Portfolio Optimization Project

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Abstract

This report aims to construct portfolios using diverse allocation strategies and evaluate their performance, focusing on the constituents of the $S\&P100$ index. Two performance analyses are carried out: the first covering the period from May 2021 to May 2022, and the second maintaining the weights derived during the initial period throughout May 2022 and May 2023. Additional analyses are proposed, including backtesting with rebalancing, and a comparison with a real $S\&P100$ ETF.

Point 1

The first model relies on Markowitz Portfolio Theory. Using the canonical Mean-Variance analysis of portfolios (where the optimization was carried out numerically, to satisfy our long-only constraint) we generated the portfolio frontier. From this set of portfolios we picked two particular portfolios:

1. **Portfolio A**: the one minimizing the Varince on the observed period. Mathematically, the optimization carried out by portfolio frontier is the following, first defining the set of long-only portfolios (LOP), then defining the portfolio frontier (PF) and finally the minimization:

$$LOP = \{\mathbf{w} \in \mathbb{R}^n : 0 \leq \mathbf{w}_i \leq 1 \ \forall i = 1, \dots, n\}$$

$$PF = \{\mathbf{w} \in \text{LOP} : \forall \mathbf{v} \in \text{LOP s.t. } \sigma(\mathbf{w}) = \sigma(\mathbf{v}),\ \mu(\mathbf{w}) \geq \mu(\mathbf{v})\}$$

$$\mathbf{w_{MVP}} = \{w : \min_{\mathbf{w} \in \text{PF}} \ \sigma^2_{Port}\}$$

2. **Portfolio B**: the one maximizing the Sharpe Ratio, the optimization carried out is the following

$$\mathbf{w_{SR}} = \{w : \max_{\mathbf{w} \in \text{PF}} \ \text{SR}(\mathbf{w})\}$$

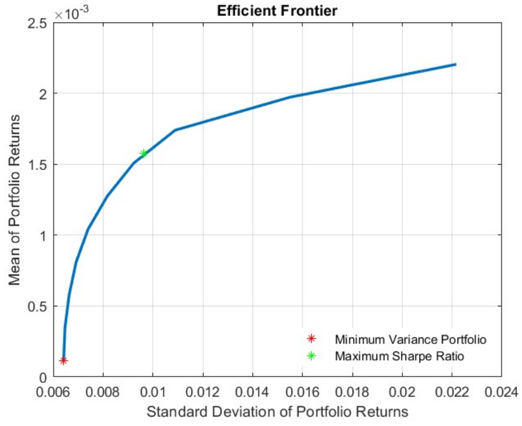
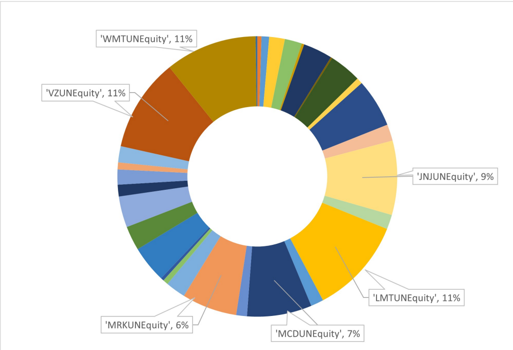
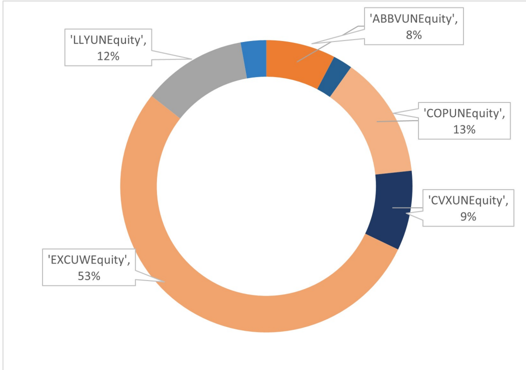


Figure 1: Portfolio frontier for 100 $S\&P$ assets

The constituents of our portfolios are illustrated in the figures below.



(a) Portfolio A weights



(b) Portfolio B weights

Figure 2: Portfolios composition

Point 2

We extend the analysis of the efficient frontier by imposing a set of additional constraints. These constraints are intended to refine our portfolio optimization process and align it with specific sector exposures.
The comprehensive set of constraints includes:

- The long-only portfolios constraint, i.e. $0 \leq \mathbf{w}_i \leq 1$
- The overall exposure of companies belonging to the *"Consumer Discretionary"* sector must be greater than 15%.
- The overall exposure of companies belonging to the *"Industrials"* sector must be less than 5%.
- The weights of companies belonging to sectors composed of fewer than 5 companies are constrained to be null.

Under these combined constraints, we proceed with the computation of the efficient frontier using the Mean-Variance analysis. The objective is to find portfolios that balance risk and return while adhering to the specified sector exposure requirements.

1. **Portfolio C** is constructed by minimizing the variance within the modified efficient frontier. The optimization problem is formulated as follows:

$$\min_{w \in PF} \sigma^2_{\text{Port}}$$

2. **Portfolio D** is constructed by maximizing the Sharpe Ratio within the modified efficient frontier.The optimization problem is formulated as follows:

$$\max_{w \in PF} \text{SR}(w)$$

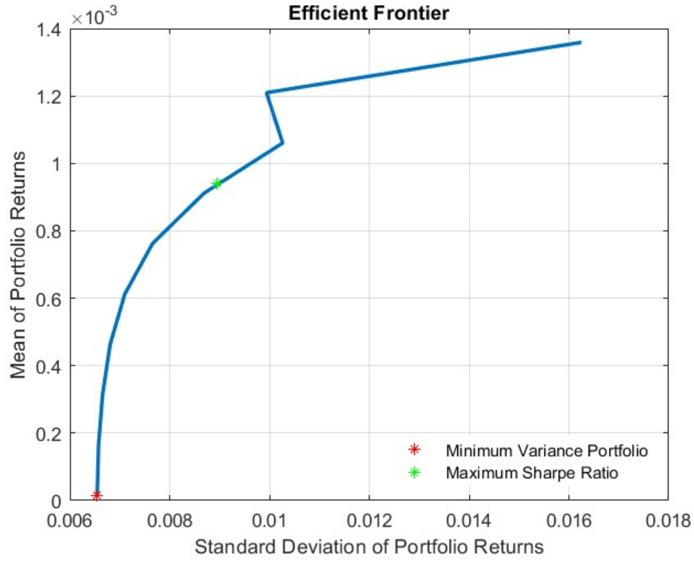
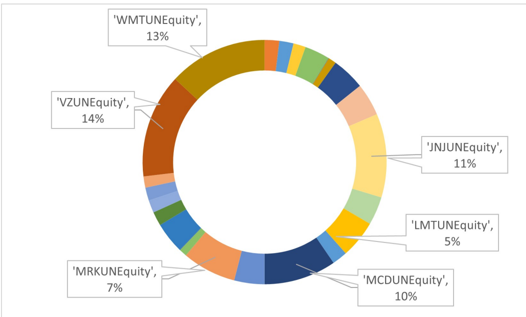
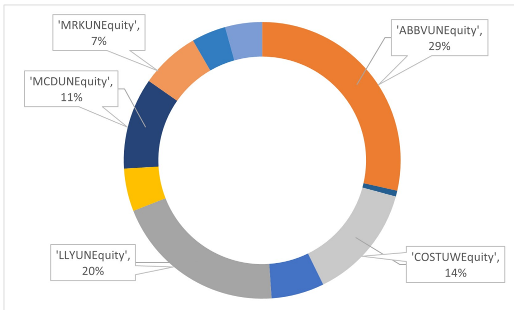


Figure 3: Portfolio frontier for 100 $S\&P$ assets with additional constraint

The constituents of our portfolios are illustrated in the figures below.



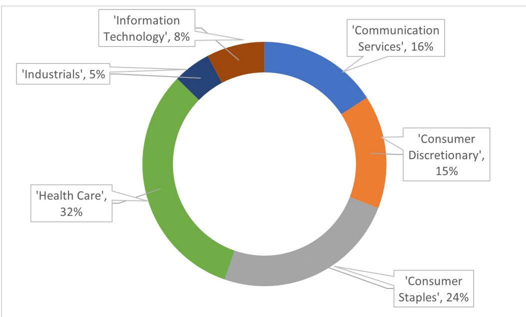
(a) Portfolio C weights



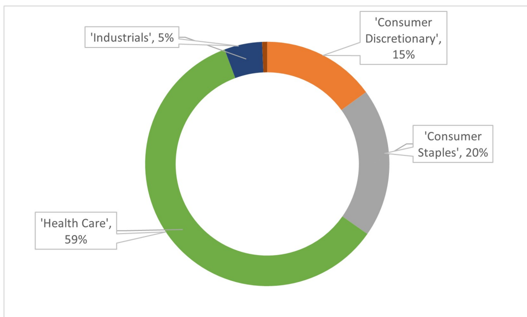
(b) Portfolio D weights

Figure 4: weights for the two previous portfolios

The upcoming figure illustrates the comprehensive exposure across various sectors, revealing the fulfillment of constraints.



(a) Portfolio C weights by sector



(b) Portfolio D weights

Figure 5: weights for the two previous portfolios by sector

Point 3

To enrich our analysis, we employ a resampling method to compute the efficient frontier. This technique involves generating multiple portfolios by randomly selecting different subsets of assets. The resulting figure showcases the diversity of portfolios and the associated risk-return profiles. We execute the resampling method of the point 1 and 2. Similar to the Point 1, we identify and save two specific portfolios under the resampling method: **Portolio E** is the MV Portfolio and **Potfolio G** is the Max SR Portfolio.

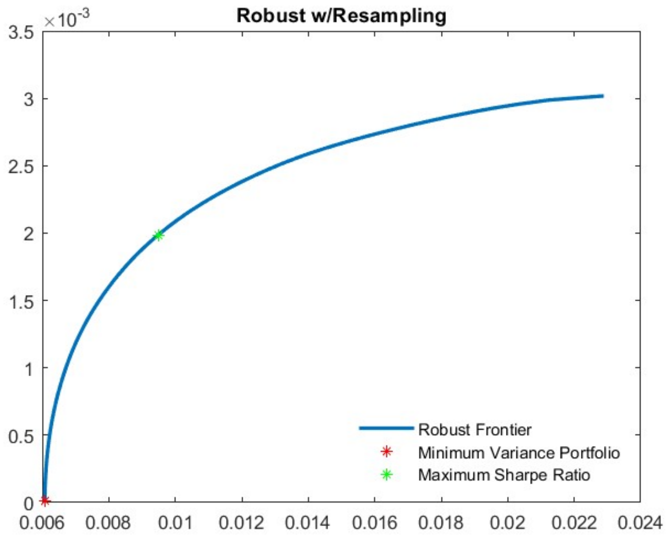
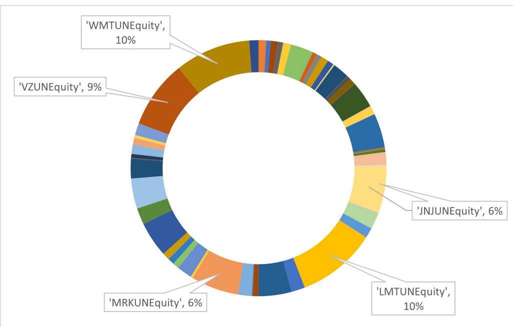
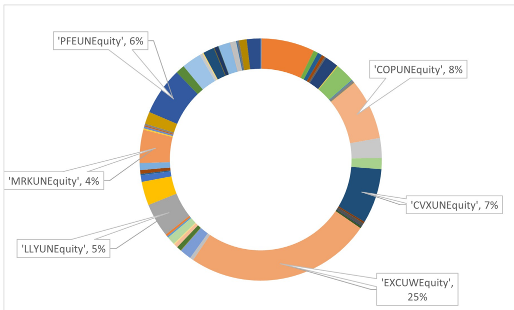


Figure 6: Robust Portfolio frontier for 100 $S\&P$ assets of the Point 1



(a) Portfolio E weights



(b) Portfolio G weights

Figure 7: Portfolios composition of Point 1 with resampling method

Similar to the Point 2, we identify and save two specific portfolios under the resampling method: **Portolio F** as the MV Portfolio and **Potfolio H** as the Max SR Portfolio.

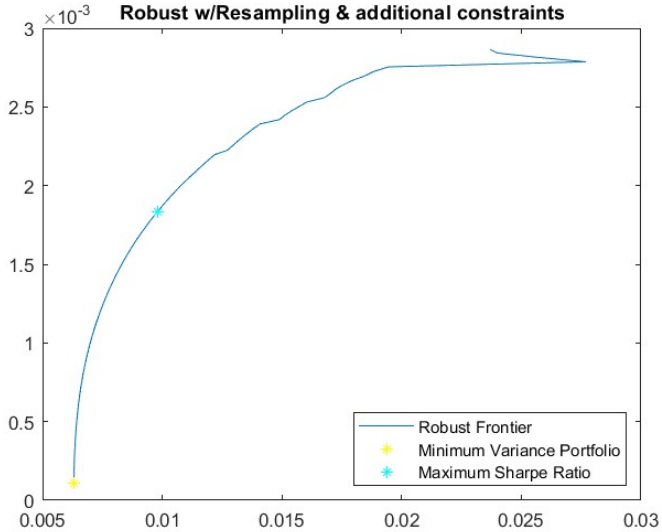
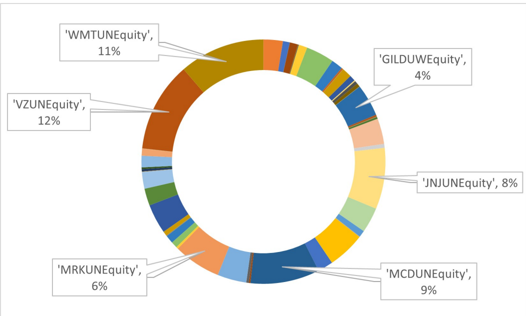
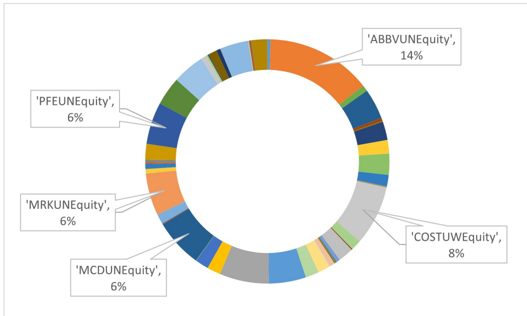


Figure 8: Robust Portfolio frontier for 100 $S\&P$ assets of the Point 2

The inclusion of these portfolios contributes to a more comprehensive understanding of the trade-offs between risk and return.



(a) Portfolio F weights



(b) Portfolio H weights

Figure 9: Portfolios composition of Point 2 with resampling method

Point 4

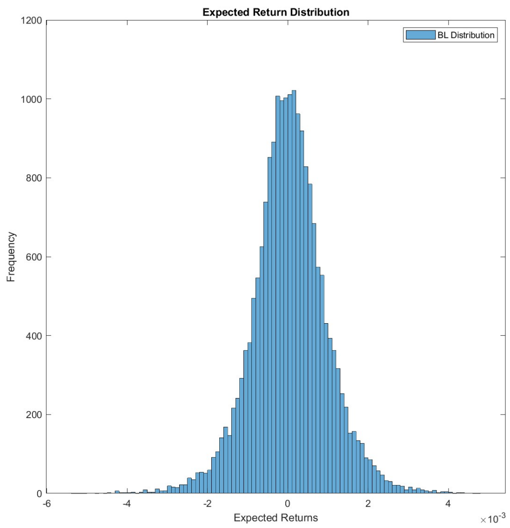
We now employ the Black-Litterman model to compute the portfolio frontier, taking a Bayesian approach to asset allocation. Specifically, it combines a prior estimate of returns (for example, the market-implied returns) with views on certain assets, to produce a posterior estimate of expected returns.
The market views we suppose are:

1. Expected annual return for companies in the **"Consumer Staples"** sector: **7%**.

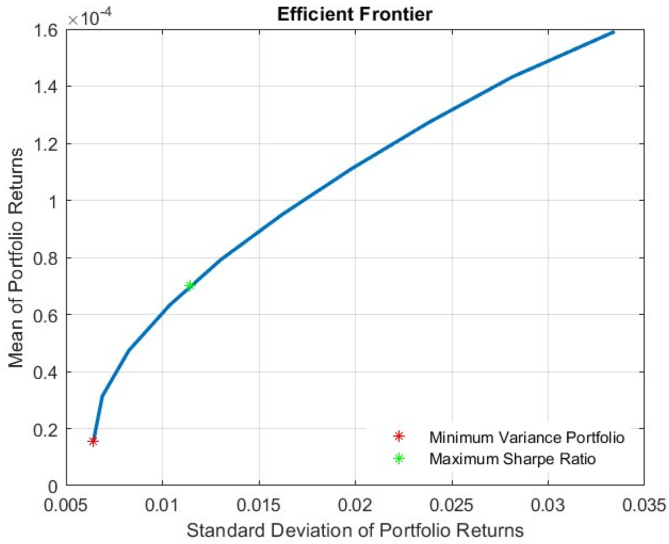
2. Expected annual return for companies in the **"Healthcare"** sector: **3%**.

3. Expected outperformance of **"Communication Services"** sector over **"Utilities"** sector by **4%**.

With these views, we calculate the adjusted expected returns for each asset and determine the efficient frontier under standard constraints.



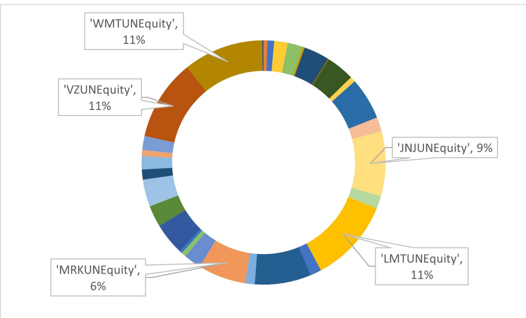
(a) Expected return distribution



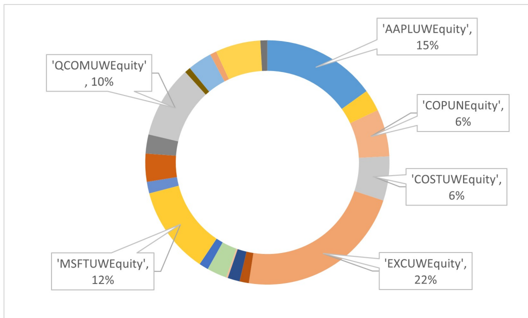
(b) Portfolio frontier for 100 $S\&P$ assets

Figure 10: Result with Black-Litterman

Two key portfolios are identified: **Portfolio I** as the Minimum Variance Portfolio with Black-Litterman and **Portfolio L** as the Maximum Sharpe Ratio Portfolio with Black-Litterman.



(a) Portfolio I weights



(b) Portfolio L weights

Figure 11: Portfolios composition with Black-Litterman

Point 5

At this point, we decided to compute the Maximum Diversified portfolio and the Maximum Entropy one under the following constraints:

1. The standard constraints from the points above.
2. For companies belonging to the *"Financials"* sector, weights have to be between 0.1% and 2%.
3. For companies belonging to the *"Industrials"* sector, weights have to be between 0.5% and 1%.

Firstly, we obtained **Portfolio M**, the maximum diversified one, by maximizing the Diversification Ratio DR given by the following formula:

$$\mathrm{DR}(w) = \frac{w^T \sigma}{\sigma_P}$$

On the other hand, **Portfolio N** was computed by maximizing the Entropy function

$$Entropy(w) = -\sum_{i=1}^{n} w_i \log(w_i)$$

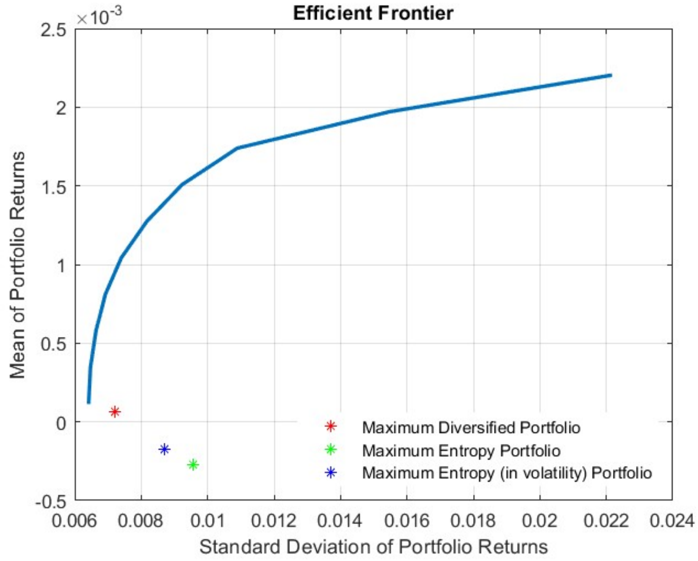
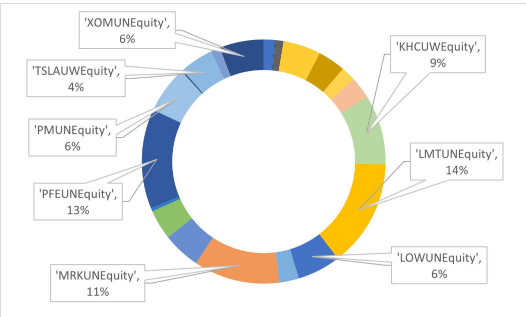
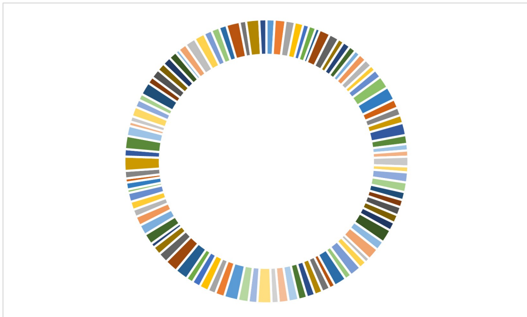


Figure 12: Portfolio frontier for 100 $S\&P$ assets



(a) Portfolio M weights

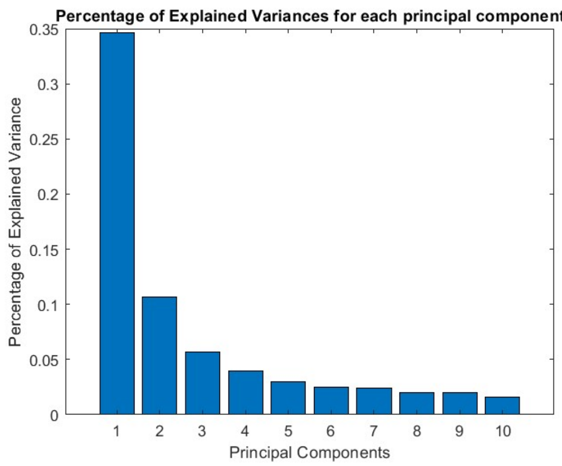


(b) Portfolio N weights

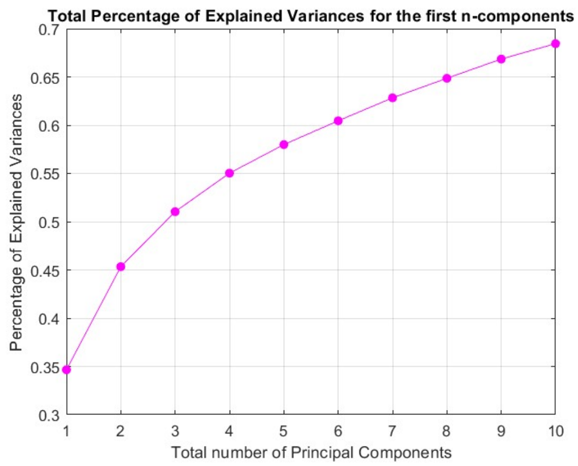
Figure 13: weights for the two previous portfolios

Point 6

Continuing our analysis, we tried to apply Principal Component Analysis (PCA) with 10 factors, aiming to get a portfolio (**Portfolio P**) that maximizes its expected returns while having a volatility lower or equal than $\sigma_{tgt} = 0.7\%$.
PCA allows us to identify uncorrelated factors that capture the most significant variation in the original data. These factors are linear combinations of the original variables..



(a) Percentage of the explained variance for each principal component



(b) Total Percentage of the explained variances for the first 10 components

Figure 14: Results PCA

The portfolio composition we get is the following one (Figure 15)

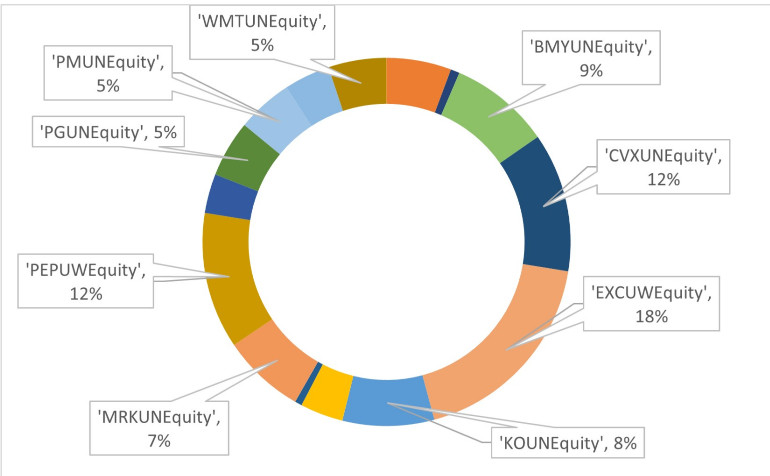


Figure 15: Portfolio P component

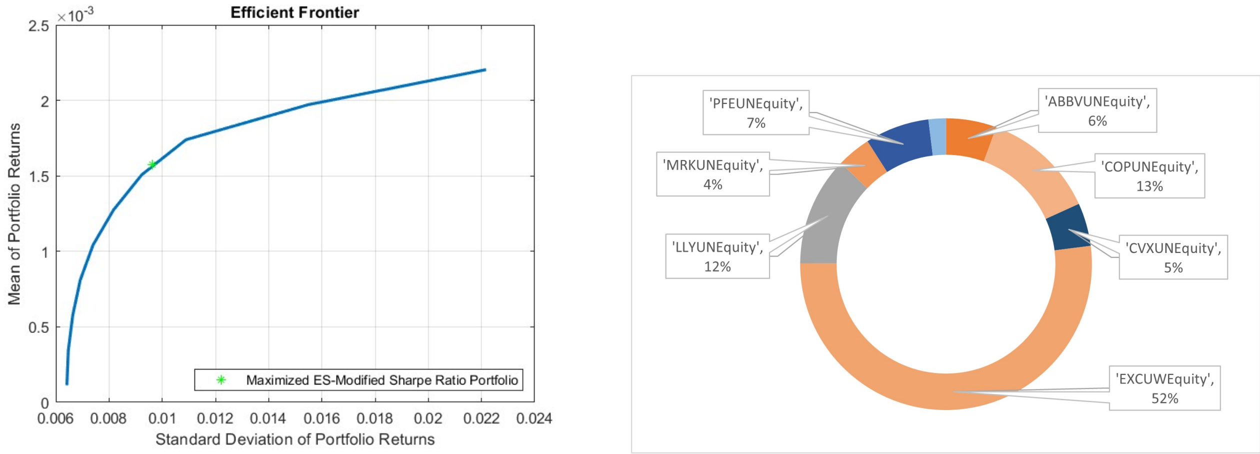
Point 7

Finally, the last portfolio we created, **Portfolio P**, is obtained using the Variance-Covariance method to maximize the ES-Modified Sharpe Ratio obtained with the following formula:

$$SR = \frac{ExpectedReturns}{ES^p}$$

where ES is the Expected Shortfall computed using p=0.05 with:

$$ES^p = \mu + \sigma \frac{\phi(\Phi^{-1}(1-p))}{p}$$



(a) Portfolio frontier for 100 $S\&P$ assets

(b) Portfolio Q component

Figure 16: ES-Modified Sharpe Ratio result

Point 8

In this section, we compared all the portfolios with reference to the time interval chosen for the optimization. We added the Equally Weighted (EW) portfolio to the mix, which is theoretically equally to the Max Entropy in asset weights portfolio.

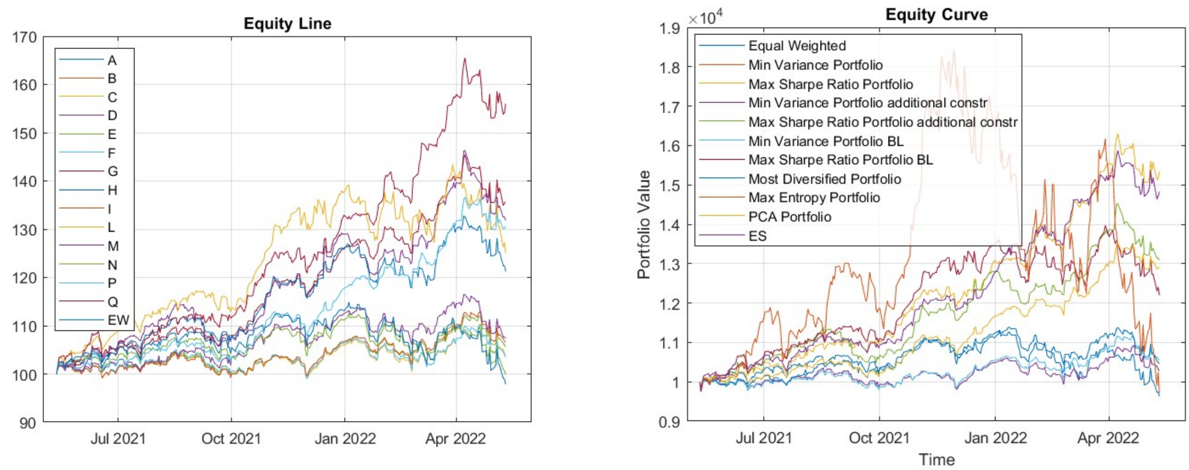
Table 1: Performance metrics

	AnnRet	AnnVol	Sharpe	MaxDD	Calmar
A	0.0631	0.1011	0.6240	-0.0563	1.1197
B	0.5372	0.1518	3.5390	-0.0757	7.0970
C	0.0358	0.1033	0.3466	-0.0627	0.5714
D	0.3054	0.1422	2.1472	-0.0994	3.0735
E	0.0540	0.1018	0.5314	-0.0582	0.9294
F	0.0328	0.1039	0.3156	-0.0614	0.5336
G	0.3423	0.1260	2.7152	-0.07222	4.7394
H	0.2043	0.1319	1.5482	-0.0862	2.3701
I	0.0630	0.1011	0.6234	-0.0564	1.1185
L	0.2408	0.1792	1.3436	-0.1276	1.8870
M	0.0711	0.1132	0.6279	-0.0778	0.9132
N	-0.0019	0.1367	-0.0140	-0.11377	-0.0168
P	0.2944	0.1131	2.6032	-0.0504	5.8362
Q	0.5372	0.1518	3.539	-0.0757	7.097
EW	-0.0208	0.1506	-0.1384	-0.1486	-0.1403

The Sharpe Ratios for portfolios B and Q consistently emerge as top performers across multiple metrics, demonstrating their effectiveness in achieving high returns with controlled volatility. This result aligns with expectations, as these portfolios are designed to maximize the ratio between return and volatility without additional constraints. They once again excel in the Calmar Ratio metric (7.0970 and 7.097, respectively), reinforcing their ability to deliver strong risk-adjusted returns relative to their maximum drawdown. The emphasis on optimizing the Sharpe Ratio makes these portfolios attractive for investors seeking a balance between risk and return.

On the other hand, portfolio N, which maximizes entropy in asset volatility, stands out as the worst performer. Entropy maximization in this context implies a strategy that diversifies the volatility of assets but, in this case, it comes at the cost of a negative return. This outcome highlights the trade-off between diversification and returns, emphasizing that a purely volatility-focused strategy may not always lead to favorable investment outcomes.

It's remarkable that the Equally Weighted (EW) portfolio also shows suboptimal performance across all metrics, including experiences the deepest maximum drawdown (-14.86%) and negative returns. The negative values in both return and Sharpe Ratio indicate that a simple equal-weighted strategy may not be the most effective in this specific market environment or short time period.



(a) Equity Line

(b) Equity Line with rebalancing

Figure 17: Final result

In the figure on the right we proceeded to do the backtesting using the proprietary function in MatLab, which gives also the possibility of rebalancing the portfolios, we chose to rebalance the portfolios once a month.

Portfolio B with monthly rebalancing emerges as a strong performer, delivering high total returns, favorable risk-adjusted performance (Sharpe ratio), and relatively lower volatility. Portfolios with additional constraints also demonstrate robust performance across multiple metrics.
As in the previous analysis, the Equally weighted portfolio result weak in this specific market condition, showing a negative total return.

Portfolio N with monthly rebalancing tends to accentuate market conditions characterized by increased volatility. This inclination results in higher returns during bullish market phases but also leads to more significant drawdowns during adverse periods.

Ultimately, it is essential to take into account practical implications, as portfolios characterized by higher turnover, such as Portfolio M and Portfolio N may experience elevated transaction costs. This consideration is significant because higher turnover can lead to more frequent buying and selling of assets, potentially resulting in higher expenses related to trading activities.
In addition to evaluating the performance of our optimized portfolios, it is insightful to compare them against a real existing portfolio. For this purpose, we consider the iShares S&P 100 ETF (OEF). The performance metrics for OEF are presented in Table (2).

	AnnRet	AnnVol	Sharpe	MaxDD	Calmar
S&P 100 ETF	-0.022184	0.18252	-0.12155	-0.17999	-0.12325

Table 2: Performance Metrics of iShares S&P 100 ETF (OEF)

We observe that portfolios B and Q outshine the chosen benchmark, underscoring their appeal to investors seeking enhanced risk-adjusted returns.

Part B

In this final section, we evaluated our portfolio performances in the next time period, from May 2022 to May 2023. What we can see is that on average, portfolios perform worse than the last time interval, and that is something we expected (look at Table 4).
It is interesting to note that the SharpeRatio maximizing portfolios are the ones achieving worse performances: this is something that has to be expected, mainly because those portfolios can suffer from "overfitting" to market conditions, given that the overall market outlook has changed (Table 3).
The best performing portfolios, at this time, is surprisingly the portfolio D. This may be due to the fact that the additional constraints made sure that we wouldn't invest in stocks that performed badly during the year considered.
It's also interesting to note how the two worst performing portfolios, N and EW, are now achieving positive returns:

- in case of the EW Portfolio, we can generally say that the performances depend on the average return of the market. So, while last year the market had negative returns on average, in this next period we are experiencing somewhat of a bull market;
- when it comes to the portfolio N, we must look at this table:

Table 3: Comparison between the two periods

	Sum of the mean of returns	**Sum of the variance of returns**
First period	-0.0276	0.0329
Second period	0.0008	0.0407

Given that this portfolio maximizes entropy in asset volatility, it makes sense that in a period of increased volatility there is a chance to obtain higher returns, with the drawback of having one of the highest drawdowns.

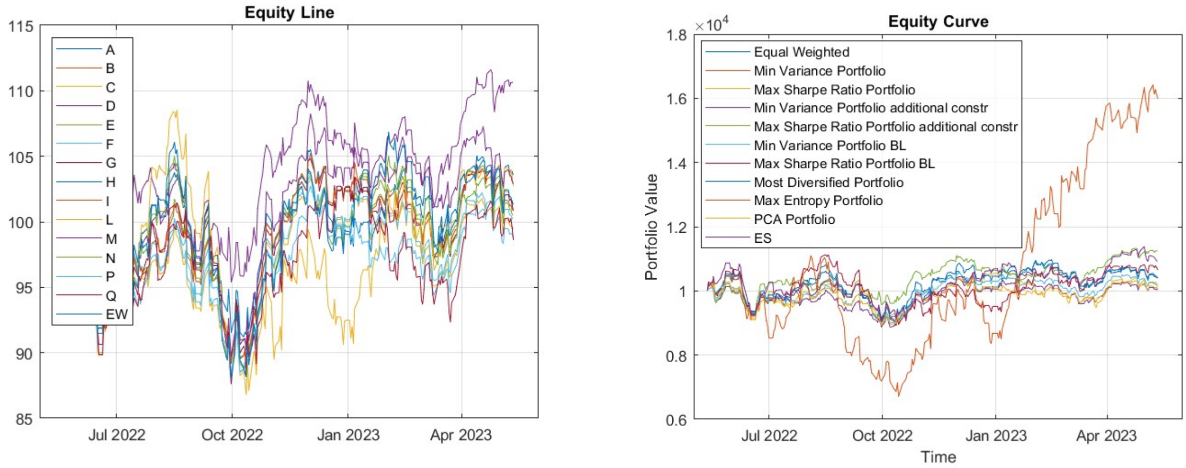
Table 4: Performance metrics in the new period

	AnnRet	**AnnVol**	**Sharpe**	**MaxDD**	**Calmar**
A	0.0081	0.1430	0.0567	-0.1231	0.0658
B	-0.0136	0.1953	-0.0698	-0.1805	-0.0755
C	-0.0055	0.1397	-0.0396	-0.1241	-0.0445
D	0.1029	0.1618	0.6362	-0.0879	1.1712
E	0.0126	0.1449	0.0872	-0.1239	0.1020
F	0.0015	0.1412	0.0107	-0.1247	0.0121
G	0.0077	0.1666	0.0462	-0.1406	0.0547
H	0.0341	0.1686	0.2027	-0.1210	0.2823
I	0.0080	0.1430	0.0562	-0.1231	0.0652
L	0.0264	0.2330	0.1133	-0.2004	0.1317
M	0.0274	0.1670	0.1639	-0.1414	0.1937
N	0.0042	0.1837	0.0227	-0.1625	0.0257
P	-0.0129	0.1505	-0.0859	-0.1311	-0.0986
Q	-0.0136	0.1952	-0.0698	-0.1805	-0.0755
EW	0.0127	0.1999	0.0633	-0.1711	0.0740

Furthermore, by looking at this table, we can notice that the factual hypothesis that were theorized in the Black-Litterman approach were not satisfied by the market conditions: generally, both portfolios I and L suffered lower returns, nevertheless greater than 0.
If our hypothesis were to be verified, then we would have observed better performance from those two portfolios.

On top of that, the portfolio L is the one having the highest volatility and drawdown, meaning that the stocks to which was given more weight in the calibration phase suffered from increased volatility.

To conclude our analysis, we take a look to portfolios B and Q: from those results, we can discern that in the period considered the two methods used to compute the SharpeRatio have the same behaviour.



(a) Equity Line with the following period

(b) Equity Line with rebalancing

Figure 18: Final result

We procedeed to do the rebalancing also for this time period, and what we obtained is quite surprising, regarding portfolio N: it achieves massive returns and drawdowns, it may be due to the fact that this next period considered is more volatile, so, as we pointed out before, we are experiencing better performance in portfolio N.
Going in-depth with this analysis, this portfolio is achieving the highest SharpeRatio (1.16) of all the rebalanced portfolio, hedging-out the portfolio D by almost 50%.

The second best performing portfolio, in agreement to what we experienced in the previous point, is portfolio D.
In order to discover the underlying motives to this behaviour, we noticed that those sectors that were excluded from the constraints realized negative returns in this last period, while the ones to which was given more weight experienced a year of higher returns.

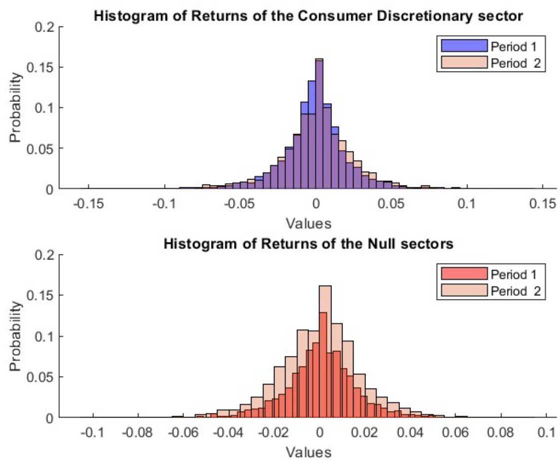


Figure 19: Sector-wise comparison of returns

There are two main takeaways from this plot:

- stocks in the consumer discretionary sector have had better returns in the last period;
- the distribution of the "null" sectors in the last period is more skewed on the left than before, indicating lower returns on average.

There is something quite interesting happening with portfolio C, that is the worst performing portfolio, despite obtaining a positive return with rebalancing (0.09 Sharpe Ratio): this phenomena may be related to the fact, in periods of increased volatility, minimizing it, with the additional constraints, may lead to very low return.
Furthermore, we have the opportunity to draw a comparison with an actual ETF, such as iShares S&P 100.

	AnnRet	AnnVol	Sharpe	MaxDD	Calmar
ETF	0.056363	0.22069	0.25539	-0.19339	0.29145

Table 5: Performance Metrics for ETF

As expected the performances are way better with respect to our portfolios for obvious reasons,this superior performance can be attributed to the ETF's diverse holdings and strategic allocation across the S&P 100 index, providing a more stable and optimized investment structure. Consequently, the notable outperformance is a result of the ETF's ability to capture the overall market trends and mitigate individual stock risks inherent in our portfolios which, as said before, are calibrated considering a different period.

Conclusion

Having analyzed a number of portfolios, we can conclude that the perfect strategy **does not** exist: it is all profoundly linked to the market conditions, to the investor's risk adversion and one's skillset.
One thing is certain: rebalancing portfolios surely improves the performance of whatever strategy, mainly because doing so the weights are adapted to the newfound market conditions.
In fact, every single portfolio, in the rebalancing stage, has outperformed their "buy and hold" version.
Obviously, in this case there are two drawbacks, that in this analysis weren't included:

- one major factor is given by transaction costs. Even a few percentage points paid in fees, commissions and buy-sell cost could drastically change the perspective on the rebalancing argument, making it less advantageous;

- other factors include market timing risk, tax implications and increased need for monitoring. In some way, frequent rebalancing could lead to less flexibility in the asset allocation, making it more sensible to market movements. The need of additional monitoring constitutes also a cost, while in some states an excessive rebalancing could lead to unwanted capital gain taxes.