Elements of Maths for Biology

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20210927

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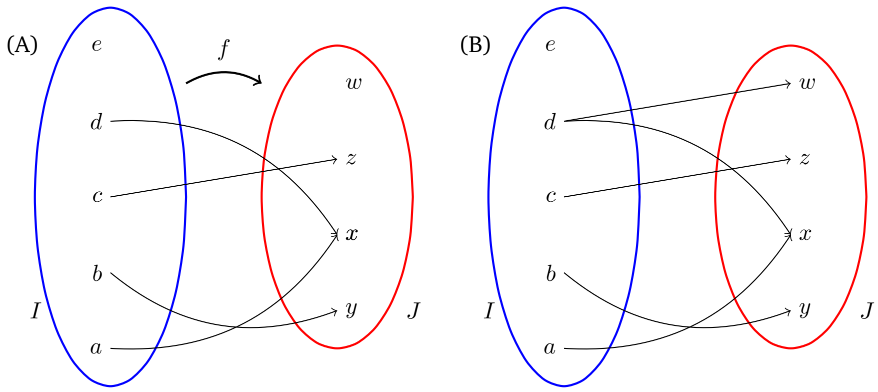

Figure 1: Functions. (A) Function f. (B) Not a function.

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1 README

This document is intended to serve both for training and for future reference. As a reference document, you may find it useful for the first biomaths class (3BS, Fall semester) and for linear algebra (3BIM, Winter Semester).

When important concepts are encoutered for the first time, they highlighted in **bold** next to their definition. Exercises are important, they can introduce theory or techniques that will be prove useful. We tried to make the examples as complete as possible. This means that they are long, you could probably solve them faster.

2 Fonctions, maps

A **function** is a relation, denoted in general f, that associate an element x belonging to a **domain** I, and at most an element y of the **image** J. The domain I and J are sets, usually $I, J \in \mathbb{R}$.

function
domain
image

A **map** is a relation that associate *each* element of its domain to exactly one element of its image. Maps and functions are related but slightly different concepts. A function f is a map if it is defined for all elements of of its domain I. A map is always a function, but the term can also be used when the domain or the image are not numbers (Figure 1).

map

The **graph** of a function f, denoted $\mathcal{G}(f)$ is the set of all pairs $(x, f(x))$ in the $I \times J$ plane. For real-valued functions, the graph is represented in the Cartesian plane. **graph**

Functions are not numbers. Do not confuse

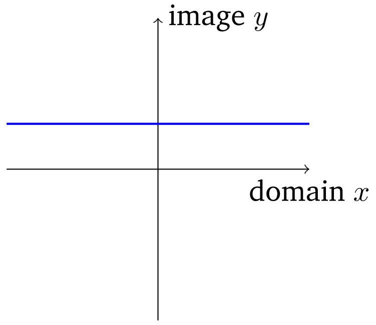
- f the function
- $f(x)$ the evaluation of f at element x; $f(x)$ is an element of the image (usually a number)
- $\mathcal{G}(f)$ the graph of f.

Consequently, never write

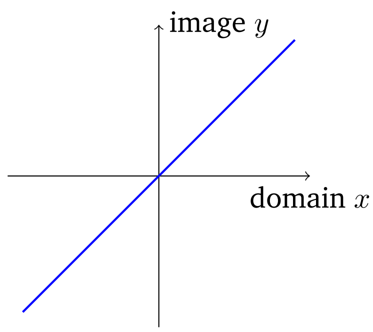
- $f(x)$ is increasing... but write f is increasing...
- $f(x)$ is decreasing... but write f is decreasing...
- $f(x)$ is continuous... but write f continuous...

2.1 Usual maps

- $f : \mathbb{R} \to \mathbb{R}$, with $x \to k$, $k \in \mathbb{R}$ constant; $x \to x$, identity map.

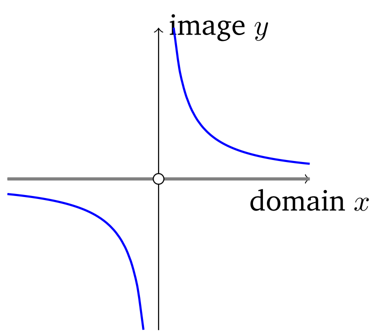


constant map

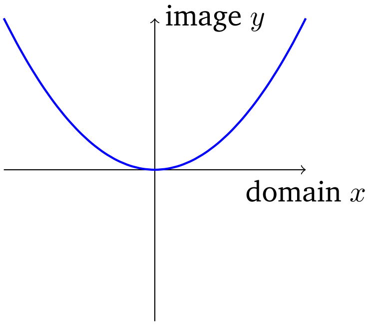


identity map

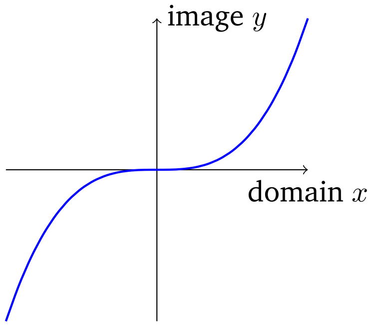
- $f : \mathbb{R} \backslash \{0\} \to \mathbb{R}$, with $x \to \frac{1}{x}$, inverse.



- $f : \mathbb{R} \to \mathbb{R}$, with $x \to x^2$, parabola; $x \to x^3$, cubic map.

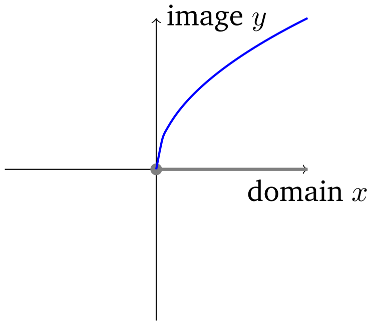


parabola

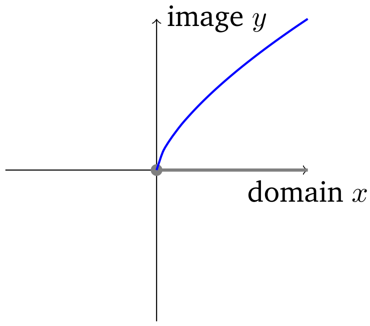


cubic map

- $f : \mathbb{R}^+ \to \mathbb{R}$, with $x \to \sqrt{x} = x^{\frac{1}{2}}$, square root; more generally with $x \to x^{\frac{p}{q}} = \sqrt[q]{x^p}$, fractional power.

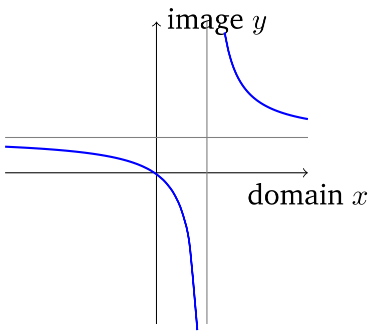


square root

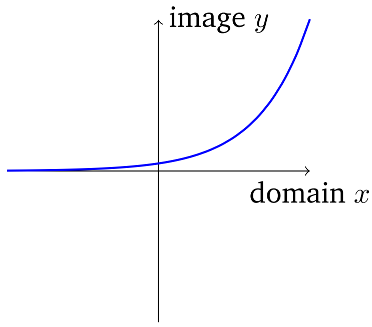


fractional root $(p = 2, q = 3)$

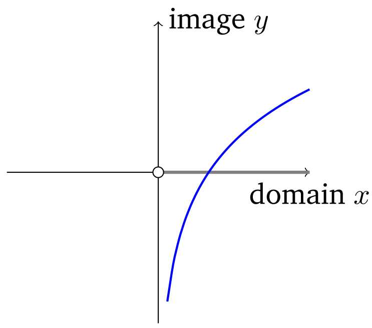
- $f : \mathbb{R}\backslash\{-d/c\} \to \mathbb{R}$, with $x \to \frac{ax+b}{cx+d}$.



- $f : \mathbb{R} \to \mathbb{R}$, with $x \to \exp(x)$, exponential.

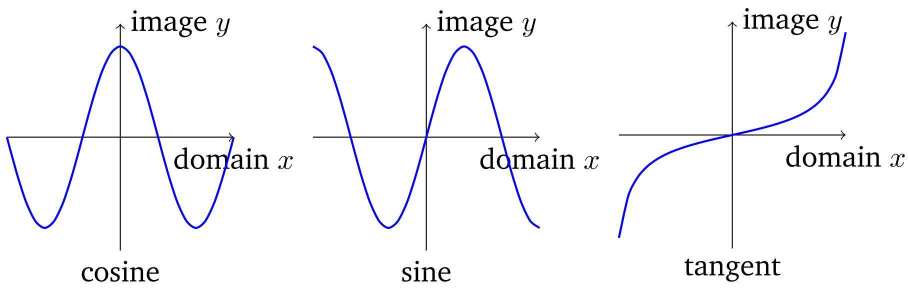


- $f : \mathbb{R}^+ \backslash \{0\} \to \mathbb{R}$, with $x \to \ln(x)$, natural logarithm.

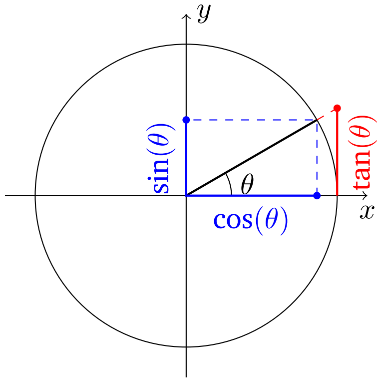


On logarithms: For $a, b > 0$, n positive integer, $\ln(ab) = \ln(a) + \ln(b)$, $\ln(a^n) = n \ln(a)$, $\ln(a/b) = \ln(a) - \ln(b)$.

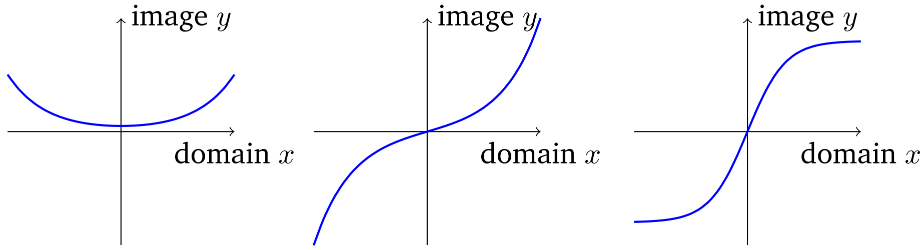
- $f : \mathbb{R} \to \mathbb{R}$, with $x \to \cos(x)$, cosine; $x \to \sin(x)$, sine; $x \to \tan(x)$, tangent.



On trigonometric functions. In the diagram below is shown the relationship between sine, cosine and tangent, of a angle θ.



- $f : \mathbb{R} \to \mathbb{R}$, with $x \to \cosh(x) = \frac{1}{2}(e^x + e^{-x})$, hyperbolic cosine; $x \to \sinh(x) = \frac{1}{2}(e^x - e^{-x})$, hyperbolic sine; $x \to \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$, hyperbolic tangent.



hyperbolic cosine hyperbolic sine hyperbolic tangent

2.2 Exercises on functions

3 Derivatives

We call the **derivative** of the function $f : I \to J$ $(I, J \subset \mathbb{R})$,, at point $a \in I$ the limit, if it exists, derivative

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

The derivative is denoted $f'(a)$. An alternative representation of the limit is obtained by setting $h = x - a$,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

If the derivative exists for all elements $a \in I$, we say that **differentiable** on I. differentiable

- If f is differentiable on I, and $f'(x) > 0$, then f is strictly increasing on I.
- If f is differentiable on I, and $f'(x) < 0$, then f is strictly decreasing on I.

However, if f is strictly increasing, it does not mean that $f'(x) > 0$. For example the function f with $f(x) = x^3$ is strictly increasing on $\mathbb{R}$, but $f'(0) = 0$. Where the derivative exists, we can define the derivative function $f' : I \to \mathbb{R}$ of f.

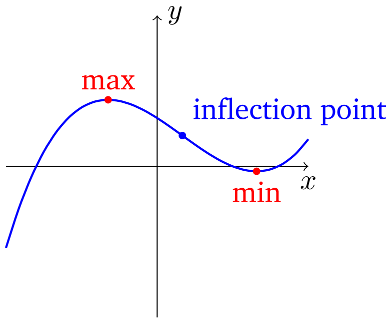


Figure 2: Extrema, inflection points of the polynomial $f(x) = (x + 0.8)(x - 0.5)(x - 0.8)$.

The **second derivative** of a function f, denoted f'' is the derivative of f', where defined. If $f''(x)$ exists and $f''(x) > 0$ for all $x \in I$, we say that f is **convex** (U-shaped). If $f'(x) = 0$ and $f''(x) > 0$, the point x is a **minimum**. If $f'(x)$ and $f''(x) < 0$, the point x is a **maxmimum**. Maxima and minima are **extrema**. If $f''(0) = 0$, the point x is an **inflection point** (Figure 2).

second derivative
convex
minimum
maxmimum
extrema
inflection point

3.1 List of common derivatives

The derivative is linear. If f and g are differentiable on I, and $a \in \mathbb{R}$,

- $(f + g)' = f' + g'$.
- $(af)' = a(f')$.
- $(af + g)' = a(f') + g'$.

The derivative follow the **rule of composed functions**. If $g : I \to J$ and $f : J \to K$, then $f \circ g$ is function $x \to f(g(x))$. If f and g are differentiable, the derivative

rule of composed functions

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

Example 1 Let $f : x \to x^2$ and $g : x \to 3x + 1$, two differentiable functions, with $f'(x) = 2x$ and $g'(x) = 3$. The derivative of the composed function $f \circ g$ at x is

$$f'(g(x))g'(x) = f'(3x + 1)g'(x) = 2(3x + 1) \cdot 3 = 6(3x + 1) = 18x + 6.$$

The derivative could have been obtained by computing the composed function $f(g(x)) = (3x + 1)^2 = 9x^2 + 6x + 1$.

Example 2 Compute the derivative of $f : x \to \sin(1/x)$. The function f is composed of a sine and an inverse function. To compute the derivative, we decomposed the function f as $f(x) = g(h(x))$ with $g(x) = \sin(x)$ and $h(x) = 1/x$. The derivatives

$g'(x) = \cos(x)$ and $h'(x) = -1/x^2$.

$$f'(x) = g'(h(x))h'(x) = \cos(1/x)\Big(\frac{-1}{x^2}\Big) = -\frac{\cos(1/x)}{x^2}.$$

Example 3 A function f is bijective (invertible) if there exists a function, denoted f^{-1}, such that $f \circ f^{-1} = f^{-1} \circ f$ is the identity map. If f is differentiable and invertible, what is the derivative of f^{-1}?

We apply the derivative to $f(f^{-1})$. Given that $f(f^{-1}(x)) = x$ by definition, we have $\big(f(f^{-1})\big)' = 1$, and

$$\begin{aligned}
\big(f(f^{-1})\big)'(x) &= f'(f^{-1}(x))(f^{-1})'(x), \\
&= 1, \\
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))}.
\end{aligned}$$

Take for instance $f(x) = x^2$ on $x \in (0, 1]$. The inverse is $f^{-1}(x) = \sqrt{x}$. The derivative of f is $f(x) = 2x$ and the derivative

$$f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{2(\sqrt{x})}.$$

Function	Derivative	Note
x^a	ax^{a-1}	$a \in \mathbb{R}$
$\frac{1}{x}$	$\frac{-1}{x^2}$	
$x^{\frac{1}{2}}$	$\frac{1}{2x^{\frac{1}{2}}}$	
$\ln(x)$	$\frac{1}{x}$	
e^x	e^x	
$\cosh(x)$	$\sinh(x)$	
$\sinh(x)$	$\cosh(x)$	
$\cos(x)$	$-\sin(x)$	
$\sin(x)$	$\cos(x)$	
$\dfrac{u(x)}{v(x)}$	$\dfrac{v(x)u'(x) - u(x)v'(x)}{v^2(x)}$	
$u(x)v(x)$	$u'(x)v(x) + u(x)v'(x)$	

3.2 Exercises on derivatives

4 Integrals and primitives

5 Complex numbers

A complex number is a number that can be expressed in the form $a + ib$, where a and b are real numbers, and the symbol i is called **imaginary unit**. The imaginary unit satisfies the equation $i^2 = -1$. Because no real number satisfies this equation, this number is called *imaginary*. imaginary unit

For the complex number $z = a + ib$, a is called the **real part** and b is called the **imaginary part**. The real part of z is denoted $\Re(z)$ (`\Re` in LaTeX) or just $\mathrm{Re}(z)$. The imaginary part of z denoted $\Im(z)$ (`\Im` in LaTex) or just $\mathrm{Im}(z)$. The set of all complex numbers is denoted $\mathbb{C}$ (`\mathbb{C}` in LaTeX). real part imaginary part

We need complex numbers for solving polynomial equations. The fundamental theorem of algebra asserts that a polynomial equation of with real or complex coefficients has complex solutions. These polynomial equations arise when trying to compute the eigenvalues of matrices, something we need to do to solve linear differential equations for instance.

Arithmetic rules that apply on real numbers also apply on complex numbers, by using the rule $i^2 = -1$: addition, subtraction, multiplication and division are associative, commutative and distributive.

Let $u = a + ib$ and $v = c + id$ two complex numbers, with real coefficients a, b, c, d. Then

- $u + v = a + ib + c + id = (a + c) + i(b + d)$.
- $uv = (a + ib)(c + id) = ac + iad + ibc + i^2bd = ac - bd + i(ad + bc)$.
- $\frac{1}{v} = \frac{1}{c+id} = \frac{c-id}{(c-id)(c+id)} = \frac{c-id}{c^2+d^2} = \frac{c}{c^2+d^2} - i\frac{d}{c^2+d^2}$.
- $u = v$ if and only if $a = c$ and $b = d$.

  It follows from the rule on i that

- $\frac{1}{i} = -i$. (Proof: $\frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$.)

Multiplying by the imaginary unit i is equivalent to a counterclockwise rotation by $\pi/2$ (Figure 3)

$$ui = (a + ib)i = ia + i^2b = -b + ia.$$

Let $z = a+ib$ a complex number with real a and b. The **conjugate** of z, denoted $\bar{z}$, is conjugate

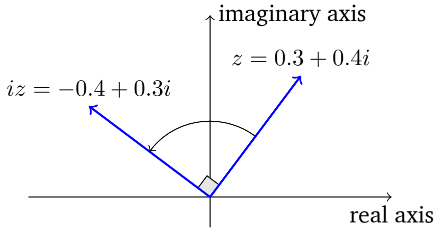


Figure 3: Rotation in the complex plane. Multiplication by i is a 90 degree counter-clockwise rotation.

$a - ib$. The conjugate of the conjugate of z is z (*reflection*, Figure 3). The **modulus** of z, denoted $|z|$ is $\sqrt{z\bar{z}}$. The product $z\bar{z} = (a+ib)(a-ib) = a^2+b^2+i(-ab+ab) = a^2+b^2$. The modulus is the complex version of the absolute value, for if z (i.e. $b = 0$), $|z| = \sqrt{a^2} = |a|$. It is always a real, positive number, and $|z| = 0$ if and only if $z = 0$. The modulus also has the property of being the *length* of the complex number z, if a and b are the sides of a rectangular triangle, then $|z|$ is the hypotenuse.

modulus

When simplifying a ratio involving a complex v at the denominator, it is important to convert it to a real number by multiplying the ratio by $\bar{v}/\bar{v}$. For instance, if $v \neq 0$,

$$\frac{u}{v} = \frac{u\bar{v}}{u\bar{v}} = \frac{u\bar{v}}{|v|^2}.$$

The denominator $|v|^2$ is always a positive real number.

By allowing complex values, nonlinear functions of real numbers like exponentials, logarithms and trigonometric functions can have their domain extended to all real and complex numbers. The most useful extension is the exponential function. Recall that the exponential function e^x, where $e \approx 2.71828$ is Euler's constant, satisfies the relation $e^{x+y} = e^x e^y$. This remains true for complex numbers. The **Euler's formula** relates the exponential of a imaginary number with trigonometric functions. For a real number y,

Euler's formula

$$e^{iy} = \cos(y) + i\sin(y).$$

Therefore, for any complex number $z = a + ib$, the exponential

$$e^z = e^{a+ib} = e^a e^{ib} = e^a\big(\cos(b) + i\sin(b)\big).$$

Tips on complex numbers

- If x is real, ix is pure imaginary. If y is imaginary, iy is real.
- $|i| = 1$. For any real θ, $|e^{i\theta}| = 1$.
- $|z_1 z_2| = |z_1||z_2|$.

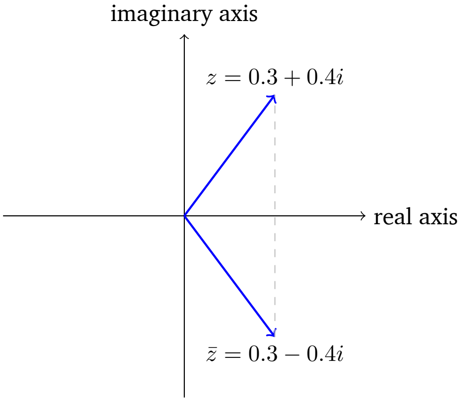


Figure 4: Complex plane

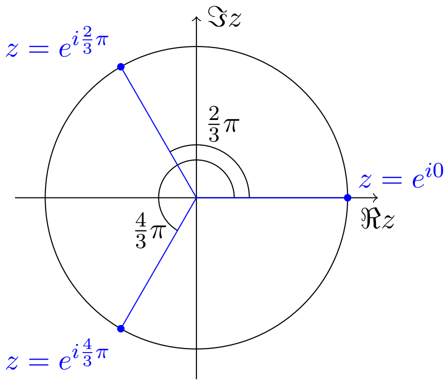


Figure 5: The roots of $z^3 - 1$.

- In particular, $|iz| = |i||z| = |z|$. (Multplying by i is a rotation in the complex plane, it does not change the modulus.)

5.1 Roots of a complex number

For complex numbers, the equation $z^n = 1$ has n solutions. They are called the **root of unity**. For $n = 2$, we have the well-known roots $z = \pm 1$, which are real. What are the roots of $z^3 = 1$? To find them, we express z in polar coordinates: $z = re^{i\theta}$. Then root of unity

$$z^3 = (re^{i\theta})^3 = r^3 e^{i3\theta} = 1.$$

The equation implies that z has modulus 1, so $r = 1$. The remaining term $e^{i3\theta} = 1$ implies that 3θ is a multiple of 2π because $e^{i\omega} = 1$ if and only if $\omega = 2k\pi$, for some integer k. Therefore $\theta = \frac{2}{3}k\pi$, for $k = 0, 1, 2, \ldots$. How many distinct points do we have? Clearly, $k = 3$ is equivalent to $k = 0$: $e^{i\frac{2}{3}3\pi} = e^{i2\pi} = ei0$. In the same way $k = 4$ is equivalent to $k = 1$, and so on. Therefore, there are exactly three distinct solutions for θ: $0, \frac{2}{3}\pi, \frac{4}{3}\pi$ (Figure 5).

5.2 Exercises on complex numbers

Exercice 1 Let the complex number $z = 2 + 3i$. Compute $\bar{z}$, $|z|$, $|\bar{z}|$ (compare with $|z|$), z^2, $\Re(\bar{z})$, $\Im(\bar{z})$, $\frac{z+\bar{z}}{2}$, $\frac{z-\bar{z}}{2}$, $-z$, iz.

Exercice 2 Any complex number can be represented in **polar form**: $z = r(\cos(\theta) + i\sin(\theta))$.

polar form

- Show that $|z| = r$
- Show that $z = re^{i\theta}$
- Conclude that for any complex number z, $|z| = 1$ if and only if z can be expressed as $z = e^{i\theta}$ for a real θ.

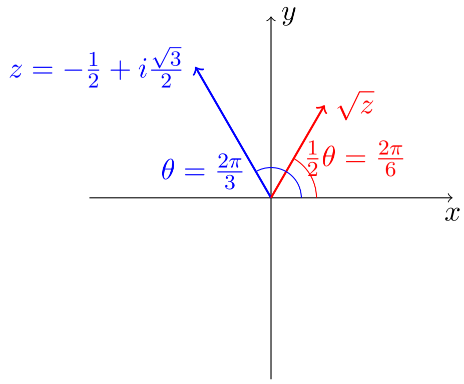
Exercice 3 Using Euler's formula, show that $\cos(a)\cos(b)-\sin(a)\sin(b) = \cos(a+b)$. *(Use the property that $e^{ia+ib} = e^{ia}e^{ib}$ and apply Euler's Formula)*.

Exercice 4 Show Euler's identity: $e^{i\pi} = -1$.

Exercice 5 What are the roots of the equation $z^6 = 1$?

Exercice 6 For a complex z, find necessary and sufficient conditions for e^{zt}, $t > 0$, to converge to 0.

Exercice 7 Let the complex number $z = a + ib$ with real a and b. Compute $\sqrt{z}$ (that is, express $s = \sqrt{z}$ as $s = \alpha + i\beta$, with real α and β)



6 Matrices in dimension 2

6.1 Eigenvalues of a 2×2 matrix

A 2×2 matrix A is an array with 2 rows and 2 columns:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

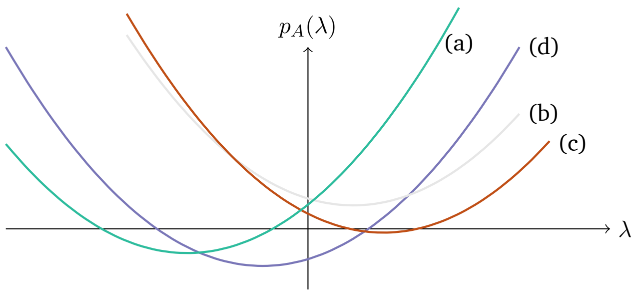


Figure 6: Graph of the characteristic polynomial. (*a, green*) polynomial with two negative roots, $p(\lambda) = 0.08 + 0.8\lambda + \lambda^2$. (*b, gray*) polynomial with two complex roots, $p(\lambda) = 0.1 - 0.3\lambda + \lambda^2$. (*c, red*) polynomial with two positive roots, $p(\lambda) = 0.05 + 0.5\lambda + \lambda^2$. (*d, purple*) polynomial with a negative and a positive root, $p(\lambda) = -0.1 - 0.3\lambda + \lambda^2$.

Usually, the **coefficients** a, b, c, d are real numbers. The **identity** matrix is the matrix

coefficients
identity

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The **determinant** of A, denoted $\det A$ or $|A|$ is the number $ad - bc$. The **trace** of A, denoted $\operatorname{tr} A$, is the sum of the main **diagonal** of A: $a + d$.

determinant
trace
diagonal

The **characteristic polynomial** of A is the second order polynomial in λ obtained by computing the determinant of the matrix $A - \lambda I$ (Figure 6),

characteristic polynomial

$$\det(A - \lambda I) = (a - \lambda)(d - \lambda) - bc = ad - bc - \lambda(a + d) + \lambda^2.$$

The characteristic polynomial $p_A(\lambda)$ of A can be expressed in terms of its determinant and trace:

$$p_A(\lambda) = \det A - \operatorname{tr} A\lambda + \lambda^2.$$

The **eigenvalues** of A are the roots of the characteristic polynomial. By the fundamental theorem of algebra, we know that the characteristic polynomial has exactly two roots, counting multiple roots. These roots can be real, or complex. The eigenvalues of A are calculated using the quadratic formula:

eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\left(\operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4\det A}\right).$$

From this formula, we can classify the eigenvalues of A. Let

$$\Delta = (\operatorname{tr} A)^2 - 4\det A$$

the **discriminant** of the quadratic formula. The two eigenvalues of A are real if and

discriminant

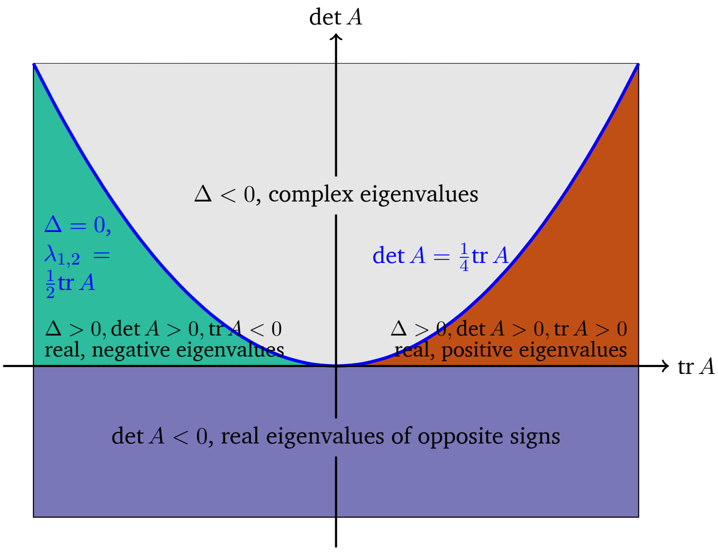


Figure 7: Properties of the eigenvalues of a 2×2 matrix A with respect to det A and tr A.

only if $\Delta \geq 0$, i.e. tr $A)^2 \geq 4 \det A$ Then we have the following properties (Figure 7):

1. $\Delta < 0$, complex eigenvalues
   - The two eigenvalues are complex conjugate: $\lambda_1 = \bar{\lambda}_2$
   - Their real part $\Re(\lambda) = \frac{1}{2}\text{tr } A$.
2. $\Delta = 0$, there is a single root of multiplicity 2: $\lambda = \frac{1}{2}\text{tr } A$.
3. $\Delta > 0, \det A > 0$, real, distinct eigenvalues of the same sign.
   - tr $A > 0$ and det $A > 0$. Then $\lambda_{1,2}$ are distinct and positive.
   - tr $A < 0$ and det $A > 0$. Then $\lambda_{1,2}$ are distinct and negative.
4. det $A < 0$, real distinct eigenvalues of opposite sign.
   - $\lambda_1 < 0 < \lambda_2$.
5. det $A = 0$ one of the eigenvalue is zero, the other eigenvalue is tr A.

6.1.1 Exercises on eigenvalues

Exercice 8 Properties of the eigenvalues of 2×2 matrices. For each 2×2 matrix, compute the determinant, the trace, and the discriminant, and determine whether the eigenvalues are real, complex, distinct, and the sign (negative, positive, or zero) of the real parts.

$$A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} -1 & 2 \\ 1/2 & 2 \end{pmatrix}.$$

6.2 Matrix-vector operations

A matrix defines a linear transformation between vector spaces. Given a vector x, the product Ax is vector composed of linear combinations of the coefficients of x. For a matrix 2×2, the vector x must be a vector of size 2, and the product Ax is a vector of size two. If $x = (x_1, x2)^t$ (the ^{t} stands for the transpose, because x must be a column vector), and $A = [a_{ij}]_{i=1,2,\, j=1,2}$, then

$$Ax = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}.$$

Successive linear transformations can be accomplished by applying several matrices. Given two matrices A, B, the matrix product $C = AB$ is also a matrix. The matrix C is the linear transformation that first applies B, then A. Matrix product is *not* commutative is general: $AB \neq BA$. *(If B means 'put on socks' and A means 'put on shoes', then BA does not have the expected result.)* The product of two matrices $A = [a_{ij}]_{i=1,2,\, j=1,2}$ and $B = [b_{ij}]_{i=1,2,\, j=1,2}$ is

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}.$$

The **sum of two matrices** $A + B$ is performed element-wise: $A + B = [a_{ij} + b_{ij}]_{i=1,2,\, j=1,2}$. The **sum of two vectors** is defined similarly. Addition is commutative. Matrix operations are associative and distributive.

sum of two matrices

sum of two vectors

$$\begin{aligned} A + B &= B + A, \\ A(B + C) &= AB + BC, \\ A(BC) &= (AB)C. \end{aligned}$$

Matrices and vectors can be multiplied by a scalar value (real or complex). **Multiplication by a scalar** is associative, distributive, and commutative. The result of the multiplication by a scalar is to multiply each coefficient of the matrix or vector by

Multiplication by a scalar

the scalar. For example, if λ, μ are scalars,

$$\begin{aligned}
\lambda A &= A(\lambda I) = A\lambda, \\
\lambda(A+B) &= \lambda A + \lambda B, \\
(\lambda A)B &= \lambda(AB) = A(\lambda B), \\
(\mu+\lambda)A &= \mu A + \lambda A, \\
\mu(\lambda A) &= (\mu\lambda)A, ...
\end{aligned}$$

The product between two column vectors is not defined, because the sizes do not match. However, we can define the **scalar product** between two column vectors x, y in the same way matrix product is defined:

scalar product

$$x^t y \equiv x_1 y_1 + x_2 y_2.$$

If the vectors are complex-valued, we need also to conjugate the transposed vector x^t. The conjugate-transpose is called the **adjoint** and is denoted *. Thus, if x is complex-valued, the adjoint x^* is the row vector $(\bar{x}_1, \bar{x}_2)$. The scalar product for complex-valued vectors is denoted $x^* y$. Since this notation also works for real-valued vector, we will used most of the time.

adjoint

Two vectors are **orthogonal** if their scalar product is 0. In the plane, this means that they are oriented at 90 degree apart. Orthogonal vectors are super important because they can be used to build orthogonal bases that are necessary for solving all sorts of *linear problems*.

orthogonal

6.2.1 Exercises on Matrix-vector and matrix-matrix operations

Exercice 9 Compute matrix-vector product

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

What is the transformation given by this matrix.

Exercice 10 Compute the matrix-matrix product

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Can you tell what transformation this is?

Exercice 11 Now compute the product of the same matrices, but in the inverse order

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Compare with the solution found in the previous exercise. What is this transformation?

Exercice 12 Find the matrix that takes a vector $x = (x_1, x_2)^t$ and returns $(ax_1, bx_2)^t$.

Exercice 13 Find the matrix that takes a vector $x = (x_1, x_2)^t$ and returns $(x_2, x_1)^t$.

Exercice 14 Find the matrix that takes a vector $x = (x_1, x_2)^t$ and returns $(x_2, 0)^t$.

Exercice 15 Compute the successive powers $A, A^2, A^3, ...$, for a diagonal matrix A:

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

Exercice 16 Compute the scalar product $x^* y$ between $x = (1 + 2i, 1 - i)^t$ and $y = (0.5 - i, -0.5)^t$.

Exercice 17 Now compute the scalar product $y^* x$ and compare with the result with the previous exercise.

Exercice 18 Compute the scalar product between $z = (z_1, z_2)^t$ and itself, if z is a complex-valued vector. What can you say about the result?

Tips on eigenvalues Some matrices have special shapes that make it easier to compute the determinant, and the eigenvalues. These are called eigenvalue-revealing shapes.

- Diagonal matrices have their eigenvalues on the diagonal.
- **Triangular matrices**, i.e. matrices that have zeros above (lower-triangular matrix) or below (upper-triangular matrix) the main diagonal have also their eigenvalues on the diagonal. Triangular matrices
- A matrix with a row or a column of zeros has its determinant equal to zero. This implies that one of its eigenvalues is 0.

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7 Eigenvalue decomposition

In many applications, it is useful to decompose a matrix into a form that makes it easier to operate complex operations on. For instance, we might want to compute the powers of a matrix A: A^2, A^3, A^4. Multiplying matrices are computationally intensive, especially when the size of the matrix becomes large. The **power of a**

matrix is $A^k = AA...A$, k times. The zeroth power is the identity matrix: $A^0 = I$. power of a matrix

The **inverse** of a matrix A, denoted by A^{-1} is the unique matrix such that $AA^{-1}A^-1A = I$. The notation is self-consistent with the positive powers of A. The inverse does not always exist. A matrix is **invertible** if and only if its determinant is not 0. If A and B are invertible, then AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$. inverse invertible

The **eigenvalue decomposition** is a decomposition of the form $A = XDX^{-1}$, where D is a diagonal matrix, and X is an invertible matrix. If there exists such a decomposition for A, then computing powers of A becomes easy: eigenvalue decomposition

$$
\begin{aligned}
A^k &= (XDX^{-1})^k = XDX^{-1}\,XDX^{-1}\,...XDX^{-1},\\
&= XD(X^{-1}X)D(X^{-1}X)D...(X^{-1}X)DX^{-1},\\
&= XD^kX^{-1}.
\end{aligned}
$$

The eigenvalue decomposition does not always exists, because it is not always possible to find an invertible matrix X. When it exists, though, the columns of the matrix X is composed of the eigenvectors of A. When A is a 2×2 matrix, it is enough to find 2 linearly independent eigenvectors x and y for the matrix

$$X = \left(\begin{array}{c|c} x_1 & y_1 \\ x_2 & y_2 \end{array}\right)$$

to be invertible.

7.1 Eigenvectors

The **eigenvectors** of a matrix A are the *nonzero* vectors x such that for an eigenvalue λ of A, eigenvectors

$$Ax = \lambda x.$$

If x is an eigenvector, so is any αx for any scalar value α. If there are two linearly independent eigenvectors x and y associated to an eigenvalue, $\alpha x + \beta y$ is also an eigenvector. There is at least one eigenvector for each distinct eigenvalue, but there may be more than one when the eigenvalue is repeated.

Example 4 Distinct, real eigenvalues The matrix

$$A = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}$$

is upper-triangular; this is one of the eigenvalue-relealing shapes. The eigenvalues are -1 and 1. These are distinct eigenvalues, so each eigenvalue possesses a single eigenvector. The eigenvector x associated to $\lambda_1 = -1$ is found by solving the eigensystem

$$Ax = (-1)x.$$

The unknown quantity x appears on both sides of the equation. We can find a simpler form by noting that multiplying a vector by the identity matrix is neutral: $(-1)x = (-1)Ix$. The eigenproblem becomes

$$\begin{aligned} Ax &= (-1)Ix, \\ Ax - (-1)Ix &= 0, \\ \big(A - (-1)I\big)x &= 0, \end{aligned}$$

that is, the eigenvector is a nonzero solution of the linear system $\big(A - \lambda I\big)x = 0$. In general, if a matrix B is invertible, the only solution to $Bx = 0$ is $x = 0$ (the vector of zeroes). But, by construction, $A - \lambda I$ cannot be invertible if λ is an eigenvalue: its determinant is exactly the characteristic polynomial evaluated at one of its roots, so it is zero. This is why the eigensystem has nonzero solutions. Now, because $A - \lambda I$ is not invertible, this means that a least one of its rows is a linear combination of the others. For 2×2 matrices, this implies that the two rows are colinear, or redundant. For our example, the eigensystem reads

$$\begin{aligned} \begin{pmatrix} -1-(-1) & -2 \\ 0 & 1-(-1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & -2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

we immediately see that the two rows $(0, -2)$ and $(0, 2)$ are colinear, with a factor -1. This leads to an underdetermined system: $0x_1 + -2x_2 = 0$. The solution is $x_2 = 0$ and we can take x_1 to be any value, save 0. We choose $x = (1, 0)^t$.

For the eigenvalue $\lambda_2 = +1$, the eigensystem reads:

$$\begin{aligned} \begin{pmatrix} -1-(+1) & -2 \\ 0 & 1-(+1) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} -2 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Again, the second row $(0,0)$ can be neglected, and the solution is $-2y_1 + 2y_2 = 0$, or $y_1 = y_2$. It is customary to choose an eigenvector with norm 1. The **norm** of a complex-valued vector $y = (y_1, y_2)^t$ is norm

$$||y|| = \sqrt{y^* y} = \sqrt{\bar{y}_1 y_1 + \bar{y}_2 y_2} = \sqrt{|y_1|^2 + |y_2|^2}.$$

Here, the eigenvector is $y = (y_1, y_1)^t$, so $||y|| = \sqrt{|y_1|^2 + |y_1|^2} = \sqrt{2}\sqrt{|y_1|^2} = \sqrt{2}|y_1|$. Taking $||y|| = 1$ solves $|y_1| = 1/\sqrt{2}$. This means that we could take a negative, or a complex value for y_1, as long as the $|y_1| = 1/\sqrt{2}$. Going for simplicity, we take $y_1 = 1/\sqrt{2}$.

Example 5 Complex eigenvalues

The matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is *not* diagonal, so we have to compute the eigenvalues by hand. The trace of A is zero, the determinant is $0 - (1)(-1) = 1$, and the discriminant is -4. A negative discriminant implies complex eigenvalues,

$$\lambda_{1,2} = \frac{1}{2}\left(0 \pm \sqrt{-4}\right) = \pm i.$$

For the eigenvalue $\lambda_1 = +i$, the eigensystem reads:

$$\begin{pmatrix} -(+i) & -1 \\ 1 & -(+i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The two rows $(-i, 1)$ and $(1, -i)$ should be colinear, but this is not obvious with the complex coefficients. Multiplying the first row by i gives $i(-i, -1) = (-i^2, -i) = (-(-1), -i) = (1, -i)$, the second row, ok. Having confirmed that the system is indeed underdetermined, we can week a solution to $-ix_1 - x_2 = 0$. Solving for $x_2 = -ix_1$, we obtain the eigenvector $x = (x_1, -ix_2)^t$. Normalization of x imposes

$$||x|| = \sqrt{|x_1|^2 + |-ix_1|^2} = \sqrt{|x_1|^2 + |x_1|^2} = \sqrt{2}|x_1| = 1.$$

As in the previous example, we can choose $x_1 = 1/\sqrt{2}$.

The second eigenvectors, associated $\lambda_2 = -i$, solves the eigensystem

$$\begin{pmatrix} -(-i) & -1 \\ 1 & -(-i) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The first row yields $iy_1 - y_2 = 0$, so $y = (y_1, iy_2)^t$. A normalized eigenvector can be $y = (1/\sqrt{2}, i/\sqrt{2})^t$. We could also have chosen $y = (i/\sqrt{2}, -1/\sqrt{2})^t$.

Example 6 Repeated eigenvalues 1

The matrix

$$\begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$$

is lower-trianglar, with repeated eigenvalues on the diagonal, $\lambda_{1,2} = -1$. The eigenvectors associated with -1 satisfy the eigenproblem

$$\begin{pmatrix} -1-(-1) & 0 \\ 2 & -1-(-1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The first row vanishes, and the second row means that $x_1 = 0$, leaving for instance $x_2 = 1$, and $x = (0, 1)^t$. There are no other linearly independent eigenvectors. This is not always the case, repeated eigenvalues can have more than one independent eigenvector, as in the next example.

Example 7 Repeated eigenvalues 2

The matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

is diagonal, with repeated eigenvalues on the diagonal, $\lambda_{1,2} = -1$. The eigenvectors associated with -1 satisfy the eigenproblem

$$\begin{pmatrix} -1-(-1) & 0 \\ 0 & -1-(-1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Now, the two rows vanished, leaving no condition at all on x_1 and x_2. This means that all the vectors are eigenvectors! How many linearly independent eigenvectors can we find? Vectors of size 2 live in a vector space of dimension 2; we can find at most 2 linearly independent vectors. We can choose for instance the canonical basis: $x = (1, 0)^t$ and $y = (0, 1)^t$.

Tips on eigenvalue decomposition

- A 2×2 matrix (or any square matrix) admits an eigenvalue decomposition if all the eigenvalues are distinct. For 2×2 matrices, eigenvalues are distinct if and only if the discriminant $\Delta \neq 0$.
- If the matrix has a repeated eigenvalue, it will admit an eigenvalue decomposition if the number of (linearly independent) eigenvectors is equal to the number of times the eigenvalue is repeated. The number of eigenvectors is called geometric multiplicity, and the number of repeats is called algebraic multiplicity.
- The eigenproblem should be underdetermined; you should always be able to eliminate at least one row by linear combination. If you cannot, this means that there is a error, possibly an incorrect eigenvalue, or a arithmetic mistake in computing $A - \lambda I$.
- Because eigenvalues are in general complex, the eigenvectors will also be complex.
- The eigenvector matrix X needs to be inverted. When the eigenvectors can be chosen so that they are orthogonal and normalized, the inverse $X^{-1} = X^*$ (i.e. the conjugate transpose of X). Symmetric matrices have orthogonal eigenvalues, so this class of matrices are especially easy to diagonalise.
- Eigenvalue decomposition and invertibility are two different concepts. A matrix can be invertible without admitting an eigenvalue decomposition, and vice versa.
- When a matrix does not admit an eigenvalue decomposition, it still can be triangularised. One such triangularisation is the Jordan decomposition: $A = P(D + S)P^{-1}$, where P is invertible, D is the diagonal matrix of eigenvalues, and S is a **nilpotent** matrix, i.e. a nonzero matrix such that $S^k = 0$ for $k \geq k_0 > 1$. nilpotent

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7.2 Exercises on eigenvalues decomposition

Exercice 19 Find, if there is any, an eigenvalue decomposition of

$$A = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

To compute X^{-1}, you can use the fact that because A is real and symmetrical, the eigenvectors are orthogonal, meaning that $X^{-1} = X^t$, if the eignevectors are normalized.

8 Linearisation of functions $\mathbb{R}^2 \to \mathbb{R}^2$

Nonlinear systems of ordinary differential equations are used to describe the **dynamics** (evolution in time) of concentration of biochemical species, population densities in ecological systems, of the electrophyiology of neurons. dynamics

Two dimensional systems are described by two ordinary differential equations (ODEs)

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1, x_2), \\ \frac{dx_2}{dt} &= f_2(x_1, x_2), \end{aligned}$$

The variables x_1, x_2 are functions of time: $x_1(t)$, $x_2(t)$, and f_1, f_2 are the derivatives. We define the two-dimensional vectors $\boldsymbol{x} = (x_1, x_2)^t$ (here we will use **bold** for vectors), and $\boldsymbol{f} = (f_1, f_2)^t$. The ODEs can now be represented in vector format,

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}).$$

Here we assume that there exists a point in the 2D plane $\bar{\boldsymbol{x}}$ such that the derivative $\boldsymbol{f}(\bar{\boldsymbol{x}}) = 0$. This point is called a **steady state** because the derivatives are all zeros; the steady state is therefore a solution to the system of ODE. steady state

We are interested in how $\boldsymbol{f}$ is behaving around the steady state. To do that we linearize the function $\boldsymbol{f}$ at the steady state. **Linearisation** is a first-order expansion. For a function from $\mathbb{R}^2 \to \mathbb{R}^2$, a first-order expansion around a point $\boldsymbol{x}_0$ is Linearisation

$$\boldsymbol{f}(\boldsymbol{x}) \approx \boldsymbol{f}(\boldsymbol{x}_0) + \boldsymbol{D}\boldsymbol{f}(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0)$$

When expanding around a steady state, the constant term $\boldsymbol{f}(\bar{\boldsymbol{x}}) = 0$. In the second term, $\boldsymbol{D}\boldsymbol{f}$ is a 2×2 matrix, called the Jacobian matrix, and often denoted $\boldsymbol{J}$. The

Jacobian matrix for the function $\boldsymbol{f}$ is defined as

Jacobian matrix

$$\boldsymbol{J} = \boldsymbol{D}\boldsymbol{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}.$$

When evaluated at a steady state, the Jacobian matrix can provide information on the dynamics of the nonlinear ODE system. More precisely, the eigenvalues of the Jacobian matrix can determine whether the steady state is stable (attracts solutions) or is unstable. **Linearisation around a steady state means computing the Jacbian matrix at the steady state.**

Example 8 Linearisation around a steady state

The Lotka-Volterra equations is a classical ODE system mathematical biology. The equations reads

$$\frac{dx}{dt} = ax - xy,$$
$$\frac{dy}{dt} = xy - by,$$

for a, b positive constants. The solution vector is $\boldsymbol{x} = (x, y)^t$ and the derivatives are $f_1(x, y) = ax - xy$ and $f_2 = xy - by$. We first look for steady states

$$f_1 = ax - xy = 0, \quad f_2 = xy - by.$$

If x and y are not zero, we have $x = b$ and $y = a$. If $x = 0$, the second equation implies $y = 0$. If $y = 0$, the first equation implies $x = 0$. Therefore there are two steady states, $\bar{\boldsymbol{x}} = (b, a)^t$ and $\hat{\boldsymbol{x}} = (0, 0)^t$.

We have the following derivatives

$$\frac{\partial f_1}{\partial x}(x, y) = a - y,$$
$$\frac{\partial f_1}{\partial y}(x, y) = -x,$$
$$\frac{\partial f_2}{\partial x}(x, y) = y,$$
$$\frac{\partial f_2}{\partial y}(x, y) = x - b,$$

The Jacobian matrix is

$$\boldsymbol{J} = \begin{pmatrix} a-y & -x \\ y & x-b \end{pmatrix}.$$

Evaluated at the steady state $\bar{\boldsymbol{x}} = (b,a)^t$ and $\hat{\boldsymbol{x}} = (0,0)^t$, the Jacobian matrices are

$$\boldsymbol{J}(\bar{\boldsymbol{x}}) = \begin{pmatrix} 0 & -b \\ a & 0 \end{pmatrix}, \quad \boldsymbol{J}(\hat{\boldsymbol{x}}) = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix}.$$

8.1 Exercises on linearisation

Exercice 20 Let the function $\boldsymbol{f} = (f_1, f_2)^t$, with

$$f_1(x,y) = -dx + x\,\text{exp}(-axy), \quad f_2(x,y) = x - y,$$

$d < 1$, a, d positive. Find the steady states (by solving the equations $f_1 = 0, f_2 = 0$). Compute the Jacobian matrix, and evaluate the Jacobian matrix at each steady state.

Exercice 21 Compute the Jacobian matrices of each of the following functions of (x,y). All parameters are constants. You do not need to compute the steady states just the matrices.

- van der Pol oscillator

$$f_1(x,y) = \mu\big((1-x^2)y - x\big),\ f_2(x,y) = y.$$

- Two-compartment pharmacokinetics

$$f_1(x,y) = a - k_{12}x + k_{21}y - k_1 x,\ f_2(x,y) = k_{12}x - k_{21}y.$$

- SI epidemiological model

$$f_1(x,y) = -\beta xy,\ f_2(x,y) = \beta xy - \gamma y.$$

9 Solution of systems of linear differential equations in dimension 2

Linear differential equations have linear derivative parts, which can be represented in matrix-vector format

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{A}\boldsymbol{x}(t),$$

for a vector $\boldsymbol{x}$ square matrix $\boldsymbol{A}$. For initial conditions $\boldsymbol{x}(t) = \boldsymbol{x}_0$, the **solution of the linear system of ODEs** is

solution of the linear system of ODEs

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0.$$

If we have at our disposal an eigenvalue decomposition of $\boldsymbol{A} = \boldsymbol{X}\boldsymbol{D}\boldsymbol{X}^{-1}$, the **exponential of the matrix** is

exponential of the matrix

$$\begin{aligned} e^{\boldsymbol{A}t} &= \boldsymbol{X}e^{\boldsymbol{D}t}\boldsymbol{X}^{-1}, \\ &= \boldsymbol{X}\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}\boldsymbol{X}^{-1}. \end{aligned}$$

Therefore, the long-time behavior of the exponential is controlled by the eigenvalues $\lambda_{1,2}$.

Example 9 Solution of a linear system of ODEs

Consider the linear system of ODEs given by the Lotka-Volterra model linearised at its nonzero steady state $\bar{\boldsymbol{x}} = (b, a)^t$ is

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 & -b \\ a & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}. \tag{1}$$

This system approximates the nonlinear version near the steady state. In this linear system, variables (x, y) are deviations from the steady state; their solutions are "centered" around 0. To solve this linear system, we will diagonalise the matrix

$$A = \begin{pmatrix} 0 & -b \\ a & 0 \end{pmatrix}.$$

The goal is to go slowly through every step once for this system. In general it is not necessary to solve the system completely by hand; knowledge of the eigenvalues is often sufficient in many applications.

We have $\det A = 0 - a(-b) = ab > 0$, $\operatorname{tr} A = 0$ and $\Delta = 0 - 4ab = -4ab < 0$. The eigenvalues are therefore complex conjugates: $\lambda_{1,2} = \pm i\sqrt{ab}$. Distinct eigenvalues means that A is diagonalisable. The eigenvector associated to $\lambda_1 = i\sqrt{ab}$ is given by the system

$$\left(\begin{array}{cc|c} -i\sqrt{ab} & -b & 0 \\ a & -i\sqrt{ab} & 0 \end{array} \right)$$

We have from the first row $-i\sqrt{ab}x = by$. Letting $x = b$ and $y = -i\sqrt{ab}$, we obtain the non-normalized eigenvector $\tilde{x}_1 = (b, -i\sqrt{ab})^t$. Normalization is done by dividing by

$$||\tilde{x}|| = \sqrt{b^2 + (-i\sqrt{ab})^2} = \sqrt{b^2 + ab},$$

to obtain the first eigenvector

$$x = \begin{pmatrix} \frac{b}{\sqrt{b^2+ab}} \\ \frac{-i\sqrt{ab}}{\sqrt{b^2+ab}} \end{pmatrix} = \begin{pmatrix} \frac{b}{\sqrt{b}\sqrt{b+a}} \\ \frac{-i\sqrt{a}\sqrt{b}}{\sqrt{b}\sqrt{b+a}} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{b}}{\sqrt{b+a}} \\ \frac{-i\sqrt{a}}{\sqrt{b+a}} \end{pmatrix}.$$

The second eigenvector is computed the same way (watch out for the slightly different signs!). The eigenproblem for the eigenvalue $\lambda = -i\sqrt{ab}$ is

$$\left(\begin{array}{cc|c} +i\sqrt{ab} & -b & 0 \\ a & +i\sqrt{ab} & 0 \end{array} \right)$$

Given that the only change is $-i \to +i$, the second eigenvector is

$$x_2 = \begin{pmatrix} \frac{\sqrt{b}}{\sqrt{b+a}} \\ \frac{i\sqrt{a}}{\sqrt{b+a}} \end{pmatrix}.$$

The solution to the linear ODE is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \boldsymbol{X} e^{\boldsymbol{D}t} \boldsymbol{X}^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

with

$$\boldsymbol{X} = \frac{1}{\sqrt{b+a}} \begin{pmatrix} \sqrt{b} & \sqrt{b} \\ -i\sqrt{a} & i\sqrt{a} \end{pmatrix}, \quad \boldsymbol{D} = \begin{pmatrix} +i\sqrt{ab} & 0 \\ 0 & -i\sqrt{ab} \end{pmatrix}$$

The **inverse of a** 2×2 **matrix** with coefficients a, b, c, d is

inverse of a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

This is conditional to $\det = ad - bc \neq 0$, of course. With this formula, the inverse of

$\boldsymbol{X}$ is

$$\boldsymbol{X}^{-1} = \frac{1}{\sqrt{b+a}} \frac{1}{\det \boldsymbol{X}} \begin{pmatrix} i\sqrt{a} & -\sqrt{b} \\ i\sqrt{a} & \sqrt{b} \end{pmatrix}.$$

The determinant $\det \boldsymbol{X} = \frac{i\sqrt{b}\sqrt{a}}{b+a} + \frac{i\sqrt{a}\sqrt{b}}{b+a} = 2i\frac{\sqrt{ab}}{b+a}$. The inverse reduces to

$$\frac{1}{\sqrt{b+a}} \frac{a+b}{2i\sqrt{ab}} \begin{pmatrix} i\sqrt{a} & -\sqrt{b} \\ i\sqrt{a} & \sqrt{b} \end{pmatrix} = \frac{-i\sqrt{b+a}}{2\sqrt{ab}} \begin{pmatrix} i\sqrt{a} & -\sqrt{b} \\ i\sqrt{a} & \sqrt{b} \end{pmatrix} = \frac{\sqrt{b+a}}{2\sqrt{ab}} \begin{pmatrix} \sqrt{a} & i\sqrt{b} \\ \sqrt{a} & -i\sqrt{b} \end{pmatrix}.$$

We have now obtained the eigenvalue decompostion of $\boldsymbol{A} = \boldsymbol{X}\boldsymbol{D}\boldsymbol{X}^{-1}$. To solve the linear ODE, we need to compute the product

$$\begin{aligned}
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \boldsymbol{X} e^{\boldsymbol{D}t} \boldsymbol{X}^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \\
&= \frac{1}{\sqrt{b+a}} \begin{pmatrix} \sqrt{b} & \sqrt{b} \\ -i\sqrt{a} & i\sqrt{a} \end{pmatrix} \begin{pmatrix} e^{+i\sqrt{ab}t} & 0 \\ 0 & e^{-i\sqrt{ab}t} \end{pmatrix} \frac{\sqrt{b+a}}{2\sqrt{ab}} \begin{pmatrix} \sqrt{a} & i\sqrt{b} \\ \sqrt{a} & -i\sqrt{b} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \\
&= \frac{1}{2\sqrt{ab}} \begin{pmatrix} \sqrt{b} & \sqrt{b} \\ -i\sqrt{a} & i\sqrt{a} \end{pmatrix} \begin{pmatrix} e^{+i\sqrt{ab}t} & 0 \\ 0 & e^{-i\sqrt{ab}t} \end{pmatrix} \begin{pmatrix} \sqrt{a} & i\sqrt{b} \\ \sqrt{a} & -i\sqrt{b} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \\
&= \frac{1}{2\sqrt{ab}} \begin{pmatrix} \sqrt{b}e^{+i\sqrt{ab}t} & \sqrt{b}e^{-i\sqrt{ab}t} \\ -i\sqrt{a}e^{+i\sqrt{ab}t} & i\sqrt{a}e^{-i\sqrt{ab}t} \end{pmatrix} \begin{pmatrix} \sqrt{a}x_0 + i\sqrt{b}y_0 \\ \sqrt{a}x_0 - i\sqrt{b}y_0 \end{pmatrix}.
\end{aligned}$$

To simplify the last steps of the calculation, we will introduce the following notation. Using Euler's formula, $e^{\pm i\sqrt{ab}t} = \cos(\sqrt{ab}t) \pm i\sin(\sqrt{ab}t)$. Let $c = \cos(\sqrt{ab}t)$, $s = \sin(\sqrt{ab}t)$, and $C_1 = \sqrt{a}x_0 + i\sqrt{b}y_0$, $C_2 = \sqrt{a}x_0 - i\sqrt{b}y_0$. The solution reads

$$\begin{aligned}
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \frac{1}{2\sqrt{ab}} \begin{pmatrix} \sqrt{b}e^{i\sqrt{ab}t}C_1 + \sqrt{b}e^{-i\sqrt{ab}t}C_2 \\ -i\sqrt{a}e^{i\sqrt{ab}t}C_1 + i\sqrt{a}e^{-i\sqrt{ab}t}C_2 \end{pmatrix}, \\
&= \frac{1}{2\sqrt{ab}} \begin{pmatrix} \sqrt{b}(c+is)C_1 + \sqrt{b}(c-is)C_2 \\ -i\sqrt{a}(c+is)C_1 + i\sqrt{a}(c-is)C_2 \end{pmatrix}, \\
&= \frac{1}{2\sqrt{ab}} \begin{pmatrix} \sqrt{b}c(C_1+C_2) + i\sqrt{b}s(C_1-C_2) \\ \sqrt{a}s(C_1+C_2) + i\sqrt{a}c(-C_1+C_2) \end{pmatrix}, \\
&= \frac{1}{2\sqrt{ab}} \begin{pmatrix} 2\sqrt{ab}\cos(\sqrt{ab}t)x_0 - 2b\sin(\sqrt{ab}t)y_0 \\ 2a\sin(\sqrt{ab}t)x_0 + 2\sqrt{ab}\cos(\sqrt{ab}t)y_0 \end{pmatrix}, \\
&= \begin{pmatrix} \cos(\sqrt{ab}t)x_0 - \sqrt{b/a}\sin(\sqrt{ab}t)y_0 \\ \sqrt{a/b}\sin(\sqrt{ab}t)x_0 + \cos(\sqrt{ab}t)y_0 \end{pmatrix}.
\end{aligned}$$

And that's it! We have obtained a solution to the linear ODE (Figure 8).

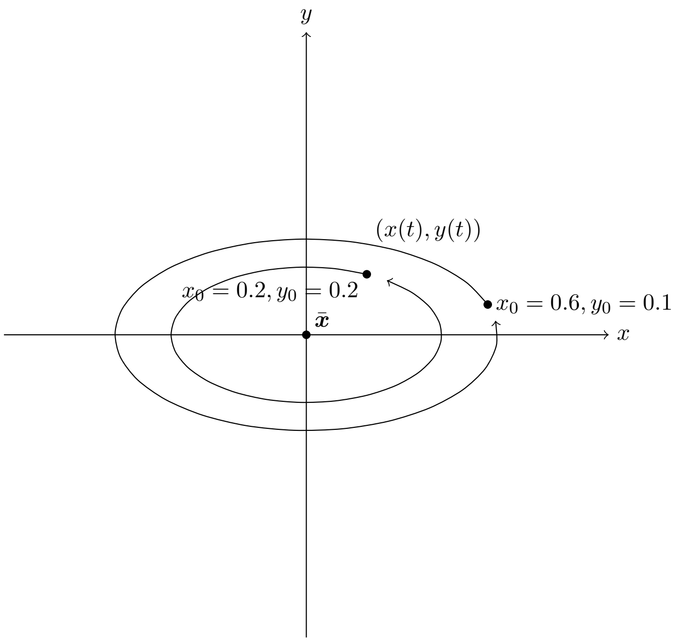

Figure 8: Solution of the linear system of ODEs (1), with $a = 0.1$, $b = 0.4$.

10 Glossary

French	English	Note
dérivable	differentiable	
matrice jacobienne	Jacobian matrix	
rang	rank	
noyau	kernel	notation: ker
ensemble	set	
espace vectoriel	vector space	
sous-espace vectoriel	linear subspace	
valeur propre	eigenvalue	
vecteur propre	eigenvector	
sous-espace propre	eigenspace	
décomposition en valeurs propres	eigenvalue decomposition	
décomposition en valeurs singulières	singular value decomposition	
valeur singulière	singular value	
trace	trace	
déterminant	determinant	det
base	basis	
application linéaire	linear map	
application	map	
dimension	dimension	
moindres carrés	least-squares	
produit scalaire	scalar product	
Vect	Span	
famille libre	linearly independent set	
famille génératrice	spanning set	