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Lucas's signal-extraction model

A finite state exposition with aggregate real shocks

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This is a version of Lucas's (1972) 'Expectations and the Neutrality of Money' with finite supports for the exogenous random variables and with aggregate real shocks. The finite supports simplify the proofs and permit solutions for examples to be computed. The presence of aggregate shocks permits the model to display output-aggregate demand and output-inflation correlations that more closely duplicate those found in some data sets, correlations that have been interpreted as evidence against the model.

1. Introduction

This is a simplified version of Robert E. Lucas's (1972) 'Expectations and the Neutrality of Money'. It departs from the original in three ways. First, individuals care only about leisure when young and consumption when old rather than leisure when young and consumption when old and young. This is an innocuous simplification that has been used by others. Second, changes in the supply of money are used to finance government consumption rather than to pay interest on money. This makes the policies studied more closely resemble actual policies.[1] Third, the supports for the probability distributions of the exogenous random variables are finite, rather than infinite and uncountable.

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[1] The analysis and results are similar for versions in which changes in the money supply either finance lump-sum transfers to the old of each generation or are used to pay interest on money as in Lucas's original paper.

This allows simpler mathematics to be used and permits equilibria to be computed for examples because it makes the stationary equilibrium a finite-dimensional vector, rather than a function defined on an interval.[2]

The substantive contribution of the paper concerns the role of aggregate real shocks – actually, aggregate supply shocks. I emphasize that the assumption that supply shocks average out across isolated markets is an inessential appendage to Lucas's theory of how aggregate demand impinges on output. Indeed, a one-market version of the model, in which the supply shock is by definition an aggregate shock, requires fewer informational restrictions. The magnitude of aggregate supply shocks does, however, play a crucial role in determining the model's implications for correlations. The larger such shocks are, the smaller the implied simple correlations between total output and the aggregate demand shock and between total output and the inflation rate (or innovations in the price level). In fact, while the former remains positive in the presence of aggregate supply shocks, the latter tends to be negative. Although these effects of aggregate supply shocks seem obvious, they need to be emphasized because the appearance of precisely such correlations in some data sets has contributed to the view that Lucas's theory is not important [see, for example, Barro (1989, pp. 2–4)].

The exposition is organized as follows. The model is set out in section 2. Equilibrium is defined and existence and uniqueness of a stationary equilibrium is established in section 3. Sections 4–6 describe properties of the stationary equilibrium. Section 4 contrasts the model's implications for different policy rules with its implications for the time series correlation between output and the growth rate of money under a given policy rule. Section 5 discusses output–inflation correlations. Section 6 describes the equilibria for two numerical examples and uses them to explore the consequences of the presence of aggregate supply shocks for output-on-money and output-on-inflation regressions. Section 7 concludes.

2. The model

The model is defined over integer dates $t \geq 1$ and is one of two-period-lived overlapping generations. There is a single consumption good per date, and there are no intertemporal technologies.

Each two-period-lived person maximizes $\mathrm{E}[u(c) + v(c')]$, where E denotes expected value, c is consumption of leisure when young, and c' is consumption of the good when old. I assume that government consumption does not affect the way individuals rank alternative distributions of private consumption. The

[2] For a version that simplifies the model by using linear approximations to first-order conditions, see McCallum (1984).

functions u and v are twice continuously differentiable and increasing, and their derivatives satisfy

$$u'' < 0, \qquad v'' < 0, \qquad u'(0) = v'(0) = \infty, \tag{1}$$

$$g' > 0 \quad \text{where} \quad g(x) \equiv xv'(x). \tag{2}$$

Assumption (2) is equivalent to the assumption that c and c' are gross substitutes. Each two-period-lived person is endowed with $w > 0$ units of leisure when young and with a (constant returns to scale) technology that permits units of leisure to be converted one-for-one into units of the current consumption good.[3]

There are two exogenous random processes: one for the size of each generation and one for the proportional change in the stock of money. I call generation t those who are young at t and old at $t+1$, and let N_t be the size of generation t. I let M_{t-1} be the stock of money brought into period t by generation $t-1$, and assume that $M_t = x_t M_{t-1}$ with $x_t \geq 1$. Both N_t and x_t are identically and independently distributed (over time and of each other) with $\ln(N_t) = \beta + \Delta i \equiv \ln N^i$ with probability π_i $(i = 1, 2, \ldots, I)$ and $\ln(x_t) = \lambda + \Delta j \equiv \ln x^j$ with probability θ_j $(j = 1, 2, \ldots, J)$, where Δ is a positive scalar, $\lambda + \Delta \geq 0$, and ln is the natural logarithm.[4]

The following information assumptions are crucial. At t, people do not directly observe N_t and x_t. They know previous realizations and the distributions of N_t and x_t. Moreover, generation t appears at t simultaneously with any new time t information. This implies that there cannot be explicit risk-sharing arrangements between generations $t-1$ and t.

The above supports for N_t and x_t are chosen so that they permit several different (N_t, x_t) pairs to give rise to the same ratio $x_t/N_t \equiv z_t$. This permits there to be incomplete information in this discrete version of the model. There will be different (N_t, x_t) pairs that give rise to a single z_t if $I \geq 2$ and $J \geq 2$ and if, as I have assumed, the supports for x_t and N_t satisfy $\ln N^{i+1} - \ln N^i = \ln x^{j+1} - \ln x^j$. It follows that $\ln z_t = (\lambda - \beta) + \Delta k \equiv \ln z^k$ with probability $\phi_k \equiv \sum_{j=1}^{J} \theta_j \pi_{i-k}$, where $\pi_i = 0$ if $i \notin \{1, 2, \ldots, I\}$ and where $k \in \{1 - I, 2 - I, \ldots, J - 1\}$, a set which contains $I + J - 1$ elements and which we denote by Z.

Later we will need the probability that $\ln N_t = \beta + \Delta i$ conditional on $\ln z_t = (\lambda - \beta) + \Delta k$. This conditional probability is equal to the probability

[3]The model can also be interpreted as a pure-exchange model with w an endowment of the good and c consumption of the good. What is lost under this pure-exchange interpretation is the identification of saving as output.

[4]Note that $M_t - M_{t-1}$ is the time t nominal deficit. In fact, since nominal GNP at t will turn out to be equal to M_t, it follows that $(M_t - M_{t-1})/M_t = 1 - x_t^{-1}$ is the ratio of the deficit to GNP. (For x_t near unity, $1 - x_t^{-1}$ is well-approximated by $\ln x_t$.)

that $\ln N_t = \beta + \Delta i$ and $\ln x_t = \lambda + \Delta(i+k)$ conditional on $\ln z_t = (\lambda - \beta) + \Delta k$. It is $(\pi_i \theta_{i+k})/\phi_k \equiv \phi_{ik}$, where $\theta_{i+k} = 0$ if $i + k \notin \{1, 2, \ldots, J\}$. The last assumption is a further restriction on the distributions of N_t and x_t.

Let $F(j, k) = \sum_{i=1}^{j} \phi_{ik}$, the probability that $N_t \leq N^j$ conditional on $z_t = z^k$. The $F(j, k)$ are assumed to satisfy

$$F(j-1, k+1) \leq F(j, k) \leq F(j, k+1) \quad \text{for all } j \text{ and } k. \tag{3}$$

The first inequality says that the probability that $x_t \leq x^h$ conditional on $z_t = z^k$ is nonincreasing in z^k, while the second says that the probability that $N_t \leq N^j$ conditional on $z_t = z^k$ is nondecreasing in z^k. [It is easy to verify that uniform distributions for $\ln N_t$ and $\ln x_t$ – namely, $\pi_i = 1/I$ and $\theta_j = 1/J$ – satisfy (3)].

To complete the model, I assume that each member of generation 0, the old at $t = 1$, maximizes $\mathrm{E}v(c')$ and is endowed with $M_0/N_0 > 0$ units of money.

3. Existence and uniqueness of stationary equilibrium

We start with a description of individual choice problems. At date t, a young person's choices of leisure when young, c_t, and a distribution for consumption when old, c'_{t+1}, are constrained by

$$c_t \leq w - y_t, \qquad y_t \geq m_t/p_t, \qquad c'_{t+1} \leq m_t/p_{t+1}, \tag{4}$$

where y_t is production, m_t is nominal money holdings held from t to $t+1$, and p_t is the time t price level. The first inequality says that consumption of leisure when young is bounded above by the difference between the endowment and production. The second says that purchases of money are bounded above by output. And the third says that consumption of the good when old is bounded above by the value of money acquired when young. Our preference assumptions imply that maximizing choices satisfy these constraints at equality so that we can say, more simply, that each young person at t chooses y_t to maximize

$$W(y_t) = u(w - y_t) + \mathrm{E}[v(y_t p_t/p_{t+1})], \tag{5}$$

where the expectation is taken conditional on p_t and the history of the policy and population processes up through and including $t - 1$, but not t. In particular, if we denote the vector $(M_0, x_1, x_2, \ldots, x_t, N_1, N_2, \ldots, N_t)$ by q_t, then the expectation is conditional on (q_{t-1}, p_t). As for old people at $t = 1$, they simply offer for sale in the aggregate M_0, the amount of money they begin with at $t = 1$.

Given these individual choice problems, we can state a definition of equilibrium.

Definition. Let Q_t be the set of possible realizations for the policy and population processes up to and including t with typical element q_t. An equilibrium is a sequence of functions $\{p_t, y_t\}$, with p_t and y_t each mapping Q_t to a real number, such that for all $t \geq 1$ and all $q_t \in Q_t$,

(i) $y_t = y_t(q_t)$ maximizes $W(y_t)$ when $p_t = p_t(q_t)$ and when the expectation is taken over the distribution of $p_{t+1} = p_{t+1}(q_{t+1})$ conditional on (q_{t-1}, p_t), and

(ii) $N_t y_t(q_t) = M_t / p_t(q_t)$.

Note that condition (i) is individual optimization under rational expectations and (ii) is market clearing, the equating of the supply of money at t to the demand on the part of the young at t.

Existence of a stationary equilibrium is established by a version of a guess-and-verify argument. The conjecture, or guess, is that there is an equilibrium that is stationary in the sense that y_t depends on q_t only by way of z_t and that this dependence is such that z_t/y_t is strictly increasing in z_t. (Recall that $z_t = x_t/N_t$.) In other words, the conjecture is that there is an equilibrium in which the function $y_t(q_t)$ takes the form $y(z_t)$ where $z_t/y(z_t)$ is strictly increasing. Since z_t takes on one of $I + J - 1$ values, the range of the function y is simply a vector in R^{I+J-1}. Later we will denote this range by y and its kth component, $y(z^k)$, by y^k, where $k \in Z = \{1 - I, 2 - I, \ldots, J - 1\}$.

Proposition 1. There exists one and only one equilibrium satisfying $y_t(q_t) = y(z_t) > 0$ and $z_t/y(z_t)$ strictly increasing.

Proof. The proof proceeds as follows. We first show that if $y_t = y(z_t)$ and satisfies the above strictly increasing property and if p_t is constructed to satisfy equilibrium condition (ii), then the conditioning information (q_{t-1}, p_t) is equivalent to the conditioning information (q_{t-1}, z_t). Then, using that equivalence, we show that there exists a $y(z_t)$ function that satisfies equilibrium condition (i) and the strictly increasing property. Finally, we establish uniqueness of $y(z_t)$.

To see that the conditioning information (q_{t-1}, p_t) is equivalent to the conditioning information (q_{t-1}, z_t), note that a price level function that satisfies equilibrium condition (ii) satisfies $p_t = M_0 x_1 x_2, \ldots, x_{t-1} z_t / y_t \equiv \zeta(q_{t-1}) z_t / y(z_t)$. It follows that if $z_t/y(z_t)$ is strictly monotone in z_t, then the mapping from (q_{t-1}, p_t) to (q_{t-1}, z_t) is one-to-one and onto.

With p_t finite, the optimizing choice of y_t is the unique solution to the first-order condition, $W'(t) = 0$. Multiplying by y_t, we can write this condition as

$$f(y_t) = \mathrm{E}[g(y_t p_t / p_{t+1})], \tag{6}$$

where $f(x) \equiv xu'(w-x)$ and where, as above, $g(x) \equiv xv'(x)$. Note that the argument of f is output or saving and that of g is consumption when old. Now substituting for $y_t p_t / p_{t+1}$ using equilibrium condition (ii), we can rewrite (6) as

$$f(y_t) = \mathrm{E}[g(y_{t+1}/z_{t+1} N_t)]. \tag{7}$$

Then, imposing the conjecture, (7) becomes

$$f[y(z_t)] = \mathrm{E}\{g[y(z_{t+1})/z_{t+1} N_t]\}. \tag{8}$$

Now we use the above equivalence of conditioning information to describe the distribution of the argument of g in (8) conditional on (q_{t-1}, z_t). It follows from the serial independence assumptions that q_{t-1} is not relevant and that z_t is relevant only for predicting N_t, which is independent of z_{t+1}. Therefore, conditional on $z_t = z^k$, $y(z_{t+1})/z_{t+1} N_t = y(z^h)/z^h N^i$ with probability $\phi_h \phi_{ik}$ for each (k, h, i) in the set $Z \times Z \times \{1, 2, \ldots, I\}$. (Recall that ϕ_h is the unconditional probability that $z_{t+1} = z^h$ and that ϕ_{ik} is the probability that $N_t = N^i$ conditional on $z_t = z^k$.)

It follows that (8) holds if and only if there exists $y \in R^{I+J-1}$ with $y^k \in (0, w)$, such that, for all $k \in Z$,

$$f(y^k) = \sum_i \phi_{ik} \left[\sum_h \phi_h g(y^h/z^h N^i) \right]. \tag{9}$$

Eqs. (9) are simply $I + J - 1$ simultaneous equations. Lemma 1 shows, using a fixed point argument that relies on assumption (1), that (9) has a solution satisfying $y^k \in (0, w)$. Lemma 2 uses assumptions (1)–(3) and shows that the solution satisfies the strictly increasing property which in our vector notation is equivalent to $z^{k+1}/y^{k+1} > z^k/y^k$. That there is only one equilibrium satisfying the conjecture is established in Lemma 3, which shows that there is only one solution to (9). Lemmas 1–3 and their proofs are given in appendix A. □

4. Total output–money regressions

In this section, we discuss the implications of the stationary equilibrium for the least squares regression of the logarithm of total output, $\ln(N_t y_t)$, on the first difference of the logarithm of the stock of money, $\ln x_t$. We discuss the sign of the regression coefficient in a time series under a given policy, and we describe how the regression varies across different policies. We begin with two propositions, proved in appendix A, about the equilibrium function $y(z_t)$.

Proposition 2. *Assume (1)–(3). Any solution to (9) satisfies $y^{k+1} \geq y^k$ and with strict inequality for $k \in \{1-I, 2-I, \ldots, -1\}$ or $k \in \{J-I, J-I+1, \ldots, J-1\}$.*

Proposition 3. *Consider two economies that differ only with regard to the mean rate of money creation λ. If $\lambda_1 > \lambda_2$, then the corresponding equilibrium of y functions satisfy $y_1^k \leq y_2^k$ for all k and with strict inequality for some k.*

Proposition 2 says that the equilibrium y function is weakly increasing and is strictly increasing over parts of its domain. It suggests that the regression coefficient in a regression of $\ln(N_t y_t)$ on $\ln x_t$ is positive. Since N_t and x_t are independent, the regression coefficient is the same as that in the regression of $\ln y_t$ on $\ln x_t$. Proposition 2 implies that the support of y_t for a given x_t is weakly increasing in x_t. For the case of uniform distributions for x_t and N_t, this implies that the regression coefficient is positive. I have not been able to show that it is positive in general, although Proposition 2 suggests that that may well be true. In section 6, some population regressions for examples are presented. They are computed using the fact that $\text{prob}\{y_t = y^k \text{ and } x_t = x^i\} = \text{prob}\{z_t = z^k \text{ and } x_t = x^i\} = \text{prob}\{N_t = N^{i-k} \text{ and } x_t = x^i\} = \pi_{i-k}\theta_i$, so that $\mathrm{E}(\ln y_t \ \ln x_t) = \sum_i \sum_k (\pi_{i-k}\theta_i \ln y^k \ln x^i)$.

Proposition 3 implies that the support of $\ln y_t$ for a given $\ln x_t - \lambda$ is decreasing in λ. Thus, roughly speaking, it implies that the samples under two different policies identical except for λ do not lie on the same regression line. In particular, one would mistakenly overpredict total output under the λ_1 policy by extrapolating from the regression implied by experience under the λ_2 policy, and vice versa.

Proposition 3 also implies an inverse association between the average rate of money creation and average total output. As Lucas (1973) remarks in 'Some International Evidence on Output–Inflation Tradeoffs', a potential source for checking such a prediction is data for different countries. However, as he also notes, such a test would be weak because average output across countries is affected by all sorts of things. He, therefore, focuses on cross-section second-moment observations and reports the following regularities in the data he examines: (a) no tendency for the variance of the logarithm of real output about trend to be proportional to the variance of the first difference of the logarithm of nominal output and (b) a tendency in regressions of the deviation of the logarithm of real output from trend on itself lagged and on the first difference of the logarithm of nominal output for the regression coefficient of the latter regressor to approach zero as the variance of that regressor gets large. For the case in which $\ln x_t$ is uniformly distributed, we can show that the stationary equilibrium of the model in this paper is consistent with these regularities.

As regards (a), since in the model there is no trend in output and since nominal output is the money stock, we need to examine the relationship between the variance of $\ln(N_t y_t)$ and that of $\ln x_t$. First, note that if $\ln x_t$ is uniformly distributed, then its variance is equal to $\Delta^2(J+1)(J-1)/12$. [This follows from $\sum_{i=1}^{K} i^2 = K(K+1)(2K+1)/6$.] Next, note that if $\ln x_t$ is uniformly distributed, then $\phi_{ik} = \pi_i$ for $k \in \{0, 1, 2, \ldots, J-I\}$ so that the r.h.s. of (9) does not depend on k for such k. Hence, $y^k = y_J$, a constant, for such k and $y_t = y_J$ with probability $[IJ - I(I-1)]/IJ$, which is increasing in J and approaches one as $J \to \infty$. Moreover, it follows from $z^{k+1}/y^{k+1} > z^k/y^k$ that the support for $\ln y_t$ is contained in $[\ln y_J - (I-1)\Delta, \ln y_J + (I-1)\Delta]$. Therefore, the variance of $\ln y_t$ is bounded and, in fact, approaches zero as $J \to \infty$. Thus, there is no tendency for the variance of $\ln(N_t y_t)$ to be proportional to the variance of $\ln x_t$.

As regards (b), since in the model output is serially independent and nominal output is the money stock, the comparable regression coefficient for the model is simply the coefficient of $\ln x_t$ in a regression of $\ln(N_t y_t)$ on $\ln x_t$ or, equivalently, in a regression of $\ln y_t$ or $\ln x_t$. The same facts used to show consistency with (a) imply that that coefficient approaches zero as $J \to \infty$.

5. Output–inflation correlations

One version of the 'Phillips curve' is a positive correlation in time series between output and the inflation rate. It turns out that the model so far examined seems unlikely to produce such a correlation.

From equilibrium condition (ii), $\ln p_t = \ln M_t - \ln(N_t y_t)$. Therefore, the logarithm of the (gross) inflation rate between $t-1$ and t can be written as

$$\rho_t \equiv \ln p_t - \ln p_{t-1} = Y_{t-1} - Y_t + \ln x_t, \tag{10}$$

where $Y_t = \ln(N_t y_t)$. Noting that Y_t and Y_{t-1} are independent, we have

$$\mathrm{C}(\rho_t, Y_t) = \mathrm{C}(\ln x_t, Y_t) - \mathrm{V}(Y_t), \tag{11}$$

where C stands for covariance and V for variance. There is no presumption that $\mathrm{C}(\rho_t, Y_t)$ is positive. One source tending to make it negative is the contribution of N_t. High N_t tends to produce high total output and a low price level.

In Lucas (1972), data for output and inflation are interpreted as aggregates across regions, where each region is an economy of the kind so far described. In particular, the regions are isolated; they do not trade with one another and people in one region do not observe the price in any other region. It follows that the equilibrium for a region does not depend on the pattern of cross-region correlations for x_t and N_t. In other words, we are free to impose any assumptions about cross-region correlations of the policy and population realizations without disturbing the equilibrium of Proposition 1. As regards the policy realization, I follow Lucas and assume a common policy realization for all regions.

For now, let the argument k denote region k, so that total output across regions at t is $\sum_k y_t(k)\, N_t(k)$. Following Lucas, define a price index by dividing total 'nominal output' by total real output as just defined. 'Nominal output' in each region is the value of output in the region in terms of currency, which is simply $M_t(k)$, given our assumption that all of output is sold by the young for currency. Thus, the price index at t is $\sum_k M_t(k)/\sum_k y_t(k)\, N_t(k)$. It follows that if $x_t(k) = x_t$ (outcomes for x_t are identical across regions), then the first difference of the logarithm of the price index is again given by (10), but with $Y_t = \ln \sum_k y_t(k)\, N_t(k)$. Therefore, the covariance between inflation and the logarithm of total output is again given by (11).

In the next section, I report total output–inflation correlations, as implied by (11), under various specifications for the cross-region correlation of the N_t realizations. One of these is the specification Lucas adopted: two regions and N_t realizations that are perfectly negatively correlated.

6. Two examples

The stationary equilibrium for each of two single-region economies is obtained using the contraction mapping computational routine outlined in appendix B. Then, treating each economy as composed of several identical regions, we report regressions for different specifications for cross-region correlations of the outcomes for N_t, while assuming that all regions experience the same outcome for x_t.

The examples are identical except for the policy process. The specification for the first, called example A, is $u(x) = v(x) = x^{1/2}$, $w = 1.0$, $\beta = \lambda = 0$, $\Delta = 0.05$, $\pi_i = 1/I$, $\theta_j = I/J$, $I = 2$, $J = 2$. The specification for the second, called example B, is identical except that $J = 4$. Note that example A has an average money creation rate of about 7.5 percent, while B has one of 12.5 percent and greater variance. The stationary equilibrium per-person outputs are reported in table 1.

Table 1

Per-person outputs for two example economies.

Example A		Example B	
k	y^k	k	y^k
−1	0.4772	−1	0.4649
0	0.4813	0	0.4690
1	0.4854	1	0.4690
		2	0.4690
		3	0.4731

Table 2

Population time series least squares regressions of the logarithm of total output, Y_t, on the growth rate of the money stock, $\ln x_t$: $Y_t = a_0 + a_1 \ln x_t$.

	Example A			Example B		
Cross-region specification	a_0	a_1	R^2	a_0	a_1	R^2
1 region	– 0.67	0.17	0.04	– 0.69	0.05	0.02
2 regions, N_t perfectly negatively correlated	0.02	0.17	1.00	0.00	0.05	0.90
2 regions, N_t independent	0.02	0.17	0.08	0.00	0.05	0.03
4 regions, N_t independent	0.72	0.17	0.15	0.70	0.05	0.06

Table 3

Population time series least squares regressions of the logarithm of total output, Y_t, on the inflation rate, ρ_t: $Y_t = a_0 + a_1 \rho_t$.

	Example A			Example B		
Cross-region specification	a_0	a_1	R^2	a_0	a_1	R^2
1 region	– 0.64	– 0.26	0.20	– 0.67	– 0.10	0.07
2 regions, N_t perfectly negatively correlated	0.02	0.20	0.96	0.00	0.05	0.89
2 regions, N_t independent	0.05	– 0.14	0.08	0.02	– 0.03	0.01
4 regions, N_t independent	0.73	– 0.03	0.00	0.70	0.01	0.00

Average per capita output is lower in example B. This is expected given the higher money creation rate in example B. Notice also that the range for per capita output is the same for the two examples up to four significant digits, thus implying a lower variance of real output for example B despite its higher variance in the growth rate of the money stock.

Table 2 reports population parameters for the time series least squares regression of the logarithm of total real output, $Y_t \equiv \ln \sum N_t y_t$ (the summation being over regions), on the logarithm of x_t. Notice that the regression coefficients of x_t are smaller in example B. Notice also that although the coefficients of x_t in table 2 are not sensitive to the cross-region specification of N_t outcomes [this would follow directly from independence between N_t and x_t if the dependent variable were $\sum \ln(N_t y_t)$], the degree of explanatory power of the regression varies greatly (R^2 is the fraction of the variance of the dependent variable accounted for by the regression).

Table 3 reports population parameters for the time series least squares regression of the logarithm of total real output at t on the logarithm of gross inflation between $t-1$ and t, defined as described above. Here the regression coefficient is very sensitive to the cross-region specification. The results are consistent with the surmise that the regression coefficient is decreasing in the variance of *aggregate* real shocks.

One interesting feature of tables 2 and 3 is the low R^2 in all cases except when there is no aggregate real shock – the second row in the tables. This suggests that the Lucas theory should not be rejected because regressions of the types reported in those tables yield insignificant results.

7. Conclusion

I have emphasized that the assumption that there are no aggregate supply shocks is an extraneous assumption in Lucas (1972). Nothing about how the model economy works depends on that assumption. Moreover, a version with aggregate supply shocks has at least two desirable features relative to one without such shocks. First, the informational restrictions are less severe. Even if the economy is made up of regions, as long as there is an aggregate component to supply shocks, it is not necessary to assume that people in one region cannot observe prices in other regions. Second, the version without aggregate shocks implies that simple regressions between real output and policy realizations should fit perfectly except for having a misspecified functional form. Data do not come close to such perfect fits. As is obvious and as is illustrated in section 6, the presence of aggregate real shocks reduces the fit of such regressions. Finally, although the presence of aggregate supply shocks weakens or eliminates the positive Phillips curve correlation, this too may be desirable because such positive correlations are far from ubiquitous.

Appendix A: Proofs

Lemma 1. *Assume (1). There exists $y \in R_{++}^{I+J-1}$ with $y^k < w$ that satisfies (9).*

Proof. We will apply Brouwer's Fixed Point Theorem: Any continuous mapping of a closed and bounded convex set in Euclidean space into itself maps at least one point into itself.

First, we define a mapping so that the assumptions of this theorem hold. Then we show that a fixed point of the mapping is a solution of (9) with $y^k \in (0, w)$. It is convenient here to write (9) as $f(y^k) = \sum_i \phi_{ik} G_i(y)$, where $G_i(y) \equiv \sum_h \phi_h g(y^h/z^h N^i)$.

Note that by assumption (1), for all $x \in R_{++}$, the inverse of f, denoted $f^{-1}(x)$, exists, is increasing, and is such that $f^{-1}(x) \in (0, w)$.

Now, for each k, choose j so that $\phi_{jk} > 0$ and pick $e_k \in (0, w)$ so that $(\phi_{jk}\phi_k/N^j z^k)v'(e_k/N^j z^k) > u'(w - e_k)$. [By (1), in particular $v'(x) \to \infty$ as $x \to 0$, this can be done.] Then, since f^{-1} is increasing, for $y^k = e^k$ and $y^h \in (0, w)$, $h \neq k$,

$$f^{-1}\left[\sum_{i=1}^{I} \phi_{ik} G_i(y)\right] \geq f^{-1}[\phi_{jk}\phi_k(e_k/N^j z^k)v'(e_k/N^j z^k)]$$

$$> f^{-1}[e_k u'(w - e_k)] = e_k. \tag{i}$$

Now let $S = \times_k[e_k, w]$. Obviously, S is a closed and bounded convex set in R^{I+J-1}. For any $s \in S$, define the mapping $H(s) = (h_{1-I}(s), h_{2-I}(s), \ldots, h_{J-1}(s)) \in R^{I+J-1}$ by

$$h_k(s) = \max\left\{e_k, f^{-1}\left[\sum_{i=1}^{I} \phi_{ik} G_i(s)\right]\right\}. \tag{ii}$$

[The idea of using the maximum in defining H comes from Manuelli (1986), where it is used in a different model. One reason to use it here is that the mapping cannot be defined on $[0, w]$ since $v'(0)$ is not defined.] It follows that $h_k(s) \in [e_k, w)$, so that $H(s) \in S$ for any $s \in S$. Since G_i is continuous and the maximum of continuous functions is continuous, H is a continuous mapping. Thus, H has a fixed point; that is, there exists $\bar{s} \in S$ such that, for all $k \in Z$,

$$\bar{s}_k = \left\{e_k, f^{-1}\left[\sum \phi_{ik} G_i(\bar{s})\right]\right\}. \tag{iii}$$

To show that $\bar{s}$ satisfies (9), it suffices to show that the maximum in (iii) is not e_k. Suppose to the contrary. Then for some k, $\bar{s}_k = e_k \geq f^{-1}[\sum \phi_{ik} G_i(\bar{s})]$. But this inequality violates (i). □

Lemma 2. Assume (1)–(3). If $y \in R^{I+J-1}_{++}$ *with* $y^k < w$ *satisfies* (9), *then* (a) $y^{k+1} \geq y^k$ *and* (b) $z^{k+1}/y^{k+1} > z^k/y^k$.

Proof. Here it is convenient to let $G(N_i) \equiv \sum_h \phi_h g(y^h/z^h N^i)$. Since (9) is satisfied, it follows that

$$[f(y^{k+1}) - f(y^k)]/f(y^{k+1}) = \delta_k \bigg/ \left[\sum_{i=1}^{I} \phi_{i,k+1} G(N_i)\right], \tag{i}$$

where $\delta_k = \sum \phi_{i,k+1} G(N_i) - \sum \phi_{ik} G(N_i)$, the difference between the r.h.s.'s of the $(k+1)$th and kth equations of (9). We will derive inequalities for both sides of (i) that together imply the conclusions. We start with the r.h.s. of (i).

Using the identity that connects densities and cumulative distributions – namely, that for any $(a_1, a_2, \ldots, a_I)$, $\sum_{i=1}^{I} \phi_{ik} a_i = \sum_{i=1}^{I-1} F(a_i - a_{i+1}) + a_I$ – $\delta_k = \sum_{i=1}^{I-1} [F(i,k+1) - F(i,k)][G(N_i) - G(N_{i+1})]$. Since $N^{i+1} > N^i$, (2) implies that $G(N_i) > G(N_{i+1})$. Therefore, the second inequality of (3) implies $\delta_k \geq 0$ and conclusion (a). Using $G(N_i) > G(N_{i+1})$ and the first inequality of (3), we have

$$\begin{aligned} \delta_k &\leq \sum_{i=1}^{I-1} [F(i,k+1) - F(i-1,k+1)][G(N_i) - G(N_{i+1})] \\ &= \sum_{i=1}^{I-1} \phi_{i,k+1} G(N_i)[G(N_i) - G(N_{i+1})]/G(N_i). \end{aligned} \tag{ii}$$

Applying the mean value theorem to $\ln G(x)$, we get $\ln G(N_i) - \ln G(N_{i+1}) = -(\partial \ln G/\partial \ln x)(\ln N^{i+1} - \ln N^i)$. It follows from (2) that $-(\partial \ln G/\partial \ln x) \in (0,1)$. Therefore, $\ln[G(N_i)/G(N_{i+1})] < \ln N^{i+1} - \ln N^i = \Delta = \ln(z^{k+1}/z^k)$. Hence, $G(N_i)/G(N_{i+1}) < z^{k+1}/z^k$ or $[G(N_i) - G(N_{i+1})]/G(N_i) = 1 - G(N_{i+1})/G(N_i) < 1 - z^k/z^{k+1}$. By (ii), this implies $\delta_k < (1 - z^k/z^{k+1}) \times \sum_{i=1}^{I} \phi_{i,k+1} G(N_i)$ or

$$\delta_k \Big/ \left[\sum_{i=1}^{I} \phi_{i,k+1} G(N_i) \right] < 1 - z^k/z^{k+1}. \tag{iii}$$

Now, turning to the l.h.s. of (i), we apply the mean value theorem to $\ln f(x)$, we get $\ln f(y^{k+1}) - \ln f(y^k) = (\mathrm{d}\ln f/\mathrm{d}\ln x)(\ln y^{k+1} - \ln y^k)$. Since (1) implies that $\mathrm{d}\ln f/\mathrm{d}\ln x > 1$, conclusion (a) implies that $\ln[f(y^{k+1})/f(y^k)] \geq \ln(y^{k+1}/y^k)$. Therefore,

$$1 - y^k/y^{k+1} \leq 1 - f(y^k)/f(y^{k+1}) = [f(y^{k+1}) - f(y^k)]/f(y^{k+1}). \tag{iv}$$

Inequalities (i), (iii), and (iv) give conclusion (b). □

Lemma 3. Assume (1)–(3). If $y_1 \in R_{++}^{I+J-1}$ *and* $y_2 \in R_{++}^{I+J-1}$ *satisfy (9), then* $y_1 = y_2$.

Proof. Assume to the contrary that $y_1 \neq y_2$. Let $a_i^k = \ln y_i^k$, let $\max_k |a_1^k - a_2^k| = |a_1^j - a_2^j| = \delta_j > 0$, and consider the jth equation of (9). Since y_1 and y_2 are solutions,

$$|\ln f(y_1^j) - \ln f(y_2^j)| = |\ln H_j(a_1) - \ln H_j(a_2)|, \tag{i}$$

where $H_j(a) = \sum_{i=1}^{I} \phi_{ij} \sum_h \phi_h g(y^h/z^h N^i)$.

Applying the mean value theorem to the l.h.s. of (i), as in the proof of Lemma 2, we have

$$\ln f(y_1^j) - \ln f(y_2^j) > \delta_j. \tag{ii}$$

Applying the mean value theorem to the r.h.s. of (i), we get

$$\begin{aligned} |\ln H_j(a_1) - \ln H_j(a_2)| &= \left| \sum_{h\in Z} (\partial \ln H_j/\partial \ln y^h)(a_1^h - a_2^h) \right| \\ &\le \sum |\partial \ln H_j/\partial \ln y^h||a_1^h - a_2^h| \\ &\le \delta_j \sum |\partial \ln H_j/\partial \ln y^h| < \delta_j, \end{aligned} \tag{iii}$$

where the last inequality follows from (2). Obviously, (i), (ii), and (iii) are inconsistent. □

Proposition 2. Assume (1)–(3). Any solution to (9) satisfies $y^{k+1} \ge y^k$ and with strict inequality for $k \in \{1-I, 2-I, \ldots, -1\}$ or $k \in \{J-I, J-I+1, \ldots, J-1\}$.

Proof. The weak inequality was proved in Lemma 2. From the expression for δ^k in the proof of Lemma 2, it also follows that $y^{k+1} > y^k$ for any k for which there exists $i \in \{1, 2, \ldots, I-1\}$ such that $F(i, k+1) > F(i, k)$. For $k \in \{1-I, 2-I, \ldots, -1\}$ we have $F(-k, k+1) > 0$ and $F(-k, k) = 0$, while for $k \in \{J-I, J-I+1, \ldots, J-1\}$ we have $F(J-k-1, k+1) = 1$ and $F(J-k-1, k) < 1$. □

Proposition 3. Consider two economies that differ only with regard to the mean rate of money creation λ. If $\lambda_1 > \lambda_2$, then the corresponding equilibrium of y functions satisfy $y_1^k \le y_2^k$ for all k and with strict inequality for some k.

Proof. By the implicit function theorem applied to (9), the Proposition 1 solution for y^k is a differentiable function of λ, where the derivative of $\ln y^k$ w.r.t. λ, $\mathrm{d}\ln y^k/\mathrm{d}\lambda$, is obtained as follows.

Eqs. (9) are equivalent to $\ln f(y^k) - \ln \sum_{i=1}^I \phi_{ik} G_i(y) \equiv B_k(\ln y, \lambda) = 0$, where $G_i(y)$ is defined as in Lemma 1 – namely, $G_i(y) = \sum_h \phi_h g[y^h/N^i \mathrm{e}^{(\lambda-\beta+\Delta h)}]$. We can differentiate totally to obtain $Bx = c$, where $B = [b_{kh}]$ is $(J+I-1) \times (J+I-1)$ with $b_{kh} = \partial B_k(\ln y, \lambda)/\partial \ln y^h$, $x = [x_k]$ with $x_k = -\mathrm{d}\ln y^k/\mathrm{d}\lambda$, and $c = [c_k]$ with $c_k = \partial B_k(\ln y, \lambda)/\partial\lambda$. Theorem 4.c.3 of Takayama (1985) is as follows. Let B be an $n \times n$ matrix with $b_{ii} > 0$ for all i and $b_{ij} \le 0$ for $i \ne j$. Then there exists a unique $x \ge 0$ such that $Bx = c$ for every $c \ge 0$ if and only if B has a dominant diagonal. We now show that B and c satisfy these sign conditions and that B has a dominant diagonal.

It follows from (2) that $\partial \ln \sum_{i=1} \phi_{ik} G_i(y)/\partial \ln y^j \in (0, 1)$. Therefore, $b_{kj} < 0$ for $j \neq k$. Since $\partial \ln f(y^k)/\partial \ln y^k > 1$, we also have $b_{kk} > 0$. Moreover, (1) and (2) together imply $b_{kk} + \sum_{j \neq k} b_{kj} > 0$. Therefore, $b_{kk} > \sum_{j \neq k}(-b_{kj}) = \sum_{j \neq k} |b_{kj}|$, which is diagonal dominance. Finally, (2) also implies that $\partial \ln \sum_{i=1}^{I} \phi_{ik} G_i(y)/\partial \lambda \in (-1, 0)$, which implies that $c_k > 0$.

Since all the hypotheses of the above stated theorem hold, it follows that $\mathrm{d} \ln y^k/\mathrm{d}\lambda \leq 0$. And since the right-hand side of each of eqs. (9) is decreasing in λ, $\mathrm{d} \ln y^k/\mathrm{d}\lambda < 0$ for at least some k. $\square$

Appendix B: The computational procedure for the section 6 examples

The procedure involves iterating on the following mapping, which is closely related to H in the proof of Lemma 1. Let $S = \times_k[\ln e_k, \ln w]$, where e_k is defined in the proof of Lemma 1. For any s in S, let $T(s)$ in R^{I+J-1} be defined by

$$t_k(s) = \Gamma\left[\ln \sum_{i=1}^{I} \phi_{ik} G_i(\mathrm{e}^s)\right],$$

where $\mathrm{e}^s \in R^{I+J-1}$ with kth component e^{s_k}, G_i is the function defined in Lemma 1, $t_k(s)$ is the kth component of $T(s)$, and Γ is the inverse of the function $\ln[\mathrm{e}^x u'(w - \mathrm{e}^x)]$, i.e., $\Gamma(r)$ is the value of x that is the solution to $\ln[\mathrm{e}^x u'(w - \mathrm{e}^x)] = r$. (Here e^x is the exponential function.)

Using the mean value theorem as in the proof of Lemma 2, it can be shown that assumptions (1)–(3) imply that T is a contraction mapping on S, thereby assuring that the iterative procedure converges to a fixed point of T, say s^*. It is obvious from the definition of T that the vector y whose kth component is the exponential of the kth component of s^* satisfies (9).

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