MIS 381N Homework 1

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Problem 1

We know that $(AB)^{-1} = B^{-1}A^{-1}$ so we also know that $(AB)^T = B^T A^T$.

We also know that the multiplicative identity I aka AA^{-1} is symmetric, so $I^T = I$.

By setting $B = A^T$ and multiplying to each side, we see:

$$
\begin{aligned}
(A^{-1})^T \times A^T &= (A^T)^{-1} \times A^T && (1)\\
(AA^{-1})^T &= I && (2)\\
I^T &= I && (3)\\
I &= I && (4)\\
& && (5)
\end{aligned}
$$

This proves that $(A^{-1})^T = (A^T)^{-1}$.

Problem 2

There are four different types of loans - First mortgage, Second mortgage, Home improvement, Personal overdraft - which are represented respectively by x_1,x_2,x_3,and x_4 .

Based on the given information, we know that:

$$
\begin{cases}
x_1 + x_2 + x_3 + x_4 & = 250,000,000\\
0.45x_1 - 0.55x_2 & = 0\\
x_2 & = 62,500,000\\
0.14x_1 + 0.2x_2 + 0.2x_3 + 0.1x_4 & = 150,000,000
\end{cases} \tag{6}
$$

We can use this to create three matrices:

$$
A = \begin{bmatrix} 1 & 1 & 1 & 1\\ .45 & -.55 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0.14 & 0.2 & 0.2 & 0.1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 250,000,000\\ 0\\ 62,500,000\\ 37,500,000 \end{bmatrix} \tag{7}
$$

We can then solve for the four loan amounts by setting $A\mathbf{x} = \mathbf{y}$ aka $\mathbf{x} = A^{-1}\mathbf{y}$.

```
y = c(250000000,0,62500000,37500000)
A = matrix(c(1,.45,0,.14,1,-.55,1,.2,1,0,0,.2,1,0,0,.1),4,4)
x = solve(A,y)
x_mat = matrix(c(x),4,1)
print(x_mat)
```

```
##          [,1]
## [1,] 76388889
## [2,] 62500000
## [3,] 31944444
## [4,] 79166667
```

Problem 3

We want to choose x_1,x_2,x_3,x_4

to maximize

$$1.5x_1 + 2.5x_2 + 3x_3 + 4.5x_4$$

subject to the constraints

$$\begin{cases} 2x_1 + 4x_2 + 3x_3 + 7x_4 & \leq 100000 \\ 3x_1 + 2x_2 + 3x_3 + 4x_4 & \leq 50000 \\ 2x_1 + 3x_2 + 2x_3 + 5x_4 & \leq 60000 \\ x_1, x_2, x_3, x_4 & \geq 0 \end{cases} \tag{8}$$

In matrix notation, we can set this up as:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1.5 \\ 2.5 \\ 3 \\ 4.5 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 4 & 3 & 7 \\ 3 & 2 & 3 & 4 \\ 2 & 3 & 2 & 5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 100,000 \\ 50,000 \\ 60,000 \end{bmatrix} \tag{9}$$

We can then maximize $\mathbf{C}^T\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $x \geq 0$

Problem 4

Part a

```
A = matrix(nrow=20, ncol=20)
A[1,1] = 1
for(j in 1:20){
   for(i in 1:20){
     if (j > i){
        A[i,j] = i/j
      }
     else if (j == i){
       A[i,j] = i/j
      }
     else{
       A[i,j] = j/i
     }
   }
```

```
}
A
```

```
##             [,1]      [,2]      [,3]      [,4]      [,5]      [,6]
##  [1,] 1.00000000 0.5000000 0.3333333 0.2500000 0.2000000 0.1666667
##  [2,] 0.50000000 1.0000000 0.6666667 0.5000000 0.4000000 0.3333333
##  [3,] 0.33333333 0.6666667 1.0000000 0.7500000 0.6000000 0.5000000
##  [4,] 0.25000000 0.5000000 0.7500000 1.0000000 0.8000000 0.6666667
##  [5,] 0.20000000 0.4000000 0.6000000 0.8000000 1.0000000 0.8333333
##  [6,] 0.16666667 0.3333333 0.5000000 0.6666667 0.8333333 1.0000000
##  [7,] 0.14285714 0.2857143 0.4285714 0.5714286 0.7142857 0.8571429
##  [8,] 0.12500000 0.2500000 0.3750000 0.5000000 0.6250000 0.7500000
##  [9,] 0.11111111 0.2222222 0.3333333 0.4444444 0.5555556 0.6666667
## [10,] 0.10000000 0.2000000 0.3000000 0.4000000 0.5000000 0.6000000
## [11,] 0.09090909 0.1818182 0.2727273 0.3636364 0.4545455 0.5454545
## [12,] 0.08333333 0.1666667 0.2500000 0.3333333 0.4166667 0.5000000
## [13,] 0.07692308 0.1538462 0.2307692 0.3076923 0.3846154 0.4615385
## [14,] 0.07142857 0.1428571 0.2142857 0.2857143 0.3571429 0.4285714
## [15,] 0.06666667 0.1333333 0.2000000 0.2666667 0.3333333 0.4000000
## [16,] 0.06250000 0.1250000 0.1875000 0.2500000 0.3125000 0.3750000
## [17,] 0.05882353 0.1176471 0.1764706 0.2352941 0.2941176 0.3529412
## [18,] 0.05555556 0.1111111 0.1666667 0.2222222 0.2777778 0.3333333
## [19,] 0.05263158 0.1052632 0.1578947 0.2105263 0.2631579 0.3157895
## [20,] 0.05000000 0.1000000 0.1500000 0.2000000 0.2500000 0.3000000
##             [,7]      [,8]      [,9]     [,10]      [,11]      [,12]
##  [1,] 0.1428571 0.1250000 0.1111111 0.1000000 0.09090909 0.08333333
##  [2,] 0.2857143 0.2500000 0.2222222 0.2000000 0.18181818 0.16666667
##  [3,] 0.4285714 0.3750000 0.3333333 0.3000000 0.27272727 0.25000000
##  [4,] 0.5714286 0.5000000 0.4444444 0.4000000 0.36363636 0.33333333
##  [5,] 0.7142857 0.6250000 0.5555556 0.5000000 0.45454545 0.41666667
##  [6,] 0.8571429 0.7500000 0.6666667 0.6000000 0.54545455 0.50000000
##  [7,] 1.0000000 0.8750000 0.7777778 0.7000000 0.63636364 0.58333333
##  [8,] 0.8750000 1.0000000 0.8888889 0.8000000 0.72727273 0.66666667
##  [9,] 0.7777778 0.8888889 1.0000000 0.9000000 0.81818182 0.75000000
## [10,] 0.7000000 0.8000000 0.9000000 1.0000000 0.90909091 0.83333333
## [11,] 0.6363636 0.7272727 0.8181818 0.9090909 1.00000000 0.91666667
## [12,] 0.5833333 0.6666667 0.7500000 0.8333333 0.91666667 1.00000000
## [13,] 0.5384615 0.6153846 0.6923077 0.7692308 0.84615385 0.92307692
## [14,] 0.5000000 0.5714286 0.6428571 0.7142857 0.78571429 0.85714286
## [15,] 0.4666667 0.5333333 0.6000000 0.6666667 0.73333333 0.80000000
## [16,] 0.4375000 0.5000000 0.5625000 0.6250000 0.68750000 0.75000000
## [17,] 0.4117647 0.4705882 0.5294118 0.5882353 0.64705882 0.70588235
## [18,] 0.3888889 0.4444444 0.5000000 0.5555556 0.61111111 0.66666667
## [19,] 0.3684211 0.4210526 0.4736842 0.5263158 0.57894737 0.63157895
## [20,] 0.3500000 0.4000000 0.4500000 0.5000000 0.55000000 0.60000000
##            [,13]      [,14]      [,15]     [,16]      [,17]      [,18]
##  [1,] 0.07692308 0.07142857 0.06666667 0.0625000 0.05882353 0.05555556
##  [2,] 0.15384615 0.14285714 0.13333333 0.1250000 0.11764706 0.11111111
##  [3,] 0.23076923 0.21428571 0.20000000 0.1875000 0.17647059 0.16666667
##  [4,] 0.30769231 0.28571429 0.26666667 0.2500000 0.23529412 0.22222222
##  [5,] 0.38461538 0.35714286 0.33333333 0.3125000 0.29411765 0.27777778
##  [6,] 0.46153846 0.42857143 0.40000000 0.3750000 0.35294118 0.33333333
##  [7,] 0.53846154 0.50000000 0.46666667 0.4375000 0.41176471 0.38888889
##  [8,] 0.61538462 0.57142857 0.53333333 0.5000000 0.47058824 0.44444444
```

```
##  [9,] 0.69230769 0.64285714 0.60000000 0.5625000 0.52941176 0.50000000
## [10,] 0.76923077 0.71428571 0.66666667 0.6250000 0.58823529 0.55555556
## [11,] 0.84615385 0.78571429 0.73333333 0.6875000 0.64705882 0.61111111
## [12,] 0.92307692 0.85714286 0.80000000 0.7500000 0.70588235 0.66666667
## [13,] 1.00000000 0.92857143 0.86666667 0.8125000 0.76470588 0.72222222
## [14,] 0.92857143 1.00000000 0.93333333 0.8750000 0.82352941 0.77777778
## [15,] 0.86666667 0.93333333 1.00000000 0.9375000 0.88235294 0.83333333
## [16,] 0.81250000 0.87500000 0.93750000 1.0000000 0.94117647 0.88888889
## [17,] 0.76470588 0.82352941 0.88235294 0.9411765 1.00000000 0.94444444
## [18,] 0.72222222 0.77777778 0.83333333 0.8888889 0.94444444 1.00000000
## [19,] 0.68421053 0.73684211 0.78947368 0.8421053 0.89473684 0.94736842
## [20,] 0.65000000 0.70000000 0.75000000 0.8000000 0.85000000 0.90000000
##           [,19] [,20]
##  [1,] 0.05263158  0.05
##  [2,] 0.10526316  0.10
##  [3,] 0.15789474  0.15
##  [4,] 0.21052632  0.20
##  [5,] 0.26315789  0.25
##  [6,] 0.31578947  0.30
##  [7,] 0.36842105  0.35
##  [8,] 0.42105263  0.40
##  [9,] 0.47368421  0.45
## [10,] 0.52631579  0.50
## [11,] 0.57894737  0.55
## [12,] 0.63157895  0.60
## [13,] 0.68421053  0.65
## [14,] 0.73684211  0.70
## [15,] 0.78947368  0.75
## [16,] 0.84210526  0.80
## [17,] 0.89473684  0.85
## [18,] 0.94736842  0.90
## [19,] 1.00000000  0.95
## [20,] 0.95000000  1.00
```

Part b

```
all.equal(t(A),A)
```

```
## [1] TRUE
```

Part c

```
C = solve(A)
C
```

```
##               [,1]          [,2]          [,3]          [,4]
##  [1,]  1.333333e+00 -6.666667e-01  1.110223e-16 -7.401487e-17
##  [2,] -6.666667e-01  2.133333e+00 -1.200000e+00  1.480297e-16
##  [3,] -1.413082e-32 -1.200000e+00  3.085714e+00 -1.714286e+00
##  [4,]  4.979293e-32 -2.044220e-16 -1.714286e+00  4.063492e+00
##  [5,] -7.524744e-32 -3.140025e-16  8.635068e-16 -2.222222e+00
##  [6,]  1.557983e-31  4.821318e-16 -9.735802e-16  1.965793e-15
```

```
##  [7,] -3.053043e-31 -1.833291e-16  3.586874e-16  3.347749e-16
##  [8,]  1.848893e-31 -3.483053e-18 -2.351061e-16 -1.980115e-15
##  [9,]  1.248003e-31 -3.093501e-18  2.381996e-16  2.001495e-15
## [10,]  7.269317e-17 -1.481689e-16 -5.787629e-16  1.085180e-16
## [11,] -1.529717e-16 -1.517075e-17  1.863474e-15 -4.043005e-15
## [12,]  7.964643e-17  1.592929e-16 -1.449855e-15  3.569319e-15
## [13,] -3.119139e-31  1.870931e-16 -1.496745e-17 -6.885028e-16
## [14,]  5.974151e-31 -1.880857e-16 -1.389551e-17  4.059474e-16
## [15,] -4.183917e-31 -9.262150e-19  4.167967e-16  2.445208e-16
## [16,]  5.719331e-17 -3.440281e-16  4.714378e-16 -2.246926e-15
## [17,] -1.180633e-16  7.075629e-16 -1.659423e-15  2.334304e-15
## [18,]  6.066576e-17 -3.647661e-16  9.714237e-16 -9.822259e-16
## [19,]  7.778034e-32 -7.309161e-19  7.309161e-19  1.611670e-15
## [20,] -3.034080e-31 -1.423363e-17  1.423363e-17 -1.366428e-15
##                [,5]          [,6]          [,7]          [,8]
##  [1,]  7.401487e-17  3.700743e-17 -7.401487e-17  3.700743e-17
##  [2,] -2.812565e-16  1.924387e-16  1.480297e-16 -1.184238e-16
##  [3,]  1.998401e-16 -1.617754e-16 -1.903239e-16  4.948423e-16
##  [4,] -2.222222e+00  1.015061e-15  5.004815e-16 -2.297985e-15
##  [5,]  5.050505e+00 -2.727273e+00 -3.083953e-16  3.016667e-15
##  [6,] -2.727273e+00  6.041958e+00 -3.230769e+00 -4.360149e-15
##  [7,] -1.339100e-15 -3.230769e+00  7.035897e+00 -3.733333e+00
##  [8,]  4.942452e-15 -5.673893e-15 -3.733333e+00  8.031373e+00
##  [9,] -3.078033e-15 -2.883143e-15  8.315330e-15 -4.235294e+00
## [10,] -1.182568e-15  8.826134e-15 -1.324477e-14  4.883312e-15
## [11,]  4.022777e-15 -6.634674e-15  1.472574e-14 -8.783864e-15
## [12,] -3.241465e-15  5.670826e-15 -2.354291e-14  2.504431e-14
## [13,]  6.970556e-16 -5.751778e-15  2.542115e-14 -3.232114e-14
## [14,] -7.999845e-16  7.041054e-15 -1.892866e-14  2.627046e-14
## [15,] -4.779269e-16 -5.301655e-15  8.471162e-15 -1.470088e-14
## [16,]  3.131632e-15 -7.119319e-17 -4.297638e-16  5.275242e-15
## [17,] -8.546314e-16 -6.699787e-17 -9.706521e-16 -1.830186e-15
## [18,] -4.560059e-16  4.498332e-16  7.130975e-15 -9.748958e-15
## [19,] -1.324420e-15 -3.819767e-15 -5.090099e-15  1.626727e-14
## [20,]  8.540177e-16  4.440892e-15 -1.423363e-15 -6.291264e-15
##                [,9]         [,10]         [,11]         [,12]
##  [1,] -1.110223e-16  1.480297e-16 -2.590520e-16  1.664003e-31
##  [2,]  4.440892e-16 -1.628327e-16  2.960595e-17  8.881784e-17
##  [3,] -5.709718e-16 -8.659740e-16  1.636786e-15 -7.993606e-16
##  [4,]  2.488309e-15  9.375217e-16 -4.807442e-15  3.454027e-15
##  [5,] -3.067131e-15 -2.181196e-15  6.369764e-15 -4.317534e-15
##  [6,] -2.496837e-15  9.307240e-15 -9.055693e-15  7.532436e-15
##  [7,]  4.184687e-15 -1.131061e-14  1.494531e-14 -2.080786e-14
##  [8,] -4.235294e+00 -1.421085e-15 -6.687461e-15  1.984992e-14
##  [9,]  9.027864e+00 -4.736842e+00 -8.463818e-15  3.641050e-15
## [10,] -4.736842e+00  1.002506e+01 -5.238095e+00 -1.168656e-14
## [11,] -3.135288e-15 -5.238095e+00  1.102277e+01 -5.739130e+00
## [12,] -5.057983e-15 -3.005132e-15 -5.739130e+00  1.202087e+01
## [13,]  1.965440e-14 -1.554904e-14  9.934111e-15 -6.240000e+00
## [14,] -2.404122e-14  2.238567e-14 -2.808084e-14  1.270844e-14
## [15,]  1.726465e-14 -5.379457e-15  1.268915e-14 -2.252184e-14
## [16,] -3.302669e-15 -1.532390e-14  1.264808e-14  2.028311e-14
## [17,]  1.274594e-16  3.212630e-15  3.490426e-15 -3.057718e-14
## [18,]  2.685882e-15  1.189165e-14 -2.803009e-14  4.085226e-14
```

```
## [19,] -6.244947e-15 -4.096054e-15  6.765359e-15 -8.735909e-15
## [20,]  1.622634e-15  1.708035e-16  7.401487e-15 -1.024821e-14
##               [,13]         [,14]         [,15]         [,16]
##  [1,] -2.220446e-16  1.110223e-16  2.220446e-16 -3.700743e-17
##  [2,]  3.552714e-16 -3.996803e-16 -1.776357e-16 -2.812565e-16
##  [3,] -2.474211e-16  2.188725e-16 -1.141944e-16  7.708120e-16
##  [4,] -1.268826e-15  2.608143e-16  9.022765e-16 -2.735026e-15
##  [5,]  1.211152e-15  6.111834e-16 -2.512020e-15  3.072738e-15
##  [6,] -5.398945e-15  3.582216e-15 -3.726623e-17  7.080583e-16
##  [7,]  1.993904e-14 -1.330730e-14 -3.188333e-17 -1.673875e-15
##  [8,] -2.363948e-14  1.700078e-14  9.125598e-16 -4.702121e-16
##  [9,]  1.055812e-14 -1.161610e-14 -3.093501e-17  4.210254e-15
## [10,] -4.671841e-15  1.165039e-14  1.716811e-15 -1.746584e-14
## [11,]  6.396999e-16 -1.468781e-14  1.241473e-15  2.873087e-14
## [12,] -6.240000e+00  6.950962e-15 -4.884209e-15 -4.207649e-15
## [13,]  1.301926e+01 -6.740741e+00 -6.004086e-15 -1.575004e-14
## [14,] -6.740741e+00  1.401788e+01 -7.241379e+00  4.029698e-15
## [15,]  2.294790e-14 -7.241379e+00  1.501669e+01 -7.741935e+00
## [16,] -3.361013e-14  4.495068e-14 -7.741935e+00  1.601564e+01
## [17,]  2.919800e-14 -2.321721e-14  3.153802e-14 -8.242424e+00
## [18,] -1.974019e-14  5.836258e-15 -1.293946e-14  5.999834e-15
## [19,]  4.017115e-15 -7.560596e-15 -1.385817e-15  2.937405e-14
## [20,]  5.124106e-16  1.041902e-14  1.076062e-14 -3.416071e-14
##               [,17]         [,18]         [,19]         [,20]
##  [1,] -7.401487e-17  1.110223e-16  1.110223e-16  1.480297e-16
##  [2,]  5.033011e-16 -1.332268e-16 -1.051815e-30 -2.516506e-16
##  [3,] -1.199041e-15 -8.564578e-17  4.758099e-16 -6.661338e-17
##  [4,]  2.615192e-15  3.313046e-16 -1.903239e-16 -4.440892e-16
##  [5,] -1.149473e-15 -1.149473e-15  5.046468e-17 -1.513940e-16
##  [6,] -6.386500e-15  2.580686e-15 -1.392825e-15  3.461101e-15
##  [7,]  1.505292e-14 -5.356399e-15 -7.771561e-16 -2.634360e-15
##  [8,] -1.364660e-14  4.775265e-15  8.045852e-15 -3.869671e-15
##  [9,]  1.132221e-15  3.715294e-15 -1.157588e-14  4.661906e-15
## [10,]  8.208416e-15  8.152766e-15 -6.341349e-15  2.237141e-15
## [11,] -6.983602e-15 -3.570689e-14  3.066261e-14 -7.737082e-15
## [12,] -1.034303e-14  3.890221e-14 -3.398557e-14  1.033608e-14
## [13,]  2.552164e-14 -2.322093e-14  1.503587e-14 -6.070371e-15
## [14,] -3.215421e-14  3.300383e-14 -1.283945e-14  2.350327e-15
## [15,]  3.617796e-14 -5.504681e-14  3.357344e-14 -1.782038e-15
## [16,] -8.242424e+00  4.034049e-14 -3.113226e-14  2.689713e-15
## [17,]  1.701472e+01 -8.742857e+00  2.236258e-14 -1.364632e-14
## [18,] -8.742857e+00  1.801390e+01 -9.243243e+00 -2.158274e-14
## [19,] -2.395650e-14 -9.243243e+00  1.901317e+01 -9.743590e+00
## [20,]  2.863806e-14 -3.871547e-14 -9.743590e+00  1.025641e+01
```

```
ident = diag(20)
product = A%*%C
all.equal(product, ident)
```

```
## [1] TRUE
```

Part d

```
d = c(1,2,3,4,5,6,7,8,9,10,10,9,8,7,6,5,4,3,2,1)
d
```

```
##  [1]  1  2  3  4  5  6  7  8  9 10 10  9  8  7  6  5  4  3  2  1
```

Part e

```
x = solve(A, C%*%d)
x
```

```
##                [,1]
##  [1,] -3.404684e-15
##  [2,]  9.118632e-15
##  [3,]  1.652012e-14
##  [4,] -4.618528e-14
##  [5,]  2.220446e-14
##  [6,] -2.295600e-14
##  [7,]  7.320412e-14
##  [8,] -5.305386e-14
##  [9,] -2.481203e+01
## [10,]  2.006424e+01
## [11,]  3.581375e+01
## [12,] -3.006263e+01
## [13,] -3.736996e-04
## [14,] -2.772044e-04
## [15,] -2.099688e-04
## [16,] -1.619541e-04
## [17,] -1.269228e-04
## [18,] -1.008779e-04
## [19,]  9.505933e+01
## [20,] -1.000629e+02
```