CS 70, Summer 2014 — Homework 1

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Sources: http://comet.lehman.cuny.edu/sormani/teaching/induction.html

Problem 1

Table 1: $P \wedge (Q \vee P) \equiv P \wedge Q$

P	Q	$Q \vee P$	$P \wedge (Q \vee P)$	$P \wedge Q$
T	T	T	T	T
T	F	T	T	F
F	T	T	F	F
F	F	F	F	F

(a) *Not Equivalent*

Table 2: $(P \Rightarrow Q) \Rightarrow R \equiv P \Rightarrow (Q \Rightarrow R)$

P	Q	R	$(P \Rightarrow Q)$	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$	$(Q \Rightarrow R)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	F	F	T	F	T	T
F	F	T	T	T	T	T
F	T	F	T	F	T	F
F	T	T	T	T	T	T

(b) *Not Equivalent*

Table 3: $(P \Rightarrow Q) \Rightarrow (P \Rightarrow R) \equiv P \Rightarrow (Q \Rightarrow R)$

P	Q	R	$(P \Rightarrow Q)$	$(P \Rightarrow R)$	$(P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$	$(Q \Rightarrow R)$	$P \Rightarrow (Q \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	F	T	T	T
F	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	T	F	T	T	T	F	T
F	T	T	T	T	T	T	T

(c) *Equivalent*

Table 4: $(P \wedge \neg Q) \Leftrightarrow (\neg P \vee Q) \equiv (Q \wedge \neg P) \Leftrightarrow (\neg Q \vee P)$

P	Q	$(P \wedge \neg Q)$	$(\neg P \vee Q)$	$(P \wedge \neg Q) \Leftrightarrow (\neg P \vee Q)$	$(Q \wedge \neg P)$	$(\neg Q \vee P)$	$(Q \wedge \neg P) \Leftrightarrow (\neg Q \vee P)$
T	T	F	T	F	F	T	F
T	F	T	F	F	F	T	F
F	T	F	T	F	T	F	F
F	F	F	T	F	F	T	F

(d) *Equivalent*

Problem 2

(a) (I) $\forall x(P(x) \Rightarrow B(x))$

(II) $\forall x(U(x) \Rightarrow \neg F(x))$

(III) $\forall x(O(x) \Rightarrow N(x))$

(IV) $\forall x(B(x) \Rightarrow F(x))$

(V) $\forall x(K(x) \Rightarrow P(x))$

(VI) $\forall x(N(x) \Rightarrow U(x))$

(b) (I) $\forall x(\neg B(x) \Rightarrow \neg P(x))$

(II) $\forall x(F(x) \Rightarrow \neg U(x))$

(III) $\forall x(\neg N(x) \Rightarrow \neg O(x))$

(IV) $\forall x(\neg F(x) \Rightarrow \neg B(x))$

(V) $\forall x(\neg P(x) \Rightarrow \neg K(x))$

(VI) $\forall x(\neg U(x) \Rightarrow \neg N(x))$

(c) If a person wears kid gloves, they go to a party, brush their hair, look fascinating, are tidy, have self control, and aren't opium-eaters.

$\forall x(K(x) \Rightarrow P(x) \Rightarrow B(x) \Rightarrow F(x) \Rightarrow \neg U(x) \Rightarrow \neg N(x) \Rightarrow \neg O(x))$

Problem 3

(a) $\forall x \exists y\ (xy \geq x^2)$ True
$Case\ 0 : x = 0 \rightarrow 0 \geq 0$
$Case\ 1 : \forall x \in \mathbb{R}^+\ \exists y\ ((y \geq x) \Rightarrow (xy \geq x^2))$
$Case\ 2 : \forall x \in \mathbb{R}^-\ \exists y\ ((y \leq x) \Rightarrow (xy \geq x^2))$

(b) $\exists y \forall x\ (xy \geq x^2)$ False
$\forall x \in \mathbb{R}\ (x^2 \geq 0)$
$Case\ 0 : y = 0, \forall x \neq 0\ (0 < x^2) \Rightarrow \neg(xy \geq x^2)$
$Case\ 1 : \exists y \in \mathbb{R}^+\ \forall x\ ((x < 0) \Rightarrow (xy < 0) \Rightarrow \neg(xy \geq x^2))$
$Case\ 2 : \exists y \in \mathbb{R}^-\ \forall x\ ((x > 0) \Rightarrow (xy < 0) \Rightarrow \neg(xy \geq x^2))$

(c) $\neg\forall x \exists y\ (xy > 0 \Rightarrow y > 0)$ False
$\neg\forall x \exists y\ (\neg(xy > 0) \vee (y > 0))$
$\exists x \forall y\ \neg(\neg(xy > 0) \vee (y > 0))$
$\exists x \forall y\ ((xy > 0) \wedge \neg(y > 0))$
$\exists x \forall y\ ((xy > 0) \wedge (y \leq 0))$

False whenever y is greater than 0 ($y > 0$) which disagrees with $\forall y$

Problem 4

(a) $\neg\forall x\ \exists y\ (P(x) \Rightarrow \neg Q(x,y)) \equiv \exists x\ \forall y\ (P(x) \wedge Q(x,y))$ Original Statement
$\exists x\ \forall y\ \neg(P(x) \Rightarrow \neg Q(x,y)) \equiv$ factor in negation
$\exists x\ \forall y\ \neg(\neg P(x) \vee \neg Q(x,y)) \equiv$ Implication to Or
$\exists x\ \forall y(P(x) \wedge Q(x,y)) \equiv$ Demorgan's Law

Equivalent

(b) $\forall x\ \exists y\ (P(x) \Rightarrow Q(x,y)) \equiv \forall x\ (P(x) \Rightarrow (\exists y\ Q(x,y)))$ Original Statement
$\forall x\ \exists y\ (\neg P(x) \vee Q(x,y)) \equiv$ Implication to Or
$\forall x\ ((\exists y\ \neg P(x)) \vee (\exists y\ Q(x,y))) \equiv$ Distribution of quantifier
$\forall x\ (\neg(\forall y\ P(x)) \vee (\exists y\ Q(x,y))) \equiv$ Factor out negation
$\forall x\ ((\forall y\ P(x)) \Rightarrow (\exists y\ Q(x,y))) \equiv$ Or to Implication
$\forall x\ (P(x) \Rightarrow (\exists y\ Q(x,y))) \equiv$ Obvious

Equivalent

(c) $\forall x\ \exists y\ (Q(x,y) \Rightarrow P(x)) \equiv \forall x\ (\exists y\ Q(x,y) \Rightarrow P(x))$ Original Statement
$\forall x\ \exists y\ (\neg Q(x,y) \vee P(x)) \equiv$ Implication to Or
$\forall x\ ((\exists y\ \neg Q(x,y)) \vee (\exists y\ P(x))) \equiv$ Distribution of quantifier
$\forall x\ (\neg(\forall y\ Q(x,y)) \vee (\exists y\ P(x))) \equiv$ Factor out negation
$\forall x\ ((\forall y\ Q(x,y)) \Rightarrow (\exists y\ P(x))) \equiv$ Or to Implication
$\forall x\ ((\forall y\ Q(x,y)) \Rightarrow P(x)) \equiv$ Obvious
$\forall x\ ((\forall y\ Q(x,y)) \Rightarrow P(x)) \not\equiv \forall x\ (\exists y\ Q(x,y) \Rightarrow P(x))$ Invalid

Not Equivalent

Problem 5

(a) $\forall n \in \mathbb{N}\ (n\ odd \Rightarrow n^2 + 2n\ odd)$ Original Statement
Assume n is odd
$\forall n \in \mathbb{N}\ \exists k \in \mathbb{Z}\ (n\ odd \Rightarrow n = 2k + 1)$ Definition of an odd number
$\forall n \in \mathbb{N}\ \exists d \in \mathbb{Z}\ (n^2 + 2n = 2d + 1 \Rightarrow n^2 + 2n\ odd)$ $n^2 + 2n$ is odd if d exists
Substitution
$(2k + 1)^2 + 2(2k + 1) = (4k^2 + 4k + 1) + 4k + 2 = (4k^2 + 8k + 2) + 1 = 2(2k^2 + 4k + 1) + 1$
$d = 2k^2 + 4k + 1$ d Exists

True, direct proof.

(b) $\forall n \in \mathbb{N}\ (n^2 + 7n + 1\ odd)$ Original Statement
Case 1: *n is odd*
$\forall n \in \mathbb{N}\ \exists k \in \mathbb{Z}\ (n\ odd \Rightarrow n = 2k + 1)$ Definition of an odd number
$\forall n \in \mathbb{N}\ \exists d \in \mathbb{Z}\ (n^2 + 7n + 1 = 2d + 1 \Rightarrow n^2 + 7n + 1\ odd)$ $n^2 + 7n + 1$ is odd if d exists
Substitution
$(2k + 1)^2 + 7(2k + 1) + 1 = (4k^2 + 18k + 8) + 1 = 2(2k^2 + 9k + 4) + 1$
$d = (2k^2 + 9k + 4)$ d Exists

$Therefore\ \forall n \in \mathbb{N}\ (n\ odd \Rightarrow n^2 + 7n + 1\ odd)$ $(n^2 + 7n + 1)$ is odd whenever n is odd

Case 2: *n is even*
$\forall n \in \mathbb{N}\ \exists k \in \mathbb{Z}\ (n\ even \Rightarrow n = 2k)$ Definition of an even number
$\forall n \in \mathbb{N}\ \exists d \in \mathbb{Z}\ (n^2 + 7n + 1 = 2d + 1 \Rightarrow n^2 + 7n + 1\ odd)$ $n^2 + 7n + 1$ is odd if d exists
Substitution
$(2k)^2 + 7(2k) + 1 = (4k^2 + 14k) + 1 = 2(2k^2 + 7k) + 1$
$d = (2k^2 + 7k)$ d Exists

$Therefore\ \forall n \in \mathbb{N}\ (n\ even \Rightarrow n^2 + 7n + 1\ odd)$ $(n^2 + 7n + 1)$ is odd whenever n is even

True, Proof by Cases

(c) $\forall\ a, b \in \mathbb{R}\ (a + b \leq 10 \Rightarrow (a \leq 7 \vee b \leq 3))$ Original Statement
$\forall\ a, b \in \mathbb{R}\ (\neg(a \leq 7 \vee b \leq 3) \Rightarrow \neg(a + b \leq 10))$ Contrapositive
$\forall\ a, b \in \mathbb{R}\ ((\neg(a \leq 7) \wedge \neg(b \leq 3)) \Rightarrow \neg(a + b \leq 10))$ Demorgan's Law
$\forall\ a, b \in \mathbb{R}\ (((a > 7) \wedge (b > 3)) \Rightarrow (a + b > 10))$ Applied Negation

True, Proof by Contrapositive

(d) $\forall r \in \mathbb{R}\ (r\ irrational \Rightarrow r+1\ irrational)$ Original Statement
$Assume\ \neg(r+1\ irrational) \equiv (r+1\ rational)$
$\forall r \in \mathbb{R}\ \exists\ a, b \in \mathbb{Z}\ (r+1\ rational \Leftrightarrow r+1 = \frac{a}{b})(b \neq 0)$ definition of a rational number
$r = \frac{a}{b} - 1 = \frac{a}{b} - \frac{b}{b} = \frac{a-b}{b}$ Solving for r which gives a rational number

Assuming $\neg(r+1\ rational)$ r is rational which is a proof by contrapositive

True, Proof by Contrapositive

(e) $\forall n \in \mathbb{N}\ (10n^2 > n!)$ Original Statement
Case 1: $n = 6$
$10(6)^2 > 6!$
$360 \not> 720$ Invalid

False, Proof by Counterexample

Problem 6

(a) $\forall n \geq 1 \ \sum_{i=1}^{n} \frac{1}{i(i+1)} = \dfrac{n}{n+1}$ Original Statement

$Proof$: By induction on n

$Base\ case$: $n = 1 \rightarrow \frac{1}{1(1+1)} = \frac{1}{1+1} = \frac{1}{2}$ True

$Inductive\ Hypothesis$: $\forall k \geq 1 \ \sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$

$Let\ n = k+1$

$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{(k+1)+1}$ Substitute

$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{(k+1)}{(k+2)}$ Simplify

$\sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} =$ Expand series

$\frac{k}{(k+1)} + \frac{1}{(k+1)(k+2)} =$ Substitute inductive hypothesis

$\frac{(k^2+2k)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} =$ Cross multiply

$\frac{(k^2+2k+1)}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} =$ Simplify and factor

$\frac{(k+1)}{(k+2)} = \frac{(k+1)}{(k+2)}$ Valid

(b) $\forall n \in \mathbb{N}\ (5|(8^n - 3^n))$ Original Statement

$Proof$: By induction on n

$Base\ case$: $(n = 0) \rightarrow (5|(8^0 - 5^0)) = (5|0)$ True

$Inductive\ hypothesis$: $\forall k \in \mathbb{N}\ (5|(8^k - 3^k))$

$let\ n = k+1$

$(5|(8^{k+1} - 3^{k+1})) = (5|(8 \times 8^k - 3 \times 3^k))$ Expand

$(5|(8 \times 8^k - 3 \times 3^k)) = (5|((8 \times 8^k) - (8 \times 3^k) + (8 \times 3^k) - (3 \times 3^k)))$

$(5|(8 \times 8^k - 3 \times 3^k)) = (5|(8 \times (8^k - 3^k) + 5 \times 3^k))$ Group factors

Our hypothesis states that 5 divides $(8^k - 3^k)$ so $\exists d \in \mathbb{Z}\ ((8^k - 3^k) = 5d)$

$(5|(8 \times 8^k - 3 \times 3^k)) = \exists d \in \mathbb{Z}\ (5|(8 \times 5d + 5 \times 3^k))$

$(5|(8 \times 8^k - 3 \times 3^k)) = \exists d \in \mathbb{Z}\ (5|5(8d + 3^k))$ Valid

Problem 7

$\forall r > 0\ \forall k > 0\ \exists m\ (\frac{1}{n_1} + ... + \frac{1}{n_k} = r)$ where m is the number of ways to make r with k and $n_1, ..., n_k$ are some positive integers

allow $m = f(r, k)$ to be a function that takes in a r and k and returns m (amounts of ways to make r with k elements)

$Base\ Case:\ (k = 1)$
$f(r, 1) \rightarrow \frac{1}{\frac{1}{r}} = r$ $\qquad n_1 = \frac{1}{r}$
$f(r, 1) = 1$ $\qquad m$ exists

$Proof$: Induction on k

$Inductive\ Hypothesis$: $\exists m\ (\frac{1}{n_1} + ... + \frac{1}{n_k} = r)$ where m is the number of ways to make r with k and $n_1, ..., n_k$ are some positive integers

$let\ k = k + 1$

$(\frac{1}{n_1} + ... + \frac{1}{n_k} + \frac{1}{n_{k+1}} = r)$ where $\frac{1}{n_{k+1}}$ is the largest term.

$r \leq (k + 1) \times \frac{1}{n_{k+1}}$
$\frac{r}{k+1} \leq \frac{1}{n_{k+1}}$ $\qquad$ Divide both sides by $(k + 1)$

$\frac{k+1}{r} \geq n_{k+1}$ $\qquad$ Inverse

This shows that n_{k+1} is bounded by $\frac{k+1}{r}$ and therefore finite.
this implies $(\frac{1}{n_1} + ... + \frac{1}{n_k} = r - \frac{1}{n_{k+1}})$ where $\frac{1}{n_{k+1}}$ and r are finite which produce $r\prime$ also finite.

$(\frac{1}{n_1} + ... + \frac{1}{n_k} = r\prime)$ $\qquad$ Valid

Problem 8

Postponed!!

Problem 9

(a) Incorrect, another base case is needed
If we add another base case where $n = 1$ then
Suppose $n = 1$. If $\max(x, y) = 1$ and $x, y \in \mathbb{N}$, then $x = 1 \vee y = 1$
hence $(x \leq y) \vee (x > y)$
This shows that the claim is false due to the fact that $\max(x, y) = n$ is true even when $x > y$

(b) Incorrect, The inductive step did not prove $n + 1 < 2^{n+1}$.

(c) Incorrect, Proof by contrapositive would lead you to assume that $n^2 + 1$ is not a multiple of 3 which would imply that $2n + 1$ is not a multiply of three. This example however did a proof of converse which isn't enough to prove the claim.