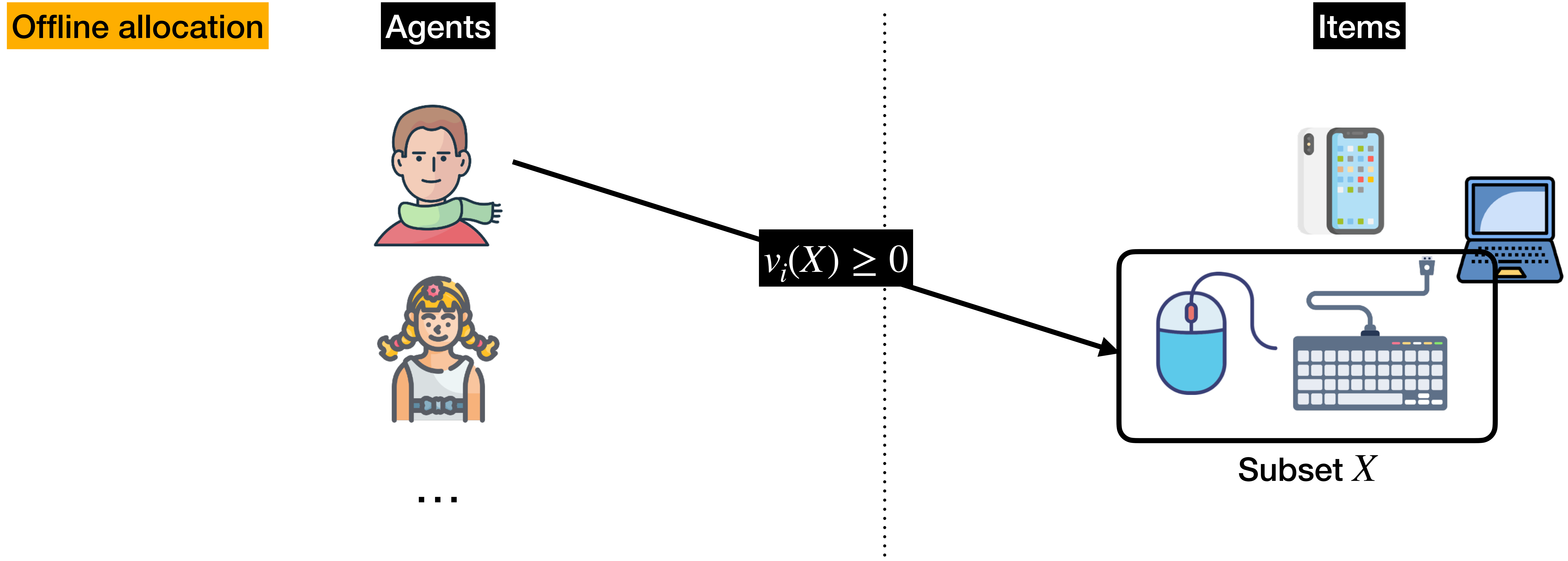
Online algorithms for combinatorial auctions

Samuel Boïté

Supervised by J. Correa, A. Cristi — Universidad de Chile, Santiago

Combinatorial auctions

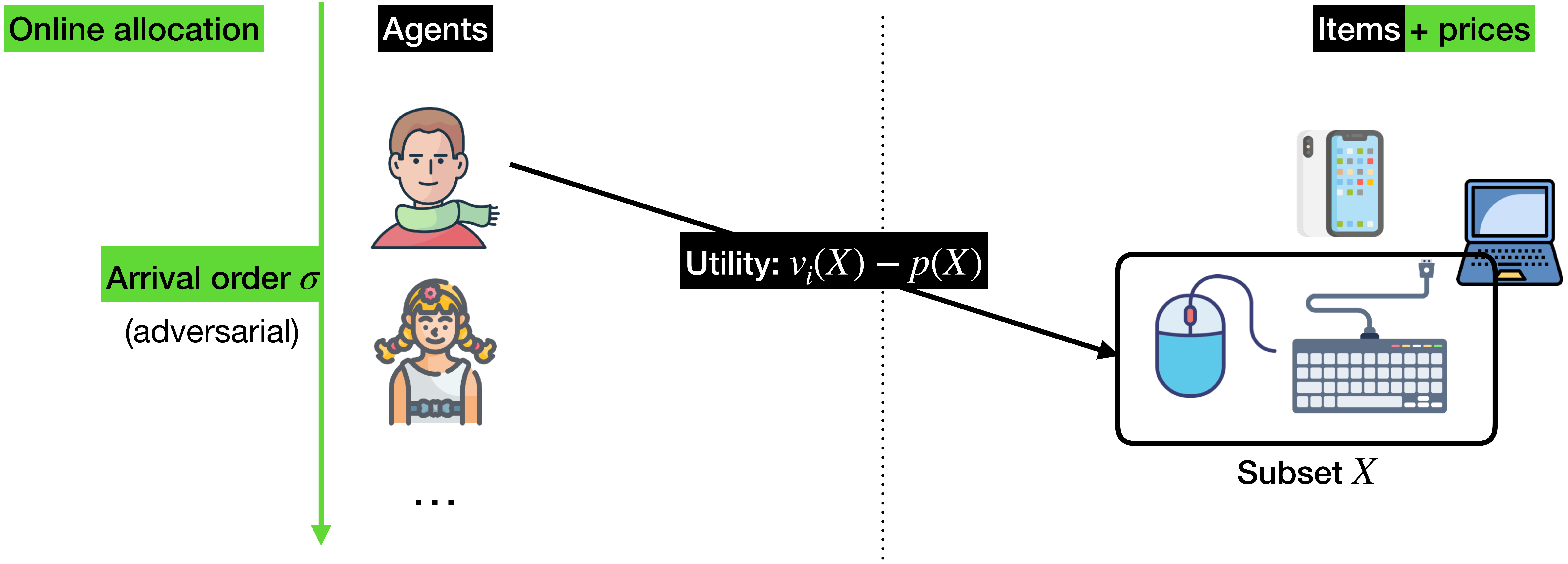
An introduction



Offline optimal allocation: OPT, welfare $v(OPT)$.

Combinatorial auctions

An introduction

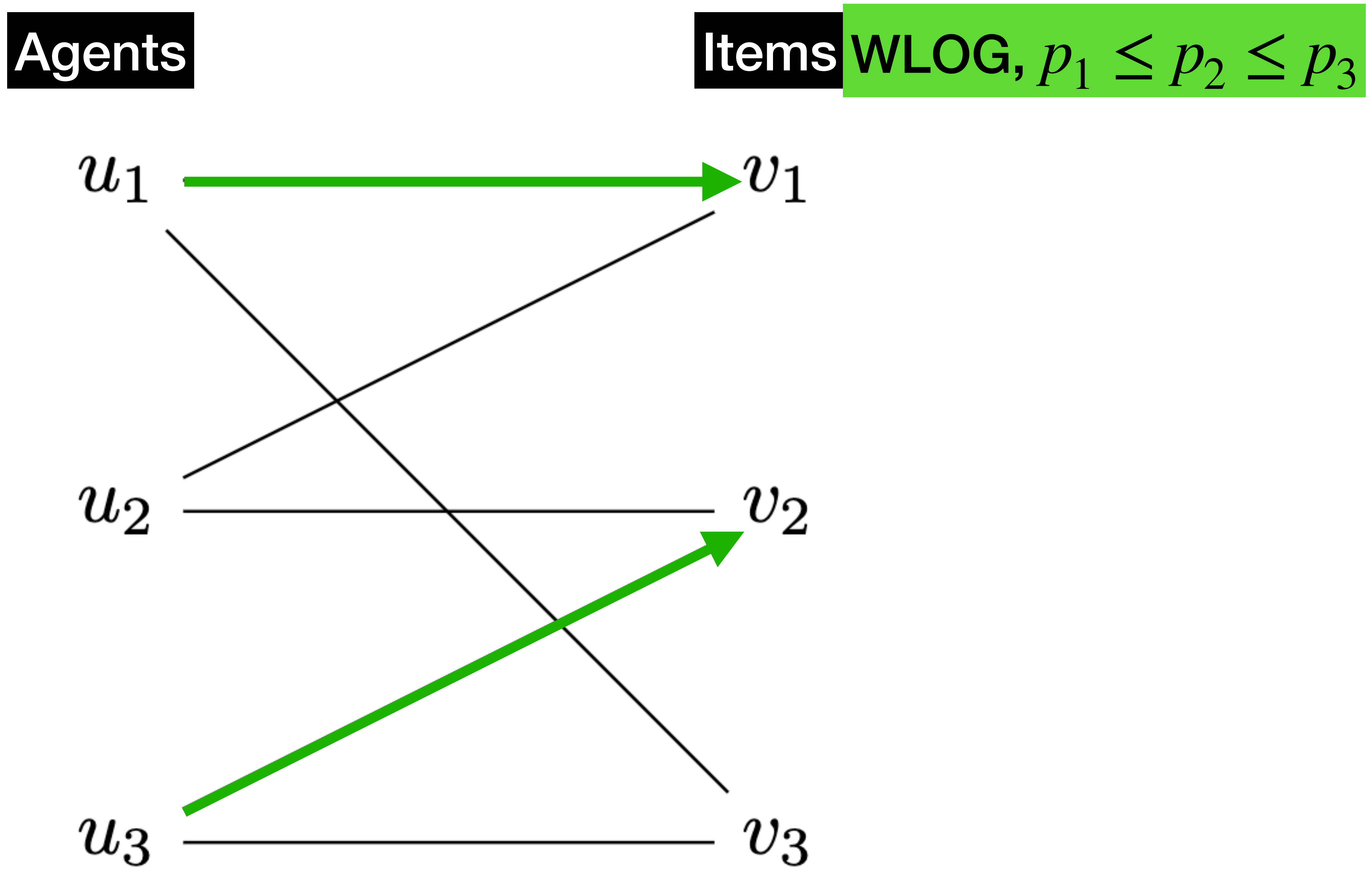


Online allocation: $ALG(p)$, welfare $v(ALG(p))$.

Competitive ratio?

Warm-up

An important example



$v(OPT) = 3$ $v(ALG) = 2$

Warm-up

With less agents

$X \subset M$	$\{a\}$	$\{b\}$	$\{a,b\}$
$v_1(X)$	1	1	2
$v_2(X)$	2	2	$[2;3)$

WLOG, $p_a \leq p_b$. If $p_b \leq 1$, $\quad 2 - p_a - p_b \geq 1 - p_a \geq 1 - p_b$.

If $p_b > 1\ldots$

$v(OPT) = 3$ $\quad$ $v(ALG) = 2$

Conclusion: $CR \leq 2/3$.

Valuation classes

- *Additive* when $v(S \sqcup T) = v(S) + v(T)$.
- *Fractionally subadditive* when $v = \max a_i$ with $(a_i)_{1 \le i \le n}$ additive.
- *Subadditive* when $v(S \cup T) \le v(S) + v(T)$.

What is the worst possible deterministic instance we can build?

(with posted price mechanisms)

Conjecture: CR = 2/3

2 items case

Towards a general proof

Prop. With 2 items & subadditive valuations, CR is 2/3.

CR is even 1 when OPT is unique.

Proof. (When OPT is unique)

For item x, we note $\max x := \max_i v_i(x)$.

Then, $v(OPT)$ can only be of the forms: $v_i(\{a, b\}),\ \max a + \max b,\ \max a + \max{}_2 b\ldots$

In each case, we set explicit prices, we verify that they work.

Proof. (When OPT is non-unique)

We have an equation like $v(OPT) = v_k(b) + v_i(a) = v_k(a) + v_j(b)$.

We rewrite $v_i(a) + v_j(b) = 2\,v(OPT) - \big(v_k(a) + v_k(b)\big) \geq \frac{2}{3} v(OPT)$.

Putting prices $p_a = v_i(a) - \varepsilon$ and $p_b = v_j(b) - \varepsilon$ works.

Beyond the 2 items case

- In general, we can have $v(ALG) < v(OPT)$ without ties.

$X \subset M$	any item	any pair	M
$v_1(X)$	1	1	2
$v_2(X)$	1/2	1/2	1/2

To have $v(ALG) = v(OPT)$, we need $\forall x,\ \ p_x > 1/2$.

But then $v_1(a) + p_b + p_c > 2 = v_1(M)$.

- Proof scheme: $v(ALG) = v(OPT)$ requires a lot of conditions on valuations & prices.
- Contradicting them yields $v(ALG) \geq 2/3 \cdot v(OPT)$.

Let's sum things up

- CR ≤ 2/3.
- CR = 2/3 for 2 items + subadditive (and even 1 when OPT unique).
- As soon as 3 items, we often have CR < 1 without ties.
- Full proof for 3 items would be tedious and uninformative.
- Time to introduce some simplifications!

Introducing simplifications

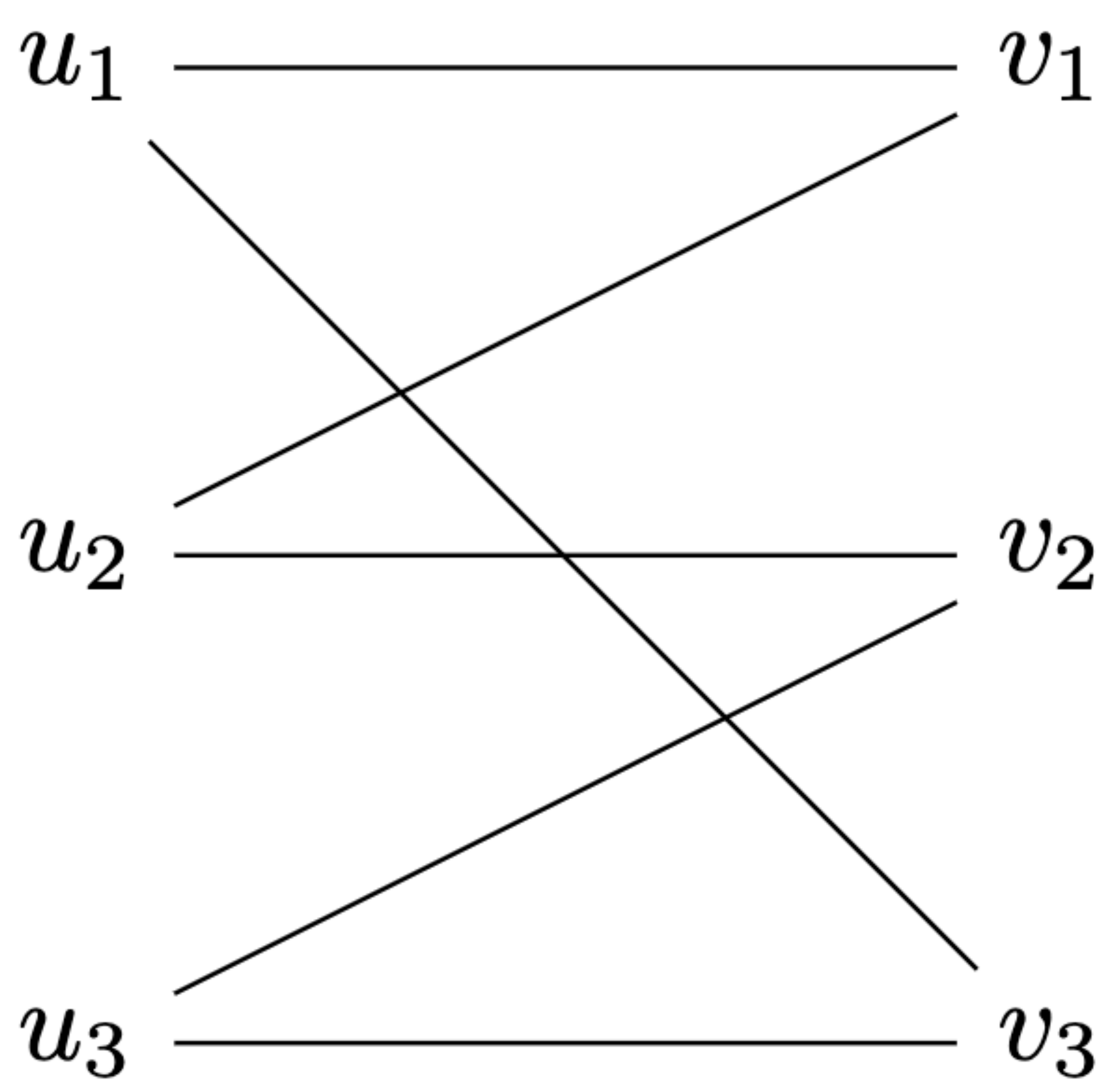
"Max-min greedy matching"

Eden et al., 2020

CR > .51

- Valuations only take values 0 and 1.
- Buying multiple items is not allowed.
- Always contains one perfect matching.

Prices $p \equiv$ priorities π over items.



First results

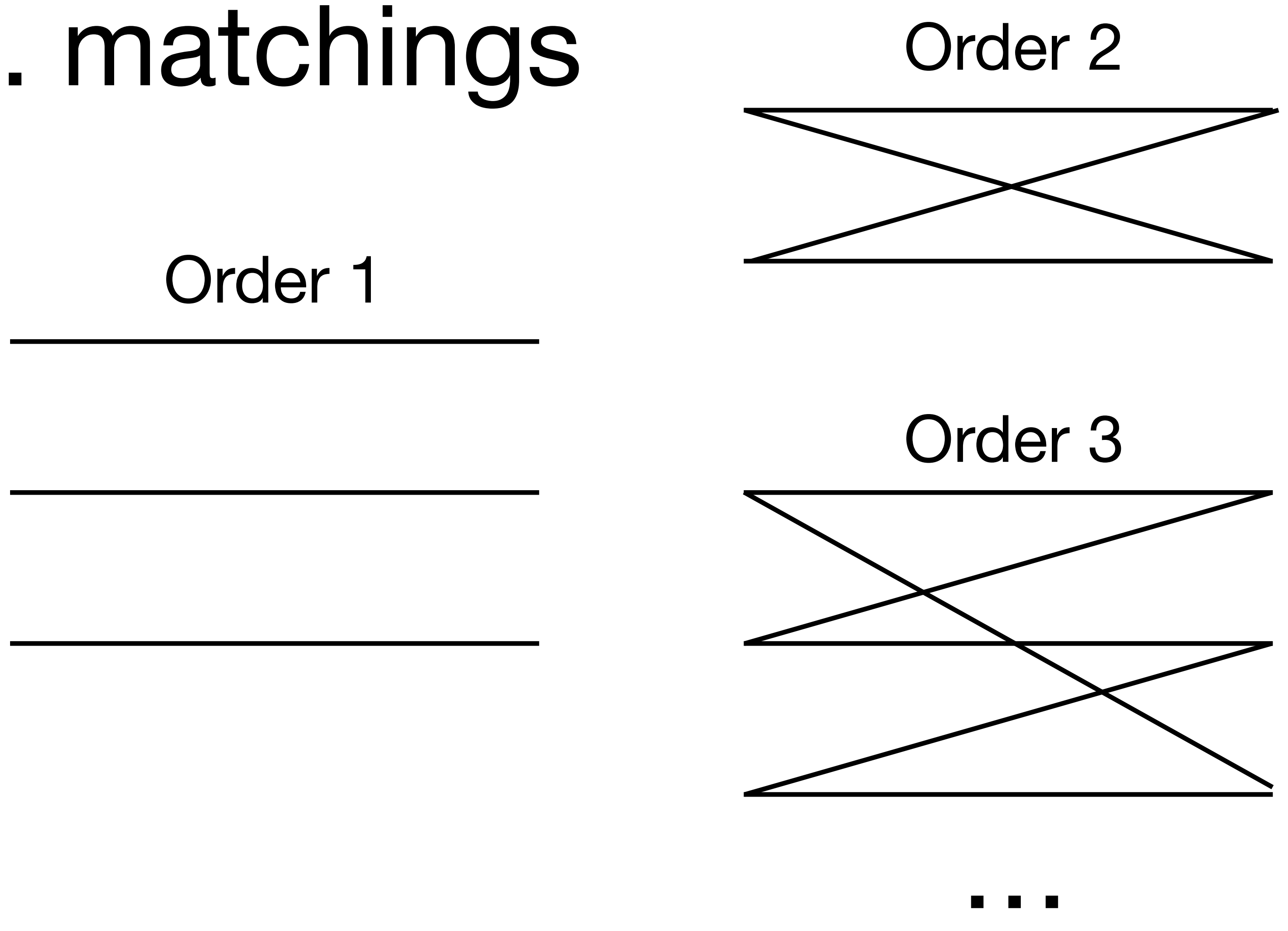
Prop. When OPT is unique, CR = 1.

Interesting case: $\mathscr{G} \supset$ two perfect matchings.

2/3 case: $\mathscr{G} =$ two disjoint perfect matchings.

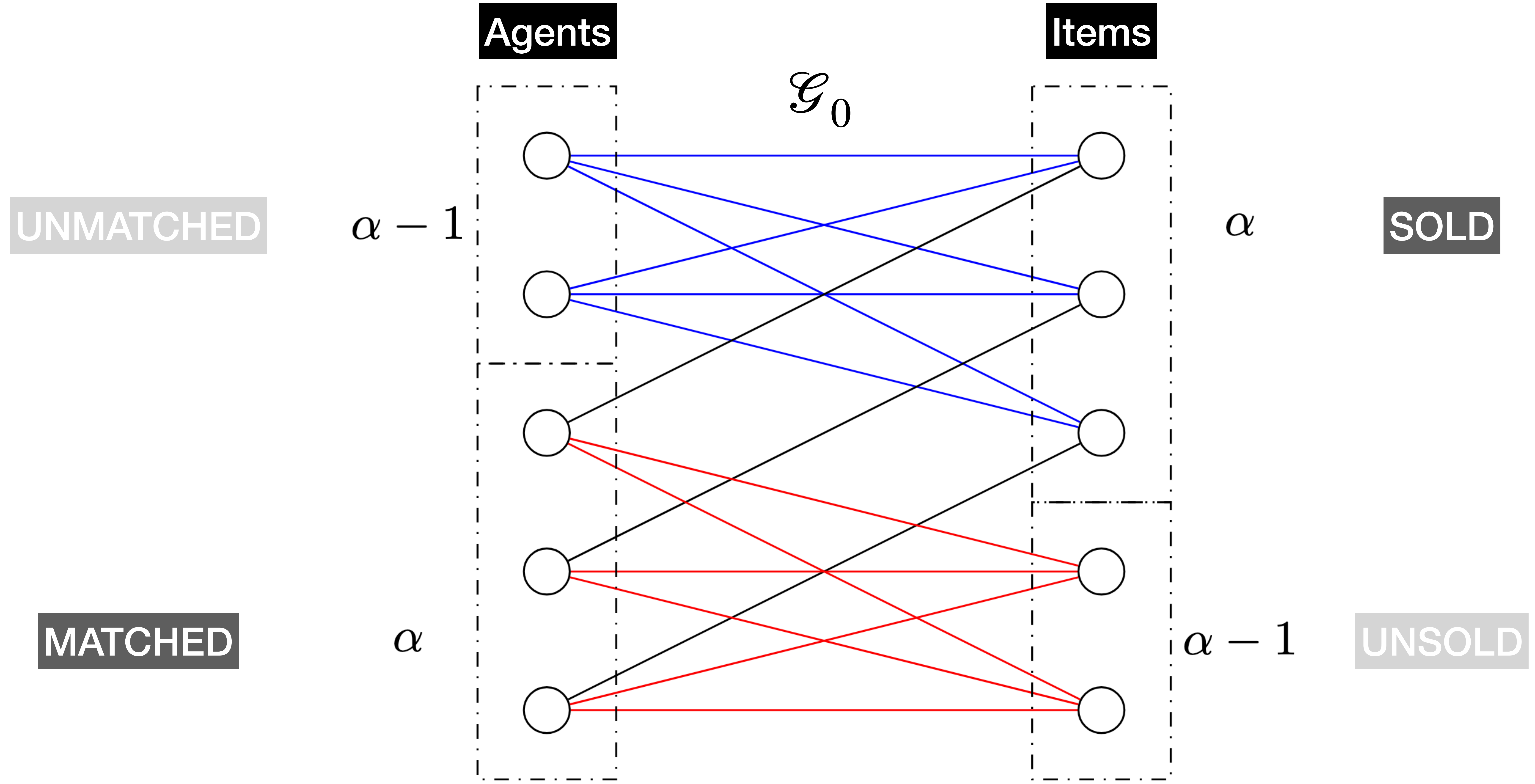
Def. $\mathscr{G}_p$: graph of utility-maximizing items.

Prop. Finding p such that $\mathscr{G}_p$ is the union of 2 perf. matchings is enough to guarantee a CR of 2/3.



Removing all edges that do not belong in a perfect matching?

Removing all edges that do not belong in a perfect matching?

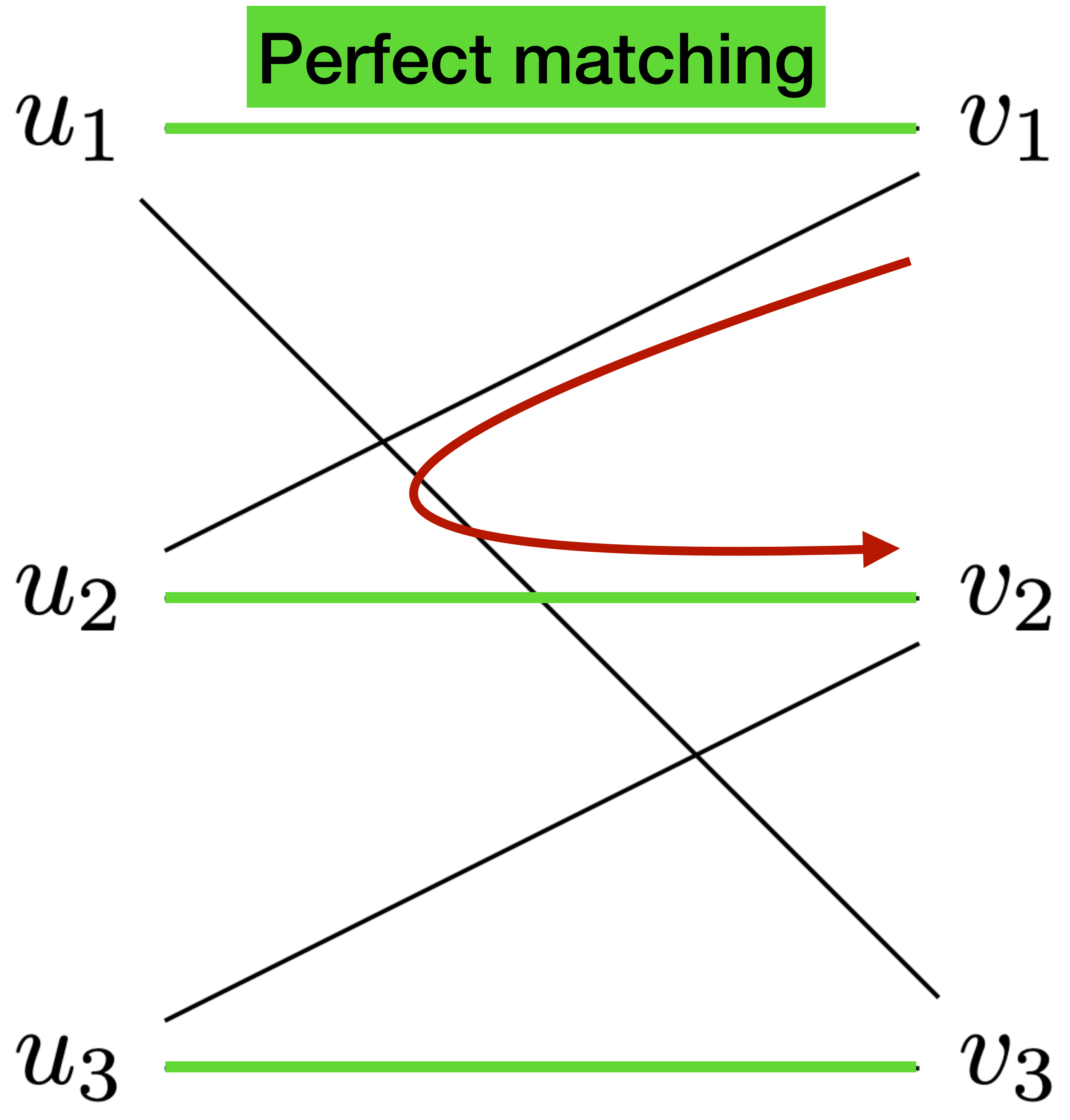


We cannot guarantee more than $\alpha/(2\alpha - 1) \sim 1/2$.

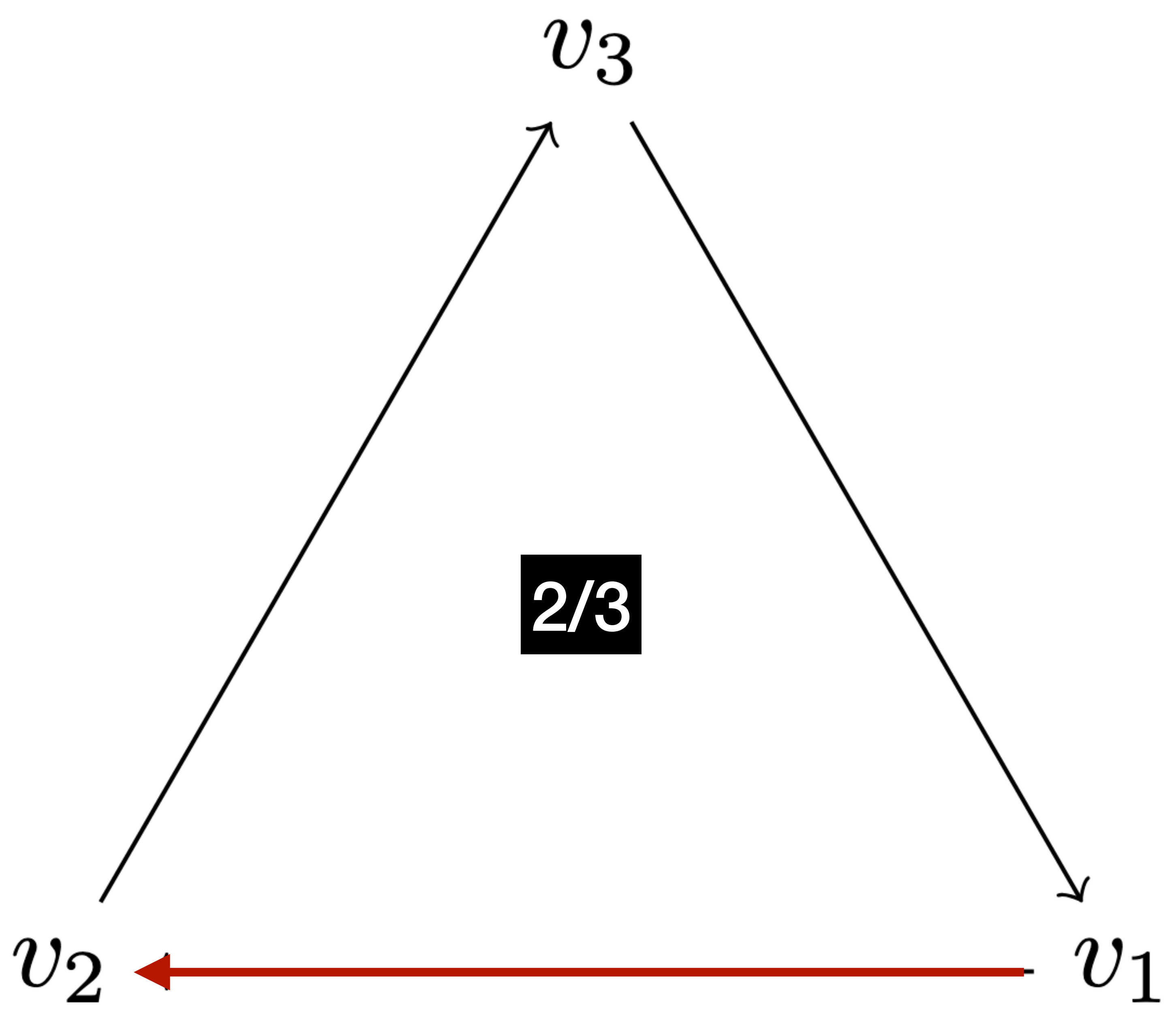
Also demonstrates that pricing items based on number of buyers is not enough.

"Spoiling graph"

Definition



(a) $\mathcal{G}_0$

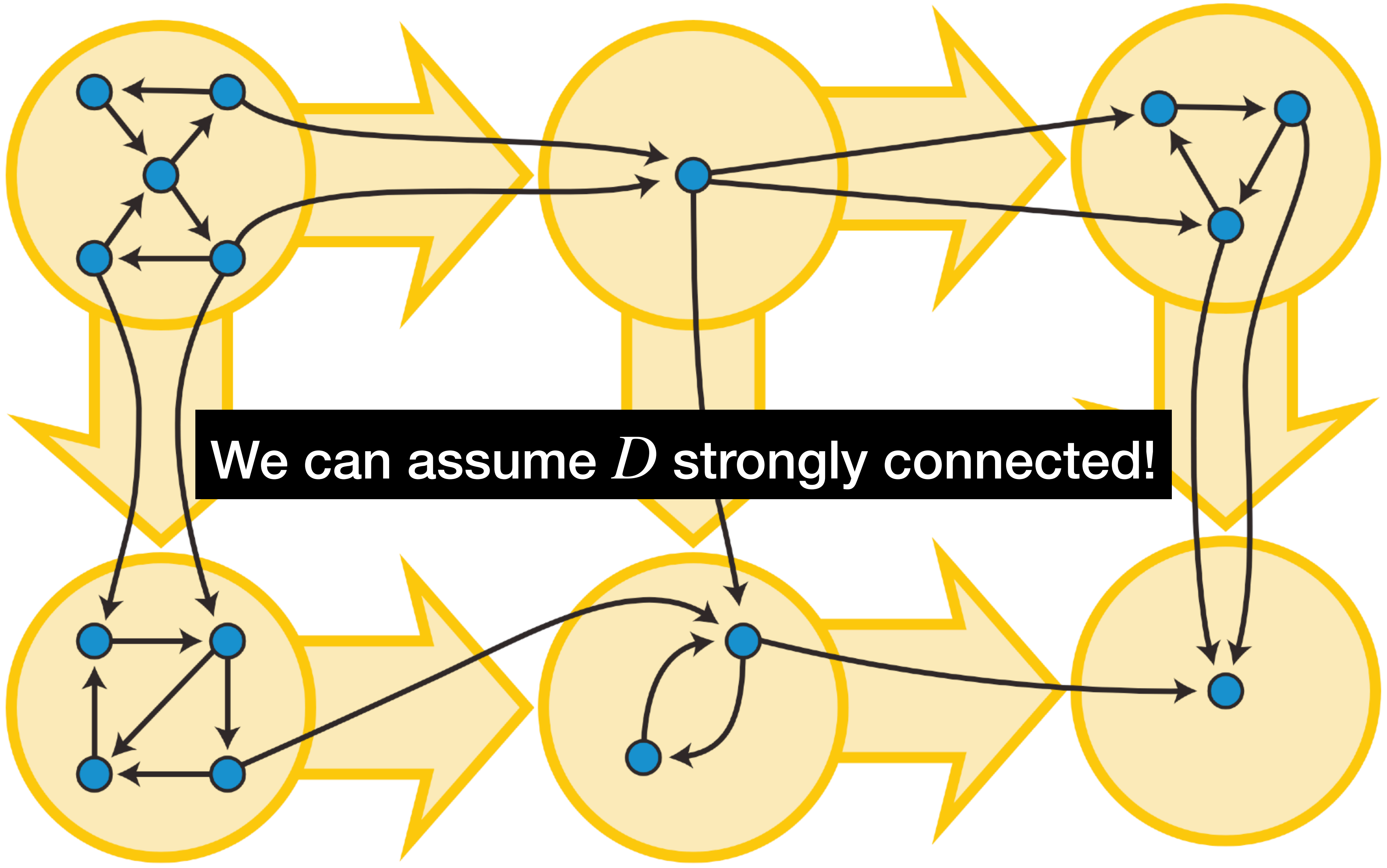


(b) Spoiling graph D

"Spoiling graph"

First observations

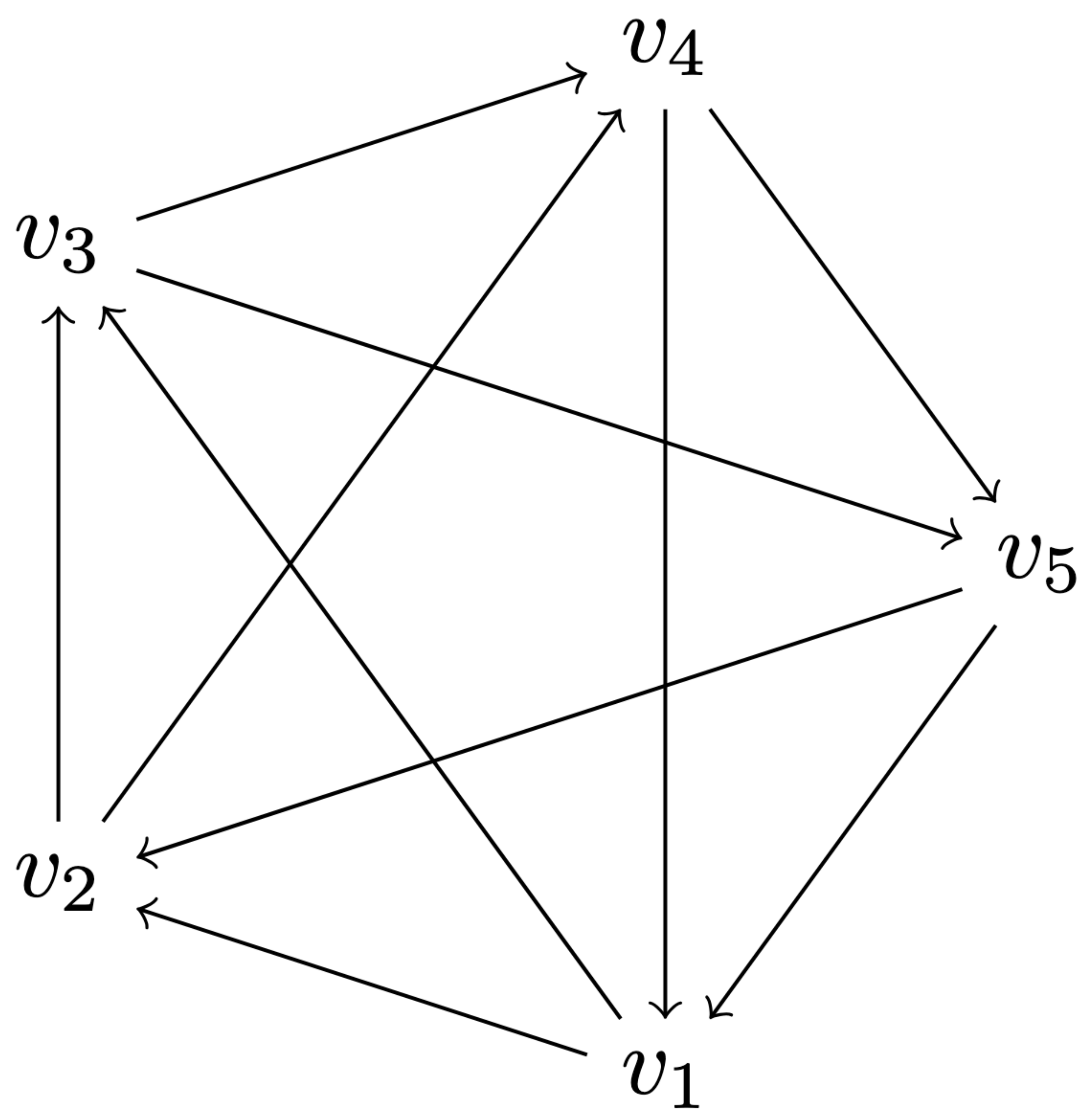
- Acyclic graph $\Longrightarrow$ we can guarantee $v(OPT)$.
  - Alternative proof of CR = 1 when OPT is unique!
- Graph condensation:



"Spoiling graph"

Feedback set approach

- Is removing $1/3$ of the edges enough to make D acyclic?
- No!



Imagine we removed at most $(n-3)/2$ nodes.

Consider two consecutive nodes u and v: the arc goes from u to v.

We formed a directed cycle.

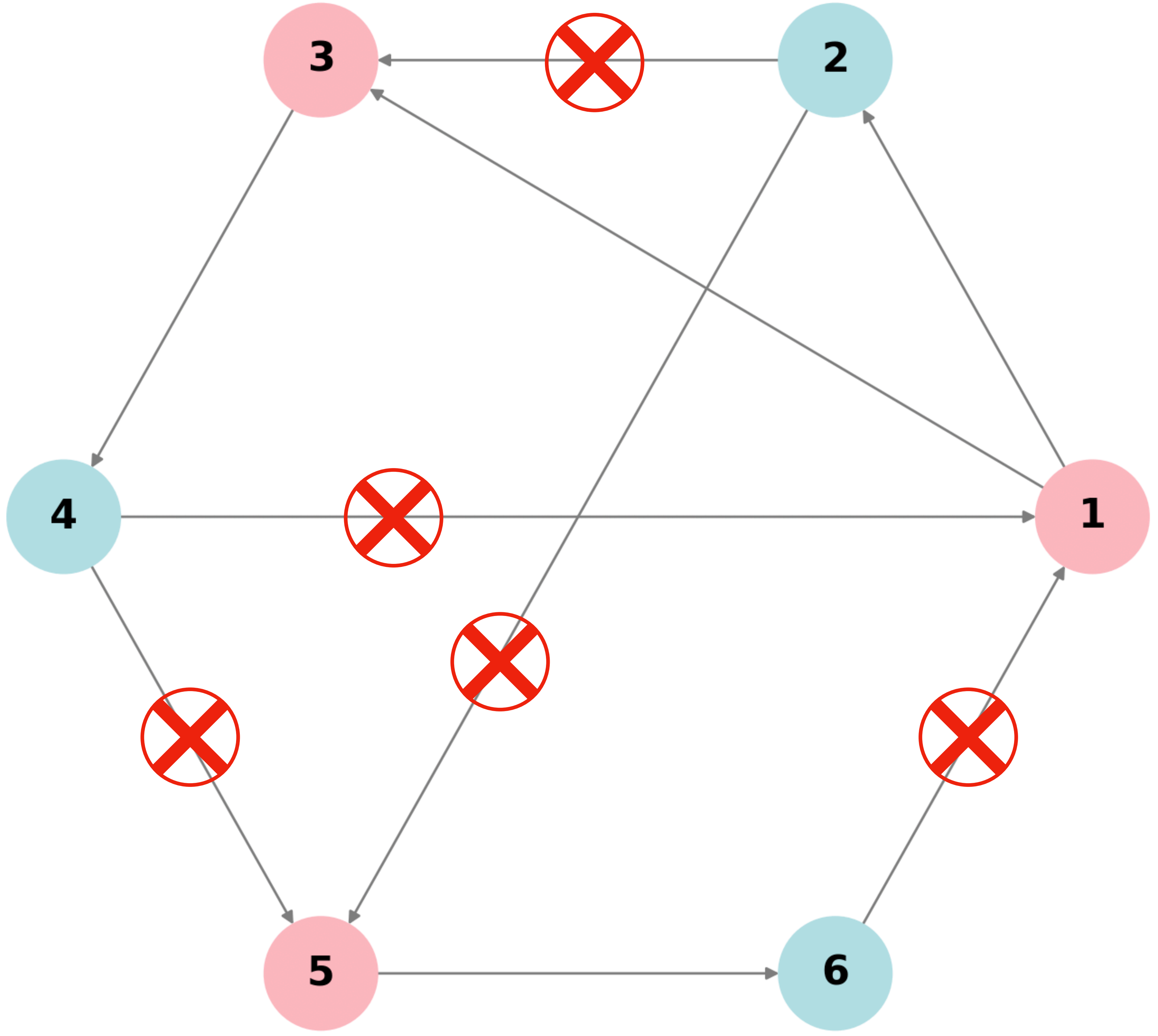
We have to remove at least 1/2 of the edges!

“Spoiling graph”

Hamiltonian case

- Hamiltonian: there is a cycle containing all the items.
- Eden et al.’s approach:

Odd is cheaper



CR ≥ 5/9

Conclusion

- CR $\leq$ 2/3.
- Conjecture: CR = 2/3 for comb. auctions & max-min greedy matching.
- Proven for simple cases.
- For max-min greedy matching, many natural approaches fail.
- Beating 5/9 on Hamiltonian graphs would be nice towards general answer!