PHY 375 - Problem Set 5

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Problem 1

Let us consider light travelling through air ($n_{\text{air}} = 1.00$) hitting a glass surface ($n_{\text{glass}} = 1.60$) with an incident angle $\theta_i = 30°$.

Part a

We are now to calculate the reflection and transmission coefficients for both p-polarized and s-polarized light.

First we have the equations for the reflection coefficients

$$\mathrm{r}_{\parallel} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} \tag{1}$$

$$\mathrm{r}_{\perp} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} \tag{2}$$

Then we have the equations for the transmission coeffiients

$$\mathrm{t}_{\parallel} = \frac{2\sin\theta_t \cos\theta_i}{\sin(\theta_i + \theta_t)} \tag{3}$$

$$\mathrm{t}_{\perp} = \frac{2\sin\theta_t \cos\theta_i}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)} \tag{4}$$

Using these four equations we can now calculate all the values for both types of light incident on the glass. Before we continue though, we need to calculate the value of θ_t. We can find this easily enough using Snell's Law

$$n_i \sin\theta_i = n_t \sin\theta_t \quad \Longrightarrow \quad \sin\theta_t = \frac{n_i \sin\theta_i}{n_t} \quad \Longrightarrow \quad \theta_t = \sin^{-1}\left(\frac{n_i \sin\theta_i}{n_t}\right)$$

If we insert the known values, namely n_i, n_t, and θ_i we can calculate θ_t.

$$\theta_t = \sin^{-1}\left(\frac{\sin(30°)}{1.60}\right) = 18.2°$$

Now that we have a value for θ_t we can now calculate the four values for the reflection and transmission coefficients. First we'll consider the two values for p-polarized light using equations 1 and 3.

$$\mathrm{r}_{\parallel} = \frac{\tan(30° - 18.2°)}{\tan(30° + 18.2°)} = \frac{\tan(11.8°)}{\tan(48.2°)} = \frac{0.209}{1.118} = 0.187$$

and

$$\mathrm{t}_{\parallel} = \frac{2\sin(18.2°)\cos(30°)}{\sin(30° + 18.2°)} = \frac{2\sin(18.2°)\cos(30°)}{\sin(48.2°)} = \frac{2 \times 0.312 \times 0.866}{0.745} = 0.725$$

Now on to the s-polarized light where we'll use equations 2 and 4

$$\mathrm{r}_{\perp} = -\frac{\sin(30° - 18.2°)}{\sin(30° + 18.2°)} = -\frac{\sin(11.8°)}{\sin(48.2°)} = -\frac{0.204}{0.745} = -0.274$$

and

$$\mathrm{t}_{\perp} = \frac{2\sin 18.2° \cos 30°}{\sin(30° + 18.2°)\cos(30° - 18.2°)} = \frac{2\sin 18.2° \cos 30°}{\sin(48.2°)\cos(11.8°)} = \frac{2 \times 0.312 \times 0.866}{0.745 \times 0.979} = 0.741$$

This gives us the following four values

$$\mathrm{r}_{\parallel} = 0.187$$

$$\mathrm{t}_{\parallel} = 0.725$$

$$\mathrm{r}_{\perp} = -0.274$$

$$\mathrm{t}_{\perp} = 0.741$$

Problem 2

Let us consider the Fresnel equation for the reflective coefficient for perpendicular light

$$\mathrm{r}_\perp = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t} \tag{5}$$

From Snell's Law we know that $n_i \sin\theta_i = n_t \sin\theta_t$. Now if we multiply equation 5 by $\frac{n_i \sin\theta_i}{n_i \sin\theta_i}$, which we can do since it is just a fancy reprisentation of 1, we can get both the numerator and denominator in a more useful form.

$$\mathrm{r}_\perp = \frac{n_i \sin\theta_i}{n_i \sin\theta_i} \times \left(\frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t}\right) = \frac{n_i^2 \sin\theta_i \cos\theta_i - n_i n_t \sin\theta_i \cos\theta_t}{n_i^2 \sin\theta_i \cos\theta_i + n_i n_t \sin\theta_i \cos\theta_t} \tag{6}$$

Now, since $n_i \sin\theta_i = n_t \sin\theta_t$ we can replace $n_i \sin\theta_i$ with $n_t \sin\theta_t$ in equation 6 to get

$$\mathrm{r}_\perp = \frac{n_i n_t \sin\theta_t \cos\theta_i - n_i n_t \sin\theta_i \cos\theta_t}{n_i n_t \sin\theta_t \cos\theta_i + n_i n_t \sin\theta_i \cos\theta_t} \tag{7}$$

If we use the trigonometric identity that says $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$ we can see that the equation above will become

$$\mathrm{r}_\perp = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)} \tag{8}$$

Since addition is reflexive, aka $a + b = b + a$, and $\sin(-\theta) = -\sin\theta$ we can rewrite equation 8 in the desired form of

$$\mathrm{r}_\perp = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

Problem 3

Let us consider what happens to $r_{\parallel}$ as θ_i goes from $0°$ to $90°$ for two ases, $n_i = 1$ and $n_t = 1.33$ and the opposite. These two cases are displayed in the following two figures. Note Brewster's Angle is the angle such that

$$\theta_b = \theta_i + \theta_t = \frac{\pi}{2}$$

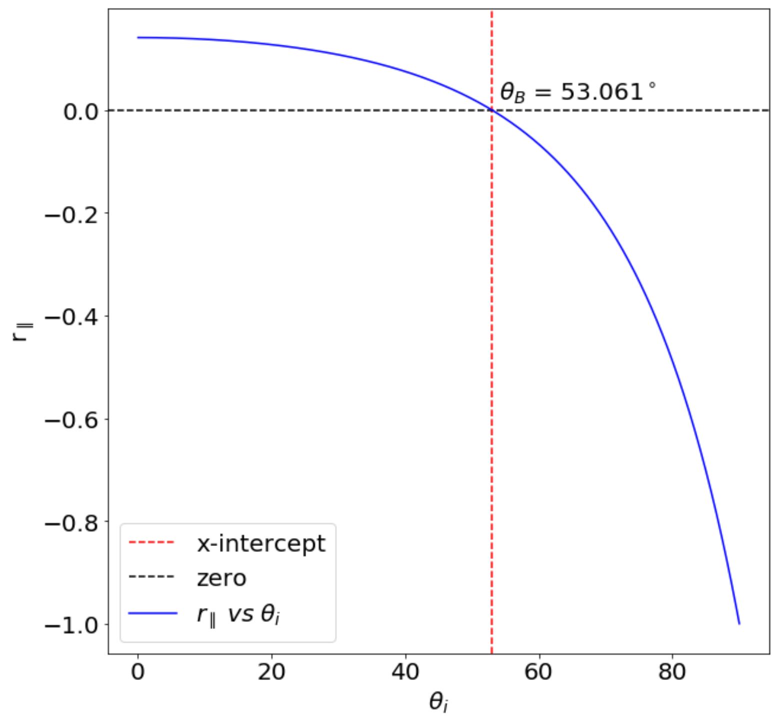

Figure 1: Plot of $r_{\parallel}$ versus θ_i where $n_i > n_t$.

The above figure gives the plot we were looking for. Both it and the plot on the next page were calculated using python. As shown on the plot Brewster's Angle for the case where $n_i > n_t$ is $\theta_{brewster} = 53.1°$.

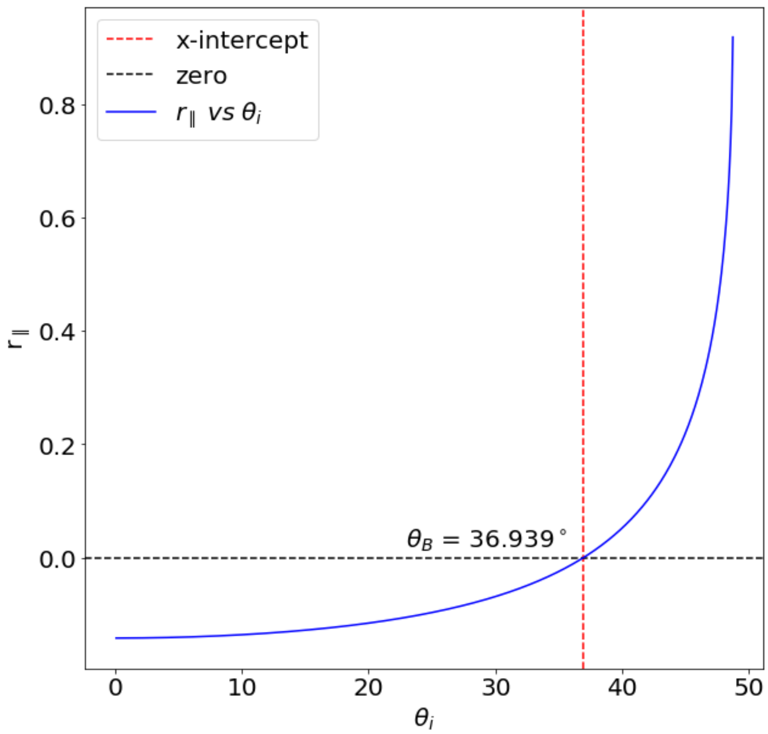

Figure 2: Plot of $r_\parallel$ versus θ_i where $n_i < n_t$.

This figure shows that Brewster's Angle for the case $n_i < n_t$ is $\theta_{brewster} = 36.9°$. These two plots give us the information we needed in order to solve Problem 3.

Problem 4

Let us consider the case of a laser whose radius $r = 1$mm which carries a power $P = 6$KW. We note now that this would be one insanely pwerful laser, and we will leave it to the end of the problem to calculate it's power relative to say, the Death Star.

First, we must consider the equations which should be brought to bear on this problem.

$$I = \langle \vec{s} \rangle = \frac{\text{Power}}{\text{Area}} = \left(\frac{\epsilon_0 \, E_0^2}{2} \right) c \quad \text{and} \quad B = \frac{E}{c}$$

Since Area is $A = \pi \, r^2 = \pi \, (1 \times 10^{-3} \, \text{m})^2 = \pi \times 10^{-6} \, \text{m}^2$, we can now calculate I using the above relation.

$$I = \frac{P}{A} = \frac{6 \times 10^3 \, W}{\pi \times 10^{-6} \, \text{m}^2} = \frac{6}{\pi} \times 10^9 \, \frac{W}{\text{m}^2} = 1.91 \times 10^9 \, \frac{W}{\text{m}^2}$$

With our value for $I = 1.91 \times 10^9 \, \frac{W}{\text{m}^2}$ now calculated, we can return to our set of useful equations to find out what the electric and magnetic fields of the laser beam are. First we will deal with the electric field.

$$I = \left(\frac{\epsilon_0 \, E_0^2}{2} \right) c = 1.91 \times 10^9 \, \frac{W}{\text{m}^2}$$

When we solve this equation for E_0 we get

$$E_0 = \sqrt{\frac{2 \times 1.91 \times 10^9 \, \frac{W}{\text{m}^2}}{\epsilon_0 \, c}} = \sqrt{\frac{2 \times 1.91 \times 10^9 \, \frac{W}{\text{m}^2}}{8.854 \times 10^{-12} \frac{C^2 \, s^2}{kg \, \text{m}^3} \, c \, \frac{\text{m}}{s}}} = 1.199 \times 10^6 \frac{\frac{\sqrt{W}}{\text{m}}}{\frac{C\sqrt{s}}{\sqrt{kg}\text{m}}}$$

Now to validate our answer we will do dimensional analysis on the units of our electric field to validate that they indeed are $\frac{N}{C}$.

$$\frac{\frac{\sqrt{W}}{\text{m}}}{\frac{C\sqrt{s}}{\sqrt{kg}\text{m}}} = \frac{\frac{\sqrt{\frac{kg\text{m}^2}{s^3}}}{\text{m}}}{\frac{C\sqrt{s}}{\sqrt{kg}\text{m}}} = \frac{\frac{\sqrt{kg}}{s\sqrt{s}}}{\frac{C\sqrt{s}}{\sqrt{kg}\text{m}}} = \frac{\sqrt{kg}}{s\sqrt{s}} \times \frac{\sqrt{kg}\text{m}}{C\sqrt{s}} = \frac{kg\text{m}}{Cs^2} = \frac{N}{C}$$

Thus we can be assured that the value of E_0 is correct, or at least that it has correct units.

Now on to calculating the value of B_0. Again, we go back to our set of useful equations and see that $B_0 = \frac{E_0}{c}$. Thus all we must do is divide our answer for E_0 by c to get the value of B_0.

$$E_0 = 1.199{\times}10^6 \frac{N}{C} \quad \text{and so} \quad B_0 = \frac{1.199 \times 10^6}{c} = 4{\times}10^{-3} \frac{N}{C\frac{\mathrm{m}}{s}} = 4{\times}10^{-3} \frac{N\,s}{C\,\mathrm{m}} = 4mT$$

We now have the value for both the electric and magnetic field, along with the irradiance of the laser.

As promised we will now compare the value of the irradiance to what would be emitted by the death star. The Death Star was able to destroy an Earth-sized planet in a few seconds, and since the binding energy of the earth is 2.24×10^{32} Joules we are going to estimate that it took the Death Star 10 esconds to destroy the planet. This gives us a power value for the laser emitted by the Death Star of 2.24×10^{31} Watts. Let us assume that the laser beam emitted by the Death Star is 100m wide, and so we can calculate the surface area of the laser as

$$A_{\text{death star}} = \pi(100\mathrm{m})^2 = 10000\pi\,\mathrm{m}^2$$

This will give us a value for the irradiance of the laser

$$I_{\text{death star}} = \frac{2.24 \times 10^{31}\,W}{10000\pi\,\mathrm{m}^2} = 7.13 \times 10^{26} \frac{W}{\mathrm{m}^2}$$

Now, comparing this with our laswer we get that the ratio of the two is

$$\frac{I_{laser}}{I_{\text{death star}}} = \frac{1.91 \times 10^9}{7.13 \times 10^{26}} = 2.68 \times 10^{-18}$$

This means that while our laser is quite powerful, we don't have to worry about it destroying the planet any time soon. In fact, it would take our laser

$$\frac{30\,s}{2.68 \times 10^{-18}} = 1.12 \times 10^{19}\,s = 8.52 \times 10^{12}\ \text{years}$$

to destroy the earth.

The important physics about the Death Star and the Earth came from `https://medium.com/starts-with-a-bang/the-physics-of-the-death-star-c21ccc58ade9#.7vjyr761o`

Problem 5

We now consider why a fish, or any observer underwater, will see a cone-framed image of what is above the surface surrounded by darkness. Let us first remember Snell's Law which states that

$$n_i \sin\theta_i = n_t \sin\theta_t$$

Using Snell's Law we can define the Crittical angle for the transmission between two medii as the angle θ_c such that $\sin\theta_c = 1 \implies \theta_c = \frac{\pi}{2}$. If we use this definition in conjunction with Snell's law we can get an expression for θ_c between any two medii

$$n_i \sin\theta_c = n_t \sin(\frac{\pi}{2}) = n_t \implies \sin\theta_c = \frac{n_t}{n_i} \implies \theta_c = \sin^{-1}\left(\frac{n_t}{n_i}\right)$$

We can use this in the case of water and air, noting that $n_{\text{water}} = 1.3374$ and $n_{\text{air}} = 1.0028$, to find the crittical angle where we have total internal reflection. The reason why this angle defines the cone seen from underwater, it's twice this angle but that's beside the point, is that light incident on the water at any angle greater than 0° relative to the origin will have a transmitted portion in the water, and since there can be no light coming from the surface at less than 0° the crittical angle defines the bound of the cone of light. So the 180° hemisphere above the water is compressed down into a cone whose angle is twice θ_c. The reason why the area outside the cone is so dark is taht it is lit only by that light which reflects off the surface of the water from below and comes back down, so it must be much darker as there are no light sources of great magnitude, or any amazing mirrors underwater.

The angle θ_c for water to air is $\sin^{-1}\left(\frac{1.0028}{1.3374}\right) = 48.57°$. This means that the angle of the cone will be

$$\theta_{\text{cone}} = 2 \times \theta_c = 97.15°$$