DATE NO

DYNAMICAL SYSTEMS

Methods of solving systems of ordinary differential equations

- Distinct Root.
- Repeated Root.
- Complex Root.

① Find the solutions of the differential equation.

$dx_1/dt = -4x_1 + x_2 + x_3$

$dx_2/dt = x_1 + 5x_2 - x_3$

$dx_3/dt = x_2 - 3x_3$

$$\begin{pmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \end{pmatrix} = \begin{bmatrix} -4 & 1 & 1 \\ 1 & 5 & -1 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$dx/dt = Ax \qquad |A - \lambda \underline{I}| = 0$

Using $\det(A - \lambda \underline{I}) = 0$

$$A - \lambda \underline{I} = \begin{bmatrix} -4 & 1 & 1 \\ 1 & 5 & -1 \\ 0 & 1 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} -4-\lambda & 1 & 1 \\ 1 & 5-\lambda & -1 \\ 0 & 1 & -3-\lambda \end{bmatrix}$$

$$|A - \lambda \underline{I}| = \begin{vmatrix} -4-\lambda & 1 & 1 \\ 1 & 5-\lambda & -1 \\ 0 & 1 & -3-\lambda \end{vmatrix} = 0$$

$$-4-\lambda \begin{vmatrix} 5-\lambda & -1 \\ 1 & -3-\lambda \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ 0 & -3-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 5-\lambda \\ 0 & 1 \end{vmatrix} = 0$$

$-4-\lambda\left[5-\lambda(-3-\lambda) - 1(-1)\right] - \left[-3-\lambda\right] + 1 = 0$

$56 + 8\lambda - 4\lambda^2 + 14\lambda + 2\lambda^2 - \lambda^3 + 3 + \lambda + 1 = 0$

$60 + 23\lambda - 2\lambda^2 - \lambda^3 = 0$

$\lambda_1 = -4 \quad \lambda_2 = 5, \quad \lambda_3 = -3$

Taking $\lambda_1 = -4$

Using $(A - \lambda \underline{I})v_1 = 0$

let $v_1 = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$

$\Rightarrow (A + 4\underline{I})v_1 = 0$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 9 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$k_2 + k_3 = 0$ ——— ①

$k_1 + 9k_2 - k_3 = 0$ ——— ②

$k_2 + k_3 = 0$ ——— ③

$k_2 = -k_3$ ——— ④

$k_1 = -9k_2 + k_3$ ——— ⑤

Plugging ④ into ⑤

$k_1 = 10k_3$ ——— ⑥

$v_1 = \begin{pmatrix} 10 \\ -1 \\ 1 \end{pmatrix} k_3 = \begin{pmatrix} 10k_3 \\ -k_3 \\ k_3 \end{pmatrix}$

choosing $k_3 = 1$

$v_1 = \begin{pmatrix} 10 \\ -1 \\ 1 \end{pmatrix} \qquad x_1(t) = e^{-4t}\begin{pmatrix} 10 \\ -1 \\ 1 \end{pmatrix}$

For $\lambda_2 = 5$

$\Rightarrow (A - 5\underline{I})v_2 = 0$

let $v_2 = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$

$$\begin{bmatrix} -9 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -8 \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-9l_1 + l_2 + l_3 = 0$ ——— ①

$l_1 - l_3 = 0$ ——— ②

$l_2 - 8l_3 = 0$ ——— ③

$l_1 = l_3$ ——— ④

$l_2 = 8l_3$ ——— ⑤

$v_2 = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} l_3 \\ 8l_3 \\ l_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix} l_3$

choosing $l_3 = 1$

$v_2 = \begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix} \qquad x_2(t) = e^{5t}\begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix}$

For $\lambda_3 = -3$

$\Rightarrow (A + 3\underline{I})V_3 = 0$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & 8 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-m_1 + m_2 + m_3 = 0$ ——— (1)

$m_1 + 8m_2 - m_3 = 0$ ——— (2)

$m_2 = 0$ ——— (3)

$-m_1 + m_3 = 0$

$m_1 = m_3$ ——— (4)

$$V_3 = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} m_1 \\ 0 \\ m_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} m_1$$

choosing $m_1 = 1$

$$V_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad x_3(t) = e^{-3t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$x(t) = c_1 e^{-4t}\begin{pmatrix} 10 \\ -1 \\ 1 \end{pmatrix} + c_2 e^{5t}\begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix} + c_3 e^{-3t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

(2)

Find the solution of the equation.

$dx_1/dt = x_1 - x_2 + 4x_3$

$dx_2/dt = 3x_1 + 2x_2 - x_3$

$dx_3/dt = 2x_1 + x_2 - x_3$

$$\begin{pmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \end{pmatrix} = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$dx/dt = Ax \qquad |A - \lambda\underline{I}| = 0$

Using $\det(A - \lambda\underline{I}) = 0$

$$A - \lambda\underline{I} = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{bmatrix}$$

$$|A - \lambda\underline{I}| = \begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ 2 & -1-\lambda \end{vmatrix} + 4\begin{vmatrix} 3 & 2-\lambda \\ 2 & 1 \end{vmatrix} = 0$$

$= (1-\lambda)\left[(2-\lambda)(-1-\lambda)+1\right] + \left[3(-1-\lambda) + 2\right] + 4\left[3 - 2(2-\lambda)\right] = 0$

$= (1-\lambda)(-2-2\lambda+\lambda+\lambda^2+1) + (-3-3\lambda+2) + 4(3-4+2\lambda) = 0$

$= (1-\lambda)(\lambda^2-\lambda-1) + (-3\lambda-1) + 4(2\lambda-1) = 0$

$= \lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

(characteristic polynomial.)

Using the factor theorem

$\lambda_1 = -2, \ \lambda_2 = 3 \quad \lambda_3 = 1$

Taking $\lambda_1 = -2$

The corresponding eigenvectors for the eigenvalue $\lambda_1 = -2$ is obtained as follows

Using $(A - \lambda\underline{I})V_1 = 0$

let $V_1 = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$

$\Rightarrow (A + 2\underline{I})V_1 = 0$

$$= \begin{bmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3k_1 - k_2 + 4k_3 = 0$ ——— (1)

$3k_1 + 4k_2 - k_3 = 0$ ——— (2)

$2k_1 + k_2 + k_3 = 0$ ——— (3)

From eqn (1)

$k_2 = 3k_1 + 4k_3$ ——— (4)

From eqn (2)

$k_3 = 3k_1 + 4k_2$ ——— (5)

Plugging (5) into (4)

$k_2 = 3k_1 + 4(3k_1 + 4k_2)$

$k_2 = 3k_1 + 12k_1 + 16k_2$

$k_1 = -k_2$ ——— (6)

Put (6) into (5)

$k_3 = 3(-k_2) + 4k_2$

$k_3 = k_2$

$$v_1 = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} -k_2 \\ k_2 \\ k_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} k_2$$

$k_2 = 0$ or 1

choosing $k_2 = 1$

$$\therefore v_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \quad x_1(t) = e^{-2t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 3$

$(A - \lambda_2 I)v_2 = 0$

let $v_2 = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$

$$= \begin{bmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2m_1 - m_2 + 4m_3 = 0$ —— (1)

$3m_1 - m_2 - m_3 = 0$ —— (2)

$2m_1 + m_2 - 4m_3 = 0$ —— (3)

$m_3 = 3m_1 - m_2$ —— (4)

$m_2 = -2m_1 + 4m_3$ —— (5)

put (5) into (4)

$m_3 = 3m_1 - (-2m_1 + 4m_3)$

$m_3 = 3m_1 + 2m_1 - 4m_3$

$m_3 = m_1$ —— (6)

put (6) into (5)

$m_2 = 2m_1$

$$v_2 = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} m_1 \\ 2m_1 \\ m_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} m_1$$

choosing $m_1 = 1$

$$v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad x_2(t) = e^{3t}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

For $\lambda_3 = 1$

$(A - \lambda_3 I) = 0$

let $v_3 = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$

$$= \begin{bmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-q_2 + 4q_3 = 0$ —— (1)

$3q_1 + q_2 - q_3 = 0$ —— (2)

$2q_1 + q_2 - 2q_3 = 0$ —— (3)

$q_2 = 4q_3$ —— (4)

$q_1 = -q_3$ —— (5)

$$v_3 = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} -q_3 \\ 4q_3 \\ q_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} q_3$$

since the solution is linear and linearly independent we can apply principle of superposition.

choosing $q_3 = 1$

$$v_3 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} \quad x_3(t) = e^{t}\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$$

$x(t) = C_1x_1(t) + C_2x_2(t) + C_3x_3(t)$

$$x(t) = C_1e^{-2t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + C_2e^{3t}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + C_3e^{t}\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$$

04/07/22

REPEATED ROOTS

A solution to a differential equation must satisfy the functional properties and the functional properties includes a ~~f.~~ solution space (solution set).

$$\frac{dx}{dt} = ax \Rightarrow \int \frac{dx}{x} = \int a\,dt$$

$\int \frac{1}{x}\,dx = \int a\,dt$

$\ln|x| = at + k$

$x = e^{at}.e^{k}$

$x(t) = e^{at}.c$

$\frac{dx}{dt} = ae^{at}.c$

$ae^{at} \cdot c = ae^{at} \cdot c$

$\frac{dx}{dt} = Ax$

$x(t) = e^{At} \cdot e^{c}$

let $A = e^{c}$

$x(t) = Ae^{At}$

setting $A = v$

$\Rightarrow x(t) = e^{At} \cdot v$

$$e^{At} v = \left[I + At + \frac{A^2t^2}{2!} \dots + \frac{A^n t^n}{n!} + \dots\right] v$$

$$\frac{d}{dt}(e^{At} v) = \frac{d}{dt}\left[I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots + \frac{A^n t^n}{n!} + \dots\right] v$$

$$= \left[A + A^2 t + \frac{A^3t^2}{2!} + \dots \frac{A^n t^{n-1}}{(n-1)!} + \dots\right] v$$

$$= A\left[I + At + \dots + \frac{A^{n-1}t^{n-1}}{(n-1)!} + \dots\right] v$$

$$= A(e^{At}) v$$

→ Since we are working in matrices it becomes an identity matrix.

$$\Rightarrow \frac{d}{dt}(e^{At} v) = A(e^{At} v)$$

setting $e^{At} v = e^{(A-\lambda I)t} \cdot e^{\lambda I t} \cdot v$

$$\Rightarrow e^{\lambda I t} = \left[I + \lambda I t + \frac{\lambda^2 I^2 t^2}{2!} + \dots + \frac{\lambda^n I^n t^n}{n!} + \dots\right]$$

$$= \left[1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \dots + \frac{\lambda^n t^n}{n!} + \dots\right]$$

$$\Rightarrow e^{\lambda I t} = e^{\lambda t}.$$

Similarly

$e^{(A-\lambda I)t} v$

$$= \left[I + (A-\lambda I)t + \frac{(A-\lambda I)^2 t^2}{2!} + \dots + \frac{(A-\lambda I)^n t^n}{n!}\right] v$$

$$= \left[v + v(A-\lambda I)t + \frac{v(A-\lambda I)^2 t^2}{2!} + \dots + \frac{v(A-\lambda I)^n}{n!}\right.$$

$$\therefore e^{At} v = e^{\lambda t}\left[v + v(A-\lambda I)t + \dots + \frac{v(A-\lambda I)^n t^n}{n!}\right.$$

Find all the ~~system of~~ solution of the system of the differential equation

$dx_1/dt = x_1 + x_2$

$dx_2/dt = x_2$

$dx_3/dt = 2x_3$

Solution

$$\begin{pmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$(A - \lambda I)$ $\quad |A - \lambda I| = 0$

$$(A-\lambda I) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} - 1\begin{vmatrix} 0 & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(1-\lambda)(2-\lambda)\right] = 0$$

$$(1-\lambda)\left[2 - 3\lambda + \lambda^2\right] = 0$$

$\cancel{2 - 3\lambda - \lambda^3}$ $\quad \lambda = 1 \quad \lambda_2 = 1 \quad \lambda_3 = 2$

To find the corresponding eigenvector for $\lambda = 1$, we have:

$(A - \lambda I) v_1 = 0$

let $v_1 = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$k_2 = 0$

$k_3 = 0$

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The solution cannot be trivial therefore we take

$e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $e_3 \begin{pmatrix}0\\0\\1\end{pmatrix}$

in this case since k_1 is the only one not known.

$V_1 = \begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} k_1$

choosing $k_1 = 1$

$V_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$

$\therefore x_1(t) = e^t \begin{pmatrix}1\\0\\0\end{pmatrix}$

To obtain the eigenvector for $\lambda_2 = 1$ We use $(A-\lambda \underline{I})^2 V = 0$ but $(A-\lambda \underline{I})V \neq 0$ since it has been proven not to be trivial.

let $V_2 = \begin{pmatrix}m_1\\m_2\\m_3\end{pmatrix}$

$$\begin{bmatrix}0&1&0\\0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}m_1\\m_2\\m_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}m_1\\m_2\\m_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$m_3 = 0$

You can choose either e_1 or e_2 but since we chose e_1 for V_1 we cannot choose e_1 again

$V_2 = \begin{pmatrix}m_1\\m_2\\m_3\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$e^{\lambda t} = [V + (A-\lambda \underline{I})tV]$

$$x_2(t) = e^t\left[\begin{pmatrix}0\\1\\0\end{pmatrix} + t\begin{bmatrix}0&1&0\\0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\right]$$

$$x_2(t) = e^t\begin{bmatrix}t\\1\\0\end{bmatrix}$$

For $\lambda_3 = 2$

$$\begin{bmatrix}-1&1&0\\0&-1&0\\0&0&0\end{bmatrix}\begin{bmatrix}n_1\\n_2\\n_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$-n_1 + n_2 = 0$

$-n_2 = 0$

$n_1 = n_2$

$n_2 = 0$

$n_1 = 0$

$V_3 = \begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$x_3(t) = e^{2t}\begin{pmatrix}0\\0\\1\end{pmatrix}$

$x(t) = C_1x_1(t) + C_2x_2(t) + C_3x_3(t)$

$$x(t) = C_1e^t\begin{bmatrix}1\\0\\0\end{bmatrix} + C_2e^t\begin{bmatrix}t\\1\\0\end{bmatrix} + C_3e^{2t}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$x(t) = e^t\left[C_1\begin{pmatrix}1\\0\\0\end{pmatrix} + C_2\begin{pmatrix}t\\1\\0\end{pmatrix}\right] + C_3e^{2t}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

3 repeated Roots

$dx_1/dt = 2x_1 + x_2 + 3x_3$

$dx_2/dt = 2x_2 - x_3$

$dx_3/dt = 2x_3$

$$\begin{bmatrix}dx_1/dt\\dx_2/dt\\dx_3/dt\end{bmatrix} = \begin{bmatrix}2&1&3\\0&2&-1\\0&0&2\end{bmatrix}$$

$A-\lambda \underline{I}$ $\qquad |A-\lambda \underline{I}| = 0$

$$A-\lambda \underline{I} = \begin{bmatrix}2&1&3\\0&2&-1\\0&0&2\end{bmatrix} - \begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}$$

$$|A-\lambda \underline{I}| = \begin{bmatrix}2-\lambda&1&3\\0&2-\lambda&-1\\0&0&2-\lambda\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$(2-\lambda)(2-\lambda)(2-\lambda) = 0$

$\lambda_1 = \lambda_2 = \lambda_3 = 2$

For $\lambda_1 = 2$

$|A - 2I|v_1 = 0$

let $v_1 = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-k_3 = 0$

$k_3 = 0$ ——— (1)

$k_2 = -3k_3$ ——— (2)

Putting $k_3 = 0$ into (2)

$k_2 = 0$

In order not to obtain a trivial matrix we choose one of the natural base form of the 3×3 matrix

$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Since k_1 is the only unknown we choose e_1

$\therefore v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $x_1(t) = e^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

For $\lambda_2 = 2$

$(A - \lambda I)^2 v_2 = 0$

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-l_3 = 0$

$l_3 = 0$

Since l_3 is the only known parameter we choose the natural base form, e_2

$\therefore v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$x_2(t) = e^{\lambda t}[v + (A - \lambda I)vt]$

$$= e^{2t}\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t\right]$$

$$x_2(t) = e^{2t}\left[\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}\right] = e^{2t}\begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda_3 = 2$

$(A - \lambda I)^3 v_3 = 0$

where $v_3 = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since we don't know any of the parameters and v_1 has taken the natural base form e_1 and v_2 has taken the natural base form e_2, then we choose the remaining natural base form.

$v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$x_3(t) = \left[v + (A - \lambda I)vt + \frac{(A - \lambda I)^2 t^2 v}{2}\right]$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^2/2$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 3t \\ -t \\ 0 \end{bmatrix} + \begin{bmatrix} -t^2/2 \\ 0 \\ 0 \end{bmatrix}$$

DATE NO

$$x_3(t) = e^{2t}\begin{bmatrix} -t^2/2 + 3t \\ -t \\ 1 \end{bmatrix}$$

$$x(t) = c_1x_1(t) + c_2x_2(t) + c_3x_3(t)$$

$$x(t) = c_1e^{2t}\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2e^{2t}\begin{bmatrix}t\\1\\0\end{bmatrix} + c_3e^{2t}\begin{bmatrix}-t^2/2 + 3t \\ -t \\ 1\end{bmatrix}$$

$$x(0) = \begin{bmatrix}1\\2\\1\end{bmatrix}$$

$$\begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}c_1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\c_2\\0\end{bmatrix} + \begin{bmatrix}0\\0\\c_3\end{bmatrix}$$

$c_1 = 1 \quad c_2 = 2 \quad c_3 = 1$

$$x(t) = e^{2t}\begin{bmatrix}1\\0\\0\end{bmatrix} + 2e^{2t}\begin{bmatrix}t\\1\\0\end{bmatrix} + e^{2t}\begin{bmatrix}-t^2/2 + 3t \\ -t \\ 1\end{bmatrix}$$

$$x(t) = e^{2t}\begin{bmatrix}1 + 5t - \tfrac{1}{2}t^2 \\ 2 - t \\ 1\end{bmatrix}$$

Other method

$(A - \lambda I)V_2 = V_1$

$(A - \lambda I)V_3 = V_2$.

COMPLEX ROOTS

Eg: Solve the initial-value problems.

$dx_1/dt = x_1$

$dx_2/dt = x_2 - x_3$

$dx_3/dt = x_2 + x_3$

$$x(0) = \begin{pmatrix}1\\1\\1\end{pmatrix} \qquad dx/dt = Ax$$

$$\begin{pmatrix}dx_1/dt\\dx_2/dt\\dx_3/dt\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&-1\\0&1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

$$A - \lambda I = \begin{bmatrix}1&0&0\\0&1&-1\\0&1&1\end{bmatrix} - \lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix}1-\lambda & 0 & 0\\0&1-\lambda&-1\\0&1&1-\lambda\end{vmatrix} = 0$$

$$1-\lambda\begin{vmatrix}1-\lambda & -1\\1 & 1-\lambda\end{vmatrix}$$

$1-\lambda[(1-\lambda)(1-\lambda) - 1(-1)]$

$1-\lambda[1 - 2\lambda + \lambda^2 + 1]$

$1-\lambda[2 - 2\lambda + \lambda^2]$

$2 - 2\lambda + \lambda^2 - 2\lambda + 2\lambda^2 - \lambda^3$

$-\lambda^3 + 3\lambda^2 - 4\lambda + 2$

$\lambda_1 = 1, \quad \lambda_2 = 1+i \quad \lambda_3 = 1-i$

For $\lambda_1 = 1$

$$\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}\begin{pmatrix}m_1\\m_2\\m_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$-m_3 = 0$ —— (1)

$m_2 = 0$ —— (2)

$m_3 = 0$ —— (3)

Hence we can't have a trivial solution we choose one of the natural base form e_1 since m_1 is the only unknown.

$$V_1 = \begin{pmatrix}m_1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}m_1$$

$$x_1(t) = e^t\begin{pmatrix}1\\0\\0\end{pmatrix}$$

For $\lambda_2 = 1+i$

$(A - \lambda I)V_2 = 0$

$$\begin{pmatrix}-i&0&0\\0&-i&-1\\0&1&-i\end{pmatrix}\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$-in_1 = 0 \quad n_1 = 0$ —— (1)

$-in_2 - n_3 = 0$ —— (2)

$n_2 - in_3 = 0$ —— (3)

$n_2 = in_3$

$$V_2 = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ in_3 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} n_3$$

choosing $n_3 = 1$

$$V_2 = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} = \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} i\right]$$

$$X_2(t) = e^{(1+i)t}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} i\right)$$

but $e^{\theta i} = \cos\theta + i\sin\theta$

$e^{-\theta i} = \cos\theta - i\sin\theta$

$e^{(1+i)} = e^t \cdot e^{ti}$

$e^{ti} = \cos(t) + i\sin t$

$$X_2(t) = e^t\left[(\cos(t) + i\sin(t)) \times \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} i\right)\right]$$

$$X_2(t) = e^t\left[\begin{pmatrix} 0 \\ 0 \\ \cos(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \cos t \\ 0 \end{pmatrix} i + \begin{pmatrix} 0 \\ 0 \\ \sin t \end{pmatrix} i + i^2 \begin{pmatrix} 0 \\ \sin t \\ 0 \end{pmatrix}\right]$$

but $i^2 = -1$

$$X_2(t) = e^t\left[\begin{pmatrix} 0 \\ 0 \\ \cos t \end{pmatrix} + \begin{pmatrix} 0 \\ \cos t \\ 0 \end{pmatrix} i + \begin{pmatrix} 0 \\ 0 \\ \sin t \end{pmatrix} i - \begin{pmatrix} 0 \\ \sin t \\ 0 \end{pmatrix}\right]$$

$$X_2(t) = e^t\left[\begin{pmatrix} 0 \\ -\sin t \\ \cos t \end{pmatrix} + \begin{pmatrix} 0 \\ \cos t \\ \sin t \end{pmatrix} i\right]$$

$$X_2(t) = e^t\begin{pmatrix} 0 \\ -\sin t \\ \cos t \end{pmatrix} + e^t\begin{pmatrix} 0 \\ \cos t \\ \sin t \end{pmatrix} i$$

$$X_2^1(t) = C_2 e^t\begin{pmatrix} 0 \\ -\sin t \\ \cos t \end{pmatrix}$$

$$X_2^2(t) = C_3 e^t\begin{pmatrix} 0 \\ \cos t \\ \sin t \end{pmatrix}$$

The last eigenvector doesn't need to be solved since λ_2 and λ_3 are conjugate of each other so there one takes the real part of the eigenvector found and the other the imaginary.

$$X(t) = e^t\left[C_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2\begin{pmatrix} 0 \\ -\sin t \\ \cos t \end{pmatrix} + C_3\begin{pmatrix} 0 \\ \cos t \\ \sin t \end{pmatrix}\right]$$

$$X(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = C_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + C_3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$C_1 = C_2 = C_3 = 1$

$$X(t) = e^t\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\sin t \\ \cos t \end{pmatrix} + \begin{pmatrix} 0 \\ \cos t \\ \sin t \end{pmatrix}\right]$$

$$X(t) = e^t\begin{pmatrix} 1 \\ \cos t - \sin t \\ \cos t + \sin t \end{pmatrix}$$

TRY OUT

Find the general solution of the given system of differential equation

(1) $dx_1/dt = x_1$

(2) $dx_2/dt = 3x_1 + x_2 - 2x_3$

(3) $dx_3/dt = 2x_1 + 2x_2 + x_3$

$$\begin{pmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

$dx/dt = Ax \qquad |A - \lambda I| = 0$

$$A - \lambda I = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 3 & 1-\lambda & -2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$1-\lambda\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

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$1-\lambda\left[(1-\lambda)(1-\lambda)-2(-2)\right] = 0$

$1-\lambda\left[1-2\lambda+\lambda^2+4\right] = 0$

$1-\lambda\left[5-2\lambda+\lambda^2\right] = 0$

$5-2\lambda+\lambda^2-5\lambda+2\lambda^2-\lambda^3 = 0$

$-\lambda^3+3\lambda^2-7\lambda+5 = 0$

$\lambda_1 = 1 \quad \lambda_2 = 1+2i \quad \lambda_3 = 1-2i$

For $\lambda_1 = 1$

$(A-I)v = 0$

let $v_1 = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & -2 \\ 2 & 2 & 0 \end{pmatrix}\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$3m_1 = +2m_3$ —— (1)

$2m_1 = -2m_2$

$m_1 = m_2$ —— (2)

$+\frac{3}{2}m_1 = m_3$ —— (3)

$$v = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} m_1 \\ -m_1 \\ +\frac{3}{2}m_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ +3/2 \end{pmatrix} m_1$$

$$x_1(t) = e^t\begin{pmatrix} 1 \\ -1 \\ +3/2 \end{pmatrix}$$

For $\lambda_2 = 1+2i$

$(A - (1+2i)I)v_2 = 0$

$$\begin{pmatrix} -2i & 0 & 0 \\ 3 & -2i & -2 \\ 2 & 2 & -2i \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-2in_1 = 0$

$n_1 = 0$ —— (1)

$-2in_2 - 2n_3 = 0$

$2n_2 - 2in_3 = 0$

$n_2 = in_3$ —— (2)

$-in_2 = n_3$ —— (3)

$$v_2 = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ in_3 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} n_3$$

choosing $n_3 = 1$

$$v_2 = \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} i\right]$$

$$x_2(t) = e^{(1+2i)t}\left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} i\right]$$

but $e^{\theta i} = \cos\theta + i\sin\theta$

$e^{(1+2i)t} = e^t \cdot e^{2ti}$

$e^{2ti} = \cos(2t) + i\sin(2t)$

$$x_2(t) = e^t(\cos(2t) + i\sin(2t)) \times \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} i\right]$$

$$x_2(t) = e^t\left[\begin{pmatrix} 0 \\ 0 \\ \cos(2t) \end{pmatrix} + \begin{pmatrix} 0 \\ \cos(2t) \\ 0 \end{pmatrix} i + \begin{pmatrix} 0 \\ 0 \\ \sin(2t) \end{pmatrix} i + i^2\begin{pmatrix} 0 \\ \sin(2t) \\ 0 \end{pmatrix}\right]$$

$$x_2(t) = e^t\left[\begin{pmatrix} 0 \\ -\sin(2t) \\ \cos(2t) \end{pmatrix} + \begin{pmatrix} 0 \\ \cos(2t) \\ \sin(2t) \end{pmatrix} i\right]$$

$$x_2(t) = e^t\begin{pmatrix} 0 \\ -\sin(2t) \\ \cos 2t \end{pmatrix} + e^t\begin{pmatrix} 0 \\ \cos(2t) \\ \sin(2t) \end{pmatrix} i$$

$$x_2^1(t) = e^t\begin{pmatrix} 0 \\ -\sin(2t) \\ \cos(2t) \end{pmatrix}$$

$$x_2^2(t) = e^t\begin{pmatrix} 0 \\ \cos(2t) \\ \sin(2t) \end{pmatrix}$$

$$x(t) = e^t\left[c_1\begin{pmatrix} 1 \\ -1 \\ +3/2 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ -\sin(2t) \\ \cos(2t) \end{pmatrix} + c_3\begin{pmatrix} 0 \\ \cos(2t) \\ \sin(2t) \end{pmatrix}\right].$$

(2)

$dx_1/dt = x_1 + x_3$

$dx_2/dt = x_2 - x_3$

$dx_3/dt = -2x_1 - x_3$

$$\begin{pmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \end{pmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$dx/dt = Ax \quad |A-\lambda I| = 0$

$$A-\lambda I = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & -1 \\ -2 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$1-\lambda \begin{vmatrix} 1-\lambda & -1 \\ 0 & -1-\lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & 1-\lambda \\ -2 & 0 \end{vmatrix}$$

$1-\lambda[(1-\lambda)(-1-\lambda)] + 1[-(-2)(1-\lambda)]$

$1-\lambda[-1 \cancel{-\lambda} \cancel{+\lambda} + \lambda^2] + 1[2-2\lambda]$

$-1+\lambda^2+\lambda-\lambda^3+2-2\lambda$

$-\lambda^3+\lambda^2-\lambda+1$

$\lambda_1 = 1 \quad \lambda_2 = i \quad \lambda_3 = -i$

For $\lambda_1 = 1$

$(A-I)V = 0$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -2 & 0 & -2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$a_3 = 0$ —— (1)

$-2a_1 = 2a_3$

$a_1 = -a_3$

but $a_3 = 0$

$a_1 = 0$ —— (2)

Since a_2 is the only unknown and we can't have a trivial solution we there select a natural form

$$e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad V_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$X_1(t) = e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_2 = i \quad (A-iI)V_2 = 0$

$$\begin{pmatrix} 1-i & 0 & 1 \\ 0 & 1-i & -1 \\ -2 & 0 & -1-i \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$(1-i)b_1 + b_3 = 0$

$-(1-i)b_1 = b_3$ —— (1)

$(1-i)b_2 = b_3$ —— (2)

$b_1 = -\dfrac{b_3}{(1-i)}$ —— (3)

$b_2 = \dfrac{b_3}{(1-i)}$ —— (4)

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} -b_3/(1-i) \\ b_3/(1-i) \\ b_3 \end{pmatrix} = \begin{pmatrix} -1/(1-i) \\ 1/(1-i) \\ 1 \end{pmatrix} b_3$$

$$\left[-\frac{1}{2} - \frac{1}{2}i, \ \frac{1}{2} + \frac{1}{2}i, \ 1 \right]$$

$$e^{it} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}-\frac{1}{2}i \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2}+\frac{1}{2}i \\ 0 \end{pmatrix}$$

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18/07/22

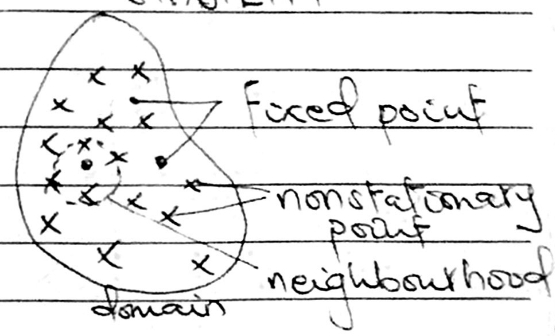
STABLE CRITICAL POINTS

Let x_1 be a critical points of an autonomous system and let $x = x(t)$ denote the solution that satisfies the initial condition $x(0) = x_0$, where $x_0 \neq x_1$. We say that x_1 is a stable critical point where, given any radius $\rho > 0$, there is a corresponding radius $r > 0$ such that if the initial position x_0 satisfies $|x_0 - x_1| < r$, then the corresponding solution $x(t)$ satisfies $|x(t) - x_1| < \rho$ for all $t > 0$. If in addition, $\lim_{t \to \infty} x(t) = x_1$ whenever $|x_0 - x_1| < r$, we call x_1 asymptotically stable critical point

UNSTABLE CRITICAL POINTS

Let x_1 be a critical point of an autonomous system and let $x = x(t)$ denote the solution that satisfies the initial condition $x(0) = x_0$, where $x_0 \neq x_1$. We say that x_1 is an unstable critical point when there is a disk of radius $\rho > 0$ with the property that for any $r > 0$, there is at least one initial position that satisfies $|x_0 - x_1| < r$, yet the corresponding solution $x(t)$ satisfies $|x(t) - x_1| \geq \rho$ for at least one $t > 0$

FIXED POINTS AND ITS STABILITY



If the non stationary point moving toward the fixed point then it converges (asymptotically stable) and vice versa for it ~~af~~ unstable fixed points.

Finding the fixed points of scalar differential equation.
Suppose $dx/dt = F(x,t)$, to find the fixed / equilibrium point $dx/dt = 0$ and solve for x.

$\frac{dx}{dt}$ = slope or gradient

$\Delta x = x_2 - x_1$

$x_2 - x_1 = 0 \qquad x_2 = x_1$

~~Eg Find~~ Stability of fixed point of a scalar D.E.

Let $\frac{dx}{dt} = F(x), \ \forall x \in \mathbb{R}$ —(2)

then $\frac{dx}{dt} = 0$ yields a fixed point x^*.

To determine the stability of the points of equation (2) Find $\frac{dF(x)}{dx}$ and evaluate it at a fixed point x^*.

DATE NO

(1) If $\frac{dF(x^*)}{dx} < 0$, then x^* is said to be asymptotically stable.

(2) If $\frac{dF(x^*)}{dx} > 0$, then x^* is an unstable fixed point

(3) If $\frac{dF(x^*)}{dx} = 0$, then test fails.

Eg

Given that $dx/dt = rx(1 - x/k)$ $\forall\, r, k > 0$ and $X_{max} = k$, find, (a) the fixed points of the above equation

(b) determine the stability of each fixed point in (a) above.

Soln

(a) To find the fixed point put $\frac{dx}{dt} = 0$

$$\Rightarrow rx\left(1 - \frac{x}{k}\right) = 0$$

$$\Rightarrow rx = 0$$

$$x_1^* = 0$$

or

$$1 - \frac{x}{k} = 0 \qquad -\frac{x}{k} = -1$$

$$x_2^* = k$$

The fixed point are $x_1^* = 0$ or $x_2^* = k$

(b) Let $F(x) = rx\left(1 - \frac{x}{k}\right)$

$$F(x) = rx - rx^2/k$$

$$\frac{dF}{dx} = r - \frac{2rx}{k}$$

At $x_1^* = 0$

$$f'(0) = r - \frac{2(0)r}{k} = r > 0$$

Since $F'(0) > 0$ it implies that $x_1^* = 0$ is an unstable fixed point.

At $x_2^* = k$

$$F'(k) = r - \frac{2\cancel{k}r}{\cancel{k}}$$

$$= r - 2r = -r < 0$$

Since $F'(k) < 0$ it implies that $x_2^* = k$ is asymptotically stable.

IRT

Determine the stability of fixed point(s) of the equation

$\frac{dx}{dt} = \sin(x)$, $\forall\, x \in \mathbb{R}$

Soln

To find the fixed point we put $dx/dt = 0$

$$\Rightarrow \sin(x) = 0$$

$$\sin^{-1}(\sin(x)) = \sin^{-1}(0)$$

$x = n\pi$, $\forall$ is an integer

let $F(x) = \sin(x)$

$$\frac{dF}{dx} = \cos(x)$$

At $x = 0$

$$F'(0) = \cos(0)$$

SYSTEM OF ODES

Given that $dx/dt = xy - y^2$ and $dy/dt = 2x^2y + y$

Find the fixed points of the above system of ODEs.

Sol^n

To find the fixed point

$\frac{dx}{dt} = 0 = \frac{dy}{dt}$

put $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$

$xy - y^2 = 0$ ——— ①

$2x^2y + y = 0$ ——— ②

From eqn ①

$y(x-y) = 0$

$y = 0$ or $x - y = 0$

$y = 0$ or $x = y$ ——— ③

put $y = 0$ into eqn ②

$2(x^2)(0) + 0 = 0$

$0 = 0$

$(x^*, y^*) = (x, 0)$ is a fixed point, where $x \in \mathbb{R}$.

putting $x = y$ into eqn ②

$2x^2(x) + x = 0$

$2x^3 + x = 0$

$x(2x^2 + 1) = 0$

$x = 0$ or $2x^2 + 1 = 0$

$2x^2 = -1$

$x^2 = \frac{-1}{2}$ $\quad \sqrt{x^2} = \pm\sqrt{\frac{-1}{2}}$

$x_{1,2} = \pm \frac{i}{\sqrt{2}}$

$x_{1,2} = \pm \frac{1}{2}\sqrt{2}\,i$ rejected

$(x^*, y^*) = (0, 0)$ is a fixed point

STABILITY OF LINEAR SYSTEM OF ODES

Consider linear system of ODEs below:

$\frac{dx}{dt} = ax + by$ ——— ①

$\frac{dy}{dt} = cx + dy$ ——— ②

we can see that $(x^*, y^*) = (0, 0)$ is a fixed/equilibrium of eqn ① and ②

Eqn ① and ② can be written as:

$\frac{dx}{dt} = Ax$

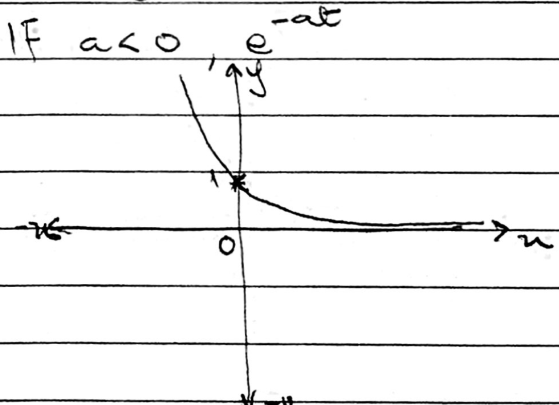
where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $x = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

$\frac{dx}{dt} = \begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix}$

$x = c_1 x_1 e^{\lambda_1 t} + c_2 x_2 e^{\lambda_2 t}$

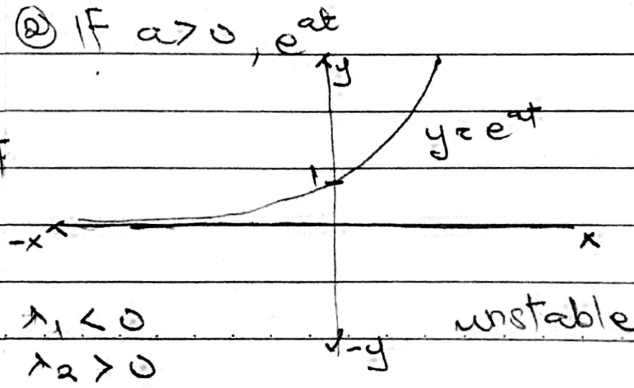
e^{at}

① If $a < 0$, e^{-at}



Asymptotically stable.

② If $a > 0$, e^{at}



$\lambda_1 < 0$

$\lambda_2 > 0$

unstable.

$|A-\lambda I| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$

$\Rightarrow (a-\lambda)(d-\lambda) - bc = 0$

$\Rightarrow ad - a\lambda - d\lambda + \lambda^2 - bc = 0$

$\Rightarrow \lambda^2 - (a+d)\lambda + ad - bc = 0$

$\lambda^2 - (\text{trace}(A))\lambda + \det(A) = 0$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

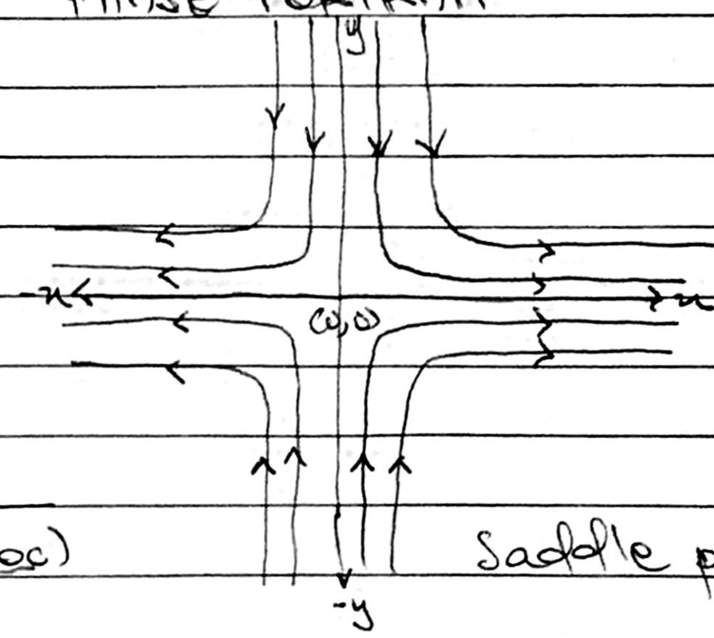
$$\lambda_{1,2} = \frac{-\{-(a+d)\} \pm \sqrt{[-(a+d)^2] - 4(1)(ad-bc)}}{2(1)}$$

$$\lambda_{1,2} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(\det A)}}{2}$$

~~$\lambda_{1,2}$~~

If trace $(A) < 0$ and $(\text{trace}(A))^2 > 4\det(A)$, then the eigenvalue are real but have negative signs. The fixed $(x^*, y^*) = (0,0)$ is asymptotically stable.
Thus, $\lambda_1 < 0$ and $\lambda_2 < 0$

PHASE PORTRAIT

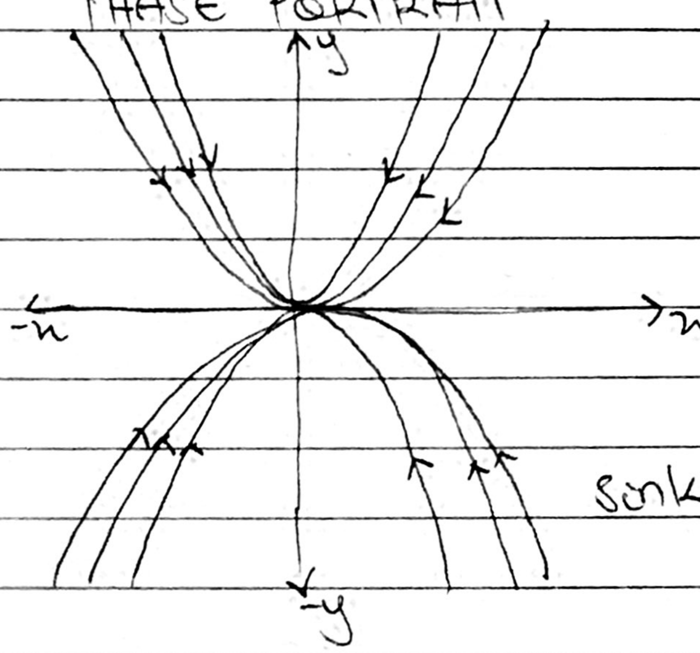


Saddle point

The fixed point $(x^*, y^*) = (0,0)$ is said to be a saddlepoint.
The fixed point $(x^*, y^*) = (0,0)$ is an unstable fixed point.

If trace > 0 (ie $\tau > 0$) and $\tau^2 > 4\Delta$, then two eigenvalues are real positive real numbers. Thus, $\lambda_1 > 0$ and $\lambda_2 > 0$. The fixed point $(x^*, y^*) = (0,0)$ is unstable fixed point. This fixed point is termed as **SOURCE**

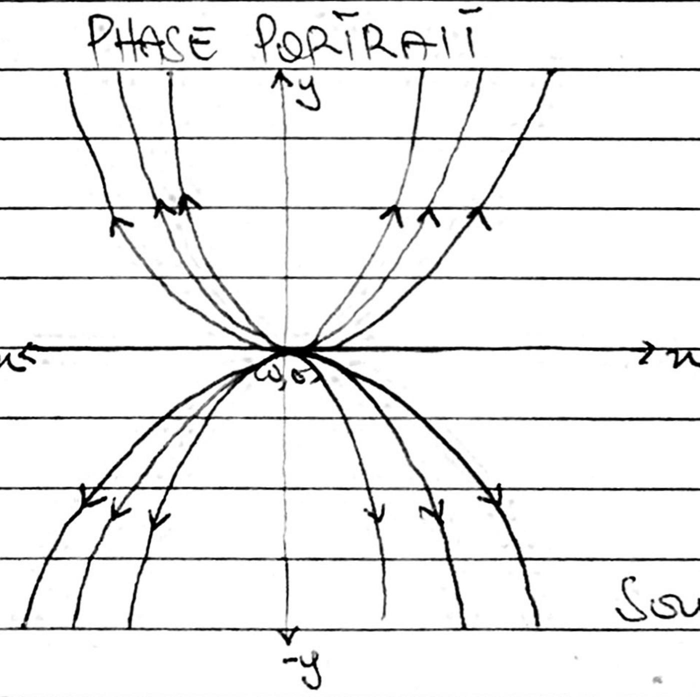
PHASE PORTRAIT



Sink

$\lambda_1 < 0$ and $\lambda_2 < 0$

PHASE PORTRAIT



Source

$\lambda_1 > 0$ and $\lambda_2 > 0$

$\lambda_{1,2} = \tau \pm \sqrt{\tau^2 + 4\Delta}$

If $\det(A) < 0$, then eigenvalues have opposite
Thus $\lambda_1 > 0$ and $\lambda_2 < 0$

COMPLEX ROOTS

If trace $(A) < 0$ and $4\Delta > \tau^2$, then eigenvalues are complex with

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negative real part.

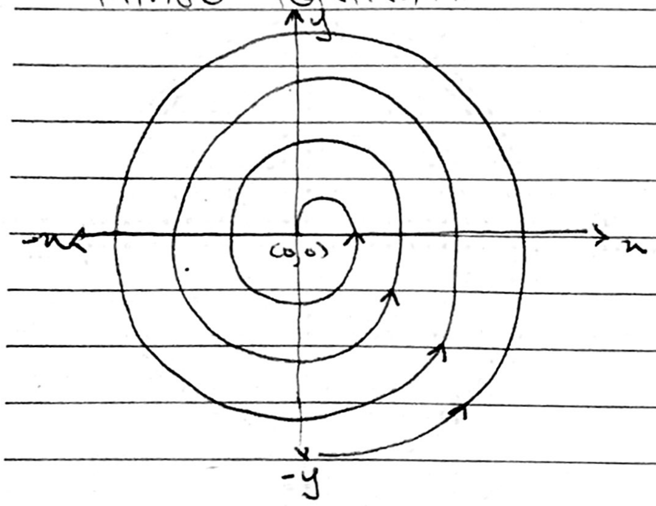
$\lambda_{1,2} = -\alpha \pm \beta i$

$\lambda_1 = -\alpha + \beta i$

$\lambda_2 = -\alpha - \beta i$

The fixed point $(x^*, y^*) = (0,0)$ is said to be **SPIRAL SINK**

PHASE PORTRAIT

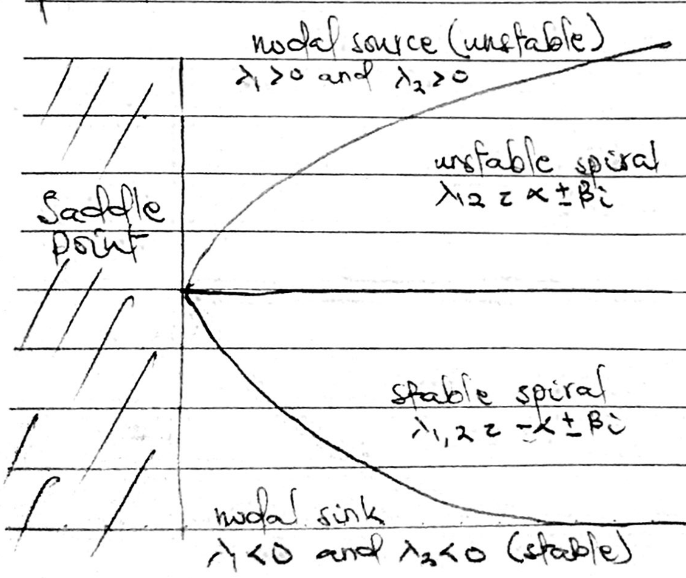


The fixed point $(x^*, y^*) = (0,0)$ is asymptotically stable since real part is negative.

(5) if $\tau > 0$ and $4\Delta > \tau^2$, then eigenvalues are complex numbers with positive real parts.

That is, $\lambda_{1,2} = \alpha \pm \beta i$ the fixed point is called **SPIRAL SOURCE**.

The fixed point $(x^*, y^*) = (0,0)$ is said to be unstable fixed point.



Example

Consider the system

$\dot{x} = x + y$

$\dot{y} = 4x - 2y$

Determine whether the fi[xed] point of the system is stab[le]

Soln

$\frac{dx}{dt} = x + y \qquad \frac{dy}{dt} = 4x - 2y$

$\frac{dx}{dt} = 0 \qquad \frac{dy}{dt} = 0$

$x + y = 0 \qquad 4x - 2y = 0$

$x = -y \qquad 2x = y$

$(x^*, y^*) = (0,0)$

$\frac{dx}{dt} = Ax$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = 0$$

$1-\lambda(-2-\lambda) - 4 = 0$

$-2 - \lambda + 2\lambda + \lambda^2 - 4 = 0$

$\lambda^2 + \lambda - 6 = 0$

$\lambda = 2 \qquad \lambda = -3$

CLASSIFICATION OF FIXED POINTS OF SYSTEM OF LINEA[R] ODEs

SPECIAL CASES (Borderline cases

$\frac{dx}{dt} = ax + by, \quad \frac{dy}{dt} = cx + dy$

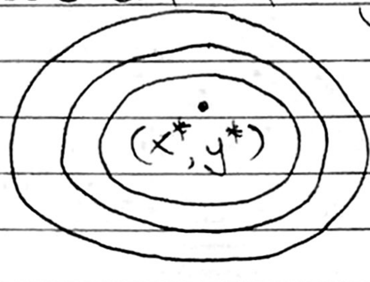
$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

(1) If $\tau = 0$, then

$\lambda_{12} = \pm\sqrt{4\Delta}\, i$

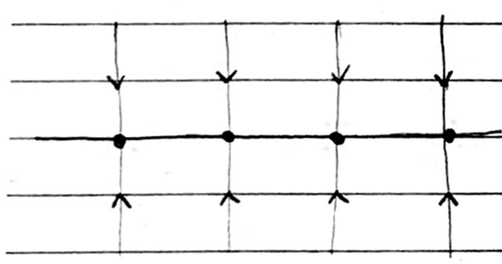
$\lambda_{12} = \pm \beta i$, purely complex

The fixed point $(x^*, y^*) = (0,0)$ is centre, which is stable but not asymptotic.



$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

If $\Delta = 0$, at least one of the eigenvalues is zero. The fixed point $(x^*, y^*) = (0,0)$ is not an isolated (unisolated fixed point). There is either a whole of fixed points or a plane of fixed points.

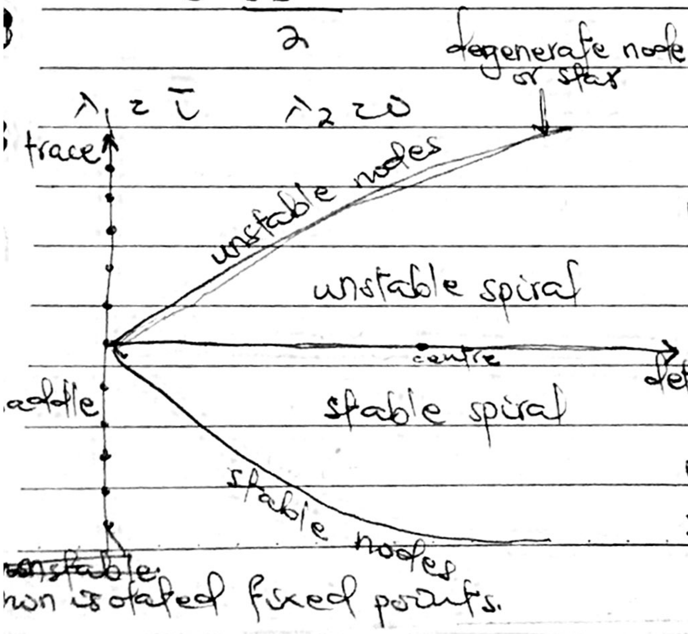


$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta = 0}}{2}$$

$$= \frac{\tau \pm \sqrt{\tau^2}}{2}$$

$$= \frac{\tau \pm \tau}{2}$$

$\lambda_1 = \tau \qquad \lambda_2 = 0$



③ If $\tau^2 - 4\Delta = 0$

$$\lambda_{1,2} = \frac{\tau \pm 0}{2}$$

$\lambda_1 = \tau \qquad \lambda_2 = \tau \qquad \lambda_1 = \lambda_2$

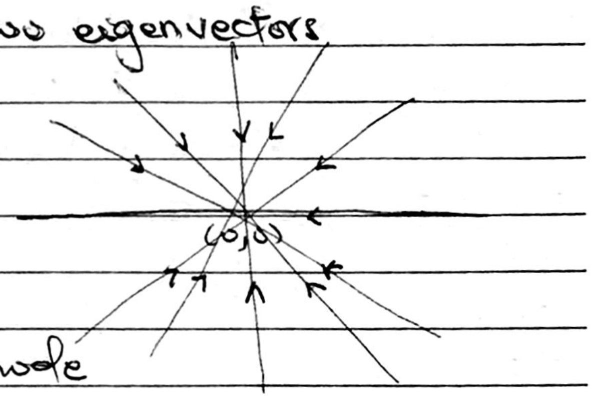
The fixed point $(x^*, y^*) = (0,0)$ is said to be degenerate node or star.

Here there can be two linearly independent eigenvector which corresponds to $\lambda_1 = \lambda_2$

OR

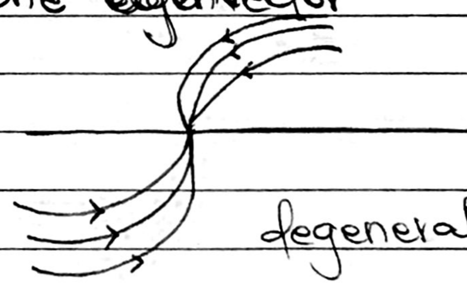
Only one linearly independent eigenvectors which corresponds to $\lambda_1 = \lambda_2$

A. Two eigenvectors



Star node

B. One eigenvector



degenerate

Defn

A matrix A is hyperbolic if none of it's eigenvalues has 0 real parts.

Similarly a system of ODEs $dx/dt = Ax$ is said to be hyperbolic if none of it's eigenvalues has 0 real part. and on the other hand if any of its eigenvalues is zero then we term it as Non-Hyperbolic.

eg — When a fixed point is either node (sink or source) or spiral (sink or source) or saddle point.

However if the fixed point of the system of ODEs has the real part of it eigenvalues to be zero is called non hyperbolic.

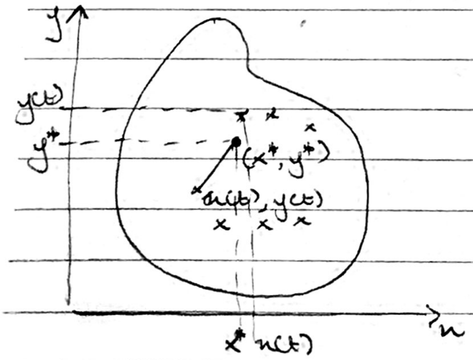
FIXED POINTS AND LINEARIZATION OF SYSTEM OF NON LINEAR EQUATIONS

Consider the system below

$$\frac{dx}{dt} = f(x,y) \quad \text{—— (1)}$$

$$\frac{dy}{dt} = g(x,y) \quad \text{—— (2)}$$

and suppose that (x^*, y^*) is the fixed point of equations (1) and (2), thus $f(x^*, y^*) = 0$ and $g(x^*, y^*) = 0$



Let $u(t) = x(t) - x^*$

$v(t) = y(t) - y^*$

$$\Rightarrow x(t) = u(t) + x^* \quad \text{—— (3)}$$

$$y(t) = v(t) + y^* \quad \text{—— (4)}$$

Differentiating each term in equation (3) and (4)

$$\left.\begin{aligned} \frac{dx}{dt} &= \frac{du}{dt} + 0 = \frac{du}{dt} \\ \frac{dy}{dt} &= \frac{dv}{dt} \end{aligned}\right\} \quad (5)$$

Plugging equations (3) – (5) into equations (1) and (2) yields

$$\frac{du}{dt} = f(u(t) + x^*, v(t) + y^*) \quad \text{—— (6)}$$

$$\frac{dv}{dt} = g(u(t) + x^*, v(t) + y^*) \quad \text{—— (7)}$$

Using Taylor's series approximation in R.H.S of equation (6) and (7) we have $f(x,y)$ about $(x,y) = (a,b)$

$$f(x,y) = f(a,b) + \left[\frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(ab)(y-b)\right] + \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(a,b)(x-a)(y-b) + \frac{\partial^2 f}{\partial y^2}(a,b)(y-b)^2\right] +$$

$$\frac{1}{3!}\left[\frac{\partial^3 f}{\partial x^3}(ab)(x-a)^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}(x-a)^2(y-b) + 3\frac{\partial^3 f}{\partial x \partial y^2}(a,b)(x-a)(y-b)^2 + \frac{\partial^3 f}{\partial y^2}(a,b)(y-b)^3\right] + \dots$$

$$f(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(ab)(y-b) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(ab)(x-a)^2$$

$$+\frac{\partial^2 f(ab)}{\partial x \partial y}(x-a)(y-b) + \frac{1}{2}\frac{\partial^2 f(ab)}{\partial y^2}(y-b)^2 + \frac{1}{6}\frac{\partial^3 f(ab)}{\partial x^3}(x-a)^3$$

$$+\frac{1}{2}\frac{\partial^3 f(ab)}{\partial x^2 \partial y}(x-a)^2(y-b) + \frac{1}{2}\frac{\partial^3 f(ab)}{\partial x \partial y^2}(x-a)(y-b)^2 + \frac{1}{6}\frac{\partial^3 f(ab)}{\partial y^3}(y-b)^3 + \dots$$

$$\frac{du}{dt} = f(u(t)+x^*, v(t)+y^*)$$

$$\frac{dv}{dt} = g(u(t)+x^*, v(t)+y^*)$$

Using the taylor's series

$$\frac{du}{dt} = f(x^*,y^*) + \frac{\partial f(x^*,y^*)}{\partial x}(x-x^*)$$

$$+ \frac{\partial f(x^*,y^*)}{\partial y}(y-y^*) + \frac{1}{2!}\left[\frac{\partial^2 f(x^*y^*)}{\partial x^2}(x-x^*)^2 + 2\frac{\partial^2 f(x^*y^*)}{\partial x \partial y}(x-x^*)(y-y^*) + \frac{\partial^2 f(x^*,y^*)}{\partial y^2}(y-y^*)^2\right] + O(x^3, x^2y, xy^2, y^3).$$

By linearization, we ignore the quadratic and higher power terms.

$$\frac{du}{dt} = f(x^*,y^*) + \frac{\partial f(x^*,y^*)}{\partial x}u + \frac{\partial f}{\partial y}(x^*,y^*)v + O(\ )$$

Since (x^*,y^*) is a fixed point, then $f(x^*,y^*) = 0$

$$\frac{du}{dt} = u\frac{\partial f}{\partial x}(x^*,y^*) + v\frac{\partial f}{\partial y}(x^* y^*)$$

Similarly

$$\frac{dv}{dt} = u\frac{\partial g}{\partial x}(x^*,y^*) + v\frac{\partial g}{\partial y}(x^*,y^*) \quad (9)$$

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}_{(x^*,y^*)} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\dot{u} = Au$$

where $A = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$ is called Jacobian matrix and should not contain non-linear terms.

HARTMAN – GROBMAN THEOREM

Fixed point is hyperbolic

1. $\lambda_1 > 0$ and $\lambda_2 > 0$
2. $\lambda_1 < 0$ and $\lambda_2 < 0$
3. $\lambda_{12} = -\alpha \pm \beta i$
4. $\lambda_{12} = \alpha \pm \beta i$
5. $\lambda_1 > 0$ and $\lambda_2 < 0$

Example: Given that

$$\frac{dx}{dt} = x^2y - x^2$$

$$\frac{dy}{dt} = x^2 - 2xy - y^2$$

(a) find all the fixed point
(b) classify each fixed point

Soln

(a) $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$

$(x^*, y^*) = (0, 0)$ is one of the fixed point

DATE NO

$x^2y - x^2 = 0$ ——— ①

$x^2 - 2xy - y^2 = 0$ ——— ②

$x(xy - x) = 0$

$x = 0 \quad xy - x = 0$

putting $x = 0$ into eqn ②

$(0)^2 - 2(0)y - y^2 = 0$

$-y^2 = 0 \quad y = 0$

$(x^*, y^*) = (0, 0)$

$J(x, y) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix}$

$x^2y - x^2 = f$

$x^2 - 2xy - y^2 = g$

$\frac{\partial f}{\partial x} = 2xy - 2x$ — Non linear

$\frac{\partial f}{\partial y} = x^2$ — Non linear

$\frac{\partial g}{\partial x} = 2x - 2y$ — linear

$\frac{\partial g}{\partial y} = -2x - 2y$ — linear.

$\therefore J(x, y) = \begin{bmatrix} 0 & 0 \\ 2x - 2y & -2x - 2y \end{bmatrix}$

At $(x^*, y^*) = (0, 0)$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$\left\| \begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix} \right\| = 0$

$-\lambda(-\lambda) = 0$

$\lambda^2 = 0$

$\lambda_1 = 0 \quad \lambda_2 = 0$

non hyperbolic.

08/08/22

If $dy/dx = \cos^2 x + 3/2 \sin(x) - 3/2 \quad \forall\ 0 \le x \le 2\pi$. Find all the fixed points.

Soln.

$$\cos^2 x + \frac{3}{2}\sin(x) - \frac{3}{2} = 0$$

but $\cos^2 x = 1 - \sin^2(x)$ according to trig identities.

$$1 - \sin^2(x) + \frac{3}{2}\sin(x) - \frac{3}{2} = 0$$

$$-\sin^2(x) + \frac{3}{2}\sin(x) - \frac{1}{2} = 0$$

$$\sin^2(x) - \frac{3}{2}\sin(x) + \frac{1}{2} = 0$$

let $y = \sin(x)$

$$y^2 - \frac{3}{2}y + \frac{1}{2} = 0$$

$$2y^2 - 3y + 1 = 0$$

$y = 1/2 \quad y = 1$

but $y = \sin(x)$

$\sin(x) = 1/2$

$x = \sin^{-1}(1/2)$

$\sin(30) = 1/2 \quad \sin(150) = 1/2$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}.$$

$\sin(x) = 1 \quad x = \sin^{-1}(1)$

$x = 90° = \pi/2$

The fixed point are $\pi/6, \pi/2, 5\pi/6$.

Let $f(x) = \cos^2 x + \frac{3}{2}\sin x - \frac{3}{2}$

$f(x) = [\cos x]^2 + 3/2 \sin(x) - 3/2$

$f'(x) = 2 \cdot -\sin(x)(\cos(x))^{2-1} + 3/2 \cos(x)$

$f'(x) = -2\sin(x)\cos(x) + 3/2\cos(x)$

At $\pi/6 = x^*$

$f'(x) = -2\sin(\pi/6)\cos(\pi/6) + 3/2\cos(\pi/6)$

$f'(\pi/6) = -2 \cdot 1/2 \cdot \sqrt{3}/2 + 3/2 \cdot \sqrt{3}/2$

$$f'(\pi/6) = \frac{\sqrt{3}}{4}$$

Since $f'(\pi/6) > 0$ it implies the $x^* = \pi/6$ is an unstable fixed point.

$f'(5\pi/6) = -2\sin(5\pi/6)\cos(5\pi/6) + 3/2\cos(5\pi/6)$

$f'(5\pi/6) = -2 \cdot 1/2 \cdot -\sqrt{3}/2 + 3/2 \cdot -\sqrt{3}/2$

$$f'(5\pi/6) = -\frac{\sqrt{3}}{4}$$

Since $f'(5\pi/6) < 0$ it implies the $x^* = 5\pi/6$ is a asymtotically stable fixed point.

Region A $x < 90°$

$x = 60 = \pi/3$

$f(x) = \cos^2(x) + 3/2 \sin(x) - 3/2$

$f(\pi/3) = \cos^2(\pi/3) + 3/2 \sin(\pi/3) - 3/2$

$= 0.049$

$f(\pi/3) > 0$ implies that $f(x)$ is increasing in region A.

Region B $x > 90$

$x = 120 = 2\pi/3$

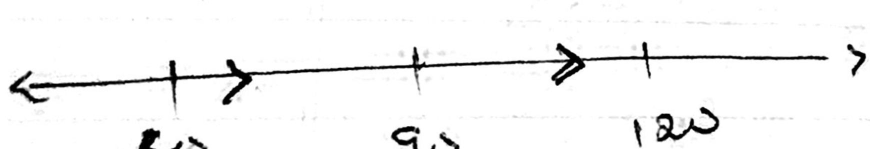
$f(x) = \cos^2(x) + 3/2 \sin(x) - 3/2$

$f(2\pi/3) = \cos^2(2\pi/3) + 3/2 \sin(2\pi/3) - 3/2$

$x = 0.049$

Since $f(2\pi/3) > 0$ implies that $f(x)$ is increasing in region B

$x^* = \pi/2$ is half stable Fixed point.



BIFURCATION

It is the change in the ~~differential e~~ qualitati solution of the different as the values of the parameter are been varie

Method

① Analytic - laydown proced

② Qualitative

③ Quantitative

↳ numerical method and grap

ONE DIMENSION

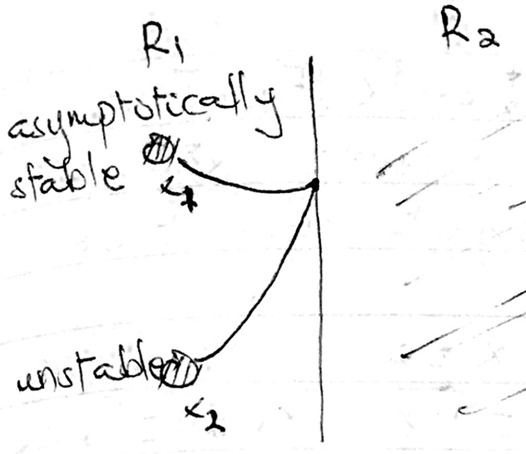
① Saddle-node or Turning point or Tangent bifurcation

② Transcritical bifurcation.

③ Pitchfork bifurcation.

$\frac{dx}{dt} = (x, \alpha)$, One dimension

$\frac{dx}{dt} = (x, y, \alpha)$, Two or more



bifurcation point

They collide and vanish at the bifurcation point

Two fixed points (unstable and asymtotically stable) created for some values of the parameter and these 2 fixed points collide at the bifurcation point and finally disappear / destroyed at other values of the parameter

Transcritical bifurcation, if the fixed point cross the bifurcation point they change stability respectively. ie if in R_1 x_1^* is asymtotically stable and x_2^* is unstable and both cross the bifurcation point x_1^* becomes unstable and x_2^* becomes asymtotically stable.

SADDLE POINT BIFURCATION

Normal form

$dx/dt = \alpha \pm x^2 \quad \forall \alpha, x \in \mathbb{R}$

Eg

Discuss the bifurcation of

$dx/dt = \alpha + x^2, \quad \forall x, \alpha \in \mathbb{R}$

Soln

① Find the fixed points of the given differential equation which will depend on the values of the parameter.

② Use the values or domains of parameter to divide its domain into 2 distinct subregions.

③ In each region, find the stability of the fixed point and draw your conclusion.

Soln

put $dx/dt = 0$

$\alpha + x^2 = 0 \quad \sqrt{x^2} = \pm\sqrt{-\alpha}$

$x_{1,2} = \pm\sqrt{-\alpha}$

Region 1 $\alpha > 0$

$x_{1,2} = \pm\sqrt{\alpha}\, i \notin \mathbb{R}$

Here, the two fixed points are destroyed or they disappear.

Region 2 $\alpha < 0$

$x_{1,2} = \pm\sqrt{-(-\alpha)} = \pm\sqrt{\alpha} \in \mathbb{R}$

In this region, two fixed points are created. check for the stability of the two fixed point

$dx/dt = \alpha + x^2$

let $f(x) = \alpha + x^2$, $f'(x) = 2x$

At $x_1^* = \sqrt{\alpha}$

$f'(\sqrt{\alpha}) = 2\sqrt{\alpha} > 0$, it implies $x_1^* = \sqrt{\alpha}$ is an unstable fixed point.

At $x_2^* = -\sqrt{\alpha}$, $f'(-\sqrt{\alpha}) = -2\sqrt{\alpha} < 0$, it implies $x_2^* = -\sqrt{\alpha}$ is asymptotically stable.

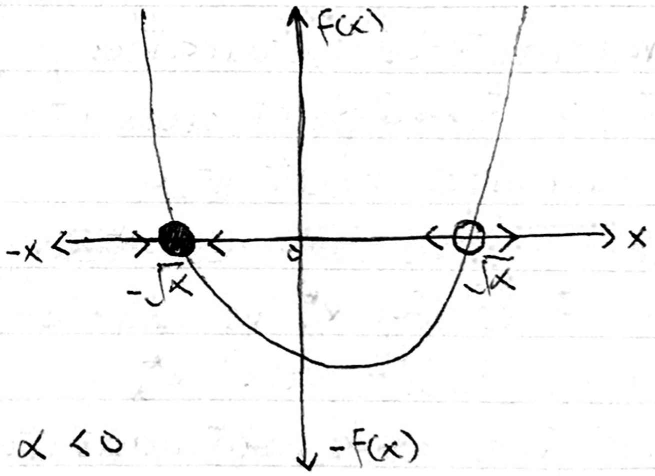
At $\alpha = 0$

$x_{1,2} = \pm\sqrt{-\alpha} \quad x_{1,2} = \pm\sqrt{0} = 0$

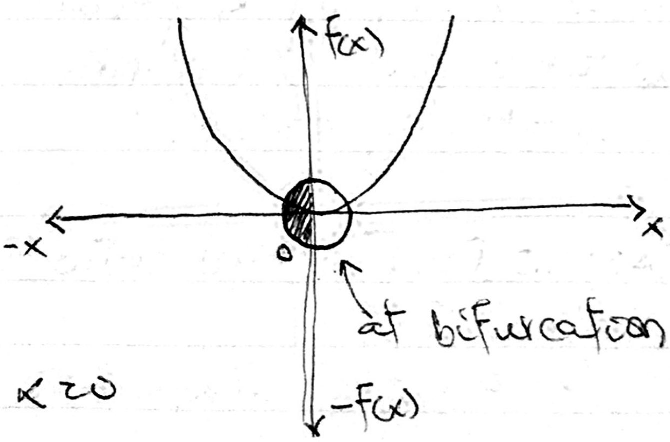
$x^* = 0$ is a bifurcation point $f(x) = \alpha + x^2$

$x_1^* = \sqrt{\alpha} \quad x_2^* = -\sqrt{\alpha}$

Case 1

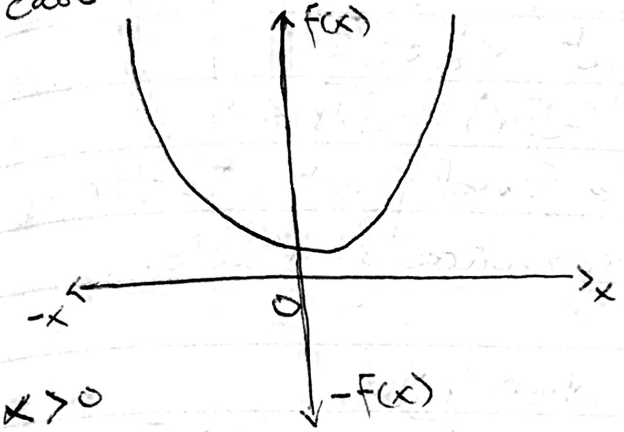


$\alpha < 0$



$\alpha = 0$

case 3



$\alpha > 0$

α – Bifurcation parameter

Dynamical systems will have some specific dynamical behaviour, If α changes, dynamical behaviour of system may change drastically – This transition point is called a Bifurcation

Bifurcation Point

Point where there is a transition in the stability or fixed point behaviour of the dynamical system specified using the transition occurs and the fixed point at which the transition occurs. (x, α)

Eg 1

Discuss the bifurcation of

$$\frac{dx}{dt} = r + x - \ln|1+x| \quad f(r,x) \in \mathbb{R} \quad (1)$$

Soln

let $g(x) = \ln|1+x|$

Expand $g(x)$ using taylor series about $x=0$

$$g(x) = g(0) + g'(0)x + \frac{g''(0)x^2}{2!} + \dots$$

$g(x) = \ln|1+x|$

$g(0) = \ln|1+0| = 0$

$g'(x) = \frac{1}{1+x} = (1+x)^{-1}$

$g'(0) = 1$

$g''(x) = \frac{-1}{(1+x)^2} = -(1+x)^{-2}$

$g''(0) = -1$

$g'''(x) = \frac{2}{(1+x)^3} = 2(1+x)^{-3}$

$g'''(0) = 2$

$$g(x) = 0 + x + \frac{(-1)x^2}{2!} + \frac{2x^3}{3!} \dots$$

For saddle node bifurcation we truncate

at ~~x=2~~ x^2

$$g(x) = x - \frac{x^2}{2} \quad \text{—— (2)}$$

Substitute eqn(2) into (1)

$$\frac{dx}{dt} = r + x - (x - x^2/2)$$

$$\frac{dx}{dt} = r + \frac{x^2}{2}$$

For fixed points $dx/dt = 0$

$$r + \frac{x^2}{2} = 0 \qquad 2r + x^2 = 0$$

$$x = \pm\sqrt{-2r}$$

when $r < 0$

$$x_{12}^* = \pm\sqrt{-2(-r)}$$

$$= \pm\sqrt{2r}$$

$x_1^* = \sqrt{2r}$ and $x_2^* = -\sqrt{2r}$

Let $f(x) = r + \frac{x^2}{2}$

$f'(x) = x$, when $x_1^* = \sqrt{2r}$

$f'(\sqrt{2r}) = \sqrt{2r} > 0$

Since $\sqrt{2r} > 0$ it implies that the fixed point $x_1^* = \sqrt{2r}$ is unstable.

at $x_2^* = -\sqrt{2r}$

$f'(-\sqrt{2r}) = -\sqrt{2r} < 0$

The fixed point $x_2^* = -\sqrt{2r}$ is asymtotically stable.

when $r > 0$

$x_{12} = \pm\sqrt{-2r} \Rightarrow \pm i\sqrt{2r}$

Hence the fixed points are destroyed.

when $r = 0$

$x_{1,2} = \pm\sqrt{2(0)} = 0$

$x_{1,2} = 0$

$f'(0) = 0$

The test failed.

Because the test failed when $r = 0$, we have to use the sign test.

when $x > 0$

let chose $x = 1$

$\frac{dx}{dt} = \frac{1}{2} \longrightarrow$ increasing

when $x < 0$, choosing $x = -1$

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$\frac{dx}{dt} = \frac{1}{2}$ $\longrightarrow$ Increasing

We can see that when $r<0$ two fixed points, $x_1^* = \sqrt{2r}$ and $x_2^* = -\sqrt{2r}$ are created, and collide at $r=0$ and are destroyed when $r>0$

Hence $dx/dt = r + x - \ln|1+x|$ $\forall (r,x) \in \mathbb{R}$ undergoes a saddle node bifurcation.

Discuss the bifurcation

$\frac{dx}{dt} = r - x - e^{-x}$, $\forall (r, x) \in \mathbb{R}$

By truncation of the taylor series about $x=0$

$g(x) = g(0) + g'(0)x + \frac{g''(0)x^2}{2!} + \dots$

$g(x) = e^{-x}$

$g(0) = 1$

$g'(x) = -e^{-x} = -1 = g'(0)$

$g''(x) = e^{-x} = 1 = g''(0)$

$g(x) = 1 - x + \frac{x^2}{2!}$

$\frac{dx}{dt} = r - x - (1 - x + x^2/2)$

$\frac{dx}{dt} = r - 1 \mp \frac{x^2}{2}$

for fixed points $dx/dt = 0$

$r - 1 \mp \frac{x^2}{2} = 0$

$2r - 2 \mp x^2 = 0$

$x^2 = +2r \mp 2$, $x = \pm\sqrt{-2+2r}$

when $r < 0$

$x_{1,2}^* = \pm\sqrt{-2+(-2r)}$

$= \pm\sqrt{-2 \mp 2r}$

$x_1^* = \sqrt{-2 \mp 2r}$ and $x_2^* = -\sqrt{-2 \mp 2r}$

let $f(x) = r - 1 \mp \frac{x^2}{2}$

$f'(x) = -x$

when $x_1^* = \sqrt{-2 \mp 2r}$

$f'(\sqrt{-2 \mp 2r}) = -\sqrt{-2 \mp 2r} > 0$

Since

when $x_2^* = -\sqrt{-2-2r}$

$f'(-\sqrt{-2-2r}) = \sqrt{-2-2r}$

$= i\sqrt{2+2r} \notin \mathbb{R}$

Hence the fixed points are destroyed.

when $r > 0$

$x_{12}^* = \pm\sqrt{-2+2r}$

$x_{12}^* = \pm\sqrt{2(r-1)}$

$f'(\sqrt{2(r-1)}) = -\sqrt{2(r-1)} < 0$

Since it is less than zero it implies $x_1^* = \sqrt{2(r-1)}$ is asymptotically stable.

$f'(-\sqrt{2(r-1)}) = \sqrt{2(r-1)} > 0$

Since it is greater than zero it implies $x_2^* = -\sqrt{2(r-1)}$ is an unstable fixed point.

when $r = 1$

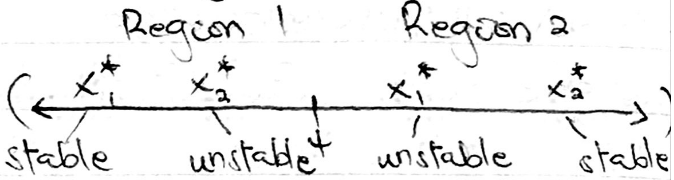
$x_{1,2} = 0$

$r = 1$ is the bifurcation

Hence $dx/dt = r - x - e^{-x}$ $\forall (r, x) \in \mathbb{R}$ undergoes a saddle node bifurcation.

TRANSCRITICAL BIFURCATION

It occurs when there is an exchange of stabilities between two fixed points. Fixed points will exists at any point.



$x = 4$

Prototype / standard of transcritical bifurcation is of the form

$$\frac{dx}{dt} = \alpha x \pm x^2, \ \forall (\alpha, x) \in \mathbb{R}$$

Eg 1

Discuss the bifurcation

$$\frac{dx}{dt} = \alpha x - x^2, \ \forall (\alpha, x) \in \mathbb{R}$$

Soln

Find the fixed points

$\alpha x - x^2 = 0$

$\cancel{\alpha x = x^2}$ $x(\alpha - x) = 0$

$x = 0$ or $x = \alpha$

We can see that the bifurcation point is when $\alpha = 0$, that is $x_{1,2} = 0$

Region 1 $\alpha > 0$

We have two fixed points

$x_1^* = 0$ and $x_2^* = \alpha$

Let $f(x) = \alpha x - x^2$

$f'(x) = \alpha - 2x$

$f'(0) = \alpha - 2(0) = \alpha$

$\Rightarrow f'(0) > 0$, it implies that $x_1^* = 0$ is unstable fixed point.

At $x_2^* = \alpha$

$f'(\alpha) = \alpha - 2(\alpha)$

$f'(\alpha) = -\alpha < 0$

$f'(\alpha) < 0$, it implies that $x_2^* = \alpha$ is asymptotically stable.

Region 2: $\alpha < 0$

We have two fixed points they are:

$x_1^* = 0$ and $x_2^* = -\alpha$

At $x_1^* = 0$

$f'(0) = -\alpha - 2(0)$

$f'(0) = -\alpha < 0$

$f'(0) < 0$ it implies that $x_1^* = 0$ is asymptotically stable.

At $x_2^* = -\alpha$

$f'(-\alpha) = -\alpha - 2(-\alpha)$

$f'(-\alpha) = -\alpha + 2\alpha$

$f'(-\alpha) = \alpha > 0$, it implies that $x_2^* = -\alpha$ is unstable fixed point.

Since the two fixed points exchange stabilities for $\alpha > 0$ and $\alpha < 0$, then $\frac{dx}{dt} = \alpha x - x^2$ undergoes transcritical bifurcation at $\alpha = 0$

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Discuss the bifurcation of

$$\frac{dx}{dt} = rx - \ln(1+x)$$

$\forall (r, x) \in \mathbb{R}$

$$g(x) = g(0) + g'(0) + \frac{g''(0)}{2!} + \ldots$$

$$g(x) = \ln(1+x)$$

$$g(0) = \ln(1) = 0$$

$$g'(x) = \frac{1}{1+x} = (1+x)^{-1}$$

$$g'(0) = 1$$

$$g''(x) = \frac{-1}{(1+x)^2} = -(1+x)^{-2}$$

$$g''(0) = -1$$

Truncate the taylor series as a quadratic form.

$$\frac{dx}{dt} = rx - \left(x - \frac{x^2}{2}\right)$$

$$\frac{dx}{dt} = rx - x + \frac{x^2}{2}$$

To find the fixed points put $dx/dt = 0$

$$rx - x + \frac{x^2}{2} = 0$$

$$x(r - 1 + x/2) = 0$$

$x = 0$ or $r - 1 + \frac{x}{2} = 0$

$$2r - 2 + x = 0$$

$$r = 1$$

$$g'''(x) = \frac{+2}{(1+x)^3} = 2(1+x)^{-3}$$

$$g'''(0) = 2$$

$$g(x) = 0 + x - \frac{x^2}{2!} + \frac{2x^3}{3!}$$

$$g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

$$\frac{dx}{dt} = rx - \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)$$

$$\frac{dx}{dt} = rx - x + \frac{x^2}{2} - \frac{x^3}{3}$$

$x_1^* = 0$ and $x_2^* = 2(1-r)$

Region $r < 1$

$$f(x) = (r-1)x + x^2/2$$

$$f'(x) = r - 1 + x$$

At $x_1^* = 0$

$$f'(0) = r - 1$$

Asymptotically stable.

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At $x_2^* = 2(1-r)$

$f'(2(1-r)) = r - 1 + 2 - 2r$

$= 1 - r$

The fixed point x_2^* is ~~a~~ unstable.

Region $r > 1$

At $x_1^* = 0$

$f'(0) = r - 1$

The fixed point x_1^* is ~~asymtotically~~ unstable.

At $x_2^* = 2(1-r)$

$f'(2(1-r)) = r - 1 + 2 - 2r$

$= 1 - r$

The fixed point x_2^* is ~~un~~stable. asymtotically.

$\therefore$ The DE undergoes a transcritical bifurcation at $r = 1$

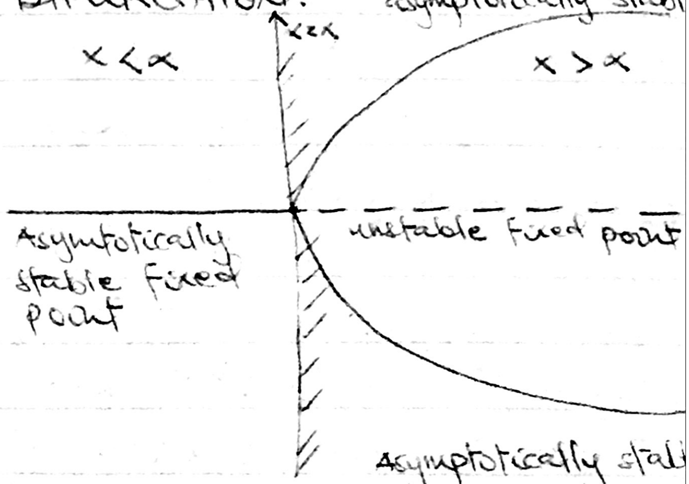
$$\frac{dx}{dt} = (r-1)x * \frac{x^2}{2} + \frac{x^3}{3}$$

put $dx/dt = 0$

22/08/22

PITCHFORK BIFURCATION

① SUPERCRITICAL PITCHFORK BIFURCATION.



Supercritical Pitchfork bifurcatio

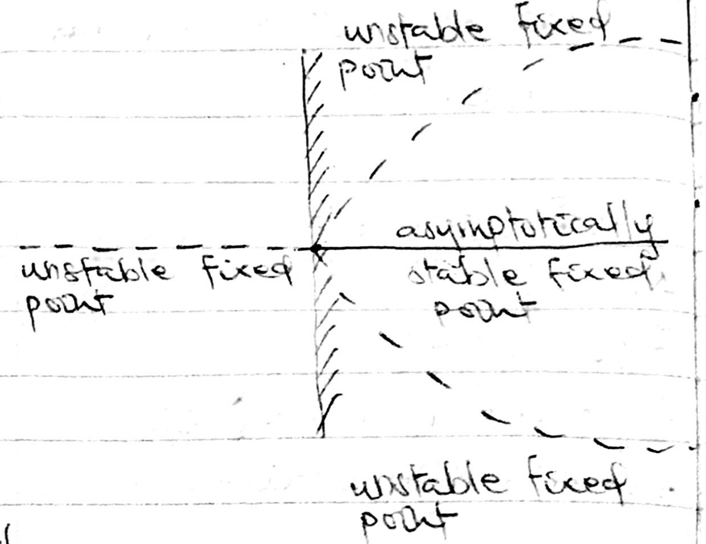
In pitchfork bifurcation, there's always symmetry no matter the type. In all cases, one fixed point is bifurcated one one side of the bifurcation and two ~~additional~~ fixed points with the previous fixed point are created in the other region.

SUPERCRITICAL

An asymptotically stable fixed point is observed on

one point of the bifurcation point and two additional fixed which are asymptotically stable fixed point are created together with previous fixed point.
In this part of the bifurcation region the asymptotically stable fixed point becomes unstable.

pitchfork bifurcation borrows ideas from saddle node and transcritical bifurcation



SUBCRITICAL PITCHFORK BIFURCATION

In subcritical pitchfork bifurcation one unstable fixed point is observed on one side of the bifurcation point. But on the other side two ~~asymptotically stable~~ unstable fixed point are created together with previous fixed point. In this region the unstable fixed point becomes asymptotically stable.
In other words the

Subcritical pitchfork.

Eg Given that $dx/dt = \lambda x - x^3$, $\forall (x, \lambda) \in \mathbb{R}$.
Discuss the bifurcation of the equation.

soln

First check for symmetry
x and $-x$

$dx/dt = \lambda x - x^3$ —— ①

putting $x = -x$ into eqn ①

$$\frac{dx}{dt}(-x) = \lambda(-x) - (-x^3)$$

$-dx/dt = -\alpha x + x^3$

$dx/dt = \alpha x - x^3$ ——— ②

$\therefore dx/dt$ is symmetric

put $dx/dt = 0$, to find the fixed point

$\alpha x - x^3 = 0$, $x(\alpha - x^2) = 0$

$x = 0$ or $\alpha - x^2 = 0$

$x_{2,3} = \pm\sqrt{\alpha}$

$x_{2,3} = 0$

This implies $\alpha = 0$ is the bifurcation point.

Region $\alpha < 0$

$x_1^* = 0$

$x_{2,3} = \pm\sqrt{\alpha}$

$x_{2,3} = \pm\sqrt{\alpha}\, i \notin \mathbb{R}$

x_2 and x_3 are destroyed in this region.

Let $f(x) = \alpha x - x^3$

$f'(x) = \alpha - 3x^2$

At $x_1^* = 0$

$f'(0) = \alpha - 3(0)^2$

$f'(0) = \alpha$, since $\alpha < 0$ in this region, it implies that $f'(0) < 0$. Hence $x_1^* = 0$ is asymptotically stable.

Region 2: $\alpha > 0$

$x_1^* = 0$

$x_{2,3} = \pm\sqrt{\alpha}$ $x_2 = \sqrt{\alpha}$ $x_3 = -\sqrt{\alpha}$

Two additional fixed points $x_2^* = \sqrt{\alpha}$ and $x_3 = -\sqrt{\alpha}$ are created in this region together with $x_1^* = 0$.

$f'(x) = \alpha - 3x^2$

At $x_1^* = 0$

$f'(0) = \alpha - 3(0)^2 = \alpha$

Since $\alpha > 0$ in this region, it implies that $f'(0) > 0$, Hence $x_1^* = 0$ is an unstable fixed.

At $x_2^* = \sqrt{\alpha}$

$f'(\sqrt{\alpha}) = \alpha - 3(\sqrt{\alpha})^2$

$f'(\sqrt{\alpha}) = \alpha - 3\alpha$

$f'(\sqrt{\alpha}) = -2\alpha$, since $\alpha > 0$, it implies that $f'(\sqrt{\alpha}) < 0$, hence $x_2^* = \sqrt{\alpha}$ is asymptotically stable

At $x_3^* = -\sqrt{\alpha}$

$f'(-\sqrt{\alpha}) = \alpha - 3(-\sqrt{\alpha})^2$

$f'(-\sqrt{\alpha}) = -2\alpha$, since $\alpha > 0$ in this region it implies that $x_3^* = -\sqrt{\alpha}$ is asymptotically stable.

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Since $x_1^* = 0$ is asymptotically stable for $\alpha < 0$ and becomes unstable fixed point for $\alpha > 0$ together with two additional asymptotically stable fixed point, it implies $dx/dt = \alpha x - x^3$ undergoes supercritical pitchfork bifurcation at $\alpha = 0$

29/08/22

Given that $dx/dt = \mu_1 + \mu_2 x - x^2$ $\forall \mu_1, \mu_2, x \in \mathbb{R}$. Discuss the bifurcation of the above differential equation.

Soln

$$\frac{dx}{dt} = \mu_1 + \mu_2 x - x^2$$

Put $dx/dt = 0$ to fixed the fixed points.

$$\mu_1 + \mu_2 x - x^2 = 0$$

$a = -1$ $\quad b = \mu_2$ $\quad c = \mu_1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-\mu_2 \pm \sqrt{\mu_2^2 - 4(-1)(\mu_1)}}{2(-1)}$$

$$= \frac{-\mu_2 \pm \sqrt{\mu_2^2 + 4(\mu_1)}}{-2}$$

$$= \frac{-\mu_2 \pm \sqrt{\mu_2^2 + 4\mu_1}}{-2}$$

$$x_2^* = \frac{\mu_2 + \sqrt{\mu_2^2 + 4\mu_1}}{2}$$

$$x_1^* = \frac{-\mu_2 + \sqrt{\mu_2^2 + 4\mu_1}}{-2}$$

Perfect and Imperfect bifurcation on one parameter

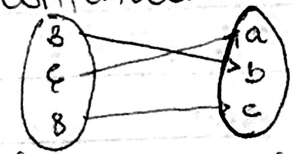
$\frac{dx}{dt} = f(x, \alpha)$, $\forall x \in \mathbb{R}$ is a parameter is Perfect vice versa

However if $dx/dt = f(x,\alpha)$ $\forall x \in \mathbb{R}$ where α is a parameter is said imperfect bifurcation if it cannot be classified under saddle node, transcritical or pitchfork bifurcation.

Homeomorphism

The map/function is onto, one

-to-one and inverse continuous.



Range = {a b c}

domain Codomain

Range = Codomain set, the map / function is an onto

Under homeomorphism an equation is a well-posed equation.

onto — solution

one-to-one = unique

For inverse to exist then the function should be one-to-one and onto.

DIFFEOMORPHISM

(near-identity)

$$\frac{dx}{dt} = \mu_1 + \mu_2 x - x^2$$

Use the near-identity diffeomorphism.

$$\frac{dx}{dt} = \mu_1 + \mu_2 x - x^2 \quad \text{——— } ①$$

Soln

Let $y(t) = x(t) - \alpha$ ——— ②

$$\frac{dy}{dt} = \frac{dx}{dt} \quad \text{——— } ③$$

putting eqns ② and ③ onto eqn ①

$$\frac{dy}{dt} = \mu_1 + \mu_2(y+\alpha) - (y+\alpha)^2$$

$$\frac{dy}{dt} = \mu_1 + \mu_2 y + \mu\alpha - (y^2 + 2\alpha y + \alpha^2)$$

$$\frac{dy}{dt} = \mu_1 + \mu_2 y + \mu\alpha - y^2 - 2\alpha y - \alpha^2$$

$$\frac{dy}{dt} = -y^2 + (\mu_2 - 2\alpha)y + \mu_1 + \alpha\mu_2 - \alpha^2 \quad ④$$

$$ay^2 + \overset{\to 0}{by} + c$$

$$\mu_2 - 2\alpha = 0$$

$$\mu_2 = 2\alpha \qquad \alpha = \mu_2/2 \quad ⑤$$

In order to obtain a saddle node bifurcation we set the coefficient to zero

putting ⑤ onto ④

$$\frac{dy}{dt} = -y^2 + \left(\mu_2 - 2\left(\frac{\mu_2}{2}\right)\right)y + \mu_1 + \left(\frac{\mu_2}{2}\right)\mu_2 - \left(\frac{\mu_2}{2}\right)^2$$

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$$\frac{dy}{dt} = -y^2 + \mu_1 + \frac{\mu_2^2}{2} - \frac{\mu_2^2}{4}$$

$$\frac{dy}{dt} = \mu_1 + \frac{1}{4}\mu_2^2 - y^2$$

let $\beta = \mu_1 + 1/4\,\mu_2^2$

$$\frac{dy}{dt} = \beta - y^2$$

put $dy/dt = 0$ to find the fixed point.

$$\beta - y^2 = 0$$

$$\beta = y^2$$

$$y^*_{1,2} = \pm\sqrt{\beta}$$

Conversion of Higher Scalar Differential Equation To A System of ODEs.

Eg Change $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2+3$ to a system of ODEs.

Soln

Let $u_1(x) = y(x)$ ——— ②

and $u_2(x) = dy/dx$ ——— ③

Differentiating the terms on the sides of equation ②

$$\frac{du_1}{dx} = \frac{dy}{dx} \quad \text{——— ④}$$

Plugging eqn ③ into ④

$$\frac{du_1}{dx} = u_2 \quad \text{——— ⑤}$$

Similarly, differentiating the terms on each side of equation ③ yields

$$\frac{du_2}{dx} = \frac{d^2y}{dx^2} \quad \text{——— ⑥}$$

From eqn ① we have

$$\frac{d^2y}{dx^2} = 5\frac{dy}{dx} - 6y \quad \text{——— ⑦}$$

Substituting eqns ③, ② and ⑦ into eqn ⑥ yields

$$\frac{du_2}{dx} = 5u_2 - 6u_1$$

Summary

$$\frac{du_1}{dx} = u_2$$

$$\frac{du_2}{dx} = -6u_1 + 5u_2$$

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$$\begin{pmatrix} du_1/dx \\ du_2/dx \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 0-\lambda & 1 \\ -6 & 5-\lambda \end{vmatrix} = 0$$

$$-\lambda(5-\lambda) + 6 = 0$$

$$\lambda_1 = 3 \quad \lambda_2 = 2$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 V_1 e^{\lambda_1 t} + C_2 V_2 e^{\lambda_2 t}$$

HOPF BIFURCATION

1. Supercritical hopf bifurcation
2. Subcritical hopf bifurcation.

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0 \quad \text{—— } (1)$$

$$\left.\begin{aligned} &\text{let } y(x) = e^{mx} \\ &\frac{dy}{dx} = me^{mx} \\ &\frac{d^2y}{dx^2} = m^2 e^{mx} \end{aligned}\right\} (2)$$

$$m^2 e^{mx} - 5me^{mx} + 6e^{mx} = 0$$

$$e^{mx}[m^2 - 5m + 6] = 0$$

$$\text{but } e^{mx} \neq 0$$

$$\text{Thus } m^2 - 5m + 6 = 0$$

$$m_1 = 3, \quad m_2 = 2$$

$$y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$y(x) = C_1 e^{3x} + C_2 e^{2x}$$

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