Problem 3

① Show $h_{ij} = \frac{1}{n} + \frac{(x_i - \bar{x})(x_j - \bar{x})}{S_{xx}}$ and $h_{ii} = \frac{1}{n} + \frac{(x_i - \bar{x})^2}{S_{xx}}$

$H = X(X^TX)^{-1}X^T$

In simple linear regression, $X = \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$.

$h_{ii} = (1 \ \ x_i)\,(X^TX)^{-1}\begin{pmatrix} 1 \\ x_i \end{pmatrix}$

$X^TX = \begin{pmatrix} n & n\bar{x} \\ n\bar{x} & \sum_{i=1}^n x_i^2 \end{pmatrix}$ $\quad (X^TX)^{-1} = \frac{1}{n\sum_{i=1}^n x_i^2 - n^2\bar{x}^2}\begin{pmatrix} \sum_{i=1}^n x_i^2 & -n\bar{x} \\ -n\bar{x} & n \end{pmatrix}$

$= \frac{1}{n\sum_{i=1}^n (x_i - \bar{x})^2}\begin{pmatrix} \sum_{i=1}^n x_i^2 & -n\bar{x} \\ -n\bar{x} & n \end{pmatrix}$

$= \frac{1}{S_{xx}}\begin{pmatrix} \frac{1}{n}\sum_{i=1}^n x_i^2 & -\bar{x} \\ -\bar{x} & 1 \end{pmatrix}$

$\therefore h_{ii} = (1 \ \ x_i) \cdot \frac{1}{S_{xx}} \cdot \begin{pmatrix} \frac{1}{n}\sum_{i=1}^n x_i^2 & -\bar{x} \\ -\bar{x} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ x_i \end{pmatrix}$

$= \frac{1}{S_{xx}}\left(\frac{1}{n}\sum_{i=1}^n x_i^2 + x_i^2 - 2\bar{x}x_i\right)$

$= \frac{1}{S_{xx}}\left[\left(\frac{1}{n}\sum_{i=1}^n x_i^2 - \bar{x}^2\right) + (x_i^2 - 2\bar{x}x_i + \bar{x}^2)\right]$

$= \frac{1}{n} + (x_i - \bar{x})^2 / S_{xx}$

$\therefore h_{ii} = \frac{1}{n} + \frac{(x_i - \bar{x})^2}{S_{xx}}$

Similarly, for h_{ij}. $h_{ij} = \frac{1}{S_{xx}}(1 \ \ x_i)\begin{pmatrix} \frac{1}{n}\sum_{i=1}^n x_i^2 & -\bar{x} \\ -\bar{x} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ x_j \end{pmatrix}$

$= \frac{1}{S_{xx}}\left(\frac{1}{n}\sum_{i=1}^n x_i^2 + x_ix_j - \bar{x}x_i - \bar{x}x_j\right)$

$= \frac{1}{S_{xx}}\left[\left(\frac{1}{n}\sum_{i=1}^n x_i^2 - \bar{x}^2\right) + (x_ix_j - \bar{x}x_i - \bar{x}x_j + \bar{x}^2)\right]$

$= \frac{1}{n} + \frac{(x_i - \bar{x})(x_j - \bar{x})}{S_{xx}}$

② Show $\sum_{j=1}^{n} h_{ij} = 1$

$$\sum_{j=1}^{n} h_{ij} = \sum_{j=1}^{n} \frac{1}{n} + \frac{(x_i - \bar{x})(x_j - \bar{x})}{S_{xx}}$$

$$= 1 + \frac{(x_i - \bar{x})}{S_{xx}} \cdot \sum_{j=1}^{n} (x_j - \bar{x})$$

$$= 1 + \frac{(x_i - \bar{x})}{S_{xx}} \cdot \left(\sum_{j=1}^{n} x_j - n\bar{x} \right)$$

$$= 1 + \frac{(x_i - \bar{x})}{S_{xx}} \cdot 0$$

$$= 1.$$