PARTIAL DIFFERENTIAL EQUATIONS

BASIC CONCEPTS OF PARTIAL DIFFERENTIAL EQUATIONS [1]

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Recommended Textbook:*Partial Differential Equations-An introduction by Walter A. Strauss*[2]

The study of partial differential equations (PDEs) started in the 18th century in the work of Euler, d'Alembert, Lagrange and Laplace as a central tool or more generally, as the principal mode of analytical study of models in the physical science. Beginning in the middle of the 19th century, particularly with the work of Riemann, PDEs also became an essential tool in other branches of mathematics and today they are used in a almost all disciplines. For example they are used in the natural or physical sciences (such as physics, Chemistry, Biology, Earth science, Meteorology) and engineering disciplines (such as computer science, artificial intelligence), as well as in the social sciences (such as economics, psychology, sociology, political science). Physicists, engineers, statisticians, Actuarists, operations research analysts, and economists use PDEs extensively to explain a system, study the effects of different components, and to make predictions about behaviour.

Mathematical Models, Conservation and Constitute Laws

The notion of a *mathematical model* is understood as a mathematical problem whose solution describes the behaviour of the studied system. In general, a mathematical model is a simplified mathematical description of a real-world problem. In our case, we will deal with models described by partial differential equations, that is, differential equations with two or more independent variables. Studying natural, technical, economical, biological, chemical and even social processes, we observe two main tendencies: the tendency to achieve a certain *balance* between causes and consequences, and the tendency to break this balance. Thus, as a starting point for the derivation of many mathematical models, we usually will use some law or principle that expresses such a balance between the so called *state quantities* and *flow quantities* and their spatial and time changes.

Let us consider a medium (body, liquid, gas, solid substance, etc.) that fills a domain.

$$\Omega \subset \mathbb{R}^N$$

Here N denotes the spatial dimension. In real situations, usually, $N=3$, in simplified models, $N=2$ or $N=1$. We denote by

$$u = u(\boldsymbol{x}, t), \boldsymbol{x} \in \Omega, t \in [0, T) \subset [0, +\infty)$$

the state function (scalar, vector or tensor) of the substance considered at a point $\mathbf{x}$ and time t. In further considerations, we assume u to be a scalar function. The flow function (vector function, in general) of the same substance will be denoted by

$$\phi = \phi(\mathbf{x}, t), \ \mathbf{x} \in \Omega, \ t \in [0, T) \subset [0, +\infty)$$

The density of sources is usually described by a scalar function

$$f = f(\mathbf{x}, t), \ \mathbf{x} \in \Omega, \ t \in [0, T) \subset [0, +\infty)$$

Let $\Omega_B \in \Omega$ be an arbitrary inner subdomain of Ω. The integral

$$U(\Omega_B, t) = \int_{\Omega_B} u(\mathbf{x}, t) d\mathbf{x}$$

represents the total amount of the quantity considered in the *balance domain* Ω_B at time t. The integral

$$\int_{t_1}^{t_2} \int_{\Omega_B} u(\mathbf{x}, t) \, d\mathbf{x} \, dt$$

then represents the total amount of the quality in Ω_B and in the time interval $[t_1, t_2] \subset [0, T)$. (The set $\Omega_B \ X \ [t_1, t_2]$ is called the space-time balance domain.)

In particular, if the state function $u(\mathbf{x}, t)$ corresponds to the mass density $\varrho(\mathbf{x}, t)$, then the integral

$$m(\Omega_B, t) = \int_{\Omega_B} \varrho(\mathbf{x}, t) d\mathbf{x}$$

represents the mass of the substance in the balance domain Ω_B at time t, and the value

$$m(\Omega_B, t_1, t_2) = \int_{t_1}^{t_2} \int_{\Omega_B} \varrho(\mathbf{x}, t) d\mathbf{x} dt$$

corresponds to the total mass of the substance contained in Ω_B during the time interval $[t_1, t_2]$.

If we denote by $\partial\Omega_B$ the boundary of the balance domain Ω_B, then the boundary integral (surface integral in $\mathbb{R}^3$, curve integral in $\mathbb{R}^2$)

$$\Phi(\partial\Omega_B, t) = \int_{\partial\Omega_B} \phi(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) dS$$

 represents the amount of the quantity "flowing through" the boundary $\partial\Omega_B$ in the direction of the outer normal $\mathbf{n}$ during the time interval $[t_1, t_2]$.

As we have stated above, the distribution of sources is usually described by a function $f = f(\mathbf{x}, t)$ corresponding to the source density at a point $\mathbf{x}$ and time t. The integral

$$F(\Omega_B, t_1, t_2) = \int_{t_1}^{t_2} \int_{\Omega_B} f(\mathbf{x}, t) d\mathbf{x} dt$$

then represents the total source production in Ω_B during the time interval $[t_1, t_2]$.

Evolution Conservation Law

If the time evolution of the system has to be taken into account, we speak about an *evolution* process. To derive a balance principle for such a process, we choose an arbitrary balance domain $\Omega_B \subset \Omega$ and an arbitrary time interval $[t_1, t_2] \subset [0, \infty)$. For simplicity, we consider a scalar state function $u = u(x, t)$, a vector flow function $\phi = \phi(x, t)$, and a scalar source function $f = f(x, t)$.

The basic balance law says that the change of the total amount of the quantity u contained in Ω_B between times t_1 and t_2 must be equal to the total amount flowing across the boundary $\partial\Omega_B$ from time t_1 to t_2, and to the increase (or decrease) of the quantity produced b sources (or sinks) inside Ω_B during the time interval $[t_1, t_2]$. In the language of mathematics, we write it as

$$\int_{\Omega_B} u(x, t_2)dx - \int_{\Omega_B} u(x, t_1)dx \tag{1}$$

$$= -\int_{t_1}^{t_2} \int_{\partial\Omega_B} \phi(x, t) \cdot n(x)dSdt + \int_{t_1}^{t_2} \int_{\Omega_B} f(x, t)dxdt.$$

The minus sign in front of the first term on the right-hand side of Eq. (1) corresponds to the fact that the flux is understood as positive in the outward direction. Equation (1) represents the evolution conservation law in its integral (global) form.

If we assume u to have continuous partial derivative with respect to t, we can write the difference $u(x, t_2)$- $u(x, t_1)$ as $\int_{t_1}^{t_2} \frac{\partial}{\partial t} u(x, t)dt$ and change the order of the integration on the left-hand side of Equation (1), then it follows that:

$$\int_{t_1}^{t_2} \int_{\Omega_B} \frac{\partial}{\partial t} u(x, t)dxdt = -\int_{t_1}^{t_2} \int_{\partial\Omega_B} \phi(x, t) \cdot n(x)dSdt + \int_{t_1}^{t_2} \int_{\Omega_B} f(x, t)dxdt \tag{2}$$

Since the time interval $[t_1, t_2]$ has been chosen arbitrarily, we can come (under the assumption of continuity in the tie variable of all functions involved) to the expression

$$\int_{\Omega_B} \frac{\partial}{\partial t} u(x, t)dx = -\int_{\partial\Omega_B} \phi(x, t) \cdot n(x)dS + \int_{\Omega_B} f(x, t)dx \tag{3}$$

Now, if we assume ϕ to be continuously differentiable in the spatial variables, we can use the Divergence Theorem, according to which we can write

$$\int_{\partial\Omega_B} \phi(x, t) \cdot n(x)dS = \int_{\Omega_B} div\phi(x, t)dx \tag{4}$$

If we substitute this relation into Equation (3) and assume the continuity of all functions in spatial variables, we will get the following:

$$\frac{\partial}{\partial t} u(x, t) + div\phi(x, t) = f(x, t) \tag{5}$$

Since the balance domain has been arbitrary as well. Equation (5) is a local version of (1) and expresses the *conservation law in its differential (local) form.* It is a single equation for two unknown functions u and ϕ. The sources f are usually given, however, they can depend on x, t also via the quantity u, that is, we can have $f = f(x, t, u(x, t))$. Thus the conservation law alone is not sufficient for the construction of a mathematical model.

Stationary Conservation Law

Sometimes we are not interested in the time evolution of the system considered. We say that we study the *stationary state or behaviour of the system*. It means that we suppose all quantities to be time-independent (they have zero time derivatives). In such cases, we use simplified versions of conservative laws. In particular, the global form of the stationary conservation law is given by

$$\int_{\partial \Omega_B} \phi(x) \cdot n(x) dS = \int_{\Omega_B} f(x) dx, \tag{6}$$

and its local version has the form

$$div\phi(x) = f(x) \tag{7}$$

(under the assumptions of continuity of all functions involved and their spatial derivatives).

Conservation Law in One Dimension

In some situations, we an assume that all significant changes proceed only in one direction (for instance, in modeling the convection in a wide tube, when we are not interested in the situation near the tube walls; or, conversely in modeling the behaviour of a thin string or a thin bar with constant cross-section). In such cases, we can reduce our model to one spatial dimension. Since the corresponding one-dimensional basic conservation law differs in some minor points from the general one (eg., ϕ is now a scalar function), we state it here explicitly.

Let us consider a tube with constant cross-section A, an arbitrary segment $a \leq x \leq b$, a time interval $[t_1, t_2]$, and a quantity with density u.
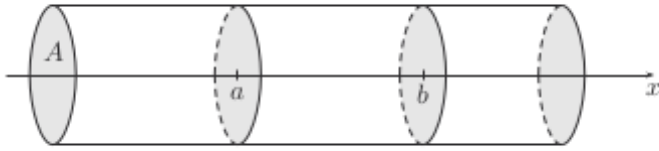

Figure 1: An Isolated tube with cross-section; the quantities considered change only in the direction of the x-axis.

The conservation law says, again, that the change of quantity in the spatial segment $[a, b]$ between times t_1 and t_2 equals the total flow at the point $x = a$ decreased by the total flow at the point $x = b$ from time t_1 to t_2, and to the total source balance in $[a, b]$ and $[t_1, t_2]$:

$$\int_a^b u(x, t_2) dx - \int_a^b u(x, t_1) dx \tag{8}$$

$$= \int_{t_1}^{t_2} (\phi(a, t) - \phi(b, t)) dt + \int_{t_1}^{t_2} \int_a^b f(x, t) dx dt.$$

This equation represents the *one-dimensional conservation law in its integral (global) form* (1). If the functions u and ϕ are smooth enough, we can proceed similarly to the general, multidimensional case, and obtain the differential formulation.

To be specific, if u has continuous partial derivative with respect to t and ϕ has continuous partial derivative with respect to x, equation 8 reduces to the form

$$\int_{t_1}^{t_2} \int_a^b [u_t(x,t) + \phi_x(x,t) - f(x,t)]dxdt = 0 \qquad (9)$$

Since the interval $[a,b]$ and $[t_1, t_2]$ have been closely arbitrarily, the integrand must be identically equal to zero, thus

$$u_t(x,t) + \phi_x(x,t) = f(x,t) \qquad (10)$$

Equation (10) is a local version of (9) and expresses the *one-dimensional conservation law in its differential (local) form* (cf. (1.4)).

If we model a one-dimensional stationary phenomenon, we use the stationary version of the previous conservation law, that is

$$\phi_x(x) = f(x) \qquad (11)$$

which is actually an ordinary differential equation.

Constitutive Laws

In particular processes and phenomenon, the state and flow functions (quantities) have their concrete terms and notation. For example, for the description of thermodynamic processes, we usually use some of the following parameters:

State parameters		**Flow parameters**	
1	Density	1	Velocity
2	Pressure	2	Momentum
3	Temperature	3	Tension
4	Specific internal energy	4	Heat Flux
5	Entropy		

The problem of mutual dependence or independence of these parameters is very complicated and it is connected with the choice of the mathematical model. Relations between the state quantity and the relevant flow quantity are usually based on the generalization of experimental observations and depend on the properties of the particular medium or material. They are usually called *constitutive laws* or *material relations*.

We end this section with a few examples of fundamental physical processes and their mathematical models.

Example 1 (Convection and Transport). *The one-dimensional convection model describes the drift of a contaminant in a tube with a flowing liquid of constant speed. Here the state quantity is the concentration u of the contaminant, the flow quantity is its flux ϕ. Both these quantities are related by the constitutive law:*

$$\phi = cu$$

with a constant $c > 0$ corresponding to the velocity of the flowing liquid. Substituting this relation into the local conservation law, we obtain the transportequation

$$u_t + cu_x = f(x,t)$$

Example 2 (**Diffusion**). *The diffusion processes can be characterized by the state quantity u corresponding to the concentration of the diffusion substance and by its flux ϕ (the flow quantity). In the case, the constitutive law is so called Fick's law*

$$\phi = -kgradu.$$

which describes the fact that molecules tend to move from places of higher concentration to places with lower concentration. The material constant $k > 0$ is called the diffusion coefficient. This law together with the conservation law yields the diffusion equation

$$u_t - k\Delta u = f.$$

Example 3 (Heat Flow). *In the model of the heat flow, the state quantity u corresponds to the temperature, and its multiple $c\rho u$ describes the density of the internal heat energy. Here c is the specific heat capacity and μ the mass density (material constant) of the medium considered. The flow quantity is represented by the heat flux ϕ. The corresponding constitutive law is so called Fourier's law, which expresses the fact that heat flux is directly proportional to the temperature gradient:*

$$\phi = -kgradu.$$

The constant $K > 0$ represents the heat (or thermal) conductivity (a material constant). Inserting the constitutive law directly into the conservation law (which is nothing else but the heat conservation law), we obtain the heat equation

$$u_t - k\Delta u = f$$

(Here, $k = \frac{K}{c\rho}$ is called the thermal diffusivity.) Notice that the heat and diffusion equations are identical! For detailed derivation see Section 5.1

Example 4 (**Wave Emotion**). *The wave motion is described by th mass density of the medium and its displacement u (state quantities), and by its inner tension and momentum (flow quantities). The simplest model- the wave equation- has the form*

$$u_{tt} - c^2\Delta u = f,$$

where the constant c corresponds to the speed of the wave propagation. Here we use the conservation of mass and the conservation of momentum (Newton's law of motion), but we cannot use the standard scheme (to substitute the corresponding constitutive law into the conservation law). The derivation of the wave equation requires slightly different arguments and is treated in detail in Section 4.1

Example 5 (**Stationary Processes**). *The models of stationary phenomena can be derived separately (for example, as states with minimal energy), or we can understand them as special cases of the corresponding dynamical processes with time-independent quantities. Using the latter approach, the three previous equations are reduced to the Poisson (or Laplace, for f = 0) equation*

$$\Delta u = f$$

All the above are treated in detail in through out the course.

References

W.A. Strauss. *Partial Differential Equations: An Introduction.* Wiley,
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