Q4

a. if $\begin{cases} x_i = x' \text{ for } i = 1, \cdots, n-1 \\ x_i = x'' \text{ for } i = n \end{cases}$, then $\bar{x} = \dfrac{(n-1)x' + x''}{n}$

$$S_{xx} = \sum (x_i - \bar{x})^2 = \sum x_i^2 - 2\bar{x}x_i + \bar{x}^2 = \sum x_i^2 - 2\bar{x}\sum x_i + n\bar{x}^2 = \sum x_i^2 - 2n\bar{x}^2 + n\bar{x}^2 = \sum x_i^2 - n\bar{x}^2$$

$$\therefore S_{xx} = \sum_{i=1}^{n-1} x'^2 + x''^2 - n \cdot \left[\frac{(n-1)x' + x''}{n} \right]^2$$

$$= (n-1)x'^2 + x''^2 - \frac{1}{n}\left[(n-1)x' + x''\right]^2$$

$$= (n-1)x'^2 + x''^2 - \frac{1}{n}\left[(n-1)^2 x'^2 + 2(n-1)x'x'' + x''^2\right]$$

$$= \frac{n-1}{n}\left(nx'^2 + \frac{n}{n-1}x''^2 - (n-1)x'^2 - 2x'x'' - \frac{1}{n-1}x''^2 \right]$$

$$= \frac{n-1}{n}\left(x''^2 + x'^2 - 2x'x'' \right)$$

$$= \frac{n-1}{n} \cdot (x' - x'')^2$$

b. $(x_i - \bar{x})(x_n - \bar{x}) = (x' - \bar{x})(x'' - \bar{x})$

$$= \left(x' - \frac{(n-1)x' + x''}{n}\right)\left(x'' - \frac{(n-1)x' + x''}{n}\right)$$

$$= \frac{(nx' - nx' + x' - x'')(nx'' - nx' + x' - x'')}{n^2}$$

$$= \frac{-(x' - x'')[(n-1)(x' - x'')]}{n^2}$$

$$= \frac{-(n-1)(x' - x'')^2}{n^2}$$

$$(x_n - \bar{x})^2 = \left(x'' - \frac{(n-1)x' + x''}{n}\right)^2$$

$$= \left(\frac{nx'' - nx' + x' - x''}{n}\right)^2$$

$$= \left[\frac{(1-n)}{n}(x' - x'')\right]^2$$

$$= \left(\frac{n-1}{n}\right)^2 (x' - x'')^2$$

(c) From Q3, $h_{in} = \frac{1}{n} + \frac{(x_i - \bar{x})(x_n - \bar{x})}{S_{xx}}$

$$= \frac{1}{n} + \frac{(n-1)(x' - x'')^2}{n^2} \cdot \frac{1}{(x' - x'')^2 \left(\frac{n-1}{n}\right)}$$

$$= \frac{1}{n} - \left(\frac{n-1}{n^2}\right)\left(\frac{n}{n-1}\right)$$

$$= \frac{1}{n} - \frac{1}{n}$$

$$= 0.$$

$$h_{nn} = \frac{1}{n} + \frac{(x_n - \bar{x})^2}{S_{xx}}$$

$$= \frac{1}{n} + \left(\frac{n-1}{n}\right)(x' - x'')^2 \cdot \frac{1}{(x' - x'')^2 \cdot \left(\frac{n-1}{n}\right)}$$

$$= \frac{1}{n} + \left(\frac{n-1}{n}\right)^2 \cdot \left(\frac{n}{n-1}\right)$$

$$= \frac{1}{n} + \frac{n-1}{n}$$

$$= 1$$