CS 70, Summer 2014 — Homework 5

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July 31, 2014

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Sources:

Problem 1

(a) If we have two decks of 52 cards we have 104! ways to order if we ignore repeats. Since there are 2 decks, we have 2 of each card so we have to divide by 2! for every unique card since the order of the repeats don't matter.

$$\frac{104!}{(2!)^{52}} = \frac{104!}{2^{52}}$$

(b) If we have a binary string that is of length 66. it must have 34 bits that is a "1" for it to have a majority of ones. So what we can do is find all the ways to set up the string and substract the ones that have an equal amount of 0's and 1's. Then we can divide by 2 because half would be majority 0's.
The amount of equal bit strings would have 33 o's and 33 1's. so $\binom{66}{33}$

$$\frac{2^{66} - \binom{66}{33}}{2}$$

(c) We have 8 balls and 24 bins. So for each ball we have 24 bins to place the ball in. So each ball will have 24 choices.
24^8 ways

(d) We can use the star and bars method. There would be 23 bars and 8 stars. So we would have $(23 + 8)!$ ways to arrange the string and we would have 23 and 8 repeats of bars and stars. So we have

$$\frac{31!}{23! \times 8!} = \binom{31}{8}$$

(e) If we place 1 ball into each bin we statisfy the requirement for each of the 5 bins having at least 1 ball. We now have 3 balls left. We can now place each one of these balls in any of the bins. So each ball has 5 choices. $5^3 = 125$ choices

(f) We have 30 students which is 15 pairs. Lets first line up the 30 students. There are 30! ways to line them up. We then pair up Student 1 and 2, 3 and 4 and so forth. Since we don't care if student 1 is paired with 2 or if 2 is paired with 1. We divided by 2! for each pair. Since we also don't care the order of when the pairs were picked we also divide by 15!.
$\dfrac{30!}{15! \times 2^{15}}$ ways

Problem 2

(a) if we have p beads that could be at most k colors with the string having more than 1 color that would mean that we have k choices for each of the p beads or k^p and since we can't have the same color for all the string we have to substract k ways for the string have all the same color. $k^p - k$ ways.

(b) From the problem we can see that every necklace is p shifts from being identical to itself meaning that $(p-1)$ necklaces are equivalent to it. This can be said for all necklaces then. Every necklace has $(p-1)$ equivalent necklaces that can be shifted to obtain. This would mean that the amount of non-equivalent necklaces is just the total amount of ways to make a necklace divided by p or

$\frac{k^p - k}{p}$ non-equivalent necklaces.

(c) let $a = k$ or the amount of colors which can't be 0 since you can't have 0 colors in a necklace. We know that the amount of ways we can make a unique necklace is

$$\frac{k^p - k}{p} \Rightarrow k^p - k = p \times n$$

which shows that the amount of ways to make a unique necklace is divisible by p. If we take the modulo of both sides we get
$k^p - k = p \times n \bmod p$
$k^p - k \equiv 0 \bmod p$
$k^p \equiv k \bmod p$
$k^{p-1} \equiv 1 \bmod p$
$a^{p-1} \equiv 1 \bmod p$

Problem 3

(a) Let's say we have two groups, A with n people and B with m people and then we choose a people from A and b from B. The amount of ways we do this is less than or equal to if we just combined A and B and chose $a+b$ people. If we pick $a+b$ people from A and B then for each choice everyone from A and B is considered giving us much more choices rather than choosing just from A or just from B.

(b) Let's have a bin N with n balls. Let's choose a balls from N and place them in a bin A. N now has $n-a$ balls. After this we then choose $b-a$ balls from N and place them in bin B. Bin N now has $n-a-(b-a)$ or $n-b$ balls and Bin B has $b-a$ balls.
N: $n-b$ balls
A: a balls
B: $b-a$ balls

Let's also consider if bin N has n balls and then we choose b balls from it and place it in B. N now has $n-b$ balls and B has b balls. Then from B we choose a balls and place it in A. B now has $b-a$ balls and A has a balls.
N: $n-b$ balls
A: a balls
B: $b-a$ balls
These are equivalent arguments

(c) Let's say we have a village of hunters and gatherers with n people. We need to seperate the village into hunters and gatherers and each one needs a leader. First let's choose a people from the village to be gatherers. $\binom{n}{a}$ We now have a gatherers and $n-a$ hunters. We now choose 1 from each group to be the leader of their respective groups. $\binom{a}{1}\binom{n-a}{1} = a(n-a)$
Total ways: $a(n-a)\binom{n}{a}$
This would be equivalent to if we chose a hunting leader from n people $\binom{n}{1}$and then chose a gathering leader from $n-1$ people. $\binom{n-1}{1}$ We then choose the rest of the gathering group that aren't leaders $(a-1)$ from $n-2$ people. $\binom{n-2}{a-1}$
Total Ways: $\binom{n}{1}\binom{n-1}{1}\binom{n-2}{a-1} = n(n-1)\binom{n-2}{a-1}$

Problem 4

(a) $\binom{2n}{2} = 2\binom{n}{2} + n^2$

$$\frac{2n!}{(2n-2!)2!} = \frac{2n \times (2n-1)}{2!} = \frac{4n^2-2n}{2!} = \frac{2n^2+2n^2-2n}{2!} = \frac{2n^2-2n}{2!} + n^2 =$$
$$2 \times \frac{n^2-n}{2!} + n^2 = 2 \times \frac{n(n-1)}{2!} + n^2 = 2 \times \frac{n!}{(n-2)!2!} + n^2 = 2\binom{n}{2} + n^2$$

(b) Let's say we have a group A with $2n$ people in a line and we want to choose 1 pair from the group. This would be $\binom{2n}{2}$. Now let's say we spilt the line in half so each side of the line would have n people. We can now choose 2 people from the left side of the line. $\binom{n}{2}$ 2 people from the right side $\binom{n}{2}$ or one from each side. $\binom{n}{1}\binom{n}{1}$. Now let's add up all these possibilities
$\binom{n}{2} + \binom{n}{2} + \binom{n}{1}\binom{n}{1} = 2\binom{n}{2} + n^2$
So the amount of ways you could select 2 people from $2n$ is equivalent to $2\binom{n}{2} + n^2$

Problem 5

Lets first calculate the amount of ways we can get 4 distinct numbers on the dice.
Out of 6 numbers lets choose 4 to be the distinct numbers, hence $\binom{6}{4}$. So we can either have the 2 repeating numbers to be different numbers or the same giving us 2 2-of-a-kinds or a single 3-of-a-kind. The amount of ways we can have 6 numbers in which two are distinct pairs is $\binom{4}{2}\frac{6!}{2!2!}$. The amount of ways we can have 6 numbers in which there is one three-of-a-kind is $\binom{4}{1}\frac{6!}{3!}$. Since we can have either of theses we add them up to one another.

amount of ways to have exactly 4 numbers $= \binom{6}{4}\left(\binom{4}{2}\frac{6!}{2!2!} + \binom{4}{1}\frac{6!}{3!}\right) = 15 \times (6 \times 180 + 4 \times 120)$

$$= \frac{23400}{6^6} = 50.15\%$$

I would play this game. In the long run I would win more money due to the fact that the win chance is greater than 50%.

Problem 6

(a)

Box 1: $P = 1/4$
Red, Blue $= (1/4) \times (2/3) \times (1/2) = 1/12$
Red, Red $= (1/4) \times (2/3) \times (1/2) = 1/12$
Blue, Blue $= (1/4) \times (1/3) \times (0/2) = 0$
Blue, Red $= (1/4) \times (1/3) \times (2/2) = 1/12$

Box 2: $P = 3/4$
Red, Blue $= (3/4) \times (1/3) \times (2/3) = 1/4$
Red, Red $= (3/4) \times (1/3) \times (0/2) = 0$
Blue, Blue $= (3/4) \times (2/3) \times (1/2) = 1/4$
Blue, Red $= (3/4) \times (2/3) \times (1/2) = 1/4$

(b) P[balls have different colors] = P[Red,Blue] + P[Blue,Red]
P[balls have different colors] = [1/12 + 1/4] + [1/12 + 1/4]
P[balls have different colors] = 8/12 = 2/3

Problem 7

If treat the 3 discrete math books as 1 unit we then have 10 books with 4 about COBOL and 5 about underwater basket weaving. So the amount of ways to arrange the books with the discrete math books always together is

$\dfrac{10!}{5! \times 4!} = 1260$ ways

And the amount of ways to arrange the books normally is

$\dfrac{12!}{3!4!5!}$ so the probability is

$$\frac{\frac{10!}{5!4!}}{\frac{12!}{3!4!5!}} = \frac{3!}{12 \times 11} = \frac{1}{22} = 4.54\%$$

Problem 8

(a) If you roll a dice and then another dice, the probability to roll that the same number is 1/6.

(b) The possiblities for dice being rolled and a sum of 4 or less if produced is
(1,1) (1,2) (1,3)
(2,1) **(2,2)**
(3,1)
so the probabilty for doubles such that a sum of 4 or less was produced is $(2/6) = (1/3)$.

(c) if our two dice land on different numbers then it had to have been 6 ways for the first dice to land and 5 ways for the second dice. So there is 30 ways the dice could have made different numbers. Next we calculate the number of ways 6 could have appeared. If the first dice resulted in a 6 then there are 5 ways. If the second dice was a 6 then there are also 5 ways. Since the events are independant the total ways that we could have a 6 is 10. So the probabilty of 6 appearing when we have two different dice outcomes is 10/30 or 1/3.

Problem 9

(a) After we add the \$5 bill to the bag and shake it up, we now assume a randomized bill pull. We know that the probability of both of the bills being a 5 is 1/3 due to the fact that we know that one of the bills is a 5.
The probabilty that we pull out a 5 for our first draw is the probabilty of us pulling a bill times the probabilty of it being a 5.

$P(First\ 5) = 1/2 \times 1/1 + 1/2 \times 1/3 = 2/3$
$P(Both\ 5) = 1/3$
$P(Second\ 5 \mid First\ 5) = \dfrac{P(Both\ 5)}{P(First\ 5)} = \dfrac{1/3}{2/3} = 1/2$

(b) We have a total of 6 sides, 3 are heads and 3 are tails and we have a 50-50 chance of pulling either. So when we find out we have heads there are 1 in 2 ways this can happen. 1 way is that we got either of the 2 sides of the HH coin each with a probabilty of 1/3 giving us a total of 2/3 of getting the HH coin. The other way is that we got the Head side of the normal coin with a probability of 1/3. So if we got the HH coin with a probability of 2/3 then our chance of getting another Heads is 1/1. So the chance of this is 2/3. If we got the heads side of the normal coin then our probabilty for another heads is 0.
The total probability of getting another heads is 2/3.