This short description gives context to my questions at the bottom. We wish to solve the ODE

$$\frac{dq}{dt} = P_{V(q)}(U(q))$$

where q are the positions of the circles, V the set of allowed velocities and $U(q)$ is a vector of desired velocities.

Recent results show that [2] if, for each time t, the constraint set $Q(t)$ of allowed positions satisfies a geometric condition that guarantees local well-definedness of the projection P_Q, then there exists a unique, absolutely continuous solution to the above ODE. The condition is prox regularity. Define the projection onto closed set S as

$$P_S(y) = \{z \in S \mid d_S(y) = |y - z|\}$$

then the proximal normal cone $N(S, x)$ at x is

$$N(S, x) = \{v \mid \text{ there exists } \alpha > 0, \quad P_S(x + \alpha v)\}.$$

Finally,

Definition 0.1. *Let $S \subset \mathbb{R}^n$ be closed. Then S is η-prox-regular if for all $y \in S, x \in \partial S$ and unit vector $v \in N(S, x)$,*

$$B(x + \eta v, \eta) \cap S = \emptyset$$

The key result of [4] is that Q is prox regular. They verify a standard regularity condition for the C^2 constraint functions $f_i(q)$ that define Q. Namely, the gradients $\nabla_q f_i(q)$ must be positively linearly independent.

Thus the key ingredient both for the theory and numerics is a description of the gradient vectors $\nabla_q f_i(q)$. In the case of circles, this is easy to compute directly from the distance function $f_i(q) = |q_i - q_j| - 2R$ or geometrically: moving two circles apart on a line through their centers always increases the distance between them so the gradient points in this direction.

The distance between two ellipses requires solving a quartic [1], so an explicit formula for the gradient is difficult. A good model problem is the distance between an point and an ellipse (though this still requires solving a quadratic equation [3]). Let $d(g)$ be the distance between an ellipse with configuration $g \in SE(2)$ (orientation preserving rotations and translations) and the origin. Then how do we compute compute $\nabla_g d(g)$ if we don't have an explicit formula for d? More generally, what does the 'gradient' mean on a Lie group? Which directional derivative points in the direction of steepest ascent?

References

[1] Y-K Choi, Wenping Wang, Yang Liu, and M-S Kim. Continuous collision detection for two moving elliptic disks. *IEEE Transactions on Robotics*, 22(2):213–224, 2006.

[2] Jean Fenel Edmond and Lionel Thibault. Relaxation of an optimal control problem involving a perturbed sweeping process. *Mathematical programming*, 104(2-3):347–373, 2005.

[3] L. MAISONOBE. Quick computation of the distance between a point and an ellipse. `http://www.spaceroots.org/documents/distance/distance-to-ellipse.pdf`, 2003. [Online; accessed March 6, 2019].

[4] Bertrand Maury and Juliette Venel. A discrete contact model for crowd motion. *ESAIM: Mathematical Modelling and Numerical Analysis*, 45(1):145–168, 2011.