Energy Finance Project

Group 10

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1 Introduction

This study endeavors to illustrate the calibration process of the Heath-Jarrow-Morton (HJM) model, with a specific focus on German power swaps throughout the monitoring period spanning the fourth quarter of 2024. Initially, comprehensive details pertaining to the model and its dynamics will be provided. Subsequently, we will explore the admissible conditions for the model parameters to ensure that forward prices exhibit martingale properties.
Following this, the calibration process will commence by optimizing the parameters based on option prices. The objective is to minimize the disparity between model-derived prices and observed market prices.

1.1 HJM Model

The electricity and gas markets actively engage in the trading of forward contracts, commonly referred to as swaps, wherein the commodity is delivered over a specified period. Let $F(t, \tau_1, \tau_2)$ denote the price at time t for a swap contract with delivery spanning the interval $[\tau_1, \tau_2]$. Notably, these swap contracts are typically transacted over the time period $t \in [0, \tau_1)$.

Observably, a swap contract with both initiation and termination of delivery at time u is equivalent to a forward contract. Consequently, $F(t, u, u)$ represents the forward price at time t for a contract with delivery at time $u \geq t$.

The continuous no-arbitrage condition implies that any model applicable to swaps with arbitrary delivery periods $[\tau_1, \tau_2]$ must emanate from a forward dynamics perspective. Consequently, we introduce an extension of the forward dynamics. Assuming, under the risk-neutral measure Q, the price dynamics for swap contracts based on the Heath-Jarrow-Morton (HJM) approach is given by:

$$
\begin{aligned}
F(t,\tau_1,\tau_2) = & F(0,\tau_1,\tau_2)(\exp\Bigg(\int_0^t A(u,\tau_1,\tau_2)\,du \\
& + \sum_{k=1}^{p}\int_0^t \Sigma_k(u,\tau_1,\tau_2)\,dW_k(u) \\
& + \sum_{j=1}^{n}\int_0^t \Upsilon_j(u,\tau_1,\tau_2)\,dJ_j(u)\Bigg))
\end{aligned} \tag{1}
$$

where $A(t,\tau_1,\tau_2)$, $\Sigma_k(t,\tau_1,\tau_2)$, and $\Upsilon_k(u,\tau_1,\tau_2)$, $k = 1,\ldots,p$, $j = 1,\ldots,n$, are real-valued continuous functions defined for $0 \le t \le \tau_1 \le \tau_2 \le T$, with T serving as an upper bound for delivery times in the market.
Additionally, it is assumed that the functions Σ_k are positive, and the initial forward curve $F(t,\tau_1,\tau_2)$ is a real-valued and continuous function for $0 \le t \le \tau_1 \le \tau_2 \le T$.
In our framework we assume $\mathbf{p} = \mathbf{2}$ and $\mathbf{n} = \mathbf{0}$ so the previous formula becomes:

$$
F(t,\tau_1,\tau_2) = F(0,\tau_1,\tau_2)(\exp\left(\int_0^t A(u,\tau_1,\tau_2)\,du\right) + \sum_{k=1}^{p=2}\int_0^t \Sigma_k(u,\tau_1,\tau_2)\,dW_k(u)) \tag{2}
$$

2 Admissible range for the model parameters/Conditions on the drift

Starting from formula 2, and with the additional assumption for $\Sigma_k(t,\tau_1,\tau_2)$, the admissible range for the model parameters would be:

$$
\begin{aligned}
&\Sigma_k(t,\tau_1,\tau_2) && \in [0,\infty] \quad \text{for } k = 1,2 \\
&A(t,\tau_1,\tau_2) && \in \mathbf{R}
\end{aligned} \tag{3}
$$

As we said before, since we need the explicit dynamics of the forward under the risk-neutral probability Q, the price has to be a martingale. This entails a condition on A, and we refer to Proposition 6.3 (Benth 2008) which allows us to find the admissible conditions for the model parameters.
We consider the general **Drift Condition** which must hold in order to avoid arbitrage dynamics for the individual swap contracts:

$$
\begin{aligned}
&\int_0^t A(u,\tau_1,\tau_2)\,du + \frac{1}{2}\sum_{k=1}^{p}\Sigma_k^2(u,\tau_1,\tau_2)\,du + \sum_{j=1}^{n}\int_0^t \Upsilon_j(u,\tau_1,\tau_2)\,d\gamma(u) \\
&+ \int_0^t\int_{\mathbb{R}} \left(e^{\Upsilon_j(u,\tau_1,\tau_2)z} - 1 - \Upsilon_j(u,\tau_1,\tau_2)z\,\mathbf{1}_{\{|z|<1\}}\right)\nu_j(dz,du) = 0
\end{aligned} \tag{4}
$$

In our framework the formula becomes:

$$
\int_0^t \left(A(u,\tau_1,\tau_2) + \frac{1}{2}\Sigma_1^2 + \frac{1}{2}\Sigma_2^2\right) du = 0 \tag{5}
$$

Under the drift condition, for every $t \le \tau_1$, the swap price $F(t) \triangleq (t,\tau_1,\tau_2)$ in (2) has for $t \le \tau_1$ the following Q-dynamics

$$
\frac{dF(t)}{F(t)} = \Sigma_1 dW_1(t) + \Sigma_2 dW_2(t) \tag{6}
$$

For example, to model the sigmas we could based our analysis on [Kiesel, Schindlmayer and Borger (2006)] paper, they model the volatility terms as follows:

$$\Sigma_1 = \sigma_1 e^{-\alpha(\tau_1 - t)}$$

$$\Sigma_2 = \sigma_2$$

The motivation for this model is that Σ_1 mimics the volatility term structure arising from a mean reversion model, while the second volatility models the non-stationary part.
We will base our analysis by imposing the two sigmas constant, time-dependent and using time dependent volatility function.

3 Calibration

3.1 Σ_1 and Σ_2 constant

In order to calibrate the two-factor model to market data, we need to estimate the parameters

$$\phi = (\Sigma_1, \Sigma_2)$$

such that the model fits the market behaviour. Since we have modelled under a risk-neutral measure, we need to find risk-neutral parameters, which can be observed using option-implied parameters. First of all, we need to compute the discount factors in our maturity dates, in our Matlab code named `datesExpiry`. To solve this task we build the function `interpol`, that performs a linear interpolation not directly on the discount factors but on the corresponding zero rates, as market practice.

At this point, performs the calibration of a two-factor HJM (Heath-Jarrow-Morton) model using the Black-76' formula (eq. 7) for pricing. The model is calibrated using the least squares non-linear optimization (`lsqnonlin`) to minimize the difference between market prices and model prices. What are the prices of the option written on swaps in our model?
Suppose a call option written on a swap contract with delivery period $[\tau_1, \tau_2]$ has exercise time $T \leq \tau_1$ and strike K. The option price at time t is then given as

$$C(t; T, K, \tau_1, \tau_2) = e^{-r(T-t)} \left\{ F(t, \tau_1, \tau_2) \mathcal{N}(d_1) - K \mathcal{N}(d_2) \right\} \tag{7}$$

where

$$d_1 = d_2 + \sqrt{\sum_{k=1}^{p} \int_t^T \Sigma_k^2(s, \tau_1, \tau_2)\, ds}$$

$$d_2 = \frac{\ln\left(\frac{F(t,\tau_1,\tau_2)}{K}\right) - 0.5 \sum_{k=1}^{p} \int_t^T \Sigma_k^2(s, \tau_1, \tau_2)\, ds}{\sqrt{\sum_{k=1}^{p} \int_t^T \Sigma_k^2(s, \tau_1, \tau_2)\, ds}}$$

The calibration involves finding optimal values for `sigma_1` and `sigma_2`. We assume the following initial values for the parameters:

Σ_1	Σ_2
0.3	0.05

Table 1: Initial values of Σ_1, Σ_2

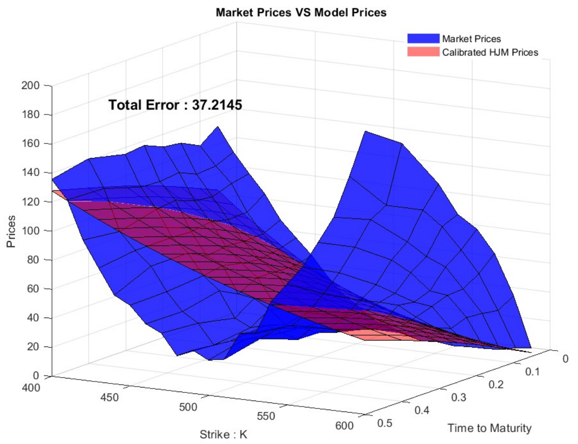
Please be aware that these initial values also apply to subsequent calibrations.
For this first calibration we use constant Σ_1 and Σ_2 to calibrate the model. It means that we can define a $\hat{\Sigma}$ function

$$\hat{\Sigma} = \sqrt{(\Sigma_1^2 + \Sigma_2^2)}$$

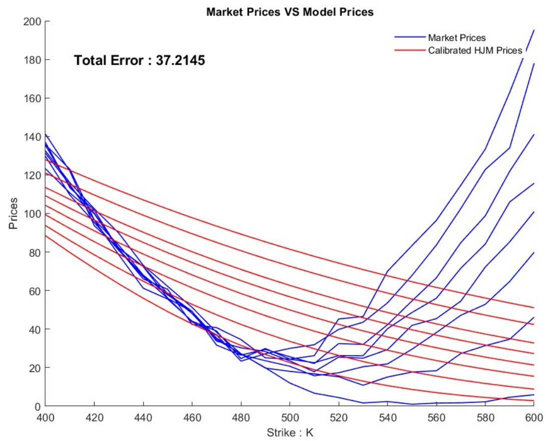
and so in formula 7 the d_1 vary as

$$d_1 = d_2 + \hat{\Sigma}\sqrt{(T - t_0)}$$

The calibrated model is used to compute prices, and the error between the market and model prices is calculated.



(a) 3D comparison



(b) 2D comparison

Figure 1: Market vs Model prices

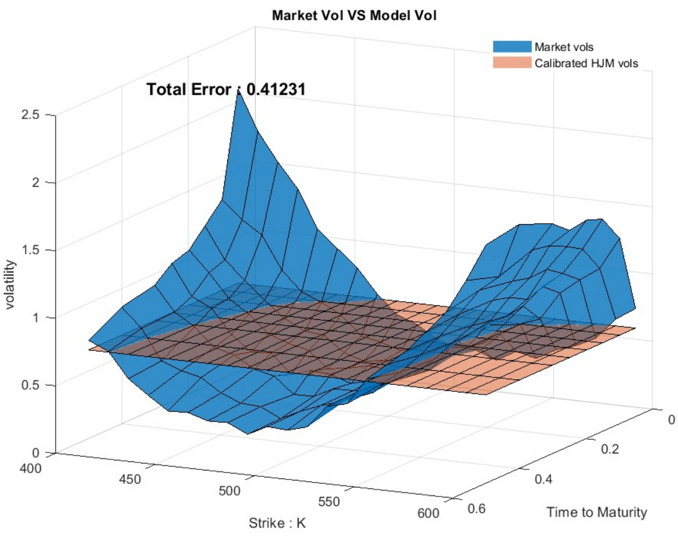


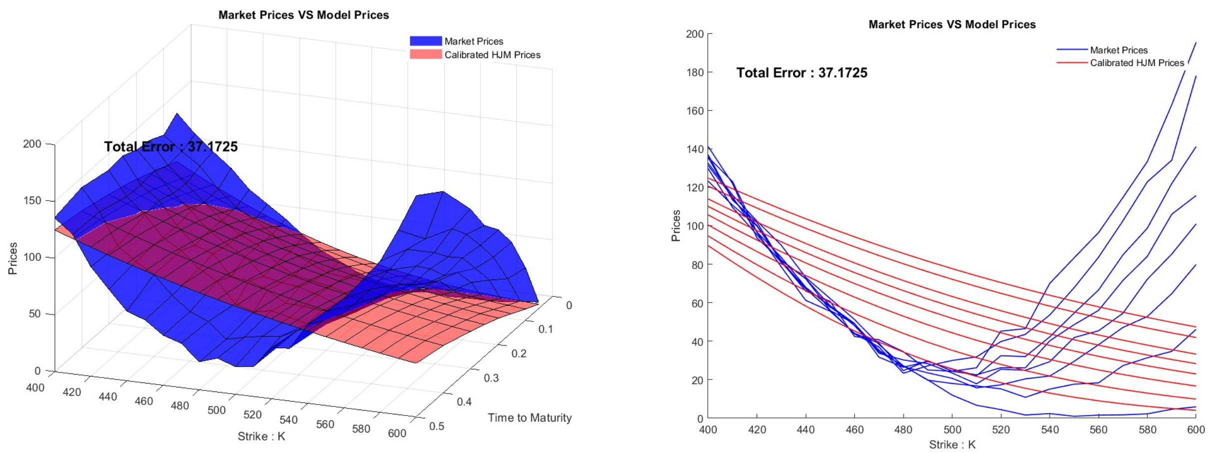
Figure 2: Market vs Model volatility

The model does not align well with market prices, leading to a substantial error. A Root Mean Square Error of 37.21$ indicates a significant deviation between the model and observed market prices.

A simplistic model might fail to incorporate crucial factors or intricate relationships present in the market, leading to a significant discrepancy between the model-generated prices and actual market prices.
More precisely, the model neglects the fact that volatility can exhibit notable variations. When examining volatility surfaces, we find that the one generated from calibrating our model bears a striking resemblance to a simplistic plane. This is in stark contrast to the observed volatility, which resembles a characteristic paraboloid shape.

3.2 Σ_1 time-dependent

Now we attempt to calibrate the model on the 4Q24 German option prices by considering a generic time-dependent $\Sigma_1(t, t_1, t_2)$. This implies that `sigma_1` will be a vector of size 8, corresponding to the various time-to-maturity values considered.



(a) 3D comparison

(b) 2D comparison

Figure 3: Market vs Model prices

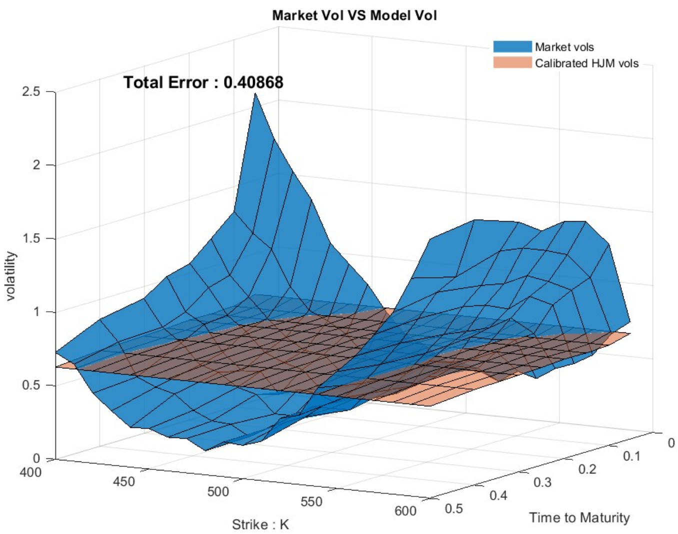


Figure 4: Market vs Model volatility

In this framework, we are working with a time dependent Σ_1, changing its value whenever the option we calibrate the model on has a different maturity.

The calibration process aims to identify the values of $\hat{\Sigma}$ within the integral that minimize the error, thus the error's magnitude will hinge on the distinct values $\hat{\Sigma}$ assumes from time t to maturity T. If N represents the number of distinct values $\hat{\Sigma}$ can acquire from time t to T, within our models

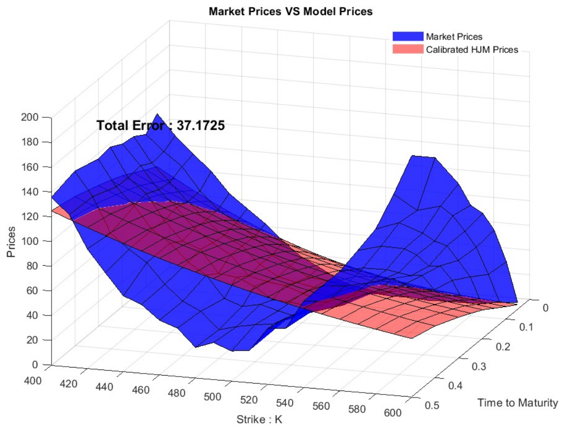
$$N = max(\#\Sigma_1, \#\Sigma_2)$$

With Σ_1 capable of taking 8 distinct values, $\hat{\Sigma}$ can also accommodate 8 different values. Employing $\hat{\Sigma}$ or not will not alter the number of distinct values $\hat{\Sigma}$ can take, implying that only the integral of Σ_1 is relevant. Σ_2's sole function is to diminish the value of Σ_1 within the integral by a constant amount.

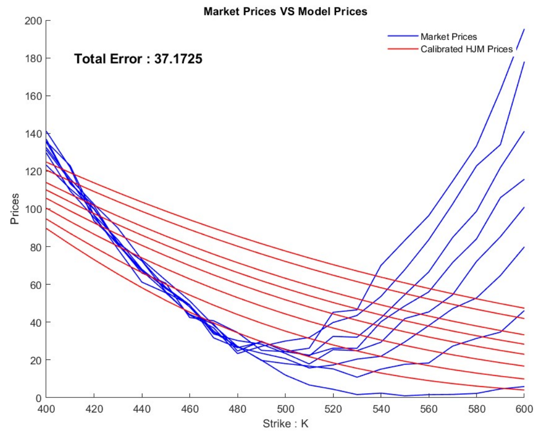
The model still fails to consider the fact that volatility can exhibit significant variations based on the strike price, not just the time to maturity. So this leads to the notable error seen on the graph. The volatility surface is still far from resembling the shape of the observed volatility surface.

3.3 Σ_1 and Σ_2 time-dependent

Now we endeavor to calibrate the model using the 4Q24 German option prices, taking into account generic time-dependent volatility functions $\Sigma_1(t, t_1, t_2)$ and $\Sigma_2(t, t_1, t_2)$. This implies that `sigma_1` and `sigma_2` will be vectors of size 8, corresponding to the various time-to-maturity values considered.



(a) 3D comparison



(b) 2D comparison

Figure 5: Market vs Model prices

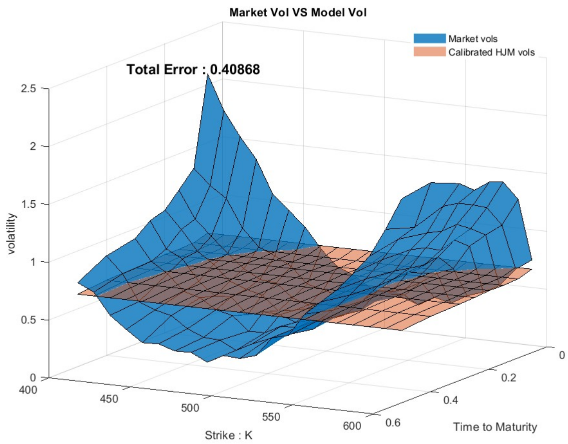


Figure 6: Market vs Model volatility

As mentioned in Section 3.2,

$$N = max(\#\Sigma_1, \#\Sigma_2)$$

Even if $\#\Sigma_2 = 8$, this will not alter N from the preceding point, N remaining 8 as before. This implies that the newly introduced Σ_2 does not enhance the complexity of our model, consequently yielding the same error and volatility surface as in the preceding scenario.

3.4 Time-dependent volatility function

We tried adding a new time dependent parameter α in our model,

$$\sigma_1(t) = \Sigma_1 e^{-\alpha(\tau_1 - t)}$$
$$\sigma_2 = \Sigma_2$$

However, the inclusion of an additional time-dependent parameter didn't reduce the error, as it does not augment the model's complexity. Hence, we opted to introduce a novel parameter with genuine utility. A parameter β, which scales both Σ_1 and Σ_2, that effectively modulates volatility in response to the strike price.

$$\sigma_1(t, K) = \Sigma_1(t)\beta(K)$$
$$\sigma_2(K) = \Sigma_2\beta(K)$$

Calibration is performed on the HJM model with a deterministic time-dependent volatility function. Σ_1 varies with time, while Σ_2 remains constant. Additionally, a variable β is introduced to capture differences in volatility across different strike prices. The volatility formula is given by:

$$\hat{\Sigma} = \sqrt{(\Sigma_1^2 + \Sigma_2^2)\beta} \tag{8}$$

So referring to equation (7) the Call formula vary in d_1 as follows:

$$d_1 = d_2 + \hat{\Sigma} \cdot (T - t_0)$$

In order to calibrate the two-factor model to market data, we need to estimate the parameters

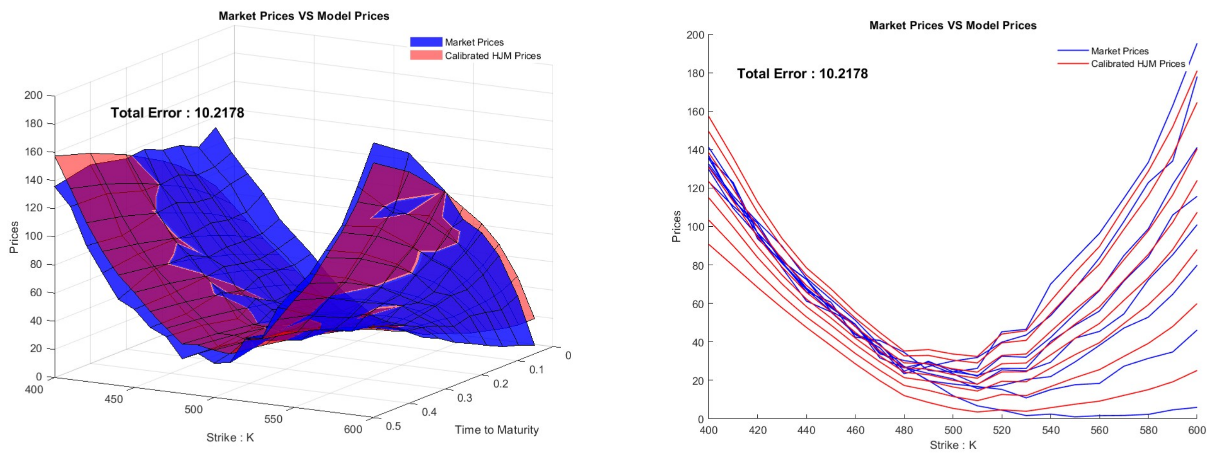
$$\phi = (\Sigma_1, \Sigma_2, \beta)$$

The calibration involves finding optimal values for `sigma_1`, `sigma_2`, and `beta`. We assume the following initial values for the parameters:

Σ_1	Σ_2	β
0.5	0.1	0.5

Table 2: Initial values of Σ_1, Σ_2, and β.

So Σ_1 denote a space of dimension 8, Σ_2 a space of dimension 2, and β a space of dimension 21, determined by the count of distinct strikes in the dataset.



(a) 3D comparison

(b) 2D comparison

Figure 7: Market vs Model prices

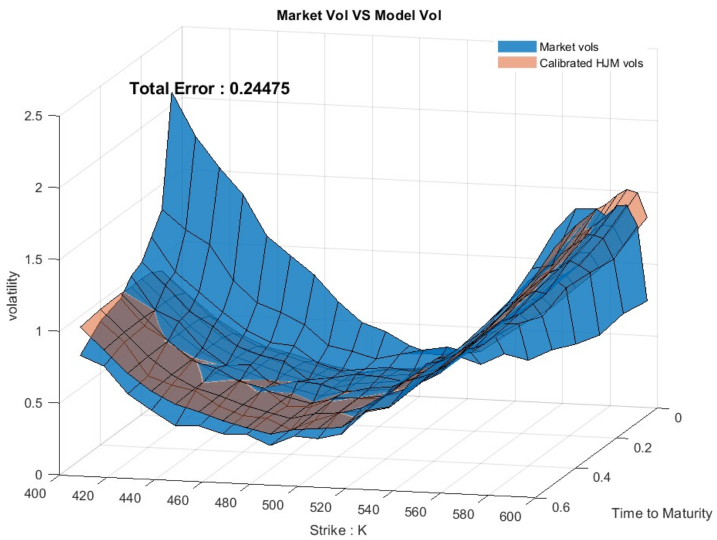
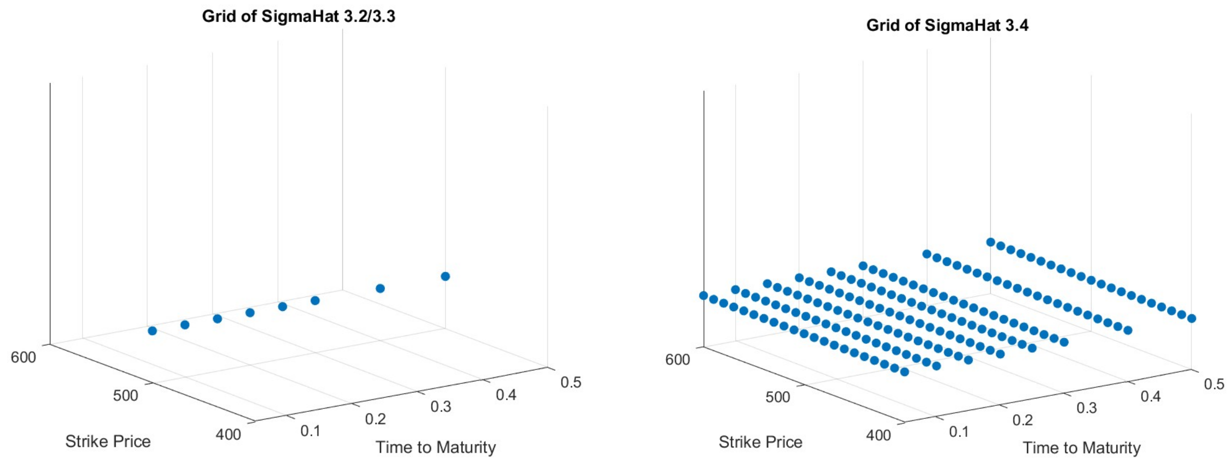


Figure 8: Market vs Model volatility

By incorporating a novel strike-dependent parameter Beta, we can effectively augment the model's complexity. In fact, $\hat{\Sigma}$ can now accommodate $N = 8 \cdot 21 = 168$ distinct values, significantly expanding the model's flexibility.



(a) grid without β

(b) grid using β

Figure 9: $\hat{\Sigma}$ grid

The error has been substantially reduced to 0.24$ as $\hat{\Sigma}$ is now freely adaptable within the entirety of the grid. The volatility surface generated by the model now closely resembles the observed volatility surface.

4 Option Pricing Using Calibrated Model

To price a down-and-in call option with maturity of 6 months, strike $K = 500$, and barrier $L = 450$, we employ the HJM models calibrated in the third section.
We start our analysis by considering the Payoff of a down-and-in call option:

$$\text{Payoff}(T) = \begin{cases} \max(S_T - K, 0) & \text{if } \min(S_t) \leq L \; \forall t \in [0, T] \\ 0 & \text{else} \end{cases} \tag{9}$$

Then, for each of the models that were calibrated, we use this formula to generate multiple paths for our forward price:

$$\frac{dF(t)}{F(t)} = (\Sigma_1(t) dW_1(t) + \Sigma_2(t) dW_2(t))\beta(K), \qquad \text{K} = 500 \tag{10}$$

That once discretized, it becomes:

$$F(t+1) = F(t) \cdot (1 + (\Sigma_1(t)\sqrt{dt}Z_1(t) + \Sigma_2(t)\sqrt{dt}Z_2(t)) \cdot \beta(K)) \tag{11}$$

with $dt = \frac{T}{N}$ (N is the monitoring period, chosen to be daily, hence equal to 252) and Z_i being a realization of two independent standard normal variables.
Having introduced this framework, we proceeded to price the options using Montecarlo method, obtaining:

There is quite a big difference in price between the first three models and the last one, but this was to be expected: in fact, the realized volatility for the first three models is around 65%,

Model	Price
Model 3.1	43.98
Model 3.2	40.67
Model 3.3	40.82
Model 3.4	8.34

Table 3: MC price for the 4 different models.

while the last one is in the neighbourhoods of 29%.
If we were to trust one of the four models, we would trust the last one, due to the fact that the model is able to encapsulate the market volatility surface better that the other ones.
In order to further more justify this point, we decided to use a closed formula to price our barrier option.
We start by considering the closed formula for a down-and-in call option:

$$C(T, K, B, S_0) = S_0(\frac{L}{S_0})^{2\lambda}N(y) - Ke^{-rT}(\frac{L}{S_0})^{2\lambda-2}N(y - \sigma\sqrt{T}) \tag{12}$$

with

$$\lambda = \frac{r + \frac{\sigma^2}{2}}{\sigma^2} \quad y = \frac{\log(\frac{L^2}{S_0 K})}{\sigma\sqrt{T}} + \sigma\sqrt{T} \tag{13}$$

where T is the time to maturity, r is the risk free rate, K is the strike price of the call option, L is the barrier and N(x) being the cumulative distribution function of a standard gaussian.
Once we had this equation in order to use this formula in our case, we reasoned in the following way:

- The dynamics given by formula (10) are equivalent to the ones given by a Black-and-Scholes model, from whom the previous formula was derived, with zero drift and diffusion term equal to the sum of the two Brownian motions with modified volatility given by Σ_k. This is true in our case, since the two Brownian motions are assumed to be independent, hence the sum of two independent gaussian random variables is given by a gaussian random variable whose mean and variance are given by the sum of the factors' mean and variance.

- We had to change this formula taking into account that the underlying in this case is the forward price:

  - We start by considering Black-76 formula to price a call option in case a forward F is the underlying:

    $$C = e^{-rT}\left[FN(d_1) - KN(d_2)\right] \tag{14}$$

    Where:

    $$d_1 = \frac{\ln\left(\frac{F}{K}\right) + \left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$
    $$d_2 = d_1 - \sigma\sqrt{T}$$

  - By considering this formula, we do the same steps that are required to prove the closed formula for the barrier option, exchanging the Black-and-Scholes formula with Black-76'. In fact, we consider the absorbed process:

    $$F_L(t) = F(\min(t, \tau_L)),\ where\ \ \tau_L = inf(t >= 0)|F_t = H \tag{15}$$

of which we know the density, as given in Proposition 18.3 of Arbitrage Theory in Continuous Time, Bjork.

– We finally get the formula:

$$C(T,K,B,F) = Fe^{-rT}(\frac{L}{F})^{2\lambda}N(y) - Ke^{-rT}(\frac{L}{F})^{2\lambda-2}N(y-\sigma\sqrt{T}) \tag{16}$$

with

$$\lambda = \frac{r+\frac{\sigma^2}{2}}{\sigma^2} \quad y = \frac{\log(\frac{L^2}{FK})}{\sigma\sqrt{T}} + \sigma\sqrt{T} \tag{17}$$

As σ, we used the sigma-hat that were computed by our models.
We used our closed formula for two things.
Initially, we wanted to check whether our Montecarlo pricing algorithm was accurate enough, obtaining the final result given by this figure:

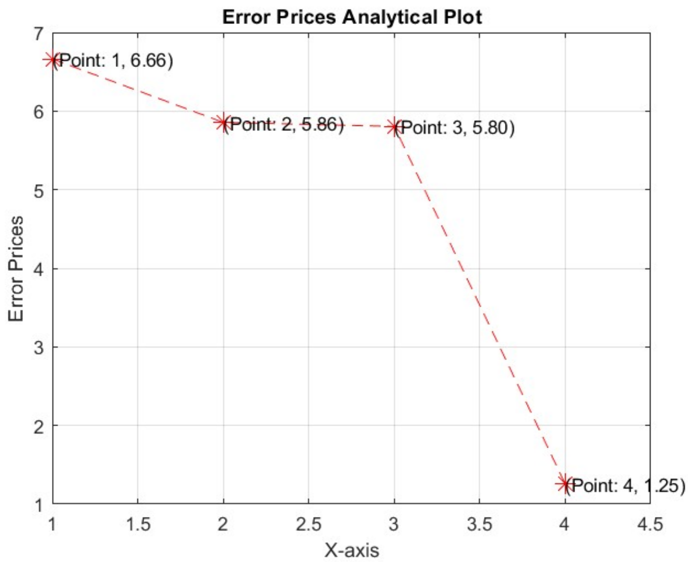


Figure 10: MC vs Closed-form solution

As we can see, the more complex the model, the better the Montecarlo simulation becomes. In case of the fourth model, the price difference is lower due to the fact that the contract price is also lower, around 8.
Then, we finally wanted to check the effectiveness of our model versus the market, that is, checking the price difference in the analytical formula considering model volatility versus market volatility..
The previous volatility surface could suggest that we will get a high error for the first three models, while the error in the case of the fourth model would be lower.

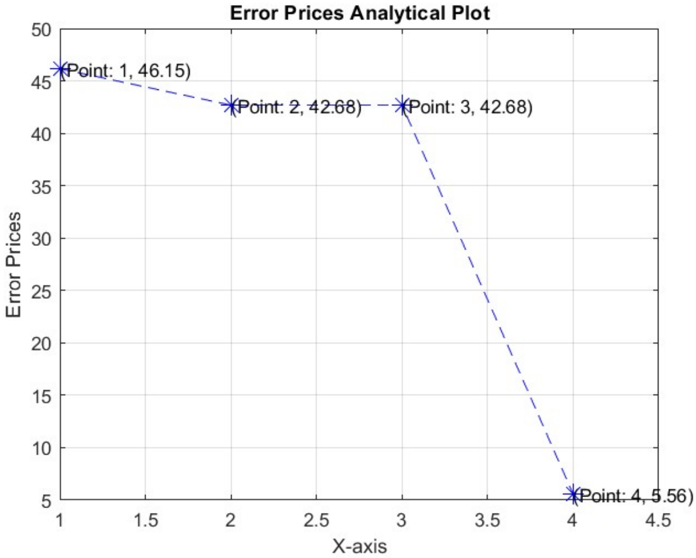


Figure 11: Market vs Model

This is indeed what we got, mainly because the realized volatility for the last model is around 29% as we said before, while the market realized a volatility of less than 23%.

5 Conclusion

While the HJM model offers easy implementation and quick calibration due to its assumption of normality in log-returns, its application in risk metrics and derivative pricing has limitations. Market complexities like jumps, volatility smiles, and clustering are neglected, prompting the need for stochastic volatility and jump dynamics, which complicate calibration, especially in a multivariate context.

Drawbacks include reliance on historical futures prices for calibration, potentially leading to inaccurate replication of quoted options. Illiquid option power markets often necessitate historical calibration, and the model inherits spot log-return characteristics from the forward market, lacking direct consideration of spot quotations.

Despite these limitations, the HJM framework is widely used for simulating futures and spot prices, serving as a benchmark. Future research may explore innovative approaches, such as incorporating jumps or stochastic volatility in a multi-dimensional setting, balancing mathematical and numerical tractability in academia and industry. In light of our analysis, it's safe to say that the market data, as mentioned earlier, could have benefited from a more complex model, as used in the last part of section 3, or perhaps by integrating more factors instead of the two considered.