Linear models

How do they work?

Samuel Robinson, Ph.D.

October 8, 2020

Motivation

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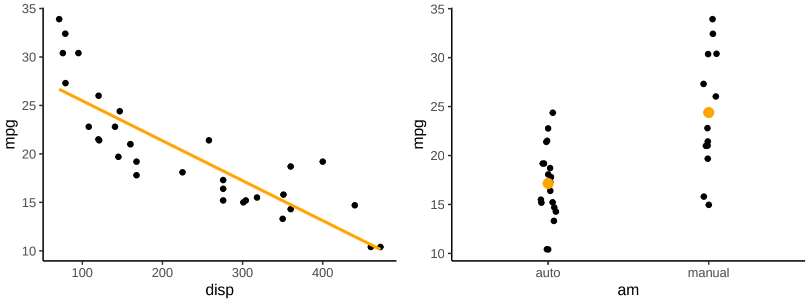

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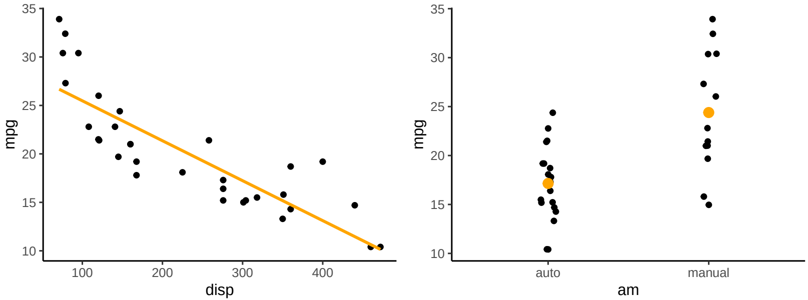
- *I have 2+ groups of data, and I want to know whether the means are different*



Model terminology

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$$\hat{y} = b_0 + b_1 x_1 + b_2 x_2 ... + b_i x_i$$
$$y \sim Normal(\hat{y}, \sigma)$$

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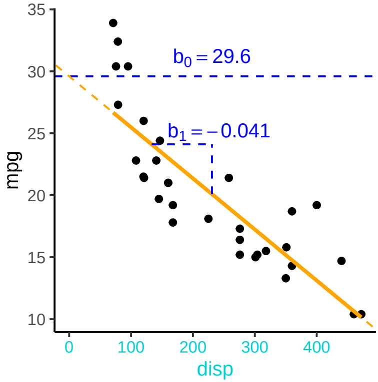

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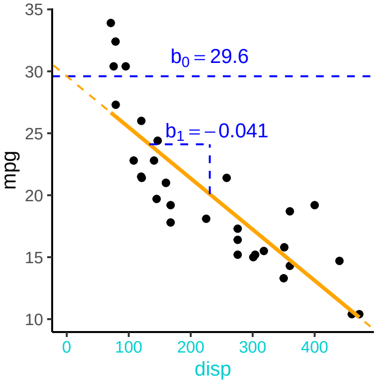

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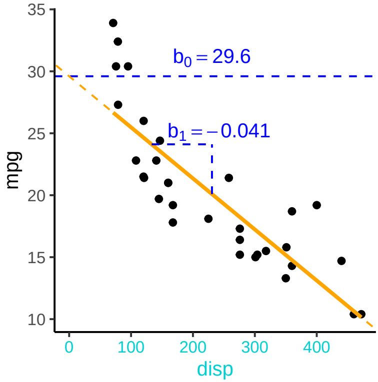

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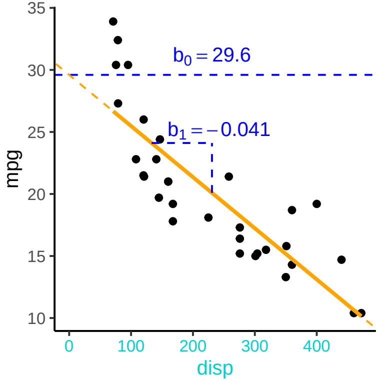

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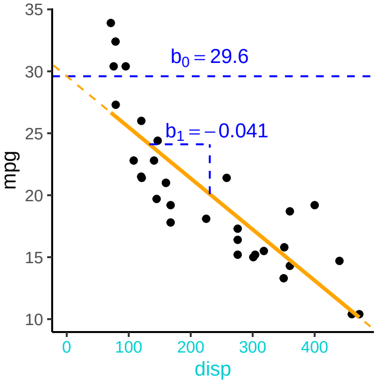

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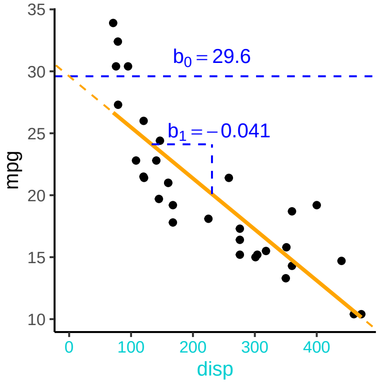

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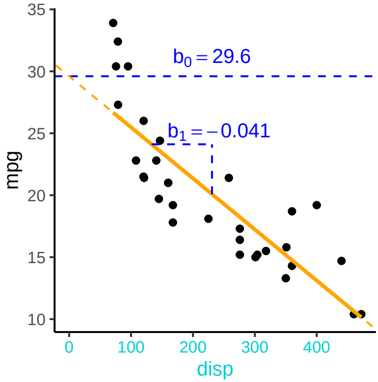

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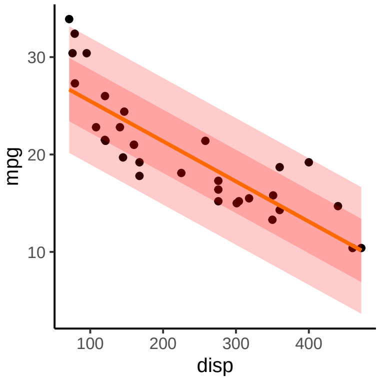

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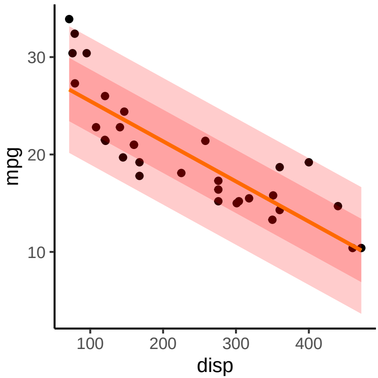

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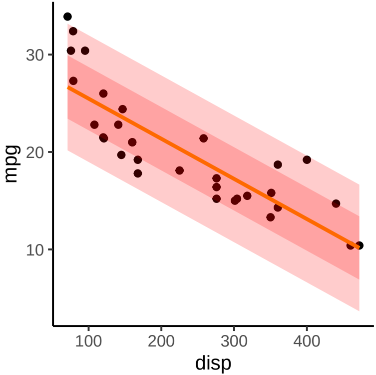


- σ is the "leftover" or "residual" variance
- i.e. variation between samples that the model couldn't explain
- Since $y \sim Normal(\hat{y}, \sigma)$, this means that points are normally distributed around the *entire line* of $\hat{y}$

How do I get R to fit this model?

`lm` is one of the main functions used for linear modeling:

```
#Formula structure: y ~ x
mod1 <- lm(mpg ~ disp, #mpg depends on disp
          data = mtcars) #Name of the dataframe containing mpg & disp
summary(mod1)
```

```
##
## Call:
## lm(formula = mpg ~ disp, data = mtcars)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -4.8922 -2.2022 -0.9631  1.6272  7.2305
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 29.599855   1.229720  24.070  < 2e-16 ***
## disp        -0.041215   0.004712  -8.747 9.38e-10 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.251 on 30 degrees of freedom
## Multiple R-squared:  0.7183, Adjusted R-squared:  0.709
## F-statistic: 76.51 on 1 and 30 DF,  p-value: 9.38e-10
```

For a detailed breakdown of `lm`'s output, click here

Simulate data

Now that we know how linear models work, we can simulate our own data:

```
#Parameters:

b0 <- 1 #Intercept
b1 <- 2 #Slope
sigma <- 3 #SD

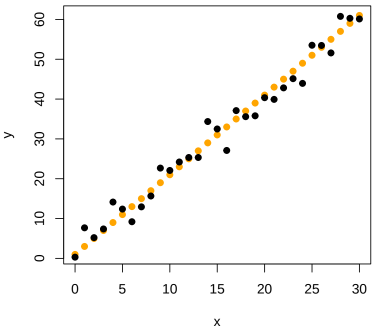
#Make up some data:

x <- 0:30 #Predictor values

#Predicted y values
pred_y <- b0 + b1*x

#Add "noise" around pred_y
actual_y <- rnorm(n = length(pred_y),
                  mean = pred_y,
                  sd= sigma)
```

```
#Plot the data we just made
plot(x,pred_y,col='orange',pch=19,
     ylab='y')
points(x,actual_y,col='black',pch=19)
```



Fit a model from simulated data

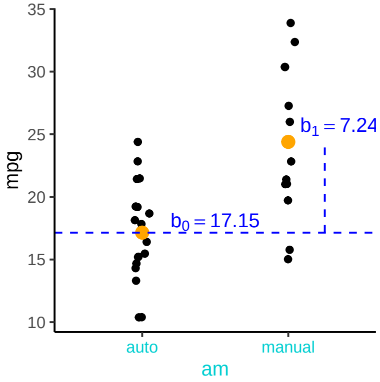
How does R do at finding the coefficients?

Remember: $b_0 = 1, b_1 = 2, \sigma = 3$

```
#Put the simulated data into a dataframe
fakeDat <- data.frame(x = x, y = actual_y, pred = pred_y)
mod1sim <- lm(y ~ x, data = fakeDat) #Fit a linear model
summary(mod1sim)
```

```
##
## Call:
## lm(formula = y ~ x, data = fakeDat)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -5.7568 -1.7623 -0.2176  1.9419  5.3572
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  2.02974    1.00445   2.021   0.0526 .
## x            1.92670    0.05751  33.499   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.864 on 29 degrees of freedom
## Multiple R-squared:  0.9748, Adjusted R-squared:  0.9739
## F-statistic:  1122 on 1 and 29 DF,  p-value: < 2.2e-16
```

What about categorical data?



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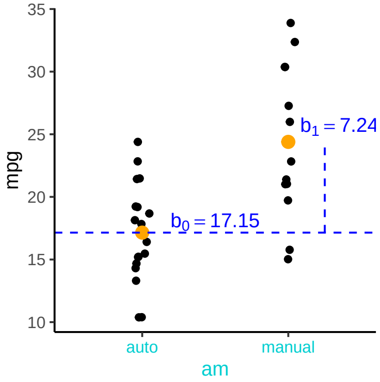

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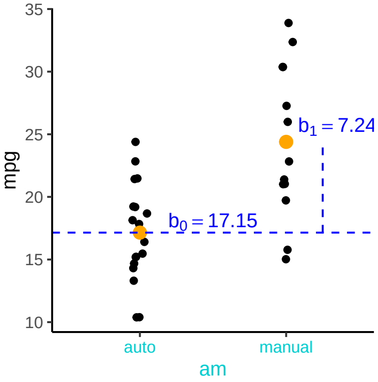

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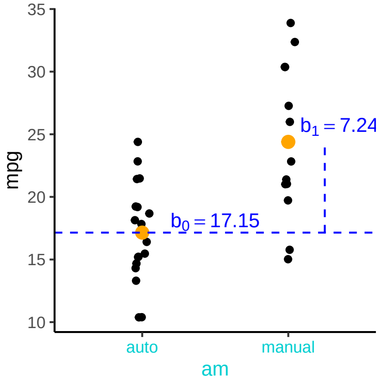

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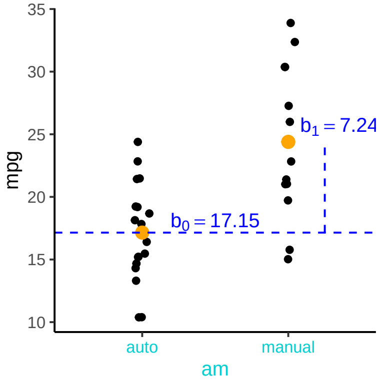

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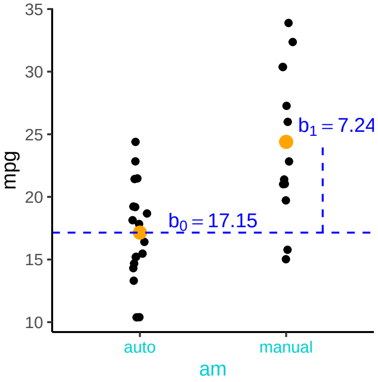

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How do I get R to fit this model?

Syntax is exactly the same for this model

```
#Formula structure: y ~ x
mod2 <- lm(mpg ~ am, #mpg depends on am
           data = mtcars) #Name of the dataframe containing mpg & am
summary(mod2)
```

```
##
## Call:
## lm(formula = mpg ~ am, data = mtcars)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -9.3923 -3.0923 -0.2974  3.2439  9.5077
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   17.147      1.125  15.247 1.13e-15 ***
## am             7.245      1.764   4.106 0.000285 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 4.902 on 30 degrees of freedom
## Multiple R-squared:  0.3598, Adjusted R-squared:  0.3385
## F-statistic: 16.86 on 1 and 30 DF,  p-value: 0.000285
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- Use `lm` to fit a model to the data you just simulated
  - How does R do at guessing your coefficients?