# Distributed optimal SVFB cooperative control of multi-agents systems

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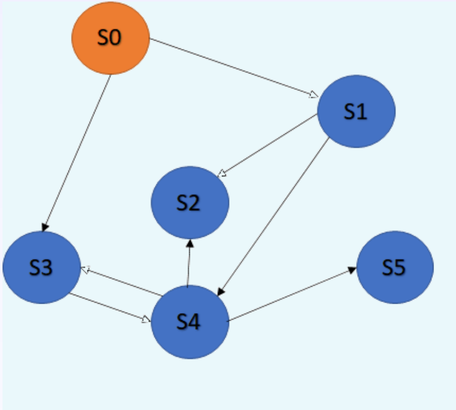
# General formulation

- Here we will focus on CPS described as multi-agents systems modeled by means of directed graphs (digraphs)




- Each node $S_i$ of the graph is a dynamical LTI system
- Edges are used to model communication between agents

# General formulation

We will consider multi-agents control problems (generically called here *synchronization problems*) where the CPS is made up of

- 1 *leader node* $S_0$
- $N$ *follower nodes* $S_i$ $(i = 1, 2, \dots, N)$
- The agents cooperate in order to perform a task possibly dictated (to some extent) by the leader node
- In order to perform the assigned task the follower nodes exploit information shared on the communication network represented by the digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$
- Examples: autonomous vehicles platooning, unmanned air vehicles (UAVs) formation control, control of autonomous mobile robots team, ...

# General formulation: cooperative tracking problems

- In this part we will mainly focus on *cooperative tracking problem* where the task to be performed is dictated by the leader node $S_0$ which generates the desired target trajectory (acting as a command generator exosystem)

- We will use the term *cooperative regulator problem* when the consensus is sought among agents in the absence of a leader node

- Assumption: the N follower agent are identical

# Reference book

B5 *Cooperative Control of Multi-Agent Systems*, F. Lewis et al., Springer London , 2013

# Agents mathematical models

- The dynamics of the **leader node** $S_0$ is described by

$$\dot{x}_0 = Ax_0, \;\; y_0 = Cx_0 \tag{1}$$

where $x_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^p$

- The dynamics of the N identical **follower nodes** $S_i$ is described by

$$\dot{x}_i = Ax_i + Bu_i, \;\; y_i = Cx_i \tag{2}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_0 \in \mathbb{R}^p$ and $i \in \mathcal{N} = \{1, 2, \ldots, N\}$

- The triple $(A, B, C)$ is *stabilizable* and *detectable*

# Communication network modeling

- The **follower nodes** $S_i$ share information through a communication network represented as a digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ with $N$ nodes $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ and a set of edges (arcs) $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$
- The **adjacency matrix** associated to $\mathcal{G}$ is $\mathcal{A} = [a_{ij}] \in \mathbb{R}^N$
- $a_{ij} > 0$ **is the weight for edge** $(v_j, v_i)$ implying that node $i$ can get information from node $j$ (node $j$ is a neighbor of node $i$)
- The neighbor set of node i is denoted as $\mathcal{N}_i = \{j | a_{ij} > 0\}$
- We assume there is no self-loop ($a_{ii} = 0, \forall i$)
- To describe connection between the leader node and the graph $\mathcal{G}$ of follower nodes we define the **augmented graph** $\overline{\mathcal{G}} = \{\overline{\mathcal{V}}, \overline{\mathcal{E}}\}$ where $\overline{\mathcal{V}} = \{v_0, v_1, \ldots, v_N\}$, $\overline{\mathcal{E}} \subset \overline{\mathcal{V}} \times \overline{\mathcal{V}}$

# Cooperative state variables feedback (SBVF) control

## Fundamental assumptions

- All the **state variables** of the agents are assumed to be measurable ($C = I$ in equation (2))
- The **leader node** $S_0$ can only be observed by a small subset of the follower nodes
- **If node $i$ observes the leader**, node $i$ is said to be **pinned to the leader**. Therefore an edge $(v_0, v_i)$ **exists** in the augmented graph $\overline{\mathcal{G}}$ with **weight** $g_i$ **(pinning gain)**
- **The control law of each agent can only use the local neighborhood information** of that agent, according to the graph topology (fully distributed control scheme)
- There exists at least **one directed path from the leader node to every follower node**.

# Cooperative state variables feedback (SBVF) control

## Local controller at each node $i$

### Neighborhood tracking error of node $i$

vector

$$\varepsilon_i = \sum_{j=1}^{N} a_{ij}(x_j - x_i) + g_i(x_0 - x_i) \tag{3}$$

### SVFB control protocol for each node $i$

$$u_i = cK\varepsilon_i \tag{4}$$

- coupling gain: $c > 0$
- feedback gain matrix: $K \in \mathbb{R}^{m \times n}$

# Cooperative state variables feedback (SBVF) control

## Closed-loop control system

### Closed-loop system of each node $i$

$$\dot{x}_i = Ax_i + cBK\left(\sum_{j=1}^{N} a_{ij}(x_j - x_i) + g_i(x_0 - x_i)\right) \tag{5}$$

### Global closed-loop system dynamics

$$\dot{x} = (I_N \otimes A - c(L+G) \otimes BK)x + (c(L+G) \otimes BK)\underline{x}_0 \tag{6}$$

- $x = col(x_1, x_2, \ldots, x_N) \in \mathbb{R}^{nN}$, $\underline{x}_0 = col(x_0, x_0, \ldots, x_0) \in \mathbb{R}^{nN}$
- Laplacian matrix of $\mathcal{G}$, $L = [l_{ij}] = D - \mathcal{A}$, $D = diag(d_1, d_2, \ldots, d_N)$, $d_i$ in-degree of node $i$
- Pinning matrix $G = diag(g_1, g_2, \ldots, g_N)$, $\otimes$ is the Kronecker product

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix}$$

$$A \otimes B = \begin{bmatrix} 1 \cdot 0 & 1 \cdot 5 & 2 \cdot 0 & 2 \cdot 5 \\ 1 \cdot 6 & 1 \cdot 7 & 2 \cdot 6 & 2 \cdot 7 \\ & & & \end{bmatrix}$$

# Cooperative state variables feedback (SBVF) control

## Disagreement error $\delta$

### Local disagreement error at each node $i$

$$\delta_i(t) = x_i(t) - x_0(t) \tag{7}$$

### Global disagreement error and global disagreement error dynamics

$$\delta(t) = x(t) - \underline{x}_0(t) = col(\delta_1, \delta_2, \ldots, \delta_N) \tag{8}$$

$$\dot{\delta} = \dot{x} - \dot{\underline{x}}_0 = A_c\delta \tag{9}$$

$$A_c = I_N \otimes A - c(L+G) \otimes BK \tag{10}$$

# Cooperative state variables feedback (SBVF) control

## Objective and solution

### Objective of the cooperative tracking problem

The cooperative tracking problem is solved if

$$\lim_{t \to \infty} \delta(t) = 0 \tag{11}$$

### Cooperative tracking problem solution

The global disagreement error converges to 0 if and only if matrix

$$A_c = I_N \otimes A - c(L + G) \otimes BK \tag{12}$$

is *Hurwitz* (i.e., it has all the eigenvalues with strictly negative real part)

## Polinomi [ modifica | modifica wikitesto ]

Dato un polinomio reale:

$$p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

la matrice di Hurwitz corrispondente al polinomio $p$ è la matrice quadrata di dimensione $n \times n$ data da:

$$H = \begin{pmatrix} a_1 & a_3 & a_5 & \dots & \dots & \dots & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & & & & \vdots & \vdots & \vdots \\ 0 & a_1 & a_3 & & & & \vdots & \vdots & \vdots \\ \vdots & a_0 & a_2 & \ddots & & & 0 & \vdots & \vdots \\ \vdots & 0 & a_1 & & \ddots & & a_n & \vdots & \vdots \\ \vdots & \vdots & a_0 & & & \ddots & a_{n-1} & 0 & \vdots \\ \vdots & \vdots & 0 & & & & a_{n-2} & a_n & \vdots \\ \vdots & \vdots & \vdots & & & & a_{n-3} & a_{n-1} & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & a_{n-4} & a_{n-2} & a_n \end{pmatrix}$$

Nel 1895 Adolf Hurwitz ha stabilito (criterio di Routh-Hurwitz) che un polinomio è stabile (ovvero le radici hanno parte reale strettamente negativa) se e solo se tutti i minori principali di guida della matrice di $H(p)$ sono positivi:

$$\Delta_1(p) = \begin{vmatrix} a_1 \end{vmatrix} = a_1 > 0$$

$$\Delta_2(p) = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = a_2 a_1 - a_0 a_3 > 0$$

$$\Delta_3(p) = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = a_3 \Delta_2 - a_1 (a_1 a_4 - a_0 a_5) > 0$$

e così via. I minori $\Delta_k(p)$ sono detti determinanti di Hurwitz.

# Cooperative state variables feedback (SBVF) control

## Closed-loop CPS eigenvalues

- The following Lemma provides useful insight about the eigenvalues of the closed-loop multi-agents system

**Lemma 1 (closed-loop eigenvalues**

$$eig(A_c) = \bigcup_{i=1}^{N} eig(A - c\lambda_i BK) \tag{13}$$

where $\lambda_i,\ i = 1, \ldots, N$ are the eigenvalues of the matrix $L + G$

- Lemma 1 can be easily proved by analyzing the structure of matrix $A_c$ in (12)

## Controller design (I)

The following crucial considerations are directly obtained from Lemma 1:

- The closed-loop dynamics of the whole multi-agents system depends both on the local controller parameters $K$ and $c$ and on the eigenvalues of matrix $L + G$ accounting for the communication network effect

- Stability of the single agent dynamics does not imply stability of the multi-agents systems

$$(A - BK) \text{ Hurwitz} \not\Rightarrow A_c \text{ Hurwitz} \tag{14}$$

?

- The coupling parameter $c$ is introduced to cope with the effect of $\lambda_i$ on the global multi-agent system dynamics

# Cooperative state variables feedback (SBVF) control

## Controller design (II)

### Theorem 1 (Cooperative controller design)

Consider the local distributed control protocols given in equation (4). Design the SVFB control gain $K$ as

$$K = R^{-1}B'P \tag{15}$$

any positive definite matrix

where $P$ is the unique positive definite solution of the algebraic Riccati equation (ARE)

$$A'P + PA + Q - PBR^{-1}B'P = 0 \tag{16}$$

Then

$$\lim_{t\to\infty} \delta(t) = 0 \tag{17}$$

if

$$c \geq \frac{1}{2\min_{i\in\mathcal{N}} Re(\lambda_i)} \tag{18}$$

# Proof of Theorem 1: preliminary results control

## Lyapunov equation

**Result 1 (Stability of LTI systems)**

Given an LTI systems described by the state equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{19}$$

the system is asymptotically stable if and only if the *Lyapuonov equation*

$$A * P + PA + = -Q, \;\; Q > 0 \tag{20}$$

has a solution $P$, symmetric and positive definite Then

$$\lim_{t\to\infty} \delta(t) = 0 \tag{21}$$

if

$$c \geq \frac{1}{2\min_{i\in\mathcal{N}} Re(\lambda_i)} \tag{22}$$

# Proof of theorem 1

## PRELIMINARY RESULTS

- LYAPUNOV EQ. and stability of a single LTI agent

Given an LTI sys described by:

$$\dot{x} = Ax(t) + Bu(t) \qquad (1)$$

The sys is asymptotically stable (i.e $A$ is Hurwitz $\triangleq \forall\, eig(A)<0$)

$\iff$

$A^T P + PA = -Q, Q>0 \; (2)$ has a solution

$P > 0$ which symmetric

- Optimal linear quadratic ctrl of LTI sys (LQ ctrl)

Take into consideration:

$$\dot{x} = Ax(t) + Bu(t)$$

and assume that all the state variables are physically measured

Compute the state feedback gain $K$ (s.t $u = -Kx$) which solves the following optimization problem:

$$u(t) = \underset{u(t)}{\arg\min} \underbrace{\left[ \frac{1}{2} \int_0^\infty x^T(t)\, Q\, x(t) + u^T(t)\, R\, u(t) \right] dt}_{J} \qquad (3)$$

s.t

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$J$ is given by the weighted sum of the 2-norm (energy) of the states and the 2-norm of

- Why are we interested in minimizing $J$?

(i) by minimizing the energy of $x(t)$

$$\Rightarrow \begin{cases} \cdot \text{ shorter transient} \\ \cdot \text{ damping oscillations} \end{cases}$$

(ii) by minimizing the energy of $u(t) \Rightarrow$ minimize the energy from our actuators

- Matrices $Q > 0$, $R > 0$ are selected in order to look for the best tradeoff btw (i) and (ii)

- $Q$ and $R$ are user defined parameter

- Typically $Q > 0$, $R > 0$ are selected as diagonal matrix

$$Q = \begin{bmatrix} q_1 & & \emptyset \\ & \ddots & \\ \emptyset & & q_n \end{bmatrix}$$

- $q_i$ is weighting the energy of $x_i$

$$R = \begin{bmatrix} r_1 & & & \emptyset \\ & r_2 & & \\ & & \ddots & \\ \emptyset & & & r_p \end{bmatrix}$$

- $r_j$ is weighting the energy of input $u_j$

**Result 2: Solution to optimal LQ ctrl problem**

The optimal sol. to problem (3) is given by:

$$u(t) = -K \cdot x(t)$$

where:

$$K = R^{-1} B^T P$$

with $P > 0$ obtained as sol. to the algebraic Riccati eq.:

$$A^T P + PA + Q - P \cdot B \cdot R^{-1} \cdot B^T \cdot P = \emptyset \qquad (4)$$

<u>REMARK 1</u>: Problem 3 is cvx opt. problem $\Rightarrow$ unique global optimum which is exactly obtained by setting:

$$u(t) = -K x(t)$$

<u>REMARK 2</u>: $K = R^{-1} B^T P$ applied to the single agent asymptotic stability of local closed loop agent