Problem 2:

a. Ridge Regression: $\hat{\beta}_\lambda = \underset{\beta}{\text{argmin}} \left[\sum_{i=1}^{n} (y_i - \hat{\beta}_i)^2 + \lambda \sum_{j=1}^{n} \hat{\beta}_j^2 \right]$

$\frac{\partial \text{Loss}}{\partial \beta_i} = -2(y_i - \hat{\beta}_i) + 2\lambda \hat{\beta}_i = 0. \Leftrightarrow \hat{\beta}_i (2 + 2\lambda) = 2y_i \rightarrow \hat{\beta}_i = \frac{y_i}{1+\lambda} \Rightarrow \hat{\beta}_i^R(\lambda) = \frac{\hat{\beta}_i}{1+\lambda}$

b. Lasso Regression: $\hat{\beta}_\lambda = \underset{\beta}{\text{argmin}} \left[\sum_{i=1}^{n} (y_i - \hat{\beta}_i)^2 + \lambda \sum_{j=1}^{n} |\beta_j| \right]^2.$

Considering $\frac{\partial |\beta_i|}{\partial \beta_i} = S_i.$ $\begin{cases} S_i = \text{sign}(\beta_i) \text{ when } \beta_i \neq 0. \\ S_i \in [-1,1] \text{ if } \beta_j = 0. \end{cases}$

① when $\beta_i > 0$

$\frac{\partial \text{Loss}}{\partial \beta_i} = -2(y_i - \beta_i) + \lambda = 0 \Rightarrow \beta_i = y_i - \frac{\lambda}{2}$

$y_i - \frac{\lambda}{2} > 0 \Rightarrow y_i > \frac{\lambda}{2}$

③ when $\beta_i < 0$.

$\frac{\partial \text{Loss}}{\partial \beta_i} = -2(y_i - \beta_i) - \lambda = 0 \Rightarrow \beta_i = y_i + \frac{\lambda}{2}$

$y_i + \frac{\lambda}{2} \cancel{>0} < 0 \Rightarrow y_i < -\frac{\lambda}{2}$

So $\hat{\beta}_i^L(\lambda) = \begin{cases} y_i - \frac{\lambda}{2} & \text{if } y_i > \frac{\lambda}{2} \\ y_i + \frac{\lambda}{2} & \text{if } y_i < -\frac{\lambda}{2} \\ 0 & \text{if } |y_i| \leq \frac{\lambda}{2} \end{cases}$

② when $\beta_i = 0$

$-2(y_i - \cancel{\beta_i}^{0}) + S_i\lambda = 0.$

$-2y_i + \lambda S_i = 0.$

$y_i = \frac{\lambda S_i}{2}.$ $\leftarrow \because S_i \in [-1,1]$

$|y_i| \leq \frac{\lambda}{2}$

c. When shrinking LS estimator, ridge does not tend to shrink LS estimator to zeros expect when λ is very large. Ridge tends to shrink LS estimator to smaller value. But the lasso tends to force some coefficients to equal zero. For example, in the example above, when $|y_i| = |\hat{\beta}_i^{OLS}| \leq \frac{\lambda}{2}$, lasso will shrink them to be 0.