EXPERIMENT 5: HARMONIC OSCILLATOR PART 1: SPRING OSCILLATOR

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Analysis of Damped and Undamped Vertical Oscillations

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Abstract:

This report examines the motion of a spring-mass system oscillating vertically in both damped and undamped cases. Theoretical and Experimental frequencies are determined and compared from equations and detailed analysis respectively. To inspect the relative motion of the system as it oscillates up and down, the spring and mass are hung from a Force Sensor that records Voltages over time. Analysis of Voltage vs Time plots for both free and damped oscillation reveal the experimental frequencies of oscillation, which are compared to their theoretical values evaluated mathematically. A deeper examination into the damped case can provide additional information about the system, such as the damping time τ, quality factor Q, and damping term b. To damp oscillation, the system is dropped through an aluminum tube. Embedded metal in the mass interacts with the tube resulting in a damping force. Critical to this report is determining if that force is proportional to the system's velocity. Therefore, it must be shown that there is no discernible relation between the ratios of successive oscillation amplitudes. Experimental frequencies were $f_0 = (0.684 \pm 0.003)\text{Hz}$ and $f_{\text{damped}} = (0.681 \pm 0.002)\text{Hz}$. Theoretical frequencies were $f_0 = (0.685 \pm 0.001)\text{Hz}$ and $f_{\text{damped}} = (0.681 \pm 0.002)\text{Hz}$. All frequencies were consistent. $\tau = (12 \pm 1)\text{seconds}$, $b = (0.029 \pm 0.005)\text{kg/s}$ and $Q = 26 \pm 4$. The damping force was proportional to velocity.

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Introduction

The Force acting on an object undergoing Simple Harmonic Motion is described by Hooke's Law $F = -kx$. Newton's Second Law also defines Force as $F = ma$. (If a damping force $F = -bv$ is added to the system, Newton's second law becomes $F = ma - bv$). Setting these equal produces a solvable differential equation. When solved, one may completely describe the motion of the object in SHM. The purpose of this experiment is to determine frequencies of free and damped oscillatory motion theoretically and analytically and compare, determine the damping time and quality factor for a damped oscillation, and to prove that the damping force added to the system is proportional to the system's velocity. To determine each of these, I released a mass-spring system hung vertically from a Force Sensor and added a damping force by releasing through an aluminum tube. Detailed analysis of Voltage readings over time provide all necessary information.

Methods

Before oscillations, the spring constant needed to be measured. Suspend Pasco Scientific Force Sensor vertically above the ground using a Pasco Scientific Stand. Hang spring from Force sensor hook. Hang five separate masses (50, 100, 150, 200, 250 g) and measure the distance from the ground to the bottom of the spring using a Starrett Meter Stick. Create a plot of the Tension (mg) vs the distance measured (in m) and take the magnitude of the slope of the linear fit to be the spring constant. Weigh the large unknown mass with embedded magnets using Ohaus Dial-O-Gram Balance, then attach to vertical spring. Plug same Force Sensor into Pasco 850 Universal Interface (DAQ). Choose User Defined Sensor in Volts. Create a Table with columns of Time and Voltage from the Sensor. Set frequency to 20 Hz and Continuous Mode. Release the mass downward and allow the system to oscillate for about 10 seconds, and Record. Stop recording after 1 more minute. Arrange an Aluminum Tube with length 45.75 cm, inner diameter 5.150 cm, and outer diameter 6.320 cm so that the mass will oscillate through it. Repeat the same process to record time and voltage data for a minute. Create Voltage vs Time plots for both free and damped oscillations. A few measures were taken to avoid systematic error. String was used to connect the mass to the spring and the spring to the Force Sensor to eliminate possible spring rotation. The system was allowed to oscillate for 10-20 seconds before recording data to eliminate drift in the Force Sensor recordings.

Analysis

The definition for the frequency of an undamped harmonic oscillator is Equation 5.1:[1] $f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$. If a damping force $F = -bv$ is added to the system, then Newton's second law yields Equation 5.2:[1] $m\ddot{x} = -kx - b\dot{x}$. Solving this differential equation for x yields Equation 5.3:[1] $x(t) = Ae^{i\omega t}$. Plugging this back into Equation 5.2:[1] and solving for ω results in Equation 5.5:[1] $\omega = \frac{ib}{2m} \pm \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$. If we plug ω back into Equation 5.3,[1] we can separate the exponent and produce Equation 5.6:[1] $x(t) = Ae^{i\omega_{\mathrm{d}} t}e^{-bt/2m}$ where the damping time $\tau \equiv \frac{2m}{b}$ and $\omega_{\mathrm{d}} \equiv \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$. The damping term is thus $b = \frac{2m}{\tau}$. Since $f_d = \frac{\omega_d}{2\pi}$, we produce Equation 5.8:[1] $f_d = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = f_0\sqrt{1 - \frac{b^2}{4km}}$. Equation 5.10[1] defines $f_d \equiv f_0\sqrt{1 - \frac{1}{4Q^2}}$ so we recover Equation 5.11:[1] $Q = \frac{\sqrt{km}}{b} = \frac{\tau}{2}\sqrt{\frac{k}{m}}$ or $\sqrt{\frac{k}{m}} = \frac{2Q}{\tau}$. Plugging this back in to Equation 5.1[1] gives $f_0 = \frac{Q}{\pi\tau}$. Note: subscript 0 stands for free oscillation, and d for damped oscillation.

To determine measured frequencies f_0 (free oscillation) and f_{damped} (damped oscillation), we need to look at the time when various peaks in Voltage occurred from Figures 2 and 3. The frequency can be calculated by dividing the difference in the peak number by the time between the peaks. By using various combinations of peaks, a set of frequencies for both free and damped oscillations can be collected. The combinations used were peaks 1&3; 1&4; 1&5; ... 1&14. 1&2 was not used because the peaks are too close. The average of this set is f_{best} while the uncertainty δf is calculated from Equation ii.13.[1] The decay time τ can be estimated by using Equation 5.13[1] for various combinations in the ratio of peak voltages, taking the average, and using Equation ii.13[1] to calculate uncertainty.

The Mass of the Hanging Weight: $M = (0.17370 \pm 0.00005)$ kg

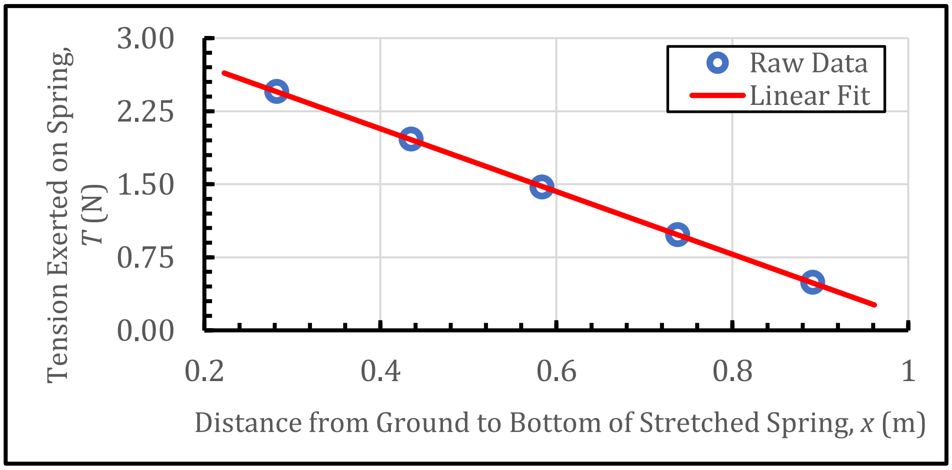

Figure 1: Determining Spring Constant using Hooke's Law. Different masses were hung vertically from a spring, and the distance x from the ground to the bottom of the stretched spring was measured (x-axis). The y-axis is the Tension Force mg exerted on the spring by each mass. The blue circles are the raw data for each mass and the red line is the linear fit to the data with equation $T = ax + b$ where $a = (-3.22 \pm 0.01)$ N/m and $b = (3.358 \pm 0.006)$ N. The magnitude of the slope a corresponds to the spring constant k. So $|a| = k = (3.22 \pm 0.01)$ N/m. Uncertainties in a, b, and k come from regression analysis.

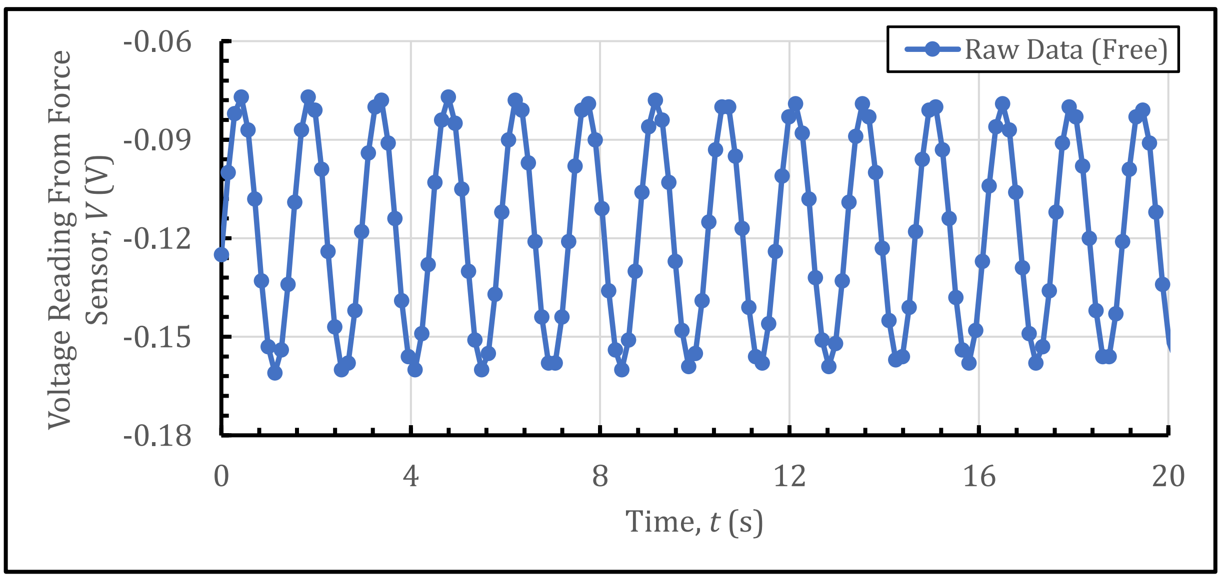

Figure 2: Free Oscillation of a Mass hung on a Vertical Spring. The spring was suspended vertically from force sensor, and the mass M was attached to the bottom of the spring. The system was released and began to oscillate freely. The Force Sensor recorded voltages over time. The blue points are the raw

data and the blue lines connect the points to show a sine wave resemblance. The frequency f_0 was determined using peak 1 as the base. Values for f_0 were calculated by taking the difference in peak # 3-14 and 1 and dividing that by the time difference between the peaks (12 total). The best value for f_0 and its uncertainty calculated using method described above. $f_0 = (0.684 \pm 0.003)$ Hz.

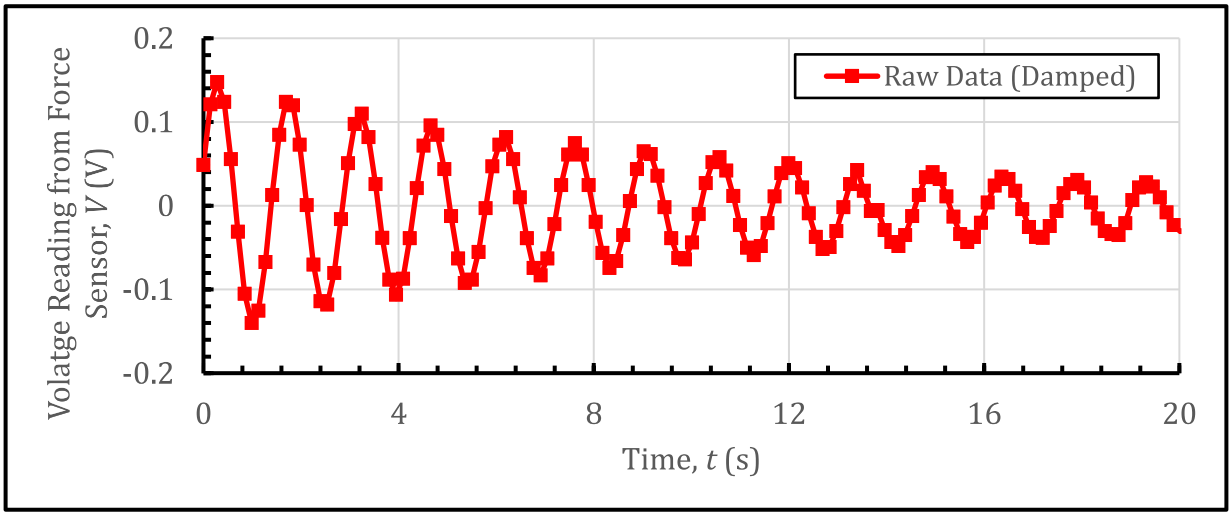

Figure 3: Damped Oscillation of a Mass hung on a Vertical Spring. The same conditions as described in Figure 2, except the system oscillated through an aluminum tube. The magnets in the mass interact with the aluminum tube resulting in a damping force and damped oscillations. The red square points are the raw data (plus 0.108 V to center the oscillations at 0) and the red lines connect the points to show a damped sine wave resemblance. The frequency f_{damped} was determined using the exact same method as for f_0. $f_{\text{damped}} = (0.681 \pm 0.002)$ Hz.

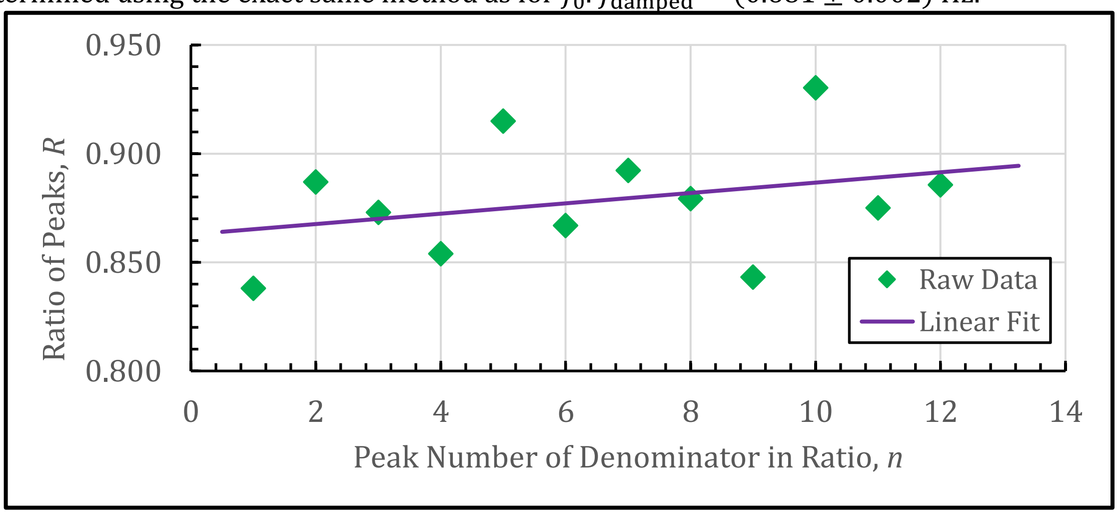

Figure 4: Ratios of Voltage Values in Consecutive Peaks from Damped Oscillation. Each green data point represents the ratio: (Voltage of peak $n + 1$)/(Voltage of peak n), where n is an integer resembling peaks from 1 to 12 in Figure 3. The purple line is the linear fit to the data with the equation $R = an + b$ where $a = (0.002 \pm 0.002)$ and $b = (0.86 \pm 0.02)$. Since the slope a is consistent with zero, we can conclude that there is no discernible trend and that the damping force is indeed proportional to velocity, $F = -bv$. Thus, $m\ddot{x} = -kx - b\dot{x}$ is an accurate model to describe damped simple harmonic motion within this experiment.

τ_{best} is determined from the average of 12 τ values derived from the 12 ratios R in Figure 4 and Equation 5.13:[1] $\tau = -\frac{\mathrm{T}}{\ln[\frac{\mathrm{V(t+T)}}{\mathrm{V(t)}}]}$ where $\frac{\mathrm{V(t+T)}}{\mathrm{V(t)}}$ are the values of each ratio R, and $\mathrm{T} = \frac{1}{\mathrm{f_d}}$ ($\mathrm{f_d}$ is the measured damped frequency from Figure 3). Uncertainty derived from Equation ii.13.[1] Therefore, $\tau = (12 \pm 1)$ seconds. To determine Q, we need a value for b, the damping term, which can be solved for in Equation 5.7[1] to get $b = \frac{2m}{\tau} = (0.029 \pm 0.005)$ kg/s. Then we can plug in values into Equation 5.11[1] to calculate $Q = (26 \pm 4)$. Q is then used to calculate predicted f_d using Equation 5.10.[1] A summary of frequencies is shown in Table 1. All uncertainties for all values derived below.

Oscillation Type	Predicted Frequency (Hz)	Measured Frequency (Hz)
Free (f_0)	0.685 ± 0.001	0.684 ± 0.003
Damped (f_d)	0.7 ± 0.1	0.681 ± 0.002

Table 1: Predicted and Measured Frequencies for both Free and Damped Oscillations. Predicted f_0 calculated from Equation 5.1.[1] Predicted f_d calculated from Equation 5.10.[1] Measured f_0 calculated in Figure 2. Measured f_d calculated in Figure 3.

Uncertainty Propagations (Ordered for Sequential Calculation)

Measured Frequencies, f_0 and f_d (Equation ii.13[1]):

$\delta f_0 = \frac{\sigma_{f_0}}{\sqrt{N}}$ and $\delta f_d = \frac{\sigma_{f_d}}{\sqrt{N}}$ where $N = 12$ in both cases and σ is the standard deviation of the set.

Damping Time, τ (Equation ii.13[1]):

$\delta\tau = \frac{\sigma_\tau}{\sqrt{N}}$ where $N = 12$ and σ is the standard dev. All τ values gathered from ratios R in Figure 4.

Damping Term, $b = \frac{2m}{\tau} = 2x$ $(x = \frac{m}{\tau})$ (Equations ii.21[1] and ii.23[1]):

$$\delta b = 2\delta x = 2b_{\text{best}}\sqrt{(\frac{\delta m}{m_{\text{best}}})^2 + (\frac{\delta\tau}{\tau_{\text{best}}})^2}$$

Q-Factor, $Q = \frac{\sqrt{km}}{b} = \frac{\sqrt{x}}{b} = \frac{y}{b}$ $(x = km, y = \sqrt{x})$ (Equations ii.23[1] and ii.24[1]):

$$\delta Q = Q_{\text{best}}\sqrt{(\frac{\delta y}{y})^2 + (\frac{\delta b}{b})^2} = Q_{\text{best}}\sqrt{(\frac{\delta x}{2x})^2 + (\frac{\delta b}{b})^2} = Q_{\text{best}}\sqrt{(\frac{\delta k}{2k_{\text{best}}})^2 + (\frac{\delta m}{2m_{\text{best}}})^2 + (\frac{\delta b}{b_{\text{best}}})^2}$$

Predicted Frequency, $f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = A\sqrt{x}$ $\left(x = \frac{k}{m}\right)$ (Equations ii.23[1] and ii.24[1]):

$$\delta f_0 = \frac{f_{0,\text{best}}}{2}(\frac{\delta x}{x}) = \frac{f_{0,\text{best}}}{2}\sqrt{(\frac{\delta k}{k_{\text{best}}})^2 + (\frac{\delta m}{m_{\text{best}}})^2}$$

Predicted Frequency, $f_d = f_0\sqrt{1 - \frac{1}{4Q^2}} = f_0\sqrt{x} = f_0 y$ $(x = 1 + AQ^{-2}, A = -\frac{1}{4}, y = \sqrt{x})$

$$\delta f_d = f_{d,\text{best}}\sqrt{(\frac{\delta f_0}{f_0})^2 + (\frac{\delta y}{y})^2} = f_{d,\text{best}}\sqrt{(\frac{\delta f_0}{f_0})^2 + (\frac{\delta x}{2x})^2} = f_{d,\text{best}}\sqrt{(\frac{\delta f_0}{f_{0,\text{best}}})^2 + (\frac{\delta Q}{Q_{\text{best}}})^2}$$

Conclusions

The goal of the experiment was to show consistency in comparing frequencies gathered theoretically and experimentally and to show the damping force is proportional to velocity, thus proving Equation 5.2[1] accurately models the damped oscillation in this experiment. Experimental frequencies were $f_0 = (0.684 \pm 0.003)$Hz and $f_\mathrm{d} = (0.681 \pm 0.002)$Hz. Theoretical frequencies were $f_0 = (0.685 \pm 0.001)$Hz and $f_\mathrm{d} = (0.681 \pm 0.002)$Hz. Given their uncertainties, the predicted and measured free and damped oscillation frequencies are consistent when compared. Since there is no discernible relation (the slope in Figure 4 $a = (0.002 \pm 0.002)$ is consistent with zero) in the ratio of peak voltages in the damped case, we can conclude the damping force is proportional to the system's velocity, thus Equation 5.2[1] was applied correctly. An important systematic uncertainty is the nonlinear motion of the oscillating system after it is dropped. This causes the system to return to a height less than it should, thus producing a smaller value for τ. To avoid this systematic error, a machine could be used to release the mass without any interference in the horizontal direction.

References

[1] Campbell, W.C. *et al*. Physics 4AL: Mechanics Lab Manual (ver. April 3, 2017). (University of California Los Angeles, Los Angeles, California).