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Recovery Risk

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Financial Engineering final project

Abstract

This study updates Moody's Loss Given Default (LGD) research for bank loans, in particular we look to secondary market price quotes of bank loans one month after the time of default - allowing markets to process the default news and revalue the debt - with the aim of analyze the risks associated with a financial institution's credit portfolio considering recovery risk on top of default and migration risk.

The aim of this project is to implement the MC simulation with at least 100,000 scenarios in the case of stochastic recovery rate and compute the corresponding VaR.
In order to do that we try to calibrate the distribution of the recovery rate to rating agency data, then we simulate the stochastic recovery rate through the Credit Capital Model and at the end we valuate the Credit portfolio VaR by varying the different parameters of the model.

The project focuses on developing a comprehensive framework to assess, monitor these risks, ultimately enhancing the institution's risk measure capabilities and overall financial stability.

Contents

Chapter 1

Defaulted bank loans distribution calibration

The US syndicated loan market is a great market continuously growing, this trend benefits investors who gain diversification by having access to different instrument type relative to public debt. This trend also shifts the default risk management burden from the originating institutions to investors.
One of the main problem in default studies is that there is no assurance of capturing information on all defaulted bank loans and no study has done so. In effect, this somewhat skews our sample towards larger obligors. By their nature, bank loans are a private contract between the borrower and the lending institution. Exceptions to this include syndicated loans, and separately, those loans packaged into CBOs, which become more public. This study continues Moody's analysis of the bank loan market to the benefit of investor risk management understanding.

1.1 Dataset description

From the paper Gupton (2000) we start our analysis by calibrating a beta distribution to the empirical distribution of defaulted bank loans (Sr. Secured and Sr. Unsecured) and discuss the similarities and differences of the two distributions. In particular the above data are set out in more detail below.

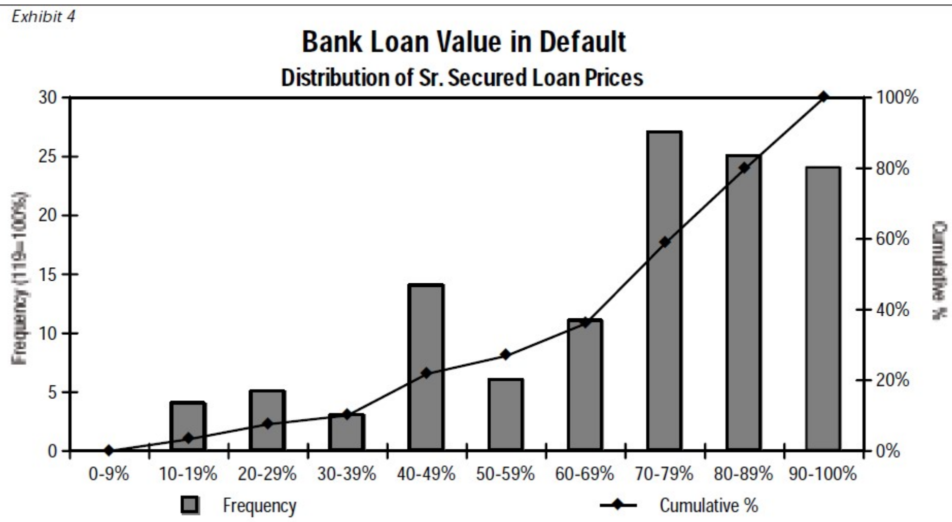


Figure 1.1: Distributions of secured bank loan valuations in default - both a histogram of recovery rates and a cumulative distribution curve.

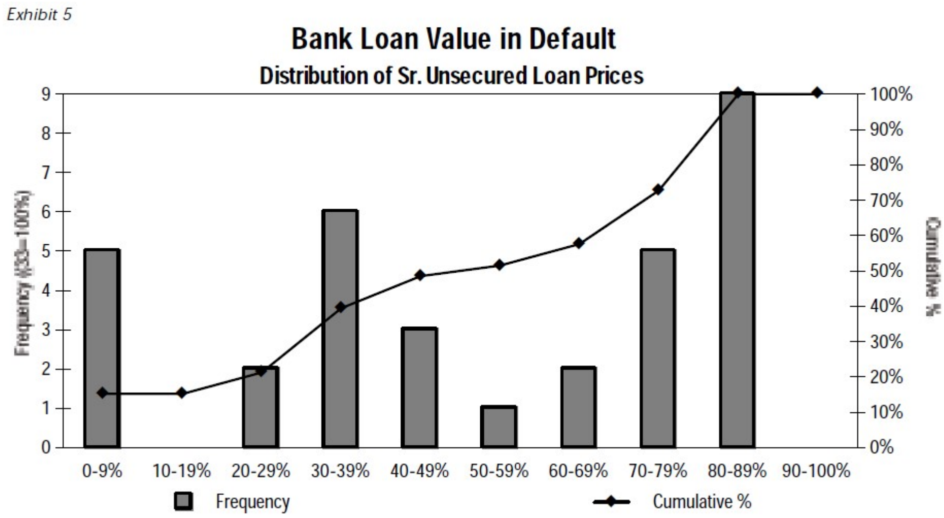


Figure 1.2: Distributions of unsecured bank loan valuations in default - both a histogram of recovery rates and a cumulative distribution curve.

Exhibit 6

Descriptive Statistics for the Time to Default Resolution

Bank Loans	Count	Average	Median	Maximum	10th Percentile	Minimum	Standard Deviation
Sr. Secured	119	$69.5	$74.0	$98.0	$39.2	$15.0	$22.5
Sr. Unsecured	33	$52.1	$50.0	$88.0	$5.8	$5.0	$28.6
Long Term Public Debt (of these same Bank Loan Borrowers)							
Sr. Secured	6	$59.1	$49.0	$98.5	$30.0	$0.1	$32.6
Sr. Unsecured	51	$45.1	$44.0	$104.8	$16.0	$0.5	$25.7
Sr. Sub	55	$29.4	$24.0	$98.0	$4.0	$0.5	$23.6
Sub	32	$29.1	$29.3	$87.5	$4.5	$0.5	$20.6
Jr. Sub	5	$10.8	$12.5	$20.8	$3.7	$1.5	$7.2

Figure 1.3: Tabulation of additional statistics for the bank loans illustrated in the previous figures

In this project we are going to use only the data from 1.3 since the complete data of the other figures are very difficult to download from the Moody's website for non-financial institutions. Actually, we could have done some reverse engineering from the figure in order to obtain the data but we chose the "conservative" way.

1.2 Beta distribution

The Beta distribution is a versatile probability distribution that could be used to model probabilities in different scenarios. In our case it is useful since it is defined on the interval $[0, 1]$ and obviously it's scalable on the interval $[0, 100]$. From the data set (Figure 1.3) we have the average and the standard deviation and so we use the method of moments in order to estimate the parameters α and β.

We use the first two moments as follows:

$$\hat{\alpha} = \left(\frac{\bar{x}(1 - \bar{x})}{s^2} \right) \bar{x}$$

$$\hat{\beta} = \left(\frac{\bar{x}(1 - \bar{x})}{s^2} \right) (1 - \bar{x})$$

We get:

Secured Loan Prices distr.

$$\hat{\alpha_1} = 2.2151 \quad \hat{\beta_1} = 0.9721$$

Unsecured Loan Prices distr.

$$\hat{\alpha_2} = 1.0686 \quad \hat{\beta_2} = 0.9824$$

1.3 Similarities of the two betas

The summary statistics are the mean, standard deviation, index of skewness and index of kurtosis (Table 1.1).

Table 1.1: Betas properties

	Secured	Unsecured
α	2.2151	1.0686
β	0.9721	0.9824
$\mathbb{E}[X]$	0.6950	0.5210
$Var[X]$	0.0506	0.0818
mode	1.0000	1.0000
skewness	-0.5493	-0.0519
kurtosis	-0.4081	-1.1816

First of all we can observe from the pdfs of the betas (Figure 1.4) that the mode of the two distributions, i.e the x corresponding to the peak of the density, is 1 and so they are unimodal distributions. We can also say that because the pdfs are monotonically increasing, hence the mode will always be on the right end.
The skewness of the beta distribution has a close form:

$$\gamma_1 = \frac{2(\beta-\alpha)\sqrt{\alpha+\beta+1}}{(\alpha+\beta+1)\sqrt{\alpha\beta}}$$

As we can see from the table the skewness is negative in both cases: as expected for secured loans the distribution is left centered, which means that the frequencies of high recovery rate is greater while for unsecured loans the skewness is close to zero and so the distribution is almost centered. The kurtosis of the beta distribution has also close form:

$$Kurt = \frac{6[(\alpha-\beta)^2(\alpha+\beta+1)-\alpha\beta(\alpha+\beta+2)]}{\alpha\beta(\alpha+\beta+2)(\alpha+\beta+3)}$$

A distribution with a negative kurtosis value indicates that the distribution has lighter tails than the normal distribution.

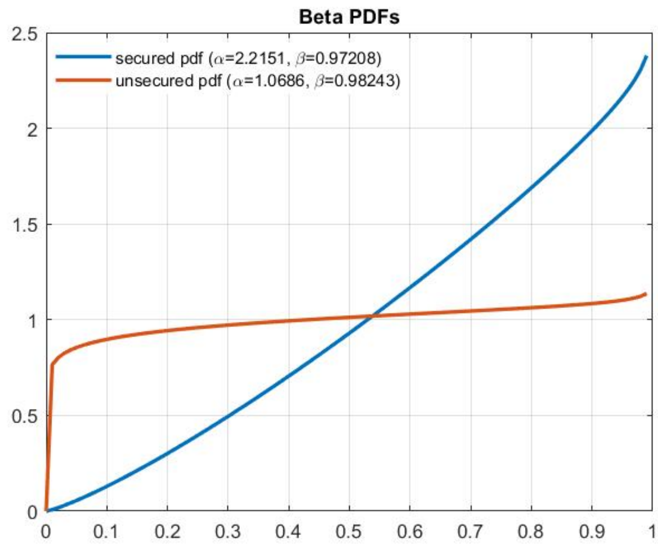


Figure 1.4: Betas PDF

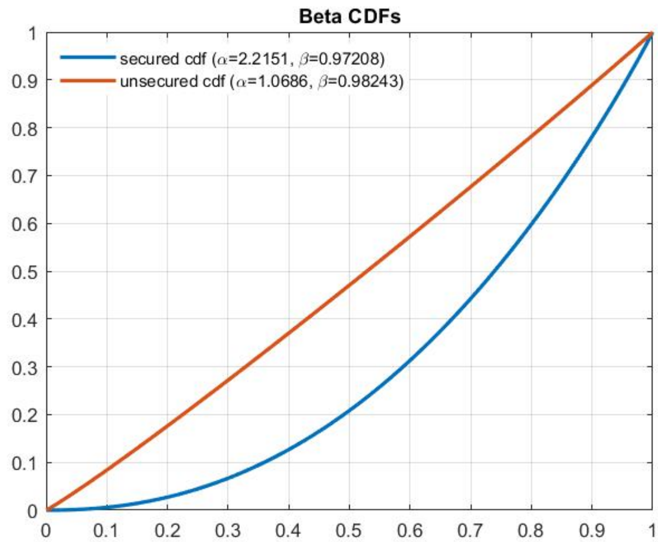


Figure 1.5: Betas CDF

Chapter 2

MC simulation with stochastic recovery rate

Since the default rate distribution does not resemble a concentrated distribution around a common average default rate (as predicted by independence), we now introduce a simple model which more closely resembles the empirical observations and considers default correlation.

2.1 Credit capital model

First of all we are going to evaluate the stochastic recovery rate through the credit capital model. The credit capital model (Frye (2000)) uses the conditional approach suggested by Michael Gordy and Christopher Finger. The distinctive feature of the credit model presented here is that an **economic downturn causes damage to the value of collateral**. When systematic collateral damage enters the credit model, the capital allocated to a well-collateralized loan rises by a multiple.
Taking collateral damage into account complicates a credit capital model. However, the results of the model can be well approximated by a function of expected loss alone.
The variables in the model depend on a systematic risk factor (X), a random variable representing the good years and bad years of the economy, when X exceeds zero, obligor j tends to prosper.
Suppose a face value of 1$ for each j obligor, at the end of a <u>one-year analysis horizon</u>, the value

of collateral is a random number who depends via term called "loading", q_j:

$$Collateral_j = \mu_j(1 + \sigma_j C_j)$$

$$C_j = q_j X + \sqrt{1 - q_j^2} Z_j$$

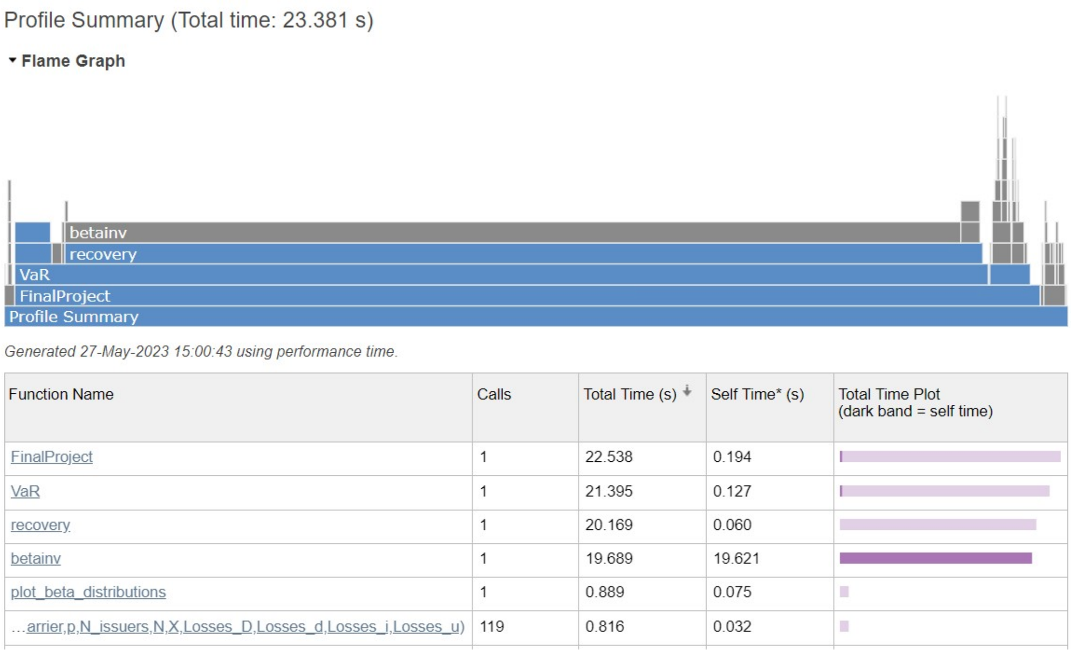
where X and Z_j have independent standard normal distributions.
The Recovery for each obligor is:

$$Recovery_j = \beta^{-1}(\phi(C_j))$$

The beta should have mean and variance of the previous distribution, in particular from now on we will analyse only the Secured loans type.

2.2 Recovery values simulation

We implement the simulation including the genaration of 100000 economic scenario for 119 obligors. We immediately notice that the main limitation of this simulation in MATLAB is the computational time required by the **betainv function** to run 100,000 scenarios x 119 obligors. If we try to run the recovery rate simulation in the base case credit portfolio VaR (for the computation methodology look at 3.2) the measure execution time will be approximatly as show in Figure 2.1



Function Name	Calls	Total Time (s)	Self Time* (s)	Total Time Plot (dark band = self time)
FinalProject	1	22.538	0.194	
VaR	1	21.395	0.127	
recovery	1	20.169	0.060	
betainv	1	19.689	19.621	
plot_beta_distributions	1	0.889	0.075	
…arrier,p,N_issuers,N,X,Losses_D,Losses_d,Losses_i,Losses_u)	119	0.816	0.032	

Figure 2.1: running time of the MATLAB code

To mitigate these limitations, it may be necessary to explore alternative computational methods or utilize more powerful computing resources. Indeed in Chapter 4.1 we will be looking for a fast approximation to the inverse CDF of the Beta distribution.

Chapter 3

Credit Portfolio VaR

The overall financial condition of the obligor, A_j, also depends on the systematic risk factor via a positive loading, p_j:

$$A_j = p_j X + \sqrt{1 - q_j^2} X_j$$

where X_j, the idiosyncratic variable i.e. a r.v. which affects the fortunes of obligor j and nothing else, has standard normal distributions iid independent of X and Z_j.

The parameter p plays an important role in the asset equation, it controls how much the systematic factor affects the set of issuers. If an economy has a low value of p (near zero), issuers have little connection to the state of the economy. In such an economy, each issuer finds its independent, idiosyncratic factor to be far more important than the common factor. Therefore the prosperity (or default) of one firm has little connection to the prosperity (or default) of other firms.

On the other hand, if an economy has a high value of p, each issuer is strongly tied to the general economy. An economy with a high level of p is a highly cyclical economy, and any year that X takes a low value will be a bad year for many issuers. Thus, an economy having a large value of p will have a severe credit cycle.

The correlation between two obligors depends on their loadings on the systematic risk factor X:

$$Corr\left(A_j, A_k\right) = Cov\left(p_j X + \sqrt{1 - q_j^2} X_j, p_k X + \sqrt{1 - q_k^2} X_k\right) = p_j p_k$$

Theorem 3.1 (Default condition in Credit capital model)**.** *In Credit Capital Model default occurs iff*

$$D_j = I_{A_j<\phi(PD_j)}$$

where PD_j represents the probability of default for obligor j.

If default occurs, the bank can recover, properly discounted and net of foreclosure expenses, no more than the loan amount:

$$Recovery_j = min[1, Collateral_j]$$

i.e.

$$LGD_j = max[0, 1 - Collateral_j]$$

The model allows collateral value to exceed exposure, but it does not allow the bank to recover more than exposure! If default occurs and collateral value exceeds exposure, the bank has no loss. For simplicity, the model includes losses due only to default and not due to downgrade or changes in pricing spreads. The amount lost to obligor j is then the product of the default event and the loss given default:

$$Loss_j = D_j \cdot LGD_j \tag{3.1}$$

Capital models are used by some banks to target the credit ratings they receive from rating agencies. A bank that targets an investment grade rating might wish to hold enough credit capital to absorb the loss that arises in 99.90% of Monte Carlo simulation runs. We may speak of a target solvency of 99.90% or of a target insolvency of 0.10%.

3.1 Migration Risk

Now, after analysing the impact of the stochastic recover rate, we must also study the impact of downgrade event in credit rating. For the migration risk we are referring to the following transition matrix (Assigment Case 3 RM)

Table 3.1: simplified rating transition matrix

	IG	HY
IG	52.81 %	46.19%
HY	35.00%	60.00%

We are considering the starting point to be at IG rate so we study four different scenarios:

1. Default, with probabilities simulated before
2. Down grade, with 46.19% probability
3. Unchanged grade, with 52.81% probability
4. Up Grade, with probability zero since we are in the highest grade

The face value of the loan one year from now, in the case of down grade and unchanged grade, are evaluated as follow:

$$FV = e^{-r_{1y}(T_{1y}-t_0)} \cdot 100$$

In the case of default we compute the face value as:

$$FV = Recovery \cdot 100$$

The loss value for each obligor is:

$$Loss_i = \frac{FV_i - \mathbb{E}[FV]}{N_{issuer}}$$

At the end the VaR will be:

$$VaR = Q_\alpha[TotalLosses]$$

with $\alpha = 0.10\%$

3.2 VaR Simulation

We evaluate the Credit Portfolio VaR with confidence level 99.9% ($\alpha = 0.01\%$) and 1y holding period, including migration risk. Any parameter required by the MC simulation are fixed (case 3 RM) except the standardized asset correlation, in particular we study the following case:

1. **Base Case**, we fix $p = q = 50\%$
2. **CreditMetrics assumption**, $p = 50\%$ and $q = 0\%$
3. **CreditRisk+ assumption**, In this case the recovery rate is not stochastic anymore, indeed $\sigma = 0\%$ and $p = 50\%$. The formula for the recovery rate is now:

$$Recovery_i = min\{Collateral_i, 0\}$$

4. **Credit Portfolio VaR in dependence of the sensitivity** of obligor's financial condition to systematic risk factor p in the range [0,0.5], keeping σ and μ as in the base case and assuming $q = p$

Results for the main set of simulations are displayed in Table 3.2.

Table 3.2: VaR Results

	Credit VaR
Base Case	11.90 %
CreditMetrics	5.61 %
CreditRisk+	5.21 %

The sensitivity parameter, q, controls the strength of the effect of the systematic factor on recovery. This role is parallel to the role of p in the asset equation; Note that

$$Corr(A_j, X) = p$$

and

$$Corr(R_j, X) = q$$

The effect of the recovery risk on VaR depends on the sensitivity of the obligor's collateral to systematic risk and the stochasticity of the recovery process. If the collateral value is highly correlated with systemic factors or if the recovery process is uncertain and unpredictable, recovery risk can significantly increase the potential loss in a portfolio.

3.2.1 Credit Portfolio VaR in dependence of p

We compute the the Credit portfolio Var w.r.t the following p grid:

$$p_{grid} = [0 : 0.1 : 0.5]$$

The results are shown in the following figure.

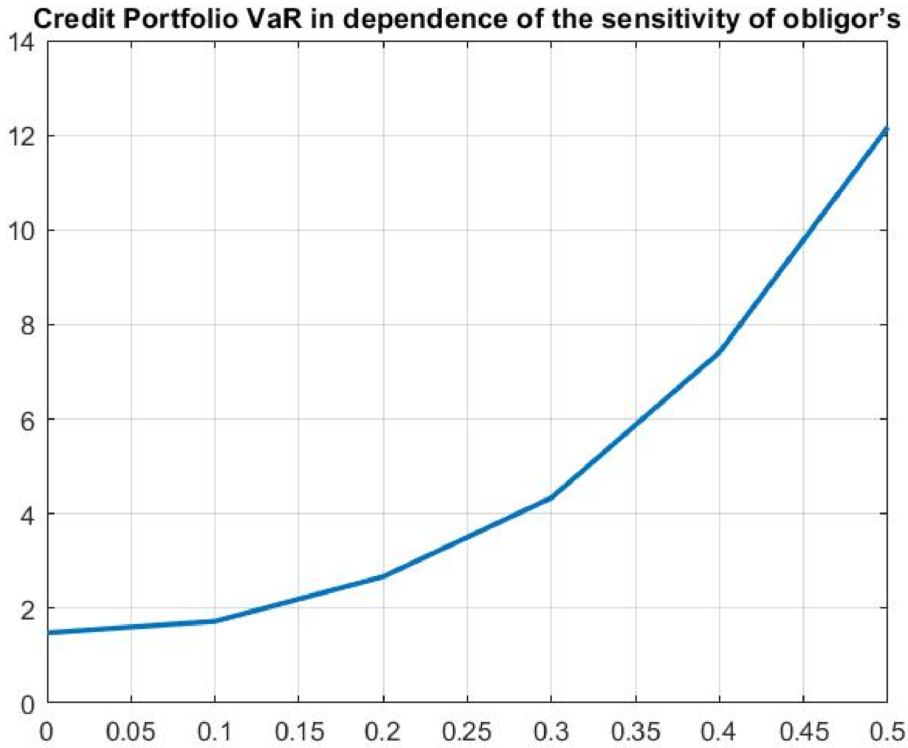


Figure 3.1: Chart of the Credit Portfolio VaR in dependence of the sensitivity of obligor's

Below a certain p the influence of the sensitivity of obligor seems to be negligible in the evaluation of the Credit VaR. We now focus our attention on the interval $[0 : 0.025 : 0.15]$ in order to show our thesis.

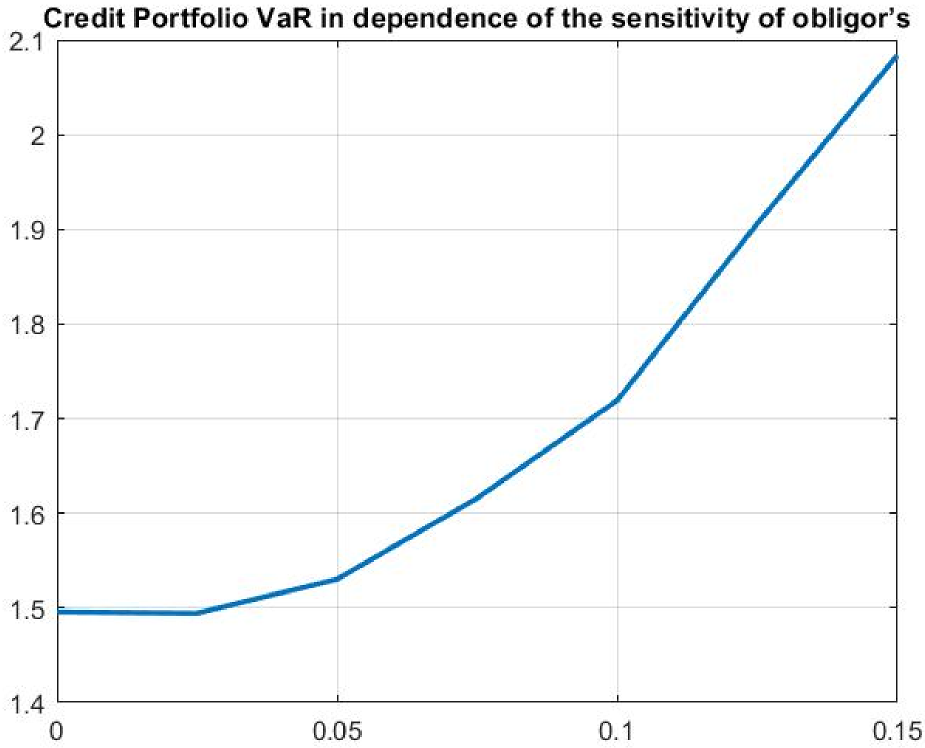


Figure 3.2: Chart of the Credit Portfolio VaR in dependence of the sensitivity of obligor's in $[0, 0.15]$ interval

Increments:

Table 3.3: VaR using the the two distributions and errors

p_{inf}	p_{sup}	$increments = VaR(p_{sup}) - VaR(p_{inf})$
0%	2.5%	-0.0014348%
2.5%	5%	0.035909%
5%	7.5%	0.085502%
7.5%	10%	0.10333%
10%	12.5%	0.18672%
12.5%	15%	0.17779%

Based on the increment vector the p value under which the VaR increment is negligible, p_{lim}, is between $[0.1, 0.125]$ (we fix the threshold value increment at 0.1%). In the chapter 4.2 we will propose a similar result in order to compute this p_{lim} exploiting the lower computational time.

Chapter 4

Model limitation and possible solutions

The main theoretical limitations of this model is that it is driven by a single systematic risk factor rather than by a multitude of correlation parameters. This simplification of course lacks a great deal of detail, but it is appropriate for studying general influences such as an economic downturn. The present model departs from Gordy by allowing recovery, as well as default, to depend on the state of the systematic risk factor.

4.1 A beta-like distribution

The main numerical limitation of this approach is the computational time of the **betainv function** in MATLAB. In order to mitigate to this problem we propose to substitute the beta distribution with a Kumaraswamy distribution (*Kumaraswamy distribution* 2006). It is similar to the Beta distribution, but much simpler to use especially in simulation studies since its probability density function, cumulative distribution function and quantile functions can be expressed in closed form.

The PDF for the Kumaraswamy distribution $K(a, b)$ is:

$$f(x|a, b) = abx^{a-1}(1 - x^a)^{b-1}$$

and the CDF is

$$F(x|a, b) = 1 - (1 - x^a)^b$$

The question that immediately arises in our mind is how to find parameters (a,b) in order to get a good approximation. If you have a $Beta(\alpha, \beta)$ distribution that you want to approximate with a $K(a, b)$ distribution, how do you pick a and b?

We reject the Method of Moments since the mean and variance are not linear. A possible approach is present here "A beta-like distribution" (Cook 2009) where the author take a beta distribution, in our case the beta distribution corresponding to the secured loans, set $a = \alpha$ and match the mode in order to get b.

The Kumaraswamy parameters are:

$$a = \alpha$$

$$b = \frac{1}{a}\left(1 + \frac{a-1}{mode^a}\right)$$

As the author says the method is OK but not excellent. Indeed the distribution is able to absorb the shape of the distribution but not pefectly approximate the initial beta distribution.

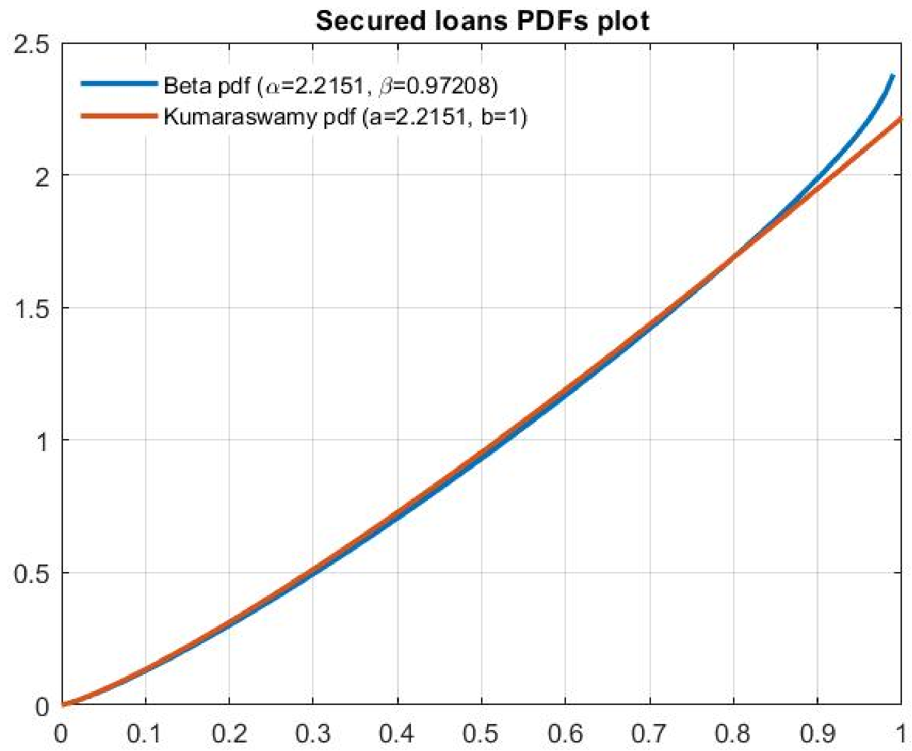


Figure 4.1: Beta and Kuma pdf

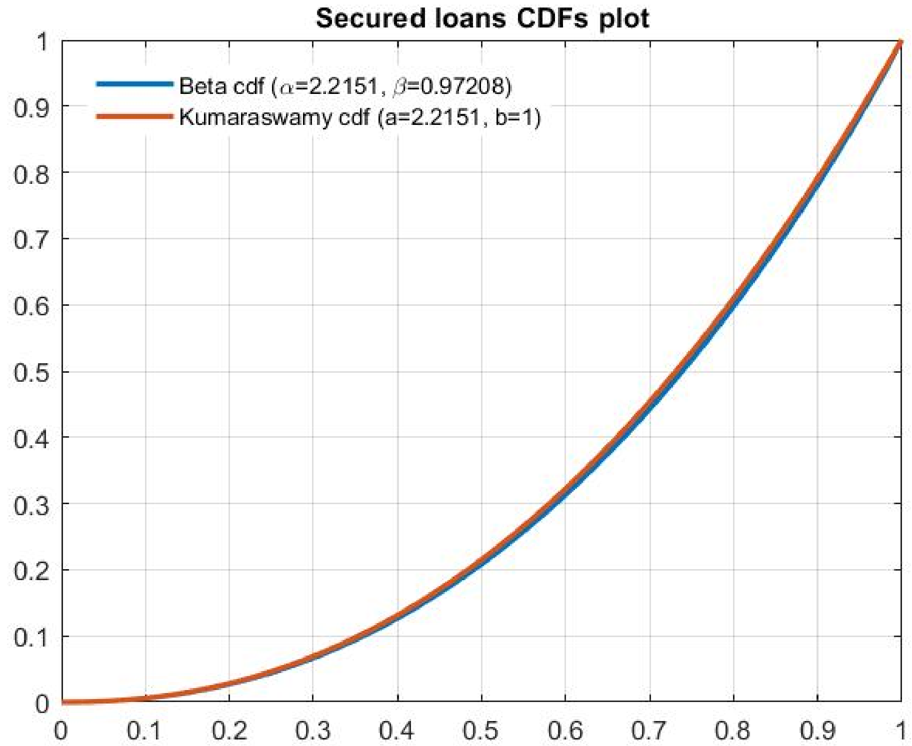
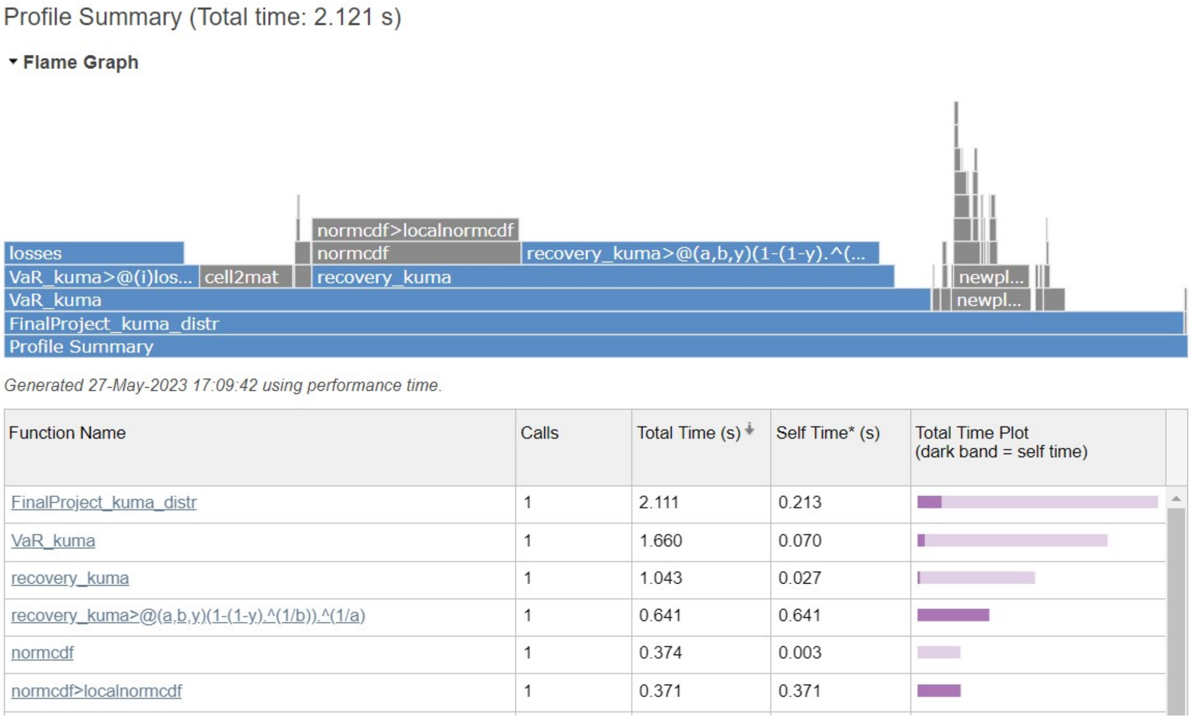


Figure 4.2: Beta and Kuma cdf

The inverse CDF is:

$$F^{-1}(x|a,b) = 1 - (1 - x^{\frac{1}{a}})^{\frac{1}{b}}$$

The results are quite impressive, starting from the time computing. If we run the same code as Section 2.2, now we get the following result (Figure 4.3), 2 sec vs 23 sec is a very appreciated result!



Function Name	Calls	Total Time (s)	Self Time* (s)	Total Time Plot (dark band = self time)
FinalProject_kuma_distr	1	2.111	0.213	
VaR_kuma	1	1.660	0.070	
recovery_kuma	1	1.043	0.027	
recovery_kuma>@(a,b,y)(1-(1-y).^(1/b)).^(1/a)	1	0.641	0.641	
normcdf	1	0.374	0.003	
normcdf>localnormcdf	1	0.371	0.371	

Figure 4.3: running time of the MATLAB code with Kumaraswamy distribution

The errors are shown in the following Table 4.1

Table 4.1: VaR using the the two distributions and errors

	VaR (beta distr.)	VaR (kuma. distr.)	error
$p = q = 50\%$	11.90%	11.7490%	0.1502%
$p = 50\%, q = 0\%$	5.61%	5.70%	0.0915%
$\sigma = 0\%, p = 50\%$	5.21%	5.20%	0.0094%

4.2 VaR in dependence of p with Kuma.

We compute the the Credit portfolio VaR with respect to the previous Credit VaR chart, considering a with a much lower step:

$$p_{grid} = [0 : 0.01 : 0.5]$$

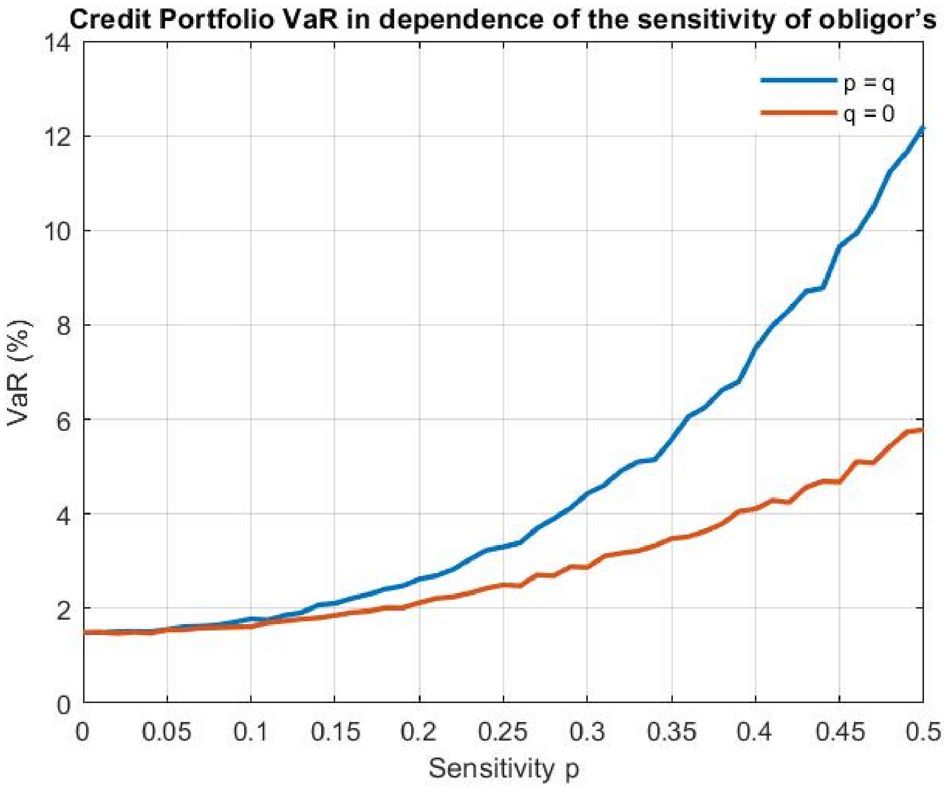


Figure 4.4: Chart of the Credit Portfolio VaR in dependence of the sensitivity of obligor's

In this case we also plot the case $q = 0$ and notice that, as seen in chapter 3.2.1, the p_{lim} belongs to the interval $[0.1, 0.125]$ since before that point the curves coincide.

References

Cook, John D. (2009). *A beta-like distribution.* URL: https://www.johndcook.com/blog/2009/11/24/kumaraswamy-distribution.

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