A Non-commutative Extension of the Sewing Lemma and its Application for Quantum Mechanics

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Abstract

In this thesis, we explore the classical Sewing Lemma and introduce Young's integration theory, which allows us to compute integrals of the form

$$\int f\,dg,$$

where f and g are Hölder continuous functions with exponents α and β, provided that $\alpha + \beta > 1$.

We then present a non-commutative extension of the Sewing Lemma, enabling solutions to differential equations of the form

$$U_t = U_0 + \int_0^t U_s\,dA_s,$$

where U and A are matrix-valued functions. The solution to these equations takes the form of an "integral product"

$$U_t = \prod_0^t e^{dA_s},$$

a concept we will introduce in the thesis.

These equations generalize the fourth postulate of quantum mechanics, which states that the time evolution of a quantum system is described by the time evolution operator $U(t, t_0)$, solving the initial value problem:

$$i\frac{d}{dt}U(t, t_0) = H(t)U(t, t_0), \quad U(t_0, t_0) = 1,$$

where $H(t)$ is the self-adjoint Hamiltonian operator governing the system.

Finally, we examine the implications of this generalisation in quantum computing, with potential applications in quantum machine learning.