MODELING

Value function
Q-factor
Optimality equation

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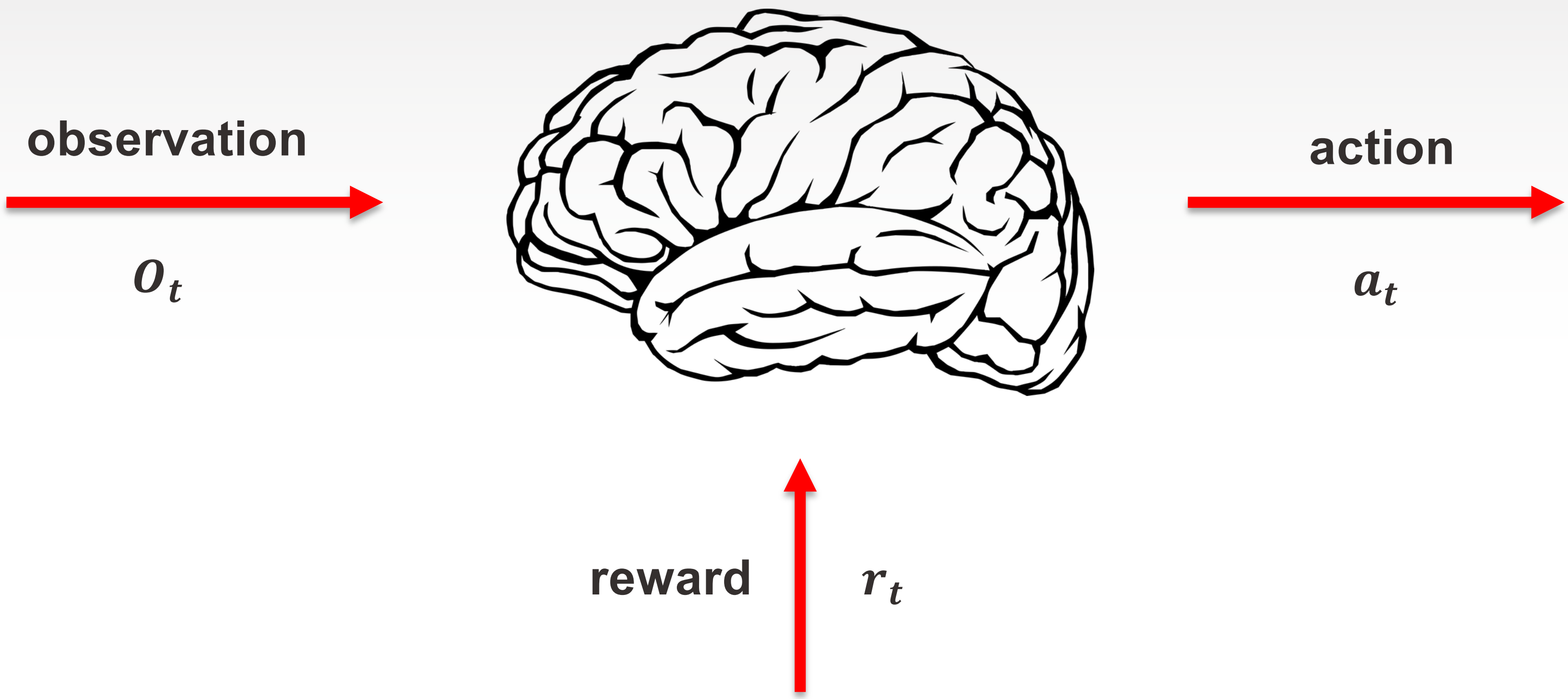
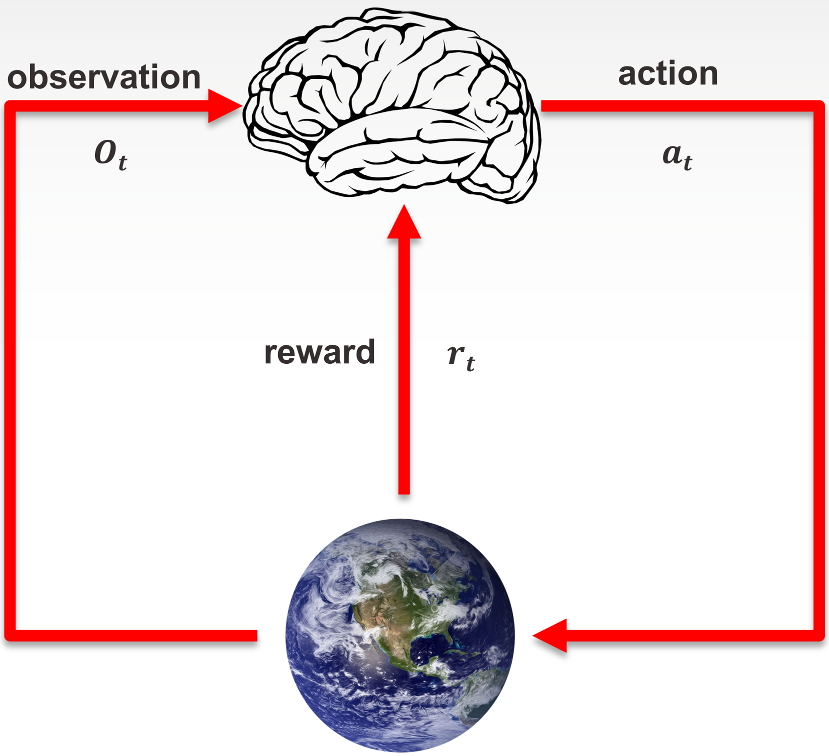
TRANSITION FUNCTION AND POLICIES

Agent and Environment



Agent and Environment



- At each step t the agent
  - Executes action a_t
  - Receives observation O_t
  - Receives scalar reward r_t
- The environment
  - Executes action a_t
  - Receives observation O_{t+1}
- t increments

History and State

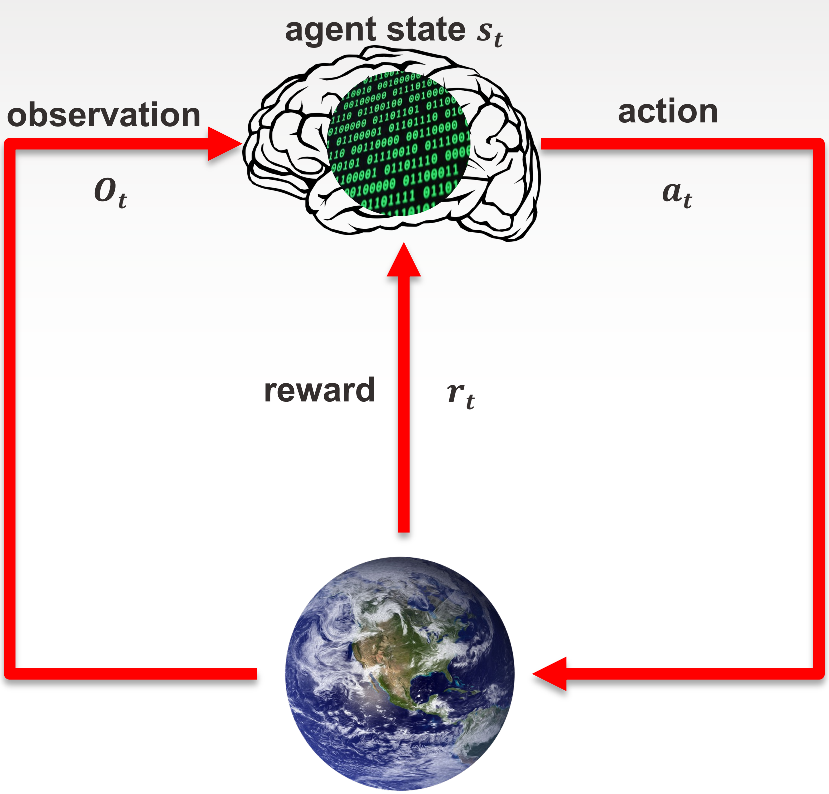
- The history is the sequence of observations, actions, rewards

$$H_t = O_0, r_0, a_0, \dots, a_{t-1}, O_t, r_t$$

- What happens next depends on the history
  - The agent selects actions
  - The environment selects observations
- State is the information used to determine what happens next
- State a function of the history

$$s_t = f(H_t)$$

Agent State



- Agent state s_t is the agent's internal representation
  - Whatever information the agent uses to pick the next action
  - Or/and to calculate reward
- Information used by reinforcement learning algorithms

Rewards

- Reward r_t is a scalar feedback signal
- Indicates how well agent is doing at step t
- The agent's job is to maximize cumulative reward
- All goals described by the maximization of expected cumulative reward

State

- Information state contains all useful information from the history
  - Markov state
- State s_t is Markov if and only if

$$\mathbb{P}[s_{t+1} \mid s_t] = \mathbb{P}[s_{t+1} \mid s_0, \ldots, s_t]$$

  - "The future is independent of the past given the present"
- The history H_t is Markov
  - The state is a sufficient statistic of the future

State Space

- State space $\mathcal{S}$
  - Set of all possible states
- Discrete or continuous
  - Usually discrete and finite
  - Continuous
    - Discretize
- Challenge
  - In practice extremely high cardinality
- Inventory management: 0, 1, 2, 3, …
  - 3 SKU's
    - $\mathbb{Z}\ x\ \mathbb{Z}\ x\ \mathbb{Z}$
- Drone: All possible x,y,z coordinates
  - Continuous but discretized in practice
- Tic-tac-toe: $\{-1,0,1\}^9$

Actions

- $\mathcal{A}$ – action space
- Usually discrete and finite
  - If continuous, discretize
- $a_t \in \mathcal{A}$
- Goal of RL: find a_t for each time *t*
- Depends on state
  - $a_t = a_t(s_t)$

- Inventory
  - $\mathcal{A} = \{0,1,2,\ldots,M\}$
- Drone
  - $\mathcal{A}$ = {stay, up, down, left, right, …}
  - 27 options
- Tic-tac-toe
  - $\mathcal{A} = \{1,2,\ldots,9\}$

Markov Chains

- s_{t+1} depends on s_t
- Key to identify $p(s_{t+1}|s_t)$
- Inventory
  - $p(s_{t+1} = 10|s_t = 5)???$
  - Ordered 5, demand = 0
  - Ordered 8, demand = 3
  - If we know action, we can calculate the transition probability

$$p(s_{t+1}|s_t, a_t)$$

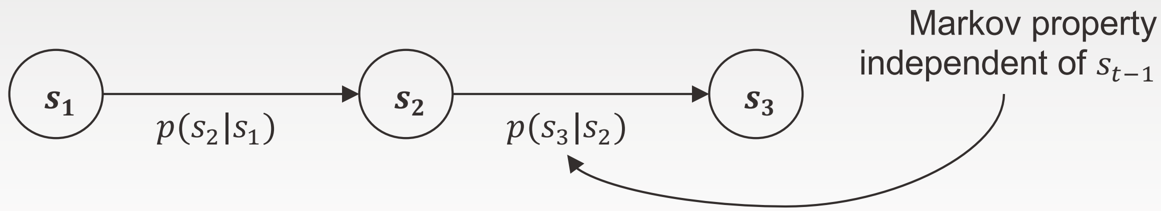
  - $p(s_{t+1} = 10|s_t = 5, a_t = 8) = p(D = 3)$

- Markov chains *P*
- Transition matrix in RL
  - P^a
- Every action function gives a transition function

Transitions

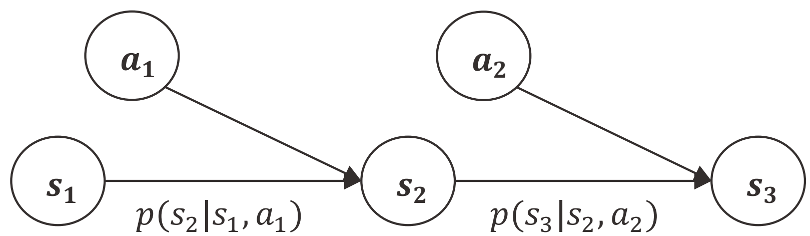
$$P_{ij} = p(s_{t+1} = i | s_t = j)$$

matrix



$$P_{ij}^{k} = P_{ijk} = p(s_{t+1} = i | s_t = j, a_t = k)$$

tensor



Policy

- Permissible family of action functions
  - Denoted by π
  - Policy again function of state
- Restrict action functions
  - Computational tractability
  - Business and operational requirements
    - Order only 0 or Q
- Policy usually parametric π_θ
  - Goal to find best set of parameters

$$\pi_\theta(s_t) = \theta^T \cdot s_t$$

Better: neural network

Stochastic Policies

- Flip a coin based on certain probability to select an action
- Find probability distribution over all possible actions
- Drone
  - 27 dimensional probability vector
  - Neural networks
    - Input state vector
    - Output 27 dimensional probability
      - Obtained through softmax
- Continuous actions
  - Normal distribution with mean and standard deviations depending on state

Stochastic Policies

- Deterministic/pure policy special stochastic policy
- Stochastic policies more general
- Algorithmically much better to handle
  - Continuous in parameters
  - Differentiable in parameters
- Can be mathematically shown
  - Given a stochastic policy there is a deterministic policy of the same reward
- On the paper stochastic policies do not yield anything new

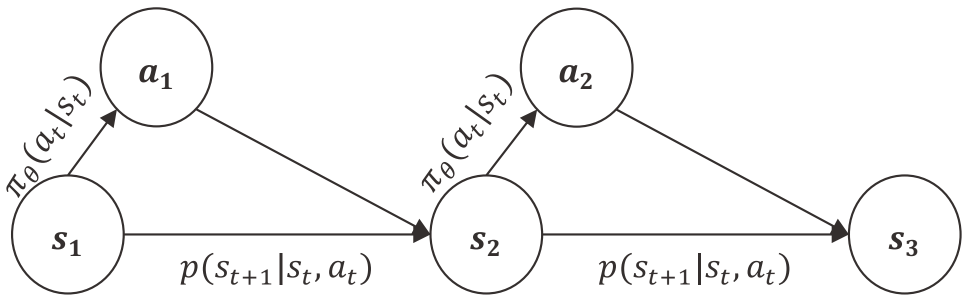
Objective Function

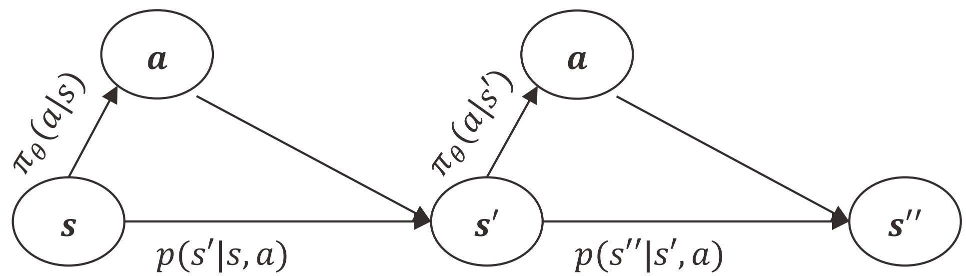
- Find policy that maximizes total reward
  - $\max_{\pi} E[\sum_{t=0}^{T} \gamma^t \cdot r(s_t, a_t)]$
- Optimization to find a function
- Future reward is discounted
  - Future money is discounted
- $0 < \gamma \leq 1$
  - Well defined also for $T = \infty$

$$s_0, \pi(s_o), r_0, s_1, \pi(s_1), r_1, s_2, \pi(s_2), r_2, s_3, \pi(s_3), r_3, \ldots$$

Infinite case

- Drop t
  - Everything else the same
- s, a, $r(s,a)$, and $p(s'|s,a)$ define Markov decision process
  - Infinite version of Markov chains with actions





Objective Function

- $\tau = s_0, a_0, s_1, a_1, s_2, a_2, s_3, a_3, \ldots$
  - Trajectory, sample path, episode

$$p_\theta(\tau) = p_\theta(s_0, a_0, \ldots, s_T, a_T) = p(s_0) \prod_{t=0}^{T} \pi_\theta(a_t|s_t) p(s_{t+1}|s_t, a_t)$$

  - Policy/actions parametric with respect to θ
- Inventory
  - P(10,3; 5,7; 1,0; 0,10) = p(state = 10) π_θ(3|10) p(D=8) π_θ(7|5) p(D=11) π_θ(0|1) p(D=1) π_θ(10|0)

Objective Function

$$\theta^* = \arg\max_\theta E_{\mathcal{T}\sim p_\theta(\mathcal{T})}\left[\sum_t \gamma^t r(s_t, a_t)\right]$$

$$= \arg\max_\theta \sum_{t=0}^{T} E_{(s_t,a_t)\sim p_\theta(s_t,a_t)}[\gamma^t r(s_t, a_t)]$$

- Each sample path has probability
  - Distribution over sample paths
- Expectation over all sample paths

Putting it Together

- State and action spaces
- State, action (policy)
- Transition function
- Reward
- Discount factor
- Initial state (or distribution)

- Stochastic policies computationally tractable
  - But not needed on the paper

- Given policy
  - Markov chain
- Transition probabilities can be viewed as tensor
- Another view
  - Markov chain with states (state,action)

VALUE FUNCTION AND Q-FACTOR

Total Reward

- Initial state $E_{s_0 \sim p(s_0)}$
- $\sum_{t=0}^{T} \gamma^t \cdot r(s_t, a_t) = r(s_0, a_0) + \gamma \underbrace{\sum_{t=1}^{T} \gamma^{t-1} \cdot r(s_t, a_t)}$

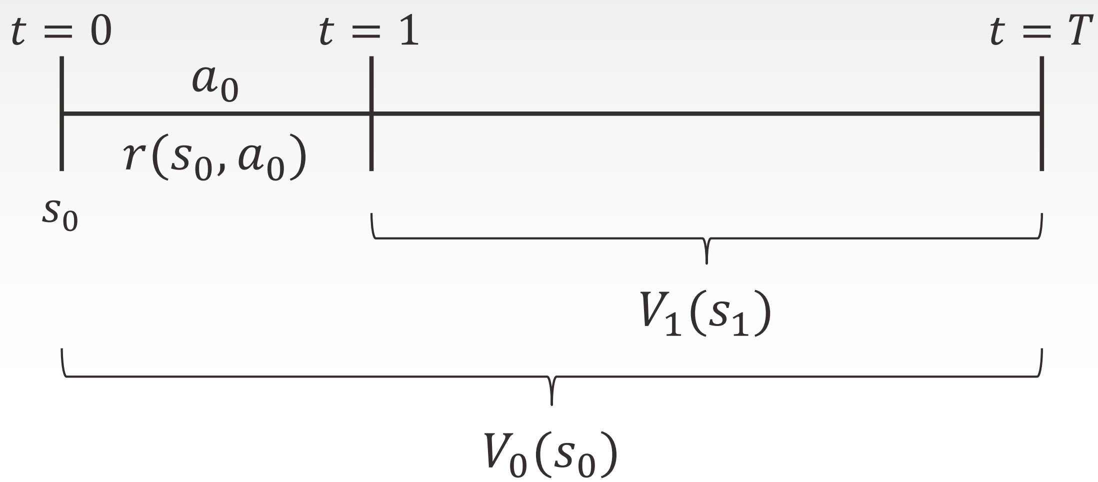
  - Same as original
  - Start in a different state

- Let $V_t(s_t)$ = maximum expected total reward from time *t* to the end given that at time *t* we are in state s_t
- Value function

Equivalent Expressions for Expectation

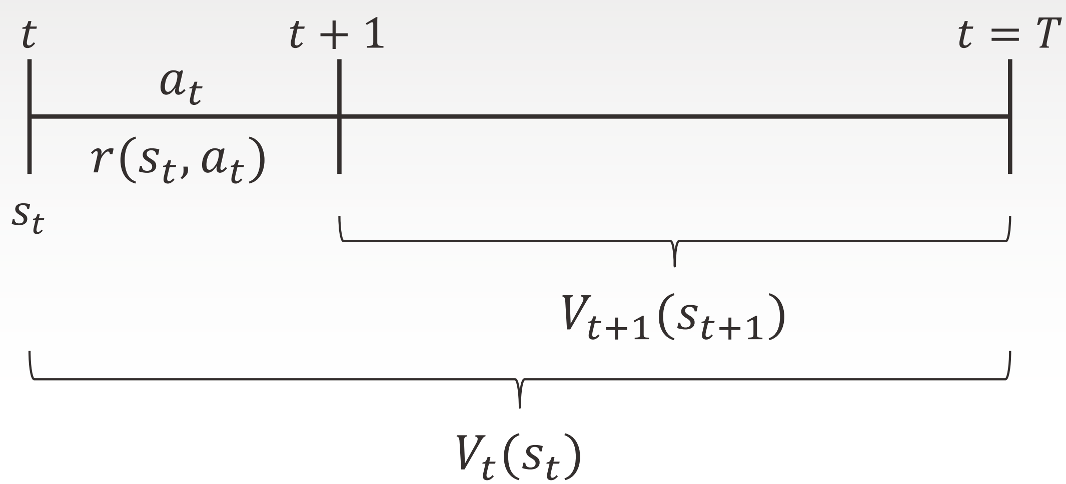
- Given s_t and policy π
- $a_t \sim \pi(a_t|s_t)$ - sampled action based on policy
- Next state
  - $s_{t+1} \sim p(s_{t+1}|s_t, a_t)$
- Entire process
  - $a_t \sim \pi_\theta(a_t|s_t)$, $s_{t+1} \sim p(s_{t+1}|s_t, a_t)$ - sampled next state based on sampled action
  - Equivalently $(s_{t+1}, a_{t+1}) \sim p_\theta(s_{t+1}, a_{t+1}|s_t, a_t)$

Optimality Equation



$$V_0(s_0) = \max_{a_0 \in \mathcal{A}} [r(s_0, a_0) + \gamma E_{s_1 \sim p(s_1|s_0,a_0)} V_1(s_1)]$$

Optimality Equation



$$V_t(s_t) = \max_{a_t \in \mathcal{A}} [r(s_t, a_t) + \gamma E_{s_{t+1} \sim p(s_{t+1}|s_t, a_t)} V_{t+1}(s_{t+1})]$$

Value of Policy

- Let $V_t^\pi(s_t)$ = expected total reward of policy π from time *t* to the end given that at time *t* we are in state s_t

$$V_t^\pi(s_t) = \sum_{t'=t}^{T} \gamma^{t'} E_{a_{t'} \sim \pi(a_{t'}|s_{t'}), s_{t'+1} \sim p_\theta(s_{t'+1}|s_{t'}, a_{t'})}[r(s_{t'}, a_{t'})|s_t]$$

$$V_t^\pi(s_t) = E_{a_t \sim \pi(a_t|s_t), s_{t+1} \sim p(s_{t+1}|s_t, a_t)}[r(s_t, a_t) + \gamma V_{t+1}^\pi(s_{t+1}|s_t)]$$

- s_{t+1} function of a_t, s_t and stochastic environment

$$V_t^\pi(s_t) = E_{\substack{a_t \sim \pi(a_t|s_t) \\ s_{t+1} \sim p(s_{t+1}|s_t, a_t)}}[r(s_t, a_t) + \gamma V_{t+1}^\pi(s_{t+1})]$$

Everywhere $\pi = \pi_\theta$

Stationarity

- An optimal policy
  - Depends on *t* (π_t or π_{θ_t})
- Optimal value function depends on *t*
  - V_t
- In practice for computational tractability
  - Policy does not depend on *t*
  - Value function and other subsequent functions do not depend on *t*
  - Our standard assumption

Optimal Policy

- $V(s_t) = \max_{\pi} V^{\pi}(s_t)$
- RL is about

$$\arg\ max_{\theta} V^{\pi_{\theta}}(s_o)$$

$$\arg\ max_{\theta} E_{s_o \sim p(s_o)} V^{\pi_{\theta}}(s_o)$$

- Omitting discounting factor
- Omitting t in V_t

Example

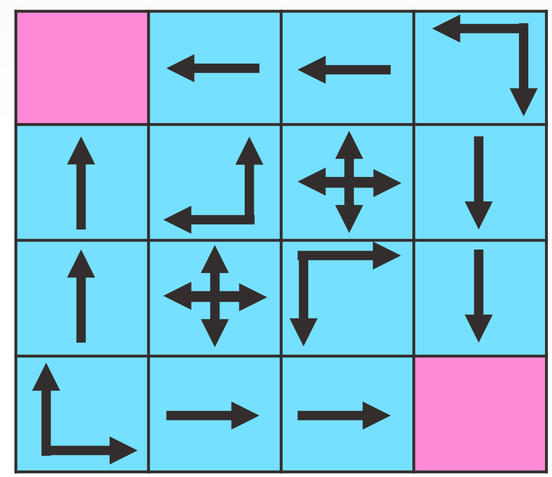
- Reward of -1, 4 actions, reward of 0 at corner points

Random policy

0	-14	-20	-22
-14	-18	-22	-20
-20	-22	-18	-14
-22	-20	-14	0

Optimal policy

0	-1	-2	-3
-1	-2	-3	-2
-2	-3	-2	-1
-3	-2	-1	0



M. Harmon, S. Harmon: Reinforcement Learning: a Tutorial

Q-factor

- $\sum_{t=0}^{T} E_{(s_t,a_t)\sim p_\theta(s_t,a_t)}[r(s_t,a_t)]$
- $Q^\pi(s_t,a_t) = \sum_{t'=t}^{T} E_{(s_{t'},a_{t'})\sim p_\theta(s_{t'},a_{t'})}[r(s_{t'},a_{t'})]$

$$Q^\pi(s_0,a_0) = r(s_0,a_0) + E_{\substack{a_1\sim\pi(a_1|s_1)\\ s_1\sim p(s_1|s_0,a_0)}} [[r(s_1,a_1) + \ \dots |s_1] \, |s_0,a_0]$$

$$Q^\pi(s_t,a_t) = \sum_{t'=t}^{T} E_{(s_{t'},a_{t'})\sim p_\theta(s_{t'},a_{t'})} r(s_{t'},a_{t'})$$