276A HW3 SS

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$\boxed{\text{Problem 1a}}$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $x_t | z_{0:t} \sim \mathcal{N}(\mu_{t|t}, \Sigma_{t|t})$

$x_{t+1} = A \cdot x_t + w_t$, $z_t = H \cdot x_t + v_t$, $w_t \sim \mathcal{N}\left(0, \begin{bmatrix} \frac{1}{100} & 0 \\ 0 & 1 \end{bmatrix}\right)$

Prediction: $v_t \sim \mathcal{N}(0, 10^2)$

$$\mu_{t+1|t} = A \cdot \mu_{t|t}$$

$$\Sigma_{t+1|t} = A \cdot \Sigma_{t|t} A^T + W = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \Sigma_{t|t} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{100} & 0 \\ 0 & 1 \end{bmatrix}$$

Kalman Gain:

$$K_{t+1|t} = \Sigma_{t+1|t} \cdot H^T \cdot \left(H \Sigma_{t+1|t} \cdot H^T + V\right)^{-1}$$

$$= \Sigma_{t+1|t} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \Sigma_{t+1|t} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 100\right)^{-1}$$

Update:

$$\mu_{t+1|t+1} = \mu_{t+1|t} + K_{t+1|t} \cdot \left(z_{t+1} - H \mu_{t+1|t}\right)$$

$$= \mu_{t+1|t} + K_{t+1|t} \cdot \left(z_{t+1} - \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \mu_{t+1|t}\right)$$

$$\Sigma_{t+1|t} = \left(I - K_{t+1|t} \cdot H\right) \cdot \Sigma_{t+1|t}$$

$$= \left(I - K_{t+1|t} \cdot \begin{bmatrix} 1 & 0 \end{bmatrix}\right) \cdot \Sigma_{t+1|t}$$

$\boxed{\text{Problem 1b}}$

$$\pi^{(k)}(x) := \frac{P_h(z_{t+1}|x) \cdot P_a(x|\mu_{t|t}^{(k)})}{\int P_h(z_{t+1}|s) \cdot P_a(s|\mu_{t|t}^{(k)}) \cdot ds}$$

$P_h(z_{t+1}|x)$, $z_{t+1} \sim \mathcal{N}(\begin{bmatrix} 1 & 0 \end{bmatrix} \cdot x_t, 100)$

$P_a(x|\mu_{t|t})$, $x \sim \mathcal{N}\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mu_{t|t}, \begin{bmatrix} \frac{1}{100} & 0 \\ 0 & 1 \end{bmatrix}\right)$

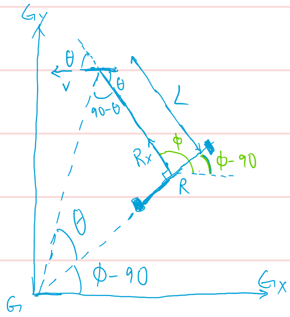
$$\pi^{(k)}(x) = \frac{1}{\eta_{t+1}} \Big[\exp\Big(-\tfrac{1}{2}(z_{t+1} - \begin{bmatrix} 1 & 0 \end{bmatrix} x_t)^T \cdot \tfrac{1}{100} \cdot (z_{t+1} - \begin{bmatrix} 1 & 0 \end{bmatrix} x_t\Big)$$

$$\cdot \exp\Big(-\tfrac{1}{2}(x - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mu_{t|t}) \cdot \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix} \cdot (x - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mu_{t|t})\Big)\Big]$$

where $\eta_{t+1} = \int \exp\Big(-\tfrac{1}{2}(z_{t+1} - \begin{bmatrix} 1 & 0 \end{bmatrix} s)^T \tfrac{1}{100} (z_{t+1} - \begin{bmatrix} 1 & 0 \end{bmatrix} x_t\Big)$

$$\cdot \exp\Big(-\tfrac{1}{2}(s - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mu_{t|t}) \cdot \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix} (s - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \mu_{t|t})\Big)\Big]$$

Problem 2a



r : distance b/w frame R and G

$\tan\theta = \frac{L}{r}$, $r = \frac{L}{\tan\theta}$

V_b : velocity of R (center of mass)

$V_b = V\cos(\theta)$, suppose V is scalar.

$$\dot{x} = V_b \cdot \cos(\phi) = V \cdot \cos(\theta) \cdot \cos(\phi)$$

$$\dot{y} = V_b \cdot \sin(\phi) = V \cdot \cos(\theta) \cdot \sin(\phi)$$

$$\dot{\phi} = W_R = \frac{V_b}{r} = V \cdot \cos(\theta) \frac{\tan(\theta)}{L} = \frac{V \cdot \sin(\theta)}{L}$$

Problem 2b

let $\tau = t - t_0$

$$x(t) = x(t_0) + \int_{t_0}^{t} \dot{x}\, ds$$

$$= x(t_0) + \int_{t_0}^{t} V \cdot \cos(\theta) \cdot \cos(\phi)\, ds$$

$$= x(t_0) + \tau \cdot V \cdot \cos(\theta) \cdot \operatorname{sinc}\left(\frac{w_{t_0} \cdot \tau}{2}\right) \cdot \cos\left(\phi_{t_0} + \frac{w_{t_0} \cdot \tau}{2}\right)$$

$$y(t) = y(t_0) + \int_{t_0}^{t} \dot{y}\, ds$$

$$= y(t_0) + \int_{t_0}^{t} V\cos(\theta) \cdot \sin(\phi)\, ds$$

$$= y(t_0) + \tau \cdot V \cdot \cos(\theta) \cdot \operatorname{sinc}\left(\frac{w_{t_0} \cdot \tau}{2}\right) \cdot \sin\left(\phi_{t_0} + \frac{w_{t_0} \cdot \tau}{2}\right)$$

$$\phi(t) = \phi(t_0) + \int_{t_0}^{t} \dot{\phi}\, ds$$

$$= \phi(t_0) + \int_{t_0}^{t} \frac{V \cdot \sin(\theta)}{L}\, ds$$

$$= \phi(t_0) + \tau \cdot w_{t_0}$$

where $w_{t_0} = \frac{V\sin(\theta)}{L}$

Problem 2c

Control inputs are perturbed by additive Gaussian noise:

$V + w_v$, $\theta + w_\theta$; $w_v \sim N(0, \sigma_v^2)$, $w_\theta \sim N(0, \sigma_\theta^2)$

EKF: $\quad s \sim N(\mu_{t|t}, \Sigma_{t|t})$

EKF: $s \sim \mathcal{N}(\mu_{t|t}, \Sigma_{t|t})$

$$A_t := \frac{\partial a}{\partial s}(\mu_{t|t}, u_t, 0), \quad \text{where } u_t = (v, \theta),\ s = (x, y, \phi)$$

$$= \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} & \frac{\partial y}{\partial \phi} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial \phi} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial \phi} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -\tau\cdot(v+w_v)\cdot\cos(\theta+w_\theta)\operatorname{sinc}\left(\frac{w_{t_0}\cdot\tau}{2}\right)\cdot\sin\left(\phi_{t_0}+\frac{w_{t_0}\cdot\tau}{2}\right) \\ 0 & 1 & \tau\cdot(v+w_v)\cdot\cos(\theta+w_\theta)\operatorname{sinc}\left(\frac{w_{t_0}\cdot\tau}{2}\right)\cdot\cos\left(\phi_{t_0}+\frac{w_{t_0}\cdot\tau}{2}\right) \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q_t = \frac{\partial a}{\partial w}(\mu_{t|t}, u_t, 0), \quad \text{where } w = (w_v, w_\theta)$$

$$= \begin{bmatrix} \frac{\partial x}{\partial w_v} & \frac{\partial x}{\partial w_\theta} \\ \frac{\partial y}{\partial w_v} & \frac{\partial y}{\partial w_\theta} \\ \frac{\partial \phi}{\partial w_v} & \frac{\partial \phi}{\partial w_\theta} \end{bmatrix}$$

$$= \begin{bmatrix} \tau\cos(\theta+w_\theta)\cdot\operatorname{sinc}\left(\frac{w_{t_0}\cdot\tau}{2}\right)\cos\left(\phi_{t_0}+\frac{w_{t_0}\cdot\tau}{2}\right) & -\tau(v+w_v)\cdot\sin(\theta+w_\theta)\cdot\operatorname{sinc}\left(\frac{w_{t_0}\cdot\tau}{2}\right)\cos\left(\phi_{t_0}+\frac{w_{t_0}\cdot\tau}{2}\right) \\ \tau\cos(\theta+w_\theta)\cdot\operatorname{sinc}\left(\frac{w_{t_0}\cdot\tau}{2}\right)\sin\left(\phi_{t_0}+\frac{w_{t_0}\cdot\tau}{2}\right) & -\tau(v+w_v)\cdot\sin(\theta+w_\theta)\cdot\operatorname{sinc}\left(\frac{w_{t_0}\cdot\tau}{2}\right)\cos\left(\phi_{t_0}+\frac{w_{t_0}\cdot\tau}{2}\right) \\ \frac{\sin(\theta+w_\theta)}{L} & \frac{(v+w_v)\cdot\cos(\theta+w_\theta)}{L} \end{bmatrix}$$

Mean and cov propagation:

$$\mu_{t+1|t} = a(\mu_{t|t}, u_t, 0)$$

$$\Sigma_{t+1|t} = A_t\cdot\Sigma_{t|t}A_t^T + Q_t\cdot\begin{bmatrix}\sigma_v^2 & 0 \\ 0 & \sigma_\theta^2\end{bmatrix}\cdot Q_t^T$$

Problem 2d

$(x_b, y_b) :=$ beacon position in frame $\{G\}$

$(x_w, y_w) :=$ position of front wheel in $\{G\}$

$(x_R, y_R) :=$ position of frame $\{R\}$ in $\{G\}$

$$x_w = x_R + L\cdot\cos(\phi)$$

$$y_w = y_R + L\sin(\phi)$$

(i) distance $d := \sqrt{(x_w - x_b)^2 + (y_w - y_b)^2}$

(ii) $J = \frac{\partial d}{\partial s} = \left[\frac{\partial d}{\partial x} \quad \frac{\partial d}{\partial y} \quad \frac{\partial d}{\partial \phi}\right]$

$$\frac{\partial d}{\partial x} = \frac{1}{2d}\cdot 2\cdot(x_R + L\cos(\phi) - x_b) = \frac{x_R + L\cos(\phi) - x_b}{d}$$

$$\frac{\partial d}{\partial y} = \frac{y_R + L\sin(\phi) - y_b}{d} \qquad \left(\text{similar to } \frac{\partial d}{\partial x}\right)$$

$$\frac{\partial d}{\partial \phi} = \frac{1}{2d}\cdot\Big[2(x_R + L\cos(\phi) - x_b)\cdot(-L\sin(\phi)) + 2(y_R + L\sin(\phi) - y_b)\cdot(L\cos(\phi))\Big]$$

$$= \frac{L}{d}\Big[\cos(\phi)(y_R + L\sin(\phi) - y_b) - \sin(\phi)(x_R + L\cos(\phi) - x_b)\Big]$$

$$= \frac{L}{d}\Big[\cos(\phi)(y_R - y_b) - \sin(\phi)(x_R - x_b)\Big]$$