Problem #3

(a) $\log \pi_{ik} = \alpha_k^T x_i - \log z$

$\log z = \alpha_k^T x_i - \log \pi_{ik}$

$\exp(\log z) = \exp(\alpha_k^T x_i - \log \pi_{ik})$

$z = \exp(\alpha_k^T x_i) / \pi_{ik}$ for $k = 1, 2, \cdots, K$

$\because \sum_{k=1}^{K} \pi_{ik} = 1$

$\therefore z = \sum_{k=1}^{K} \exp(\alpha_k^T x_i)$

(b) $\log \pi_{ik} = \alpha_k^T x_i - \log z$

$\exp(\log \pi_{ik}) = \exp(\alpha_k^T x_i - \log z)$

$\pi_{ik} = \exp(\alpha_k^T x_i) / z$

(c) $\log\left(\frac{\pi_{ik}}{\pi_{i1}}\right) = \beta_k^T x_i$

$\log(\pi_{ik}) - \log(\pi_{i1}) = \beta_k^T x_i$, for $k = 2, \cdots, K$

$\log \pi_{ik} = \beta_k^T x_i + \log \pi_{i1}$, for $k = 2, \cdots, K$

$\pi_{ik} = \exp(\beta_k^T x_i) \cdot \pi_{i1}$, for $k = 2, \cdots, K$

$\because \pi_{ik} = \exp(\alpha_k^T x_i) / z$ from 3b

$\therefore \exp(\beta_k^T x_i) \cdot \pi_{i1} = \exp(\alpha_k^T x_i) / z$

$\exp(\beta_k^T x_i) = \exp(\alpha_k^T x_i) / z \cdot \pi_{i1}$

$\beta_k^T x_i = \alpha_k^T x_i - \log(z \cdot \pi_{i1})$

$\beta_k^T = \alpha_k^T - \log(z \cdot \pi_{i1})$

$\beta_k^T = \alpha_k^T - (\log(z) + \log \pi_{i1})$

Also from previous: $\log \gamma_{ik} = \alpha_k^T x_i - \log Z$

$\therefore$ for $k = 1$: $\log \gamma_{i1} = \alpha_1 x_i - \log Z$

$\log \gamma_{i1} + \log Z = \alpha_1 x_i$

$\therefore \beta_k^T = \alpha_k^T - \alpha_1 x_i$

$\therefore \beta_k = \alpha_k - \alpha_1 x_i$, where $\alpha_1 x_i$ is a constant