

Thermoelasticity: Concepts, Applications, and Modeling

ME287 Project Group Report

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MS/MEng Students, Introduction to Continuum Mechanics
December 2025

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1 Background

Thermoelasticity describes how temperature changes and mechanical deformation interact in materials. Thermoelastic effects appear quite readily in a wide range of materials, from standard engineering metals such as aluminum, titanium and steels, to ceramics, polymers, and even fiber-reinforced composites. Metals commonly exhibit thermal expansion and thermally induced residual stresses in applications like welding, turbine blades, and aerospace structures exposed to aerodynamic heating. Polymers show large thermal expansion coefficients and strong temperature-dependent stiffness. Composite laminates often experience warping or residual stress due to mismatched thermal expansion between fibers and matrix while ceramics and glass are highly sensitive to thermal gradients that produce thermal shocking and cracking. These examples demonstrate that thermoelastic coupling is not a rare or exotic effect, but rather a central part of material response whenever temperature change occurs.

At its core, thermoelasticity arises because temperature changes alter the equilibrium spacing of atoms in a solid. When a material is free to expand or contract, a temperature change produces a stress-free thermal strain. However, when the material is partially or fully constrained, these thermal strains lead to internal stresses that can cause structural distortion, buckling, fatigue, or fracture. The coupling also works in the opposite direction: mechanical deformation changes the internal energy and can generate or absorb heat. In dynamic scenarios, such as rapid heating or high-frequency loading, the interaction between temperature and deformation can produce propagating thermoelastic waves, thermal damping, or localized temperature rise due to mechanical work. Overall, the qualitative picture is that temperature and deformation are not independent fields. Each field influences the other through the material's atomic structure and energy storage mechanisms.

Mathematically, thermoelasticity couples the mechanical and thermal fields through both constitutive relations and balance laws. In the simplest small-strain setting, the classically Duhamel-Neumann form expresses the stress as the elastic stress minus a thermal strain contribution proportional to the temperature change, typically written as

$$\boldsymbol{\sigma} = \mathbf{C} : (\boldsymbol{\varepsilon} - \alpha \Delta T \mathbf{I})$$

The thermal strain term $\alpha \Delta T \mathbf{I}$ encodes the idea that a temperature change produces a stress-free dilation. Thermoelasticity also modifies the first law of thermodynamics: the heat equation includes mechanical power, while the Clausius-Duhem inequality constrains admissible constitutive choices to ensure positive entropy production. Standard models use Fourier's law for heat flux, but when the classical assumption of infinite thermal signal speed becomes invalid—at short times or micro-scales—generalized laws such as the Cattaneo-Vernotte relation introduce thermal relaxation time and lead to hyperbolic temperature evolution. In finite-strain thermoelasticity, the deformation gradient is often multiplicatively decomposed into thermal and elastic parts to represent a temperature-dependent stress-free configuration. These mathematical ingredients form the backbone of how thermoelastic effects enter continuum mechanics.

The development of thermoelasticity spans nearly two centuries. Early foundations were laid in the 19th century: Duhamel first postulated coupling between temperature fields and elastic deformation, Neumann developed the linear thermoelastic theory for homogeneous materials, and Kelvin contributed to early studies on thermal effects in metals. In the mid-20th century, interest surged due to aerospace and nuclear appli-

cations, prompting researchers like Biot and Weiner to formalize and standardize the theory within modern continuum mechanics. By the late 20th century, limitations of classical Fourier-based thermoelasticity became apparent, particularly for high-rate or small-scale phenomena. This period saw major advancements: Cattaneo and Vernotte introduced thermal relaxation time; Lord and Shulman developed the first generalized thermoelastic model with finite heat propagation speed; and Green, Lindsay, and Naghdi expanded the framework into a family of extended theories capable of capturing wave-like thermal transport. These contributions collectively shaped today's understanding of thermoelastic coupling in engineering and physics.

2 Constitutive Analysis

Symbol	Definition
\mathbf{F}	Deformation gradient
θ	Temperature field
$\mathbf{g} = \nabla\theta$	Temperature gradient
ψ	Helmholtz free energy
η	Entropy
\mathbf{T}	Cauchy stress tensor
\mathbf{T}_{RR}	Second Piola Stress
\mathbf{q}	Heat flux vector (spatial form)
ρ	Mass density in the current configuration
ρ_R	Mass density in the reference configuration
$J = \det \mathbf{F}$	Volume ratio (current volume / reference volume)
$\mathbf{C} = \mathbf{F}^\top \mathbf{F}$	Right Cauchy–Green deformation tensor
\mathbf{U}	Right stretch tensor (from polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$)
\mathbf{R}	Rotation tensor (polar decomposition)
\mathbf{D}	Rate-of-deformation tensor ($\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^\top)$)
\mathbf{L}	Velocity gradient ($\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$)
\mathbf{Q}	Arbitrary proper orthogonal tensor (rigid rotation)
ψ_R	Free energy per reference volume ($\psi_R = \rho_R \tilde{\psi}$)
η_R	Entropy per reference volume ($\eta_R = \rho_R \tilde{\eta}$)
\mathbf{q}_R	Heat flux per reference volume ($\mathbf{q}_R = \rho_R \tilde{\mathbf{q}}$)

Table 1: Definitions of variables used in the thermoelastic constitutive framework.

2.1 FINITE DEFORMATION THERMO ELASTICITY

2.1.1 Constitutive Relations

Propose the following constitutive relations

$$\psi = \hat{\psi}(\mathbf{F}, \theta, \mathbf{g}), \quad \mathbf{T} = \hat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{g}), \quad \mathbf{q} = \hat{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{g}), \quad \eta = \hat{\eta}(\mathbf{F}, \theta, \mathbf{g})$$

ψ : Helmholtz free energy, η : entropy, \mathbf{T} : Cauchy stress, \mathbf{q} : heat flux

2.1.2 Frame Indifference

$$\mathbf{F}^* = \mathbf{Q}\mathbf{F}, \quad \mathbf{T}^* = \mathbf{Q}\mathbf{F}, \quad \mathbf{q}^* = \mathbf{q}, \quad \mathbf{g}^* = \mathbf{g}$$

$$\begin{cases} \hat{\psi}(\mathbf{F}, \theta, \mathbf{g}) = \hat{\psi}(\mathbf{Q}\mathbf{F}, \theta, \mathbf{g}) \\ \hat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{g}) = \mathbf{Q} \hat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{g}) \mathbf{Q}^\top \\ \hat{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{g}) = \hat{\mathbf{q}}(\mathbf{Q}\mathbf{F}, \theta, \mathbf{g}) \\ \hat{\eta}(\mathbf{F}, \theta, \mathbf{g}) = \hat{\eta}(\mathbf{Q}\mathbf{F}, \theta, \mathbf{g}) \end{cases}$$

Let $\mathbf{Q}(t) = \mathbf{R}^\top(t)$ from polar decomposition:

$$\hat{\psi}(\mathbf{R}^\top \mathbf{F}, \theta, \mathbf{g}) = \hat{\psi}(\mathbf{R}^\top \mathbf{R}\mathbf{U}, \theta, \mathbf{g}) = \hat{\psi}(\mathbf{U}, \theta, \mathbf{g}) \equiv \tilde{\psi}(\mathbf{C}, \theta, \mathbf{g})$$

$$\mathbf{R}^\top \hat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{g}) \mathbf{R} = \hat{\mathbf{T}}(\mathbf{U}, \theta, \mathbf{g}) \equiv \tilde{\mathbf{T}}(\mathbf{C}, \theta, \mathbf{g})$$

$$\hat{\mathbf{q}}(\mathbf{R}^\top \mathbf{F}, \theta, \mathbf{g}) = \tilde{\mathbf{q}}(\mathbf{C}, \theta, \mathbf{g}), \quad \hat{\eta}(\mathbf{R}^\top \mathbf{F}, \theta, \mathbf{g}) = \tilde{\eta}(\mathbf{C}, \theta, \mathbf{g})$$

2.1.3 Thermodynamic Restrictions

Second law of Thermodynamics:

$$\rho(\dot{\psi} + \eta \dot{\theta}) - \mathbf{T} : \mathbf{D} + \frac{1}{\theta} \mathbf{q} \cdot \mathbf{g} \leq 0$$

Kinematic identity:

$$\begin{aligned} \dot{\mathbf{C}} &= \dot{\mathbf{F}}^\top \mathbf{F} + \mathbf{F}^\top \dot{\mathbf{F}} = \mathbf{F}^\top (\mathbf{L}^\top + \mathbf{L}) \mathbf{F} = 2 \mathbf{F}^\top \mathbf{D} \mathbf{F} \\ \mathbf{D} &= \frac{1}{2} \mathbf{F}^{-\top} \dot{\mathbf{C}} \mathbf{F}^{-1} \\ \mathbf{T} : \mathbf{D} &= \frac{1}{2} \mathbf{T} : \mathbf{F}^{-\top} \dot{\mathbf{C}} \mathbf{F}^{-1} = \frac{1}{2} \mathbf{F}^{-1} \mathbf{T} \mathbf{F}^{-\top} : \dot{\mathbf{C}} \end{aligned}$$

And so, in the new referential,

$$\begin{aligned} \mathbf{T}_R &= J \mathbf{T} \mathbf{F}^{-\top} \\ \mathbf{T}_{RR} &= \mathbf{F}^{-1} \mathbf{T}_R = J \mathbf{F}^{-1} \mathbf{T} \mathbf{F}^{-\top} \\ \therefore \quad \mathbf{T} : \mathbf{D} &= \frac{1}{2} J^{-1} \mathbf{T}_{RR} : \dot{\mathbf{C}} \end{aligned}$$

Now if we define energy per reference volume:

$$\psi_R = \rho_R \tilde{\psi}(\mathbf{C}, \theta, \mathbf{g}), \quad \eta_R = \rho_R \tilde{\eta}(\mathbf{C}, \theta, \mathbf{g}) \quad \mathbf{q}_R = \rho_R \tilde{\mathbf{q}}(\mathbf{C}, \theta, \mathbf{g}) \quad J = \rho_R / \rho$$

If we rewrite the 2nd law in referential form:

$$J^{-1}(\dot{\psi}_R + \eta_R \dot{\theta} - \frac{1}{2} \mathbf{T}_{RR} : \dot{\mathbf{C}} + \frac{1}{\theta} \mathbf{q}_R \cdot \mathbf{g}) \leq 0$$

Expand free energy via chain rule:

$$\dot{\psi}_R = \frac{\partial \tilde{\psi}_R}{\partial \mathbf{C}} : \dot{\mathbf{C}} + \frac{\partial \tilde{\psi}_R}{\partial \theta} \dot{\theta} + \frac{\partial \tilde{\psi}_R}{\partial \mathbf{g}} \cdot \dot{\mathbf{g}}$$

$$\left(\frac{\partial \tilde{\psi}_R}{\partial \mathbf{C}} - \frac{1}{2} \mathbf{T}_{RR} \right) : \dot{\mathbf{C}} + \left(\frac{\partial \tilde{\psi}_R}{\partial \theta} + \tilde{\eta}_R \right) \dot{\theta} + \frac{1}{\theta} \mathbf{q}_R \cdot \mathbf{g} + \frac{\partial \tilde{\psi}_R}{\partial \mathbf{g}} \cdot \dot{\mathbf{g}} \leq 0$$

Thermodynamic consistency requires:

$$\mathbf{T}_{RR} = 2 \frac{\partial \tilde{\psi}_R(\mathbf{C}, \theta)}{\partial \mathbf{C}}, \quad \eta_R = - \frac{\partial \tilde{\psi}_R(\mathbf{C}, \theta)}{\partial \theta}$$

Note that T_{RR} and η_R are independent of \mathbf{g} , and thus we can update our constitutive relations as a function of just \mathbf{C} and θ .

From there we can get the dissipation term:

$$\frac{1}{\theta} \mathbf{q}_R \cdot \mathbf{g} + \frac{\partial \tilde{\psi}_R}{\partial \mathbf{g}} \cdot \dot{\mathbf{g}} \leq 0$$

2.1.4 Entropy balance in the referential form

$$\dot{\psi}_R + \eta_R \dot{\theta} = \frac{1}{2} \mathbf{T}_{RR} : \dot{\mathbf{C}} - \text{div}(\mathbf{q}_R) + q_R$$

If we define the specific free energy

$$\dot{\psi}_R + \eta_R \dot{\theta} = \dot{\epsilon}_R - \theta \dot{\eta}_R$$

$$\therefore \dot{\epsilon}_R = \frac{1}{2} \mathbf{T}_{RR} : \dot{\mathbf{C}} + \theta \dot{\eta}_R$$

And finally we can have the new entropy balance

$$\boxed{\theta \dot{\eta}_R = \dot{\epsilon}_R - \frac{1}{2} \mathbf{T}_{RR} : \dot{\mathbf{C}} = -\text{div}(\mathbf{q}_R) + q_R}$$

2.1.5 Energy balance in the referential form

By definition

$$\theta \dot{\eta}_R = \theta \frac{\partial \eta_R(\mathbf{C}, \theta)}{\partial \mathbf{C}} : \mathbf{C} + \theta \frac{\partial \eta_R(\mathbf{C}, \theta)}{\partial \theta} \dot{\theta}$$

Which leads to

$$\theta \dot{\eta}_R = \frac{1}{2} \theta \mathbf{M}(\mathbf{C}, \theta) : \mathbf{C} + c(\mathbf{C}, \theta)$$

The new version of the energy balance is thus

$$\boxed{c(\mathbf{C}, \theta) \dot{\theta} = -\text{div}(\tilde{\mathbf{q}}_R(\mathbf{C}, \theta, \mathbf{g})) + \frac{1}{2} \theta \mathbf{M}(\mathbf{C}, \theta) : \dot{\mathbf{C}} + q_R}$$

2.1.6 Material Moduli

Establish the Gibbs and Maxwell relation:

$$\begin{cases} \dot{\psi}_R = \frac{\partial \tilde{\psi}_R}{\partial \mathbf{C}} \cdot \mathbf{C} + \frac{\partial \tilde{\psi}_R}{\partial \theta} \cdot \theta = \frac{1}{2} \mathbf{T}_{RR} : \dot{\mathbf{C}} - \tilde{\eta}_R \dot{\theta} \\ \frac{\partial T_{RR}}{\partial \theta} = \frac{\partial^2 \tilde{\psi}_R}{\partial \mathbf{C} \partial \theta} = \frac{\partial \tilde{\eta}_R}{\partial \mathbf{C}} \end{cases}$$

Then, we can introduce

Elasticity tensor:

$$\mathbb{C}(\mathbf{C}, \theta) = 2 \frac{\partial^2 \tilde{\psi}_R}{\partial \mathbf{C}^2}$$

Stress-temperature modulus:

$$\mathbf{M}(\mathbf{C}, \theta) = \frac{\partial \mathbf{T}_{RR}}{\partial \theta} = -2 \frac{\partial \tilde{\eta}_R}{\partial \mathbf{C}} = 2 \frac{\partial^2 \tilde{\psi}_R}{\partial \mathbf{C} \partial \theta}$$

Heat capacity:

$$c(\mathbf{C}, \theta) = \frac{\partial \tilde{\epsilon}(\mathbf{C}, \theta)}{\partial \theta} = \frac{\partial \tilde{\psi}_R(\mathbf{C}, \theta)}{\partial \theta} + \tilde{\eta}_R(\mathbf{C}, \theta) + \frac{\partial \tilde{\eta}_R(\mathbf{C}, \theta)}{\partial \theta} \theta = \theta \frac{\partial \tilde{\eta}_R(\mathbf{C}, \theta)}{\partial \theta} = -\frac{\partial^2 \tilde{\psi}_R(\mathbf{C}, \theta)}{\partial \theta^2}$$

2.2 SMALL-STRAIN THERMOELASTICITY

2.2.1 Constitutive relation

$$\mathbf{H} = \mathbf{F} - \mathbf{1} = \nabla u, \quad \mathbf{E}^G = \frac{1}{2}(\mathbf{C} - \mathbf{1}) \approx \boldsymbol{\varepsilon}$$

$$\mathbf{C} \approx \mathbf{1} + 2\boldsymbol{\varepsilon}, \quad J \approx 1$$

Momentum balance:

$$\rho \ddot{\mathbf{u}} = \operatorname{div} \mathbf{T} + \mathbf{b}$$

Energy balance:

$$c \dot{\theta} = -\operatorname{div} \mathbf{q} + \frac{1}{2} \theta \mathbf{M} : \dot{\boldsymbol{\varepsilon}} + q$$

We can now write new relations using small deformation instead of C

$$\begin{cases} \psi = \bar{\psi}(\boldsymbol{\varepsilon}, \theta) \\ \mathbf{T} = 2\mathbf{F}\mathbf{T}_{RR} = 2\mathbf{F} \frac{\partial \bar{\psi}(\boldsymbol{\varepsilon}, \theta)}{\partial \boldsymbol{\varepsilon}} \\ \eta = -\frac{\partial \bar{\psi}(\boldsymbol{\varepsilon}, \theta)}{\partial \theta} \\ \mathbf{q} = \bar{\mathbf{q}}(\boldsymbol{\varepsilon}, \theta, \mathbf{g}) = -K\mathbf{g} \\ \mathbf{K} = \hat{\mathbf{K}}(\boldsymbol{\varepsilon}, \theta) \end{cases}$$

2.2.2 Final equation

Energy balance:

$$c \dot{\theta} = -\operatorname{div}(\mathbf{q}) + \frac{1}{2} \mathbf{M} : \dot{\boldsymbol{\varepsilon}} + q$$

Free energy imbalance:

$$\boxed{\frac{1}{\theta} \bar{q}(\varepsilon, \theta, \mathbf{g}) \cdot \mathbf{q} + \frac{\partial \bar{\psi}_R(\varepsilon, \theta)}{\partial \mathbf{g}} \cdot \dot{\mathbf{g}} \leq 0}$$

Entropy balance:

$$\boxed{\theta \dot{\eta} = -\text{div}(\mathbf{q}) + q}$$

Evolution equation for the temperature

$$c\dot{\theta} = \text{div}(\tilde{\mathbf{K}}(\varepsilon, \theta)\mathbf{g}) + \theta \mathbf{M}(\varepsilon, \theta) : \dot{\varepsilon} + q$$

Now, if we assume c and \mathbf{K} constant

$$\boxed{c\dot{\theta} = \mathbf{K} : \text{grad}(\mathbf{g}) + \theta \mathbf{M}(\varepsilon, \theta) : \dot{\varepsilon} + q}$$

2.3 LINEAR THERMOELASTICITY

Taylor expansion at $\theta = \theta_0$, $\psi = \hat{\psi}(\mathbf{0}, \theta_0)$, $\varepsilon = \mathbf{0}$

$$\begin{aligned} \psi &= \bar{\psi}(\mathbf{0}, \theta_0) + \left(\frac{\partial \bar{\psi}(\varepsilon, \theta)}{\partial \varepsilon_{pq}} \right)_{\varepsilon=\mathbf{0}, \theta=\theta_0} \varepsilon_{pq} + \left(\frac{\partial \bar{\psi}(\varepsilon, \theta)}{\partial \theta} \right)_{\varepsilon=\mathbf{0}, \theta=\theta_0} (\theta - \theta_0) \\ &\quad + \left(\frac{\partial^2 \bar{\psi}(\varepsilon, \theta)}{\partial \varepsilon_{pq} \partial \varepsilon_{rs}} \right)_{\varepsilon=\mathbf{0}, \theta=\theta_0} \varepsilon_{pq} \varepsilon_{rs} + \left(\frac{\partial^2 \bar{\psi}(\varepsilon, \theta)}{\partial \varepsilon_{pq} \partial \theta} \right)_{\varepsilon=\mathbf{0}, \theta=\theta_0} \varepsilon_{pq} (\theta - \theta_0) + \left(\frac{\partial^2 \bar{\psi}(\varepsilon, \theta)}{\partial \theta^2} \right)_{\varepsilon=\mathbf{0}, \theta=\theta_0} (\theta - \theta_0)^2 \end{aligned}$$

If we assume that there's no free energy when $\theta = \theta_0$ and $\varepsilon = \mathbf{0}$

$$\psi = \frac{1}{2} \mathbf{M} : \mathbb{C} \varepsilon + (\theta - \theta_0) \mathbf{M} : \varepsilon - \frac{1}{2} \frac{c}{\theta_0} (\theta - \theta_0)^2$$

Which yields

$$\boxed{\begin{aligned} \mathbf{T} &= \frac{\partial \psi}{\partial \varepsilon} = \frac{1}{2} \mathbb{C} \varepsilon + \mathbf{M}(\theta - \theta_0) \\ \eta &= \frac{\partial \psi}{\partial \theta} = -\mathbf{M} : \varepsilon + \frac{c}{\theta_0} (\theta - \theta_0) \\ \rho \ddot{u} &= \text{div} \left(\frac{1}{2} \mathbb{C} \varepsilon + \mathbf{M}(\theta - \theta_0) \right) + b_0 \end{aligned}}$$

2.4 MATERIAL SYMMETRY CONSIDERATION

2.4.1 Isotropic representation theory

$$\begin{cases} \mathbb{C} = 2\mu \mathbb{I}^{sym} + \lambda \mathbf{1} \otimes \mathbf{1} \\ \mathbf{M} = \beta \mathbf{1} \\ \mathbf{K} = k \mathbf{1} \end{cases}$$

with λ and μ the Lamé moduli, β the stress-temperature modulus, \mathbf{K} thermal conductivity and $\kappa = 2\lambda + \frac{2}{3}\mu$ the isothermal bulk modulus

$$\boxed{\begin{aligned}\psi &= \mu |\varepsilon|^2 + \frac{\lambda}{2} \text{tr}(\varepsilon)^2 + \beta(\theta - \theta_0) \text{tr}(\varepsilon) - \frac{c}{2\theta} (\theta - \theta_0)^2 \\ \mathbf{T} &= 2\mu\varepsilon + \lambda \text{tr}(\varepsilon) \mathbf{1} + \beta(\theta - \theta_0) \mathbf{1} \\ \eta &= -\beta \text{tr}(\varepsilon) + \frac{c}{\theta} (\theta - \theta_0) \\ \mathbf{K} &= k\mathbf{g} \\ \rho \ddot{u} &= \text{div}(2\mu\varepsilon + \lambda \text{tr}(\varepsilon) \mathbf{1} + \beta(\theta - \theta_0) \mathbf{1}) + b_0\end{aligned}}$$

2.4.2 Redefine Moduli

$$\begin{aligned}\varepsilon &= \frac{1}{2\mu} \left(\mathbf{T} - \frac{\lambda}{2\mu + 3\lambda} (\text{tr } \mathbf{T}) \mathbf{1} \right) + \alpha(\theta - \theta_0) \mathbf{1}, \\ \alpha^{\text{def}} &= -\frac{\beta}{2\mu + 3\lambda}\end{aligned}$$

Express the stress-strain relation in terms of E , ν , α (more useful for applications)

$$\begin{aligned}\varepsilon &= \frac{1}{E} [(1 + \nu) \mathbf{T} - \nu (\text{tr } \mathbf{T}) \mathbf{1}] + \alpha(\theta - \theta_0) \mathbf{1}. \\ \mathbf{T} &= \frac{E}{1 + \nu} \left[\varepsilon + \frac{\nu}{1 - 2\nu} (\text{tr } \varepsilon) \mathbf{1} - \frac{1 + \nu}{1 - 2\nu} \alpha(\theta - \theta_0) \mathbf{1} \right]\end{aligned}$$

Express the stress-strain using specific constants of Aluminum 7075 in Table 2.

$$\varepsilon \approx 1.86 \times 10^{-11} \mathbf{T} - 4.60 \times 10^{-12} (\text{tr } \mathbf{T}) \mathbf{1} + 2.34 \times 10^{-5} (\theta - \theta_0) \mathbf{1}$$

$$\mathbf{T} \approx 5.39 \times 10^{10} [\varepsilon + 0.971 (\text{tr } \varepsilon) \mathbf{1} - 9.16 \times 10^{-5} (\theta - \theta_0) \mathbf{1}]$$

2.5 Analytical Solution

For our unit test we applied a linear ramp temperature field onto a cube and modeled it as one element. For this scenario we assumed constant q .

Energy balance:

$$q = c \dot{\theta}$$

Thermoelastic relations (isotropic, linear):

$$\mathbf{T} = \frac{E\alpha}{1 - 2\nu} (\theta - \theta_0) \mathbf{1}, \quad \varepsilon = \alpha(\theta - \theta_0) \mathbf{1}.$$

Case 1: Constrained Cube ($\varepsilon = 0$)

Temperature evolution:

$$\theta(t) = \theta_0 + \frac{q}{c} t.$$

Stress:

$$\mathbf{T}(t) = -\frac{E\alpha}{1-2\nu} [\theta(t) - \theta_0] \mathbf{1} = -\frac{E\alpha}{1-2\nu} \left(\frac{q}{c}\right) t \mathbf{1}.$$

Final result (compressive stress):

$$\boxed{\mathbf{T}(t) = -\frac{E\alpha}{1-2\nu} \frac{q}{c} t \mathbf{1}}$$

Case 2: Unconstrained Cube ($\mathbf{T} = 0$)

Free thermal strain:

$$\boldsymbol{\varepsilon}(t) = \alpha(\theta(t) - \theta_0) \mathbf{1}.$$

Displacement field (uniform expansion):

$$\mathbf{u}(x, t) = \alpha(\theta(t) - \theta_0)(x, y, z)^T.$$

New cube side length:

$$L(t) = L [1 + \alpha(\theta(t) - \theta_0)].$$

The unconstrained cube expands freely with no stress.

3 Numerical Tests

We implement a finite-strain isotropic thermoelastic material in ABAQUS/EXPLICIT via a user subroutine **VUMAT**. The model uses a multiplicative split of the deformation gradient, a Hencky (logarithmic) elastic strain, and a thermal contribution acting only on the volumetric part of the stress. Two temperature-update modes are implemented in the same **VUMAT**: a prescribed linear temperature ramp for verification on an unconstrained cube, and an adiabatic thermoelastic mode for simulations with external loads and constraints.

3.1 Kinematics and Strain Measure

At each increment and integration point, ABAQUS provides the old and new deformation gradients $\mathbf{F}_T, \mathbf{F}_\tau$, the new right stretch \mathbf{U}_τ , the old Cauchy stress $\boldsymbol{\sigma}^n$, and the internal variables **STATE_OLD**. The constitutive model assumes a multiplicative decomposition

$$\mathbf{F}_\tau = \mathbf{F}_e \mathbf{F}_\theta,$$

where \mathbf{F}_e is the elastic part and \mathbf{F}_θ is a thermal distortion tensor stored in **STATE** (initialized as \mathbf{I}). The elastic deformation gradient is

$$\mathbf{F}_e = \mathbf{F}_\tau \mathbf{F}_\theta^{-1}.$$

The elastic right Cauchy–Green tensor is $\mathbf{C}_e = \mathbf{F}_e^\top \mathbf{F}_e$. A spectral decomposition routine (**SPECTRAL**) returns eigenpairs

$$\mathbf{C}_e = \mathbf{Q} \operatorname{diag}(\lambda_1^2, \lambda_2^2, \lambda_3^2) \mathbf{Q}^\top,$$

with principal elastic stretches λ_i . The elastic Hencky strain is then

$$\mathbf{E}_e = \mathbf{Q} \operatorname{diag}(\ln \lambda_1, \ln \lambda_2, \ln \lambda_3) \mathbf{Q}^\top.$$

We split \mathbf{E}_e into volumetric and deviatoric parts,

$$\text{tr}(\mathbf{E}_e) = E_{e,11} + E_{e,22} + E_{e,33}, \quad \mathbf{E}_e^{\text{dev}} = \mathbf{E}_e - \frac{1}{3} \text{tr}(\mathbf{E}_e) \mathbf{I},$$

and the elastic Jacobian is $J_e = \det \mathbf{F}_e = \exp(\text{tr}(\mathbf{E}_e))$.

3.2 Constitutive Law and Thermal Coupling

The material is isotropic with Young's modulus E and Poisson ratio ν read from `PROPS(1:2)`. The shear and bulk moduli are

$$\mu = \frac{E}{2(1+\nu)}, \quad \kappa = \frac{E}{3(1-2\nu)}.$$

Thermal expansion is characterized by a coefficient α and a reference temperature T_{ref} (stored in `PROPS(3)` and `PROPS(4)`). A thermal coupling parameter

$$\beta = -3\kappa\alpha$$

is introduced so that the thermal effect enters only the spherical part of the stress. In addition, the specific heat at constant configuration c is passed as `PROPS(5)` and used for the adiabatic update.

Using the elastic Hencky strain, the deviatoric and spherical parts of a second-Piola-type stress \mathbf{T}_e in the intermediate configuration are

$$\mathbf{T}_e^{\text{dev}} = 2\mu \mathbf{E}_e^{\text{dev}}, \quad T_e^{\text{sph}} = \kappa \text{tr}(\mathbf{E}_e) + \beta(\theta - \theta_{\text{ref}}),$$

so that

$$\mathbf{T}_e = \mathbf{T}_e^{\text{dev}} + T_e^{\text{sph}} \mathbf{I}.$$

Verification mode (linear temperature ramp). For the unconstrained cube unit test, temperature is prescribed as a simple linear ramp in time. In this mode, the local temperature is taken as

$$\theta(t) = \theta_{\text{ref}} + \dot{\theta}_{\text{heat}} t,$$

where $\dot{\theta}_{\text{heat}}$ is a constant heating rate (e.g. 60°C/s), implemented in the VUMAT via the total explicit time `TOTAL_TIME` and the step size `DT`. This choice decouples the temperature field from the mechanics and allows a straightforward verification of pure thermal expansion: the cube is left unconstrained and should expand uniformly according to α and the imposed $\theta(t)$.

Adiabatic thermoelastic mode. For simulations with external forces and constraints, the VUMAT uses an adiabatic thermoelastic update that couples volumetric elastic strain and temperature through the specific heat. Under an adiabatic assumption (no heat flux), a first-order relation between $\text{tr}(\mathbf{E}_e)$ and θ leads to

$$\text{adiabatic} = 1 - \frac{\beta}{c} \text{tr}(\mathbf{E}_e), \quad \theta = \frac{\theta_{\text{ref}}}{\text{adiabatic}},$$

as implemented in the code. This expression is evaluated at each integration point and time step; the resulting θ is written back to `TEMP_NEW` and used inside $\mathbf{T}_e^{\text{sph}} = \kappa \text{tr}(\mathbf{E}_e) + \beta(\theta - \theta_{\text{ref}})$. In this way, volumetric elastic strain and temperature are coupled consistently in an adiabatic setting.

3.3 From Elastic Stress to Cauchy Stress and Abaqus Output

From the eigenpairs of \mathbf{C}_e we construct

$$\mathbf{U}_e^{-1} = \mathbf{Q} \operatorname{diag}(\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}) \mathbf{Q}^T, \quad \mathbf{R}_e = \mathbf{F}_e \mathbf{U}_e^{-1},$$

and compute a Kirchhoff-like Cauchy stress

$$\boldsymbol{\tau} = \frac{1}{J_e} \mathbf{R}_e \mathbf{T}_e \mathbf{R}_e^T.$$

To express the stress in the basis expected by ABAQUS, we use the polar decomposition of the total deformation \mathbf{F}_τ . Given \mathbf{U}_τ and its inverse \mathbf{U}_τ^{-1} , the total rotation is

$$\mathbf{R}_\tau = \mathbf{F}_\tau \mathbf{U}_\tau^{-1},$$

and the Cauchy stress returned to ABAQUS is

$$\boldsymbol{\sigma} = \mathbf{R}_\tau^T \boldsymbol{\tau} \mathbf{R}_\tau.$$

The components of $\boldsymbol{\sigma}$ are written into `STRESS_NEW`.

The internal variables stored in `STATE` are:

- \mathbf{F}_θ in `STATE(1:9)` (thermal distortion tensor),
- the deviatoric part of the elastic stress (Mandel-like) in `STATE(10:18)`,
- optional scalars (e.g. ν) in higher indices.

At each call, the subroutine also updates the specific internal energy `ENER_INTERN_NEW` from the stress power using a trapezoidal rule. Thus, for each integration point, the outputs sent back to ABAQUS are:

- updated Cauchy stress: `STRESS_NEW`,
- updated history variables: `STATE_NEW`,
- updated internal energy (and optionally inelastic energy).

ABAQUS/EXPLICIT then uses these to compute internal forces and advance the global solution; in the next increment they reappear as `STRESS_OLD`, `STATE_OLD`, and `ENER_INTERN_OLD`.

3.4 Analytical Solution Verification (Summary)

Two types of verification tests were performed, corresponding to the two temperature modes in the VUMAT:

- *Unconstrained cube with linear temperature ramp*: the cube is left traction-free, and temperature is ramped linearly in time using the verification mode. A non-adiabatic simulation is performed to verify the relation between temperature increase and thermal expansion. The numerical response reproduces uniform thermal expansion with negligible stress, consistent with the prescribed α and the imposed $T(t)$. The model used a single-mesh cube of 7075 Aluminum. The U1 displacements shown by Abaqus, as represented in Figure 1 show $1.414 \times 10^{-3} m$ of displacement. We hand-calculated a change in length of $1.404 \times 10^{-3} m$ validating our model.

- *Loaded, constrained configurations under adiabatic coupling:* for cases with external forces and kinematic constraints, the adiabatic thermoelastic mode is used. The resulting stress fields show the expected coupling between volumetric elastic strain and temperature, and the Cauchy stress remains symmetric and objective under superposed rigid-body motions.

These tests confirm that the VUMAT correctly represents the intended finite-strain isotropic thermoelastic behavior in both the prescribed-temperature verification setting and the adiabatic thermoelastic setting.

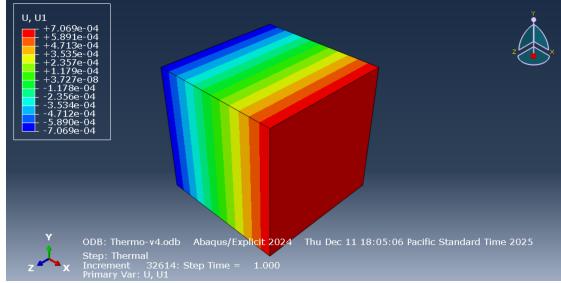


Figure 1: Unconstrained Cube Deformation

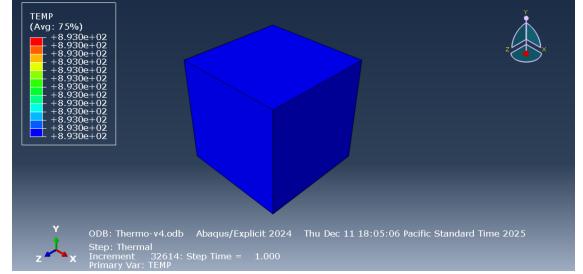


Figure 2: Unconstrained Cube Temperature Gradient

3.5 Computational analysis

3.5.1 Train Track Model

For the model that follows, we have two $10 \times 10 \times 100$ mm solid blocks that are combined to create a $20 \times 10 \times 100$ mm block. On both ends of the combined block, constraints were added so that translational and rotational movements are equal to zero. These boundary conditions can be seen in Figures 3 and 4. Four simulations were run with varying materials.

One pair of simulations tested Aluminum 7075 next to A2 Steel under compressive and tensile forces. We applied a tensile force of magnitude $10 \times 10^6 N$ and compressive force of magnitude $10 \times 10^6 N$. The deformation gradient and temperature gradient of the simulation under tension can be seen in Figures 5 and 6. The deformation gradient and temperature gradient of the simulation under compression can be seen in Figures 7 and 8. Under these loads, we saw deformations in the range of 1-2mm and temperature increases/decreases of 0-1 Kelvin. To verify our small strain assumptions, we have included SDVs 1-3 in Figures 15, 16, and 17. Showing the first 3 components of \mathbf{F} . We see that the diagonals are uniquely identity, and the off-diagonals are 0, indicating that our small $|\mathbf{H}|$ assumption is satisfied.

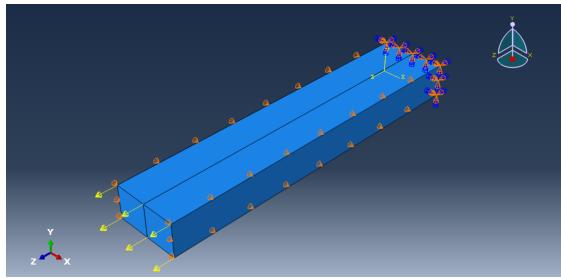


Figure 3: Train Track BC (Tensile)

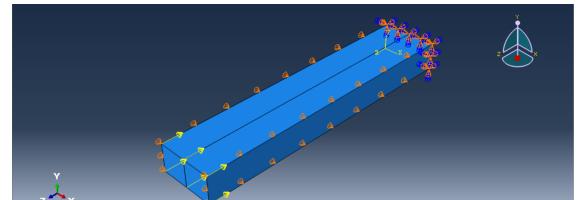


Figure 4: Train Track BC (Compressive)

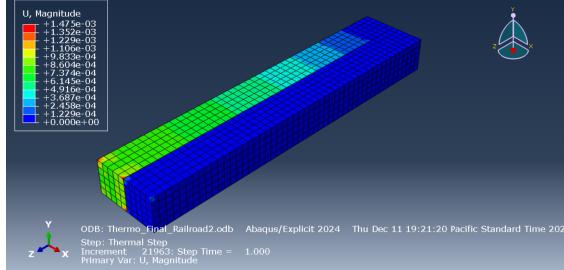


Figure 5: Al 7075 - A2 Steel Tensile Def

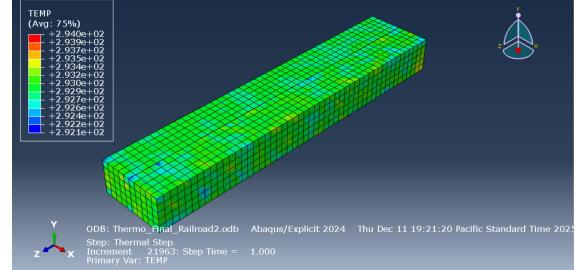


Figure 6: Al 7075 - A2 Steel Tensile Temp

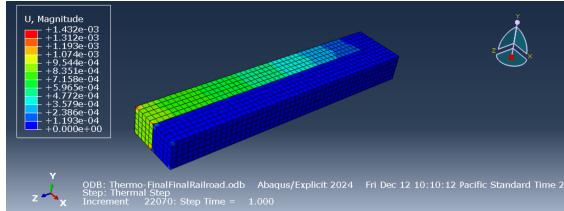


Figure 7: Al 7075 - A2 Steel Compressive Def

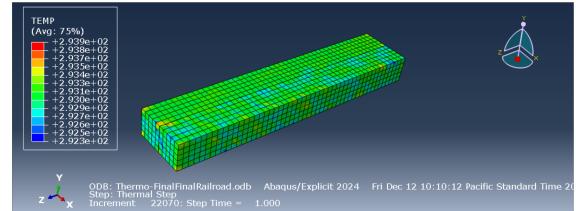


Figure 8: Al 7075 - A2 Steel Compressive Temp

The second pair of simulations was run on Aluminum 7075 next to Nitrile-Butadiene Rubber. We decided to choose a different thermoelastic material with a larger coefficient of thermal expansion to hopefully see better temperature gradients as we apply load. We applied a compressive force of magnitude $1 \times 10^5 N$. Interesting things to note include the deformation gradient and how it propagates less than in the Aluminum 7075 and A2 Steel combination. This effect can be seen in Figures 9 and 11. This is likely due to higher stiffness. This simulation saw deformations up to 15 cm and temperature increases of 7 Kelvin, clearly demonstrating a thermal coupling effect! We also applied an opposite tensile force of $1 \times 10^5 N$ and saw cooling of 7 Kelvin! Figures 10 and 12 show temperature gradient concentration in the regions where force is applied, which makes sense because stress is most concentrated here. Hip Hip Hooray!

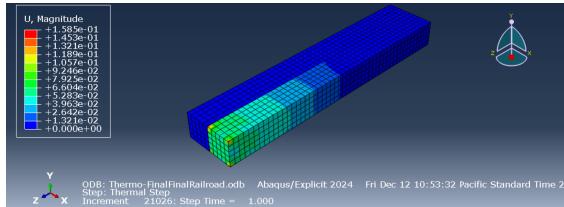


Figure 9: Al 7075 - Rubber Comp. Def

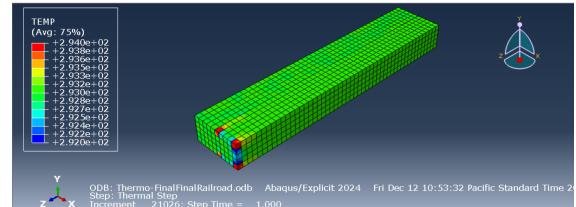


Figure 10: Al 7075 - Rubber Comp. Temp.

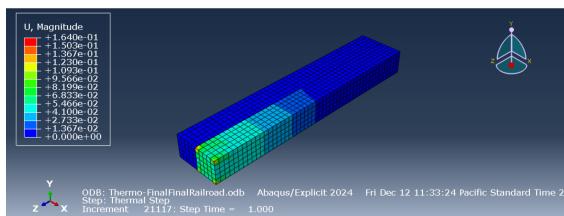


Figure 11: Al 7075 - Rubber Tens. Def.

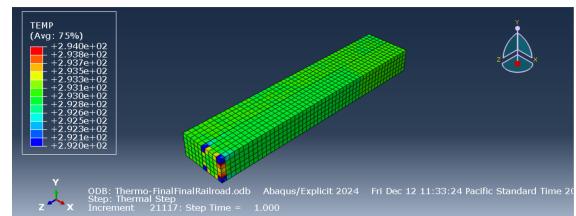


Figure 12: Al 7075 - Rubber Tens. Temp.

3.6 Mesh Refinement - Train Track

Previously, thermoelastic behavior of the train track model was simulated using a global mesh size of 2. The mesh was refined to a global size of 1.5, resulting in 6,564 elements. All other simulation conditions were held constant to enable direct comparison with the tensile simulation between Aluminum 7075 and Nitrile–Butadiene Rubber. A tensile force of $1 \times 10^{-5} N$ was applied to the left boundary, while the right boundary was fixed. For clarity, Figures 13 and 14 are compared against Figures 11 and 12. The refined mesh exhibits a larger displacement, approximately 9 cm greater than in the coarser mesh. Additionally, a larger thermal gradient is observed, with temperatures ranging from 290–294 K compared to the previous 292–294 K. This behavior is consistent with the constitutive analysis under adiabatic conditions, where tensile deformation leads to a decrease in temperature due to thermoelastic coupling. Due to time constraints, additional mesh refinements were not performed to see if this is the true final convergent solution.

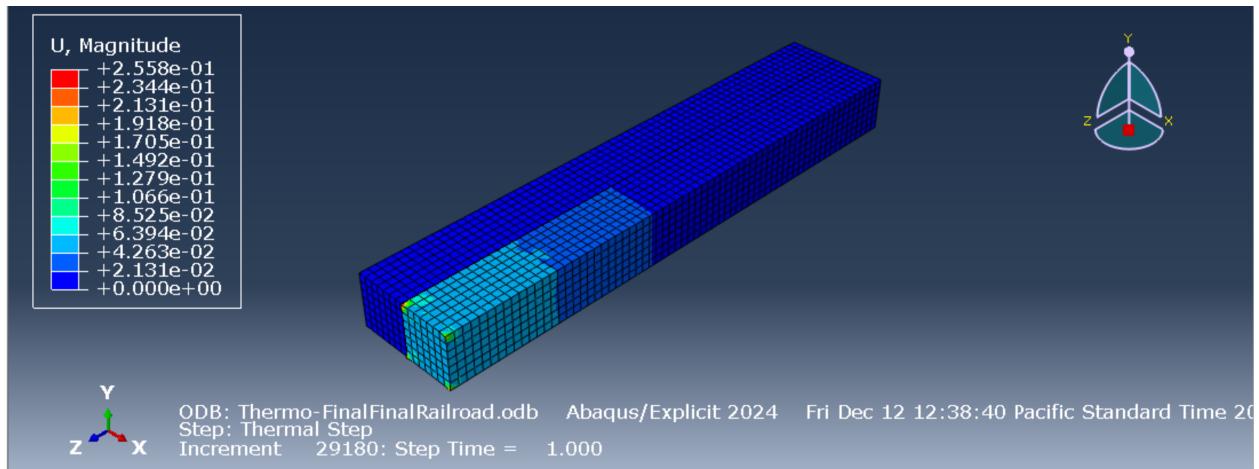


Figure 13: Mesh Refined Al 7075 - Rubber Def.

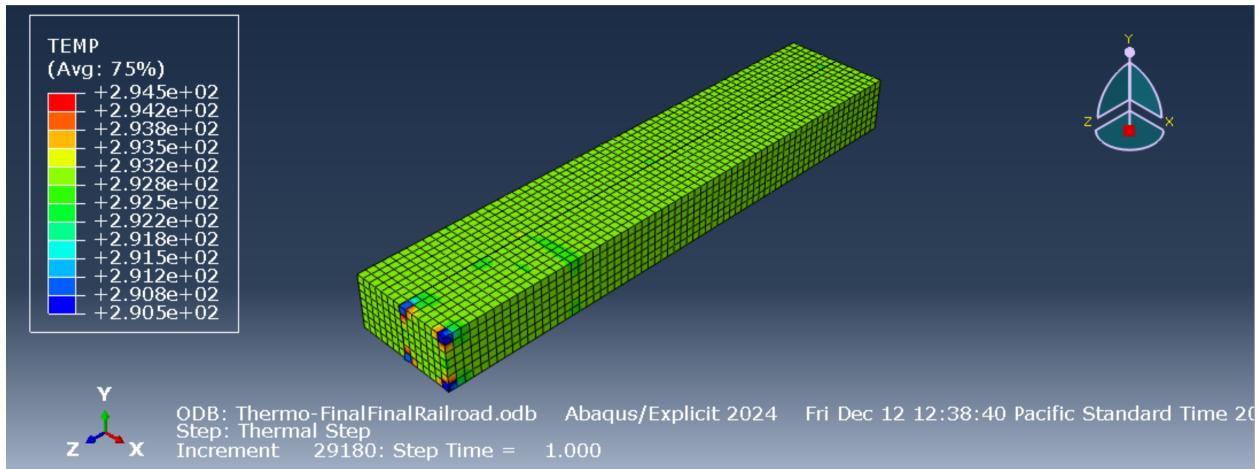


Figure 14: Mesh Refined Al 7075 - Rubber Temp.

A Appendix

A.1 Material Properties

Table 2: Thermoelastic Material Properties

7075 Aluminum

Symbol	Description	Units	Value
E	Young's modulus	Pa	71.7×10^9
ν	Poisson's ratio	—	0.33
α	Thermal expansion coefficient	1/K	2.34×10^{-5}
C_p	Specific heat	J/kg-K	960
ρ	Density	kg/m ³	2810
T_{ref}	Reference temperature	K	293
k	Thermal Conductivity	W/m-K	130

A2 Steel

Symbol	Description	Units	Value
E	Young's modulus	Pa	2.07×10^{11}
ν	Poisson's ratio	—	0.33
α	Thermal expansion coefficient	1/K	11×10^{-6}
C_p	Specific heat	J/kg-K	460
ρ	Density	kg/m ³	7861
T_{ref}	Reference temperature	K	293
k	Thermal Conductivity	W/m-K	16

Nitrile Butadiene Rubber

Symbol	Description	Units	Value
E	Young's modulus	Pa	4×10^6
ν	Poisson's ratio	—	0.41
α	Thermal expansion coefficient	1/K	150×10^{-6}
C_p	Specific heat	J/kg-K	1350
ρ	Density	kg/m ³	1000
T_{ref}	Reference temperature	K	293
k	Thermal Conductivity	W/m-K	0.25

A.2 Additional Figures

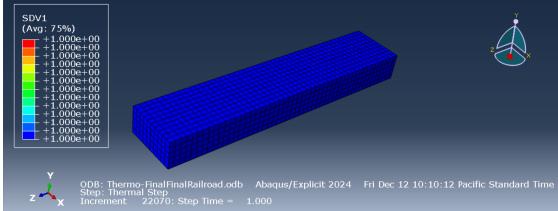


Figure 15: Train Track SDV1

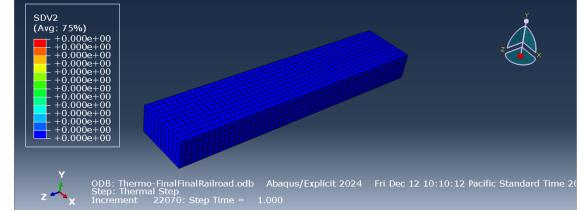


Figure 16: Train Track SDV2

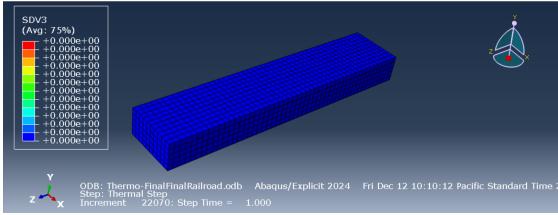


Figure 17: Train Track SDV3

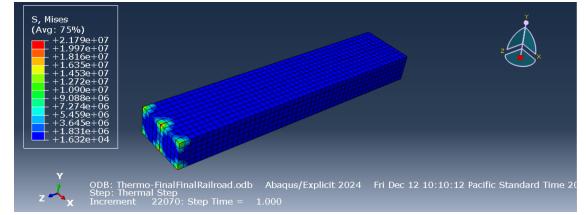


Figure 18: Train Track Mises

A.3 Abaqus VUMAT

Listing 1: Thermoelastic VUMAT implementation

```

1      real*8 F_t(3,3),F_tau(3,3),U_tau(3,3),U_inv(3,3), det_U,R_tau(3,3)
2      real*8 T_tau(3,3), Ee(3,3), det_F, E_tau(3,3), Re_tau(3,3)
3      real*8 Ftheta_t(3,3), Ftheta_tau(3,3), Fe(3,3), eigvecs(3,3), eigvals(3)
4      real*8 I_1(3,3), Te_0(3,3), stretches(3,3)
5      real*8 Ftheta_inv(3,3), Ee0(3,3), tr_Ee, dtmp
6      real*8 det_Fe, zero_m(3,3), snake, z, Ue_inv(3,3)
7      real*8 mu, kappa, dens_p, Te_sph, youngs, nu
8      real*8 spec_heat, term_adia
9      real*8 F_inv(3,3), beta, alpha, tempref, temp, dtemp
10     parameter(ONE=1., ONE_HALF=0.5, TWO=2., ZERO=0.,
11      +           ONE_THIRD=1./3., TWO_THIRD=2./3.,
12      +           THREE_HALF=1.5, THREE=3.)
13
14
15     I_1 = reshape((/1, 0, 0, 0, 1, 0, 0, 0, 1 /), (/3,3/))
16     Zero_m = reshape((/0, 0, 0, 0, 0, 0, 0, 0, 0 /), (/3,3/))
17     z=0.d0
18
19
20
21 ! Get properties defined in input file
22 !
23     youngs                  = props(01)
24     nu                      = props(02)
25     alpha = props(03)
26     tempref = props(04)
27     spec_heat = props(05) ! Add Specific Heat (c) here

```

```

28
29     mu=youngs/(2.d0*(1.d0+nu))
30     kappa=youngs/(3.d0*(1.d0-2.d0*nu))
31     beta = -3.d0*kappa*alpha
32
33
34     do km = 1,NBLOCK
35
36     !Copy the old and new Deformation gradients into F_T and F_tau
37     !respectively
38     !
39     F_T(1,1) = DEGRAD_OLD(KM,1)
40     F_T(2,2) = DEGRAD_OLD(KM,2)
41     F_T(3,3) = DEGRAD_OLD(KM,3)
42     F_T(1,2) = DEGRAD_OLD(KM,4)
43
44     F_tau(1,1) = DEGRAD_NEW(KM,1)
45     F_tau(2,2) = DEGRAD_NEW(KM,2)
46     F_tau(3,3) = DEGRAD_NEW(KM,3)
47     F_tau(1,2) = DEGRAD_NEW(KM,4)
48
49     U_tau(1,1) = STRETCH_NEW(KM,1)
50     U_tau(2,2) = STRETCH_NEW(KM,2)
51     U_tau(3,3) = STRETCH_NEW(KM,3)
52     U_tau(1,2) = STRETCH_NEW(KM,4)
53
54     if (NSHR .eq. 1) then
55         F_T(2,3) = ZERO
56         F_T(3,1) = ZERO
57         F_T(2,1) = DEGRAD_OLD(KM,5)
58         F_T(3,2) = ZERO
59         F_T(1,3) = ZERO
60
61         F_tau(2,3) = ZERO
62         F_tau(3,1) = ZERO
63         F_tau(2,1) = DEGRAD_NEW(KM,5)
64         F_tau(3,2) = ZERO
65         F_tau(1,3) = ZERO
66
67         U_tau(2,3) = ZERO
68         U_tau(3,1) = ZERO
69         U_tau(2,1) = U_tau(1,2)
70         U_tau(3,2) = ZERO
71         U_tau(1,3) = ZERO
72     else
73         F_T(2,3) = DEGRAD_OLD(KM,5)
74         F_T(3,1) = DEGRAD_OLD(KM,6)
75         F_T(2,1) = DEGRAD_OLD(KM,7)
76         F_T(3,2) = DEGRAD_OLD(KM,8)
77         F_T(1,3) = DEGRAD_OLD(KM,9)
78
79         F_tau(2,3) = DEGRAD_NEW(KM,5)
80         F_tau(3,1) = DEGRAD_NEW(KM,6)
81         F_tau(2,1) = DEGRAD_NEW(KM,7)

```

```

82      F_tau(3,2) = DEFGRAD_NEW(KM,8)
83      F_tau(1,3) = DEFGRAD_NEW(KM,9)
84
85      U_tau(2,3) = STRETCH_NEW(KM,5)
86      U_tau(3,1) = STRETCH_NEW(KM,6)
87      U_tau(2,1) = U_tau(1,2)
88      U_tau(3,2) = U_tau(2,3)
89      U_tau(1,3) = U_tau(3,1)
90  endif
91
92
93  if(total_time.eq.0 .and. step_time.eq.0) then
94    !
95    ! If first dummy step: Initialize state vars
96    !
97    State_old(km,1:9)  = (/1, 0, 0, 0, 1, 0, 0, 0, 1 /) ! Ftheta
98    State_old(km,10:18) = (/0, 0, 0, 0, 0, 0, 0, 0, 0 /) ! Dev part of Mandel
99    temp = temp_old(KM)
100
101
102  call SPECTRAL(matmul(transpose(F_tau), F_tau), eigvals, eigvecs)
103
104
105  E_tau = matmul(matmul(eigvecs,
106 + 0.5d0*reshape((/ dlog(eigvals(1)),z,z,z,dlog(eigvals(2)),z,z,z,
107 + dlog(eigvals(3)) /), (/3, 3/)), transpose(eigvecs))
108
109
110  tr_Ee = E_tau(1,1)+E_tau(2,2)+E_tau(3,3)
111
112  Ee0 = E_tau - (tr_Ee/3.d0) * I_1
113
114  T_tau = 2.d0 * mu * Ee0 + kappa * tr_Ee * I_1
115
116
117 else
118
119  ! Get state vars from last step
120
121  Ftheta_t = reshape(state_old(km,1:9), (/3, 3/))
122  Te_0 = reshape(state_old(km,10:18), (/3, 3/))
123
124  call SPECTRAL(Te_0, eigvals, eigvecs)
125
126  state_new(km,20)=0.d0
127  state_new(km,19)=nu
128
129  Ftheta_tau=Ftheta_t
130
131
132  ! Etheta=.5d0*(matmul(Ftheta_tau,transpose(Ftheta_tau))-I_1)
133
134
135  call matinv(Ftheta_tau, Ftheta_inv, dtmp)

```

```

136
137      ! Calculate Fe:
138
139      Fe=matmul(F_tau, Ftheta_inv)
140
141      call SPECTRAL(matmul(transpose(Fe), Fe), eigvals, eigvecs)
142
143
144      Ee = matmul(matmul(eigvecs,
145 + 0.5d0*reshape((/dlog(eigvals(1)),z,z,z,dlog(eigvals(2)),z,z,z,
146 + dlog(eigvals(3)) /), (/3, 3/))), transpose(eigvecs))
147
148      tr_Ee = Ee(1,1)+Ee(2,2)+Ee(3,3)
149
150      Ee0 = Ee - (tr_Ee/3.d0) * I_1
151
152
153      det_Fe=dexp(tr_Ee)
154
155      ! VERIFICATION MODE (LINEAR RAMP)
156      ! Assume you want to heat up by 60 degrees over 1.0 second
157      !heating_rate = 600.d0
158
159      ! Calculate temp based on TOTAL_TIME (Simulation Time)
160      !temp = tempref + heating_rate * (TOTAL_TIME + DT)
161      !temp = 0.d0 + heating_rate * (TOTAL_TIME + DT)
162
163      !ADIABATIC
164      term_adia = ONE - (beta / spec_heat) * tr_Ee
165      temp = tempref / term_adia
166
167
168      TEMP_NEW(KM) = temp
169      Te_sph = kappa * tr_Ee + beta*(temp - tempref)
170      Te_0 = 2.d0 * mu * Ee0
171
172
173      Ue_inv = matmul(matmul(eigvecs,
174 + reshape((/ eigvals(1)**(-.5d0),z,z,z,eigvals(2)**(-.5d0),z,z,z,
175 + eigvals(3)**(-.5d0) /), (/3, 3/))), transpose(eigvecs))
176
177      Re_tau = matmul(matmul(F_tau, Ftheta_inv), Ue_inv)
178
179
180      T_tau = matmul(Re_tau, matmul(Te_0+I_1*Te_sph,
181 + transpose(Re_tau)))/det_Fe
182
183      state_new(km,1:9) = reshape(Ftheta_tau, (/9/))
184      state_new(km,10:18) = reshape(Te_0, (/9/))      ! Dev Mandel
185
186      end if ! if(total_time.eq.0 .and. step_time.eq.0)
187
188
189      ! Do not alter anything below this line.

```

References

- [1] DesignerData, *Nitrile Butadiene Rubber*. Available at: <https://designerdata.nl/materials/plastics/rubbers/nitrile-butadiene-rubber>. Accessed on December 12, 2025.
- [2] L. Anand and S. Govindjee, *Continuum Mechanics of Solids*, Oxford University Press, Oxford, 2020.
- [3] MatWeb, *Aluminum 7075-T6 (MA7075T6) Material Data Sheet*, <https://asm.matweb.com/search/specifymaterial.asp?bassnum=ma7075t6>, Accessed: December 11, 2025.
- [4] K. Kamrin and L. Mahadevan, “Soft Catenaries,” *Journal of Fluid Mechanics*, vol. 691, pp. 165–177, 2012, doi: 10.1017/jfm.2011.466.