HW 1.

Q1. $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$

(a) $X^T A X$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 a_{11} + x_2 a_{12} & x_1 a_{12} + x_2 a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \left[x_1 (x_1 a_{11} + x_2 a_{12}) + x_2 (x_1 a_{12} + x_2 a_{22}) \right]$$

$$= \left[x_1^2 a_{11} + x_1 x_2 a_{12} + x_1 x_2 a_{12} + x_2^2 a_{22} \right]$$

$$= \left[x_1^2 a_{11} + 2 x_1 x_2 a_{12} + x_2^2 a_{22} \right]$$

(b) $$\frac{\partial X^T A X}{\partial X} = \begin{bmatrix} \dfrac{\partial X^T A X}{\partial x_1} \\[2ex] \dfrac{\partial X^T A X}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 a_{11} x_1 + 2 x_2 a_{12} \\ 2 x_1 a_{12} + 2 x_2 a_{22} \end{bmatrix}$$

$$= 2 \begin{bmatrix} a_{11} x_1 + x_2 a_{12} \\ x_1 a_{12} + x_2 a_{22} \end{bmatrix}$$

$$\because A \cdot X = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{12}x_1 + a_{22}x_2 \end{bmatrix}$$

$$\therefore \frac{\partial X^T A X}{\partial X} = 2 \cdot A \cdot X$$

Q2. (a)

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$$

(b)

$$X^T X = \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$$

$$= \begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix}$$

(c) Yes, because $X^TX_{1,2} = X^TX_{2,1}$

(d) now $X = \begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1p} \\ 1 & x_{21} & \cdots & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & & \vdots \\ 1 & x_{n1} & & \cdots & x_{np} \end{bmatrix}$ $n \times (p+1)$

$X^T = \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ x_{11} & x_{21} & \cdots & \cdots & x_{n1} \\ \vdots & \vdots & \ddots & & \vdots \\ x_{1p} & x_{2p} & & \cdots & x_{np} \end{bmatrix}$ $(p+1) \times n$

$$X^T X = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_{11} & x_{21} & \cdots & x_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1p} & x_{2p} & \cdots & x_{np} \end{bmatrix} \begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1p} \\ 1 & x_{21} & \cdots & \cdots & x_{2p} \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & x_{n1} & & \cdots & x_{np} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \sum_i^n x_{i1} & \cdots & \sum_i^n x_{ip} \\ \sum_i^n x_{i1} & \ddots & & \sum_i^n x_{i1} \cdot x_{ip} \\ \vdots & & \ddots & \\ \sum_i^n x_{ip} & & & \sum_i^n (x_{ip})^2 \end{bmatrix} \quad (p+1)(1+p)$$

Let $A = X^T X$, $A_{ij} = A_{ji}$