First, simplify $\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}$

$$= \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{S_{xx}}$$

$$= \frac{\sum_{i=1}^{n}(x_i-\bar{x})y_i - \sum_{i=1}^{n}(x_i-\bar{x})\bar{y}}{S_{xx}}$$

$$= \frac{1}{S_{xx}}\left(\sum_{i=1}^{n}(x_i-\bar{x})y_i - \bar{y}\sum_{i=1}^{n}x_i + n\bar{y}\bar{x}\right)$$

$$= \frac{1}{S_{xx}}\left(\sum_{i=1}^{n}(x_i-\bar{x})y_i - n\bar{y}\bar{x} + n\bar{y}\bar{x}\right)$$

$$= \frac{\sum_{i=1}^{n}(x_i-\bar{x})y_i}{S_{xx}}$$

$$Cov(\bar{y}, \hat{\beta}_1) = Cov\left(\bar{y}, \frac{\sum_{i=1}^{n}(x_i-\bar{x})y_i}{S_{xx}}\right)$$

lemma 1

$$= \frac{1}{S_{xx}} Cov\left(\bar{y}, \sum_{i=1}^{n}(x_i-\bar{x})y_i\right)$$

lemma 2

$$= \frac{1}{S_{xx}} \sum_{i=1}^{n} Cov(\bar{y}, (x_i-\bar{x})y_i)$$

lemma 1

$$= \frac{1}{S_{xx}} \sum_{i=1}^{n} (x_i-\bar{x}) Cov(\bar{y}, y_i)$$

$$= \frac{1}{S_{xx}} \sum_{i=1}^{n} (x_i-\bar{x}) Cov\left(\frac{1}{n}\sum_{j=1}^{n} y_j, y_i\right)$$

lemma 1 & 2

$$= \frac{1}{S_{xx}} \sum_{i=1}^{n} (x_i-\bar{x}) \cdot \frac{1}{n} \sum_{j=1}^{n} Cov(y_j, y_i)$$

$y_j \perp\!\!\!\perp y_i \ \forall i \neq j$

$$= \frac{1}{S_{xx}} \sum_{i=1}^{n} (x_i-\bar{x}) \cdot \frac{1}{n} \cdot Cov(y_i, y_i)$$

$$= \frac{1}{S_{xx}} \sum_{i=1}^{n} (x_i-\bar{x}) \cdot \frac{1}{n} Var(y_i)$$

$$= \frac{1}{S_{xx}} \cdot \frac{1}{n} \cdot \sigma^2 \cdot \sum_{i=1}^{n}(x_i-\bar{x})$$

$$= \frac{1}{S_{xx}} \cdot \frac{1}{n} \cdot \sigma^2 \cdot \left(\sum_{i=1}^{n} x_i - n\bar{x}\right)$$

$$= \frac{1}{S_{xx}} \cdot \frac{1}{n} \cdot \sigma^2 \cdot 0$$

$$= 0$$

Since $Cov(\bar{y}, \hat{\beta}_1) = 0$, $\bar{y} \perp\!\!\!\perp \hat{\beta}_1$.