Sympy Paper Printer Demo

Solution to Chapter 01 Problem 05 of Optimal Control with Aerospace Applications [1]

This back-of-the-chapter problem is all about Hohmann and bi-parabolic satellite transfers. In addition to being an interesting study in orbital maneuvers, this problem shows many tips and tricks to solving problems like this symbolically and numerically, and how to transition between the two.

5-a

First, we are asked to describe a Hohmann Transfer and derive a formula for the total velocity change.

A Hohmann transfer is a co-planer transfer between 2 circular orbits (although it extends easily to 2 coelliptic elliptical orbits when the burns are only at periapsis and apoapsis). There are 2 burns, the first to get onto the transfer orbit between the 2 circles, and the second happens 180 degrees of anomaly later. The burns are always along the velocity vector of both the pre and post orbits, and on the elliptical transfer orbit the burns are always at 0 and 180 degrees anomaly. I say anomaly because everything said before is correct for mean, true and eccentric anomalies.

We could spend more time talking about Hohmann Transfers, but lets get to evaluating values. We will start with taking the two expression the problem asked us to put our problem in, but we are going to solve for r_o and r_f in terms of those expressions. Then we add equations 1.36 and 1.41 together and do the substitution.

$$r_f = \alpha(r_o(r_o, \alpha), \alpha) r_o(r_o, \alpha)$$

$$r_o = \frac{\frac{\Delta V_{tol}}{\sqrt{(\frac{\mu}{r_o})}}^2 \mu\left(\mu, \Delta V_{tol}, \frac{\Delta V_{tol}}{\sqrt{(\frac{\mu}{r_o})}}\right)}{\Delta V_{tol}^2}$$

$$r_f = \frac{\frac{\Delta V_{tol}}{\sqrt{(\frac{\mu}{r_o})}}^2 \alpha\left(\Delta V_{tol}, \mu\left(\mu, \Delta V_{tol}, \alpha, \frac{\Delta V_{tol}}{\sqrt{(\frac{\mu}{r_o})}}\right), \alpha, \frac{\Delta V_{tol}}{\sqrt{(\frac{\mu}{r_o})}}\right) \mu\left(\mu, \Delta V_{tol}, \alpha, \frac{\Delta V_{tol}}{\sqrt{(\frac{\mu}{r_o})}}\right)}{\Delta V_{tol}^2}$$

$$\Delta V_{tol} = \Delta V_{tol} = \sqrt{\frac{\mu}{r_f}}\left(1 - \sqrt{\frac{2r_o}{r_f + r_o}}\right) + \sqrt{\frac{\mu}{r_o}}\left(\sqrt{\frac{2r_f}{r_f + r_o}} - 1\right)$$

$$\Delta V_{tol} = \frac{\left(-\sqrt{\alpha}\sqrt{\alpha + 1} + \sqrt{2}\alpha + \sqrt{\alpha + 1} - \sqrt{2}\right) \Delta V_{tol}\left(\Delta V_{tol}, \alpha, \frac{\Delta V_{tol}}{\sqrt{(\frac{\mu}{r_o})}}\right)}{\frac{\Delta V_{tol}}{\sqrt{(\frac{\mu}{r_o})}}\sqrt{\alpha}\sqrt{\alpha + 1}}$$

$$\frac{\Delta V_{tol}}{\sqrt{\left(\frac{\mu}{r_o}\right)}} = \frac{\alpha^{\frac{3}{2}}\sqrt{2\alpha+2} - \sqrt{2}\sqrt{\alpha}\sqrt{\alpha+1} + \sqrt{\alpha}\left(\alpha+1\right) - \alpha\left(\alpha+1\right)}{\alpha\left(\alpha+1\right)}$$

The problem explicitly states that we should simplify this into the most compact form. I tried various techniques in sympy to do that, but this may not be exactly what the author is looking for.

5-b

We do the same thing, but this time we sum the circular speed with the parabolic speeds.

$$\Delta V_{tol} = -\sqrt{\frac{\mu}{r_o}} - \left(\sqrt{\frac{\mu}{r_f}} - \frac{\sqrt{2}\sqrt{\mu}}{\sqrt{r_o}} - \frac{\sqrt{2}\sqrt{\mu}}{\sqrt{r_f}}\right)$$

$$\Delta V_{tol} = -\frac{\left(1-\sqrt{2}\right)\left(\sqrt{\alpha}+1\right)}{\sqrt{\alpha}}$$

5-c

We make the plot comparing the two transfer types. Note that since the problem assumes $r_f > r_o$, the lower bound of the plot must be >= 1 as the problem says.

We can see that the cutoff is about a ratio of 12, likely the 11.94 value quoted earlier in the chapter.

5-d

Here we will use a numerical root-finder to find when the two transfer techniques equal each other. Attempts to solve it symbolically failed.

$$\frac{\alpha^{\frac{3}{2}}\sqrt{2\alpha+2} - \sqrt{2}\sqrt{\alpha}\sqrt{\alpha+1} + \sqrt{\alpha}\left(\alpha+1\right) - \alpha\left(\alpha+1\right)}{\alpha\left(\alpha+1\right)} = -\frac{\left(1-\sqrt{2}\right)\left(\sqrt{\alpha}+1\right)}{\sqrt{\alpha}}$$

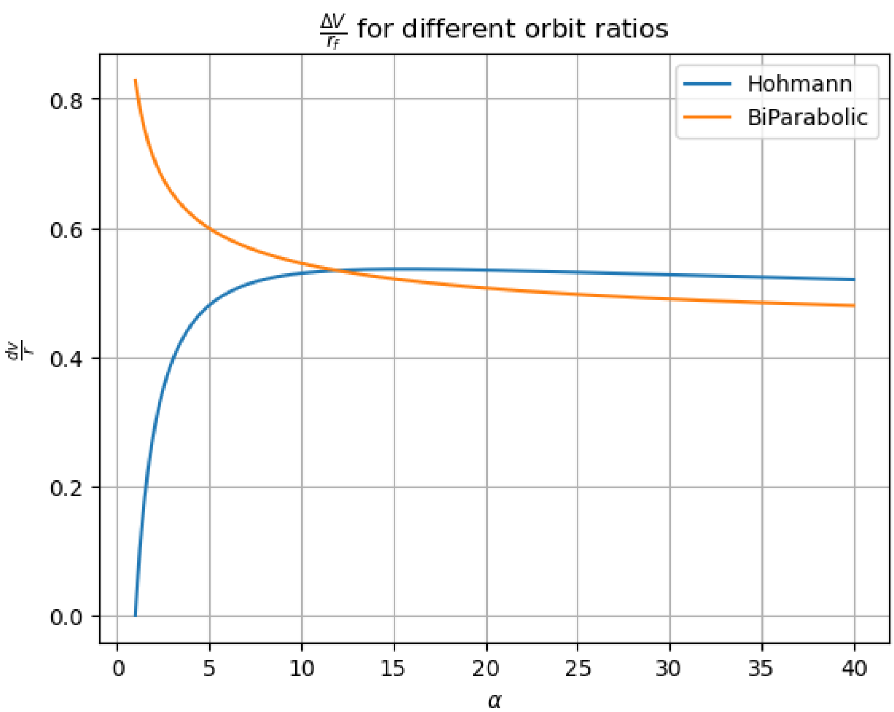
$$0 = \frac{\alpha^{\frac{3}{2}}\sqrt{2\alpha+2} - \sqrt{2}\sqrt{\alpha}\sqrt{\alpha+1} + \sqrt{\alpha}\left(\alpha+1\right) - \alpha\left(\alpha+1\right)}{\alpha\left(\alpha+1\right)} + \frac{\left(1-\sqrt{2}\right)\left(\sqrt{\alpha}+1\right)}{\sqrt{\alpha}}$$

$\alpha = 11.9387654726459$

This matches the values a few pages earlier in the textbook.

Acknowledgments

Many thanks for the Citation Style Language website for making citations so easy and simple [2]. Also thanks to the AIAA for publishing their csl file [3]. Also, this sample of the Sympy Paper Printer is made with the Sympy Paper Printer and is in the public domain as specified on its github page [4]. Please forgive, but also point out, any mistakes or problems or potential improvements. Especially with how citations are handled.



$\frac{\Delta V}{r_f}$ for different orbit ratios
Hohmann
BiParabolic
0.8
0.6
0.4
0.2
0.0
$\frac{dV}{r}$
0
5
10
15
20
25
30
35
40
α

Figure 1: png

References

[1] Longuski, J. M., Guzmán, J. J., and Prussing, J. *Optimal Control with Aerospace Applications*. Springer, 2014.

[2] Citation Style Language. https://citationstyles.org/. Accessed Dec. 31, 2022.

[3] O'Brien, P. AIAA CSL. https://github.com/citation-style-language/styles/blob/master/american-institute-of-aeronautics-and-astronautics.csl. Accessed Dec. 31, 2022.

[4] Gilbert, S. Scipy Paper Printer. https://github.com/samthegliderpilot/SimpyPaperPrinter/. Accessed Jan. 2, 2023.