PHY 375 - Homework 6

Samuel Barton

March 12, 2017

Problem 1

In this problem we are asked to find the relation between δ and the angles in the provided triangle. We first note that this triangle is isocoles and thus the two bottom angles are equal, by extension of the definition of isocoles triangles. Next we note that the angle B as labeled in the given triangle must be $\pi - \mathrm{A}$ since the sum of the four angles of any quadrilateral must be 360° and the other two angles are formed by lines normal to their edges of the triangle and thus are 90° angles.

We use Snell's Law to get the value for $\theta_1^{'}$ remembering that $n_i = 1$ and $n_t = n$

$$\sin\theta_1 = n\sin\theta_1^{'}$$

$$\theta_1^{'} = \sin^{-1}\left(\frac{\sin\theta_1}{n}\right) \tag{1}$$

We can now get the value of $\theta_2^{'}$ if we remember that the sum of the angles in a triangle must be π radians

$$\theta_1^{'} + \theta_2^{'} + \mathrm{B} = \pi$$

$$\theta_2^{'} = \pi - \mathrm{B} - \theta_1^{'}$$

Since $\mathrm{B} = \pi - \mathrm{A}$, we can simplify the above equation as follows

$$\begin{aligned}\theta_2^{'} &= \pi - \theta_1^{'} - (\pi - \mathrm{A})\\ \theta_2^{'} &= \pi - \theta_1^{'} - \pi + \mathrm{A}\\ \theta_2^{'} &= -\theta_1^{'} + \mathrm{A}\\ \theta_2^{'} &= \mathrm{A} - \theta_1^{'}\end{aligned}$$

If we now substitute in equation 1 we get

$$\theta_2^{'} = \mathrm{A} - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right) \tag{2}$$

With $\theta_2^{'}$, we can now calculate θ_2 by again using Snell's Law, except this time $n_i = n$ and $n_t = 1$.

$$n\sin\theta_2^{'} = \sin\theta_2$$
$$\theta_2 = \sin^{-1}\left(n\sin\theta_2^{'}\right)$$

$$\theta_2 = \sin^{-1}(n \sin\left[\mathrm{A} - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right] \tag{3}$$

We now calculate δ_1 and δ_2 by using the geometric principle that the opposite angles of any two intersecting lines are equal. Thus

$$\theta_1 = \delta_1 + \theta_1^{'}$$
$$\delta_1 = \theta_1 - \theta_1^{'}$$

Using equation 1 we get

$$\delta_1 = \theta_1 - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right) \tag{4}$$

and

$$\theta_2 = \delta_2 + \theta_2^{'}$$
$$\delta_2 = \theta_2 - \theta_2^{'}$$

Using equations 2 and 3 we get

$$\delta_2 = \sin^{-1}(n \sin\left[\mathrm{A} - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right] - \left[\mathrm{A} - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right]$$
$$\delta_2 = \sin^{-1}(n \sin\left[\mathrm{A} - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right] - \mathrm{A} + \sin^{-1}\left(\frac{\sin\theta_1}{n}\right) \tag{5}$$

With values for δ_1 and δ_2 we again use the principle that the sum of the angles of a triangle must be π raidans to get the angle C, where C is the third angle in the triangle $\delta_1\delta_2$C

$$\delta_1 + \delta_2 + \mathrm{C} = \pi$$

$$\mathrm{C} = \pi - \delta_1 - \delta_2 \tag{6}$$

With the value for C we can finally calculate the value δ by using the geometric theorem that the sum of the interior and exterior angles of a line relative to another line must equal π radians. In this case the lines are that of the incident ray, and the ray which exits the prism. The angle C is the exterior angle to the incident ray relative to the final ray, and the angle δ is the internal angle.

$$\delta = \pi - \mathrm{C} \tag{7}$$

Now we begin back-substitution starting with equation 6.

$$\delta = \pi - (\pi - \delta_1 - \delta_2)$$

Next we substitute in equations 4 and 5 to get the final expression for δ

$$\delta = \pi - \left[\pi - \left[\theta_1 - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right] - \left[\sin^{-1}(n \sin\left[\mathrm{A} - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right] - \mathrm{A} + \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right]\right]$$
$$\delta = \pi - \left[\pi - \theta_1 + \sin^{-1}\left(\frac{\sin\theta_1}{n}\right) - \sin^{-1}(n \sin\left[\mathrm{A} - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right] + \mathrm{A} - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right]$$

$$\delta = \pi - \left[\pi - \theta_1 - \sin^{-1}(n \sin\left[\mathrm{A} - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right] + \mathrm{A}\right]$$

$$\delta = \pi - \pi + \theta_1 + \sin^{-1}(n \sin\left[\mathrm{A} - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right] - \mathrm{A}$$

And finally we reach the value for δ

$$\delta = \theta_1 + \sin^{-1}(n \sin\left[\mathrm{A} - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right] - \mathrm{A}$$

This relation gives us δ as a function of θ_1 and A.

Problem 2

We are now to calculate the total transmittance for the light ray we just discussed in problem one passing through the same isocoles triangle. First off we must remember some important definitions

For our two surfaces we get that

$$\mathrm{T}_{\parallel} = \mathrm{T}_{\parallel_1} \times \mathrm{T}_{\parallel_2} = \mathrm{t}^2_{\parallel_1} \times \mathrm{t}^2_{\parallel_2} \tag{8}$$

$$\mathrm{T}_{\perp} = \mathrm{T}_{\perp_1} \times \mathrm{T}_{\perp_2} = \mathrm{t}^2_{\perp_1} \times \mathrm{t}^2_{\perp_2} \tag{9}$$

$$\mathrm{t}_{\parallel} = \frac{2\sin\theta_t \cos\theta_i}{\sin(\theta_i + \theta_t)} \tag{10}$$

$$\mathrm{t}_{\perp} = \frac{2\sin\theta_t \cos\theta_i}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)} \tag{11}$$

We will use the above equations for the perpendicular and parallel transmittance to generate both $\mathrm{T}_{\perp}(\theta_1)$ and $\mathrm{T}_{\parallel}(\theta_1)$.

First we consider the value of $\mathrm{T}_{\parallel}$. We insert our values for θ_i and θ_t on the first surface and get

$$\mathrm{t}_{\parallel_1} = \frac{2\sin\theta_1' \cos\theta_1}{\sin(\theta_1 + \theta_1')}$$

$$\mathrm{t}_{\parallel_1} = \frac{2\sin\left[\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right]\cos\theta_1}{\sin(\theta_1 + \sin^{-1}\left(\frac{\sin\theta_1}{n}\right))} \tag{12}$$

Now we consider the parallel component of the transmittance on the second surface where θ_i and θ_t are θ_2' and θ_2 respectively

$$\mathrm{t}_{\parallel_2} = \frac{2\sin\theta_2 \cos\theta_2'}{\sin(\theta_2 + \theta_2')}$$

$$\mathrm{t}_{\parallel_2} = \frac{2\sin\left[\sin^{-1}(n\sin\left[\mathrm{A} - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right]\right]\cos\left[\mathrm{A} - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right]}{\sin(\sin^{-1}(n\sin\left[\mathrm{A} - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right]) + \mathrm{A} - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right))} \tag{13}$$

With these we can calculate the total parallel transmittance

$$T_\parallel = \left[\frac{2\sin\left[\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right]\cos\theta_1}{\sin(\theta_1+\sin^{-1}\left(\frac{\sin\theta_1}{n}\right))}\right]^2 + \left[\frac{2\sin\left[\sin^{-1}(n\sin\left[A-\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right]\right]\cos\left[A-\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right]}{\sin(\sin^{-1}(n\sin\left[A-\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right])+A-\sin^{-1}\left(\frac{\sin\theta_1}{n}\right))}\right]^2 \tag{14}$$

With the parallel transmittance out of the way, we now calculate the perpendicular transmittance for both surfaces using the same methodology as above. On the first surface $\theta_i = \theta_1$ and $\theta_t = \theta_1'$.

$$t_{\perp_1} = \frac{2\sin\theta_1'\cos\theta_1}{\sin(\theta_1+\theta_1')\cos(\theta_1-\theta_1')}$$

$$t_{\perp_1} = \frac{2\sin\left[\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right]\cos\theta_1}{\sin(\theta_1+\sin^{-1}\left(\frac{\sin\theta_1}{n}\right))\cos(\theta_1-\sin^{-1}\left(\frac{\sin\theta_1}{n}\right))} \tag{15}$$

Now we consider the second surface, where θ_i and θ_t are θ_2' and θ_2 respectively

$$t_{\perp_2} = \frac{2\sin\theta_2\cos\theta_2'}{\sin(\theta_2'+\theta_2)\cos(\theta_2'-\theta_2)} \tag{16}$$

We cannot fit the fully expanded version of this on one line, so we split it up below.

$$t_{\perp_2} = \frac{2\sin\left[\sin^{-1}\left(n\sin\left[A-\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right]\right)\right]\cos\left[A-\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right]}{\sin\left[A-\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)+\sin^{-1}\left(n\sin\left[A-\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right]\right)\right]} \times$$
$$\frac{1}{\cos\left[A-\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)-\sin^{-1}\left(n\sin\left[A-\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right]\right)\right]}$$

We combine equations 15 and 16 according to the relation defined by equation 9 to get the value for the perpendicular transmittance

$$T_\perp = \left[\frac{2\sin\theta_2\cos\theta_2'}{\sin(\theta_2'+\theta_2)\cos(\theta_2'-\theta_2)}\right]^2 + \left[\frac{2\sin\left[\sin^{-1}\left(n\sin\left[A-\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right]\right)\right]\cos\left[A-\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right]}{\sin\left[A-\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)+\sin^{-1}\left(n\sin\left[A-\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right]\right)\right]} \times\right.$$
$$\left.\frac{1}{\cos\left[A-\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)-\sin^{-1}\left(n\sin\left[A-\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right]\right)\right]}\right]^2$$

We will see in the next problem how the parallel and perpendicular transmittance relate to the incident angle as it goes from 0° to 90°

Problem 3

In this problem we display the plots of $\delta(\theta_1)$ vs θ_1, $\mathrm{T}_{\parallel}(\theta_1)$ vs θ_1, and $\mathrm{T}_{\perp}(\theta_1)$ vs θ_1.

First we display the graph of $\delta(\theta_1)$ vs θ_1

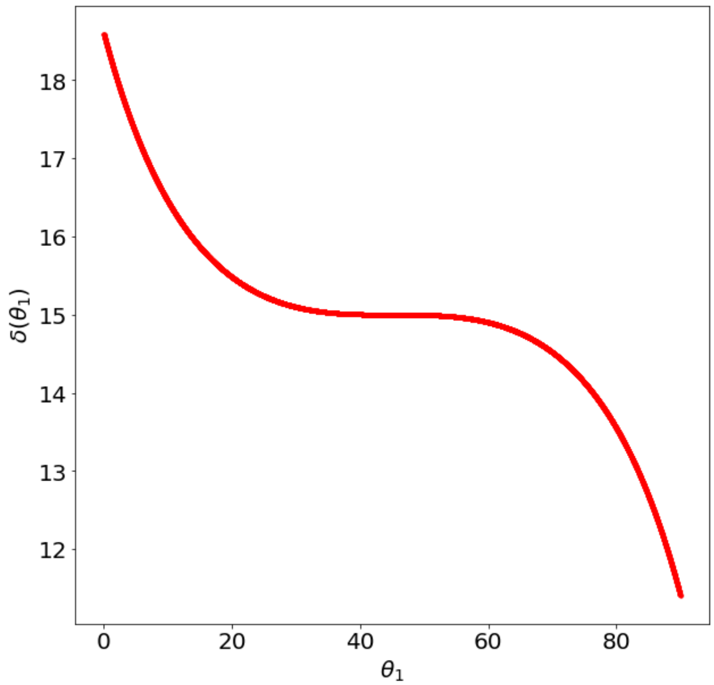

Figure 1: Plot of $\delta(\theta_1)$ versus θ_1

Next we display the graphs of both $\mathrm{T}_{\parallel}(\theta_1)$ vs θ_1, and $\mathrm{T}_{\perp}(\theta_1)$ vs θ_1.

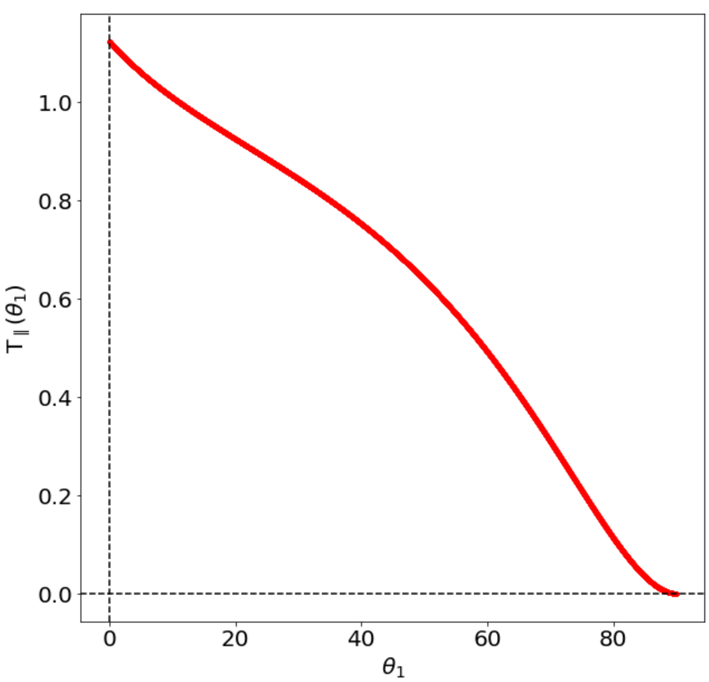

Figure 2: Plot of $\mathrm{T}_{\parallel}(\theta_1)$ versus θ_1

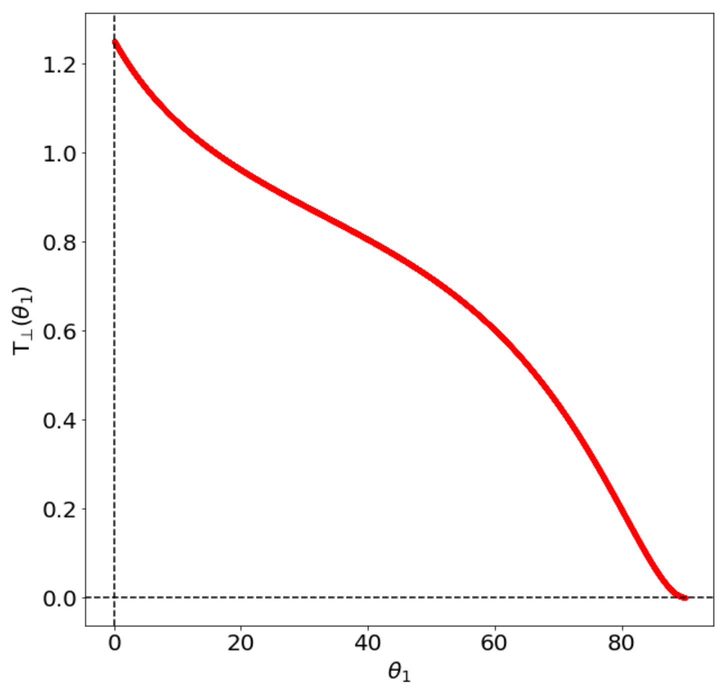

Figure 3: Plot of $\mathrm{T}_{\perp}(\theta_1)$ versus θ_1

Problem 4

In this problem we consider the case of a light source 1.2m away from a crystal ball whose diameter is 20cm which implies that R = 0.1m. We are to calculate where the image will be, both the virtual image on the left-hand side of the ball, and the actual image on the right hand side of the ball. In order to do this we also need to know that the index of refraction of the crystal ball is 1.45 and the ray is travelling through air whose index of refraction is considered to be 1 for the sake of this problem.

In order to do this we must first define some useful equations which we will need
For the first surface that the light ray is incident upon

$$\frac{n_m}{\mathrm{S}_{\mathrm{o}_1}} + \frac{n_l}{\mathrm{S}_{\mathrm{i}_1}} = \frac{n_l - n_m}{\mathrm{R}}$$
$$\frac{n_l}{\mathrm{S}_{\mathrm{i}_1}} = \frac{n_l - n_m}{\mathrm{R}} - \frac{n_m}{\mathrm{S}_{\mathrm{o}_1}}$$
$$\mathrm{S}_{\mathrm{i}_1} = \frac{n_l}{n_l - n_m}\mathrm{R} - \frac{n_l}{n_m}\mathrm{S}_{\mathrm{o}_1} \tag{17}$$

And for the second surface where the light emerges from

$$\frac{n_l}{d - \mathrm{S}_{\mathrm{i}_1}} + \frac{n_m}{\mathrm{S}_{\mathrm{i}_2}} = \frac{n_m - n_l}{\mathrm{R}}$$
$$\frac{n_m}{\mathrm{S}_{\mathrm{i}_2}} = \frac{n_m - n_l}{\mathrm{R}} - \frac{n_l}{d - \mathrm{S}_{\mathrm{i}_1}}$$
$$\mathrm{S}_{\mathrm{i}_2} = \frac{n_m}{n_m - n_l}\mathrm{R} - \frac{n_m}{n_l}(d - \mathrm{S}_{\mathrm{i}_1}) \tag{18}$$

If we insert the known values into equations 17 and 18 we get

$$\mathrm{S}_{\mathrm{i}_1} = \frac{1.45}{1.45 - 1}0.1\text{ m} - \frac{1.45}{1}1.2\text{ m} = -1.418\text{ m}$$
$$\mathrm{S}_{\mathrm{i}_2} = \frac{1}{1 - 1.45}0.1\text{ m} - \frac{1}{1.45}(0.2\text{ m} - (-1.418\text{ m})) = -1.338\text{ m}$$

The negative sign on $\mathrm{S}_{\mathrm{i}_1}$ means that there is a virtual image on the left side of the ball which sits behind the source of the light. The negative sign on $\mathrm{S}_{\mathrm{i}_2}$ means that the final image sits to the right of the second edge of the crystal. Below is a figure representing the situation we have described.