Q5

Proof : $E(\hat{\beta}_{GLS}) = \beta$ and $Cov(\hat{\beta}_{GLS}) = (X'\Sigma^{-1}X)^{-1} = \sigma^2 (X'WX)^{-1}$

$$\hat{\beta}_{GLS} = (X'WX)^{-1} X'WY$$

$$\Rightarrow \hat{\beta}_{GLS} = (X'WX)^{-1} X'W(\beta X + \epsilon)$$

$$= \beta \cancel{(X'WX)^{-1}(X'WX)} + (X'WX)^{-1} X'W \cdot \epsilon$$

$$= \beta + (X'WX)^{-1} X'W\epsilon$$

$$E[\hat{\beta}_{GLS}] = E[\beta] + E[(X'WX)^{-1} X'W\epsilon]$$

$$= \beta + (X'WX)^{-1} X'W \cancel{E[\epsilon]}^{\,0}$$ by assumption

$$= \beta$$

$\Rightarrow$ given $Var(AX) = A\,Var(X)\,A'$, we let.

$$A = (X'WX)^{-1} X'W$$

then: $Var(\hat{\beta}_{GLS}) = (X'WX)^{-1} X'W\, Var(Y|X)\, WX(X'WX)^{-1}$

$$\because Var(Y|X) = \sigma^2 W^{-1}$$

$$\therefore Var(\hat{\beta}_{GLS}|X) = \sigma^2 (X'WX)^{-1} X'WX (X'WX)^{-1}$$

$$= \sigma^2 (X'WX)^{-1}$$

$$\because Cov(X, X) = Var(X)$$

$$\therefore Cov(\hat{\beta}_{GLS}) = \sigma^2 (X'WX)^{-1}$$

Q6 (a) $\bar{y}_w$ will be an unbiased estimator for μ when:

$$E(\bar{y}_w) = \mu$$

$$E(\bar{y}_w) = E(w_1 Y_1 + w_2 Y_2) = E(w_1 Y_1) + E(w_2 Y_2)$$
$$= w_1 E(Y_1) + w_2 E(Y_2)$$
$$= w_1 \mu + w_2 \mu$$
$$= (w_1 + w_2)\mu$$

when $w_1 + w_2 = 1$, $E(\bar{y}_w) = \mu$, therefore is unbiased

(b) $Var(\bar{y}_w) = Var(w_1 Y_1 + w_2 Y_2)$

$$= w_1^2 Var(Y_1) + w_2^2 Var(Y_2)$$
$$= w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \cancelto{0}{Cov(Y_1, Y_2)}$$

$$= w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2$$

(c) $Var(\bar{y}_w)$ is minimized when $\dfrac{\partial V(\bar{y}_w)}{\partial w_i} = 0$

$$\frac{\partial V(\bar{y}_w)}{\partial w_i} = \frac{\partial (w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2)}{\partial w_1} + \frac{\partial (w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2)}{\partial w_2}$$

$$= 2\sigma_1^2 w_1 + 2\sigma_2^2 w_2$$

let $2\sigma_1^2 w_1 + 2\sigma_2^2 w_2 = 0$

$$\sigma_1^2 w_1 = -\sigma_2^2 w_2 \Rightarrow \begin{array}{l} w_1 = -(\sigma_2^2 w_2) \cdot \dfrac{1}{\sigma_1^2} \\ w_2 = -(\sigma_1^2 w_1) \cdot \dfrac{1}{\sigma_2^2} \end{array}$$

$\Rightarrow Var(\bar{y}_w)$ is minimized when $w_i \propto 1/\sigma_i^2$