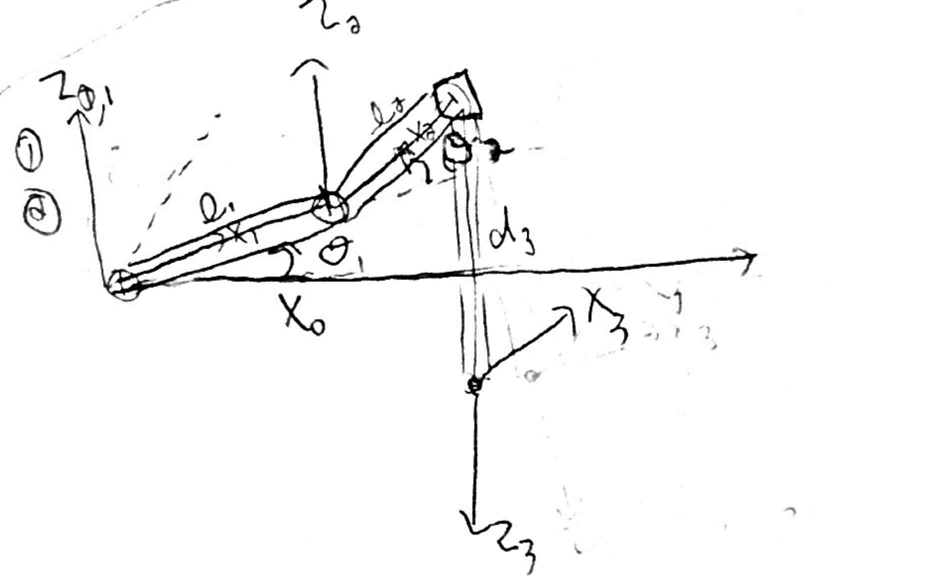
In general,

1) Draw robot
2) Pick frames
3) Write table
4) Write Transforms ${}^{i-1}_{i}T$, ${}^{world}_{tool}T$
5) Solve the inverse kinematics (given a desired tool position, $[x, y, z]$, what joint variables do we need to get there?
6) Compute Jacobian. We have Cart Space $= f($Joint Space$)$ from 5. To get velocity, $\dot{CS} = J(JS)\dot{f}(JS)$. So, J maps velocity in JS to velocity in CS.



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i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	l_1	0	θ_2
3	π	l_2	$-d_3$	0

4) $${}^{i-1}_{i}T = R_X(\alpha_{i-1})D_X(a_{i-1})R_Z(\theta_i)D_Z(d_i)$$

$$= \begin{bmatrix} c(\theta_i) & -s(\theta_i) & 0 & a_{i-1} \\ s(\theta_i)c(\alpha_{i-1}) & c(\theta_i)c(\alpha_{i-1}) & -s(\alpha_{i-1}) & -s(\alpha_{i-1})d_i \\ s(\theta_i)s(\alpha_{i-1}) & c(\theta_i)s(\alpha_{i-1}) & c(\alpha_{i-1}) & c(\alpha_{i-1})d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$d_{\hat{Q}}(x,y)$ is dist(x,y), measured along $\hat{Q}$ axis

where $a_i = d_{\hat{X}_i}(\hat{Z}_i, \hat{Z}_{i+1})$ (carefully sign!)

$\alpha_i = \text{angle}_{\hat{X}_i}(\hat{Z}_i, \hat{Z}_{i+1})$

$d_i = d_{\hat{Z}_i}(\hat{X}_{i-1}, \hat{X}_i)$

$\theta_i = \text{angle}_{\hat{Z}_i}(\hat{X}_{i-1}, \hat{X}_i)$

So, for our robot, (recall ${}^{\{i-1\}}_{\{i\}}T = \begin{bmatrix} {}^{\{i-1\}}_{\{i\}}R & {}^{\{i-1\}}P_{ORG\{i\}} \\ 0 & 1 \end{bmatrix}$)

$$ {}^{0}_{1}T = \begin{bmatrix} R_{\hat{z}}(\theta_1) & 0 \\ 0 & 1 \end{bmatrix}, \quad {}^{1}_{2}T = \begin{bmatrix} R_{\hat{z}}(\theta_2) & \begin{matrix} l_1 \\ 0 \\ 0 \end{matrix} \\ 0 & 1 \end{bmatrix} $$

by inspection, using

$$ {}^{2}_{3}T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

from formula 4, be sure to check if makes sense at every step.

So,
$$ {}^{0}_{3}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T = \begin{bmatrix} R_{\hat{z}}(\theta_1) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{\hat{z}}(\theta_2) & \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} {}^{2}_{3}T $$

$$ = \begin{bmatrix} R_{\hat{z}}(\theta_1) R_{\hat{z}}(\theta_2) & R_{\hat{z}}(\theta_1) \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} {}^{2}_{3}T $$

$$ = \begin{bmatrix} c(\theta_1+\theta_2) & -s(\theta_1+\theta_2) & 0 & c(\theta_1) l_1 \\ s(\theta_1+\theta_2) & c(\theta_1+\theta_2) & 0 & s(\theta_1) l_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \ldots $$

$$= \begin{bmatrix} c(\theta_1+\theta_2) & s(\theta_1+\theta_2) & 0 & l_2 c(\theta_1+\theta_2)+c(\theta_1)l_1 \\ s(\theta_1+\theta_2) & -c(\theta_1+\theta_2) & 0 & l_2 s(\theta_1+\theta_2)+s(\theta_1)l_1 \\ 0 & 0 & -1 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^{0}_{3}T = {}^{WORLD}_{TOOL}T$$

Could use fixed point numbers, 24-digit is sufficient, could improve precision.

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⑤ Given $[x, y, z]$, what joint variables, $[\theta_1, \theta_2, d_3]$ do we need to reach there? Obviously, $z = -d_3$, so, it comes to just the plane, so,

$$l_2 c(\theta_1+\theta_2) + c(\theta_1) l_1 = x$$
$$l_2 s(\theta_1+\theta_2) + s(\theta_1) l_1 = y$$

$$+\ (l_2 c_{12})^2 + 2 l_2 c_{12} c_1 l_1 + (c_1 l_1)^2 = x^2$$
$$(l_2 s_{12})^2 + 2 l_2 s_{12} s_1 l_1 + (s_1 l_1)^2 = y^2$$

$$l_2^2 + 2 l_2 l_1 (c_{12} c_1 + s_{12} s_1) + l_1^2 = x^2 + y^2$$
$$l_2^2 + 2 l_2 l_1 c(\theta_2) + l_1^2 = x^2 + y^2$$

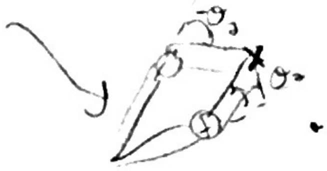
thus this is bounded in magnitude by 1,

$$c(\theta_2) = \left(\frac{x^2 + y^2 - (l_2^2 + l_1^2)}{2 l_2 l_1} \right)$$

$$\boxed{\theta_2 = \tan^{-1}(s(\theta_2), c(\theta_2))}$$

$$s(\theta_2) = \pm\sqrt{1 - c(\theta_2)}$$

This gives 2 possible solutions, , that corresp to "elbow-up" and "elbow-down"



then, $l_2\big(c(\theta_1)c(\theta_2) - s(\theta_1)s(\theta_2)\big) + c(\theta_1)l_1 = x$

$\vdots$

$x = k_1 c_1 - k_2 s_1$ $\quad k_1 = l_1 + l_2 c_2$

$y = k_1 s_1 + k_2 c_1$, $\quad k_2 = l_2 s_2$

Let $r = \sqrt{k_1^2 + k_2^2}$,

$\gamma = \tan^{-1}(k_2, k_1)$,

$k_1 = r\cos(\gamma)$, $\quad \frac{x}{r} = c(\gamma + \theta_1)$

$k_2 = r\sin(\gamma)$, $\quad \frac{y}{r} = s(\gamma + \theta_1)$

So, $\gamma + \theta_1 = \tan^{-1}\left(\frac{y}{r}, \frac{x}{r}\right) = \tan^{-1}(y, x)$

$\therefore \quad \theta_1 = \tan^{-1}(y, x) - \tan^{-1}(k_2, k_1)$

There are many ways to find θ_1 & θ_2.
This was just one of them.

⑥ we need velocities in CS, we use:

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{\ \ i}R\,{}^{i}\omega_i + \begin{bmatrix}0\\0\\\dot{\theta}_{i+1}\end{bmatrix}, $$

${}^{i+1}\omega_{i+1}$ is the angular velocity of link $i+1$ in global coords, expressed in frame $\{i+1\}$.

$$ {}^{i+1}V_{i+1} = {}^{i+1}_{\ \ i}R\left({}^{i}V_i + {}^{i}\omega_i \times {}^{i}P_{i+1}\right) + \begin{bmatrix}0\\0\\\dot{d}_{i+1}\end{bmatrix} $$

in the $\hat{Z}$-axis since that's how we chose the frames!

So, for our robot,

$$ {}^{1}\omega_1 = \begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix}, \quad {}^{1}V_1 = 0, \quad {}^{2}\omega_2 = \begin{bmatrix}0\\0\\\dot{\theta}_1+\dot{\theta}_2\end{bmatrix}, \quad {}^{2}V_2 = \begin{bmatrix}c(\theta_2) & s(\theta_2) & 0\\ -s(\theta_2) & c(\theta_2) & 0\\ 0 & 0 & 1\end{bmatrix}\left[\begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix}\times\begin{bmatrix}l_1\\0\\0\end{bmatrix}\right. $$

$$ {}^{2}V_2 = \begin{bmatrix} l_1 s(\theta_2)\dot{\theta}_1 \\ l_1 c(\theta_2)\dot{\theta}_1 \\ 0 \end{bmatrix}, \quad {}^{3}\omega_3 = {}^{3}_{2}R\,{}^{2}\omega_2 = {}^{2}\omega_2, \quad {}^{3}V_3 = \begin{bmatrix}0\\0\\\dot{d}_3\end{bmatrix} + R_x(\pi)\left({}^{2}V_2 + \begin{bmatrix}0\\0\\\dot{\theta}_1+\dot{\theta}_2\end{bmatrix}\times\begin{bmatrix}l_2\\0\\-d_3\end{bmatrix}\right) $$

$$ {}^{3}V_3 = \begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{bmatrix}\left({}^{2}V_2 + \begin{bmatrix}0 & -(\dot{\theta}_1+\dot{\theta}_2) & 0\\(\dot{\theta}_1+\dot{\theta}_2) & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}l_2\\0\\-d_3\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{bmatrix}\left(\begin{bmatrix} l_1 s(\theta_2)\dot{\theta}_1 \\ l_2(\dot{\theta}_1+\dot{\theta}_2) + l_1 c(\theta_2)\dot{\theta}_1 \\ 0\end{bmatrix}\right) $$

$$ {}^{3}V_3 = \begin{bmatrix} l_1 s(\theta_2)\dot{\theta}_1 \\ -(l_2(\dot{\theta}_1+\dot{\theta}_2) + l_2 c(\theta_2)\dot{\theta}_1 \\ \dot{d}_3 \end{bmatrix} $$