This time we rewrite (2) as

$$i_b = \frac{aQ^2+bQ+c}{dQ^2+eQ}$$

where this time the coefficients are

$$a = 2(\lambda-1)(\lambda\mu+1), \quad b = 2R(\lambda+1)(\lambda\mu-1), \quad c = e = R^2(\lambda+1)^2(\mu-1), \quad d = R(\lambda+1)(\lambda\mu-1).$$

Then $i_b$ is decreasing in $Q$ if and only if

$$\begin{aligned}\frac{\partial i_b}{\partial Q} =& \frac{(dQ^2+eQ)(2aQ+b)-(aQ^2+bQ+c)(2dQ+e)}{(dQ^2+eQ)^2} < 0\\ \iff& (db-ae)Q^2 + 2cdQ + ec > 0.\end{aligned}$$

And we see that

$$\begin{aligned} db - ae &= 2R^2(\lambda+1)^2(\lambda^2\mu^2 - 3\lambda\mu + \lambda + \mu) > 0\\ cd &= R^3(\lambda+1)^3(\mu-1)(\lambda\mu-1) > 0\\ ec &= R^2(\lambda+1)^4(\mu-1)^2 > 0.\end{aligned}$$

Hence the above inequality is true for every positive $Q$, thus $i_b$ is decreasing in $Q$.

So we can conclude that $R_{\text{eff}} = 1/i_b$ is increasing in both $R$ and $Q$, so we have monotonicity.

# 4 Cutting Wires in Continuous-Time Markov Chains

We begin with a continuous time Markov chain $(X_t)_{t\geq 0}$ with rate matrix $Q$ and a stationary distribution $\boldsymbol{\mu}$, so then $\boldsymbol{\mu}Q = 0$. The jump chain of $(X_t)$ has transition probability matrix $P$ where

$$P_{ij} = -\frac{Q_{ij}}{Q_{ii}} = \frac{Q_{ij}}{\sum_{k\sim j} Q_{ik}}.$$

Here $\sum_{k\sim j}$ denotes summing over all neighbouring vertices $k$ of $j$.

We want to see what happens to the network generated by the jump chain after we cut wires in the continuous-time chain.

## 4.1 Detailed balance in jump chain

By definition of a stationary distribution for a continuous-time Markov chain,

$$\begin{aligned}0 &= \sum_i \mu_i Q_{ij} \\ &= \sum_{i \neq j} \mu_i Q_{ij} + \mu_j Q_{jj} \\ &= \sum_{i \neq j} \mu_i Q_{ij} - \mu_j \sum_{k \neq j} Q_{jk}.\end{aligned}$$

for every state $j$. Now multiply and divide the first term by $\sum_{l \sim i} Q_{il}$ and the second by $\sum_{m \sim j} Q_{jm}$:

$$\begin{aligned}&\sum_{i \sim j} \mu_i \frac{Q_{ij}}{\sum_{l \sim i} Q_{il}} \cdot \sum_{l \sim i} Q_{il} = \mu_j \sum_{k \sim j} \frac{Q_{jk}}{\sum_{m \sim j} Q_{jm}} \cdot \sum_{m \sim j} Q_{jm} \\ \iff \quad &\sum_{i \sim j} \left( \mu_i \sum_{l \sim i} Q_{il} \right) P_{ij} = \mu_j \sum_{m \sim j} Q_{jm}.\end{aligned}$$

So $\mu_i \sum_{l \sim i} Q_{il} = -\mu_i Q_{ii}$ is proportional to the stationary distribution of $P$, say $\boldsymbol{\nu}$. Then $\nu_i := -\mu_i Q_{ii}/c$ where $c := -\sum_i \mu_i Q_{ii}$ is the normalising constant of proportionality.

If we have detailed balance for $Q$ over the edge $ij$, then $\mu_i Q_{ij} = \mu_j Q_{ji}$. It follows from this that

$$\begin{aligned}\mu_i \sum_{l \sim i} Q_{il} \frac{Q_{ij}}{\sum_{l \sim i} Q_{il}} &= \mu_j \sum_{m \sim j} Q_{jm} \frac{Q_{ji}}{\sum_{m \sim j} Q_{jm}} \\ \implies c\nu_i P_{ij} &= c\nu_j P_{ji} \\ \implies \nu_i P_{ij} &= \nu_j P_{ji}\end{aligned}$$

So we also have detailed balance for the jump chain.

## 4.2 Electric network of the jump chain

We want to find the electric network associated with our jump chain. For $i \neq j$, the electric network of the jump chain is given by

$$\begin{aligned}D_{ij} &= \sqrt{\nu_i P_{ij} \cdot \nu_j P_{ji}} = \sqrt{-\frac{1}{c}\mu_i Q_{ii} \cdot \frac{Q_{ij}}{-Q_{ii}} \cdot -\frac{1}{c}\mu_j Q_{jj} \cdot \frac{Q_{ji}}{-Q_{jj}}} \\ &= \frac{1}{c}\sqrt{\mu_i Q_{ij} \cdot \mu_j Q_{ji}}\end{aligned}$$

and similarly

$$\gamma_{ij} = \sqrt{\frac{\mu_i Q_{ij}}{\mu_j Q_{ji}}}.$$

Now let's cut a wire. Take two vertices $x$ and $y$ whose connecting edge is reversible, so $\gamma_{xy} = 1$. This will form a new continuous-time Markov chain with rate matrix $Q^*$ where

- $Q^*_{xy} = Q^*_{yx} = 0$;
- $Q^*_{xx} = Q_{xx} + Q_{xy}$;
- $Q^*_{yy} = Q_{yy} + Q_{yx}$;
- $Q^*_{ij} = Q_{ij}$ for all other edges $ij$.

The jump chain of the new network has transistion matrix $P^*$ given by

- $P^*_{xy} = P^*_{yx} = 0$;
- $P^*_{xj} = -Q_{xj}/(Q_{xx} + Q_{xy})$ for $j \neq x, y$;
- $P^*_{yj} = -Q_{yj}/(Q_{yy} + Q_{yx})$ for $j \neq x, y$;
- $P^*_{ij} = P_{ij}$ for every other edge $ij$.

### 4.3 What is the stationary distribution of $Q^*$ and $P^*$?

Let's consider the stationary distribution of $Q^*$. We know $\boldsymbol{\mu}$ is the stationary distribution for $Q$, so for every state $j$, $\sum_i \mu_i Q_{ij} = 0$. Hence for $j \neq x, y$, $\mu_j = \sum_i \mu_i Q^*_{ij}$. For $j = x$,

$$\begin{aligned}
\sum_i \mu_i Q^*_{ix} &= \sum_{i \neq x,y} \mu_i Q^*_{ix} + \mu_x Q^*_{xx} + \mu_y Q^*_{yx} \\
&= \sum_{i \neq x,y} \mu_i Q_{ix} + \mu_x (Q_{xx} + Q_{xy}) \\
&= \sum_{i \neq x} \mu_i Q_{ix} + \mu_x Q_{xy}
\end{aligned}$$

This is equal to $\sum_i \mu_i Q_{ix}$ if and only if $\mu_x Q_{xy} = \mu_y Q_{yx}$. The same condition is obtained in the case where $j = y$. In words, after cutting an edge, the stationary distribution remains the same if and only if the original chain was reversible along the edge that was cut.

Assume $\mu_x Q_{xy} = \mu_y Q_{yx}$. Now we calculate the stationary distribution of $P^*$, say $\boldsymbol{\nu}^*$. Reusing the formula for $\boldsymbol{\nu}$ but for the cut network, $\nu_i^* = \mu_i Q_{ii}^* / (\sum_i \mu_i Q_{ii}^*)$. If $i \neq x, y$,

$$\nu_i^* = \frac{\mu_i Q_{ii}}{\sum_i \mu_i Q_{ii} + \mu_x Q_{xy} + \mu_y Q_{yx}},$$

$$\nu_x^* = \frac{\mu_x (Q_{xx} + Q_{xy})}{\sum_i \mu_i Q_{ii} + \mu_x Q_{xy} + \mu_y Q_{yx}},$$

and

$$\nu_y^* = \frac{\mu_y (Q_{yy} + Q_{yx})}{\sum_i \mu_i Q_{ii} + \mu_x Q_{xy} + \mu_y Q_{yx}}.$$

# 5 Using MATLAB to check for monotonicity with bistochasticity

We want to randomly generate bistochastic networks, and with the $\gamma_{xy}$ fixed on every edge, increase the resistances while remaining bistochastic. We can then check to see whether the effective resistance has also increased.

## 5.1 The code

```
n=6;

R_eff = zeros(trials,2);
trial = 1;
while trial≤    10 %<- Number of networks to test

    %STEP 1
    %Generate matrix G_{xy} = D_{xy}*gamma_{xy} whose rows and columns sum to Const

    Const=1;

    G=rand(n).*not(eye(n));
    G=Const*bistochastic(G,0.01,200);

    % STEP 2
    % Calculate the gamma_{xy} and the initial D_{xy}

    D_init = zeros(n);
    gamma = ones(n);
```

```
    for i = 1:n
        for j = 1:n
            D_init(i,j) = sqrt(G(i,j)*G(j,i));
            if i≠j
                gamma(i,j) = sqrt(G(i,j)/G(j,i));
            end
        end
    end

    % Re-index D into a vector and sort the gammas into a matrix V such that
    % D is orthonormal to the columns of V when the network is bistochastic

    D_vec = [zeros(n*(n-1)/2,1);Const];
    ind=0;
    V = [zeros(n*(n-1)/2,2*n); -ones(1,2*n)]; % V dot D_vec will be zero for ... each column of V
    for i=1:n-1
        for j=1+i:n
            ind = ind + 1;
            D_vec(ind) = D_init(i,j);
            for k = 1:n
                if k==i
                    V(ind,k) = gamma(k,j);
                    V(ind,k+n) = gamma(j,k);
                elseif k==j
                    V(ind,k) = gamma(k,i);
                    V(ind,k+n) = gamma(i,k);
                end
            end
        end
    end

    % Step 3

    % Now find orthonormal basis for S^orthogonal
    Q = GramSchmidt(V);
    Q(:,rank(V):end) = [];
    % Projector matrix onto S^orthgonal
    P_orth = Q*Q';

    % Step 4
    % P = I - P^orth is the projector matrix onto S
    P = eye(n*(n-1)/2+1)-P_orth;

    % Find up to n*(n-1)/2+1-rank(v) columns of P with only positive components
    F = zeros(n*(n-1)/2+1,n*(n-1)/2+1-rank(V));
    count = 0;
    for i = 1:n*(n-1)/2+1
        if all(P(:,i)>0)
            count = count + 1;
            F(:,count) = P(:,i);
        end
        if count == n*(n-1)/2+1-rank(V)
            break
        end
    end

    % If there are no positive columns of P, skip this network
    if count == 0
```

```
            continue
        end

        % Step 5
        % Increase D's along those positive vectors in some way

        alpha = 1*rand(n*(n-1)/2+1-rank(V),1);
        D_vec = D_vec + F*diag(alpha);

        % Put increased D's back into a matrix
        D = zeros(n);
        ind=0;
        for i=1:n-1
            for j=1+i:n
                ind = ind + 1;
                D(i,j) = D_vec(ind);
            end
        end
        D = D+D';
        lambda = gamma.^2;
        C_init = (1+lambda)./(2*gamma).*D_init;
        C_final = (1+lambda)./(2*gamma).*D;

        R_eff(trial,1) = double(calc_R_eff(C_init,lambda));
        R_eff(trial,2) = double(calc_R_eff(C_final,lambda));

        trial = trial + 1;

    end
    R_eff
```

### 5.2 The theory

We begin with the network on the complete graph with $n$ vertices. (Note that can by used as the general form of any network of $n$ vertices by setting the conductances to zero on each edge we don't want in the network.) From the Markovian and bistochastic properties, we conclude that at every vertex $x$ we have two equations:

$$\sum_{z\sim x} D_{xz}\gamma_{xz} = \sum_{z\sim x} D_{xz}\gamma_{zx} = K \tag{3}$$

where $K$ is some constant of our choosing. Setting $G_{xy} = D_{xy}\gamma_{xy}$, we see that this is equivalent to the matrix $G = (G_{xy})$ being bistochastic up to a constant.

- Step 1 is to generate a random bistochastic matrix $G$ with zeros on the diagonals.

- In step 2, we calculate the $D_{xy} = \sqrt{G_{xy} \cdot G_{yx}}$ and $\gamma_{xy} = \sqrt{G_{xy}/G_{yx}}$ for every edge. We reorganise the elements of $D$ (which is symmetric) into a $\binom{n}{2} + 1$ dimensional vector, with the constant $K$ in the last entry. Then we can choose $2n$ vectors that will be orthogonal to the $D$ vector when each of the $2n$ equations in (3) are true.

These vectors will consist of the $\gamma_{xy}$ and are contained the the matrix $V$ in the code.

- Now denote by $S$ the subspace of $\mathbb{R}^{\binom{n}{2}+1}$ that contains all vectors $(D_{xy}, K)$ satisfying (3), i.e. $S$ is the bistochastic subspace. We see that the columns of $V$ span the subspace that is orthogonal to $S$, which was by design from step 2. We'll denote this subspace by $S^{\perp}$.

  We use Gram Schmidt orthogonalisation to find an orthonormal basis for $S^{\perp}$. This new basis is stored in the matrix $Q$. Interestingly the rank of $V$ is smaller than $2n$.

  From $Q$ we can calculate the projector matrix of $S^{\perp}$, $P^{\perp} = QQ^T$, which projects any vector onto $S^{\perp}$.

- The projector matrix of $S$ is calculated by $P = I - P^{\perp}$, where $I$ is the identity matrix. The columns of $P$ then span the subspace $S$, and our next step is to look for vectors in $S$ with only non-negative components.

  We resort to simply looking at columns of $P$ – often enough at least one of the columns has only non-negative components. Interestingly this is almost always the last column of $P$. We input all such vectors into the matrix $F$.

- The final step is to increase $D$ along these positive vectors. We choose some random $\alpha$ vector telling us how far to move along these vectors from our starting point. With the $D$ vector increased, we reorganise it back into matrix form, and calculate the conductances $C_{xy}$ and the amplifiers $\lambda_{xy}$. This is then plugged into our effective resistance solver as seen in 2.3.

Since the $D_{xy}$ have increased, the resistances of the circuit have decreased, so we expect to see the effective resistance also decrease if our conjecture is correct. Having completed hundreds of trials at various values of $n$, we are yet to find a counterexample.