HW2

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Problem 1a

$$R(t) = \begin{bmatrix} \cos\frac{t\pi}{3} & 0 & -\sin\frac{t\pi}{3} \\ 0 & 1 & 0 \\ \sin\frac{t\pi}{3} & 0 & \cos\frac{t\pi}{3} \end{bmatrix}, \quad R(t=1) = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$\theta = \|w\| = \cos^{-1}\left(\frac{tr(R)-1}{2}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\hat{w} = \log(R) = \frac{\theta}{2\sin\theta}(R-R^T)$$

$$= \frac{\pi}{3}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix} 0 & 0 & -\sqrt{3} \\ 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 \end{bmatrix} = \frac{\pi}{3}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\hat{w} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}, \quad w = [w_1 \ \ w_2 \ \ w_3]$$

$$\boxed{w = \left[0 \ \ -\frac{\pi}{3} \ \ 0\right], \ \theta = \frac{\pi}{3} \qquad \xi = \frac{w}{\|w\|} = [0 \ \ -1 \ \ 0]}$$

Problem 1b

$$q = \left[\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\cdot\xi\right]$$

$$q = \left[\cos\left(\frac{\pi}{6}\right), \sin\left(\frac{\pi}{6}\right)\cdot\frac{w}{\|w\|}\right] = \left[\frac{\sqrt{3}}{2}, \frac{1}{2}\cdot[0 \ -1 \ 0]\right]$$

$$q = \left[\frac{\sqrt{3}}{2}, 0, -\frac{1}{2}, 0\right]$$

$$q^{-1} = \frac{\bar{q}}{|q|^2} = [q_s, -q_v]\cdot\frac{1}{q_s^2 + q_v^T q_v} = \frac{1}{\frac{3}{4}+\frac{1}{4}}\cdot[q_s, -q_v]$$

$$q^{-1} = [q_s, -q_v] = \bar{q}$$

$$\boxed{q = \left[\frac{\sqrt{3}}{2}, 0, -\frac{1}{2}, 0\right], \quad q^{-1} = \left[\frac{\sqrt{3}}{2}, 0, \frac{1}{2}, 0\right]}$$

Problem 1c

$$\hat{w}(t) = \dot{R}(t)\cdot R^T(t), \quad \hat{w}(t=1) = \dot{R}(t=1)R^T(t=1)$$

$$\dot{R}(t) = \begin{bmatrix} -\sin\left(\frac{t\pi}{3}\right) & 0 & -\cos\left(\frac{t\pi}{3}\right) \\ 0 & 0 & 0 \\ \cos\left(\frac{t\pi}{3}\right) & 0 & -\sin\left(\frac{t\pi}{3}\right) \end{bmatrix}\cdot\frac{\pi}{3}$$

$$\dot{R}(t=1) = \begin{bmatrix} -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \end{bmatrix} \cdot \frac{\pi}{3} \quad ; \quad R^T(t=1) = \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$\hat{\omega}(t=1) = \frac{\pi}{3} \begin{bmatrix} -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & 0 & +\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$= \frac{\pi}{3} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\boxed{\begin{aligned} &\omega(t=1) = \frac{\pi}{3} \begin{bmatrix} 0 & -1 & 0 \end{bmatrix} && \text{(angular)} \\ &\dot{s}_w(t=1) = \dot{p}(t=1) = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} && \text{(linear)} \end{aligned}}$$

Problem 1 d

$$R = {}_wR_B$$

$${}_w\dot{R}_B = {}_wR_B \cdot \hat{\omega}_B \quad , \quad \hat{\omega}_B = R^T \cdot \dot{R} = R^T(t=1)\dot{R}(t=1)$$

$$R^T(t=1) = \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix} \quad \dot{R}(t=1) = \begin{bmatrix} -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \end{bmatrix} \cdot \frac{\pi}{3}$$

$$\hat{\omega}_B = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \frac{\pi}{3} \quad , \quad \boxed{\begin{aligned} &\omega_B = \frac{\pi}{3} \begin{bmatrix} 0 & -1 & 0 \end{bmatrix} && \text{(angular)} \\ &\dot{s}_B = 0 && \text{(linear)} \end{aligned}}$$

$\dot{s}_B$ would be 0, because the robot is rigid body and points on the robot body do not move relative to the body/robot frame.

Problem 1 e

$$P_w = T(t) P_R$$

$$P_R = T^{-1}(t) \cdot P_w = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} P_w \\ 1 \end{bmatrix}$$

$$R^T(t=1) = \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix} \quad , \quad p(t=1) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$-R^T(t=1) \cdot p(t=1) = -\begin{bmatrix} 2.23 \\ 0 \\ 0.13 \end{bmatrix}$$

$$\boxed{P_R = \begin{bmatrix} 2.268 \\ 0 \\ -7.93 \end{bmatrix}}$$

Problem 2 a   EKF $x_0 \sim N(0,1)$

$x_t \sim N(\mu_t, \Sigma_t)$ for mean; $x_t \sim N(1, \Sigma_t)$ for covariance

$x_{t+1} = a(x_t, w_t) = -0.1 \cdot x_t + \cos(x_t) + w_t \quad , w_t \sim N(0,1)$

$z_t = h(x_t, v_t) = x_t^2 + v_t \quad , v_t \sim N(0, 0.5)$

$A = \frac{da}{dx_t}\Big|_{x_t = x_t^{nom}} = -0.1 - \sin(x_t) = -0.1 - \sin(1)$

$Q = \frac{da}{dw}\Big|_{x_t = x_t^{nom}} = 1$

$H = \frac{dh}{dx_t}\Big|_{x_t = x_t^{nom}} = 2x_t = 2$

$R = \frac{dh}{dv}\Big|_{x_t = x_t^{nom}} = 1$

Prediction :

$\mu_{t+1|t} = a(\mu_{t|t}, 0) \quad ; \quad \mu_{1|0} = \cos(0) = 1$

$\Sigma_{t+1|t} = A \Sigma_{t|t} A^T + Q W Q^T = (-0.1 - \sin(1))^2 \cdot \Sigma_{t|t} + 1$

Kalman gain :

$K_{t+1|t} = \Sigma_{t+1|t} \cdot H (H \cdot \Sigma_{t+1|t} \cdot H^T + R \cdot V R^T)^{-1}$

$= 2 \cdot \Sigma_{t+1|t} \cdot (4 \Sigma_{t+1|t} + \frac{1}{2})^{-1}$

Update :

$\mu_{t+1|t+1} = \mu_{t+1|t} + K_{t+1|t} (z_{t+1} - h(\mu_{t+1|t}, 0))$

$\Sigma_{t+1|t+1} = (1 - K_{t+1|t} \cdot H) \cdot \Sigma_{t+1|t}$

$= (1 - 2 \cdot K_{t+1|t}) \cdot \Sigma_{t+1|t}$

Problem 2 b   UKF, additive noise

$x_{t+1} = a(x_t, w_t) = -0.1 \cdot x_t + \cos(x_t) + w_t \quad , w_t \sim N(0,1)$

$z_t = h(x_t, v_t) = x_t^2 + v_t \quad , v_t \sim N(0, 0.5)$

$d = 1 \; , \lambda = 0$

$\chi_{t|t}^{(0)} = x^{nom} = 1 \quad , \quad \chi_{t|t}^{(i)} = x^{nom} \pm \sqrt{1} \cdot \sqrt{\Sigma_{t|t}}$

$= 1 \pm \sqrt{\Sigma_{t|t}}$

$W_0^{(m)} = 0, \; W_i^{(m)} = \frac{1}{2 \cdot d} = \frac{1}{2} \;\Big|\; W_0^{(c)} = 2 \;, W_i^{(c)} = \frac{1}{2d} = \frac{1}{2}$

Predict:

$$\mu_{t+1|t} = \sum_{n=0}^{2d} W_i^{(m)} \cdot a(\chi_{t|t}^{(i)})$$

$$= \sum \frac{-0.1}{2}\left(1 \pm \sqrt{\Sigma_{t|t}+1}\right) + \cos\left(1 \pm \sqrt{\Sigma_{t|t}+1}\right)$$

$$\Sigma_{t+1|t} = \sum_{n=0}^{2d} W_i^{(c)} \left(\left(a\left(\chi_{t|t}^{(i)}\right) - \mu_{t+1|t}\right) \cdot \left(a(\chi_{t|t}^{(i)}) - \mu_{t+1|t}\right)^T + W\right.$$

$$= \sum_{n=0}^{2} W_i^{(c)} \left(-0.1(\chi_{t|t}^{(i)}) + \cos(\chi_{t|t}^{(i)}) - \mu_{t+1|t}\right)^2 + 1$$

Kalman Gain:

$$\chi_{t+1|t}^{(0)} = \mu_{t+1|t} \quad , \chi_{t+1|t}^{(i)} = \mu_{t+1|t} \pm \sqrt{\Sigma_{t+1|t}}$$

$$m_{t+1|t} = \sum_{i=0}^{2d} W_i^{(m)} h(\chi_{t+1|t}^{(i)}) = \sum_{i=0}^{2} W_i^{(m)} (\chi_{t+1|t}^{(i)})^2$$

$$C_{t+1|t} = \sum_{i=0}^{2d} W_i^{(c)} \cdot (\chi_{t+1|t}^{(i)} - \mu_{t+1|t}) \quad h(\chi_{t+1|t}^{(i)} - m_{t+1|t})^T$$

$$= \sum_{i=0}^{2} W_i^{(c)} \cdot (\chi_{t+1|t} - \mu_{t+1|t}) \cdot \quad (\chi_{t+1|t}^{(i)} - m_{t+1|t})^2$$

$$S_{t+1|t} = \sum_{i=0}^{2d} W_i^{(c)} \cdot \left(h(\chi_{t+1|t}^{(i)}) - m_{t+1|t}\right) \cdot \left(h(\chi_{t+1|t}^{(i)}) - m_{t+1|t}\right)^T + 0.5$$

$$= \sum_{i=0}^{2} W_i^{(c)} \cdot \left((\chi_{t+1|t}^{(i)})^2 - m_{t+1|t}\right)^2 + 0.5$$

$$K_{t+1|t} = C_{t+1|t} \cdot S_{t+1|t}^{-1}$$

Update:

$$\mu_{t+1|t+1} = \mu_{t+1|t} + K_{t+1|t}\left(z_{t+1} - m_{t+1|t}\right)$$

$$\Sigma_{t+1|t+1} = \Sigma_{t+1|t} - K_{t+1|t} S_{t+1|t} K_{t+1|t}^T$$