Introducing the notion of bond in the multi-adjoint framework*

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1 Introduction

The ability to process large amounts of data efficiently is essential for extracting useful information from databases. Formal concept analysis (known in short as FCA) has been one of the most widely used and developed mathematical theory for extracting knowledge from relational databases since its introduction in the eighties [13]. The theory deems databases as formal contexts [8], which consist of a set of objects, a set of attributes, and a relationship between these sets. FCA tools are capable of manipulating data and extracting relevant information, which is then represented using the algebraic structure of a complete lattice [?]. Several extensions of this mathematical theory have been introduced in a fuzzy environment. Among them, the multi-adjoint framework [1,4,5,11] is one of the most flexible and versatile, making it ideal for modeling real-world problems.

Various methods have been developed and utilized in FCA to simplify data processing,such as factoring and aggregating data tables [2,3,7,9,12]. These techniques enable the reduction of large tables into smaller tables, known as factors, from which important information can be extracted. Furthermore, these factors can be aggregated without altering the information. In particular, we focus on the notion of bond between formal contexts, which was originally defined in the classical setting [8] and was extended to the fuzzy framework using residuated lattices [10]. Bonds permit the aggregation of contexts while preserving the information contained in the concepts generated by each individual context.This work aims to extend the aforementioned notion to the multi-adjoint framework and analyze the conditions that enable obtaining bonds in a simpler manner.

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2 Preliminaries

In this preliminaries section, we introduce foundational notions related to FCA. Throughout this paper, the notation A^B will be employed to denote maps from B to A. Particularly, if $A = \{1, \dots, n\}$ we may use n^B to denote $\{1, \dots, n\}^B$. We now proceed to define what precisely a context is within the setting of FCA.

Definition 1. *A* context *is a tuple* $\mathbb{K} = (O, P, R)$ *such that* O *and* P *are non-empty sets (usually interpreted as objects and properties, respectively) and* R *is a relation in* $O \times P$.

In addition, the derivation operators $^\uparrow \colon 2^O \to 2^P$ and $^\downarrow \colon 2^P \to 2^O$ are defined as

$$X^\uparrow = \{a \in P \mid (x, a) \in R, \text{ for all } x \in X\}$$
$$A^\downarrow = \{x \in O \mid (x, a) \in R, \text{ for all } a \in A\}$$

for all $X \subseteq O$ and $A \subseteq P$. A *concept* is a pair $\langle X, A\rangle$ where X and A are respectively subsets of objects and properties, satisfying that $X^\uparrow = A$ and $A^\downarrow = X$, where X is called the *extent* and A is the *intent*. The set of all the concepts of the context (O, P, R) is denoted by $\mathcal{C}(O, P, R)$, and with the ordering defined by $\langle X_1, A_1\rangle \preceq \langle X_2, A_2\rangle$ if and only if $X_1 \subseteq X_2$ (equivalently $A_2 \subseteq A_1$), for all $\langle X_1, A_1\rangle, \langle X_2, A_2\rangle \in \mathcal{C}(O, P, R)$, forms a complete lattice [6] called *concept lattice*. We will refer to this lattice as the *concept lattice* of the context (O, P, R), and to the objects and properties of a concept as the *extent* and *intent*, respectively.

3 Bonds

The aim of this study is to investigate methods for aggregating contexts (factors) and constructing a new context from them. This new context has to preserve the information contained in the factors. Our objective is to study ways of joiningcontexts and constructing a new context from these. We wish for this new context to preserve the information of the factors. Throughout this section, we will pursue this idea presenting the theory of bonds in the non-fuzzyclassical case and working through an example. until finally arriving at a remark discussing when the empty set is a bond. [No creo que se entienda esto ahora en este punto sin haber definido nada.] In Section 4, we will translate these notions into the multi-adjoint framework.

From now on, we will consider a family of contexts $\{\mathbb{K}_i = (O_i, P_i, R_i)\}_{i \in \Lambda}$, with Λ an arbitrarynon-empty set of indices. We will simply denoted the derivation operators $^\uparrow$ and $^\downarrow$ in each context $\mathbb{K}_i$ as $^{\uparrow_i}$ and $^{\downarrow^i}$, respectively. For every context $\mathbb{K}_i$, we are going to simplify the notation of the derivation operators by dropping the R, that is, $^{\uparrow_{R_i}}$ and $^{\downarrow^{R_i}}$ will simply be denoted as $^{\uparrow_i}$ and $^{\downarrow^i}$, respectively. Moreover, if $R_{ij} \subseteq O_i \times P_j$ is a relation, we ~~will~~ define the mappings

$^{\uparrow_{ij}}\colon 2^{O_i} \to 2^{P_j}$ and $^{\downarrow^{ij}}\colon 2^{P_j} \to 2^{O_i}$ as

$$\begin{aligned} X^{\uparrow_{ij}} &= \{a \in P_j \mid (x,a) \in R_{ij},\ \text{for all } x \in X\} \\ A^{\downarrow^{ij}} &= \{x \in O_i \mid (x,a) \in R_{ij},\ \text{for all } a \in A\} \end{aligned}$$

for all $X \subseteq O_i$ and $A \subseteq P_j$. When we consider $X = \{x\}$ and $A = \{a\}$, we may use the notation $x^{\uparrow_{ij}}$ and $a^{\downarrow^{ij}}$ instead of $\{x\}^{\uparrow_{ij}}$ and $\{a\}^{\downarrow^{ij}}$.

Our initial step will involve defining the concept of a bond.

Definition 2. *Given two different contexts $\mathbb{K}_i$ and $\mathbb{K}_j$, a* bond *from $\mathbb{K}_i$ to $\mathbb{K}_j$ is a relation $R_{ij} \subseteq O_i \times P_j$ such that*

- *$x^{\uparrow_{ij}}$ is an intent of $\mathbb{K}_j$ for every object $x \in O_i$,*
- *$a^{\downarrow^{ij}}$ is an extent of $\mathbb{K}_i$ for every property $a \in P_j$.*

A direct consequence of the definition is that for every extent X of $\mathbb{K}_i$ and intent A of $\mathbb{K}_j$, $X^{\uparrow_{ij}}$ is an intent of $\mathbb{K}_j$ and $A^{\downarrow^{ij}}$ is an extent of $\mathbb{K}_i$. For instance,

$$X^{\uparrow_{ij}} = \inf\{x^{\uparrow_{ij}} \mid x \in X\}$$

is an intent of $\mathbb{K}_j$. [No entiendo esto, parece que se cambiara la definición del operador no?]

Now, we will~~Let's~~ examine this definition more closely through a practical example

Example 1. Consider two contexts $\mathbb{K}_1 = (O_1, P_1, R_1)$ and $\mathbb{K}_2 = (O_2, P_2, R_2)$ where

$$\begin{aligned} &O_1 = \{x_1, x_2\}, \quad && P_1 = \{a_1, a_2\}, \quad && R_1 = \{(x_1,a_1),(x_1,a_2),(x_2,a_2)\} \\ &O_2 = \{x_3, x_4\}, && P_2 = \{a_3, a_4\}, && R_2 = \{(x_3,a_3),(x_4,a_4)\} \end{aligned}$$

A bond can be visualized by placing the two contexts diagonally, one beneath the other, and the bond in the top right corner (a bond R_{ji} from $\mathbb{K}_j$ to $\mathbb{K}_i$ would be placed in the bottom left), as observed in Table 1. Given that the set of objects and the set of properties are always extent and intent of a context, respectively, we are guaranteed that the relation $R^{\top}_{12} = O_1 \times P_2$ is a bond. Indeed, for any $x \in O_1$ and $a \in P_2$, $x^{\uparrow_{12}} = P_2$ is an intent of $\mathbb{K}_2$ and $a^{\downarrow^{12}} = O_1$ is an extent of $\mathbb{K}_1$.

We can usually find other non-trivial bonds, such as

$$R^{1}_{12} = \{(x_1,a_3),(x_1,a_4),(x_2,a_3)\}$$

For this relation, $x_1{}^{\uparrow_{12}} = \{a_3, a_4\}$ and $x_2{}^{\uparrow_{12}} = \{a_3\}$ are both intents of $\mathbb{K}_2$. The first one because it is the set of properties P_2 and the second one because

$$\{a_3\}^{\downarrow^{2}\uparrow_{2}} = \{x_3\}^{\uparrow_2} = \{a_3\}$$

	a_1	a_2	a_3	a_4
x_1	×	×	×	×
x_2		×	×	
x_3			×	
x_4				×

	a_1	a_2	a_3	a_4
x_1	×	×		
x_2		×	×	
x_3			×	
x_4				×

Table 1. Tables showing the contexts $\mathbb{K}_1$ and $\mathbb{K}_2$ on the diagonal and the relations $R^1_{1\,2}$ (left table) and $R^2_{1\,2}$ (right table) of the Example 1 on the top right corner.

Likewise, $a_3{}^{\downarrow^{1\,2}} = \{x_1, x_2\}$ and $a_4{}^{\downarrow^{1\,2}} = \{x_1\}$ are extents of $\mathbb{K}_1$, the first one for being the set of objects O_1 and the second one because

$$\{x_1\}^{\uparrow_1\downarrow^1} = \{a_1, a_2\}^{\downarrow^1} = \{x_1\}$$

However, not all relations in $O_1 \times P_2$ are bonds. Consider for instance the relation $R^2_{1\,2} = \{(x_2, a_3)\}$. This relation is not a bond because $a_3{}^{\downarrow^{1\,2}} = \{x_2\}$ is not an extent of $\mathbb{K}_1$, that is,

$$\{x_2\}^{\uparrow_1\downarrow^1} = \{a_2\}^{\downarrow^1} = \{x_1, x_2\} \neq \{x_2\}$$

Another interesting case is the empty relation, $R^{\perp}_{1\,2} = \varnothing$. Because the empty set is neither an extent of $\mathbb{K}_1$ nor an intent of $\mathbb{K}_2$, this relation is not a bond from $\mathbb{K}_1$ to $\mathbb{K}_2$.

[Añadir retículos de conceptos. Es necesario aquí?] □

Remark 1. Given two contexts $\mathbb{K}_i = (O_i, P_i, R_i)$ and $\mathbb{K}_j = (O_j, P_j, R_j)$, the empty relation is not always a bond from $\mathbb{K}_i$ to $\mathbb{K}_j$ because the empty set ~~isn't~~ is not always an extent or intent of a context. However, there are cases where it is actually a bond. If the contexts $\mathbb{K}_i$ and $\mathbb{K}_j$ satisfy that for every object $x \in O_i$ and property $a' \in P_j$, there exists $a \in P_i$ and $x' \in O_j$ such that $(x, a) \notin R_i$ and $(x', a') \notin R_j$, then this guarantees that the empty set is both an extent of $\mathbb{K}_i$ and an intent of $\mathbb{K}_j$ and therefore the empty relation $R^{\perp}_{ij}$ is a bond from $\mathbb{K}_i$ to $\mathbb{K}_j$.

4 Bonds on a multi-adjoint framework

In the previous section, we tackled the non-fuzzy case. If we consider the fuzzy scenario, we need to work with a more general notion of context, where an object can have a non-binary degree of uncertainty as to whether it has a given attribute. In this section, we will ~~introduce~~ recall the notion of a multi-adjoint framework and concept in order to define a bond between contexts associated to a multi-adjoint framework~~what a bond could look like in this setting~~. ~~We will present an example and work towards Proposition~~ 1.

The first thing we need to do is define what we mean by multi-adjoin framework.

Definition 3. *A* multi-adjoint framework *is a tuple* $(L_1, L_2, Q, \&_1, \dots, \&_n)$ *where* $(L_1, \preceq_1)$ *and* $(L_2, \preceq_2)$ *are complete lattices,* $(Q, \leq)$ *is a poset and* $(\&_k, \swarrow^k, \nwarrow_k)$ *is an adjoint triple with respect to* L_1*,* L_2 *and* P*, for all* $k \in \{1, \dots, n\}$*.*

Throughout this section, a multi-adjoint framework $(L_1, L_2, Q, \&_1, \dots, \&_n)$ will be fixed. The definition of a context is defined in the same way as the non-fuzzy, adding a function which designates an adjoint triple to each pair of objects and properties. This notion is formalized in the following definition.

Definition 4. *A* context *is a tuple* $\mathbb{M} = (O, P, R, \sigma)$*, where* O *is the set of objects,* P *is the set of properties,* R *is a* Q*-fuzzy relation* $R\colon O \times P \to Q$ *and* $\sigma\colon O \times P \to \{1, \dots, n\}$ *is a mapping which associates a specific adjoint triple in the frame with any element in* $O \times P$*.*

~~Given a multi-adjoint frame,~~[ya se ha fijado arriba] The extension of the concept-forming operators are the mappings ${}^{\uparrow_\sigma}\colon L_1^O \to L_2^P$ and ${}^{\downarrow^\sigma}\colon L_2^P \to L_1^O$ ~~are~~ defined as:

$$g^{\uparrow_\sigma}(a) = \inf\{R(x, a) \swarrow^{\sigma(x,a)} g(x) \mid x \in O\}$$
$$f^{\downarrow^\sigma}(x) = \inf\{R(x, a) \nwarrow_{\sigma(x,a)} f(x) \mid a \in P\}$$

for all $g \in L_1^O$, $f \in L_2^P$ and $a \in P$, $x \in O$. ~~As usual~~ In addition, a pair $\langle g, f\rangle$ is called a *multi-adjoint concept* if equalities $g^{\uparrow_\sigma} = f$ and $f^{\downarrow^\sigma} = g$ hold. The set g is the *extent* of the concept and f is the *intent*.[Yo quitaría esto ya que se sobreentiende de lo que es un concepto no? Y metería la notación de retículo de conceptos multiadjuntos como $\mathcal{M}$]

Definition 5. *For each* $a \in P$*, the fuzzy subsets of attributes* $\phi_{a,x} \in L_1^P$ *defined, for all* $x \in L_1$*, as*

$$\phi_{a,x}(a') = \begin{cases} x & \text{if } a' = a \\ \perp_1 & \text{if } a' \neq a \end{cases}$$

will be called fuzzy-attributes.

Analogously, the fuzzy-objects are defined in the same way.

Hereon, $\{\mathbb{M}_i = (O_i, P_i, R_i, \sigma_i)\}_{i \in \Lambda}$ will be a family of context of the frame. Given a Q-fuzzy relation R_{ij} in $O_i \times P_j$ and $\sigma_{ij}\colon O_i \times P_j \to \{1, \dots, n\}$, we will define the mappings ${}^{\uparrow_{ij}}\colon L_2^{O_i} \to L_1^{P_j}$ and ${}^{\downarrow^{ij}}\colon L_1^{P_j} \to L_2^{O_i}$ as

$$g^{\uparrow_{ij}}(a) = \inf\{R_{ij}(x, a) \swarrow^{\sigma_{ij}(x,a)} g(x) \mid x \in O_i\}$$
$$f^{\downarrow^{ij}}(x) = \inf\{R_{ij}(x, a) \nwarrow_{\sigma_{ij}(x,a)} f(a) \mid a \in P_j\}$$

for each $g \in L_2^{O_i}$, $f \in L_1^{P_j}$ and $a \in P_j$, $x \in O_i$.

In Definition 2, we described a bond from $\mathbb{K}_i$ to $\mathbb{K}_j$ as a relation R_{ij} in $O_i \times P_j$, which can also be interpreted as a ~~whole~~new context (O_i, P_j, R_{ij}). This point of view is fundamental in the definition of a bond in a multi-adjoint framework.

Definition 6. *Given two different contexts $\mathbb{K}_i$ and $\mathbb{K}_j$, a* multi-adjoint bond(1) *from $\mathbb{M}_i$ to $\mathbb{M}_j$ is a context $\mathbb{M}_{ij} = (O_i, P_j, R_{ij}, \sigma_{ij})$ such that $R_{ij} \subseteq O_i \times P_j$ and $\sigma_{ij} \colon O_i \times P_j \to \{1, \dots, n\}$ satisfying that*

1. maneter el “multi-adjoint’?’

- *${\phi_{x,t}}^{\uparrow_{ij}}$ is an intent of $\mathbb{M}_j$ for every object $x \in O_i$,*
- *${\phi_{a,s}}^{\downarrow^{ij}}$ is an extent of $\mathbb{M}_i$ for every property $a \in P_j$,*

where $s \in L_1$ and $t \in L_2$.

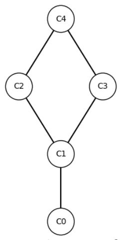
Notice that from the results in [8], given a bond $\mathbb{M}_{ij}$ from $\mathbb{M}_i$ to $\mathbb{M}_j$ and an extent X of $\mathbb{M}_{ij}$, we have that

$$X = \inf\{{\phi_{a,s}}^{\downarrow^{ij}} \mid X \subseteq {\phi_{a,s}}^{\downarrow^{ij}},\ a \in P_i, s \in L_1\}$$

[Mismo comentario que antes, utilizar la notación de g en lugar de X y tendríamos que definir los fuzzy-attributes no? La añado arriba en azul] As a consequence, X is an extent of $\mathbb{M}_i$. Dually we can show that all the intents of $\mathbb{M}_{ij}$ are intents of $\mathbb{M}_j$.

Example 2. Consider the multi-adjoint framework $([0,1]_4, [0,1]_4, [0,1]_4, \&_{\mathrm{G}}^*, \&_{\mathrm{L}}^*)$ and the contexts $\mathbb{M}_1 = (O_1, P_1, R_1, \sigma_1)$ and $\mathbb{M}_2 = (O_2, P_2, R_2, \sigma_2)$, where the relations are defined in Table 2. and $\sigma_1(x, a) = \&_{\mathrm{G}}^*$, for all $(x, a) \in O_1 \times P_1$ and $\sigma_2(x', a') = \&_{\mathrm{L}}^*$, for all $(x', a') \in O_2 \times P_2$.

R_1	a_1	a_2
x_1	0.5	0.75
x_2	0.25	1



R_2	a_3	a_4
x_3	1	0.75
x_4	0.75	1

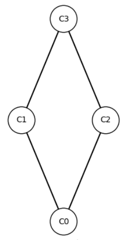


Table 2. The relations R_1 and R_2 of the contexts $\mathbb{M}_1$ and $\mathbb{M}_2$ in Example 2 together with their associated concept lattices.

Similar to Example 1, the relation $R_{1\,2}^{\top} \equiv 1$ together with any map $\sigma_{1\,2} \colon O_1 \times P_2 \to \{\&_{\mathrm{G}}^*, \&_{\mathrm{L}}^*\}$ defines a bond $\mathbb{M}_{1\,2}^{\top} = (O_1, P_2, R_{1\,2}^{\top}, \sigma_{1\,2})$ from $\mathbb{M}_1$ to $\mathbb{M}_2$ because $g_{\top} \colon O_1 \to [0,1]_4$ defined by $g_{\top}(x) = 1$ for all $x \in O_1$ is an extent of $\mathbb{M}_1$ and $f_{\top} \colon P_2 \to [0,1]_4$ defined by $f_{\top}(a) = 1$ for all $a \in P_2$ is an intent of $\mathbb{M}_2$. There is only one other non-isomorphic[1] bond from $\mathbb{M}_1$ to $\mathbb{M}_2$ in this example: the bond defined by the relation $R_{1\,2}$ on the second table of Table 3.

Consider now the empty relation $R_{1\,2}^{\perp} \equiv 0$ and any map $\sigma_{1\,2} \colon O_1 \times P_2 \to \{\&_{\mathrm{G}}^*, \&_{\mathrm{L}}^*\}$.

[1] For every map $\sigma_{1\,2} \colon O_1 \times P_2 \to \{\&_{\mathrm{G}}^*, \&_{\mathrm{L}}^*\}$, the contexts $(O_1, P_2, R_{1\,2}, \sigma_{1\,2})$ are all isomorphic in the sense that their concept lattices are the same.

	a_1	a_2	a_3	a_4
x_1	0.5	0.75	1	1
x_2	0.25	1	1	1
x_3	1	1	1	0.75
x_4	1	1	0.75	1

$R^{\top}_{1\,2}$

	a_1	a_2	a_3	a_4
x_1	0.5	0.75	1	0.75
x_2	0.25	1	1	1
x_3	1	1	1	0.75
x_4	1	1	0.75	1

$R_{1\,2}$

	a_1	a_2	a_3	a_4
x_1	0.5	0.75	0	0
x_2	0.25	1	0	0
x_3	0	0	1	0.75
x_4	0	0	0.75	1

$R^{\perp}_{1\,2}$

Table 3. The relations of the contexts $\mathbb{M}_1$ and $\mathbb{M}_2$ of Example 2 shown in the diagonal and the relations of the bonds $\mathbb{M}^{\top}_{1\,2}$ (top left), $\mathbb{M}_{1\,2}$ (top right) and $\mathbb{M}^{\perp}_{1\,2}$ (bottom) on the upper right corners.

– For any $x_i \in O_1$ and $t \in [0,1]_4$,

$$\begin{aligned}
{\phi_{x_i,t}}^{\uparrow_{1\,2}}(a) &= \inf\{R^{\perp}_{1\,2} \swarrow^{\sigma_{1\,2}(x,a)} \phi(x_i,t)(x) \mid x \in O_1\}\\
&= \inf\{0 \swarrow^{\sigma_{1\,2}(x,a)} \phi(x_i,t)(x) \mid x \in O_1\}\\
&= 0
\end{aligned}$$

Therefore, ${\phi_{x,t}}^{\uparrow_{1\,2}} \equiv 0$. This fuzzy set will be an intent of $\mathbb{K}_2$ only if $g_{\top} : O_2 \to [0,1]_L$, $g_{\top} \equiv 1$, satisfies ${g_{\top}}^{\uparrow_2} \equiv 0$. In this example we have ${g_{\top}}^{\uparrow_2}(a_3) = 0.75$ and ${g_{\top}}^{\uparrow_2}(a_4) = 0.75$. Hence, the context $\mathbb{M}^{\perp}_{1\,2} = (O_1, P_2, R^{\perp}_{1\,2}, \sigma_{1\,2})$ is not a bond from $\mathbb{M}_1$ to $\mathbb{M}_2$.

– Dually we have

$$\begin{aligned}
{\phi_{a_j,s}}^{\downarrow^{1\,2}}(x) &= \inf\{R^{\perp}_{1\,2} \nwarrow_{\sigma_{1\,2}(x,a)} \phi(a_j,s)(a) \mid a \in P_2\}\\
&= \inf\{0 \nwarrow_{\sigma_{1\,2}(x,a)} \phi(a_j,s)(x) \mid a \in P_2\}\\
&= 0
\end{aligned}$$

for any $a_j \in P_2$ and $s \in [0,1]_4$. One could also make the case that for $\mathbb{M}^{\perp}_{1\,2}$ to qualify as a bond from $\mathbb{M}_1$ to $\mathbb{M}_2$, it is necessary for ${\phi_{a,s}}^{\downarrow^{1\,2}} \equiv 0$ to serve as an extent of $\mathbb{K}_1$. Considering that ${f_{\top}}^{\downarrow^1}(x_1) = 0.5$ and ${f_{\top}}^{\downarrow^1}(x_2) = 0.25$, with $f_{\top} : P_1 \to [0,1]_4$ defined as $f_{\top} \equiv 1$, it follows that ${\phi_{a,s}}^{\downarrow^{1\,2}}$ does not fulfill the role of an intent of $\mathbb{K}_1$. Consequently, we deduce once more that $\mathbb{M}^{\perp}_{1\,2}$ is not a bond between $\mathbb{M}_1$ and $\mathbb{M}_2$.

It is interesting to note that the third concept lattice in 1 is the [suma disjunta](2) of the concept lattices $\mathcal{C}(\mathbb{M}_1)$ and $\mathcal{C}(\mathbb{M}_2)$.

2. No recuerdo cómo se llamaba a este tipo de unión de retículos. Dados dos retículos, se pueden unir colocándolos uno al lado del otro y uniendo el bottom element de cada uno con un nuevo bottom element, y haciendo lo mismo con los top elements. Roberto ha hecho referencia a este tipo de union en varias ocasiones. Si queremos hablar de esto, deberíamos introducirlo en la sección de preliminares, no?

Going forward, we will presume that Q constitutes a lattice and that every lattice under consideration is equipped with top and bottom elements, $\top_1$ and $\perp_1$ for L_1, $\top_2$ and $\perp_2$ for L_2, and $\top_3$ and $\perp_3$ for Q, respectively. In this setting we can study the relations $R^{\top}_{ij}$ and $R^{\perp}_{ij}$.

As in the classical case (Remark 1), the relation $R^{\perp}_{1\,2} \equiv \perp_3$ does not always define a bond from a context $\mathbb{M}_i$ to another context $\mathbb{M}_j$. This was the case with

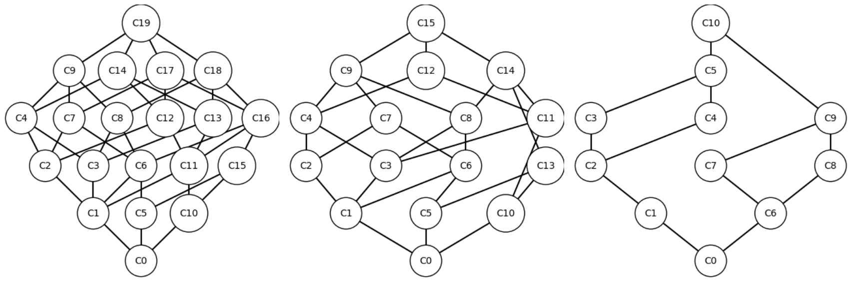


Fig. 1. The concept lattices for the three contexts shown in Table 3, in order.

Example 2. In Remark 1 we showed a characterization of when the non-fuzzy relation $R_{12}^{\perp}$ was indeed a bond, and we can show a similar property for multi-adjoint bonds. The key idea behind the next proposition is that for $\mathbb{M}_{ij}^{\perp}$ to be a bond from $\mathbb{M}_i$ to $\mathbb{M}_j$, the sets $g_{\perp}\colon O_i \to L_1$ and $f_{\perp}\colon P_j \to L_2$ defined by $g_{\perp} \equiv \perp_1$ and $f_{\perp} \equiv \perp_2$ have to be extent of $\mathbb{M}_1$ and intent of $\mathbb{M}_2$, respectively.

Proposition 1. *Consider a multi-adjoint framework $(L_1, L_2, Q, \&_1, \dots, \&_n)$ and two different contexts $\mathbb{M}_i$ and $\mathbb{M}_j$. Given any map $\sigma_{ij}\colon O_i \times P_j \to \{1, \dots, n\}$ and the relation*

$$R_{ij}^{\top}(x, a) = \top_3, \quad \text{for all } (x, a) \in O_i \times P_j$$

the context $\mathbb{M}_{ij}^{\top} = (O_i, P_j, R_{ij}^{\top}, \sigma_{ij})$ is a bond. Moreover, if for every row in R_i and column in R_j there is at least one bottom element, then the context $\mathbb{M}_{ij}^{\perp} = (O_i, P_j, R_{ij}^{\perp}, \sigma_{ij})$, where

$$R_{ij}^{\perp}(x, a) = \perp, \quad \text{for all } (x, a) \in O_i \times P_j$$

is a bond.

Remark 2. The previous result is a characterization, hence if there is a row in R_i or a column in R_j with no bottom elements, the context $\mathbb{M}_{ij}^{\perp}$ will not be a bond from $\mathbb{M}_i$ to $\mathbb{M}_j$.

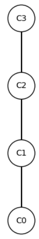
A particular case of this occurs when both $\mathbb{M}_i$ and $\mathbb{M}_j$ are normalized contexts.

Definition 7. *When Q is bounded with the top element $\top_3$ and the bottom element $\perp_3$, that is, $(Q, \leq, \perp_3, \top_3)$, a context $\mathbb{M}$ will be called* normalized *if there are no rows or columns of the Q-fuzzy relation with all values equal to bottom or with all values different from bottom.*

Corollary 1. *If the contexts $\mathbb{M}_i$ and $\mathbb{M}_j$ are normalized in the sense of Definition 7, then for any mappings $\sigma_{ij}\colon O_i \times P_j \to \{1, \dots, n\}$ and $\sigma_{ji}\colon O_j \times P_i \to \{1, \dots, n\}$, the contexts $\mathbb{M}_{ij}^{\perp} = (O_i, P_j, R_{ij}^{\perp}, \sigma_{ij})$ and $\mathbb{M}_{ji}^{\perp} = (O_j, P_i, R_{ji}^{\perp}, \sigma_{ji})$ are bonds from $\mathbb{M}_i$ to $\mathbb{M}_j$ ahd from $\mathbb{M}_j$ to $\mathbb{M}_i$, respectively.*

Example 3. We will continue working with the multi-adjoint framework defined in Example 2. Consider the contexts $\mathbb{M}_3 = (O_3, P_3, R_3, \sigma_3)$ and $\mathbb{M}_4 = (O_4, P_4, R_4, \sigma_4)$, where R_3 and R_4 are defined by the Table 4, $\sigma_3(x,a) = \&_{\mathrm{G}}^*$ for all $(x,a) \in O_3 \times P_3$ and $\sigma_4(x',a') = \&_{\mathrm{L}}^*$ for all $(x',a') \in O_4 \times P_4$.

R_3	a_1	a_2
x_1	0.75	0
x_2	0.25	0



R_4	a_3	a_4
x_3	0	0
x_4	0.5	1

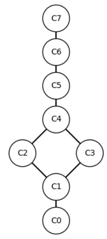
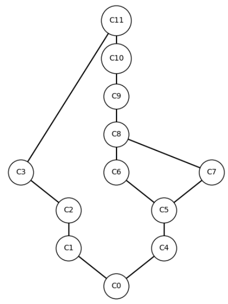


Table 4. The relations R_3 and R_4 of the contexts $\mathbb{M}_3$ and $\mathbb{M}_4$ in Example 3 together with the concept lattices of the contexts.

If we look at the rows of R_3 and the columns of R_4 we can see that there is a bottom element (the number 0) in each. Proposition 1 states that $\mathbb{M}_{3\,4}^{\perp} = (O_3, P_4, R_{3\,4}^{\perp}, \sigma_{3\,4})$ is a bond from $\mathbb{M}_3$ to $\mathbb{M}_4$ and $\mathbb{M}_{4\,3}^{\perp} = (O_4, P_3, R_{4\,3}^{\perp}, \sigma_{4\,3})$ is not a bond, for any maps $\sigma_{3\,4} \colon O_3 \times P_4 \to \{\&_{\mathrm{G}}^*, \&_{\mathrm{L}}^*\}$, $\sigma_{4\,3} \colon O_4 \times P_3 \to \{\&_{\mathrm{G}}^*, \&_{\mathrm{L}}^*\}$.

Let $\mathbb{M}$ be the context obtained by joining $\mathbb{M}_3$ and $\mathbb{M}_4$ with the relations $R_{3\,4}^{\perp}$ and $R_{4\,3}^{\perp}$, as in Table 5. If we compare its concept lattice with the third one from Fig. 1, both look like the [suma disjunta de los factores]. In the latter, the concept lattice is indeed the [suma disjunta de los factores]. The concept lattice $\mathcal{C}(\mathbb{M})$ resembles the [suma disjunta] of $\mathcal{C}(\mathbb{M}_3)$ and $\mathcal{C}(\mathbb{M}_4)$, but the bottom of $\mathcal{C}(\mathbb{M}_3)$ turns into the bottom of $\mathcal{C}(\mathbb{M})$ and the top of $\mathcal{C}(\mathbb{M}_4)$ turns into the top of $\mathcal{C}(\mathbb{M})$.

	a_1	a_2	a_3	a_4
x_1	0.75	0	0	0
x_2	0.25	0	0	0
x_3	0	0	0	0
x_4	0	0	0.5	1



	P_i	P_j
O_i	$\mathbb{M}_i$	$\mathbb{M}_{ij}^{\perp}$
O_j	$\mathbb{M}_{ji}^{\perp}$	$\mathbb{M}_j$

Table 5. On the left, the relation of the context $\mathbb{M}$ in Example 3 constructed by placing the relation R_3 in the top left, the relation $R_{3\,4}^{\perp}$ in the top right, the relation $R_{4\,3}^{\perp}$ in the bottom left and the relation R_4 in the bottom right. On the center, its concept lattice. On the right, a general way of constructing the matrix $\mathbb{M}$ for arbitrary contexts $\mathbb{M}_i$ and $\mathbb{M}_j$.

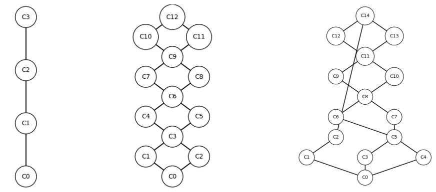
This is the case in general. Let $\mathbb{M}_i$ and $\mathbb{M}_j$ be two different contexts, and let $\mathbb{M}$(3) be the context constructed from $\mathbb{M}_i$, $\mathbb{M}_j$, $\mathbb{M}_{ij}^{\perp}$ and $\mathbb{M}_{ji^{\perp}}$, as illustrated in Table 5. Then, when comparing the concept lattices $\mathcal{C}(\mathbb{M}_i)$, $\mathcal{C}(\mathbb{M}_j)$ and $\mathcal{C}(\mathbb{M})$,

3. introduce some notation? Like $\mathbb{M}_1 \oplus \mathbb{M}_2$ or $\mathbb{M}_1 \uplus \mathbb{M}_2$. De esta forma no reutilizariamos el nombre $\mathbb{M}$ que he usado en el párrafo anterior.

if the bottom element of $\mathcal{C}(\mathbb{M}_i)$ turns into the bottom element of $\mathcal{C}(\mathbb{M})$ and if the top element of $\mathcal{C}(\mathbb{M}_j)$ turns into the top element of $\mathcal{C}(\mathbb{M})$, then the context $\mathbb{M}_{ij}^{\perp}$ is a bond from $\mathbb{M}_i$ to $\mathbb{M}_j$. Dually, if the bottom element of $\mathcal{C}(\mathbb{M}_j)$ turns into the bottom element of $\mathcal{C}(\mathbb{M})$ and if the top element of $\mathcal{C}(\mathbb{M}_i)$ turns into the top element of $\mathcal{C}(\mathbb{M})$, then the context $\mathbb{M}_{ji}^{\perp}$ is a bond from $\mathbb{M}_j$ to $\mathbb{M}_i$. It is possible to have both simultaneously, which means that both $\mathbb{M}_{ij}^{\perp}$ and $\mathbb{M}_{ji}^{\perp}$ are bonds.

A few examples of this are presented in Figure 2. For the first one, $\mathbb{M}_{ij}^{\perp}$ is a bond from $\mathbb{M}_i$ to $\mathbb{M}_j$ and also $\mathbb{M}_{ji}^{\perp}$ is a bond from $\mathbb{M}_j$ to $\mathbb{M}_j$. For the second one, $\mathbb{M}_{ji}^{\perp}$ is a bond from $\mathbb{M}_j$ to $\mathbb{M}_j = i$ but $\mathbb{M}_{ij}^{\perp}$ is not a bond from $\mathbb{M}_i$ to $\mathbb{M}_j$.

	a_1	a_2	a_3	a_4
x_1	0.75	0	0	0
x_2	0	0	0	0
x_3	0	0	0	0.25
x_4	0	0	0.25	0



	a_1	a_2	a_3	a_4
x_1	1	0	0	0
x_2	0	0.25	0	0
x_3	0	0	0.75	0
x_4	0	0	0.75	0

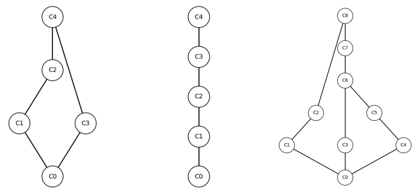


Fig. 2. Two examples of bonds joined together by the empty contexts as explained in Example 3. On the right of each table, the concept lattice for the first context, the second context, and the full context presented in the table, in order.

5 Conclusions and future work

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