Dynamic Macroeconomic Modeling with Matlab

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March 2012

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1 Introduction

The aim of the course is to simulate transitional dynamics of dynamic macroeconomic models numerically. Then, the question arises: What do we gain by simulating transitional dynamics?

- We gain additional information about the model's behavior, since often analytical derivations only deliver information about the steady state.
- It is possible to analyze particular policy reforms and conduct a pareto-ranking of different reforms.
- Numerical simulations can be used for more efficient research: The implications of small modifications of the model can be tested quickly.
- ...

For numerical solution we employ the relaxation algorithm as proposed by Trimborn et al. (2008). The algorithm is designed to solve continuous-time perfect-foresight optimization models. In particular, the algorithm can solve

- dynamic systems with any number of state variables (predetermined variables)
- models exhibiting a continuum of stationary equilibria (center-manifold)
- models exhibiting algebraic equations, e.g. equilibrium conditions or no-arbitrage conditions as frequently arise in Computable General Equilibrium models
- expected or transitory shocks (variables exhibit corners or jumps)
- non-autonomous differential equations

This algorithm (so far) cannot solve

- stochastic models
- discrete time models

Matlab and Mathematica versions of the algorithm can be downloaded at

www.relaxation.uni-siegen.de.

2 An Outline of the Theory

2.1 Numerical solution of initial value problems

Consider a linear differential equation

$$\dot{x}(t) = -\alpha x, \qquad x(0) = x_0, \qquad \alpha > 0 \tag{1}$$

A numerical solution of the differential equation is a time vector $T = \{t_0, t_1, \ldots, t_M\}$ and a corresponding vector $x = \{x_0, x_1, \ldots, x_M\}$ such that $x(t_i) \approx x_i$. A simple numerical method to compute the solution on the interval $[0, 10]$ can be constructed by discretisation. Consider a mesh of time, e.g. $T = \{0, 0.1, 0.2, \ldots, 10\}$. We discretize the differential equation on this mesh according to

$$\frac{\Delta x}{\Delta t} = -\alpha x \tag{2}$$

$$\Rightarrow \qquad \frac{x_{t_i} - x_{t_{i-1}}}{t_i - t_{i-1}} = -\alpha x_{t_{i-1}} \qquad T = \{0.1, 0.2, \ldots, 10\} \tag{3}$$

Then, the numerical solution (or an approximation) can be obtained iteratively by starting at the second mesh point and applying the difference equation from point to point according to

$$x_{t_i} \;\; = \;\; x_{t_{i-1}} - (t_i - t_{i-1})\alpha x_{t_{i-1}} \qquad T = \{0.1, 0.2, \ldots, 10\} \tag{4}$$

The comparison between numerical and analytical solution can be seen in Figure 1. Note that the numerical solution deviates from the true solution, because the linearization omits higher order terms.

This algorithm is called Explicit Euler method and can easily be generalized to any non-linear differential equation:

$$\dot{x}(t) = g(x), \qquad x(0) = x_0, \tag{5}$$

with g being sufficiently smooth. Then, the algorithm reads

$$x_i = x_{i-1} + (t_i - t_{i-1})g(x_{i-1}) \qquad i = 0.1, 0.2, \ldots, 10 \tag{6}$$

Example 1 (Solow model) *The dynamics of the Solow growth model can be summarized by the Fundamental Equation (see Barro and Sala-i-Martin, 2004, p. 30)*

$$\dot{k} = sf(k) - (n + \delta)k. \tag{7}$$

Since capital k is the only variable and capital is a state variable, for which the initial value $k(0)$ is given, we could compute transitional dynamics with the explicit Euler method.

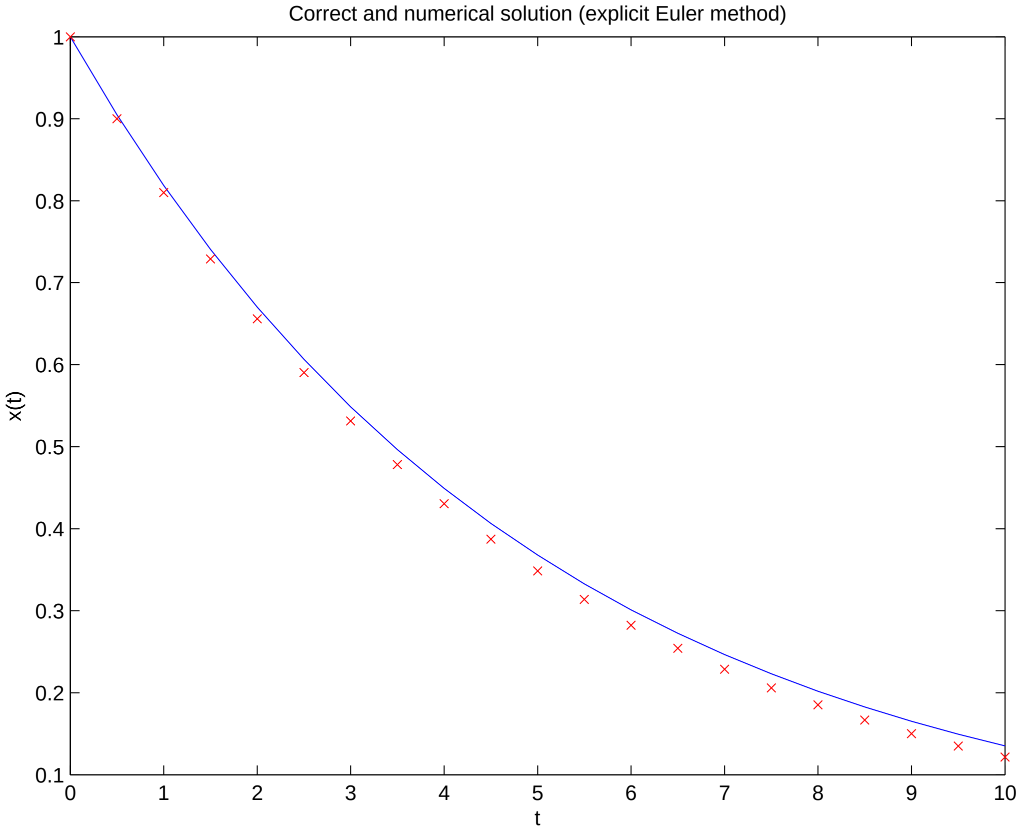

Figure 1: Correct solution (solid blue line) and numerical solution (red crosses)

The method can even be generalized to multi-dimensional differential equations. However, the initial conditions $x(0) = x_0$ must be given for all variables. Mathematicians derive iteration formulas that yield a smaller approximation error. Therefore, initial value problems can be solved with high accuracy on finite intervals, if the function g is smooth. These procedures are known as Runge-Kutta algorithms.

Example 2 (Ramsey model) *The Ramsey model (Ramsey, 1928; Cass, 1965; Koopmans, 1965) gives rise to a system of two differential equations for consumption per capita c and capital per capita k (Barro and Sala-i-Martin, 2004, Chapter 2):*

$$\dot{c} = \frac{c}{\theta}\left(\alpha k^{\alpha-1} - (\delta+\rho)\right) \tag{8}$$
$$\dot{k} = k^{\alpha} - c - (n+\delta)k, \tag{9}$$

where α denotes the elasticity of capital in production, n the population growth rate, δ the depreciation *rate, ρ the parameter for time preference and θ the inverse of the intertemporal elasticity of substitution, respectively. The interior steady state $k^* = \left(\frac{\alpha}{\delta+\rho}\right)^{\frac{1}{1-\alpha}}$ and $c^* = (k^*)^\alpha - (n+\delta)k^*$ is saddle point stable. Since k is a state variable and c is a jump variable, the model exhibits one initial boundary condition:*

$$k(0) = k_0$$

The transversality condition $\lim_{t\to\infty} k(t)\lambda(t) = 0$ with shadow price λ ensures convergence towards the interior steady state. Adjustment dynamics of the Ramsey model can be seen in Figure 2. If the

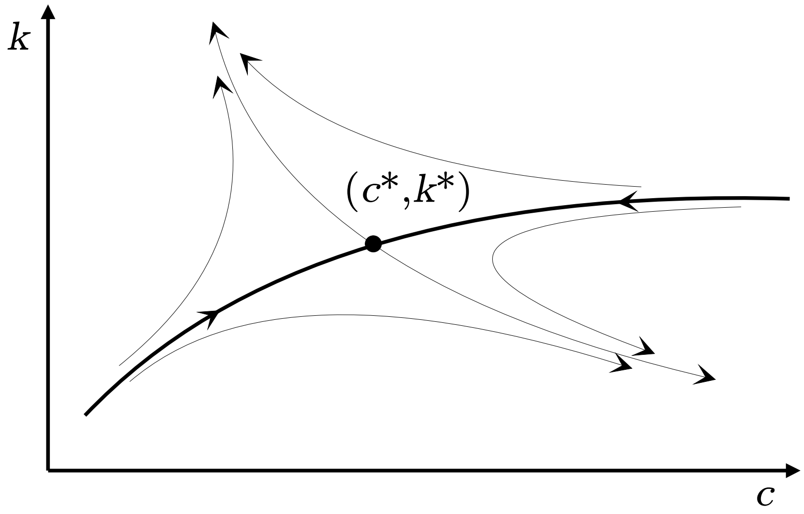

Figure 2: Phase diagram of the Ramsey model

initial value of consumption $c(0)$ is given, system (8) and (9) could be solved for intervals of time that are not too long.

2.2 Forward shooting and backward integration

Initial value solvers could be used to solve infinite-horizon problems numerically. Consider the Ramsey growth model. If an initial guess for optimal consumption $c(0)$ is provided, the system (8-9) could be solved for a finite horizon. If the solution trajectories drift in direction North-West, the initial value $c(0)$ was too high, if the solution trajectories drift in direction South-East, the initial guess was too low. Employing an iteration process, the correct value of $c(0)$ could be approximated.

This procedure is called “shooting”, since trajectories “shoot” in direction of the steady state until they pass sufficiently close. The problem is that trajectories diverge from the stable manifold (see the graphical illustration of the phase space in Figure 2). Therefore, small changes in the initial value $c(0)$ cause huge deviations from the stable manifold and, hence, the true solution. Even if one would start at the correct initial value, the solution trajectory would diverge from the stable manifold due to small numerical errors. The problem is ill conditioned.

Brunner and Strulik (2002) suggested to turn the ill conditioned problem into a well conditioned problem by inverting time. This procedure is called backward integration.

Consider a system of differential equations

$$\dot{x} = g(x) \tag{10}$$

with $x \,\epsilon\, \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$. If we introduce $\tau := -t$, we can derive

$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = -g(x) \tag{11}$$

Each vector of the vector field exactly changes its direction. E.g., for the Ramsey model Figure 2 turns into Figure 3. Hence, a stable manifold turns into an unstable manifold and vice versa. The stable solution manifold turns into an unstable manifold.

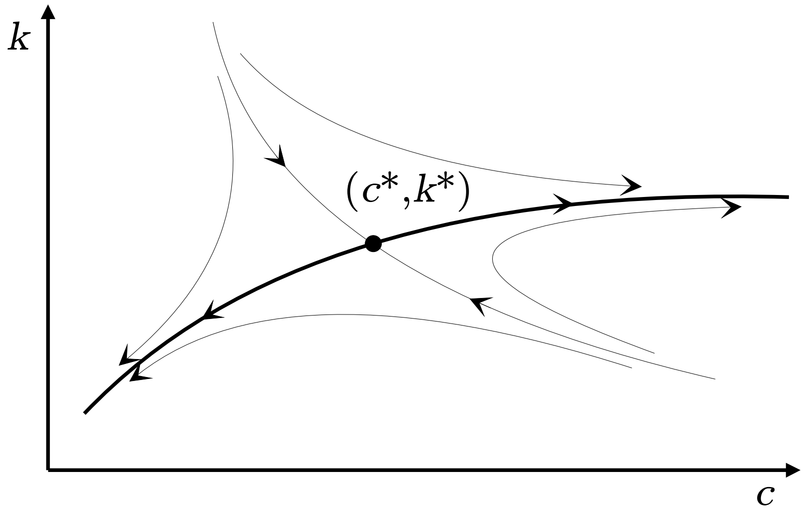

Figure 3: Phase diagram of the Ramsey model with backward looking trajectories

The idea of backward integration is to exploit the information that the economy converges towards a (unique) steady state. Therefore, the solution is traced back from a value close to the steady state to the initial condition. Looking forward, the solution converges to values in the neighborhood of the steady state in finite time. One of these values is chosen as initial value for backward integration. While the initial value $c(0)$ for the original problem is unknown, the initial value for backward integration can be found in a neighborhood of the well-known steady state. In addition, during integration the backward looking trajectories converge towards the solution manifold, instead of divergence of the forward looking trajectories. By reversing the time and thus the flow of the system, a standard forward integration procedure can be used to trace the solution trajectory back to the initial value $k(0)$. It turns out that because of the time reversal the problem is numerically stable. Finally, the trajectory is transformed back into forward looking time by a second time reversal.

Backward integration may fail if the model exhibits more than one state variable. In this case, the stable manifold is two-dimensional. Then, trajectories shooting backward from the steady state can miss the initial condition. Therefore, we apply the relaxation algorithm, which is generic with respect to the state space. This means, the algorithm treats models of different dimension conceptually in the same way.

2.3 The Generalized Problem

Economic dynamic optimization problems frequently lead to a system of differential equations potentially augmented by algebraic equations:

$$\dot{x} = f(t, x, y) \tag{12}$$

$$0 = g(t, x, y) \tag{13}$$

with $x \,\epsilon\, \mathbb{R}^{n_d}$, $y \,\epsilon\, \mathbb{R}^{n_a}$, $f : (\mathbb{R} \times \mathbb{R}^{n_d} \times \mathbb{R}^{n_a}) \to \mathbb{R}^{n_d}$ and $g : (\mathbb{R} \times \mathbb{R}^{n_d} \times \mathbb{R}^{n_a}) \to \mathbb{R}^{n_a}$. We define the total dimension of the problem as $n := n_d + n_a$. If the system is a pure system of differential equations $n_a = 0$ and $n_d = n$ would hold. Usually, the system has to be solved over an infinite horizon with boundary conditions

$$h_i(x(0), y(0)) = 0 \tag{14}$$

$$\lim_{t\to\infty} h_f(x(t), y(t)) = 0 \tag{15}$$

where $h_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ defines n_i initial boundary conditions and $h_f : \mathbb{R}^n \to \mathbb{R}^{n_f}$ defines n_f final boundary conditions such that $n_i + n_f = n_d$.

The system (12) and (13) taken together with (14) and (15) is a two-point boundary value problem, since a system of differential equations has to be solved subject to boundary conditions at the beginning and the end of the time horizon. The specific characteristics of the problem at hand are that equation (12) has to be solved for an infinite time horizon and that the solution is bound to an equation defined by (13). Frequently, the algebraic equations are differentiated with respect to time to derive a square system of differential equations. However, we do not require this for numerical solution.

Example 3 (Ramsey model) *The Ramsey model comprise two differential equations:*

$$\dot{c} = \frac{c}{\theta}\left(\alpha k^{\alpha-1} - (\delta+\rho)\right) \tag{16}$$
$$\dot{k} = k^{\alpha} - c - (n+\delta)k, \tag{17}$$

one state variable (k) and one control variable (c). Hence, we have $n = n_d = 2$*,* $n_i = 1$*,* $n_f = 1$*, and* $n_a = 0$*.*

In case of an infinite time horizon, we require the final boundary conditions to enforce convergence towards a curve of dimension n_f or less. For many economic applications the desired solution is known to converge towards a single (saddle) point. We include this case in (15). To understand the dynamics in the neighborhood of an isolated fixed point, we have to consider the Hartman Grobman theorem

Theorem 4 *(Linearization Theorem of Hartman and Grobman, Tu (1994))*
Let the nonlinear dynamic system

$$\dot{x} = f(x)$$

have a simple hyperbolic fixed point x^**. Let the Jacobian matrix* $D_x f$ *evaluated at* x^* *have* n_u *eigenvalues with positive real part and* n_s *eigenvalues with negative real part with the corresponding eigenspaces* N^u *and* N^s*, respectively (*$N^u \oplus N^s = \mathbb{R}^n$*). Then the following claims hold*

1. *In the neighborhood* U *of* $x^* \epsilon \mathbb{R}^n$ *of this equilibrium, the phase portraits of the original system and its linearization*

$$\dot{x} = D_x f|_{x^*} \cdot x$$

   are equivalent.

2. *There exists locally smooth manifolds* $W^u(x^*)$ *and* $W^s(x^*)$*, called a local unstable manifold and a local stable manifold, respectively, tangent to the linear spaces* N^u *and* N^s*, respectively.*

3. $W^s(x^*)$ *is characterized by* $||\phi(y) - \phi(x^*)|| \to 0$ *exponentially as* $t \to \infty$ *for any* $y \,\epsilon\, W^s(x^*)$*, and* $W^u(x^*)$ *is characterized by* $||\phi(y) - \phi(x^*)|| \to 0$ *exponentially as* $t \to -\infty$ *for any* $y \,\epsilon\, W^u(x^*)$*.*

For a well-defined solution the number of initial boundary conditions has to equal the dimension of the stable manifold and the number of eigenvalues with negative real part of the Jacobian matrix evaluated at the steady state.

What is a numerical solution of the problem?

A solution is a time vector $T = \{t_0, t_1, \ldots, t_M\}$ with $t_0 = 0$ and $t_M = \infty$, and a corresponding solution vector (x, y), such that $(x(t_i), y(t_i))$ denote the solution at time t_i.

2.4 The Relaxation Algorithm

"The relaxation method determines the solution by starting with a guess and improving it iteratively. As the iteration improves, the result is said to relax to the true solution." (Press et al. p. 765)

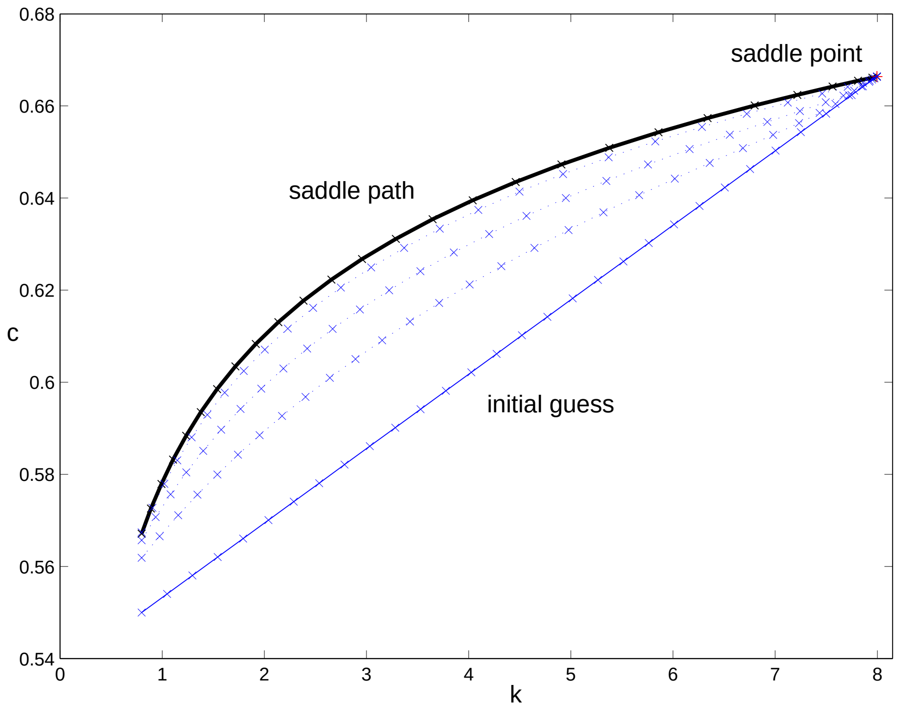

Figure 4: Relaxation in the Ramsey model

Relaxation can be seen in Figure 4. An initial (time-dependent) guess is made, which is represented in the (k, c) phase diagram. In each iteration, the initial guess is improved until it is sufficiently close to the true solution.

The principle of relaxation is to construct a large set of non-linear equations, whose solution represents the desired trajectory. This can be achieved by a discretisation of the differential equations on a mesh of points in time. This set of equations is augmented by additional algebraic equations representing equilibrium conditions or (static) no-arbitrage conditions at each mesh point. Finally, equations representing the initial and final boundary conditions are appended. The set of equations as a whole is solved simultaneously.

Constructing the set of equations

We discretize the differential equation according to

$$\dot{x} = f(x,t) \quad \Rightarrow \quad \frac{\Delta x}{\Delta t} = \frac{x_{k+1} - x_k}{t_{k+1} - t_k} \approx f\left(\frac{x_{k+1} + x_k}{2}, \frac{t_{k+1} + t_k}{2}\right). \tag{18}$$

These are $(M-1)\cdot n_d$ equations. The set of equations is augmented by n_i initial boundary conditions and n_f final boundary conditions, such that the set of equations expands to $M \cdot n_d$. Finally, algebraic equations are added for each mesh point. Hence, appending $M \cdot n_a$ equations gives $M\cdot(n_d + n_a) = M \cdot n$ equations in total. We derived a square system of non-linear equations.

Example 5 (Ramsey model) *We construct a mesh of 3 points* $T = \{0, 50, 100\}$ *and solve for* $c_0, c_{50}, c_{100}, k_0, k_{50}, k_{100}$ *numerically. For simplicity, we write the differential equations as*

$$\begin{aligned} \dot{k} &= f(c,k) \\ \dot{c} &= g(c,k) \end{aligned}$$

Then, the difference equations take the form

$$\begin{aligned} \frac{k_{50} - k_0}{50} &= f\left(\frac{c_{50} + c_0}{2}, \frac{k_{50} + k_0}{2}\right) \\ \frac{c_{50} - c_0}{50} &= g\left(\frac{c_{50} + c_0}{2}, \frac{k_{50} + k_0}{2}\right) \\ \frac{k_{100} - c_{50}}{50} &= f\left(\frac{c_{100} + c_{50}}{2}, \frac{k_{100} + k_{50}}{2}\right) \\ \frac{c_{100} - c_{50}}{50} &= g\left(\frac{c_{100} + c_{50}}{2}, \frac{k_{100} + k_{50}}{2}\right) \end{aligned}$$

The set of equations is augmented by the initial condition

$$k(0) = k_0$$

To complete the list of equations we have to fix a final boundary condition. For example, we could force consumption to equal its steady state value after 100 periods:

$$c(100) = c^*$$

Solving the set of equations

For solving the set of equations a Newton procedure is applied. The idea of the procedure can be seen in Figure 5. An initial guess for the root of the equation is needed. In a next step the value of the function for this guess is calculated. To update the initial guess the function is linearized and the root of the tangent is calculated. This root is the starting point for the next iteration. For multidimensional functions the algorithm works analogously. The procedure again needs an initial guess of the solution. The equations are evaluated for the initial guess and a multi-dimensional error function is calculated. In each iteration, the solution is updated (see Figure 4). If the error is sufficiently small, the iteration is terminated. Since the set of equations inherits a special structure from the time dependence, the set of equations can be solved using only moderate computer time.

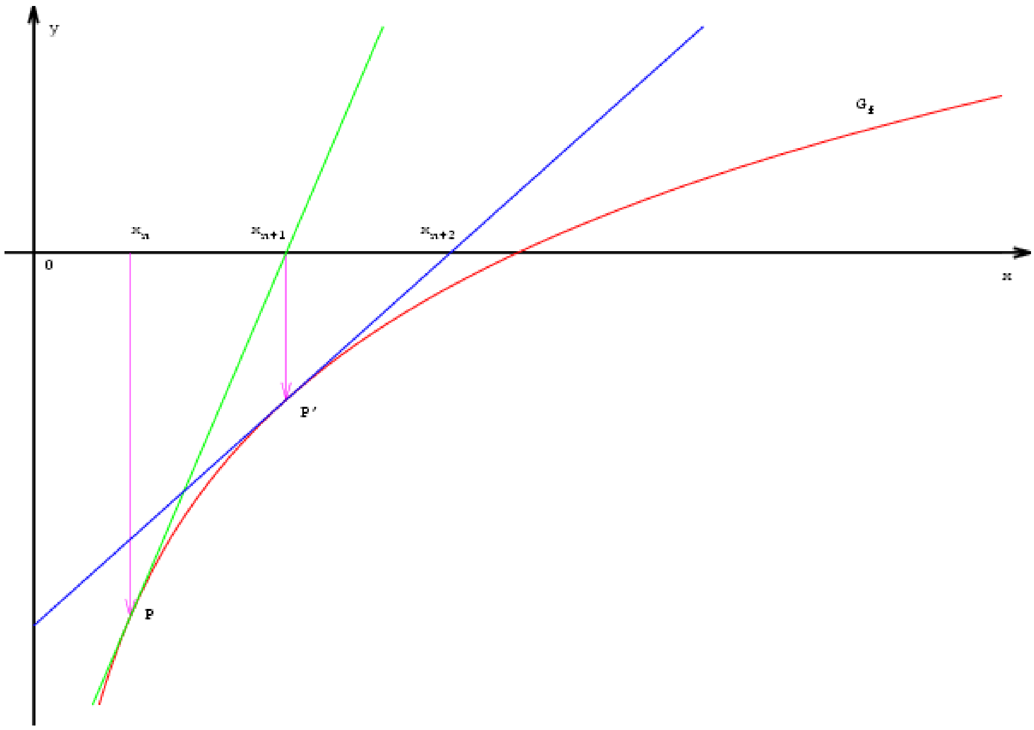

Figure 5: The Newton Procedure

Time transformation

The idea is to transform the “model-time” to the interval $[0, 1]$. Therefore, we define

$$\tau := \nu t/(1 + \nu t) \tag{19}$$

with a parameter $\nu > 0$. Then, the differential equations have to be modified according to

$$\frac{dx}{dt} = f(t,x,y) \qquad \Rightarrow \qquad \frac{dx}{d\tau} = \frac{f\left(\frac{\tau}{\nu(1-\tau)}, x, y\right)}{\nu(1-\tau)^2} \tag{20}$$

The differential equations exhibit a singularity for $\tau = 1$, but the algorithm does not need to evaluate the differential equations for $\tau = 1$ (meaning $t = \infty$). Hence, we can solve the problem for the full interval $[0, \infty)$.

The accuracy of the algorithm can be computed for the Ramsey model. If a special parametrization is chosen, the model can be solved analytically. This allows for a comparison of numerical and analytical solution. The results are shown in table 1.

Table 1: Accuracy of the relaxation algorithm for the Ramsey-Cass-Koopmans model

number of mesh points	max error c	max error k	mean error
10	$< 1.3 \cdot 10^{-2}$	$< 3.4 \cdot 10^{-2}$	$< 3.0 \cdot 10^{-3}$
100	$< 1.1 \cdot 10^{-4}$	$< 8.6 \cdot 10^{-5}$	$< 2.7 \cdot 10^{-6}$
1,000	$< 1.1 \cdot 10^{-6}$	$< 8.5 \cdot 10^{-7}$	$< 8.2 \cdot 10^{-9}$
10,000	$< 1.1 \cdot 10^{-8}$	$< 8.5 \cdot 10^{-9}$	$< 2.6 \cdot 10^{-11}$
100,000	$< 1.1 \cdot 10^{-10}$	$< 8.5 \cdot 10^{-11}$	$< 8.2 \cdot 10^{-14}$

2.4.1 Related Literature

The relaxation procedure and similar finite-difference procedures have already been employed in various fields of economics. Prominent examples comprise the solution of two point boundary value difference equations (e.g. Laffargue, 1990; Juillard et al., 1998), differential-difference equations (e.g. Boucekkine et al., 1997) as well as partial differential equations (e.g. Candler, 1999).

There are also a few applications for continuous-time optimization models in the economics literature. For instance, Oulton (1993) and Robertson (1999) employ the relaxation routine provided by Press et al. (1989) to solve a continuous-time deterministic growth model.

The most popular alternative solution methods employed in deterministic growth theory comprise backward integration (Brunner and Strulik, 2002), the finite-difference method as proposed by Candler (1999), time elimination (Mulligan and Sala-i-Martin, 1991), projection methods (e.g. Judd, 1992; Judd, 1998, Chapter 11), and the method of Mercenier and Michel (1994 and 2001). Most of the procedures and their relative advantages are described in Judd (1998) and Brunner and Strulik (2002).

3 A Short Introduction to Matlab

Matlab is a numerical computing environment and programming language, which allows for a detailed visualization of results. A detailed introduction can, for example, be found in the internet at http://www.maths.dundee.ac.uk/ ftp/na-reports/MatlabNotes.pdf.

Structure of Matlab

Commands can be entered in the Command window and are executed by pressing *enter*. E.g. entering $3 + 4$ gives the output 7, or calling the function *sinus* by entering *sin(0)* gives the output 0. Further information about a function can by displayed by calling *help function* in the command window. A more general search can be started by entering *lookfor topic* or by pressing *F1*.

A sequence of commands can be saved in an m-file and executed by calling the respective file as a command in the command window. E.g. the file *test.m* will be executed by calling *test*.

For executing m-files, the location of the file must be given to Matlab. This can be done by changing the *Current Directory* to the directory where the file is saved in, or by adding this directory as a Matlab search path (Menu *File* $\rightarrow$ *Set Path*). Note that if files with the same names are in both directories, Matlab will execute the file located in the *Current Directory*.

Entering formulas, vectors and matrices

The basic arithmetic operators are `+ - * / ^`. To calculate $\frac{4^2}{(5+2)\cdot 3}$, one must enter `4^2/((5+2)*3)`. Any variables can be assigned without prior definitions, i.e. $x = 3$ assigns the value 3 to variable x. Defining $x = [1; 2; 3]$ will construct a column vector $(1, 2, 3)^t$ whereas defining $x = [1 \quad 2 \quad 3]$ will construct a row vector $(1, 2, 3)$. Consequently, $x = [1 \quad 2 \quad 3; 4 \quad 5 \quad 6; 7 \quad 8 \quad 9]$ defines a matrix. Sequences of numbers can be constructed by the vectorization command : . E.g. $[1 : 6]$ is a vector with numbers 1 to 6, and $[1 : 2 : 7]$ is a vector with the odd numbers from 1 to 7. Componentwise products, etc. of vectors or matrices of the same size can be obtained by `.*, ./, .^` . Vectors and matrices can be transposed by `'` .

Single elements of matrices and vectors can be addressed by their index. `A(2,3)` is the element of the second row and third column of the matrix A. In the same line, `A(2,3:5)` addresses the elements 3, 4, and 5 in the second row.

If the output should be suppressed, enter a `;` at the end of the respective line.

Functions

Functions can be saved in m files in the same way as scripts. They have to be introduced according to

```
function output = functionname(x1, x2)
```

where *functionname* is the name of the function, *output* is the function output, and x_1, x_2, etc. are the input arguments. For example, the following function named *test* would yield the square of the input argument x if executed

```
function output = test(x)
    output=x*x;

>> test(3)
ans = 9
```

Exercises

1. *sin, cos, exp, sqrt, pi, inf, NaN, i*
   Calculate
$$\sqrt{2}, \quad \sqrt{-2}, \quad \frac{cos(\pi) + 2 \cdot e^0}{sin(\pi) + 1}, \quad sin(10^{10} \cdot \pi), \quad \frac{1}{0}, \quad \frac{-1}{0}, \quad \frac{0}{0}$$

2. Check the commands *clc, clear all, format, global, who, whos, which*

3. *size, eye, zeros, rand, randn, det, inv*
   Create two 3×3 matrices A and B with random entries equally distributed on the interval $[0; 10]$ and check, if
$$\begin{aligned} (A+I)(A-I) &= A^2 - I \\ (A+B)^t &= A^t + B^t \\ (AB)^t &= B^t A^t \\ \det(AB) &= \det(A) \cdot \det(B) \\ \left(A^{-1}\right)^{-1} &= A \end{aligned}$$
   holds.

4. Define the parameters $\alpha = 0.3$, $\beta = 0.4$, $\delta = 0.05$, and $A = 10$ and calculate
$$Ak^{\alpha+\beta} - \delta k$$
   for $k = 1, 1.1, 1.2, \ldots, 3$. Use vectors, i.e. define $k = 1 : 0.1 : 1.2$ etc.

5. *plot, plot3, subplot, axis, hold on, pause, close, xlabel, ylabel, title, grid, print -deps*
   Plot the *sinus* and *cosinus* functions on the interval $[0, 2\pi]$ in the same figure. Plot *sinus* with a solid blue line and *cosinus* with a dotted red line.

   Plot the helix $(sin(t), cos(t))$ on the interval $[0, 10\pi]$ in 3 dimensions. Summarize the commands in an m file. Save the figure as an *eps* file on the hard disc. (Hint: Add the commands *pause* and *close* at last.)

6. Define the function

$$f : t \to sin(e^t)$$

   in an m file and print it on the interval $[-6, 3]$.

7. *fsolve*
   Solve for x in the non-linear equation

$$x^2 = 10 \cdot \log(x)$$

   numerically. Note that there are two solutions in $(0, \infty)$.

8. *for, ODE45*
   Solve the differential equation $\dot{x} = -0.2x$ with $x(0) = 1$ numerically on the interval $[0, 10]$ employing the explicit Euler method and with the Matlab routine ODE45. Compare the numerical solution with the correct solution graphically.

4 Numerical Simulation of Macroeconomic Models

4.1 Model Preparation

To simulate a model employing the relaxation algorithm, it is necessary to fit the model to the problem as described in Section 2.3.

- Construct a square system, i.e. the number of endogenous variables and equations (differential and algebraic equations) have to coincide.
- If the variables for computation exhibit a positive long-run growth rate, they have to be scaled. This can be done by constructing ratios of variables (i.e. $k := \frac{K}{L}$) or by scale adjustment (i.e. $k := Ke^{-\gamma t}$). The latter will be explained in more detail in the section of the Lucas model.
- Choose initial conditions: These usually comprise initial values of state variables. State variables are variables, for which the model assumptions forbit a jump. Often they are easy to identify (e.g. capital in a closed economy), but sometimes there are pitfalls (e.g. Romer model).
- Choose final boundary conditions: Usually it is fine to force n_f variables to be stationary in the long-run, since then the remaining variables converge to the steady state automatically.

Write down a list of variables, starting with the state variables, then the control variables, and finally the static/algebraic variables. Write down the differential equations in the same order such that each differential equation takes the same rank in the list as its variable.

Finally, follow the instructions in the instruction manual.

4.2 The Ramsey Model

4.2.1 The model (social planner's solution)

Consider a simple neoclassical economy without technological progress. The production function is neoclassical with labor and capital as inputs according to

$$Y = K^{\alpha} L^{1-\alpha} \qquad 0 < \alpha < 1. \tag{21}$$

or

$$y = k^{\alpha} \tag{22}$$

in per capita terms. The utility function of the representative individual is of the CIES type

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma} \qquad \text{with} \quad \sigma > 0. \tag{23}$$

The economies resource constraint is given by

$$\dot{k} = y - c - (n+\delta)k \tag{24}$$

with depreciation δ and population growth rate n.

A social planner solves

$$\begin{aligned} &\max_{c(t)} \int_0^{\infty} \frac{c^{1-\sigma}}{1-\sigma} e^{(n-\rho)t} dt \\ \text{s.t.} \quad &\dot{k} = y - c - (n+\delta)k \qquad k(0) = k_0 \end{aligned} \tag{25}$$

The current-value Hamiltonian of this problem reads

$$H = \frac{c^{1-\sigma}}{1-\sigma} + \lambda(k^{\alpha} - c - (n+\delta)k) \tag{26}$$

and the necessary first-order conditions are given by

$$H_c = 0 \qquad \Leftrightarrow \qquad c^{-\sigma} = \lambda \tag{27}$$

$$H_{\lambda} = \dot{k} \qquad \Leftrightarrow \qquad \dot{k} = y - c - (n+\delta)k \tag{28}$$

$$H_k + (n-\rho)\lambda = -\dot{\lambda} \qquad \Leftrightarrow \qquad -\dot{\lambda} = (\alpha k^{\alpha-1} - (n+\delta))\lambda + (n-\rho)\lambda \tag{29}$$

and the transversality condition

$$\lim_{t\to\infty} k(t)\lambda(t) = 0. \tag{30}$$

Taking logarithms of (27) and differentiating with respect to time yields

$$-\sigma\frac{\dot{c}}{c} = \frac{\dot{\lambda}}{\lambda} \tag{31}$$

Rearranging equation (29) yields

$$\frac{\dot{\lambda}}{\lambda} = -\alpha k^{\alpha-1} + \delta + \rho \tag{32}$$

Taken together, this yields the Keynes-Ramsey rule

$$\frac{\dot{c}}{c} = \frac{\alpha k^{\alpha-1} - \delta - \rho}{\sigma}$$

The model's dynamic behavior is summarized by the dynamic system

$$\begin{aligned} \dot{c} &= c\frac{\alpha k^{\alpha-1} - \delta - \rho}{\sigma} && (33)\\ \dot{k} &= k^\alpha - c - (n+\delta)k && (34)\\ k(0) &= k_0 && (35) \end{aligned}$$

The transversality condition ensures convergence towards the interior steady state.

The interior steady state can be calculated as

$$\begin{aligned} k^* &= \left(\frac{\alpha}{\delta+\rho}\right)^{\frac{1}{1-\alpha}} && (36)\\ c^* &= (k^*)^\alpha - (n+\delta)k^* && (37) \end{aligned}$$

For adjustment dynamics to be locally unique, the Jacobian matrix evaluated at the steady state has to exhibit one eigenvalue greater than zero (unstable root) and one eigenvalue smaller than zero (stable root). The reason for this is that the two-dimensional dynamic system exhibits one initial condition, i.e. the initial value of capital. Before computing transitional dynamics, we will verify this result numerically. For an analytical calculation see Barro and Sala-i-Martin (2004, pp. 132).

To prepare the dynamic system for the relaxation algorithm we need to define one terminal condition for $t = \infty$. Since the optimal trajectory approaches a single point (c^*, k^*), this would yield two terminal equations $c(\infty) = c^*$ and $k(\infty) = k^*$. We omit one of them and force consumption to its steady state level. Capital is approaching its steady state level automatically.

4.2.2 The model (market solution)

We augment the model with taxation of capital income, labor income, and consumption. Again, we omit technological progress and assume the population to grow with constant rate n. We follow Barro and Sala-i-Martin (2004, Chapter 3) and introduce proportional taxes on wage income, τ_w, private asset income, τ_r, and consumption, τ_c. Thus the representative household's maximization problem is

$$\begin{aligned} &\max_c \int_0^\infty \frac{c^{1-\sigma}-1}{1-\sigma} e^{(n-\rho)t} dt \qquad (38) \\ \text{s.t.} \qquad \dot{a} &= (1-\tau_w)w + (1-\tau_a)ra - (1+\tau_c)c - na + T, \quad a(0) = a_0 \end{aligned}$$

whereas c denotes consumption per capita, k the capital stock per capita, T lump-sum transfers, w the wage rate, r the interest rate, σ the inverse of intertemporal elasticity of substitution, and ρ the discount factor, respectively.

The government is assumed to run a balanced budget. Therefore, government revenues equal total outlays. Government revenues are transferred back to households in a lump-sum way. Firms produce according to a Cobb-Douglas production function

$$Y = K^\alpha L^{1-\alpha}$$

whereas Y denotes the output, K the capital stock, L the amount of labor employed in production, and α the elasticity of capital in final-output production, respectively.

Since perfect competition in factor markets is assumed, firms pay the factors according to their marginal product,

$$r = \alpha k^{\alpha-1} - \delta \qquad (39)$$

$$w = (1-\alpha)k^\alpha \, . \qquad (40)$$

Solving the households optimization problem yields the Keynes-Ramsey rule

$$\frac{\dot{c}}{c} = \frac{(1-\tau_a)r-\rho}{\sigma}$$

Assuming capital market equilibrium $a = k$ yields the system

$$\dot{k} = w + rk - c - nk \qquad (41)$$

$$\frac{\dot{c}}{c} = \frac{(1-\tau_a)r-\rho}{\sigma} \, . \qquad (42)$$

with initial condition $k(0) = k_0$. Equations (39) and (40) can either be inserted into system (41) and (42), or treated as algebraic equations for the simulation.

4.2.3 Exercises

1. Solve the social planner's optimization problem numerically. Employ the parameter values $\delta = 0.05$, $\alpha = 0.33$, $\sigma = 2$, $n = 0.01$, $\rho = 0.02$. Assume that the economy initially starts with a capital stock of $k(0) = 0.1 \cdot k^*$. Plot the saving rate during transition for different parameters of σ.

2. Generate a random distribution of 20 initial capital stocks uniformly distributed on the interval $(0, k^*)$. Solve the Ramsey model for these 20 economies. Plot a scatter plot with the initial income and the growth rate the first 20 years of each economy.

3. Solve the market solution of the Ramsey model with parameter values $\tau_k = 0.25$, $\tau_w = 0.3$, and $\tau_c = 0.19$. Assume that the economy is in a steady state and that the government reduces the capital income tax by 10 percentage points. Compute the utility gain of this reform. (See next Subsection)

4. Repeat the same exercise as above, but assume that the agents in the economy anticipate this tax change five years in advance. That is, at time 0 tax rates do not change, but the information set of economic agents. (For a qualitative analysis see Figure 6 and 7)

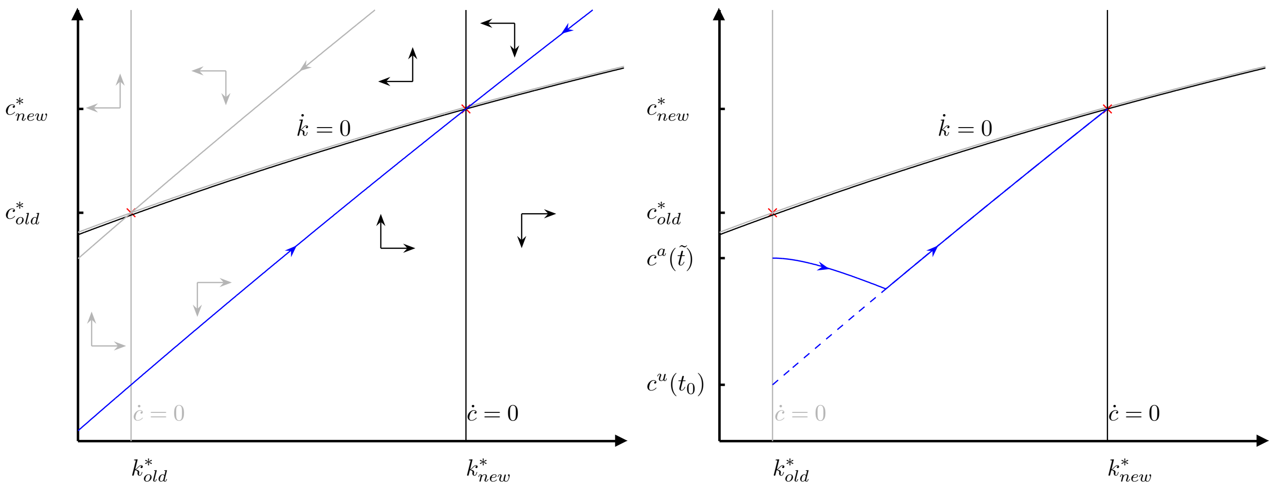

Figure 6: Phase diagram of an anticipated tax cut

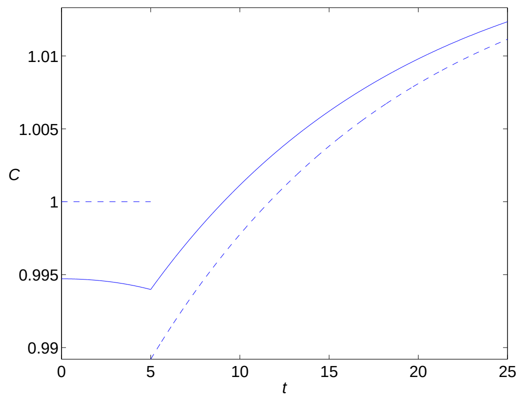

Figure 7: impulse response function for anticipated and unanticipated tax cut

4.2.4 Numerical Calculation of Utility

Consider a household that maximizes

$$\max_{c(t)} U \quad \text{with} \quad U = \int_0^\infty u(c)e^{-\rho t}dt \tag{43}$$

with per-capita consumption c, discount rate ρ, and standard utility function $u(\cdot)$. I define

$$\tilde{U}(t) = \int_0^t u(c)e^{-\rho\tau}d\tau \tag{44}$$

and assume $u(\cdot)$ to equal $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$. Moreover, I define the balanced growth rate of c to equal γ and I define the scale-adjusted variable $\tilde{c}$ as $\tilde{c} := ce^{-\gamma t}$. Then I get

$$\tilde{U}(t) = \int_0^t \frac{c^{1-\sigma}}{1-\sigma}e^{-\rho\tau}d\tau \tag{45}$$

$$= \int_0^t \frac{\tilde{c}^{1-\sigma}}{1-\sigma}e^{(\gamma(1-\sigma)-\rho)\tau}d\tau \tag{46}$$

Note that

$$\tilde{U}(0) = 0. \tag{47}$$

Differentiating with respect to time yields

$$\dot{\tilde{U}} = \frac{\tilde{c}^{1-\sigma}}{1-\sigma}e^{(\gamma(1-\sigma)-\rho)t} \tag{48}$$

For numerical simulations I can calculate $U = \tilde{U}(\infty)$ easily by including differential equation (48) together with the initial boundary condition (47) in the set of differential equations. Note that including equation (48) will add a zero eigenvalue to the set of eigenvalues of the Jacobian evaluated at a stationary point. The advantage of this procedure is that the routine *relax2e.m* provides an estimate of the relative error for the solution. Therefore, the routine provides an estimate of the relative error of U, which is very useful for welfare comparisons of different policy scenarios.

4.3 The Lucas Model

4.3.1 The model (market solution)

As a second example we want to simulate the Lucas (1988) model which is also discussed in Mulligan and Sala-i-Martin (1993), Caballe and Santos (1993), and Benhabib and Perli (1994). The analysis of the model by these authors differs in one technical aspect, namely how stationary variables are constructed for analyzing the dynamic system. While Mulligan and Sala-i-Martin (1993) and Benhabib and Perli (1994) construct stationary variables by creating ratios of endogenous variables that exhibit the same balanced growth rate, Lucas (1988) and Caballe and Santos (1993) apply scale-adjustment. The latter method consists of slowing down the motion of variables according to their respective balanced growth rates. Scale-adjustment possesses the advantage that the time paths of variables are obtained right away and, for example, utility integrals can easily be computed. Moreover, by employing scale adjustment an important characteristic of the model becomes apparent. The long-run equilibria in the scale adjusted system are not represented by an isolated fixed point, but form a center manifold of stationary equilibria. Therefore, the specific steady state to which the economy converges depends on the initial conditions, i.e. the initial endowment of physical and human capital.

Final output is produced from physical capital k and human capital h. The share u of human capital is used for final output production

$$y = A\,k^{\alpha}(uh)^{1-\alpha}h^{\gamma} \tag{49}$$

with the elasticity α of physical capital in output production, the overall productivity parameter A, and the external effect γ of human capital in final output production. Due to human capital spill over effects there are increasing returns to scale in the production sector. The remainder $1-u$ of human capital is employed to increase human capital according to

$$\dot{h} = \delta(1-u)h \tag{50}$$

with overall productivity parameter of human capital accumulation δ. A representative household maximizes intertemporal utility of consumption c

$$\max_{c} \int_0^{\infty} \frac{c^{1-\theta}}{1-\theta} e^{-\rho t} dt \tag{51}$$

with constant elasticity of intertemporal substitution σ^{-1} and discount rate ρ. The first order conditions for optimal solutions in terms of a system of differential equations read (see Benhabib and Perli, 1994, for the derivation)

$$
\begin{aligned}
\dot{k} &= A\,k^{\alpha}h^{1-\alpha+\gamma}u^{1-\alpha} - c && (52)\\
\dot{h} &= \delta(1-u)h && (53)\\
\dot{c} &= \sigma^{-1}c(\alpha A\,k^{\alpha-1}h^{1-\alpha+\gamma}u^{1-\alpha} - \rho) && (54)\\
\dot{u} &= u\left(\frac{(\gamma-\alpha)\delta}{\alpha}(1-u) + \frac{\delta}{\alpha} - \frac{c}{k}\right)\,. && (55)
\end{aligned}
$$

Balanced growth requires u, c/k as well as $k^{\alpha-1}h^{1-\alpha+\gamma}$ to be constant. The latter requirement in turn demands $(1-\alpha)\frac{\dot{k}}{k} = (1-\alpha+\gamma)\frac{\dot{h}}{h}$.

In this simple case the common balanced growth rate μ of k and c can be computed by solving the system under balanced growth assumptions:

$$\mu = \frac{1-\alpha+\gamma}{(1-\alpha+\gamma)\sigma-\gamma}\,(\delta-\rho)$$

Growth is balanced if the four variables of the system satisfy the three following equations:

$$
\begin{aligned}
1-u &= \frac{1-\alpha}{(1-\alpha+\gamma)\sigma-\gamma}(1-\rho/\delta) && (56)\\
c/k &= ((\gamma-\alpha)\psi\mu+\delta)/\alpha && (57)\\
k^{\alpha-1}h^{1-\alpha+\gamma} &= \frac{\sigma\mu+\rho}{\alpha A}u^{\alpha-1} && (58)
\end{aligned}
$$

where $\psi := (1-\alpha)/(1-\alpha+\gamma)$. We construct a stationary system by employing scale-adjustment. The transformed variables are

$$ke^{-\mu\,t}, \quad he^{-\psi\mu\,t}, \quad ce^{-\mu\,t} \quad \text{and} \quad u.$$

To avoid extra notation we continue to use the old designations of variables. The new, adjusted growth rates are reduced by the constants of adjustment, μ and $\psi\mu$, respectively. The growth rate of u remains unchanged. Therefore, the transformed system reads

$$
\begin{aligned}
\dot{k} &= A\,k^{\alpha}h^{1-\alpha+\gamma}u^{1-\alpha} - c - \mu k && (59)\\
\dot{h} &= \delta(1-u)h - \psi\mu h && (60)\\
\dot{c} &= \sigma^{-1}c(\alpha A\,k^{\alpha-1}h^{1-\alpha+\gamma}u^{1-\alpha} - \rho) - \mu c && (61)\\
\dot{u} &= u\left(\frac{(\gamma-\alpha)\delta}{\alpha}(1-u) + \frac{\delta}{\alpha} - \frac{c}{k}\right)\,. && (62)
\end{aligned}
$$

Due to scale adjustment, balanced growth solutions represented by equations (56), (57) and (58) now turn into stationary points of system (59) - (62).[1] Therefore, the system exhibits a curve of stationary equilibria (see Figure 8). This means that a unique trajectory is given by initial values of physical and human capital. However, the final steady state, to which the economy converges, is not unique but depends on the initial values of physical and human capital.

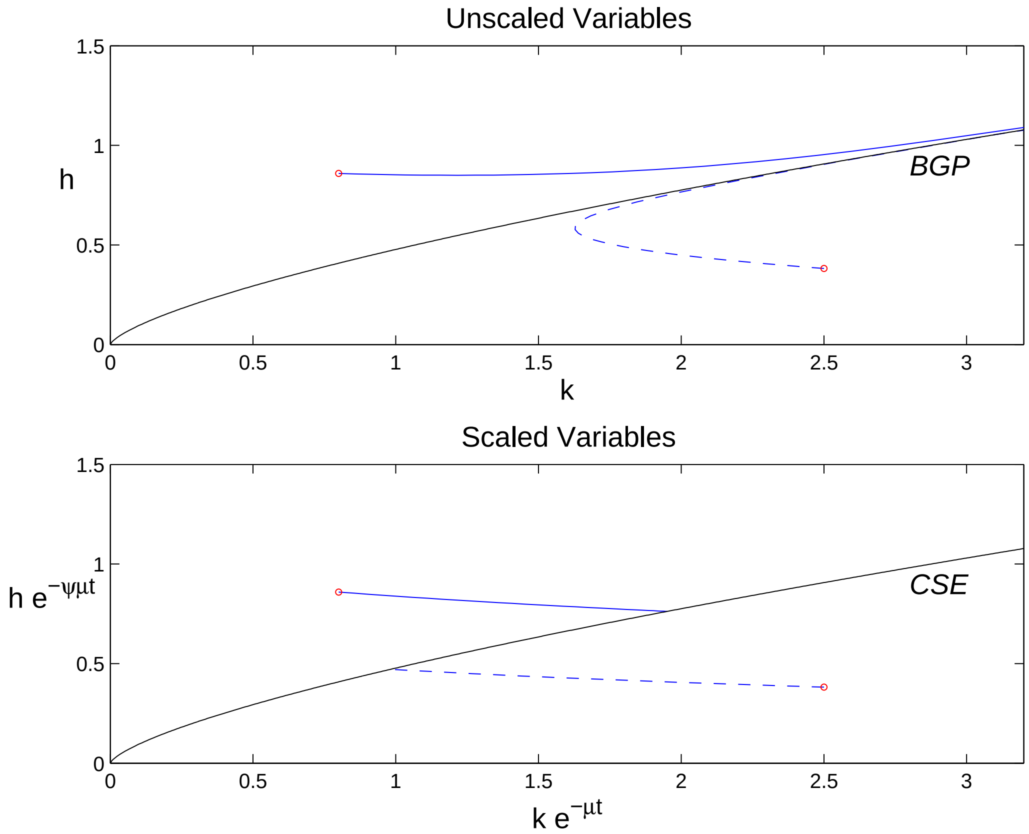

Figure 8: Phase diagram of the Lucas model

Numerical computation employing the relaxation algorithm requires the solution of the system of differential equations (59) - (62) with two initial conditions and two final conditions. The initial conditions are given by the initial values of state variables $k(0) = k_0$, $h(0) = h_0$. Final conditions should ensure convergence towards the center manifold given by equations (56), (57) and (58). However, the relaxation algorithm can only incorporate two final boundary conditions. Therefore, we incorporate

[1] It is straightforward to verify that the right hand sides of system (59) - (62) are linearly dependent, and that equations (56), (57) and (58) are the only solution.

stationary conditions for the state variables, implicitly defined by $\dot{k}(\infty) = 0$ and $\dot{h}(\infty) = 0$.

The set of valid parameter values has been investigated by Benhabib and Perli (1994). They define the subsets

$$\Theta_1 = \left\{ (A, \alpha, \gamma, \delta, \rho, \sigma) > 0 \,\middle|\, 0 < \rho < \delta \ \wedge \ \sigma > 1 + \frac{\rho(\alpha - 1)}{\delta(1 - \alpha + \gamma)} \right\} \tag{63}$$

$$\begin{aligned} \Theta_2 = \Big\{ & (A, \alpha, \gamma, \delta, \rho, \sigma) > 0 \,\Big|\, \delta < \rho < \frac{\delta(1 - \alpha + \gamma)}{\alpha - 1} \ \wedge \\ & \sigma < 1 + \frac{\rho(\alpha - 1)}{\delta(1 - \alpha + \gamma)} \Big\}. \end{aligned} \tag{64}$$

For parameter sets originating from Θ_1, the balanced growth path and trajectories converging towards the balanced growth path are unique. For parameter sets originating from Θ_2, the balanced growth path is unique, but for local transitional dynamics two different cases occur. Θ_2 can further be divided into Θ_2^A and Θ_2^B. If parameters from Θ_2^B are chosen, the balanced growth path is unstable. For parameters from Θ_2^A, the case of indeterminacy occurs, i.e. for given initial endowments of physical and human capital, there exists infinitely many trajectories converging towards the balanced growth path. We focus on the case of determinate adjustment.

4.3.2 Exercises

1. Solve adjustment dynamics for the Lucas model employing the parameter set $A = 1$, $\alpha = 0.3$, $\delta = 0.1$, $\gamma = 0.3$, $\sigma = 1.5$ and $\rho = 0.05$. Show that the economy converges towards different long-run values by simulating transitional dynamics for different initial conditions.

2. Plot the original (de-scaled) value of consumption for two different economies on the time interval $[0, 30]$.

3. Choose one particular steady state value (k^*, h^*) and compute transitional dynamics for 20 economies with (k_0, h_0) randomly chosen from $[0.9k^*, 1.1k^*]$ and $[0.9h^*, 1.1h^*]$. Plot the phase diagram (k^*, h^*) showing the transition of these economies.

4.4 The Romer (1990) Model

4.4.1 The model

Model setup

Households maximize intertemporal utility of consumption according to

$$\max_{c(t)} \int_0^\infty \frac{c^{1-\sigma}}{1-\sigma} e^{-\rho t} dt \tag{65}$$

Final output is produced in a competitive sector according to

$$Y = L^{1-\alpha-\beta} H_Y^\beta \int_0^A x(i)^\alpha di \tag{66}$$

Noting the general symmetry of intermediate goods, $x(i) = x$, we can simplify to

$$Y = L^{1-\alpha-\beta} H_Y^\beta A x^\alpha \tag{67}$$

Introducing capital as $K := Ax$ the aggregate production function can be written as

$$Y = L^{1-\alpha-\beta} H_Y^\beta A^{1-\alpha} K^\alpha \tag{68}$$

Each type of intermediate good i is produced by a monopolistic competitive firm according to

$$x(i) = k(i) \qquad \forall i \epsilon [0, A] \tag{69}$$

where $k(i)$ is capital employed by firm i. Hence, marginal costs for producing $x(i)$ are given by r. Total capital demand is given by

$$K = \int_0^A k(i) di \tag{70}$$

Capital is supplied by households who own financial wealth

$$a = K + P_A A \tag{71}$$

with number of firms (blueprints) A of value P_A.

The R&D sector is competitive and produces blueprints according to

$$\dot{A} = \eta A H_A \qquad \eta > 0, \quad A(0) = A_0 \tag{72}$$

Blueprints are sold to an intermediate firm at price P_A.

Profits of an intermediate firm

Firm i faces an intermediate demand function $p_x = \alpha x^{\alpha-1} L^{1-\alpha-\beta} H_Y^{\beta}$. Maximization of $\pi = p_x x - rx$ implies an optimal monopolistic price

$$p_x = \frac{r}{\alpha} \tag{73}$$

Hence, monopolistic firms add the markup $1/\alpha > 1$ to marginal costs r. Equilibrium profits can be written as

$$\pi = p_x x - \alpha p_x = (1-\alpha)\alpha x^{\alpha} L^{1-\alpha-\beta} H_Y^{\beta} = (1-\alpha)\alpha K^{\alpha} A^{-\alpha} L^{1-\alpha-\beta} = (1-\alpha)\alpha \frac{Y}{A} \tag{74}$$

Capital compensation

From intermediate firms we get $\frac{r}{\alpha} = \alpha x^{\alpha-1} L^{1-\alpha-\beta} H_Y^{\beta}$ and, hence,

$$r = \alpha^2 x^{\alpha-1} L^{1-\alpha-\beta} H_Y^{\beta} = \alpha^2 \left(\frac{K}{A}\right)^{\alpha-1} L^{1-\alpha-\beta} H_Y^{\beta} = \alpha^2 \frac{Y}{K} \tag{75}$$

Note that capital is underpaid relative to a competitive equilibrium.

Labor market equilibrium

Profit maximization in the final output sector implies

$$w^H = \beta \frac{Y}{H_Y} \tag{76}$$

The R&D sector is perfectly competitive. Production in this sector exhibits a positive external effect because of the 'standing on shoulders' effect. We introduce government subsidies for researchers working in the R&D sector. The government pays the fraction s_A of the R&D firm's wage bill:

$$(1-s_A) w^H = p_A \eta A \tag{77}$$

Equating both equations yields

$$\frac{p_A \eta A}{(1-s_A)} = \beta \frac{Y}{H_Y} \tag{78}$$

Capital market equilibrium

Equilibrium in the capital market requires the no-arbitrage condition to hold

$$\dot{P}_A + \pi = r P_A \tag{79}$$

In other words, the price of a blueprint P_A at time t equals the present value of discounted profits of the time interval $[t, \infty]$.

Household optimization

Household optimization yields the Keynes Ramsey rule

$$\dot{c} = \frac{c(r-\rho)}{\sigma} \tag{80}$$

Households' budget constraint is given by

$$\dot{a} = ra + w^H H_Y + w^L L + w^H H_A - c - S \tag{81}$$

with total subsidies $S = s_A w^H H_A$. Differentiating $a = K + P_A A$ with respect to time one can rewrite the equation as

$$\dot{K} = Y - c \tag{82}$$

Summary of Notation

Y	final output
K	capital
L	labor
H	human capital
A	number of blueprints
$x(i)$	amount of intermediate i
c	consumption
P_A	price of blueprint
a	household wealth
π	profit per intermediate firm
r	interest rate
w^H	wage rate of human capital
w^L	wage rate of labor
ρ	time preference rate
σ	inverse of intertemporal elasticity of consumption
α, β, η	technology parameters

Dynamic system

The dynamic system can be summarized as

$$\begin{aligned} \dot{K} &= Y - c & (83)\\ \dot{A} &= \eta A(H - H_Y)m & (84)\\ \dot{c} &= \frac{c(r-\rho)}{\sigma} & (85) \end{aligned}$$

$$\dot{P}_A = rP_A - \pi \tag{86}$$

$$\tag{87}$$

and one algebraic constraint

$$\beta \frac{Y}{H_Y} = \frac{P_A \eta A}{1 - s_A} \tag{88}$$

with $Y = L^{1-\alpha-\beta} H_Y^{\beta} A^{1-\alpha} K^{\alpha}$, $r = \alpha^2 \frac{Y}{K}$, and $\pi = (1-\alpha)\alpha \frac{Y}{A}$. Initial conditions are given by

$$K(0) = K_0 \tag{89}$$

$$A(0) = A_0 \tag{90}$$

The balanced growth rate of variables Y, K, A and c can be calculated as

$$g = \frac{\alpha\eta(1-\alpha)H - (1-s_A)\beta\rho}{\alpha - \alpha^2 + (1-s_A)\beta\sigma} \tag{91}$$

Note that it is inconvenient to use the household budget restriction as a dynamic equation, since K and A are state variables, but not a. Hence, if we want to study a specific shock, the initial state of the economy is given by K and A. If the household budget restriction would be used instead of the capital accumulation equation, households wealth $a_0 = K_0 + P_A(0)A_0$ has to be given as initial condition.

We define scale adjusted variables according to

$$\tilde{x}(t) := X(t)e^{-gt} \tag{92}$$

Then, the dynamic system modifies to (we keep the designation of variables)

$$\dot{K} = Y - c - gK \tag{93}$$

$$\dot{A} = \eta A(H - H_Y) - gA \tag{94}$$

$$\dot{c} = \frac{c(r-\rho)}{\sigma} - gc \tag{95}$$

$$\dot{P}_A = rP_A - \pi \tag{96}$$

$$\tag{97}$$

and one algebraic constraint

$$\beta \frac{Y}{H_Y} = \frac{P_A \eta A}{1 - s_A} \tag{98}$$

with $Y = L^{1-\alpha-\beta} H_Y^{\beta} A^{1-\alpha} K^{\alpha}$, $r = \alpha^2 \frac{Y}{K}$, and $\pi = (1-\alpha)\alpha \frac{Y}{A}$ and initial conditions

$$K(0) = K_0 \tag{99}$$

$$A(0) = A_0 \tag{100}$$

Note that P_A is not a growing variable.

4.4.2 Exercises

1. Simulate the transition process from a steady state with $s_A = 0$ to a steady state with $s_A = 0.02$. Plot transitional dynamics of scale adjusted variables. Which problem arises for interpretation of results? Plot original consumption without a shock and during transition of the period $[0, 30]$. Employ parameter values $\rho = 0.05$, $\sigma = 1.5$, $\eta = 0.15$, $H = 1$, $L = 1$, $\alpha = 0.4$, $\beta = 0.3$.

2. Calculate the welfare gain or loss induced by the policy measure above. Does theory tell whether welfare rises or falls?

4.5 The Jones Model

4.5.1 The model (market solution)

The presentation of the Jones (1995) model basically follows Eicher and Turnovsky (1999) who have formulated the social planner's solution of the general non-scale R&D-based growth model. For a detailed derivation of the decentralized solution see Steger (2005). As in Jones (1995), the focus here lies on the market solution. The final-output technology is given by

$$Y = \alpha_F (\phi L)^{\sigma_L} \int_0^A x(i)^{1-\sigma_L} di \tag{101}$$

where Y denotes final output, ϕ the share of labor allocated to final-output production, $x(i)$ the amount of differentiated capital goods of type i, A the number of differentiated capital goods, α_F a constant overall productivity parameter and σ_L the elasticity of labor in final-output production. Each intermediate good x_i is produced by firm i, which owns a patent on producing this good exclusively. Final output producing firms maximize profits, which yields that wages equal the marginal product $w = \sigma_L \frac{Y}{\phi L}$ and the price of one intermediate is equal to $p_i = (1 - \sigma_L)(\phi L)^{\sigma_L} x_i^{-\sigma_L}$. Noting the general symmetry among $x(i)$ and using the definition of aggregate capital $K := Ax$, the final-output technology can be written as

$$Y = \alpha_F (A\phi L)^{\sigma_L} K^{1-\sigma_L} \tag{102}$$

Intermediate firms produce one unit of intermediate from one unit of foregone consumption. Therefore, profit maximization of intermediate firms yield

$$p = p_i = \frac{r}{1-\sigma_L} \tag{103}$$

$$\pi = \pi_i = \sigma_L (1-\sigma_L) \frac{Y}{A} \tag{104}$$

Patents for new intermediate goods are produced according to the R&D technology

$$\dot{A} = J = \alpha_J A^{\eta_A} [(1-\phi)L]^{\eta_L} \tag{105}$$

with $\eta_L := \eta_L^p + \eta_L^e$, $\eta_L^p = 1$, $-1 < \eta_L^e < 0$, $\eta_A < 1$ where α_J denotes a constant overall productivity parameter, η_A the elasticity of technology in R&D and η_L the elasticity of labor in R&D. For the market solution we distinguish between the elasticity of labor observed by private firms, η_L^p, and a negative external effect of labor in research, η_L^e, caused by the 'stepping on toe' effect introduced by Jones (1995). The former elasticity equals one because of perfect competition in the research sector.

Different to Romer (1990), we assume $\eta_A < 1$, which reflects the fact that spill-overs, known as the 'standing on shoulders' effect, do not fulfill the knife-edge condition to be linear. Workers are allowed to enter the R&D sector freely, wherefore they are paid according to their marginal product

$$w = V_a \alpha_J A^{\eta_A} [(1-\phi)L]^{\eta_L - 1} \tag{106}$$

with the value of one blueprint V_a. A no arbitrage condition forces the discounted stream of profits to equal the patent's value

$$V_a(t) = \int_t^\infty \pi(\tau) e^{-\int_t^\tau r(s)ds} d\tau \tag{107}$$

Differentiating with respect to time yields

$$\dot{V}_a = rV_a - \pi \tag{108}$$

Finally, households maximize intertemporal utility according to

$$\max_c \int_0^\infty \frac{(C/L)^{1-\theta}}{1-\theta} e^{-\rho t} dt \tag{109}$$

subject to

$$\dot{K} = rK + wL - V_a \dot{A} + A\pi - C \tag{110}$$

with consumption C and the relative risk aversion equal to $\frac{1}{\theta}$. First order condition of the consumer's maximization problem is

$$\dot{C} = \frac{C}{\theta}(r - \rho - n) + nC \tag{111}$$

We transform the variables into stationary ones by expressing the system in scale adjusted variables, which are defined by $y := Y/L^{\beta_K}$, $k := K/L^{\beta_K}$, $c := C/L^{\beta_K}$, $a := A/L^{\beta_A}$, $j := J/L^{\beta_A}$ and $v_a := v/L$ with $\beta_K = \frac{1-\eta_A+\eta_L}{1-\eta_A}$, $\beta_A = \frac{\eta_L}{1-\eta_A}$. The dynamic system which governs the evolution of the economy under study, can be summarized as follows:

$$\dot{k} = y - c - \delta k - \beta_K n k \tag{112}$$

$$\dot{a} = j - \beta_A n a \tag{113}$$

$$\dot{c} = \frac{c}{\theta}[r - \delta - \rho - (1-\gamma)n] - \beta_K n c \tag{114}$$

$$\dot{v}_a = v_a[r - n] - \pi \tag{115}$$

$$\frac{\sigma_L y}{\phi} = v_a \frac{\eta_L^p j}{1-\phi} \tag{116}$$

where $y = \alpha_F (a\phi)^{\sigma_L} k^{1-\sigma_L}$, $j = \alpha_J a^{\eta_A} (1-\phi)^{\eta_L}$, $r = \frac{(1-\sigma_L)^2 y}{k}$, $\pi = \frac{\sigma_L(1-\sigma_L)y}{a}$. The (unique) stationary solution of this dynamic system corresponds to the (unique) BGP of the economy expressed in original variables.

Equations (112) and (113) are the equations of motion of (scale-adjusted) capital and technology, (114) is the Keynes-Ramsey rule of optimal consumption c, (115) shows capital market equilibrium with v_a denoting the (scale-adjusted) price of blueprints and (116) determines the privately efficient allocation of labor across final-output production and R&D.[2]

The objective is to solve the four-dimensional system of differential equations (112) - (115), taking into account the static equation (116), which must hold at all points in time. Since the steady state can only be determined numerically, the algorithm computes the steady state of the system first by applying a standard algorithm for solving non-linear equations. The choice of $k(0) = k_0$ and $a(0) = a_0$ as initial boundary conditions is obvious since k and a are the state variables.

4.5.2 Exercises

1. Solve the model with the set of parameters according to $\sigma_L = 0.6$, $\sigma_K = 0.4$, $\delta = 0.05$, $n = 0.015$, $\eta_A = 0.6$, $\eta_L = 0.5$, $\eta_L^p = 0.6$, $\rho = 0.04$, $\alpha_J = 1$ and $\gamma = 1$. Assume that the economy is in steady state and that an exogenous increase in the productivity of the research sector occurs, i.e. $\alpha_J = 1.05$. (This could, for example, be due to a new General Purpose Technology.) Calculate transitional dynamics and show how endogenous variables behave in the first 60 periods.

[2] The presence of the static efficiency condition (eq. (116)) is due to the fact that labor does neither enter final output nor R&D linearly. Hence, it is in general not possible to solve for the optimal amount of labor explicitly.

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