ECE/CS 559: Neural Networks Homework 2

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1. Question 1:

   (a) To implement $(x_1 \wedge x_2 \wedge x_3)$, I set the weights of x_1, x_2, and x_3 to $W = 1$. In this way, the result of the logical AND is true if and only if the total sum is 3. The bias is then set to -2. This ensures that if the total is 3, we get $3 - 2 = 1$, which corresponds to 1 for the sign(x) activation function. Otherwise, the result is always ≤ -1 (see Table 1).
   The architecture is shown in Figure 1.

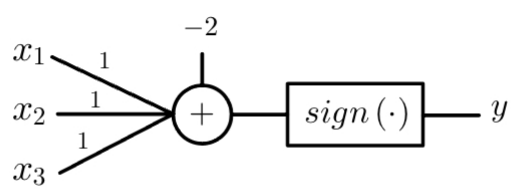

Figure 1: $(x_1 \wedge x_2 \wedge x_3)$

x_1	x_2	x_3	Sum	Bias (-2)
-1	-1	-1	-3	-5
-1	-1	1	-1	-3
-1	1	-1	-1	-3
-1	1	1	1	-1
1	-1	-1	-1	-3
1	-1	1	1	-1
1	1	-1	1	-1
1	1	1	3	1

Table 1: Truth table for x_1, x_2, x_3 with sum and bias

To consider $\neg x_3$, the sign of the weight for x_3 is inverted. This results in the maximum sum being 3 (see Table 2), which occurs when the condition $x_1 \wedge x_2 \wedge \neg x_3$ is true. In all other cases, the maximum sum will always be ≤ 1. The bias, as in the case above, is set to -2 to adjust the sum to the threshold of the sign(x) function. The architecture is shown in Figure 2.
To implement $\neg x_2 \wedge x_3$, the weight of x_1 is set to zero, as it is not considered. The weight of x_2 is set to -1 since it is negated, and the weight of x_3 is set to 1. The bias is set to -1. See Table 3, and the architecture is shown in Figure 3.

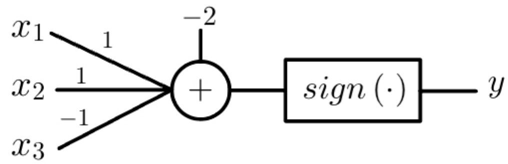

Figure 2: $x_1 \wedge x_2 \wedge \neg x_3$

x_1	x_2	$\neg x_3$	Sum	Bias (-2)
-1	-1	-1	-1	-3
-1	-1	1	-3	-5
-1	1	-1	1	-1
-1	1	1	-1	-3
1	-1	-1	1	-1
1	-1	1	-1	-3
1	1	-1	3	1
1	1	1	1	-1

Table 2: Truth table for x_1, x_2, $\neg x_3$ with sum and bias

$\neg x_2$	x_3	Sum	Bias (-1)
1	-1	0	-1
1	1	2	1
-1	-1	-2	-3
-1	1	0	-1

Table 3: Truth table for $\neg x_2$ and x_3 with sum and bias

The weights of z_1 and z_2 are both set to 1. In this case, however, the operator is OR, so the bias is set to $+1$ since the threshold of the sign(x) function must be shifted by one unit to include in the true condition the case where z_1 and z_2 are True and False, respectively, or vice versa, as the total sum is zero. See Table 4, and the architecture is shown in Figure 4.

z_1	z_2	Sum	Bias (+1)
-1	-1	-2	-1
-1	1	0	1
1	-1	0	1
1	1	2	3

Table 4: Truth table for z_1 and z_2 with sum and bias

In conclusion, it can be observed that in the case of the sign(x) activation function, the role of the bias is to shift the threshold to the left or right by k units, with bias $= \pm k$. This is necessary to ensure that the input of the function falls within the correct range of values $(-\infty, -1]$ or $[+1, +\infty)$, thereby achieving the desired behavior. The final architecture is shown in Figure 5. The neural network is a 2-2-1 feed-forward neural network.

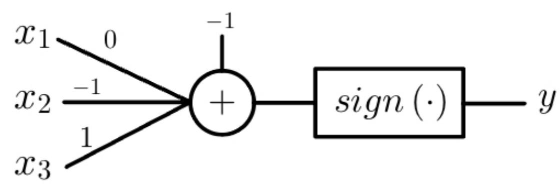

Figure 3: $\neg x_2 \land x_3$

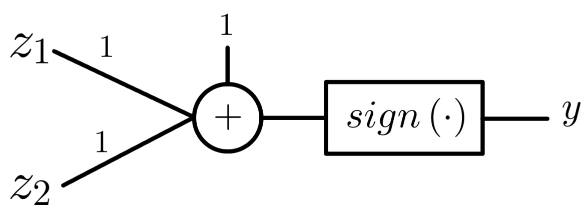

Figure 4: $z_1 \lor z_2$

(b) The input-to-output analytic equation of the neural network is:

$$y(\mathbf{x}) = \text{sign}\left(U \cdot \text{sign}(W\mathbf{x} + b) + c\right)$$

- $\mathbf{x} \in \mathbb{R}^{3\times1}$ is the input vector.
- $W \in \mathbb{R}^{2\times3}$ is the weight matrix for the first layer.
- $b \in \mathbb{R}^{2\times1}$ is the bias vector for the first layer.
- $U \in \mathbb{R}^{1\times2}$ is the weight vector for the second layer.
- $c \in \mathbb{R}$ is the bias for the second layer.
- $y \in \mathbb{R}$ is the output of the neural network.
- sign(x) is the activation function producing either -1 or +1.

The matrices and vectors are defined as:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad c = 1.$$

(c) Figure 6 shows the result of the code, demonstrating that the truth table and the input-output table match.

2. Question 2

(a) Given the Feed Forward Neural Network the matrices and vectors are defined as:

- $\mathbf{x} \in \mathbb{R}^{2\times1}$ is the input vector.
- $W \in \mathbb{R}^{3\times2}$ is the weight matrix for the first layer.
- $b \in \mathbb{R}^{3\times1}$ is the bias vector for the first layer.

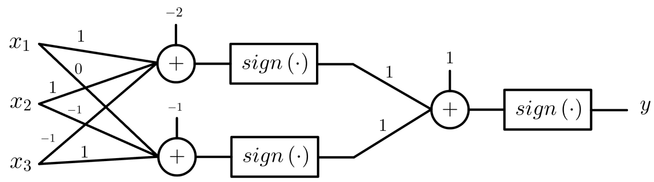

Figure 5: 2-2-1 Feed Forward Neural Network

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Print truth table for the given boolean function:
x1      x2      x3      y
-----------------------------
False   False   False   False
False   False   True    True
False   True    False   False
False   True    True    False
True    False   False   False
True    False   True    True
True    True    False   True
True    True    True    False


Print the outcome of the Neural Network:
x1      x2      x3      y
-------------------------------
-1      -1      -1      -1
-1      -1       1       1
-1       1      -1      -1
-1       1       1      -1
 1      -1      -1      -1
 1      -1       1       1
 1       1      -1       1
 1       1       1      -1
```

Figure 6: Outputted tables

- $U \in \mathbb{R}^{1\times 3}$ is the weight vector for the second layer.
- $c \in \mathbb{R}$ is the bias for the second layer.
- $y \in \mathbb{R}$ is the output of the neural network.
- $\text{step}(x)$ is the activation function producing either 0 or +1.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & -1 \\ -1 & -1 \\ 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}, \quad c = -1.5.$$

(b) The input-to-output analytic equation of the neural network is:

$$y(\mathbf{x}) = \text{step}\left(U \cdot \text{step}(W\mathbf{x} + b) + c\right)$$

(c) Figure 7 shows the output plot of all the points x, where the points are blue when $y = 0$ and red when $y = 1$.

(d) Figure 8 shows the decision boundary of the neural network.

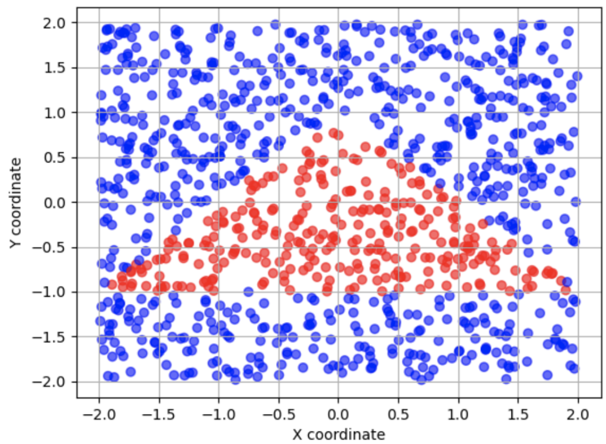

Figure 7: Plot

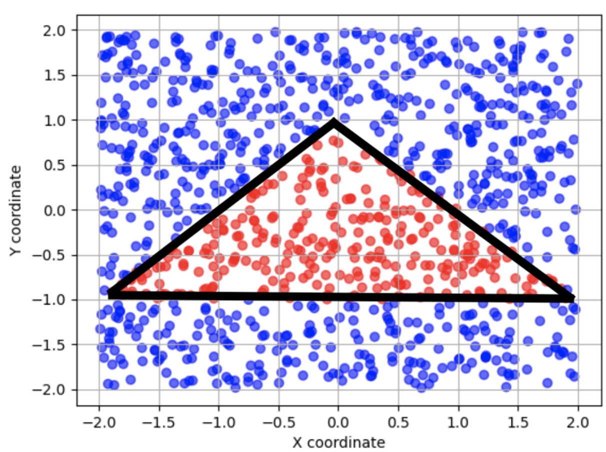

Figure 8: Plot