ECE269 - Linear Algebra

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Textbook: Matrix Analysis and Applied Linear Algebra

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October 8, 2020

1 Vector Spaces

1.1 Spaces and Subspaces

While linear algebra can be extended well to matrices, it is limiting to believe that matrices are the only mechanism in which to apply linear algebraic principles. There are many other mediums such as polynomials and functions that have similar properties to matrices that linear algebra can also be applied to. The general discussion of linear algebra thus refers to vector spaces.

A vector space involves four things (a nonempty set of vectors V, a scalar field F such as real numbers, vector addition, and scalar multiplication) and abides by the following axioms:

- $\mathbf{x} + \mathbf{y} \in V$ for all $\mathbf{x}, \mathbf{y} \in V$. This is the closure property of vector addition.
- $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$. This is associativity of vector addition.
- $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in V$. This is commutativity of vector addition.
- There is an element $\mathbf{0}$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for every $\mathbf{x} \in V$. This is the vector addition additive identity.

- There is an element $-\mathbf{x}$ such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$ for every $\mathbf{x} \in V$. This is the vector addition additive inverse.

- $\alpha\mathbf{x} \in V$ for all $\alpha \in F$ and $\mathbf{x} \in V$. This is the closure property of scalar multiplication.

- $\alpha\beta(\mathbf{x}) = \alpha(\beta\mathbf{x})$ for all $\alpha, \beta \in F$ and $\mathbf{x} \in V$. This is associativity of scalar multiplication.

- $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$ for all $\alpha \in F$ and $\mathbf{x}, \mathbf{y} \in V$. This is the distributive property of scalar multiplication.

- $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$ for all $\alpha\beta \in F$ and $\mathbf{x} \in V$. This is also the distributive property of scalar multiplication.

- $1\mathbf{x} = x$ for every $\mathbf{x} \in V$. This is the scalar multiplication multiplicative identity.

As an example of the earlier generalization from matrices, functions can be applied in the same way as the scalar multiplications. Functions can be thought of as transformations.

If there is a subset S for the vector space V over F, and S operates through the same axioms as defined previously for the vector space, then S is a subspace of V if :

- $\mathbf{x}, \mathbf{y} \in \mathbf{S} \Rightarrow \mathbf{x} + \mathbf{y} \in \mathbf{S}$

- $\mathbf{x} \in \mathbf{S} \Rightarrow \alpha\mathbf{x} \in \mathbf{S}$ for all $\alpha \in F$

For the set containing only the zero vector, this subspace is referred to as the trivial subspace. The only subspaces that are proper are either this trivial subspace or straight lines through the origin. As the dimensionality of the vector space increases, so do the proper subspace options. Another interpretation of subspace is flat surfaces through the origin, or the trivial solution.

For a given set of vectors, $S = \mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ the subspace $span(S) = \alpha\mathbf{v_1}, \alpha\mathbf{v_2}, \ldots, \alpha\mathbf{v_r}$ generated by using only linear combinations of S is the space spanned by S. If there is a vector space V which equals the space spanned by S, S spans V. Subspaces can be added together to form another subspace and the sums of their respective spans, if they span their own subspaces, spans the added together subspace.

A helpful observation is that for a given set of vectors S from a subspace V, S spans V if and only if for each vector in the subspace, there is a column vector that when multiplied with the column matrix from S, you obtain the vector in the subspace. This is deduced from the rank (number of pivots, pivotal columns, or nonzero rows) and the fact that for A$\mathbf{x} = \mathbf{b}$ to be consistent, the rank of A needs to equal the rank of A—b.

1.2 Four Fundamental Subspaces

The four fundamental subspaces associated with $\mathbf{A}_{mxn}$ are:

- The range or column space: $R(\mathbf{A}) = \{\mathbf{Ax}\}|\mathbf{x} \in R^n, \subseteq R^m$
  The spanning set for a range space are the basic columns from the pivot positions of the original matrix

- The row space or left-hand range: $R(\mathbf{A^T}) = \{\mathbf{A^Ty}\}|\mathbf{y} \in R^m, \subseteq R^n$
  The spanning set for a row space are the non-zero rows of the reduced echelon form of the original matrix

- The nullspace: $N(\mathbf{A}) = \{\mathbf{x}|\mathbf{Ax} = \mathbf{0}\}, \subseteq R^n$
  The spanning set for a nullspace is the general solution to the form **Ax=b** in terms of the free variable vectors or of the trivial solution (rank is equal to n)

- The left-hand nullspace: $N(\mathbf{A^T}) = \{\mathbf{y}|\mathbf{A^Ty} = \mathbf{0}\}, \subseteq R^m$ The spanning set for a left-hand nullspace are the last m-rank rows of the singular matrix P resulting in **PA=U** with U being in row echelon form (A—I to U—P)

The same rows and columns between two same shaped matrices yield the same nullspaces/left-hand nullspaces.

1.3 Linear Independence

1.4 Basis and Dimension

1.5 More about Rank

1.6 Classical Least-Squares

1.7 Linear Transformations

1.8 Change of Basis and Similarity

1.9 Invariant Subspaces