Answers to Exercises Chapter 9

Chapter 9

Section 9.1, page 13

1. $$\begin{array}{cc} & \begin{array}{cc} \text{d} & \text{q} \end{array} \\ \begin{array}{c} \text{d} \\ \text{q} \end{array} & \begin{bmatrix} -10 & 10 \\ 25 & -25 \end{bmatrix} \end{array}$$

2. $$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 2 & -3 & 4 \\ -3 & 4 & -5 \\ 4 & -5 & 6 \end{bmatrix} \end{array}$$

3. $$\begin{array}{cc} & \begin{array}{ccc} \text{st} & \text{sc} & \text{p} \end{array} \\ \begin{array}{c} \text{stone} \\ \text{scissors} \\ \text{paper} \end{array} & \begin{bmatrix} 0 & 5 & -5 \\ -5 & 0 & 5 \\ 5 & -5 & 0 \end{bmatrix} \end{array}$$

4. $$\begin{array}{cc} & \begin{array}{cc} \text{r4} & \text{b9} \end{array} \\ \begin{array}{c} \text{r3} \\ \text{r6} \\ \text{b7} \end{array} & \begin{bmatrix} 4 & -12 \\ 6 & -15 \\ -11 & 9 \end{bmatrix} \end{array}$$

5. $\begin{bmatrix} 4 & 3 \\ 1 & -1 \end{bmatrix}$ **6.** $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -2 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 5 & 3 & 4 & 3 \\ -2 & 1 & -5 & 2 \\ 4 & 3 & 7 & 3 \end{bmatrix}$

8. $\begin{bmatrix} -2 & 4 & 1 & -1 \\ 3 & 5 & 2 & 2 \\ 1 & -3 & 0 & 2 \end{bmatrix}$

9. a. $E(\mathbf{x}, \mathbf{y}) = \frac{13}{12}$, $v(\mathbf{x}) = \min\left\{\frac{5}{6}, 1, \frac{9}{6}\right\} = \frac{5}{6}$, $v(\mathbf{y}) = \max\left\{\frac{3}{4}, \frac{3}{2}, \frac{1}{2}\right\} = \frac{3}{2}$

b. $E(\mathbf{x}, \mathbf{y}) = \frac{9}{8}$, $v(\mathbf{x}) = \min\left\{1, \frac{3}{4}, \frac{7}{4}\right\} = \frac{3}{4}$, $v(\mathbf{y}) = \max\left\{\frac{1}{2}, \frac{5}{4}, \frac{3}{2}\right\} = \frac{3}{2}$

10. a. $E(\mathbf{x}, \mathbf{y}) = -\frac{1}{4}$, $v(\mathbf{x}) = \min\left\{\frac{4}{3}, -\frac{4}{3}, \frac{5}{3}, \frac{1}{3}\right\} = -\frac{4}{3}$, $v(\mathbf{y}) = \max\left\{\frac{1}{4}, \frac{1}{4}, -\frac{1}{2}\right\} = \frac{1}{4}$

b. $E(\mathbf{x}, \mathbf{y}) = \frac{1}{8}$, $v(\mathbf{x}) = \min\left\{1, -\frac{1}{4}, \frac{1}{2}, -\frac{1}{4}\right\} = -\frac{1}{4}$, $v(\mathbf{y}) = \max\left\{\frac{1}{4}, -\frac{3}{4}, \frac{3}{4}\right\} = \frac{3}{4}$

11. $\hat{\mathbf{x}} = \begin{bmatrix} \frac{1}{6} \\ \frac{5}{6} \end{bmatrix}$, $\hat{\mathbf{y}} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$, $v = \frac{1}{2}$

Solution:

Given $A = \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}$, graph

$$\begin{cases} z = 3(1-t) + (0)t = 3 - 3t \\ z = -2(1-t) + (1)t = -2 + 3t \end{cases}.$$

The lines intersect at $(t, z) = \left(\frac{5}{6}, \frac{1}{2}\right)$. The optimal row strategy is $\hat{\mathbf{x}} = \mathbf{x}\left(\frac{5}{6}\right) = \begin{bmatrix} 1 - \frac{5}{6} \\ \frac{5}{6} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ \frac{5}{6} \end{bmatrix}$, and the value of the game is $v = \frac{1}{2}$. By Theorem 4, the optimal column strategy $\hat{\mathbf{y}}$ satisfies $E(\mathbf{e}_1, \hat{\mathbf{y}}) = \frac{1}{2}$ and $E(\mathbf{e}_2, \hat{\mathbf{y}}) = \frac{1}{2}$ because $\hat{\mathbf{x}}$ is a linear combination of both $\mathbf{e}_1$ and $\mathbf{e}_2$. From the second of these conditions, $\frac{1}{2} = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 3c_1 - 2c_2 \\ c_2 \end{bmatrix} = c_2$. From this, $c_1 = \frac{1}{2}$ and $\hat{\mathbf{y}} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$. As a check on this work, one can compute

$$\begin{aligned} E(\mathbf{e}_1, \hat{\mathbf{y}}) &= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 3 & -2 \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2} \end{aligned}$$

12. $\hat{\mathbf{x}} = \begin{bmatrix} \frac{9}{13} \\ \frac{4}{13} \end{bmatrix}$, $\hat{\mathbf{y}} = \begin{bmatrix} \frac{8}{13} \\ \frac{5}{13} \end{bmatrix}$, $v = \frac{6}{13}$

13. $\hat{\mathbf{x}} = \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$, $\hat{\mathbf{y}} = \begin{bmatrix} \frac{4}{5} \\ \frac{1}{5} \end{bmatrix}$, $v = \frac{17}{5}$

Solution:

Given $A = \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix}$, graph

$$\begin{cases} z = 3(1-t) + (4)t = 3 + t \\ z = 5(1-t) + (1)t = 5 - 4t \end{cases}.$$

The lines intersect at $(t, z) = \left(\frac{2}{5}, \frac{17}{5}\right)$. The optimal row strategy is $\hat{\mathbf{x}} = \mathbf{x}\left(\frac{2}{5}\right) = \begin{bmatrix} 1 - \frac{2}{5} \\ \frac{2}{5} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$, and the value of the game is $v = \frac{17}{5}$. By Theorem 4, the optimal column strategy $\hat{\mathbf{y}}$ satisfies $E(\mathbf{e}_1, \hat{\mathbf{y}}) = \frac{17}{5}$ and $E(\mathbf{e}_2, \hat{\mathbf{y}}) = \frac{17}{5}$ because $\hat{\mathbf{x}}$ is a linear combination of both $\mathbf{e}_1$ and $\mathbf{e}_2$. From the first of these conditions,

$$\begin{aligned} \frac{17}{5} &= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ 1 - c_1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 5 \end{bmatrix}\begin{bmatrix} c_1 \\ 1 - c_1 \end{bmatrix} = 5 - 2c_1 \end{aligned}$$

From this, $c_1 = \frac{4}{5}$ and $\hat{\mathbf{y}} = \begin{bmatrix} \frac{4}{5} \\ \frac{1}{5} \end{bmatrix}$. As a check on this work, one can compute

$$E(\mathbf{e}_2, \hat{\mathbf{y}}) = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} \frac{4}{5} \\ \frac{1}{5} \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 1 \end{bmatrix}\begin{bmatrix} \frac{4}{5} \\ \frac{1}{5} \end{bmatrix} = \frac{17}{5}$$

14. $\hat{\mathbf{x}} = \begin{bmatrix} \frac{9}{10} \\ \frac{1}{10} \end{bmatrix}$, $\hat{\mathbf{y}} = \begin{bmatrix} \frac{3}{5} \\ 0 \\ 0 \\ \frac{2}{5} \end{bmatrix}$, $v = \frac{13}{5}$

Solution:
Columns 2 and 3 dominate column 1, so the column player will never choose column 2 or column 3. The new game is $\begin{bmatrix} 3 & * & * & 2 \\ -1 & * & * & 8 \end{bmatrix}$. Let $B = \begin{bmatrix} 3 & 2 \\ -1 & 8 \end{bmatrix}$. Graph
$$\begin{cases} z = 3(1-t) + (-1)t = -4t + 3 \\ z = 2(1-t) + \quad 8t = \quad 6t + 2 \end{cases}$$
Solve for the intersection, to get $t = .1$, and $\hat{\mathbf{x}} = \mathbf{x}(.1) = \begin{bmatrix} 1 - .1 \\ .1 \end{bmatrix} = \begin{bmatrix} .9 \\ .1 \end{bmatrix}$. The game value is $6(.1) + 2 = 2.6$. Let $\mathbf{y} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, and set

$$2.6 = E(\mathbf{e}_1, \mathbf{y}) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & 8 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

so $3c_1 + 2c_2 = 2.6$. Since $\mathbf{y}$ is a probability vector, $3c_1 + 2(1 - c_1) = 2.6$, and $c_1 = .6$. Thus, $c_2 = 1 - .6 = .4$, and the optimal column strategy $\mathbf{y}$ for the matrix game B has entries .6 and .4. The optimal column strategy $\hat{\mathbf{y}}$ for the matrix game A has four entries.

The game matrix, written as $\begin{bmatrix} 3 & * & * & 2 \\ -1 & * & * & 8 \end{bmatrix}$, shows that $\hat{\mathbf{y}} = \begin{bmatrix} .6 \\ 0 \\ 0 \\ .4 \end{bmatrix}$ and, from above, $\hat{\mathbf{x}} = \begin{bmatrix} .9 \\ .1 \end{bmatrix}$.

15. $\hat{\mathbf{x}} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$ or $\begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$ or any convex combination of these row strategies, $\hat{\mathbf{y}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $v = 2$

Solution:
Column 2 dominates column 3, so the column player C will never play column 2. The graph shows why column 2 will not affect the column play, and the graph shows that the value of the game is 2. The new game is $\begin{bmatrix} 4 & * & 2 & 0 \\ 1 & * & 2 & 5 \end{bmatrix}$. Let $B = \begin{bmatrix} 4 & 2 & 0 \\ 1 & 2 & 5 \end{bmatrix}$. The line for column 3 is $z = 2$. That line intersects the line for column 4 where $z = 0(1-t) + 5t = 2$, and $t = .4$. An optimal row strategy is $\hat{\mathbf{x}} = \begin{bmatrix} 1 - .4 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$. Another optimal row strategy is determined by the intersection of the lines for columns 1 and 3, where $z = 4(1-t) + t = 2$, $t = \frac{2}{3}$, and $\hat{\mathbf{x}} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$. It can be shown that any convex combination of these two optimal strategies is also an optimal row strategy.

To find the optimal column strategy, set $\mathbf{y} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, and set $2 = E(\mathbf{e}_1, \mathbf{y}) = \mathbf{e}_1^T B\mathbf{y}$ and $2 = E(\mathbf{e}_2, \mathbf{y}) = \mathbf{e}_2^T B\mathbf{y}$. These two equations produce $4c_1 + 2c_2 = 2$ and $c_1 + 2c_2 + 5c_3 = 2$. Combine these with the fact that $c_1 + c_2 + c_3$ must be 1, and solve the system:

$$\begin{aligned} 4c_1 + 2c_2 \qquad &= 2 \\ c_1 + 2c_2 + 5c_3 &= 2 \\ c_1 + \ c_2 + \ c_3 &= 1 \end{aligned}$$

$$\begin{bmatrix} 4 & 2 & 0 & 2 \\ 1 & 2 & 5 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$c_2 = 1, \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

This is the column strategy for the game matrix B. For A, $\hat{\mathbf{y}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

16. $\hat{\mathbf{x}} = \hat{\mathbf{y}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $v = 2$

Solution:
Row 3 is recessive to row 2, so the row player R will never play row 3. Also, column 3 dominates column 2, so the column player C will never play column 3. Thus, the game reduces:

$$A = \begin{bmatrix} 5 & -1 & 1 \\ 4 & 2 & 2 \\ -2 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 5 & -1 & 1 \\ 4 & 2 & 2 \\ * & * & * \end{bmatrix}$$
$$\to \begin{bmatrix} 5 & -1 & * \\ 4 & 2 & * \\ * & * & * \end{bmatrix}$$

Let $B = \begin{bmatrix} 5 & -1 \\ 4 & 2 \end{bmatrix}$. The row minima are -1 and 2, so the max of the minima is 2. The column maxima are 5 and 2, so the min of the maxima is 2. Thus, the value of the game is 2, and game B has a saddle point, where R always plays row 2 and C always plays column 2. For the original game, the optimal solutions are $\hat{\mathbf{x}} = \hat{\mathbf{y}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Another solution method is to check the original matrix for a saddle point and find it directly, without reducing the size of the matrix.

17. $\hat{\mathbf{x}} = \begin{bmatrix} \frac{5}{7} \\ 0 \\ \frac{2}{7} \\ 0 \end{bmatrix}, \hat{\mathbf{y}} = \begin{bmatrix} 0 \\ \frac{5}{7} \\ \frac{2}{7} \\ 0 \\ 0 \end{bmatrix}, v = \frac{3}{7}$

Solution:
Row 2 is recessive to row 3, and row 4 is recessive to row 1, so the row player R will never play row 2 or row 4. Also, column 4 dominates column 2, so the column player C will never play column 4. Thus, the game reduces:

$$A = \begin{bmatrix} 0 & 1 & -1 & 4 & 3 \\ 1 & -1 & 3 & -1 & -3 \\ 2 & -1 & 4 & 0 & -2 \\ -1 & 0 & -2 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & -1 & 4 & 3 \\ * & * & * & * & * \\ 2 & -1 & 4 & 0 & -2 \\ * & * & * & * & * \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & -1 & * & 3 \\ * & * & * & * & * \\ 2 & -1 & 4 & * & -2 \\ * & * & * & * & * \end{bmatrix}$$

Let $B = \begin{bmatrix} 0 & 1 & -1 & 3 \\ 2 & -1 & 4 & -2 \end{bmatrix}$. (If column 4 in A is not noticed as dominant, this fact will become clear after the lines are plotted for the columns of the reduced matrix.) The equations of the lines corresponding to the columns of B are

$$\begin{aligned} &\text{(column 1)} \quad z = 0(1-t) + 2t = 2t \\ &\text{(column 2)} \quad z = 1(1-t) - t = 1 - 2t \\ &\text{(column 3)} \quad z = -1(1-t) + 4t = -1 + 5t \\ &\text{(column 4)} \quad z = 3(1-t) - 2t = 3 - 5t \end{aligned}$$

The graph of $v(\mathbf{x}(t))$ as a function of t is the polygonal path formed by line 3 (for column 3), then line 2 (column 2), and then line 4 (column 4). The highest point on this path occurs at the intersection of lines 3 and 2. Solve $z = -1 + 5t$ and $z = 1 - 2t$ to find $t = \frac{2}{7}$ and $z = \frac{3}{7}$. The value of game B is $z = \frac{3}{7}$, attained when $\hat{\mathbf{x}} = \begin{bmatrix} 1 - \frac{2}{7} \\ \frac{2}{7} \end{bmatrix} = \begin{bmatrix} \frac{5}{7} \\ \frac{2}{7} \end{bmatrix}$.

Because columns 2 and 3 of B determine the optimal solution, the optimal strategy for the column player C is a convex combination $\hat{\mathbf{y}}$ of the pure column strategies $\mathbf{e}_2$ and $\mathbf{e}_3$, say, $\hat{\mathbf{y}} = \begin{bmatrix} 0 \\ c_2 \\ c_3 \\ 0 \end{bmatrix}$. Since both coordinates of the optimal row solution are nonzero, Theorem 4 shows that $E(\mathbf{e}_i, \hat{\mathbf{y}}) = \frac{3}{7}$ for $i = 1, 2$. Each condition, by itself, determines $\hat{\mathbf{y}}$. For example,

$$\begin{aligned} E(\mathbf{e}_1, \hat{\mathbf{y}}) &= \mathbf{e}_1^T B \hat{\mathbf{y}} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 & 3 \\ 2 & -1 & 4 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ c_2 \\ c_3 \\ 0 \end{bmatrix} \\ &= c_2 - c_3 = \frac{3}{7} \end{aligned}$$

Substitute $c_3 = 1 - c_2$, and obtain $c_2 = \frac{5}{7}$ and $c_3 = \frac{2}{7}$. Thus, $\hat{\mathbf{y}} = \begin{bmatrix} 0 \\ \frac{5}{7} \\ \frac{2}{7} \\ 0 \end{bmatrix}$ is the optimal column strategy for game B. For game A, $\hat{\mathbf{x}} = \begin{bmatrix} \frac{5}{7} \\ 0 \\ \frac{2}{7} \\ 0 \end{bmatrix}$ and $\hat{\mathbf{y}} = \begin{bmatrix} 0 \\ \frac{5}{7} \\ \frac{2}{7} \\ 0 \\ 0 \end{bmatrix}$, and the value of the game is still $\frac{3}{7}$.

18. $\hat{\mathbf{x}} = \begin{bmatrix} 2/3 \\ 0 \\ 0 \\ 1/3 \end{bmatrix}, \hat{\mathbf{y}} = \begin{bmatrix} 0 \\ 2/3 \\ 1/3 \\ 0 \end{bmatrix}, v = \frac{13}{3}$

Solution:
Row 2 is recessive to row 4, and row 3 is recessive to row 1, so the row player R will never play row 2 or row 3. After these rows are removed, column 4 dominates column 2, so the column player C will never play column 4. Thus, the game reduces:

$$A = \begin{bmatrix} 6 & 4 & 5 & 5 \\ 0 & 4 & 2 & 7 \\ 6 & 3 & 5 & 2 \\ 2 & 5 & 3 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 6 & 4 & 5 & 5 \\ * & * & * & * \\ * & * & * & * \\ 2 & 5 & 3 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 6 & 4 & 5 & * \\ * & * & * & * \\ * & * & * & * \\ 2 & 5 & 3 & * \end{bmatrix}$$

Let $B = \begin{bmatrix} 6 & 4 & 5 \\ 2 & 5 & 3 \end{bmatrix}$. (If column 4 in A is not noticed as dominant, this fact will become clear after the lines are plotted for the columns of the reduced matrix.) The equations of the lines corresponding to the columns of B are

$$\begin{aligned} &\text{(column 1)} \quad z = 6(1-t) + 2t = 6 - 4t \\ &\text{(column 2)} \quad z = 4(1-t) + 5t = 4 + t \\ &\text{(column 3)} \quad z = 5(1-t) + 3t = 5 - 2t \end{aligned}$$

The graph of $v(\mathbf{x}(t))$ as a function of t is the polygonal path formed by line 2 (for column 2), then line 3 (column 3), and then line 1 (column 1). The highest point on this path occurs at the intersection of lines 2 and 3. Solve $z = 4 + t$ and $z = 5 - 2t$ to find $t = \frac{1}{3}$ and $z = \frac{13}{3}$. The value of game B is $z = \frac{13}{3}$, attained when $\hat{\mathbf{x}} = \begin{bmatrix} 1 - \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$.
Because columns 2 and 3 of B determine the optimal

solution, the optimal strategy for the column player C is a convex combination $\hat{\mathbf{y}}$ of the pure column strategies $\mathbf{e}_2$ and $\mathbf{e}_3$, say, $\hat{\mathbf{y}} = \begin{bmatrix} 0 \\ c_2 \\ c_3 \end{bmatrix}$. Since both coordinates of the optimal row solution are nonzero, Theorem 4 shows that $E(\mathbf{e}_i, \hat{\mathbf{y}}) = \frac{13}{3}$ for $i = 1, 2$. Each condition, by itself, determines $\hat{\mathbf{y}}$. For example,

$$\begin{aligned} \tfrac{13}{3} &= E(\mathbf{e}_1, \hat{\mathbf{y}}) = \mathbf{e}_1^T B\hat{\mathbf{y}} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 6 & 4 & 5 \\ 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ c_2 \\ c_3 \end{bmatrix} \\ &= 4c_2 + 5c_3 = 4c_2 + 5(1 - c_2) = 5 - c_2 \end{aligned}$$

Then $c_2 = \frac{2}{3}$ and $c_3 = \frac{1}{3}$. Thus, $\hat{\mathbf{y}} = \begin{bmatrix} 0 \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$ is the optimal column strategy for game B. For game A, $\hat{\mathbf{x}} = \begin{bmatrix} \frac{2}{3} \\ 0 \\ 0 \\ \frac{1}{3} \end{bmatrix}$ and

$\hat{\mathbf{y}} = \begin{bmatrix} 0 \\ \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix}$, and the value of the game is still $\frac{13}{3}$.

19. a. Army: 1/3 river, 2/3 land; guerrillas: 1/3 river, 2/3 land; 2/3 of the supplies get through.

b. Army: 7/11 river, 4/11 land; guerrillas: 7/11 river, 4/11 land; 64/121 of the supplies get through.

20. a. Army: 7/11 river, 4/11 land; guerrillas: 9/11 river, 2/11 land.

b. The value of the game is $-36/11$. This means the army will average 36/11 casualties a day.

21. a. True. Definition.

b. True. With a pure strategy, a player chooses one particular play with probability 1.

c. False. $v(\mathbf{x})$ is equal to the *minimum* of the inner product of $\mathbf{x}$ with each of the columns of the payoff matrix.

d. False. The Minimax Theorem says only that the value of a game is the same for both players. It does not guarantee that there is an optimal mixed strategy for each player that produces this common value. It is the Fundamental Theorem for Matrix Games that says every matrix game has a solution.

e. True. By Theorem 5, row r may be deleted from the payoff matrix, and any optimal strategy from the new matrix will also be an optimal strategy for matrix A. This optimal strategy will not involve row s.

22. a. True. Definition.

b. False. A strategy is optimal only if its value equals the value of the game.

c. True. Definition.

d. False. It guarantees the existence of a solution, but it does not show how to find a solution.

e. True. By Theorem 5, the dominating column t may be deleted from the payoff matrix, and any optimal strategy from the new matrix will also be an optimal strategy for matrix A. This optimal strategy will not involve column t. (Note, however, that if a *column* is *recessive*, it may or may not be nonzero in an optimal mixed strategy. In Example 6, column 4 is recessive to column 1, but column 4 has probability 0 in the optimal mixed strategy for C. However, column 3 is also recessive to column 1, and the probability of column 3 in the optimal strategy is positive.)

23. $\hat{\mathbf{x}} = \begin{bmatrix} \frac{1}{6} \\ \frac{5}{6} \\ 0 \end{bmatrix}$, $\hat{\mathbf{y}} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$, $v = 0$

24. a. $\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$

b. $-A^T$

25. $\hat{\mathbf{x}} = \left(\dfrac{d-c}{a-b+d-c}, \dfrac{a-b}{a-b+d-c} \right)$,

$\hat{\mathbf{y}} = \left(\dfrac{d-b}{a-b+d-c}, \dfrac{a-c}{a-b+d-c} \right)$,

$v = \dfrac{ad - bc}{a-b+d-c}$

26. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $v = 1$ and $E(\mathbf{x}, \mathbf{y}) = 1$, but $\mathbf{y}$ is not optimal. There are many other possibilities.

Section 9.2, page 22

1. Let x_1 be the amount invested in mutual funds, x_2 the amount in CDs, and x_3 the amount in savings. Then $\mathbf{b} = \begin{bmatrix} 12{,}000 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} .11 \\ .08 \\ .06 \end{bmatrix}$, and

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & -2 \end{bmatrix}$.

2. Let x_1 be the number of bags of Pixie Power, and x_2 the number of bags of Misty Might. Then $\mathbf{b} = \begin{bmatrix} 28 \\ 30 \\ 20 \\ 25 \end{bmatrix}$,

$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 50 \\ 40 \end{bmatrix}$, and $A = \begin{bmatrix} 3 & 2 \\ 2 & 4 \\ 1 & 3 \\ 2 & 1 \end{bmatrix}$.

3. $\mathbf{b} = \begin{bmatrix} 20 \\ -10 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -5 \end{bmatrix}$

4. $\mathbf{b} = \begin{bmatrix} 25 \\ 40 \\ -40 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$, $A = \begin{bmatrix} 5 & 7 & 1 \\ 2 & 3 & 4 \\ -2 & -3 & -4 \end{bmatrix}$

5. $\mathbf{b} = \begin{bmatrix} -35 \\ 20 \\ -20 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} -7 \\ 3 \\ -1 \end{bmatrix}$, $A = \begin{bmatrix} -1 & 4 & 0 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix}$

6. $\mathbf{b} = \begin{bmatrix} 27 \\ -40 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} -1 \\ -5 \\ 2 \end{bmatrix}$, $A = \begin{bmatrix} 2 & 1 & 4 \\ -1 & 6 & -3 \end{bmatrix}$

7. max $= 1360$, when $x_1 = \frac{72}{5}$ and $x_2 = \frac{16}{5}$

Solution:
First, find the intersection points for the bounding lines:

(1) $2x_1 + x_2 = 32$,
(2) $x_1 + x_2 = 18$,
(3) $x_1 + 3x_2 = 24$

Even a rough sketch of the graphs of these lines will reveal that $(0, 0)$, $(16, 0)$, and $(0, 8)$ are vertices of the feasible set. What about the intersections of the lines corresponding to (1), (2), and (3)?

The graphical method will work, provided the graph is large enough and is drawn carefully. In many simple problems, even a small sketch will reveal which intersection points are vertices of the feasible set. In this problem, however, three intersection points happen to be quite close to each other, and a slight inaccuracy on a graph of size $3'' \times 3''$ or smaller may lead to an incorrect solution. In a case such as this, the following algebraic procedure will work well:

When an intersection point is found that corresponds to two inequalities, test it in the other inequalities to see whether the point is in the feasible set.

The intersection of (1) and (2) is $(14, 4)$. Test this in the third inequality: $(14) + 3(4) = 26 > 24$. The intersection point does not satisfy the inequality for (3), so $(14, 4)$ is **not** in the feasible set.

The intersection of (1) and (3) is $(14.4, 3.2)$. Test this in the second inequality: $14.4 + 3.2 = 17.6 \le 18$, so $(14.4, 3.2)$ **is** in the feasible set.

The intersection of (2) and (3) is $(15, 3)$. Test this in the first inequality: $2(15) + (3) = 33 > 32$, so $(15, 3)$ is **not** in the feasible set.

Next, list the vertices of the feasible set: $(0, 0)$, $(16, 0)$, $(14.4, 3.2)$, and $(0, 8)$. Then compute the values of the objective function $80x_1 + 65x_2$ at these points.

$(0, 0)$:	$80(0) + 65(0) = 0$
$(16, 0)$:	$80(16) + 3(0) = 1280$
$(14.4, 3.2)$:	$80(14.4) + 65(3.2) = 1360$
$(0, 8)$:	$80(0) + 65(8) = 520$

Finally, select the maximum of the objective function, which is 1360, and note that this maximum is attained at $(14.4, 3.2)$.

8. min $= \frac{154}{13}$, when $x_1 = \frac{20}{13}$ and $x_2 = \frac{18}{13}$

Solution:
First, convert the problem to a canonical (maximization) problem:

Maximize $-5x_1 - 3x_2$, subject to
(1) $-2x_1 - 5x_2 \le -10$
(2) $-3x_1 - x_2 \le -6$
(3) $-x_1 - 7x_2 \le -7$

Next, find the intersection points for the bounding lines. The intersection of the equalities for (1) and (2) is $(\frac{20}{13}, \frac{18}{13})$. Test this in the inequality (3):
$-(\frac{20}{13}) - 7(\frac{18}{13}) = -\frac{146}{13} < -7$.
This point satisfies (3), so $(\frac{20}{13}, \frac{18}{13})$ **is** in the feasible set.

The intersection corresponding to (1) and (3) is $(\frac{35}{9}, \frac{4}{9})$. Test this in (2): $-3(\frac{35}{9}) - (\frac{4}{9}) = -\frac{109}{9} < -6$, so $(\frac{35}{9}, \frac{4}{9})$ **is** in the feasible set.

The intersection corresponding to (2) and (3) is $(\frac{7}{4}, \frac{3}{4})$. Test this in (1): $-2(\frac{7}{4}) - 5(\frac{3}{4}) = -\frac{29}{4} > -10$, so $(\frac{7}{4}, \frac{3}{4})$ is **not** in the feasible set.

The vertices of the feasible set are $(0, 6)$, $(\frac{20}{13}, \frac{18}{13})$, $(\frac{35}{9}, \frac{4}{9})$, and $(7, 0)$. The values of the objective function $-5x_1 - 3x_2$ at these points are -18, $-\frac{154}{13} \approx -11.85$, $-\frac{187}{9} \approx -20.8$, and -35, respectively. The maximum value of the objective function $-5x_1 - 3x_2$ is $-\frac{154}{13}$, which occurs at $(\frac{20}{13}, \frac{18}{13})$. So the *minimum* value of the *original* objective function $5x_1 + 3x_2$ is $\frac{154}{13}$, and this occurs at $(\frac{20}{13}, \frac{18}{13})$.

9. unbounded

10. infeasible

11. a. True. Definition.

b. False. The vector $\bar{\mathbf{x}}$ must itself be feasible. It is possible for a nonfeasible vector (as well as the optimal solution) to yield the maximum value of f.

12. a. True. This is a logically equivalent version (called the *contrapositive*) of Theorem 6.

b. False. Theorem 6 says that some extreme point is an optimal solution, but not every optimal solution must be an extreme point.

13. max profit = \$1250, when $x_1 = 100$ bags of EverGreen and $x_2 = 350$ bags of QuickGreen

Solution:
First, find the intersection points for the bounding lines:

(1) $3x_1 + 2x_2 = 1200$ (fescue)
(2) $x_1 + 2x_2 = 800$ (rye)
(3) $x_1 + x_2 = 450$ (bluegrass)

The intersection of lines (1) and (2) is $(200, 300)$. Test this in the inequality corresponding to (3):
$(200) + (300) = 500 > 450$. The intersection point does not satisfy the inequality for (3), so $(200, 300)$ is **not** in the feasible set.

The intersection of (1) and (3) is $(300, 150)$. Test this in (2): $(300) + 2(150) = 600 < 800$, so $(300, 150)$ **is** in the feasible set.

The intersection of (2) and (3) is $(100, 350)$. Test this in (1): $3(100) + 2(350) = 1000 < 1200$, so $(100, 350)$ **is** in the feasible set.

The vertices of the feasible set are $(0, 0)$, $(400, 0)$, $(300, 150)$, $(100, 350)$, and $(0, 400)$. Evaluate the objective function at each vertex:

$$
\begin{array}{ll}
(0,0): & 2(0) + 3(0) = 0 \\
(400,0): & 2(400) + 3(0) = 800 \\
(300,150): & 2(300) + 3(150) = 1050 \\
(100,350): & 2(100) + 3(350) = 1250 \\
(0,400): & 2(0) + 3(400) = 1200
\end{array}
$$

The maximum of the objective function $2x_1 + 3x_2$ is \$1250 at $(100, 350)$.

14. min cost = \$25,000, when $x_1 = 2$ days and $x_2 = 6$ days

Solution:
First, find the intersection points for the bounding lines:

$$
\begin{array}{ll}
(1) & 12x_1 + 4x_2 = 48 \\
(2) & 4x_1 + 4x_2 = 32 \\
(3) & x_1 + 5x_2 = 20
\end{array}
$$

The intersection of lines (1) and (2) is $(2, 6)$. Test this in the third inequality: $(2) + 5(6) = 32 > 20$. The intersection point satisfies the inequality for (3), so $(2, 6)$ **is** in the feasible set.

The intersection of (1) and (3) is $(20/7, 24/7)$. Test this in the second inequality:
$4(20/7) + 4(24/7) = 176/7 \approx 25.14 < 32$, so this point is **not** in the feasible set.

The intersection of (2) and (3) is $(5, 3)$. Test this in the first inequality: $12(5) + 4(3) = 72 > 48$, so $(5, 3)$ **is** in the feasible set.

The vertices of the feasible set are $(20, 0)$, $(5, 3)$, $(2, 6)$, and $(0, 12)$. Evaluate the objective function at each vertex. (The values here represent thousands of dollars.)

$$
\begin{array}{ll}
(20,0): & 3.5(20) + 3(0) = 70 \\
(5,3): & 3.5(5) + 3(3) = 26.5 \\
(2,6): & 3.5(2) + 3(6) = 25 \\
(0,12): & 3.5(0) + 3(12) = 36
\end{array}
$$

The minimum cost is \$25,000, when the production schedule is $(x_1, x_2) = (2, 6)$. That is, the cost is minimized when refinery A runs for 2 days and refinery B runs for 6 days.

15. max profit = \$1180, for 20 widgets and 30 whammies

Solution:
First, find the intersection points for the bounding lines:

$$
\begin{array}{ll}
(1) & 5x_1 + 2x_2 = 200 \\
(2) & .2x_1 + .4x_2 = 16 \\
(3) & .2x_1 + .2x_2 = 10
\end{array}
$$

The intersection of (1) and (2) is $(30, 25)$. Test this in the third inequality: $.2(30) + .2(25) = 11 > 10$. The intersection point does not satisfy the inequality for (3), so $(30, 25)$ is **not** in the feasible set.

The intersection of (1) and (3) is $(100/3, 100/6)$. Test this in the second inequality:
$.2(100/3) + .4(100/6) = 13.3 < 16$, so $(100/3, 100/6)$ **is** in the feasible set.

The intersection of (2) and (3) is $(20, 30)$. Test this in the first inequality: $5(20) + 2(30) = 160 < 200$, so $(20, 30)$ **is** in the feasible set.

The vertices of the feasible set are $(40, 0)$, $(100/3, 100/6)$, $(20, 30)$, and $(0, 40)$. Evaluate the objective function at each vertex:

$$
\begin{array}{ll}
(40,0): & 20(40) + 26(0) = 800 \\
(100/3, 100/6): & 20(100/3) + 26(100/6) = 1100 \\
(20,30): & 20(20) + 26(30) = 1180 \\
(0,40): & 20(0) + 26(40) = 1040
\end{array}
$$

The maximum profit is \$1180, when $x_1 = 20$ widgets and $x_2 = 30$ whammies.

16. Take any $\mathbf{p}, \mathbf{q}$ in $\mathcal{F}$. Then $A\mathbf{p} \le \mathbf{b}$, $A\mathbf{q} \le \mathbf{b}$, $\mathbf{p} \ge \mathbf{0}$, and $\mathbf{q} \ge \mathbf{0}$. Take any scalar t such that $0 \le t \le 1$, and let $\mathbf{x} = (1-t)\mathbf{p} + t\mathbf{q}$. Then

$$A\mathbf{x} = A[(1-t)\mathbf{p} + t\mathbf{q}] = (1-t)A\mathbf{p} + tA\mathbf{q} \qquad (*)$$

by the linearity of matrix multiplication. Since t and $1-t$ are both nonnegative, $(1-t)A\mathbf{p} \le (1-t)\mathbf{b}$ and $tA\mathbf{p} \le t\mathbf{b}$. Thus, the right side of $(*)$ is less than or equal to $\mathbf{b}$. Also, $\mathbf{x} \ge 0$ because $\mathbf{p}$ and $\mathbf{q}$ have this property and the constants $(1-t)$ and t are nonnegative. Thus, $\mathbf{x}$ is in $\mathcal{F}$. So the line segment between $\mathbf{p}$ and $\mathbf{q}$ is in $\mathcal{F}$. This proves that $\mathcal{F}$ is convex.

17. Take any $\mathbf{p}$ and $\mathbf{q}$ in S, with $\mathbf{p} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Then $\mathbf{v}^T\mathbf{p} \le c$ and $\mathbf{v}^T\mathbf{q} \le c$. Take any scalar t such that $0 \le t \le 1$. Then, by the linearity of matrix multiplication (or the dot product if $\mathbf{v}^T\mathbf{p}$ is written as $\mathbf{v}\cdot\mathbf{p}$, and so on),

$$
\begin{aligned}
\mathbf{v}^T[(1-t)\mathbf{p} + t\mathbf{q}] &= (1-t)\mathbf{v}^T\mathbf{p} + t\mathbf{v}^T\mathbf{q} \\
&\le (1-t)c + tc = c
\end{aligned}
$$

because $(1-t)$ and t are both positive and $\mathbf{p}$ and $\mathbf{q}$ are in S. So the line segment between $\mathbf{p}$ and $\mathbf{q}$ is in S. Since $\mathbf{p}$ and $\mathbf{q}$ were any points in S, the set S is convex.

18. Let S be the intersection of $S_1, \ldots, S_5$, and take $\mathbf{x}$ and $\mathbf{y}$ in S. Then $\mathbf{x}$ and $\mathbf{y}$ are in S_i for $i = 1, \ldots, 5$. For any t, with $0 \le t \le 1$, and any i, with $1 \le i \le 5$, $(1-t)\mathbf{x} + t\mathbf{y}$ is in S_i because S_i is convex. Then $(1-t)\mathbf{x} + t\mathbf{y}$ is in S, by definition of the intersection. This proves that S is a convex set.

19. Let $S = \{\mathbf{x} : f(\mathbf{x}) = d\}$, and take $\mathbf{p}$ and $\mathbf{q}$ in S. Also, take t with $0 \le t \le 1$, and let $\mathbf{x} = (1-t)\mathbf{p} + t\mathbf{q}$. Then

$$
\begin{aligned}
f(\mathbf{x}) &= \mathbf{c}^T\mathbf{x} = \mathbf{c}^T[(1-t)\mathbf{p} + t\mathbf{q}] \\
&= (1-t)\mathbf{c}^T\mathbf{p} + t\mathbf{c}^T\mathbf{q} = (1-t)d + td = d
\end{aligned}
$$

Thus, $\mathbf{x}$ is in S. This shows that S is convex.

Section 9.3, page 39

1.
$$\begin{array}{c}\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M & \end{array}\\ \left[\begin{array}{cccccc|c} 2 & 7 & 10 & 1 & 0 & 0 & 20 \\ 3 & 4 & 18 & 0 & 1 & 0 & 25 \\ \hline -21 & -25 & -15 & 0 & 0 & 1 & 0 \end{array}\right]\end{array}$$

2.
$$\begin{array}{c}\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M & \end{array}\\ \left[\begin{array}{cccccc|c} 3 & 5 & 1 & 0 & 0 & 0 & 30 \\ 2 & 7 & 0 & 1 & 0 & 0 & 24 \\ 6 & 1 & 0 & 0 & 1 & 0 & 42 \\ \hline -22 & -14 & 0 & 0 & 0 & 1 & 0 \end{array}\right]\end{array}$$

3. a. x_2

b.
$$\begin{array}{c}\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \end{array}\\ \left[\begin{array}{ccccc|c} \frac{7}{2} & 0 & 1 & -\frac{1}{2} & 0 & 5 \\ \frac{3}{2} & 1 & 0 & \frac{1}{2} & 0 & 15 \\ \hline 11 & 0 & 0 & 5 & 1 & 150 \end{array}\right]\end{array}$$

c. $x_1 = 0, x_2 = 15, x_3 = 5, x_4 = 0, M = 150$

d. optimal

4. a. x_1

b.
$$\begin{array}{c}\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \end{array}\\ \left[\begin{array}{ccccc|c} 0 & 1 & 7 & 1 & 0 & 10 \\ 1 & 0 & 5 & 1 & 0 & 6 \\ \hline 0 & 0 & 28 & 5 & 1 & 47 \end{array}\right]\end{array}$$

c. $x_1 = 6, x_2 = 10, x_3 = 0, x_4 = 0, M = 47$

d. optimal

5. a. x_1

b.
$$\begin{array}{c}\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \end{array}\\ \left[\begin{array}{ccccc|c} 0 & 2 & 1 & -1 & 0 & 4 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 8 \\ \hline 0 & -2 & 0 & 3 & 1 & 48 \end{array}\right]\end{array}$$

c. $x_1 = 8, x_2 = 0, x_3 = 4, x_4 = 0, M = 48$

d. not optimal

6. a. x_2

b.
$$\begin{array}{c}\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \end{array}\\ \left[\begin{array}{ccccc|c} -11 & 0 & 1 & -\frac{4}{3} & 0 & 40 \\ 2 & 1 & 0 & \frac{1}{6} & 0 & 5 \\ \hline 8 & 0 & 0 & \frac{1}{2} & 1 & 15 \end{array}\right]\end{array}$$

c. $x_1 = 0, x_2 = 5, x_3 = 40, x_4 = 0, M = 15$

d. optimal

7. a. False. A slack variable is used to change an inequality into an equality.

b. True. Definition.

c. False. The initial basic solution will be infeasible, but there may still be a basic feasible solution.

8. a. True. Definition.

b. True. See the comment before Example 3.

c. False. The bottom entry in the right column gives the current value of the objective function. It will be the maximum value only if the current solution is optimal.

9. The maximum is 150, when $x_1 = 3$ and $x_2 = 10$.

Solution:
First, bring x_2 into the solution; pivot with row 1. Then bring x_1 into the solution; pivot with row 2. The maximum is 150, when $x_1 = 3$ and $x_2 = 10$.

$$\begin{array}{c}\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \end{array}\\ \left[\begin{array}{ccccc|c} 2 & 3 & 1 & 0 & 0 & 36 \\ 5 & 4 & 0 & 1 & 0 & 55 \\ \hline -10 & -12 & 0 & 0 & 1 & 0 \end{array}\right]\end{array}$$

$$\sim \begin{array}{c}\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \end{array}\\ \left[\begin{array}{ccccc|c} \frac{2}{3} & 1 & \frac{1}{3} & 0 & 0 & 12 \\ \frac{7}{3} & 0 & -\frac{4}{3} & 1 & 0 & 7 \\ \hline -2 & 0 & 4 & 0 & 1 & 144 \end{array}\right]\end{array}$$

$$\sim \begin{array}{c}\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \end{array}\\ \left[\begin{array}{ccccc|c} 0 & 1 & \frac{5}{7} & -\frac{2}{7} & 0 & 10 \\ 1 & 0 & -\frac{4}{7} & \frac{3}{7} & 0 & 3 \\ \hline 0 & 0 & \frac{20}{7} & \frac{6}{7} & 1 & 150 \end{array}\right]\end{array}$$

10. The maximum is 98, when $x_1 = 10$ and $x_2 = 12$.

Solution:
First, bring x_1 into the solution; pivot with row 2. Next, scale row 1 to simplify the arithmetic. Finally, bring x_2 into the solution; pivot with row 1. The maximum is 98, when $x_1 = 10$ and $x_2 = 12$.

$$\begin{array}{c}\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \end{array}\\ \left[\begin{array}{ccccc|c} 1 & 5 & 1 & 0 & 0 & 70 \\ 3 & 2 & 0 & 1 & 0 & 54 \\ \hline -5 & -4 & 0 & 0 & 1 & 0 \end{array}\right]\end{array}$$

$$\sim \begin{array}{c}\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \end{array}\\ \left[\begin{array}{ccccc|c} 0 & \frac{13}{3} & 1 & -\frac{1}{3} & 0 & 52 \\ 1 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 18 \\ \hline 0 & -\frac{2}{3} & 0 & \frac{5}{3} & 1 & 90 \end{array}\right]\end{array}$$

$$\sim \begin{array}{c}\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \end{array}\\ \left[\begin{array}{ccccc|c} 0 & 1 & \frac{3}{13} & -\frac{1}{13} & 0 & 12 \\ 1 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 18 \\ \hline 0 & -\frac{2}{3} & 0 & \frac{5}{3} & 1 & 90 \end{array}\right]\end{array}$$

$$\sim \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \\ \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 1 & \frac{3}{13} & -\frac{1}{13} & 0 & 12 \\ 1 & 0 & -\frac{2}{13} & \frac{5}{13} & 0 & 10 \\ \hline 0 & 0 & \frac{2}{13} & \frac{21}{13} & 1 & 98 \end{array}\right] \end{array}$$

11. The maximum is 56, when $x_1 = 9$ and $x_2 = 4$.

Solution:
First, bring x_2 into the solution; pivot with row 2. Then bring x_1 into the solution; pivot with row 3. The maximum is 56, when $x_1 = 9$ and $x_2 = 4$.

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M & \\ \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 2 & 1 & 0 & 0 & 0 & 26 \\ 2 & 3 & 0 & 1 & 0 & 0 & 30 \\ 1 & 1 & 0 & 0 & 1 & 0 & 13 \\ \hline -4 & -5 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M & \\ \end{array} \\ \left[\begin{array}{cccccc|c} -\frac{1}{3} & 0 & 1 & -\frac{2}{3} & 0 & 0 & 6 \\ \frac{2}{3} & 1 & 0 & \frac{1}{3} & 0 & 0 & 10 \\ \frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & 0 & 3 \\ \hline -\frac{2}{3} & 0 & 0 & \frac{5}{3} & 0 & 1 & 50 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M & \\ \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 0 & 1 & -1 & 1 & 0 & 9 \\ 0 & 1 & 0 & 1 & -2 & 0 & 4 \\ 1 & 0 & 0 & -1 & 3 & 0 & 9 \\ \hline 0 & 0 & 0 & 1 & 2 & 1 & 56 \end{array}\right] \end{array}$$

12. The maximum is 70, when $x_1 = 6$, $x_2 = 11$, and $x_3 = 1$.

Solution:
First, bring x_2 into the solution; pivot with row 3. Next, bring x_1 into the solution; pivot with row 1. Finally, bring x_3 into the solution; pivot with row 2. The maximum is 70, when $x_1 = 6$, $x_2 = 11$, and $x_3 = 1$.

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \\ \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 2 & 0 & 1 & 0 & 0 & 0 & 28 \\ 2 & 0 & 4 & 0 & 1 & 0 & 0 & 16 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 12 \\ \hline -2 & -5 & -3 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \\ \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 0 & -2 & 1 & 0 & -2 & 0 & 4 \\ 2 & 0 & 4 & 0 & 1 & 0 & 0 & 16 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 12 \\ \hline -2 & 0 & 2 & 0 & 0 & 5 & 1 & 60 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \\ \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 0 & -2 & 1 & 0 & -2 & 0 & 4 \\ 0 & 0 & 8 & -2 & 1 & 4 & 0 & 8 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 12 \\ \hline 1 & 0 & -2 & 2 & 0 & 1 & 1 & 68 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \\ \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & -1 & 0 & 6 \\ 0 & 0 & 1 & -\frac{1}{4} & \frac{1}{8} & \frac{1}{2} & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{8} & \frac{1}{2} & 0 & 11 \\ \hline 0 & 0 & 0 & \frac{3}{2} & \frac{1}{4} & 2 & 1 & 70 \end{array}\right] \end{array}$$

13. The minimum is 180, when $x_1 = 10$ and $x_2 = 12$.

Solution:
Convert this to a maximization problem for $-12x_1 - 5x_2$, and reverse the first constraint inequality. Beginning with the first tableau below, bring x_1 into the solution, using row 1 as the pivot row. Then bring x_2 into the solution; pivot with row 2. The maximum value of $-12x_1 - 5x_2$ is -180, so the minimum of the original objective function $12x_1 + 5x_2$ is 180, when x_1 is 10 and x_2 is 12.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \\ \end{array} \\ \left[\begin{array}{ccccc|c} -2 & -1 & 1 & 0 & 0 & -32 \\ -3 & 5 & 0 & 1 & 0 & 30 \\ \hline 12 & 5 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \\ \end{array} \\ \left[\begin{array}{ccccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 16 \\ 0 & \frac{13}{2} & -\frac{3}{2} & 1 & 0 & 78 \\ \hline 0 & -1 & 6 & 0 & 1 & -192 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \\ \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & -\frac{5}{13} & -\frac{1}{13} & 0 & 10 \\ 0 & 1 & -\frac{3}{13} & \frac{2}{13} & 0 & 12 \\ \hline 0 & 0 & \frac{75}{13} & \frac{2}{13} & 1 & -180 \end{array}\right] \end{array}$$

14. The minimum is 33, when $x_1 = 0$, $x_2 = 4$, and $x_3 = 7$.

Solution:
Convert this to a maximization problem for $-2x_1 - 3x_2 - 3x_3$, and reverse the first two constraint inequalities. Beginning with the first tableau below, bring x_3 into the solution, with row 2 as the pivot. Then bring x_2 into the solution; pivot with row 1. The maximum is -33, so the minimum of $2x_1 + 3x_2 + 3x_3$ is 33, when $x_1 = 0$, $x_2 = 4$, and $x_3 = 7$.

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \\ \end{array} \\ \left[\begin{array}{ccccccc|c} -1 & 2 & 0 & 1 & 0 & 0 & 0 & 8 \\ 0 & -2 & -1 & 0 & 1 & 0 & 0 & -15 \\ 2 & -1 & 1 & 0 & 0 & 1 & 0 & 25 \\ \hline 2 & 3 & 3 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \\ \end{array} \\ \left[\begin{array}{ccccccc|c} -1 & 2 & 0 & 1 & 0 & 0 & 0 & 8 \\ 0 & 2 & 1 & 0 & -1 & 0 & 0 & 15 \\ 2 & -3 & 0 & 0 & 0 & 1 & 0 & 10 \\ \hline 2 & -3 & 0 & 0 & 3 & 0 & 1 & -45 \end{array}\right] \end{array}$$

$$
\sim\begin{array}{c}
\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M \end{array}\\
\left[\begin{array}{ccccccc|c}
-\frac{1}{2} & 1 & 0 & \frac{1}{2} & 0 & 0 & 0 & 4\\
1 & 0 & 1 & -1 & -1 & 0 & 0 & 7\\
\frac{1}{2} & 0 & 0 & \frac{3}{2} & 1 & 1 & 0 & 22\\
\hline
\frac{1}{2} & 0 & 0 & \frac{3}{2} & 3 & 0 & 1 & -33
\end{array}\right]
\end{array}
$$

15. The answer matches that in Example 7. The minimum is 20, when $x_1 = 8$ and $x_2 = 6$.

Solution:
Begin with the same initial simplex tableau, bringing x_1 into the solution, with row 2 as the pivot row. Then bring x_2 into the solution; pivot with row 1. The maximum of $-x_1 - 2x_2$ is -20, so the minimum of $x_1 + 2x_2$ is 20, when $x_1 = 8$ and $x_2 = 6$.

$$
\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & M \end{array}\\
\left[\begin{array}{ccccc|c}
-1 & -1 & 1 & 0 & 0 & -14\\
1 & -1 & 0 & 1 & 0 & 2\\
\hline
1 & 2 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
$$

$$
\sim\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & M \end{array}\\
\left[\begin{array}{ccccc|c}
0 & -2 & 1 & 1 & 0 & -12\\
1 & -1 & 0 & 1 & 0 & 2\\
\hline
0 & -3 & 0 & -1 & 1 & -2
\end{array}\right]
\end{array}
$$

$$
\sim\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & M \end{array}\\
\left[\begin{array}{ccccc|c}
0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 6\\
1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 8\\
\hline
0 & 0 & \frac{3}{2} & \frac{1}{2} & 1 & -20
\end{array}\right]
\end{array}
$$

16. The maximum annual income is \$1,100, provided by \$6,000 in mutual funds, \$4,000 in CDs, and \$2,000 in savings.

Solution:
From the bottom row of the tableau, x_1 must be brought into the solution first. The ratios to consider are 12,000/1 in row 1 and 0/1 in row 2. So pivot with row 2. Next, bring x_2 into the solution; pivot with row 3 (because the ratio 0/1 is less than the ratio 12,000/2). Finally, bring x_3 into the solution; pivot with row 1. The maximum annual income of \$1,100 is provided by \$6,000 in mutual funds, \$4,000 in CDs, and \$2,000 in savings.

$$
\begin{array}{c}
\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M \end{array}\\
\left[\begin{array}{ccccccc|c}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 12{,}000\\
1 & -1 & -1 & 0 & 1 & 0 & 0 & 0\\
0 & 1 & -2 & 0 & 0 & 1 & 0 & 0\\
\hline
-.11 & -.08 & -.06 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
$$

$$
\sim\begin{array}{c}
\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M \end{array}\\
\left[\begin{array}{ccccccc|c}
0 & 2 & 2 & 1 & -1 & 0 & 0 & 12{,}000\\
1 & -1 & -1 & 0 & 1 & 0 & 0 & 0\\
0 & 1 & -2 & 0 & 0 & 1 & 0 & 0\\
\hline
0 & -.19 & -.17 & 0 & .11 & 0 & 1 & 0
\end{array}\right]
\end{array}
$$

$$
\sim\begin{array}{c}
\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M \end{array}\\
\left[\begin{array}{ccccccc|c}
0 & 0 & 6 & 1 & -1 & -2 & 0 & 12{,}000\\
1 & 0 & -3 & 0 & 1 & 1 & 0 & 0\\
0 & 1 & -2 & 0 & 0 & 1 & 0 & 0\\
\hline
0 & 0 & -.55 & 0 & .11 & .19 & 1 & 0
\end{array}\right]
\end{array}
$$

$$
\sim\begin{array}{c}
\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M \end{array}\\
\left[\begin{array}{ccccccc|c}
0 & 0 & 1 & \frac{1}{6} & -\frac{1}{6} & -\frac{1}{3} & 0 & 2{,}000\\
1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 6{,}000\\
0 & 1 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 4{,}000\\
\hline
0 & 0 & 0 & \frac{11}{120} & \frac{11}{600} & \frac{1}{150} & 1 & 1{,}100
\end{array}\right]
\end{array}
$$

17. The maximum profit is \$1180, achieved by making 20 widgets and 30 whammies each day.

Solution:
The simplex tableau below is based on the problem of the Benri Company (Exercise 15 in Section 9.2) to maximize the profit function $20x_1 + 26x_2$ subject to various amounts of labor available for the three-step production process. To begin the simplex method, bring x_2 into the solution; pivot with row 2. Then, bring x_1 into the solution; pivot with row 3. The profit is maximized at \$1180, by making 20 widgets and 30 whammies each day.

$$
\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M \end{array}\\
\left[\begin{array}{cccccc|c}
5 & 2 & 1 & 0 & 0 & 0 & 200\\
\frac{1}{5} & \frac{2}{5} & 0 & 1 & 0 & 0 & 16\\
\frac{1}{5} & \frac{1}{5} & 0 & 0 & 1 & 0 & 10\\
\hline
-20 & -26 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
$$

$$
\sim\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M \end{array}\\
\left[\begin{array}{cccccc|c}
4 & 0 & 1 & -5 & 0 & 0 & 120\\
\frac{1}{2} & 1 & 0 & \frac{5}{2} & 0 & 0 & 40\\
\frac{1}{10} & 0 & 0 & -\frac{1}{2} & 1 & 0 & 2\\
\hline
-7 & 0 & 0 & 65 & 0 & 1 & 1040
\end{array}\right]
\end{array}
$$

$$
\sim\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M \end{array}\\
\left[\begin{array}{cccccc|c}
0 & 0 & 1 & 15 & -40 & 0 & 40\\
0 & 1 & 0 & 5 & -5 & 0 & 30\\
1 & 0 & 0 & -5 & 10 & 0 & 20\\
\hline
0 & 0 & 0 & 30 & 70 & 1 & 1180
\end{array}\right]
\end{array}
$$

18. The maximum profit is \$1250, achieved when $x_1 = 100$ (bags of EverGreen) and $x_2 = 350$ (bags of QuickGreen).

Solution:
The simplex tableau below is based on the summary at the end of Example 1 in Section 9.2. To begin the simplex method, bring x_2 into the solution; pivot with row 2. Then bring x_1 into the solution; pivot with row 3. The \$1250 maximum is achieved when $x_1 = 100$ (bags of EverGreen) and $x_2 = 350$ (bags of QuickGreen).

$$\begin{array}{c}\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M \end{array}\\ \left[\begin{array}{cccccc|c} 3 & 2 & 1 & 0 & 0 & 0 & 1200 \\ 1 & 2 & 0 & 1 & 0 & 0 & 800 \\ 1 & 1 & 0 & 0 & 1 & 0 & 450 \\ \hline -2 & -3 & 0 & 0 & 0 & 1 & 0 \end{array}\right]\end{array}$$

$$\begin{array}{c}\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M \end{array}\\ \sim\left[\begin{array}{cccccc|c} 2 & 0 & 1 & -1 & 0 & 0 & 400 \\ \frac{1}{2} & 1 & 0 & \frac{1}{2} & 0 & 0 & 400 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 1 & 0 & 50 \\ \hline -\frac{1}{2} & 0 & 0 & \frac{3}{2} & 0 & 1 & 1200 \end{array}\right]\end{array}$$

$$\begin{array}{c}\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M \end{array}\\ \sim\left[\begin{array}{cccccc|c} 0 & 0 & 1 & 1 & -4 & 0 & 200 \\ 0 & 1 & 0 & 1 & -1 & 0 & 350 \\ 1 & 0 & 0 & -1 & 2 & 0 & 100 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 1250 \end{array}\right]\end{array}$$

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1. Minimize $36y_1 + 55y_2$
subject to $2y_1 + 5y_2 \geq 10$
$3y_1 + 4y_2 \geq 12$
and $y_1 \geq 0$, $y_2 \geq 0$.

2. Minimize $70y_1 + 54y_2$
subject to $y_1 + 3y_2 \geq 5$
$5y_1 + 2y_2 \geq 4$
and $y_1 \geq 0$, $y_2 \geq 0$.

3. Minimize $26y_1 + 30y_2 + 13y_3$
subject to $y_1 + 2y_2 + y_3 \geq 4$
$2y_1 + 3y_2 + y_3 \geq 5$
and $y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$.

4. Minimize $28y_1 + 16y_2 + 12y_3$
subject to $y_1 + 2y_2 \geq 2$
$2y_1 + y_3 \geq 5$
$4y_2 + y_3 \geq 3$
and $y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$.

5. The minimum is $M = 150$, attained when $y_1 = \frac{20}{7}$ and $y_2 = \frac{6}{7}$.

Solution:
The final tableau from Exercise 9 in Section 9.3 is

$$\begin{array}{c}\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & M \end{array}\\ \left[\begin{array}{ccccc|c} 0 & 1 & \frac{5}{7} & -\frac{2}{7} & 0 & 10 \\ 1 & 0 & -\frac{4}{7} & \frac{3}{7} & 0 & 3 \\ \hline 0 & 0 & \frac{20}{7} & \frac{6}{7} & 1 & 150 \end{array}\right]\end{array}$$

The solution of the dual problem is displayed by the entries in row 3 of columns 3, 4, and 6. The minimum is $M = 150$, attained when $y_1 = \frac{20}{7}$ and $y_2 = \frac{6}{7}$.

6. The minimum is $M = 98$, attained when $y_1 = \frac{2}{13}$ and $y_2 = \frac{21}{13}$.

Solution:
The final tableau from Exercise 10 in Section 9.3 is

$$\begin{array}{c}\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & M \end{array}\\ \left[\begin{array}{ccccc|c} 0 & 1 & \frac{3}{13} & -\frac{1}{13} & 0 & 12 \\ 1 & 0 & -\frac{2}{13} & \frac{5}{13} & 0 & 10 \\ \hline 0 & 0 & \frac{2}{13} & \frac{21}{13} & 1 & 98 \end{array}\right]\end{array}$$

The solution of the dual problem is displayed by the entries in row 3 of columns 3, 4, and 6. The minimum is $M = 98$, attained when $y_1 = \frac{2}{13}$ and $y_2 = \frac{21}{13}$.

7. The minimum is $M = 56$, attained when $y_1 = 0$, $y_2 = 1$, and $y_3 = 2$.

Solution:
The final tableau from Exercise 11 in Section 9.3 is

$$\begin{array}{c}\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M \end{array}\\ \left[\begin{array}{cccccc|c} 0 & 0 & 1 & -1 & 1 & 0 & 9 \\ 0 & 1 & 0 & 1 & -2 & 0 & 4 \\ 1 & 0 & 0 & -1 & 3 & 0 & 9 \\ \hline 0 & 0 & 0 & 1 & 2 & 1 & 56 \end{array}\right]\end{array}$$

The solution of the dual problem is displayed by the entries in row 4 of columns 3, 4, 5, and 7. The minimum is $M = 56$, attained when $y_1 = 0$, $y_2 = 1$, and $y_3 = 2$.

8. The minimum is $M = 70$, attained when $y_1 = \frac{3}{2}$, $y_2 = \frac{1}{4}$, and $y_3 = 2$.

Solution:
The final tableau from Exercise 12 in Section 9.3 is

$$\begin{array}{c}\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M \end{array}\\ \left[\begin{array}{ccccccc|c} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & -1 & 0 & 6 \\ 0 & 0 & 1 & -\frac{1}{4} & \frac{1}{8} & \frac{1}{2} & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{8} & \frac{1}{2} & 0 & 11 \\ \hline 0 & 0 & 0 & \frac{3}{2} & \frac{1}{4} & 2 & 1 & 70 \end{array}\right]\end{array}$$

The solution of the dual problem is displayed by the entries in row 4 of columns 4, 5, 6, and 8. The minimum is $M = 70$, attained when $y_1 = \frac{3}{2}$, $y_2 = \frac{1}{4}$, and $y_3 = 2$.

9. **a.** False. It should be $A^T\mathbf{y} \geq \mathbf{c}$.
b. True. Theorem 7.
c. True. Theorem 7.

d. False. The marginal value is zero if it is in the optimal solution. See Example 4.

10. a. True. See the comment before Theorem 7.

b. True. Theorem 7.

c. True. Theorem 7.

d. False. The coordinates of **u** and **v** are equal to one. The vectors do not have length one.

11. The minimum is 43, when $x_1 = \frac{7}{4}$, $x_2 = 0$, and $x_3 = \frac{3}{4}$.

Solution:
The dual problem is to maximize $4y_1 + 5y_2$ subject to $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \le \begin{bmatrix} 16 \\ 10 \\ 20 \end{bmatrix}$ and $\mathbf{y} \ge \mathbf{0}$. Solve the dual problem with the simplex method:

$$\begin{array}{c} \begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & M & \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 2 & 1 & 0 & 0 & 0 & 16 \\ 1 & 1 & 0 & 1 & 0 & 0 & 10 \\ 3 & 2 & 0 & 0 & 1 & 0 & 20 \\ \hline -4 & -5 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & M & \end{array} \\ \left[\begin{array}{cccccc|c} \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 8 \\ \frac{1}{2} & 0 & -\frac{1}{2} & 1 & 0 & 0 & 2 \\ 2 & 0 & -1 & 0 & 1 & 0 & 4 \\ \hline -\frac{3}{2} & 0 & \frac{5}{2} & 0 & 0 & 1 & 40 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & M & \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 1 & \frac{3}{4} & 0 & -\frac{1}{4} & 0 & 7 \\ 0 & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & 1 \\ 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 2 \\ \hline 0 & 0 & \frac{7}{4} & 0 & \frac{3}{4} & 1 & 43 \end{array}\right] \end{array}$$

The solution of the dual of the dual (the primal) is $x_1 = \frac{7}{4}$, $x_2 = 0$, $x_3 = \frac{3}{4}$, with $M = 43$.

12. The minimum is 26, when $x_1 = \frac{5}{3}$ and $x_2 = \frac{2}{3}$.

Solution:
The dual problem is to maximize $3y_1 + 4y_2 + 2y_3$ subject to $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \le \begin{bmatrix} 10 \\ 14 \end{bmatrix}$ and $\mathbf{y} \ge \mathbf{0}$. Use the simplex tableau for the dual problem:

$$\begin{array}{c} \begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & M & \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 2 & 3 & 1 & 0 & 0 & 10 \\ 2 & 1 & 1 & 0 & 1 & 0 & 14 \\ \hline -3 & -4 & -2 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & M & \end{array} \\ \left[\begin{array}{cccccc|c} \frac{1}{2} & 1 & \frac{3}{2} & \frac{1}{2} & 0 & 0 & 5 \\ \frac{3}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 1 & 0 & 9 \\ \hline -1 & 0 & 4 & 2 & 0 & 1 & 20 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & M & \end{array} \\ \left[\begin{array}{cccccc|c} \frac{1}{2} & 1 & \frac{3}{2} & \frac{1}{2} & 0 & 0 & 5 \\ 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 6 \\ \hline -1 & 0 & 4 & 2 & 0 & 1 & 20 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & M & \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 1 & \frac{5}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 2 \\ 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 6 \\ \hline 0 & 0 & \frac{11}{3} & \frac{5}{3} & \frac{2}{3} & 1 & 26 \end{array}\right] \end{array}$$

The solution of the dual of the dual (the primal) is $x_1 = \frac{5}{3}$, $x_2 = \frac{2}{3}$, with the minimum $M = 26$.

13. The minimum cost is \$670, using 11 bags of Pixie Power and 3 bags of Misty Might.

Solution:
The problem in Exercise 2 of Section 9.2 is to minimize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \ge \mathbf{b}$ and $\mathbf{x} \ge \mathbf{0}$, where $\mathbf{x}$ lists the number of bags of Pixie Power and Misty Might, and $\mathbf{c} = \begin{bmatrix} 50 \\ 40 \end{bmatrix}$, $A = \begin{bmatrix} 3 & 2 \\ 2 & 4 \\ 1 & 3 \\ 2 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 28 \\ 30 \\ 20 \\ 25 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The dual of a minimization problem involving a matrix is a maximization problem involving the transpose of the matrix, with the vector data for the objective function and the constraint equation interchanged. Since the notation was established in Exercise 2 for a minimization problem, the notation here is "reversed" from the usual notation for a primal problem. Thus, the dual of the primal problem stated above is to maximize $\mathbf{b}^T\mathbf{y}$ subject to $A^T\mathbf{y} \le \mathbf{c}$ and $\mathbf{y} \ge \mathbf{0}$. That is, maximize $28y_1 + 30y_2 + 20y_3 + 25y_4$ subject to

$$\begin{bmatrix} 3 & 2 & 1 & 2 \\ 2 & 4 & 3 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \le \begin{bmatrix} 50 \\ 40 \end{bmatrix}$$

Here are the simplex calculations for this dual problem:

$$\begin{array}{c} \begin{array}{cccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & M & \end{array} \\ \left[\begin{array}{ccccccc|c} 3 & 2 & 1 & 2 & 1 & 0 & 0 & 50 \\ 2 & 4 & 3 & 1 & 0 & 1 & 0 & 40 \\ \hline -28 & -30 & -20 & -25 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{ccccccc|c} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & M & \end{array} \\ \left[\begin{array}{ccccccc|c} 2 & 0 & -\frac{1}{2} & \frac{3}{2} & 1 & -\frac{1}{2} & 0 & 30 \\ \frac{1}{2} & 1 & \frac{3}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 10 \\ \hline -13 & 0 & \frac{5}{2} & -\frac{35}{2} & 0 & \frac{15}{2} & 1 & 300 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{ccccccc|c} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & M & \end{array} \\ \left[\begin{array}{ccccccc|c} \frac{4}{3} & 0 & -\frac{1}{3} & 1 & \frac{2}{3} & -\frac{1}{3} & 0 & 20 \\ \frac{1}{6} & 1 & \frac{5}{6} & 0 & -\frac{1}{6} & \frac{1}{3} & 0 & 5 \\ \hline \frac{31}{3} & 0 & -\frac{10}{3} & 0 & \frac{35}{3} & \frac{5}{3} & 1 & 650 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{ccccccc|c} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & M & \end{array} \\ \left[\begin{array}{ccccccc|c} \frac{7}{5} & \frac{2}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & 22 \\ \frac{1}{5} & \frac{6}{5} & 1 & 0 & -\frac{1}{5} & \frac{2}{5} & 0 & 6 \\ \hline 11 & 4 & 0 & 0 & 11 & 3 & 1 & 670 \end{array}\right] \end{array}$$

Since the original problem is the dual of the problem solved by the simplex method, the desired solution is given by the slack variables $y_5 = 11$ and $y_6 = 3$. The value of the objective is the same for the primal and dual problems, so the minimum cost is \$670. This is achieved by blending 11 bags of Pixie Power and 3 bags of Misty Might.

14. Refinery A = 2 days, refinery B = 6 days, minimum cost = \$25,000

Solution:
Express costs in thousands of dollars, let x_1 be the number of days refinery A operates, and let x_2 be the number of days refinery B operates. Then the problem in Example 2 of Section 9.2 is to minimize $3.5x_1 + 3x_2$ subject to $\begin{bmatrix} 12 & 4 \\ 4 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 48 \\ 32 \\ 20 \end{bmatrix}$. The dual problem is to maximize $48y_1 + 32y_2 + 20y_3$ subject to

$$\begin{bmatrix} 12 & 4 & 1 \\ 4 & 4 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \leq \begin{bmatrix} 3.5 \\ 3 \end{bmatrix}.$$

Use the simplex tableau for this dual problem. The first pivot is on y_1, because the entry -48 is the most negative entry in the bottom row. The first row is chosen because the ratio b_1/a_{11} is smaller than b_2/a_{21}.

$$\begin{array}{c} \begin{array}{cccccc|c} y_1 & y_2 & y_3 & y_4 & y_5 & M & \end{array} \\ \left[\begin{array}{cccccc|c} 12 & 4 & 1 & 1 & 0 & 0 & 3.5 \\ 4 & 4 & 5 & 0 & 1 & 0 & 3 \\ \hline -48 & -32 & -20 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{cccccc|c} y_1 & y_2 & y_3 & y_4 & y_5 & M & \end{array} \\ \left[\begin{array}{cccccc|c} 1 & \frac{1}{3} & \frac{1}{12} & \frac{1}{12} & 0 & 0 & \frac{7}{24} \\ 0 & \frac{8}{3} & \frac{14}{3} & -\frac{1}{3} & 1 & 0 & \frac{11}{6} \\ \hline 0 & -16 & -16 & 4 & 0 & 1 & 14 \end{array}\right] \end{array}$$

Now, two negative entries in the bottom row happen to be equal, so either y_2 or y_3 can be the next pivot. When y_2 is used, the result is

$$\sim \begin{array}{c} \begin{array}{cccccc|c} y_1 & y_2 & y_3 & y_4 & y_5 & M & \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 0 & -\frac{1}{2} & \frac{1}{8} & -\frac{1}{8} & 0 & \frac{1}{16} \\ 0 & 1 & \frac{7}{4} & -\frac{1}{8} & \frac{3}{8} & 0 & \frac{11}{16} \\ \hline 0 & 0 & 12 & 2 & 6 & 1 & 25 \end{array}\right] \end{array}$$

When y_3 is used as a pivot in the second tableau above, more work is required:

$$\sim \begin{array}{c} \begin{array}{cccccc|c} y_1 & y_2 & y_3 & y_4 & y_5 & M & \end{array} \\ \left[\begin{array}{cccccc|c} 1 & \frac{2}{7} & 0 & \frac{5}{56} & -\frac{1}{56} & 0 & \frac{29}{112} \\ 0 & \frac{4}{7} & 1 & -\frac{1}{14} & \frac{3}{14} & 0 & \frac{11}{28} \\ \hline 0 & -\frac{48}{7} & 0 & \frac{20}{7} & \frac{24}{7} & 1 & \frac{142}{7} \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{cccccc|c} y_1 & y_2 & y_3 & y_4 & y_5 & M & \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 0 & -\frac{1}{2} & \frac{1}{8} & -\frac{1}{8} & 0 & \frac{1}{16} \\ 0 & 1 & \frac{7}{4} & -\frac{1}{8} & \frac{3}{8} & 0 & \frac{11}{16} \\ \hline 0 & 0 & 12 & 2 & 6 & 1 & 25 \end{array}\right] \end{array}$$

An extra pivot operation is required because pivoting on y_3 increases M by less than pivoting on y_2. This can be seen in advance, but the situation occurs so rarely, that a rule for deciding which pivot column to choose is hardly worth remembering. Notice that if y_2 is to be the pivot variable, then the row for this pivot is the one for which the ratio b_i/a_{i2} is the smallest. (In this example, that ratio is $\frac{11}{6} \div \frac{8}{3} = \frac{11}{16}$.) If y_3 is the pivot variable, then the row for this pivot is the one for which the ratio b_i/a_{i3} is the smallest. (In this example, that ratio is $\frac{11}{6} \div \frac{14}{3} = \frac{11}{28}$.) The rule is to choose the variable for which this "smallest" ratio is larger. In this case, since $\frac{11}{16}$ is larger than $\frac{11}{28}$, y_2 is the better choice for the pivot. Since so many ratios have to be computed, it seems easier just to pick either y_2 or y_3 and calculate the next tableau.

Since the original problem is the dual of the problem solved by the simplex method, the desired solution is given by the slack variables $y_4 = 2$ and $y_5 = 6$. The value of the objective is the same for the primal and dual problems, so the minimum cost is 25 (thousand dollars). This is achieved by operating refinery A for 2 days and refinery B for 6 days.

15. The marginal value is zero. This corresponds to labor in the fabricating department being underutilized. That is, at the optimal production schedule with $x_1 = 20$ and $x_2 = 30$, only 160 of the 200 available hours in fabricating are needed. The extra labor is wasted, and so it has value zero.

16. Allocate the additional hour of labor to the shipping department, thereby increasing the profit by \$70. The profit would increase by only \$30 if the hour of labor were added to packing, and not at all if the hour were added to fabricating.

17. $\hat{\mathbf{x}} = \begin{bmatrix} \frac{2}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix}, \hat{\mathbf{y}} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, v = 1$

18. $\hat{\mathbf{x}} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \\ 0 \end{bmatrix}, \hat{\mathbf{y}} = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix}, v = \frac{1}{4}$

19. $\hat{\mathbf{x}} = \begin{bmatrix} \frac{2}{5} \\ \frac{2}{5} \\ \frac{1}{5} \end{bmatrix}, \hat{\mathbf{y}} = \begin{bmatrix} \frac{3}{7} \\ \frac{3}{7} \\ \frac{1}{7} \end{bmatrix}, v = 1$

Solution:

The game is $\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 4 \\ 3 & -1 & 1 \end{bmatrix}$. Add 3 to shift the game: $\begin{bmatrix} 4 & 5 & 1 \\ 3 & 4 & 7 \\ 6 & 2 & 4 \end{bmatrix}$. The linear programming tableau for this game is

$$\begin{array}{c} \begin{array}{cccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & M & \end{array} \\ \left[\begin{array}{ccccccc|c} 4 & 5 & 1 & 1 & 0 & 0 & 0 & 1 \\ 3 & 4 & 7 & 0 & 1 & 0 & 0 & 1 \\ 6 & 2 & 4 & 0 & 0 & 1 & 0 & 1 \\ \hline -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivots:

$$\begin{array}{c} \begin{array}{cccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & M & \end{array} \\ \left[\begin{array}{ccccccc|c} 0 & \frac{11}{3} & -\frac{5}{3} & 1 & 0 & -\frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 3 & 5 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 1 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} \\ \hline 0 & -\frac{2}{3} & -\frac{1}{3} & 0 & 0 & \frac{1}{6} & 1 & \frac{1}{6} \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{cccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & M & \end{array} \\ \left[\begin{array}{ccccccc|c} 0 & 1 & -\frac{5}{11} & \frac{3}{11} & 0 & -\frac{2}{11} & 0 & \frac{1}{11} \\ 0 & 0 & \frac{70}{11} & -\frac{9}{11} & 1 & \frac{1}{22} & 0 & \frac{5}{22} \\ 1 & 0 & \frac{9}{11} & -\frac{1}{11} & 0 & \frac{5}{22} & 0 & \frac{3}{22} \\ \hline 0 & 0 & -\frac{7}{11} & \frac{2}{11} & 0 & \frac{1}{22} & 1 & \frac{5}{22} \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{cccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & M & \end{array} \\ \left[\begin{array}{ccccccc|c} 0 & 1 & 0 & \frac{3}{14} & \frac{1}{14} & -\frac{5}{28} & 0 & \frac{3}{28} \\ 0 & 0 & 1 & -\frac{9}{70} & \frac{11}{70} & \frac{1}{140} & 0 & \frac{1}{28} \\ 1 & 0 & 0 & \frac{1}{70} & -\frac{9}{70} & \frac{31}{140} & 0 & \frac{3}{28} \\ \hline 0 & 0 & 0 & \frac{1}{10} & \frac{1}{10} & \frac{1}{20} & 1 & \frac{1}{4} \end{array}\right] \end{array}$$

The optimal solution of the primal and dual problems, respectively, are $\bar{y}_1 = \frac{3}{28}$, $\bar{y}_2 = \frac{3}{28}$, $\bar{y}_3 = \frac{1}{28}$, and $\bar{x}_1 = \frac{1}{10}$, $\bar{x}_2 = \frac{1}{10}$, $\bar{x}_3 = \frac{1}{20}$, with $\lambda = \frac{1}{4}$. The corresponding optimal mixed strategies for the column and row players, respectively, are:

$$\hat{\mathbf{y}} = \bar{\mathbf{y}}/\lambda = \bar{\mathbf{y}} \cdot 4 = \begin{bmatrix} \frac{3}{7} \\ \frac{3}{7} \\ \frac{1}{7} \end{bmatrix} \text{ and } \hat{\mathbf{x}} = \bar{\mathbf{x}}/\lambda = \bar{\mathbf{x}} \cdot 4 = \begin{bmatrix} \frac{2}{5} \\ \frac{2}{5} \\ \frac{1}{5} \end{bmatrix}$$

The value of the game with the shifted payoff matrix is $1/\lambda$, which is 4, so the value of original game is $4 - 3 = 1$.

20. $\hat{\mathbf{x}} = \begin{bmatrix} \frac{5}{16} \\ \frac{7}{16} \\ \frac{4}{16} \end{bmatrix}, \hat{\mathbf{y}} = \begin{bmatrix} 0 \\ \frac{7}{16} \\ \frac{4}{16} \\ \frac{5}{16} \end{bmatrix}, v = -\frac{1}{16}$

Solution:

The game is $\begin{bmatrix} 2 & 0 & 1 & -1 \\ -1 & 1 & -2 & 0 \\ 1 & -2 & 2 & 1 \end{bmatrix}$. Add 3 to shift the game: $\begin{bmatrix} 5 & 3 & 4 & 2 \\ 2 & 4 & 1 & 3 \\ 4 & 1 & 5 & 4 \end{bmatrix}$.

The linear programming tableau for this game is

$$\begin{array}{c} \begin{array}{ccccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & M & \end{array} \\ \left[\begin{array}{cccccccc|c} 5 & 3 & 4 & 2 & 1 & 0 & 0 & 0 & 1 \\ 2 & 4 & 1 & 3 & 0 & 1 & 0 & 0 & 1 \\ 4 & 1 & 5 & 4 & 0 & 0 & 1 & 0 & 1 \\ \hline -1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The simplex method produces

$$\begin{array}{c} \begin{array}{ccccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & M & \end{array} \\ \left[\begin{array}{cccccccc|c} \frac{49}{47} & 0 & 1 & 0 & \frac{13}{47} & -\frac{10}{47} & \frac{1}{47} & 0 & \frac{4}{47} \\ \frac{27}{47} & 1 & 0 & 0 & \frac{11}{47} & \frac{6}{47} & -\frac{10}{47} & 0 & \frac{7}{47} \\ -\frac{21}{47} & 0 & 0 & 1 & -\frac{19}{47} & \frac{11}{47} & \frac{13}{47} & 0 & \frac{5}{47} \\ \hline \frac{8}{47} & 0 & 0 & 0 & \frac{5}{47} & \frac{7}{47} & \frac{4}{47} & 1 & \frac{16}{47} \end{array}\right] \end{array}$$

The optimal solutions of the primal and dual problems, respectively, are

$\bar{y}_1 = 0, \bar{y}_2 = \frac{7}{47}, \bar{y}_3 = \frac{4}{47}, \bar{y}_4 = \frac{5}{47}$,

and

$\bar{x}_1 = \frac{5}{47}, \bar{x}_2 = \frac{7}{47}, \bar{x}_3 = \frac{4}{47}$, with $\lambda = \frac{16}{47}$

The corresponding optimal mixed strategies for the column and row players, respectively, are

$$\hat{\mathbf{y}} = \bar{\mathbf{y}}/\lambda = \bar{\mathbf{y}} \cdot \frac{47}{16} = \begin{bmatrix} 0 \\ \frac{7}{16} \\ \frac{4}{16} \\ \frac{5}{16} \end{bmatrix} \quad \text{and}$$

$$\hat{\mathbf{x}} = \bar{\mathbf{x}}/\lambda = \bar{\mathbf{x}} \cdot \frac{47}{16} = \begin{bmatrix} \frac{5}{16} \\ \frac{7}{16} \\ \frac{4}{16} \end{bmatrix}$$

The value of the game with the shifted payoff matrix is $1/\lambda$, which is $\frac{47}{16}$, so the value of original game is $\frac{47}{16} - 3 = -\frac{1}{16}$.

21. Change this "game" into a linear programming problem and use the simplex method to analyze the game. The expected value of the game is $\frac{38}{35}$, based on a payoff matrix for an investment of \$100. With \$35,000 to invest, Bob "plays" this game 350 times. Thus, he expects to gain \$380, and the expected value of his portfolio at the end of the year is \$35,380. Using the optimal game strategy, Bob should invest \$11,000 in stocks, \$9,000 in bonds, and \$15,000 in gold.

Solution:

The game is $\begin{bmatrix} 4 & 1 & -2 \\ 1 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix}$. Add 3 to shift the game: $\begin{bmatrix} 7 & 4 & 1 \\ 4 & 6 & 3 \\ 2 & 3 & 7 \end{bmatrix}$. The linear programming problem is to

maximize $y_1 + y_2 + y_3$ subject to

$$\begin{bmatrix} 7 & 4 & 1 \\ 4 & 6 & 3 \\ 2 & 3 & 7 \end{bmatrix} \le \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The tableau for this game is

$$\begin{array}{c} \begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & M \end{array} \\ \left[\begin{array}{ccccccc|c} 7 & 4 & 1 & 1 & 0 & 0 & 0 & 1 \\ 4 & 6 & 3 & 0 & 1 & 0 & 0 & 1 \\ 2 & 3 & 7 & 0 & 0 & 1 & 0 & 1 \\ \hline -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The simplex calculations are

$$\begin{array}{c} \begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & M \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & \frac{4}{7} & \frac{1}{7} & \frac{1}{7} & 0 & 0 & 0 & \frac{1}{7} \\ 0 & \frac{26}{7} & \frac{17}{7} & -\frac{4}{7} & 1 & 0 & 0 & \frac{3}{7} \\ 0 & \frac{13}{7} & \frac{47}{7} & -\frac{2}{7} & 0 & 1 & 0 & \frac{5}{7} \\ \hline 0 & -\frac{3}{7} & -\frac{6}{7} & \frac{1}{7} & 0 & 0 & 1 & \frac{1}{7} \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & M \end{array} \\ \sim \left[\begin{array}{ccccccc|c} 1 & \frac{25}{47} & 0 & \frac{7}{47} & 0 & -\frac{1}{47} & 0 & \frac{6}{47} \\ 0 & \frac{143}{47} & 0 & -\frac{22}{47} & 1 & -\frac{17}{47} & 0 & \frac{8}{47} \\ 0 & \frac{13}{47} & 1 & -\frac{2}{47} & 0 & \frac{7}{47} & 0 & \frac{5}{47} \\ \hline 0 & -\frac{9}{47} & 0 & \frac{5}{47} & 0 & \frac{6}{47} & 1 & \frac{11}{47} \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & M \end{array} \\ \sim \left[\begin{array}{ccccccc|c} 1 & 0 & 0 & \frac{3}{13} & -\frac{25}{143} & \frac{6}{143} & 0 & \frac{14}{143} \\ 0 & 1 & 0 & -\frac{2}{13} & \frac{47}{143} & -\frac{17}{143} & 0 & \frac{8}{143} \\ 0 & 0 & 1 & 0 & -\frac{1}{11} & \frac{2}{11} & 0 & \frac{1}{11} \\ \hline 0 & 0 & 0 & \frac{1}{13} & \frac{9}{143} & \frac{15}{143} & 1 & \frac{35}{143} \end{array}\right] \end{array}$$

The optimal solution of the primal and dual problems, respectively, are

$$\bar{\mathbf{y}}_1 = \frac{14}{143}, \bar{\mathbf{y}}_2 = \frac{8}{143}, \bar{\mathbf{y}}_3 = \frac{1}{11},$$

and

$$\bar{x}_1 = \frac{1}{13}, \bar{x}_2 = \frac{9}{143}, \bar{x}_3 = \frac{15}{143}, \text{ with } \lambda = \frac{35}{143}$$

The corresponding optimal mixed strategies for the column and row players, respectively, are

$$\hat{\mathbf{y}} = \bar{\mathbf{y}}/\lambda = \bar{\mathbf{y}} \cdot \tfrac{143}{35} = \begin{bmatrix} \frac{14}{35} \\ \frac{8}{35} \\ \frac{13}{35} \end{bmatrix} \text{ and}$$

$$\hat{\mathbf{x}} = \bar{\mathbf{x}}/\lambda = \bar{\mathbf{x}} \cdot \tfrac{143}{35} = \begin{bmatrix} \frac{11}{35} \\ \frac{9}{35} \\ \frac{15}{35} \end{bmatrix}$$

The value of the game with the shifted payoff matrix is $1/\lambda$, which is $\frac{143}{35}$, so the value of original game is $\frac{143}{35} - 3 = \frac{38}{35}$. Using the optimal strategy $\hat{\mathbf{x}}$, Bob should invest $\frac{11}{35}$ of the \$35,000 in stocks, $\frac{9}{35}$ in bonds, and $\frac{15}{35}$ in gold. That is, Bob should invest \$11,000 in stocks, \$9,000 in bonds, and \$15,000 in gold. The expected value of the game is $\frac{38}{35}$, based on \$100 for each play of the game. (The payoff matrix lists the amounts gained or lost for each \$100 that is invested for one year.) With \$35,000 to invest, Bob "plays" this game 350 times. Thus, he should expect to gain \$380, and the expected value of his portfolio at the end of the year is \$35,380.

22. a. Consider $\mathbf{x}$ in $\mathcal{F}$ and $\mathbf{y}$ in $\mathcal{F}^*$, and note that $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} = \mathbf{x}^T\mathbf{c}$, and $g(\mathbf{y}) = \mathbf{b}^T\mathbf{y} = \mathbf{y}^T\mathbf{b}$. Because the entries in $\mathbf{x}$ and $\mathbf{y}$ are nonnegative, the inequalities $\mathbf{c} \le A^T\mathbf{y}$ and $A\mathbf{x} \le \mathbf{b}$ lead to

$$f(\mathbf{x}) = \mathbf{x}^T\mathbf{c} \le \mathbf{x}^T A^T \mathbf{y} = (A\mathbf{x})^T\mathbf{y} = \mathbf{y}^T(A\mathbf{x}) \le \mathbf{y}^T(\mathbf{b}) = g(\mathbf{y})$$

b. If $f(\hat{\mathbf{x}}) = g(\hat{\mathbf{y}})$, then for any $\mathbf{x}$ in $\mathcal{F}$, part (a) shows that $f(\mathbf{x}) \le g(\hat{\mathbf{y}}) = f(\hat{\mathbf{x}})$, so $\hat{\mathbf{x}}$ is an optimal solution to P. Similarly, for any $\mathbf{y}$ in $\mathcal{F}^*$, $g(\mathbf{y}) \ge f(\hat{\mathbf{x}}) = g(\hat{\mathbf{y}})$, which shows that $\hat{\mathbf{y}}$ is an optimal solution to P^*.

23. a. The coordinates of $\bar{\mathbf{x}}$ are all nonnegative. From the definition of $\mathbf{u}$, λ is equal to the sum of these coordinates. It follows that the coordinates of $\hat{\mathbf{x}}$ are nonnegative and sum to one. Thus, $\hat{\mathbf{x}}$ is a mixed strategy for the row player R. A similar argument holds for $\hat{\mathbf{y}}$ and the column player C.

b. If $\mathbf{y}$ is any mixed strategy for C, then

$$E(\hat{\mathbf{x}}, \mathbf{y}) = \hat{\mathbf{x}}^T A\mathbf{y} = \frac{1}{\lambda}\left(\bar{\mathbf{x}}^T A\mathbf{y}\right) = \frac{1}{\lambda}\left[\left(A^T\bar{\mathbf{x}}\right)\cdot\mathbf{y}\right] \ge \frac{1}{\lambda}(\mathbf{v}\cdot\mathbf{y}) = \frac{1}{\lambda}$$

c. If $\mathbf{x}$ is any mixed strategy for R, then

$$E(\mathbf{x}, \hat{\mathbf{y}}) = \mathbf{x}^T A\hat{\mathbf{y}} = \frac{1}{\lambda}\left(\mathbf{x}^T A\bar{\mathbf{y}}\right) = \frac{1}{\lambda}\left[\mathbf{x}\cdot A\bar{\mathbf{y}}\right]$$
$$\leq \frac{1}{\lambda}(\mathbf{x}\cdot\mathbf{u}) = \frac{1}{\lambda}$$

d. Part (b) implies $v(\hat{\mathbf{x}}) \geq 1/\lambda$, so $v_R \geq 1/\lambda$. Part (c) implies $v(\hat{\mathbf{y}}) \leq 1/\lambda$, so $v_C \leq 1/\lambda$. It follows from the Minimax Theorem in Section 9.1 that $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are optimal mixed strategies for R and C, respectively, and that the value of the game is $1/\lambda$.