Answers to Exercises Chapter 10

Chapter 10

Section 10.1, page 9

1. **a.** Stochastic.
b. Not stochastic. Columns do not sum to 1.

2. **a.** Stochastic.
b. Not stochastic. Matrix has entries not in interval [0, 1].

3. $\mathbf{x}_3 = \begin{bmatrix} .556 \\ .444 \end{bmatrix}$ **4.** $\mathbf{x}_3 = \begin{bmatrix} .5375 \\ .4625 \end{bmatrix}$

5. 109/216 **6.** .74976 **7.** 13/36 **8.** .414

9. **a.** .53125 **b.** 0

10. **a.** .104 **b.** 0

11. **a.** 5/8 **b.** 1/8

12. **a.** .928 **b.** .128

13. $P = \begin{bmatrix} 0 & 1/3 & 0 & 1/2 & 1/2 \\ 1/3 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/3 & 0 & 1/2 & 0 \\ 1/3 & 0 & 1/2 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 & 0 \end{bmatrix}$

14. $P = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix}$

15. $P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1 & 0 & 0 \end{bmatrix}$

16. $P = \begin{bmatrix} 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 1 & 1/2 & 0 & 1/2 & 1 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \end{bmatrix}$

17. **a.** $P = \begin{bmatrix} 0 & 1/3 & 1/4 & 1/3 & 0 \\ 1/3 & 0 & 1/4 & 0 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 0 & 1/4 & 0 & 1/3 \\ 0 & 1/3 & 1/4 & 1/3 & 0 \end{bmatrix}, \mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

b. $\mathbf{x}_3 = \begin{bmatrix} .25926 \\ .11111 \\ .25926 \\ .11111 \\ .25926 \end{bmatrix}$

18. **a.** $P = \begin{bmatrix} 0 & 1/4 & 0 & 1/3 & 0 \\ 1/2 & 0 & 1/2 & 1/3 & 1/3 \\ 0 & 1/4 & 0 & 0 & 1/3 \\ 1/2 & 1/4 & 0 & 0 & 1/3 \\ 0 & 1/4 & 1/2 & 1/3 & 0 \end{bmatrix}, \mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

b. $\mathbf{x}_4 = \begin{bmatrix} .13542 \\ .25463 \\ .18171 \\ .25116 \\ .17708 \end{bmatrix}$

19. **a.** $P = \begin{bmatrix} 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/3 & 1 \end{bmatrix}, \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

b. $\mathbf{x}_4 = \begin{bmatrix} .12963 \\ 0 \\ .12963 \\ 0 \\ .43518 \\ .30556 \end{bmatrix}$

20. **a.** $P = \begin{bmatrix} 0 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 0 \end{bmatrix}, \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

b. $\mathbf{x}_3 = \begin{bmatrix} 0 \\ .625 \\ 0 \\ .25 \\ .125 \end{bmatrix}$

21. **a.** True.
b. False. The transition matrix P cannot change over time.
c. True.

22. **a.** False. The **columns** of a transition matrix for a Markov chain must sum to 1.
b. True.
c. False. The $\{i, j\}$-entry in matrix P^3 gives the probability of a move from state j to state i in exactly three moves.

23. Sunny with probability .406, cloudy with probability .145375, rainy with probability .448625.

24. a. $P = \begin{bmatrix} .58 & 0 & .47 & 0 \\ .42 & 0 & .53 & 0 \\ 0 & .29 & 0 & .31 \\ 0 & .71 & 0 & .69 \end{bmatrix}$ **b.** .43201 **25.** $\mathbf{x}_3 = \begin{bmatrix} 1/6 \\ 5/18 \\ 5/18 \\ 5/18 \end{bmatrix}$ **26.** $\mathbf{x}_4 = \begin{bmatrix} 1/8 \\ 1/8 \\ 1/2 \\ 1/8 \\ 1/8 \end{bmatrix}$

27. a. $P = \begin{bmatrix} p^2 & p(1-p) & p(1-p) & (1-p)^2 \\ p(1-p) & p^2 & (1-p)^2 & p(1-p) \\ p(1-p) & (1-p)^2 & p^2 & p(1-p) \\ (1-p)^2 & p(1-p) & p(1-p) & p^2 \end{bmatrix}$ **b.** .94206

28. $$P = \begin{bmatrix} p^3 & p^2(1-p) & p^2(1-p) & p(1-p)^2 & p^2(1-p) & p(1-p)^2 & p(1-p)^2 & (1-p)^3 \\ p^2(1-p) & p^3 & p(1-p)^2 & p^2(1-p) & p(1-p)^2 & p^2(1-p) & (1-p)^3 & p(1-p)^2 \\ p^2(1-p) & p(1-p)^2 & p^3 & p^2(1-p) & p(1-p)^2 & (1-p)^3 & p^2(1-p) & p(1-p)^2 \\ p(1-p)^2 & p^2(1-p) & p^2(1-p) & p^3 & (1-p)^3 & p(1-p)^2 & p(1-p)^2 & p^2(1-p) \\ p^2(1-p) & p(1-p)^2 & p(1-p)^2 & (1-p)^3 & p^3 & p^2(1-p) & p^2(1-p) & p(1-p)^2 \\ p(1-p)^2 & p^2(1-p) & (1-p)^3 & p(1-p)^2 & p^2(1-p) & p^3 & p(1-p)^2 & p^2(1-p) \\ p(1-p)^2 & (1-p)^3 & p^2(1-p) & p(1-p)^2 & p^2(1-p) & p(1-p)^2 & p^3 & p^2(1-p) \\ (1-p)^3 & p(1-p)^2 & p(1-p)^2 & p^2(1-p) & p(1-p)^2 & p^2(1-p) & p^2(1-p) & p^3 \end{bmatrix}$$

29. a. $P = \begin{bmatrix} 1-p & p/6 & 0 & 0 & 0 & 0 & 0 \\ p & 1-p & p/3 & 0 & 0 & 0 & 0 \\ 0 & 5p/6 & 1-p & p/2 & 0 & 0 & 0 \\ 0 & 0 & 2p/3 & 1-p & 2p/3 & 0 & 0 \\ 0 & 0 & 0 & p/2 & 1-p & 5p/6 & 0 \\ 0 & 0 & 0 & 0 & p/3 & 1-p & p \\ 0 & 0 & 0 & 0 & 0 & p/6 & 1-p \end{bmatrix}$ **b.** .19290

30. a. In order for a transition from j to $j-1$ type I molecules to occur, a type I molecule must be chosen from urn A and a type II molecule must be chosen from urn B. Since there are j type I molecules in urn A and j type II molecules in urn B, the probability of choosing a type I molecule from urn A is j/k, and so is the probability of choosing a type II molecule from urn B. Thus the transition probability is $(j/k)^2$. The probability of a transition from j to $j+1$ type I molecules in urn A is computed in a similar fashion.

b. $\begin{bmatrix} 0 & 1/25 & 0 & 0 & 0 & 0 \\ 1 & 8/25 & 4/25 & 0 & 0 & 0 \\ 0 & 16/25 & 12/25 & 9/25 & 0 & 0 \\ 0 & 0 & 9/25 & 12/25 & 16/25 & 0 \\ 0 & 0 & 0 & 4/25 & 8/25 & 1 \\ 0 & 0 & 0 & 0 & 1/25 & 0 \end{bmatrix}$ **c.** $\mathbf{x}_3 = \begin{bmatrix} .0000 \\ .0000 \\ .2304 \\ .5120 \\ .2448 \\ .0128 \end{bmatrix}$

31. a. $P = \begin{bmatrix} 0 & 1-p & p & 0 & 0 \\ p & 0 & 0 & 0 & 0 \\ 1-p & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 1 & 0 \\ 0 & 0 & 1-p & 0 & 1 \end{bmatrix}$ **b.** .192

32. a. $P = \begin{bmatrix} 0 & 0 & 0 & 1-q & 0 & 0 \\ 0 & 0 & 1-p & 0 & 0 & 0 \\ p & 1-q & 0 & 0 & 0 & 0 \\ 1-p & q & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 1 & 0 \\ 0 & 0 & 0 & q & 0 & 1 \end{bmatrix}$ **b.** .4

c. $P = \begin{bmatrix} 0 & 1-q & 0 & 0 & p & 0 & 0 & 0 \\ 1-p & 0 & 0 & q & 0 & 0 & 0 & 0 \\ p & 0 & 0 & 1-q & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-q & 0 & 0 \\ 0 & q & 0 & 0 & 1-p & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & q & 0 & 1 \end{bmatrix}$ **d.** .496

33. Suppose that P is an $m \times m$ stochastic matrix all of whose entries are greater than or equal to p. The proof proceeds by induction. Notice that the statement to be proven is thus true for $n = 1$. Assume the statement is true for n, and let $B = P^n$. Then, since $P^{n+1} = BP$, the (i, j)-entry in P^{n+1} is $\sum_{k=1}^{m} b_{ik} p_{kj}$. Since $b_{ik} \geq p$ by the induction hypothesis, $\sum_{k=1}^{m} b_{ik} p_{kj} \geq p \sum_{k=1}^{m} p_{kj}$. But $\sum_{k=1}^{m} p_{kj} = 1$ since P is a stochastic matrix, so all of the entries in P^{n+1} are greater than or equal to p.

Section 10.2, page 21

1. $P^{10} = \begin{bmatrix} .33333 & .33333 \\ .66667 & .66667 \end{bmatrix}$

$\mathbf{q} = \begin{bmatrix} .33333 \\ .66667 \end{bmatrix}$

The probability is .33333.

2. $P^{20} = \begin{bmatrix} .47059 & .47059 \\ .52941 & .52941 \end{bmatrix}$

$\mathbf{q} = \begin{bmatrix} 8/17 \\ 9/17 \end{bmatrix} \approx \begin{bmatrix} .47059 \\ .52941 \end{bmatrix}$

The probability is .47059.

3. $P^{20} = \begin{bmatrix} .21429 & .21429 & .21429 \\ .57143 & .57143 & .57143 \\ .21429 & .21429 & .21429 \end{bmatrix}$

$\mathbf{q} = \begin{bmatrix} 3/14 \\ 4/7 \\ 3/14 \end{bmatrix} \approx \begin{bmatrix} .21429 \\ .57143 \\ .21429 \end{bmatrix}$

The probability is .21429.

4. $P^{20} = \begin{bmatrix} .22308 & .22308 & .22308 \\ .32308 & .32308 & .32308 \\ .45385 & .45385 & .45385 \end{bmatrix}$

$\mathbf{q} = \begin{bmatrix} .22308 \\ .32308 \\ .45385 \end{bmatrix}$

The probability is .22308.

5. $\begin{bmatrix} 8/17 & 8/17 \\ 9/17 & 9/17 \end{bmatrix}$

6. $\begin{bmatrix} 4/21 & 4/21 & 4/21 \\ 5/21 & 5/21 & 5/21 \\ 4/7 & 4/7 & 4/7 \end{bmatrix}$

7. P is regular since all entries in P^2 are positive.

8. P is not regular because the $(2, 1)$-entry in P^n will be zero for any n.

9. a. The transition matrix is

$$P = \begin{bmatrix} 0 & 1/4 & 0 & 0 & 0 \\ 1 & 0 & 1/2 & 0 & 0 \\ 0 & 3/4 & 0 & 3/4 & 0 \\ 0 & 0 & 1/2 & 0 & 1 \\ 0 & 0 & 0 & 1/4 & 0 \end{bmatrix}$$

The $(0, 0)$-entry in P^k will be zero if k is odd while the $(0, 1)$-entry in P^k will be zero if k is even. Thus P is not regular.

b. Compute that

$$\mathbf{q} = \begin{bmatrix} 1/16 \\ 1/4 \\ 3/8 \\ 1/4 \\ 1/16 \end{bmatrix}$$

so the chain will spend the most steps in state 2, which corresponds to both urns containing 2 molecules.

10. a. The transition matrix is

$$P = \begin{bmatrix} 0 & 1/5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2/5 & 0 & 0 & 0 \\ 0 & 4/5 & 0 & 3/5 & 0 & 0 \\ 0 & 0 & 3/5 & 0 & 4/5 & 0 \\ 0 & 0 & 0 & 2/5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1/5 & 0 \end{bmatrix}$$

The $(0, 0)$-entry in P^k will be zero if k is odd while the $(0, 1)$-entry in P^k will be zero if k is even. Thus P is not regular.

b. Compute that

$$\mathbf{q} = \begin{bmatrix} 1/32 \\ 5/32 \\ 5/16 \\ 5/16 \\ 5/32 \\ 1/32 \end{bmatrix}$$

so the chain will spend the most steps in states 2 and 3, which corresponds to one urn containing 2 molecules and the other containing 3 molecules.

11. a. The transition matrix is

$$P = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1 \\ 0 & 0 & 1/2 & 0 \end{bmatrix}$$

The $(1, 1)$-entry in P^k will be zero if k is odd while the $(1, 2)$-entry in P^k will be zero if k is even. Thus P is not regular.

b. Compute that

$$\mathbf{q} = \begin{bmatrix} 1/6 \\ 1/3 \\ 1/3 \\ 1/6 \end{bmatrix}$$

so the chain will spend the most steps in states 2 and 3.

12. a. The transition matrix is

$$P = \begin{bmatrix} 0 & .2 & 0 & 0 \\ 1 & 0 & .2 & 0 \\ 0 & .8 & 0 & 1 \\ 0 & 0 & .8 & 0 \end{bmatrix}$$

The $(1,1)$-entry in P^k will be zero if k is odd while the $(1,2)$-entry in P^k will be zero if k is even. Thus P is not regular.

b. Compute that

$$\mathbf{q} = \begin{bmatrix} .02381 \\ .11905 \\ .47619 \\ .38095 \end{bmatrix}$$

so the chain will spend the most steps in state 3.

13. $\mathbf{q} = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/6 \\ 1/6 \\ 1/6 \end{bmatrix}$ **14.** $\mathbf{q} = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$

15. $\mathbf{q} = \begin{bmatrix} 3/13 \\ 3/13 \\ 3/13 \\ 4/13 \end{bmatrix}$ **16.** $\mathbf{q} = \begin{bmatrix} 1/10 \\ 1/5 \\ 2/5 \\ 1/5 \\ 1/10 \end{bmatrix}$

17. Since $\mathbf{q} = \begin{bmatrix} .1875 \\ .1875 \\ .25 \\ .1875 \\ .1875 \end{bmatrix}$, the probability is .25.

18. Since $\mathbf{q} = \begin{bmatrix} 1/7 \\ 2/7 \\ 1/7 \\ 3/14 \\ 3/14 \end{bmatrix}$, the fraction of the time is $1/7$.

19. Since

$$P\mathbf{q} = \begin{bmatrix} 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$\mathbf{q}$ is a steady-state vector for the Markov chain. Room 6 is an absorbing state for the chain—once the mouse moves into room 6 it will stay there forever.

20. $\mathbf{q} = \begin{bmatrix} 2/15 \\ 1/5 \\ 2/15 \\ 4/15 \\ 4/15 \end{bmatrix}$

21. **a.** True. **b.** True. **c.** False. See Examples 4 and 5.

22. **a.** False. See Examples 4 and 5. **b.** True. **c.** True.

23. To the nearest day, 152 are sunny, 52 are cloudy, and 161 are rainy.

24. To the nearest day, 143 are rainy.

25. The Google matrix and its steady-state vector are

$$G = \begin{bmatrix} .03 & .03 & .88 & .03 & .2 \\ .313333 & .03 & .03 & .455 & .2 \\ .313333 & .03 & .03 & .455 & .2 \\ .313333 & .455 & .03 & .03 & .2 \\ .03 & .455 & .03 & .03 & .2 \end{bmatrix},$$

$$\mathbf{q} = \begin{bmatrix} .231535 \\ .208517 \\ .208517 \\ .208517 \\ .142915 \end{bmatrix}$$

and the PageRanks are 1; 2, 3, and 4 (tied); 5.

26. The Google matrix and its steady-state vector are

$$G = \begin{bmatrix} .166667 & .45 & .2375 & .025 & .025 & .166667 \\ .166667 & .025 & .2375 & .025 & .025 & .166667 \\ .166667 & .45 & .025 & .45 & .025 & .166667 \\ .166667 & .025 & .2375 & .025 & .45 & .166667 \\ .166667 & .025 & .2375 & .45 & .025 & .166667 \\ .166667 & .025 & .025 & .025 & .45 & .166667 \end{bmatrix},$$

$$\mathbf{q} = \begin{bmatrix} .157494 \\ .110522 \\ .197265 \\ .192212 \\ .192212 \\ .150294 \end{bmatrix}$$

and the PageRanks are 3; 5 and 4 (tied); 1; 6; 2.

27. **a.** If a dominant (AA) individual is mated with a hybrid (Aa), then the dominant individual will always contribute an A. One half of the time the hybrid will also contribute an A, leading to a dominant offspring. The other half of the time, the hybrid will contribute an a, yielding a hybrid offspring.

b. If a recessive (aa) individual is mated with a hybrid (Aa), then the recessive individual will always contribute an a. One half of the time the hybrid will also contribute an a, leading to a recessive offspring. The other half of the time, the hybrid will contribute an A, yielding a hybrid offspring.

c. If a hybrid (Aa) is mated with another hybrid (Aa), then a dominant offspring will result when both hybrids contribute an A, which happens $(1/2)(1/2) = 1/4$ of the time. Likewise a recessive offspring will result when both hybrids contribute an a, which also happens $(1/2)(1/2) = 1/4$ of the time. Finally, in all other cases, a hybrid offspring will be produced, which happens $1 - 1/4 - 1/4 = 1/2$ of the time.

28. a. Let the states be ordered AA, Aa, aa. Then, by Exercise 27,

$$P = \begin{bmatrix} 1/2 & 1/4 & 0 \\ 1/2 & 1/2 & 1/2 \\ 0 & 1/4 & 1/2 \end{bmatrix}$$

b. For the matrix P in part (a),

$$\mathbf{q} = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}$$

so 25% of the offspring will be dominant, 25% will be recessive, and 50% will be hybrids.

29. a. Confirm that all entries in P^6 are strictly positive.

b. Compute that

$$\mathbf{q} = \begin{bmatrix} 1/64 \\ 3/32 \\ 15/64 \\ 5/16 \\ 15/64 \\ 3/32 \\ 1/64 \end{bmatrix}$$

so the chain spends the most steps in state 3, which corresponds to both urns containing 3 molecules. The fraction of steps the chain spends there is 5/16.

c. Compute that

$$P\mathbf{q} = \begin{bmatrix} 1-p & p/6 & 0 & 0 & 0 & 0 & 0 \\ p & 1-p & p/3 & 0 & 0 & 0 & 0 \\ 0 & 5p/6 & 1-p & p/2 & 0 & 0 & 0 \\ 0 & 0 & 2p/3 & 1-p & 2p/3 & 0 & 0 \\ 0 & 0 & 0 & p/2 & 1-p & 5p/6 & 0 \\ 0 & 0 & 0 & 0 & p/3 & 1-p & p \\ 0 & 0 & 0 & 0 & 0 & p/6 & 1-p \end{bmatrix} \begin{bmatrix} 1/64 \\ 3/32 \\ 15/64 \\ 5/16 \\ 15/64 \\ 3/32 \\ 1/64 \end{bmatrix} = \begin{bmatrix} 1/64 \\ 3/32 \\ 15/64 \\ 5/16 \\ 15/64 \\ 3/32 \\ 1/64 \end{bmatrix} = \mathbf{q}$$

so the result of part (b) does not depend on the value of p.

30. a. Confirm that all entries in P^5 are strictly positive.

b. Compute that

$$\mathbf{q} = \begin{bmatrix} 1/252 \\ 25/252 \\ 25/63 \\ 25/63 \\ 25/252 \\ 1/252 \end{bmatrix}$$

so the chain spends the most time in states 2 and 3, which corresponds to urn A containing 2 and 3 type I molecules. The fraction of time the chain spends there is 25/63.

31. Compute that $\begin{bmatrix} 1 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} q \\ 0 \\ 0 \\ 0 \\ 1-q \end{bmatrix} = \begin{bmatrix} q \\ 0 \\ 0 \\ 0 \\ 1-q \end{bmatrix}$

32. **a.** Compute that

$$\begin{bmatrix} 0 & 1/2 & 0 & 0 & 0 \\ 1 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1 \\ 0 & 0 & 0 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 1/8 \\ 1/4 \\ 1/4 \\ 1/4 \\ 1/8 \end{bmatrix} = \begin{bmatrix} 1/8 \\ 1/4 \\ 1/4 \\ 1/4 \\ 1/8 \end{bmatrix}$$

b. The average is

$$\begin{bmatrix} .12525 & .1251 & .125 & .1249 & .12475 \\ .25025 & .25025 & .25 & .24975 & .24975 \\ .25 & .25 & .25 & .25 & .25 \\ .24975 & .24975 & .25 & .25025 & .25025 \\ .12475 & .1249 & .125 & .1251 & .12525 \end{bmatrix},$$

each column of which is very close to the steady-state vector.

33. Compute that

$$\begin{bmatrix} 1/4 & 1/3 & 1/2 & 0 & 0 \\ 1/4 & 1/3 & 1/4 & 0 & 0 \\ 1/2 & 1/3 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 3/4 \\ 0 & 0 & 0 & 2/3 & 1/4 \end{bmatrix} \begin{bmatrix} 4/11 \\ 3/11 \\ 4/11 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/11 \\ 3/11 \\ 4/11 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1/4 & 1/3 & 1/2 & 0 & 0 \\ 1/4 & 1/3 & 1/4 & 0 & 0 \\ 1/2 & 1/3 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 3/4 \\ 0 & 0 & 0 & 2/3 & 1/4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 9/17 \\ 8/17 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 9/17 \\ 8/17 \end{bmatrix}$$

If the chain is equally likely to begin in each of the states, then it begins in state 1, 2, or 3 with probability $3/5$, and in state 4 or 5 with probability $2/5$. Since

$$\frac{3}{5}\begin{bmatrix} 4/11 \\ 3/11 \\ 4/11 \\ 0 \\ 0 \end{bmatrix} + \frac{2}{5}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 9/17 \\ 8/17 \end{bmatrix} = \begin{bmatrix} 12/55 \\ 9/55 \\ 12/55 \\ 18/85 \\ 16/85 \end{bmatrix}$$

the probability of the chain being in state 1 after many steps is $12/55$.

34. **a.** The characteristic polynomial of P is $\lambda^2 - (p+q)\lambda + p + q - 1 = (\lambda - 1)(\lambda - (p+q-1))$, so 1 and $p+q-1$ are the eigenvalues of P.

b. The matrix P will fail to be regular if either $p = 1$ or $q = 1$.

c. A steady-state vector for P is $\begin{bmatrix} \dfrac{q-1}{p+q-2} \\ \dfrac{p-1}{p+q-2} \end{bmatrix}$

35. **a.** The matrix P will be a stochastic matrix if $p + q \leq 1$. It will be a regular stochastic matrix if in addition $p \neq 1$ and $q \neq 1$.

b. A steady-state vector for P is

$$\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

36. Let $1 \leq k \leq m$. Then

$$\begin{aligned} |y_k| &= |p_{k1}x_1 + \ldots + p_{km}x_m| \\ &\leq p_{k1}|x_1| + \ldots + p_{km}|x_m| \end{aligned} \tag{1}$$

So

$$\begin{aligned} |y_1| + \ldots + |y_m| &\leq p_{11}|x_1| + \ldots + p_{1m}|x_m| \\ &\quad + \ldots \\ &\quad + p_{m1}|x_1| + \ldots + p_{mm}|x_m| \\ &= (p_{11} + \ldots + p_{m1})|x_1| \\ &\quad + \ldots \\ &\quad + (p_{1m} + \ldots + p_{mm})|x_m| \\ &= |x_1| + \ldots + |x_m| \end{aligned}$$

since P is a stochastic matrix. Notice that Equation (1) will be an equality if and only if each nonzero entry in $\mathbf{x}$ has the same sign.

37. **a.** Let $\mathbf{v}$ be an eigenvector of P associated with $\lambda = 1$. Let $P\mathbf{v} = \mathbf{y}$. Then, by Exercise 36,

$$|y_1| + \ldots + |y_m| \leq |v_1| + \ldots + |v_m|$$

But $P\mathbf{v} = \mathbf{v}$, so $\mathbf{y} = \mathbf{v}$ and

$$|y_1| + \ldots + |y_m| = |v_1| + \ldots + |v_m|$$

Since equality holds, each nonzero entry in $\mathbf{v}$ must have the same sign by Exercise 36.

b. By part (a), each nonzero entry in $\mathbf{v}$ must have the same sign. Since $\mathbf{v}$ is an eigenvector, $\mathbf{v} \neq \mathbf{0}$ and so must have at least one nonzero entry. Thus the sum of the entries in $\mathbf{v}$ will not be zero, so one may define

$$\frac{1}{v_1 + \ldots + v_m}\mathbf{v}$$

This vector will also be an eigenvector of P associated with $\lambda = 1$, each entry in this vector will be non-negative, and the sum of the entries in this vector will be 1. It is thus a steady-state vector for P.

38. **a.** Suppose that the graph has n vertices. Since the graph is connected and finite, each vertex must be accessible from every other vertex within n steps. Thus the n^{th} power of the transition matrix must be a positive matrix, and the transition matrix is thus regular by definition.

b. Suppose that the graph has n vertices. Let e_i be the number of edges connected to vertex i. Then a steady-state vector for the Markov chain is

$$\frac{1}{e_1 + \ldots + e_n}\begin{bmatrix} e_1 \\ \ldots \\ e_n \end{bmatrix}$$

39. **a.** Since $\mathbf{x}_0 = c_1\mathbf{q} + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$ and $\lambda_1 = 1$, Equation (2) indicates that

$$\mathbf{x}_k = P^k\mathbf{x}_0 = c_1\mathbf{q} + c_2\lambda_2^k\mathbf{v}_2 + \ldots + c_n\lambda_n^k\mathbf{v}_n$$

b. By part (a),

$$\mathbf{x}_k - c_1\mathbf{q} = c_2\lambda_2^k\mathbf{v}_2 + \ldots + c_n\lambda_n^k\mathbf{v}_n$$

and $\mathbf{x}_k \to c_1\mathbf{q}$ since $|\lambda_i| < 1$. Since $|\lambda_2|$ is the largest magnitude eigenvalue remaining, the $c_2\lambda_2^k\mathbf{v}_2$ will be the largest of the error terms and will thus govern how quickly $\{\mathbf{x}_k\}$ converges to $c_1\mathbf{q}$.

Section 10.3, page 31

1. $\{1, 3\}, \{2\}$; reducible

2. $\{1, 2, 3\}$; irreducible

3. $\{1\}, \{2\}, \{3\}$; reducible

4. $\{1, 2, 3, 4, 5\}$; irreducible

5. $\{1, 3, 5\}, \{2, 4, 6\}$; reducible

6. $\{1, 2, 3, 4\}, \{5, 6, 7\}$; reducible

7. $\{1, 2, 3, 4, 5\}, \{6\}$; reducible

8. $\{1, 3, 4\}, \{2, 5\}$; reducible

9. $\{1, 2, 3, 4\}, \{5\}$

10. $\{1\}, \{2, 3, 4, 5\}, \{6\}$

11. $\{1, 2, 3, 4\}$; irreducible

12. $\{1\}, \{2, 3\}, \{4\}$; reducible

13. Every state is reachable from every other state in two steps or fewer, so the Markov chain is irreducible. The return times are
State 1: 4
State 2: 4
State 3: 6
State 4: 6
State 5: 6

14. Every state is reachable from every other state in one step, so the Markov chain is irreducible. The return times are
State 1: 4
State 2: 4
State 3: 4
State 4: 4

15. Every state is reachable from every other state in three steps or fewer, so the Markov chain is irreducible. The return times are
State 1: 13/3
State 2: 13/3
State 3: 13/3
State 4: 13/4

16. Every state is reachable from every other state in three steps or fewer, so the Markov chain is irreducible. The return times are
State 1: 10
State 2: 5
State 3: 5/2
State 4: 5
State 5: 10

17. 4 steps.

18. $7/2$ steps.

19. $15/2$ steps.

20. $15/4$ steps.

21. a. False. It must also be possible to go from state j to state i in a finite number of steps for states i and j to communicate with each other.

b. True.

c. False. The *reciprocals* of the entries in the steady-state vector are the return times for each state.

22. a. False. See Example 2.

b. True.

c. True.

23. 2.27368 days.

24. 4.83391 days.

25. a. Since each entry in G is positive, the Markov chain is irreducible.

b. 4.31901 mouse clicks.

26. a. Since each entry in G is positive, the Markov chain is irreducible.

b. 6.34945 mouse clicks.

27. $8/3$ steps.

28. 32 steps.

29. $16/5 = 3.2$ draws.

30. 252 draws.

31. {deuce, advantage A, advantage B}, {A wins the game}, {B wins the game}

32. {tied – A serving, tied – B serving, A ahead by 1 point – A serving, B ahead by 1 point – B serving}, {A wins the game}, {B wins the game}

33. Every state is reachable from every other state in three steps or fewer, so the Markov chain is irreducible.

34. $49/15 \approx 3.26667$ steps.

35. Each dangling node forms a separate communication class for the Markov chain.

36. Suppose that state i communicates with state j and that state j communicates with state k. Thus there exist $m \geq 0$ and $n \geq 0$ such that the (i, j)-element of P^m and the (j, k)-element of P^n are both strictly positive. Now the (i, k)-element in P^{m+n} is the probability of moving from state i to state k in $m + n$ steps. This probability is greater than or equal to the probability of moving from state i to state j in m steps, and then from state j to state k in n steps. Thus the (i, k)-element in P^{m+n} is greater than or equal to the product of the (i, j)-element of P^m and the (j, k)-element of P^n. Since this product is strictly positive, the (i, k)-element in P^{m+n} must also be strictly positive. Likewise, it may be shown that there is a power of P whose (k, i)-element is strictly positive. Likewise, it may be shown that there is a power of P whose (i, k)-element is strictly positive, so state i communicates with state k.

Section 10.4, page 40

1. The communication classes are $\{1, 3\}$ and $\{2\}$. Class $\{1, 3\}$ is transient while class $\{2\}$ is recurrent. All classes have period 1.

2. The communication class is $\{1, 2, 3\}$, which must be recurrent. The class has period 1.

3. The communication classes are $\{1\}$, $\{2\}$, and $\{3\}$. Class $\{1\}$ is recurrent while classes $\{2\}$ and $\{3\}$ are transient. All classes have period 1.

4. The communication class is $\{1, 2, 3, 4, 5\}$, which must be recurrent. The class has period 3.

5. The communication classes are $\{1, 3, 5\}$ and $\{2, 4, 6\}$. Both classes are recurrent and have period 2.

6. The communication classes are $\{1, 2, 3, 4\}$ and $\{5, 6, 7\}$. Class $\{5, 6, 7\}$ is transient while class $\{1, 2, 3, 4\}$ is recurrent. Class $\{5, 6, 7\}$ has period 3 and class $\{1, 2, 3, 4\}$ has period 2.

7. The communication class is $\{1, 2, 3, 4\}$, which must be recurrent. The class has period 4.

8. The communication class is $\{1, 2, 3, 4, 5\}$, which must be recurrent. The class has period 3.

9. The communication classes are $\{1, 2, 3, 4\}$, which is transient, and $\{5\}$, which is recurrent. Both classes have period 1.

10. The communication classes are $\{2, 3, 4, 5\}$, which is transient, and $\{1\}$ and $\{6\}$, which are recurrent. All classes have period 1.

11. Ordering the states 2, 1, 3 gives the matrix

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/4 & 1/3 \\ 0 & 1/4 & 1/3 \end{bmatrix}$$

12. The matrix is already in canonical form since there is only one communication class.

13. The matrix is already in canonical form.

14. The matrix is already in canonical form since there is only one communication class.

15. Ordering the states 1, 3, 5, 2, 4, 6 gives the matrix

$$\begin{bmatrix} 0 & .4 & .8 & 0 & 0 & 0 \\ .3 & 0 & .2 & 0 & 0 & 0 \\ .7 & .6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .7 & .5 \\ 0 & 0 & 0 & .1 & 0 & .5 \\ 0 & 0 & 0 & .9 & .3 & 0 \end{bmatrix}$$

16. The matrix is already in canonical form.

17. The original transition matrix is

$$\begin{bmatrix} 1/3 & 0 & 1 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1 \end{bmatrix}$$

Ordering the states 5, 1, 2, 3, 4 gives the matrix

$$\begin{bmatrix} 1 & 0 & 1/2 & 0 & 0 \\ 0 & 1/3 & 0 & 1 & 0 \\ 0 & 1/3 & 0 & 0 & 1/2 \\ 0 & 1/3 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 0 \end{bmatrix}$$

18. The original transition matrix is

$$\begin{bmatrix} 1 & 1/2 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 1/2 & 0 \\ 0 & 0 & 1/4 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1 \end{bmatrix}$$

Ordering the states 6, 1, 2, 3, 4, 5 gives the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 1/2 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/4 & 0 & 1/2 \\ 0 & 0 & 0 & 1/4 & 1/2 & 0 \end{bmatrix}$$

19. **a.** The communication classes are $\{1, 2, 3, 4, 5\}$ and $\{6\}$. Class $\{1, 2, 3, 4, 5\}$ is transient while class $\{6\}$ is recurrent.

b. Both classes have period 1.

c. The original transition matrix is

$$\begin{bmatrix} 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/3 & 1 \end{bmatrix}$$

Ordering the states 6, 1, 2, 3, 4, 5 gives the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1/2 & 0 & 1/3 \\ 0 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 1/3 \\ 0 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 1/3 & 0 & 1 & 0 \end{bmatrix}$$

20. **a.** The communication classes are $\{1, 3, 4\}$ and $\{2, 5\}$. Class $\{1, 3, 4\}$ is transient while class $\{2, 5\}$ is recurrent.

b. Class $\{1, 3, 4\}$ has period 1 while class $\{2, 5\}$ has period 2.

c. The original transition matrix is

$$\begin{bmatrix} 0 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 0 \end{bmatrix}$$

Ordering the states 2, 5, 1, 3, 4 gives the matrix

$$\begin{bmatrix} 0 & 1 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

21. **a.** False. A Markov chain can have more than one recurrent class.

b. True.

c. False. Every Markov chain must have a *recurrent* class.

22. **a.** True. **b.** True.

c. False. A Markov chain can have more than one recurrent class.

23. It is easy to compute that $\mathbf{q} = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$ for the matrix P.

We further find that

$$P^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$P^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and}$$

$$P^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

by direct computation. Thus $P^5 = P$, $P^6 = P^2$, $P^7 = P^3$, $P^8 = P^4 = I$, and so on. So no matter the value of n, one of the four matrices P^{n+1}, P^{n+2}, P^{n+3}, and P^{n+4} will be P, one will be P^2, one will be P^3, and one will be $P^4 = I$. Therefore

$$\lim_{n\to\infty} \frac{1}{4}\left(P^{n+1} + P^{n+2} + P^{n+3} + P^{n+4}\right)$$
$$= \frac{1}{4}\left(P + P^2 + P^3 + P^4\right)$$
$$= \frac{1}{4}\left(\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

as promised in Theorem 5.

24. It is easy to compute that $\mathbf{q} = \begin{bmatrix} 1/6 \\ 1/6 \\ 1/3 \\ 1/6 \\ 1/6 \end{bmatrix}$ for the matrix P.

We further find that

$$P^2 = \begin{bmatrix} 0 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 0 \end{bmatrix},$$

$$P^3 = \begin{bmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{bmatrix},$$

$$P^4 = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \end{bmatrix}, \quad \text{and}$$

$$P^5 = \begin{bmatrix} 0 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 0 \end{bmatrix} = P^2$$

by direct computation. Thus $P^6 = P^3$, $P^7 = P^4$, $P^8 = P^5 = P^2$, and so on. So as long as $n > 1$, one of the three matrices P^{n+1}, P^{n+2}, and P^{n+3} will be P^2, one will be P^3, and one will be P^4. Therefore

$$\lim_{n\to\infty} \frac{1}{3}\left(P^{n+1} + P^{n+2} + P^{n+3}\right) = \frac{1}{3}\left(P^2 + P^3 + P^4\right)$$
$$= \frac{1}{3}\left(\begin{bmatrix} 0 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{bmatrix} + \begin{bmatrix} 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/3 & 1/3 & 1/3 & 1/3 & 1/3 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{bmatrix}$$

as promised in Theorem 5.

25. a. It is possible to go from any state to any other state in any even number of steps, so the Markov chain is irreducible with period 2.

b. $\mathbf{q} = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}$

c. One choice is $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

d. Compute that

$$P^n = AD^nA^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1/4 & 1/4 & 1/4 \\ 1/4 & -1/4 & 1/4 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1+(-1)^n & 1-(-1)^n & 1+(-1)^n \\ 2-2(-1)^n & 2+2(-1)^n & 2-2(-1)^n \\ 1+(-1)^n & 1-(-1)^n & 1+(-1)^n \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 & 1/4 & 1/4 \\ 1/2 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/4 \end{bmatrix} + (-1)^n \begin{bmatrix} 1/4 & -1/4 & 1/4 \\ -1/2 & 1/2 & -1/2 \\ 1/4 & -1/4 & 1/4 \end{bmatrix}$$

e. The second terms in the expressions for P^n and P^{n+1} will cancel each other when added, so

$$(1/2)(P^n + P^{n+1}) = \begin{bmatrix} 1/4 & 1/4 & 1/4 \\ 1/2 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/4 \end{bmatrix}$$

as promised in Theorem 5.

26. a. It is possible to go from any state to any other state in any even number of steps, so the Markov chain is irreducible with period 2.

b. $\mathbf{q} = \begin{bmatrix} p/2 \\ 1/2 \\ (1-p)/2 \end{bmatrix}$

c. One choice is $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} p & p & -1 \\ 1 & -1 & 0 \\ 1-p & 1-p & 1 \end{bmatrix}$

d. Compute that

$$P^n = AD^nA^{-1} = \begin{bmatrix} p & p & -1 \\ 1 & -1 & 0 \\ 1-p & 1-p & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ (2p-2)/2 & 0 & p \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} p+p(-1)^n & p-p(-1)^n & p+p(-1)^n \\ 1-(-1)^n & 1+(-1)^n & 1-1(-1)^n \\ (1-p)(1+(-1)^n) & (1-p)(1-(-1)^n) & (1-p)(1+(-1)^n) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} p & p & p \\ 1 & 1 & 1 \\ 1-p & 1-p & 1-p \end{bmatrix} + (-1)^n\frac{1}{2}\begin{bmatrix} p & -p & p \\ -1 & 1 & -1 \\ 1-p & -(1-p) & 1-p \end{bmatrix}$$

e. The second terms in the expressions for P^n and P^{n+1} will again cancel each other when added, so

$$(1/2)(P^n + P^{n+1}) = \begin{bmatrix} p/2 & p/2 & p/2 \\ 1/2 & 1/2 & 1/2 \\ (1-p)/2 & (1-p)/2 & (1-p)/2 \end{bmatrix}$$

as promised in Theorem 5.

27. It is easy to compute that $\mathbf{q} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$ for the matrix P.

We further find that

$$P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

by direct computation. Thus $P^4 = P$, $P^5 = P^2$, $P^6 = P^3 = I$, and so on. So no matter the value of n, one of the three matrices P^{n+1}, P^{n+2}, and P^{n+3} will be P, one will be P^2, and one will be $P^3 = I$. Therefore

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{3}\left(P^{n+1} + P^{n+2} + P^{n+3}\right) \\
&\quad = \frac{1}{3}\left(P + P^2 + P^3\right) \\
&\quad = \frac{1}{3}\left(\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right. \\
&\qquad \left. + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \\
&\quad = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}
\end{aligned}$$

as promised in Theorem 5.

28. **a.** $EP = \begin{bmatrix} .8 & 1 & 0 & 0 & 0 \\ .2 & 0 & .3 & 0 & 0 \\ 0 & 0 & .1 & 0 & .4 \\ 0 & 0 & 0 & 1 & .6 \\ 0 & 0 & .6 & 0 & 0 \end{bmatrix}$ This is the matrix P with its rows reordered in the order $1, 2, 4, 5, 3$.

b. $PE^T = \begin{bmatrix} .8 & 1 & 0 & 0 & 0 \\ .2 & 0 & 0 & 0 & .3 \\ 0 & 0 & 0 & 0 & .6 \\ 0 & 0 & 0 & .4 & .1 \\ 0 & 0 & 1 & .6 & 0 \end{bmatrix}$ This is the matrix P with its columns reordered in the order $1, 2, 4, 5, 3$.

c. $EPE^T = \begin{bmatrix} .8 & 1 & 0 & 0 & 0 \\ .2 & 0 & 0 & 0 & .3 \\ 0 & 0 & 0 & .4 & .1 \\ 0 & 0 & 1 & .6 & 0 \\ 0 & 0 & 0 & 0 & .6 \end{bmatrix}$ This is the matrix P with its columns and rows reordered in the order $1, 2, 4, 5, 3$.

29. **a.** Since any permutation of rows may be written as a sequence of row swaps, the permutation of rows may be performed by multiplying A on the left by a sequence of elementary matrices $E_1, \ldots, E_k$. By the comment on page 106, $E = E_k \cdots E_1$, and EA will be the matrix A with its rows permuted in exactly the same order in which the rows of I_n were permuted to form E.

b. By part (a), EA^T will be the matrix A^T with its rows permuted in exactly the same order in which the rows of I_n were permuted to form E. Thus $(EA^T)^T$ will be the matrix A with its columns permuted in exactly the same order in which the rows of I_n were permuted to form E, and since $(EA^T)^T = (A^T)^T E^T = AE^T$, the result follows.

c. The matrix EAE^T is $(EA)E^T$. By part (a), EA is the matrix A with its rows permuted in exactly the same order in which the rows of I_n were permuted to form E. Applying part (b) to EA, $(EA)E^T$ is the matrix EA with its columns permuted in exactly the same order in which the rows of I_n were permuted to form E. Thus EAE^T is the matrix A with its rows and columns permuted in exactly the same order in which the rows of I_n were permuted to form E.

d. Since matrix multiplication is associative, $(EA)E^T = E(AE^T)$ and it does not matter whether the rows of matrix A or the columns of matrix A are permuted first.

Section 10.5, page 50

1. $\begin{bmatrix} 3 & 2 \\ 3/2 & 2 \end{bmatrix}$ **2.** $\begin{bmatrix} 32/21 & 4/7 \\ 10/21 & 10/7 \end{bmatrix}$

3. Using reordering 2, 4, 1, 3, 5:
$\begin{bmatrix} 1075/736 & 125/368 & 185/368 \\ 25/46 & 35/23 & 15/23 \\ 105/184 & 55/92 & 155/92 \end{bmatrix}$

4. $\begin{bmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{bmatrix}$ **5.** $\begin{bmatrix} 10/21 & 3/7 \\ 5/21 & 3/14 \\ 2/7 & 5/14 \end{bmatrix}$

6. Using reordering 2, 4, 1, 3, 5:
$\begin{bmatrix} 379/736 & 181/368 & 209/368 \\ 357/736 & 187/368 & 159/368 \end{bmatrix}$

7. $3/2$ **8.** 2 **9.** $1895/736$ **10.** $1/2$

11. At state 1: $10/21$; at state 2: $5/21$; at state 3: $2/7$

12. At state 2: $209/368$; at state 4: $159/368$

13. **a.** $9/11$ **b.** $29/11$ **c.** $3/7$

14. **a.** $3/4$ **b.** 3 **c.** $1/2$

15. **a.** 1 **b.** $10/3$

16. **a.** 2 **b.** 5

17. $5/7$ **18.** $4/19$ **19.** $38/5$ **20.** $15/4$

21. **a.** False. The (i, j)-element in the fundamental matrix M is the expected number of visits to the transient state i prior to absorption, starting at the transient state j.

b. False. See Theorem 6. **c.** True.

22. **a.** True.

b. False. See the discussion prior to Example 2.

c. True.

23. $2/7$ **24.** 4.65493 **25.** $19/2$

26. $3/5$ **27.** 3.84615 **28.** $.692308$

29. Advantage A: 2.53846
Advantage B: 3.30769

30. Advantage A: .876923
Advantage B: .415385

31. 3.01105 **32.** .676796 **33.** 3.92932 **34.** .754374

35. From the results of Exercises 31 and 33, using rally point scoring led to $3.92932 - 3.01105 = .91827$ fewer rallies being played.

36. From the results of Exercises 32 and 34, team A has a higher probability of winning if the side out scoring method is used. Thus team A should favor the side out scoring method, while team B should favor the rally point scoring method.

37. a. $\{1, 2\}$ is a recurrent class; $\{3, 4, 5\}$ is a transient class.

b. The limiting matrix for $\{1, 2\}$ is $\begin{bmatrix} 2/5 & 2/5 \\ 3/5 & 3/5 \end{bmatrix}$.

c. Since there is only one recurrent class, the probability that the chain is absorbed into $\{1, 2\}$ is 1. Thus if the chain is started in any transient state, the probability of being at state 1 after many time steps is $2/5$, the probability of being at state 2 after many time steps is $3/5$, and the probability of being at state 3, 4, or 5 after many time steps is 0.

d. Since the i^{th} column of $\lim_{n\to\infty} P^n$ gives the long-range probabilities for the chain started at state i,

$$\lim_{n\to\infty} P^n = \begin{bmatrix} 2/5 & 2/5 & 2/5 & 2/5 & 2/5 \\ 3/5 & 3/5 & 3/5 & 3/5 & 3/5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

e. $P^{50} \approx \begin{bmatrix} .4 & .4 & .4 & .4 & .4 \\ .6 & .6 & .6 & .6 & .6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

38. a. $\{1, 2\}$ and $\{3, 4\}$ are recurrent classes; $\{5, 6\}$ is a transient class.

b. The limiting matrix for $\{1, 2\}$ is $\begin{bmatrix} 3/7 & 3/7 \\ 4/7 & 4/7 \end{bmatrix}$ and the limiting matrix for $\{3, 4\}$ is $\begin{bmatrix} 8/17 & 8/17 \\ 9/17 & 9/17 \end{bmatrix}$.

c. Conflating the recurrent classes into single absorbing states gives the transition matrix

$$\begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 \end{bmatrix}$$

The matrix $A = SM$ for this matrix is $A = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$. Starting at state 5, the chain will be absorbed into $\{1, 2\}$ with probability $3/4$. Thus the long-range probability of being at state 1 given that the chain starts at state 5 is $(3/4)(3/7) = 9/28$. Likewise, the long-range probabilities of being at states 2, 3, and 4 given that the chain starts at state 5 are, respectively, $(3/4)(4/7) = 3/7$, $(1/4)(8/17) = 2/17$, and $(1/4)(9/17) = 9/68$. The long-range probability of being at either state 5 or state 6 starting at state 5 is clearly 0. The long-range probabilities starting at state 6 may be found in the same manner: in numerical order, they are $(1/4)(3/7) = 3/28$, $(1/4)(4/7) = 1/7$, $(3/4)(8/17) = 6/17$, $(3/4)(9/17) = 27/68$, 0, and 0.

d. Since the i^{th} column of $\lim_{n\to\infty} P^n$ gives the long-range probabilities for the chain started at state i,

$$\lim_{n\to\infty} P^n = \begin{bmatrix} 3/7 & 3/7 & 0 & 0 & 9/28 & 3/28 \\ 4/7 & 4/7 & 0 & 0 & 3/7 & 1/7 \\ 0 & 0 & 8/17 & 8/17 & 2/17 & 6/17 \\ 0 & 0 & 9/17 & 9/17 & 9/68 & 27/68 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

e. $$P^{50} \approx \begin{bmatrix} .428571 & .428571 & 0 & 0 & .321429 & .107143 \\ .571429 & .571429 & 0 & 0 & .428571 & .142857 \\ 0 & 0 & .470588 & .470588 & .117647 & .352941 \\ 0 & 0 & .529412 & .529412 & .132353 & .397059 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

39. The result is trivially true if $n = 1$. Assume that the result is true for $n = k$; that is, $P^k = \left[\begin{array}{c|c} I & S_k \\ \hline O & Q^k \end{array}\right]$, where $S_k = S\left(I + Q + Q^2 + \ldots + Q^{k-1}\right)$. Then

$$\begin{aligned} P^{k+1} &= P^k \cdot P \\ &= \left[\begin{array}{c|c} I & S_k \\ \hline O & Q^k \end{array}\right] \cdot \left[\begin{array}{c|c} I & S \\ \hline O & Q \end{array}\right] \\ &= \left[\begin{array}{c|c} I & S + S_k Q \\ \hline O & Q^{k+1} \end{array}\right] \end{aligned}$$

But

$$\begin{aligned} S + S_k Q &= S + S(I + Q + Q^2 + \ldots + Q^{k-1})Q \\ &= S + S(Q + Q^2 + \ldots + Q^k) \\ &= S(I + Q + Q^2 + \ldots + Q^k) \\ &= S_{k+1} \end{aligned}$$

and the result is proven by induction.

Section 10.6, page 61

1. This concerns the second column of A. The initial state is $1{:}k$ (a runner on first base, k outs). For entry $(1, 2)$ of A, the probability of a transition "to state $0{:}k$" is required. Suppose that only first base is occupied and the batter does not make an out. Only a home run will empty the bases, so the $(1, 2)$-entry is p_H.

Entry $(2, 2)$: ("to state $1{:}k$") To leave a player on first base, the batter must get to first base and the player on first base must reach home plate successfully. This cannot happen according to the model, so the $(2, 2)$-entry is 0.

Entry $(3, 2)$: ("to state $2{:}k$") To leave a player on second base, the batter must get to second base and the player on first base must reach home plate successfully. This cannot happen according to the model, so the $(3, 2)$-entry is 0.

Entry $(4, 2)$: ("to state $3{:}k$") To leave a player on third base, the batter must get to third base and the player on first base must reach home plate successfully. This can happen only if the batter hits a triple, so the $(4, 2)$-entry is p_3.

Entry $(5, 2)$: ("to state $12{:}k$") To leave players on first base and second base, the batter must get to first base and the player on first base must advance to second. The desired outcome occurs when the batter either hits a single, gets a walk, or is hit by a pitch. The $(5, 2)$-entry is thus $p_W + p_1$.

Entry $(6, 2)$: ("to state $13{:}k$") This concerns the batter getting to first base and the runner on first base advancing to third base. This cannot happen according to the model, so the $(6, 2)$-entry is 0.

Entry $(7, 2)$: ("to state $23{:}k$") To leave players on second base and third base, the batter must hit a double and the runner on first base must advance only to third base. Thus the $(7, 2)$-entry is p_2.

Entry $(8, 2)$: ("to state $123{:}k$") The starting state has just one runner on base. The next state cannot have three runners on base, so the $(8, 2)$-entry is 0.

2. This concerns the fourth column of A. The initial state is $3{:}k$ (a runner on third base, k outs). For entry $(1, 4)$ of A, the probability of a transition "to state $0{:}k$" is required. Suppose that only third base is occupied and the batter does not make an out. Only a home run will empty the bases, so the $(1, 4)$-entry is p_H.

Entry $(2, 4)$: ("to state $1{:}k$") To leave a player on first base, the batter must get to first base and the player on third base must reach home plate successfully. This happens when the batter hits a single, so the $(2, 4)$-entry is p_1.

Entry $(3, 4)$: ("to state $2{:}k$") To leave a player on second base, the batter must get to second base and the player on third base must reach home plate successfully. This happens when the batter hits a double, so the $(3, 4)$-entry is p_2.

Entry $(4, 4)$: ("to state $3{:}k$") To leave a player on third base, the batter must get to third base and the player on third base must reach home plate successfully. This happens when the batter hits a triple, so the $(4, 4)$-entry is p_3.

Entry $(5, 4)$: ("to state $12{:}k$") It is impossible to leave players on first base and second base, because the runner on third cannot return to second base. The $(5, 4)$-entry is thus 0.

Entry $(6, 4)$: ("to state $13{:}k$") This concerns the batter getting to first base and the runner on third base staying at third base. This happens when the batter is walked or is hit by a pitch, so the $(6, 4)$-entry is p_W.

Entry $(7, 4)$: ("to state $23{:}k$") To leave players on second base and third base, the batter must hit a double and the runner on third base must stay at third base. This cannot happen, so the $(7, 4)$-entry is 0.

Entry $(8, 4)$: ("to state $123{:}k$") The starting state has just one runner on base. The next state cannot have three runners on base, so the $(8, 4)$-entry is 0.

3. This concerns the fifth column of A. The initial state is $12{:}k$ (runners on first base and second base, k outs). For entry $(1, 5)$ of A, the probability of a transition "to state $0{:}k$" is required. Suppose that first and second bases are occupied and the batter does not make an out. Only a home run will empty the bases, so the $(1, 5)$-entry is p_H.

Entry $(2, 5)$: ("to state $1{:}k$") To leave a player on first base, the batter must get to first base and both players on base must reach home plate successfully. This cannot happen according to the model, so the $(2, 5)$-entry is 0.

Entry $(3, 5)$: ("to state $2{:}k$") To leave a player on second base, the batter must get to second base and both players on base must reach home plate successfully. This cannot happen according to the model, so the $(3, 5)$-entry is 0.

Entry $(4, 5)$: ("to state $3{:}k$") To leave a player on third base, the batter must get to third base and the players on base must reach home plate successfully. This can happen only if the batter hits a triple, so the $(4, 5)$-entry is p_3.

Entry $(5, 5)$: ("to state $12{:}k$") To leave players on first base and second base, the batter must get to first base, the player on first base must advance to second base, and the player on second base must reach home plate successfully. The desired outcome occurs when the batter hits a single, but the runner from second will then reach home with probability .5. The $(5, 5)$-entry is thus $.5p_1$.

Entry $(6, 5)$: ("to state 13:k") This concerns the batter getting to first base and the runner on first base advancing to third base. This cannot happen according to the model, so the $(6, 5)$-entry is 0.

Entry $(7, 5)$: ("to state 23:k") To leave players on second base and third base, the batter must hit a double, in which case the runner on first base must advance to third base and the runner on second base must reach home. Thus the $(7, 5)$-entry is p_2.

Entry $(8, 5)$: ("to state 123:k") To leave runners on first, second, and third bases, the batter must reach first and the two runners must each advance one base. This happens when the batter is walked, is hit by a pitch, or hits a single but the runner on second base does not reach home. Thus the $(8, 5)$-entry is $p_W + .5p_1$.

4. This concerns the sixth column of A. The initial state is 13:k (runners on first and third bases, k outs). For entry $(1, 6)$ of A, the probability of a transition "to state 0:k" is required. Suppose that first and third bases are occupied and the batter does not make an out. Only a home run will empty the bases, so the $(1, 6)$-entry is p_H.

Entry $(2, 6)$: ("to state 1:k") To leave a player on first base, the batter must get to first base and both players on base must reach home plate successfully. This cannot happen according to the model, so the $(2, 6)$-entry is 0.

Entry $(3, 6)$: ("to state 2:k") To leave a player on second base, the batter must get to second base and both players on base must reach home plate successfully. This cannot happen according to the model, so the $(3, 6)$-entry is 0.

Entry $(4, 6)$: ("to state 3:k") To leave a player on third base, the batter must get to third base and the players on base must reach home plate successfully. This can happen only if the batter hits a triple, so the $(4, 6)$-entry is p_3.

Entry $(5, 6)$: ("to state 12:k") To leave players on first base and second base, the batter must get to first base, the player on first base must advance to second, and the player on third base must reach home plate successfully. The desired outcome occurs when the batter hits a single. The $(5, 6)$-entry is thus p_1.

Entry $(6, 6)$: ("to state 13:k") This concerns the batter getting to first base and the runner on first base advancing to third base. This cannot happen according to the model, so the $(6, 6)$-entry is 0.

Entry $(7, 6)$: ("to state 23:k") To leave players on second base and third base, the batter must hit a double, in which case the runner on first base must advance to third base and the runner on third base must reach home. Thus the $(7, 6)$-entry is p_2.

Entry $(8, 6)$: ("to state 123:k") To leave runners on first, second, and third bases, the batter must reach first base while the runner on first base advances one base and the runner on third base does not advance. This happens only when the batter is walked or is hit by a pitch. Thus the $(8, 6)$-entry is p_W.

5. This concerns the seventh column of A. The initial state is 23:k (runners on second and third bases, k outs). For entry $(1, 7)$ of A, the probability of a transition "to state 0:k" is required. Suppose that second and third bases are occupied and the batter does not make an out. Only a home run will empty the bases, so the $(1, 7)$-entry is p_H.

Entry $(2, 7)$: ("to state 1:k") To leave a player on first base, the batter must get to first base and the players on second base and third base must reach home plate successfully. The desired outcome occurs when the batter hits a single, but the runner from second will then reach home with probability .5. Thus the $(2, 7)$-entry is $.5p_1$.

Entry $(3, 7)$: ("to state 2:k") To leave a player on second base, the batter must reach second base (a "double") and the runners on second and third bases must score. The second condition, however, is automatically satisfied because of the assumption in Table 2. So the probability of success in this case is p_2. This is the $(3, 7)$-entry.

Entry $(4, 7)$: ("to state 3:k") By an argument similar to that for the $(3, 6)$-entry, the $(4, 7)$-entry is p_3.

Entry $(5, 7)$: ("to state 12:k") To leave players on first base and second base, the batter must get to first base and the player on second base must remain there while the runner on third base reaches home. This is impossible, so the $(5, 7)$-entry is 0.

Entry $(6, 7)$: ("to state 13:k") This concerns the batter getting to first base and the runner on second base advancing to third base while the runner on third base reaches home. This can happen only if the batter hits a single, with probability p_1, and the runner on second base stops at third base, which happens with probability .5 (by Table 2). Since both events are required, the $(6, 7)$-entry is the product $.5p_1$.

Entry $(7, 7)$: ("to state 23:k") To leave players on second base and third base, the batter must hit a double and the runner on second base must advance only to third base. This cannot happen, so the $(7, 7)$-entry is 0.

Entry $(8, 7)$: ("to state 123:k") To leave runners on first, second, and third bases, the batter must reach first base and the two runners must each fail to advance one base. This happens when the batter is walked or is hit by a pitch. Thus the $(8, 7)$-entry is p_W.

6. This concerns the eighth column of A. The initial state is 123:k (runners on first, second, and third bases, k outs). For entry $(1, 8)$ of A, the probability of a transition "to state 0:k" is required. Suppose that first, second, and third bases are occupied and the batter does not make an out. Only a home run will empty the bases, so the $(1, 8)$-entry is p_H.

Entry $(2, 8)$: ("to state 1:k") To leave a player on first base, the batter must get to first base and all players on base must reach home plate successfully. This cannot happen according to the model, so the $(2, 8)$-entry is 0.

Entry (3, 8): ("to state 2:k") To leave a player on second base, the batter must get to second base and all players on base must reach home plate successfully. This cannot happen according to the model, so the (3, 8)-entry is 0.

Entry (4, 8): ("to state 3:k") To leave a player on third base, the batter must get to third base and all players on base must reach home plate successfully. This can happen only if the batter hits a triple, so the (4, 8)-entry is p_3.

Entry (5, 8): ("to state 12:k") To leave players on first base and second base, the batter must get to first base, the player on first base must advance to second, and the players on second base and third base must reach home plate successfully. The desired outcome occurs when the batter hits a single, but the runner from second will then reach home with probability .5. (The runner at third reaches home automatically.) The (5, 8)-entry is thus $.5p_1$.

Entry (6, 8): ("to state 13:k") This concerns the batter getting to first base and the runner on first base advancing to third base. This cannot happen according to the model, so the (6, 8)-entry is 0.

Entry (7, 8): ("to state 23:k") To leave players on second base and third base, the batter must hit a double, in which case the runner on first base must advance to third base and the runners on second base and third base must reach home. Thus the (7, 8)-entry is p_2.

Entry (8, 8): ("to state 123:k") To leave runners on first, second, and third bases, the batter must reach first base and the two runners must each advance one base. This happens when the batter is walked, is hit by a pitch, or hits a single but the runner on second does not reach home. Thus the (8, 8)-entry is $p_W + .5p_1$.

7. $p_W = .0954785$, $p_1 = .159996$, $p_2 = .049377$, $p_3 = .00514581$, $p_H = .0291127$, $p_O = .66089$. Thus

$$A = \begin{array}{c} \begin{array}{cccccccc} 0{:}k & 1{:}k & 2{:}k & 3{:}k & 12{:}k & 13{:}k & 23{:}k & 123{:}k \end{array} \\ \begin{bmatrix} .0291127 & .0291127 & .0291127 & .0291127 & .0291127 & .0291127 & .0291127 & .0291127 \\ .255474 & 0 & .0799978 & .159996 & 0 & 0 & .0799978 & 0 \\ .049377 & 0 & .049377 & .049377 & 0 & 0 & .049377 & 0 \\ .00514581 & .00514581 & .00514581 & .00514581 & .00514581 & .00514581 & .00514581 & .00514581 \\ 0 & .255474 & .0954785 & 0 & .0799978 & .159996 & 0 & .0799978 \\ 0 & 0 & .0799978 & .0954785 & 0 & 0 & .0799978 & 0 \\ 0 & .049377 & 0 & 0 & .049377 & .049377 & 0 & .049377 \\ 0 & 0 & 0 & 0 & .175476 & .0954785 & .0954785 & .175476 \end{bmatrix} \end{array} \begin{array}{c} \\ 0{:}k \\ 1{:}k \\ 2{:}k \\ 3{:}k \\ 12{:}k \\ 13{:}k \\ 23{:}k \\ 123{:}k \end{array}$$

8. To find the complete transition matrix, combine A in Exercise 7 with the following matrices as given in the text. The data indicates that $p_O = .66089$, so $S = \begin{bmatrix} O & O & X \end{bmatrix}$, where

$$X = \begin{array}{c} \begin{array}{cccccccc} 0{:}2 & 1{:}2 & 2{:}2 & 3{:}2 & 12{:}2 & 13{:}2 & 23{:}2 & 123{:}2 \end{array} \\ \begin{bmatrix} .66089 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .66089 & .66089 & .66089 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .66089 & .66089 & .66089 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .66089 \end{bmatrix} \end{array} \begin{array}{c} \\ 0{:}3 \\ 1{:}3 \\ 2{:}3 \\ 3{:}3 \end{array}$$

Also note that matrix $B = .66089I$, where I is the 8×8 identity matrix.

9. The sum of the first column of M shows that $E[B] = 4.53933$. The first column of SM allows $E[L]$ to be computed:

$$\begin{aligned} E[L] &= 0(.34973) + 1(.33414) + 2(.23820) + 3(.07793) \\ &= 1.04433 \end{aligned}$$

Thus

$$\begin{aligned} E[R] &= E[B] - E[L] - 3 \\ &= 4.53933 - 1.04433 - 3 = .495 \end{aligned}$$

10. The model predicts (43,257)(.495) = 21,412.2 total earned runs scored. The model's error is 309.8 earned runs, or about 1.43%.

11. Bonds: $p_W = .212933$, $p_1 = .119495$, $p_2 = .0480377$, $p_3 = .00615458$, $p_H = .0609064$, $p_O = .552474$.

Ruth: $p_W = .2004$, $p_1 = .144421$, $p_2 = .0481721$, $p_3 = .0129474$, $p_H = .0679741$, $p_O = .526085$.

Williams: $p_W = .210936$, $p_1 = .157383$, $p_2 = .0537579$, $p_3 = .00727012$, $p_H = .0533484$, $p_O = .517305$.

12. Here is a table containing the results for the three players:

Player	Offensive ERA
Babe Ruth	12.302
Ted Williams	12.095
Barry Bonds	10.074

13. **a.** The sum of the second column of M will tell the expected number of batters that will come to the plate starting with a runner on first and none out.

b. The second column of SM will give the probabilities of leaving 0, 1, 2, or 3 runners on base starting with a runner on first and none out. Thus the expected number of runners left on base starting with a runner on first and none out could be calculated.

c. The expected number of runs scored using the second column data will give the expected number of runs scored starting with a runner on first and none out.

14. The sum of the column of M is 4.53933. One batter has already reached base, so $E[B] = 1 + 4.53933 = 5.53933$. The column of SM allows $E[L]$ to be computed:

$$\begin{aligned} E[L] &= 0(.06107) + 1(.35881) + 2(.41638) + 3(.16374) \\ &= 1.68279 \end{aligned}$$

Thus

$$\begin{aligned} E[R] &= E[B] - E[L] - 3 \\ &= 5.53933 - 1.68279 - 3 = .85654 \end{aligned}$$

15. The sum of the column of M is 4.53933. One batter has already reached base, so $E[B] = 1 + 4.53933 = 5.53933$. The column of SM allows $E[L]$ to be computed:

$$\begin{aligned} E[L] &= 0(.06107) + 1(.47084) + 2(.34791) + 3(.12108) \\ &= 1.52990 \end{aligned}$$

Thus

$$\begin{aligned} E[R] &= E[B] - E[L] - 3 \\ &= 5.53933 - 1.52990 - 3 = 1.00943 \end{aligned}$$

16. The sum of the column of M is 3.02622. One batter has already made an out, so $E[B] = 1 + 3.02622 = 4.02622$. The column of SM allows $E[L]$ to be computed:

$$\begin{aligned} E[L] &= 0(.48513) + 1(.31279) + 2(.16060) + 3(.04148) \\ &= .75843 \end{aligned}$$

Thus

$$\begin{aligned} E[R] &= E[B] - E[L] - 3 \\ &= 4.02622 - .75843 - 3 = .26819 \end{aligned}$$

17. If the baserunner does not attempt a steal, you expect to score .85654 runs by Exercise 14. If the runner attempts a steal and succeeds, you expect to score 1.00943 runs by Exercise 15. If the runner attempts a steal and does not succeed, you expect to score .26779 runs by Exercise 16. Thus the expected number of runs scored if a steal is attempted is $1.00943(.8) + .26779(.2) = .861102$. Attempting a steal thus increases the expected number of runs scored.

18. If the probability of a successful steal is p and a steal is attempted, the expected number of runs scored is $1.00943p + .26779(1 - p) = .26779 + .74164p$. If the baserunner does not attempt a steal, you expect to score .85654 runs by Exercise 14. Solving the inequality $.26779 + .74164p > .85654$ will give the values of p for which to attempt a steal. The solution to this inequality is $p > .79385$.