Notes for ECE269 - Linear Algebra

Chapter 1

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July 15, 2020

1 Linear Equations in Linear Algebra

This first chapter will go over the basics of linear equations and foundations of formulating systems of linear equations into networks of vectors and matrices for more substanital analysis later in the text.

1.1 Systems of Linear Equations

A linear equation is described as follows:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

A system of linear equations is one or more linear equations as decribed above involving the same variables. Two linear systems are equivalent if the solution set for the two systems is identical. Linear systems are either consistent (have one or infinitely many solutions) or inconsistent (no solution).

A matrix is shown below:

$$\begin{bmatrix} 2 & 4 & 6 & 0 \\ 1 & 3 & 5 & 1 \\ 7 & 8 & 9 & 2 \end{bmatrix}$$

This is an augmented matrix as the values the equations solve to are included as the right most column. The linear equations are representedby the other columns in the matrix, starting with the second to right most column being constant coefficients. From there, the degree of the variablesincreases by one per column. An $m\ x\ n$ matrix indicates m rows and n columns.

To solve a system of linear equations, there are three methods in simplifying system:

- Replacing an equation with the sum of itself and the multiple of another equation
- Interchanging two equations
- Multiplying an equation by a nonzero constant

Two matrices are said to be row equivalent if these operations can be used to equate one matrix to another. This translates into the two row equivalent matrices having the same solution set. If in reduced form, there is a constradiction in the solution set, then the system of equations is inconsistent (no solution).

1.2 Row Reduction and Echelon Forms

A rectangular matrix is in echelon form (or row echelon) if it has the following properties:

- All nonzero rows are above any rows of all zeros
- Each leading entry of a row is in a column to the right of the leading entry of the row above it
- All entries in a column below a leading entry are zeros

Additionally, the following properties yield a reduced row echelon form matrix:

- The leading entry in each nonzero row is 1
- Each leading 1 is the only nonzero entry in its column

The terminology of echelon comes from the "steplike" appearance of the matrix entries. There are numerous possibilities for echelon form matricies, but there is a unique reduced echelon form for a given matrix. RREF and REF refer to the reduced echelon forms. Pivot positions are those in the matrix that correspond to the positions of leading 1's in the matrix rows. The forward phase of the process is to obtain echelon form while the backward phase is to obtain reduced echelon form. Partial pivoting (selecting the pivot as the entry with the largest absolute value in a column) reduces rounding error by a computer program.

Below is a reduced echelon form matrix:

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The solutions are obtained by translating the final column as the constants, while the preceding columns are the linear systems of equations through variables. There are three variables in the above matrix due to there being four columns, and the first two variables are considered basic variables as they are explicitly written in terms of the others. The solutions can be obtained from the reduced echelon form as the basic variables are all described in terms of constants and the free variables. For the free variables, as the third one shown above, you can choose any value, and thus there are infinitely many distinct solutions to the set.

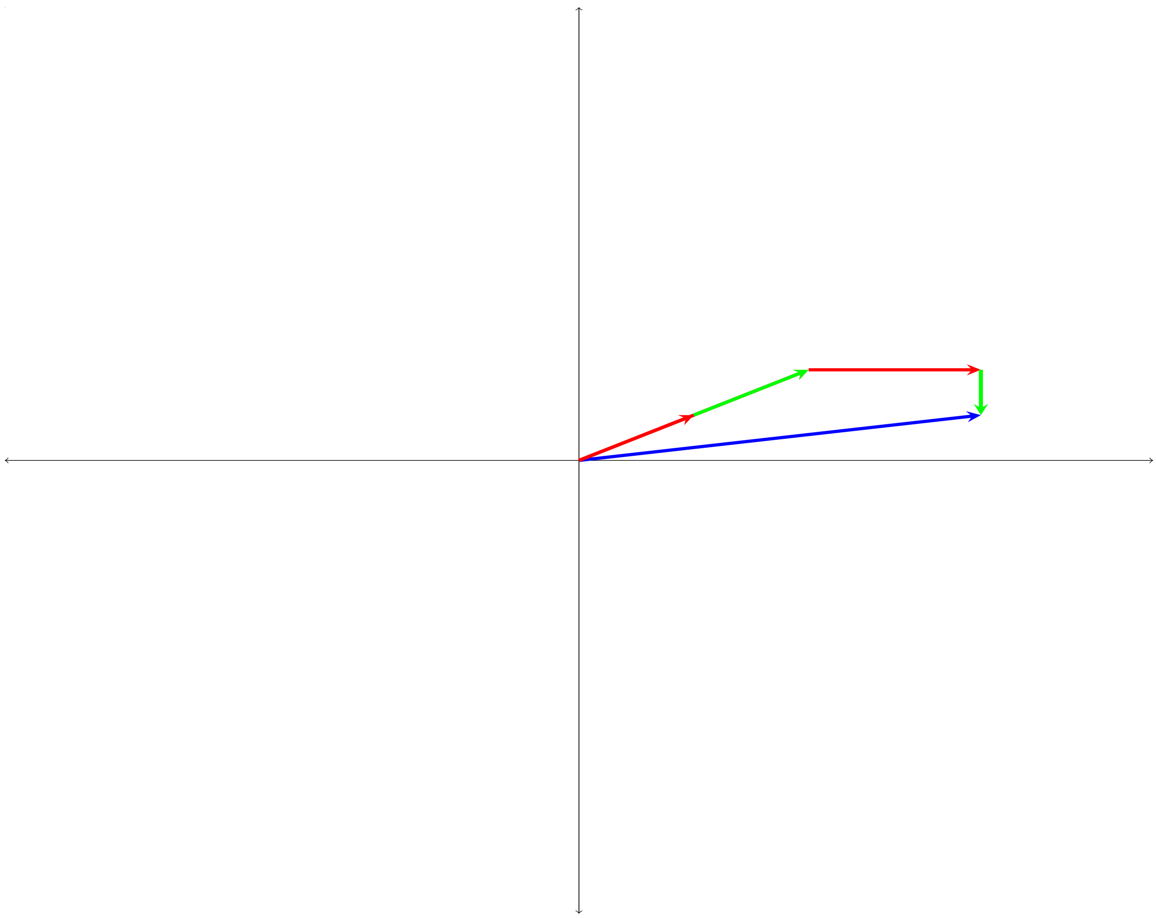

Figure 1: Figure demonstrating plotting the result from vector operations above with red being the arithmetic operations, green being scaling, and blue being the result.

1.3 Vector Equations

A matrix with only one column is considered a column vector, or simplified to vector. Examples of these are included below and can include any real numbers. Vectorsare ordered and thus the only equivalents are if they have the same components in each location.

$$\mathbf{w} = \begin{bmatrix} \mathit{w1} \\ \mathit{w2} \end{bmatrix}$$

Example operations that can be executed on vectors include basic arithmetic operations per below. Careful consideration needs to be made to ensure the operations are beingdone either elementwise or with matrix dimensionality being taken into consideration. Scalars can be multiplied with a vector to yield the scalar multiplied with each element of the matrix.

$$2 * \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} .* \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

Vectors can be in other dimensionality spaces beyond 2D. 3D matrices are represented by a third row in the column vector, and n-dimension spaces are represented by a column vector that is n-entries in length. There is a unique vector, the zero vector, whose entries are all 0 and is written as $\mathbf{0}$.

The vector **y** below is defined as the linear combination of scalar values in c (called weights) and vectors **v**.

$$\boldsymbol{y} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_n\boldsymbol{v_n} \tag{2}$$

A vector equation as shown above has the same solution set as the linear system whose augemnted matrix includes the variables **v** from above with the last term of the augmented matrix being **y**. Specific interesting examples include the scalar multiple of a variable and the zero vector which both can be shown with this format. the Span**u** is the line that includes all scalar multiples of that vector and **0**. The Span**u,v** is the plane that incorporates all of the scalar multiples of those two vectors as well as **0**. This is possible since the scalar multiple could be with 0, thus every possible value in that plane is achievable as a linear combination of the two vectors.

1.4 The Matrix Equation Ax=b

If **A** is an $m\ x\ n$ matrix, with columns $\begin{bmatrix}\mathbf{a_1} & \mathbf{a_2} & \dots & \mathbf{a_n}\end{bmatrix}$ and **x** is a weight column vector, then the product of **A** and **x** is the linear combination of the components of **A** with **x** as the weights for the components. Below shows this equation.

$$\boldsymbol{Ax} = \begin{bmatrix}\mathbf{a_1} & \mathbf{a_2} & \dots & \mathbf{a_n}\end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ \dots \\ x_n\end{bmatrix} = x_1\mathbf{a_1} + x_2\mathbf{a_2} + \ldots + x_n\mathbf{a_n}$$

The dimensionality of **x** needs to match the number of columns present in **A** for the operation to make sense. Any equation can be written in this format by taking the coefficients as the weight vector, and the variables as the $m\ x\ n$ matrix to which the scalar coefficients are applied. The equation **Ax=b** has a solution if and only if **b** is a linear combination of the columns of **A**.

The identity matrix and dot product are showcased below.

$$\begin{bmatrix}1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ x_3\end{bmatrix} = \begin{bmatrix}1\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 \\ 0\cdot x_1 + 1\cdot x_2 + 0\cdot x_3 \\ 0\cdot x_1 + 0\cdot x_2 + 1\cdot x_3\end{bmatrix} = \begin{bmatrix}x_1 \\ x_2 \\ x_3\end{bmatrix}$$

Depedning on the programming language you may want to modify the method of multiplication to increase speed. An example being that C stores matrices as rows and you would need to modify the program to interpret the data in this manner.