Complex dynamical behaviors in a discrete eco-epidemiological model with disease in prey

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Presentazione esame "Sistemi complessi"

Outline

Eco-epidemiological models.

- Ecological models $\longrightarrow$ study interactions between species and their environment.
- Epidemiological models $\longrightarrow$ study the spread of diseases in populations.

Eco-epidemiological models

Combine both aspects to study the interactions between species while considering the impact of diseases on population dynamics.

Hypoteses of the model.

1. The prey population is divided into susceptible and infected individuals.
2. In the absence of disease the prey population density grows according to a logistic curve with carrying capacity K and intrinsic growth rate r.
3. Only the susceptible prey are able to reproduce.
4. Disease spread in the prey population only. The infected populations do not recover or become immune.
5. We assume that the predator eats only the infected prey with ratio-dependent Michaelis–Menten functional response function.[1]

[1] See: J. Chattopadhyay, O. Arino, A predator–prey model with disease in the prey, Nonlinear Anal. 36 (1999) 749–766.

Discrete time eco-epidemiological model with disease in prey

Model Equations

$$S(t+1) = S(t)\exp\left\{r\left(1-\frac{S(t)+I(t)}{K}\right)-\beta I(t)\right\} \quad (1)$$

$$I(t+1) = I(t)\exp\left\{\beta S(t) - c - \frac{bY(t)}{mY(t)+I(t)}\right\} \quad (2)$$

$$Y(t+1) = Y(t)\exp\left\{\frac{kbI(t)}{mY(t)+I(t)} - d\right\} \quad (3)$$

Where $S(t)$, $I(t)$ and $Y(t)$ denote the susceptible prey, infected prey and predator populations at time t, respectively.

Parameters

- r: intrinsic birth rate of the prey population.
- K: carrying capacity for the prey population.
- β: transmission coefficient of the disease.
- c: death rate of the infected prey.
- b: predation coefficient.
- m: ratio-dependent rate. [2]
- k: coefficient in conversing prey into predator offspring.
- d: death rate of the predator.

$r, k, b, \beta, K, m, c, d > 0$.

[2] See: Xiao, Y, Chen, L: A ratio-dependent predator-prey model with disease in the prey. Appl. Math. Comput. 131, 397-414 (2002)

Basic Reproduction Number.

We can compute the *basic reproduction number*, that is the expected number of cases directly generated by one case in a population where all individuals are susceptible to infection.

$\mathcal{R}_0$: Basic Reproduction Number

$$\mathcal{R}_0 = \frac{K\beta}{c} \tag{4}$$

- K: carrying capacity for the prey population.
- β: transmission coefficient of the disease.
- c: death rate of the infected prey.

The number of equilibria depends on $\mathcal{R}_0$.

$\mathcal{R}_0 < 1$

$$\mathcal{R}_0 = \frac{K\beta}{c} < 1 \tag{5}$$

One equilibrium:

- $E_1 = (K, 0, 0)$

$\mathcal{R}_0 > 1$

$$\mathcal{R}_0 = \frac{K\beta}{c} > 1 \tag{6}$$

Always at least two equilibria:

- $E_1 = (K, 0, 0)$
- $E_2 = \left(\frac{c}{\beta}, \frac{rK}{r+K\beta}(1 - \frac{c}{K\beta}), 0\right)$

Existence of the positive endemic equilibrium.

If $kb > d$ and $mk(K\beta - c) - (kb - d) > 0$, then there exists a positive endemic equilibrium;

$E_3 = (S^*, I^*, Y^*)$

where:

- $S^* = \frac{cmk+kb-d}{mk\beta}$,
- $I^* = \frac{r}{r+K\beta}(K - S^*)$,
- $Y^* = \frac{kb-d}{md} I^*$

Stability of E_1.

Theorem 1

1. if $\mathcal{R}_0 < 1$ and $0 < r < 2$, E_1 is locally asymptotically stable;
2. if $\mathcal{R}_0 < 1$ and $r > 2$, E_1 is unstable;
3. if $\mathcal{R}_0 > 1$, E_1 is unstable.

Stability of E_2.

Theorem 2

Let $\mathcal{R}_0 > 1$, then E_2 is locally asymptotically stable if the following conditions hold:

$$bk < d, \qquad (r-4)c < 4, \qquad \frac{cr(c+2)}{4+cr} < K\beta < 1+c$$

These results are obtained by studying the eigenvalues of the Jacobian matrix evaluated at the equilibria.

$$J(E_2) = \begin{pmatrix} 1 - \frac{rS}{K} & -S(\beta + \frac{r}{K}) & 0 \\ \beta I & 1 & -b \\ 0 & 0 & e^{bk-d} \end{pmatrix}$$

Stability of E_3.

Theorem 3

Let $\mathcal{R}_0 > 1$, $kb > d$ and $mk(K\beta - c) - (kb - d) > 0$, then E_3 is locally asymptotically stable if one of the following conditions hold:

1. $\Delta \leq 0$, $P(-1) < 0$ and $-1 < \lambda_{1,2} < 1$;
2. $\Delta > 0$, $P(-1) < 0$ and $-1 < \lambda_{2,3} < 1$.

$$J(E_3) = \begin{pmatrix} 1 - \frac{r}{K}S^* & -S^*(\beta + \frac{r}{K}) & 0 \\ \beta I^* & 1 + \frac{bI^*Y^*}{(mY^*+I^*)^2} & -\frac{b(I^*)^2}{(mY^*+I^*)^2} \\ 0 & \frac{bkm(Y^*)^2}{(mY^*+I^*)^2} & 1 - \frac{bkmI^*Y^*}{(mY^*+I^*)^2} \end{pmatrix}$$

And $P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$

$$a_1 = -(J_{11} + J_{22} + J_{33}),$$

$$a_2 = J_{11}(J_{22} + J_{33}) + J_{22}J_{33} - J_{23}J_{32} - J_{12}J_{21},$$

$$a_3 = det(J(E_3)).$$

Period Doubling Bifurcation for $r > 2$.

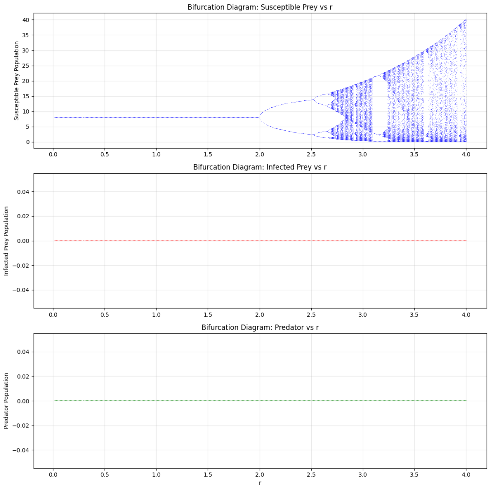

Figure: $b = 0.2, c = 0.6, d = 0.12, k = 0.1, m = 0.2, \beta = 0.05, K = 8, r \in [0.01, 4], S0 = 4, I0 = 0.5, Y0 = 0.1$

$$\mathcal{R}_0 = \frac{K\beta}{c} = \frac{8 \times 0.05}{0.6} \approx 0.67 < 1$$

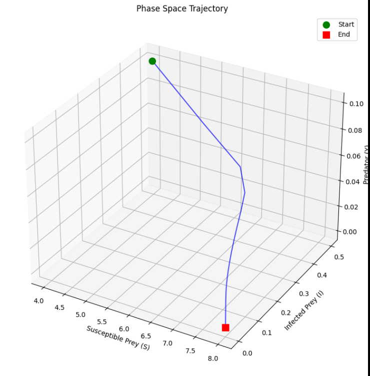

Figure: Plot of one orbit for $r = 1$.

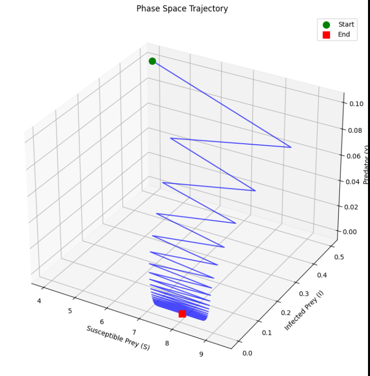

Figure: Plot of one orbit for $r = 2$.

Flip bifurcation and chaos

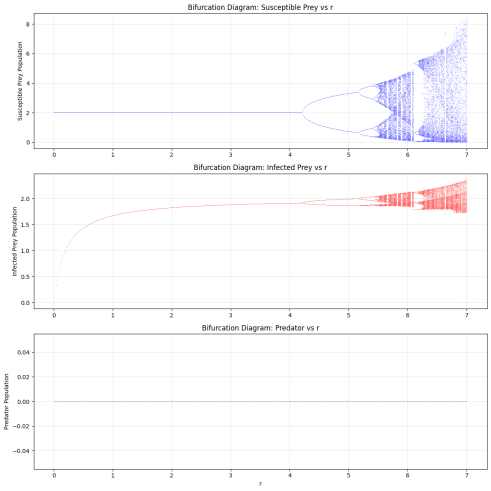

Figure: $b = 0.15, c = 0.1, d = 0.2, k = 0.2, m = 0.3, \beta = 0.05, K = 4, r \in [0.001, 7], S0 = 2, I0 = 1.5, Y0 = 1$

$$\mathcal{R}_0 = \frac{K\beta}{c} = \frac{4 \times 0.05}{0.1} = 2 > 1$$

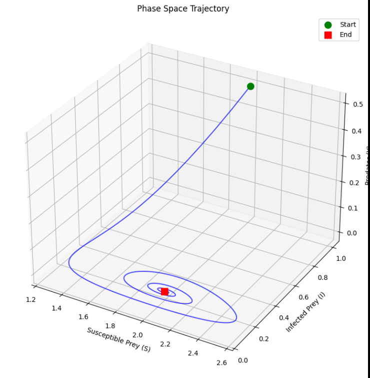

Figure: Plot of one orbit for $r = 0.02$.

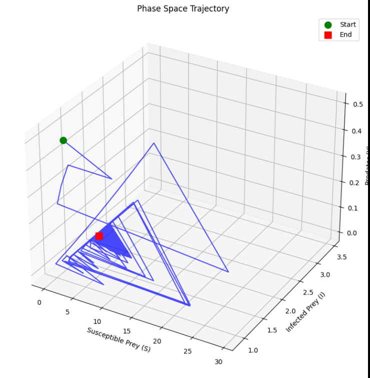

Figure: Plot of one orbit for $r = 6.05$.

A 3-period orbit appears for $r \approx 6.05$.

Comparison between Equilibria E_1 and E_2 and r.

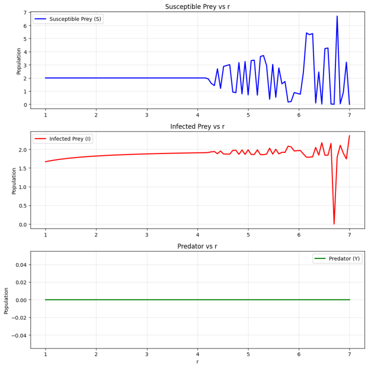

Figure: Population vs r for Equilibrium E_2.

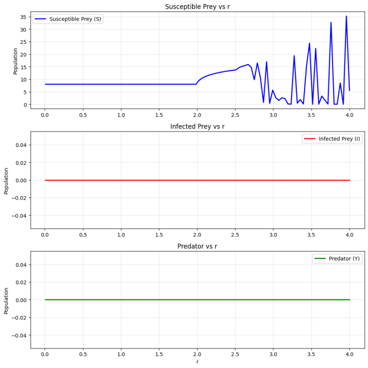

Figure: Population vs r for Equilibrium E_1.

$0 < c \leq 1$ Hopf bifurcation and chaos. $1 < c < 1.633$ S increases, I decreases. $c \geq 1.633$ flip bifurcation and chaos.

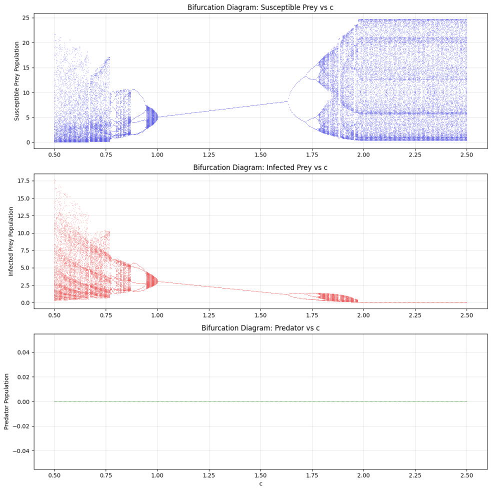

Figure: $b = 0.2, d = 0.2, k = 0.2, m = 0.5, r = 3, \beta = 0.2, K = 10, c \in [0.5, 2.5], S0 = 6, I0 = 1, Y0 = 1$

$$\mathcal{R}_0 = \frac{K\beta}{c} = \frac{10 \times 0.2}{c} = \frac{2}{c}$$

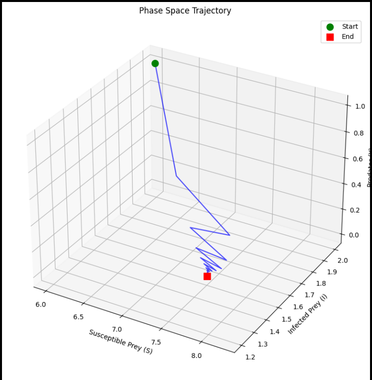

Figure: Plot of one orbit for $c = 1.5$.

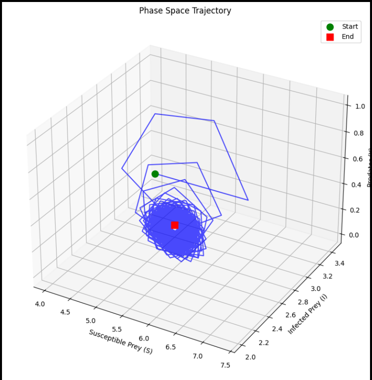

Figure: Plot of one orbit for $c = 1$.

Population vs $c \in [0.5, 2.5]$.

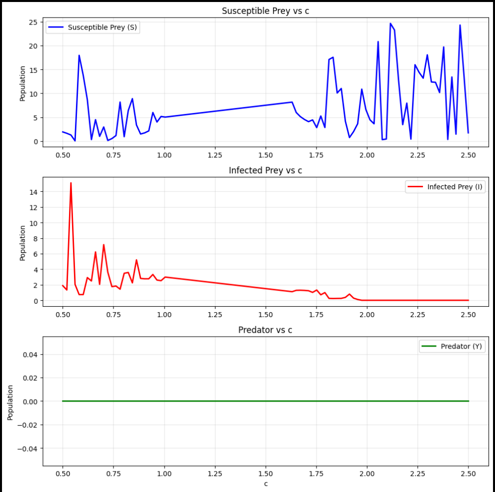

Figure: Population vs $c \in [0.5, 2.5]$.

E_2 stable for $2 < K < 7$, flip bifurcation and Hopf bifurcation for $K \geq 7$.

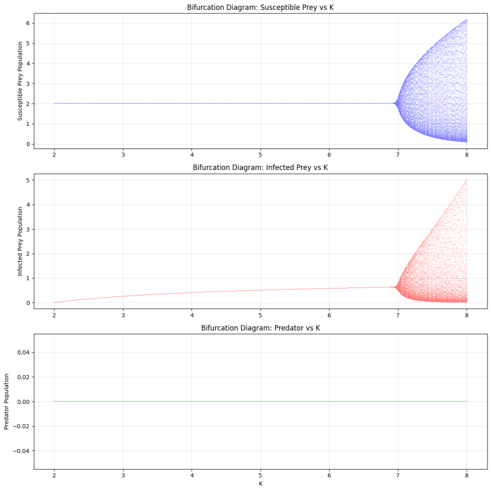

Figure: $b = 0.1, c = 0.4, d = 0.2, k = 0.2, m = 0.5, r = 0.2, \beta = 0.2, K \in [2, 8], S0 = 1, I0 = 0.5, Y0 = 0.2$

$$\mathcal{R}_0 = \frac{K\beta}{c} = \frac{K \times 0.2}{0.4} = \frac{K}{2}$$

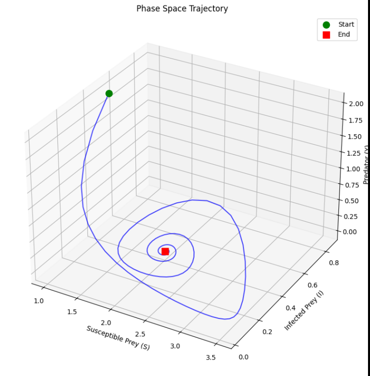

Figure: Plot of one orbit for $K = 4$.

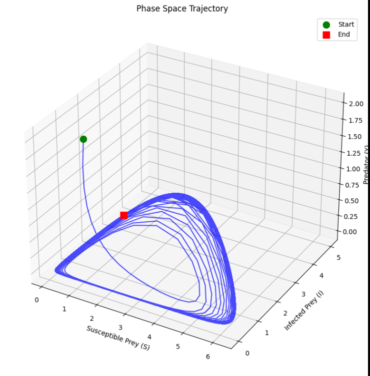

Figure: Plot of one orbit for $K = 8$.

Population vs $K \in [2, 8]$.

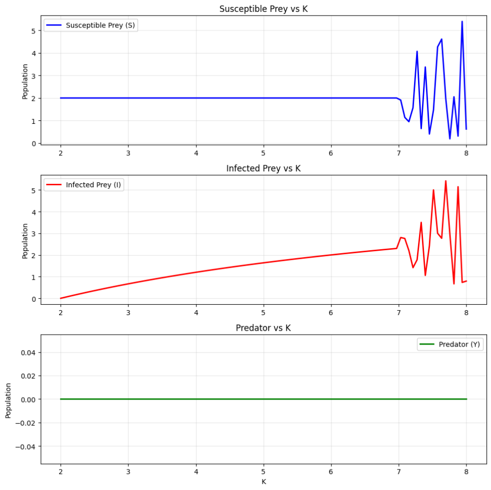

Figure: Population vs $K \in [2, 8]$.

$0.15 < b < 0.28$ Hopf bifurcation. $b \geq 0.28 < 0.57$ stable equilibrium.

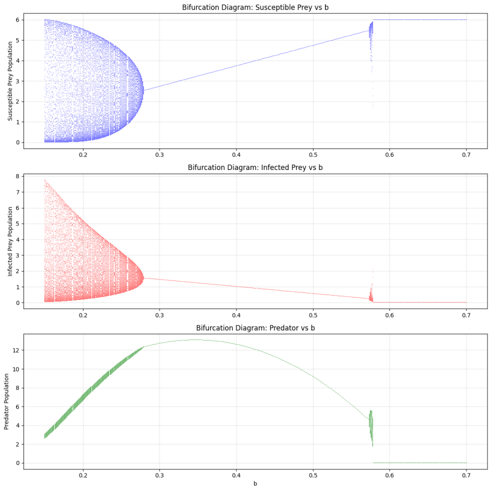

Figure: $c = 0.1, d = 0.02, k = 0.3, m = 0.4, r = 1.2, \beta = 0.25, K = 6, b \in [0.15, 7], S0 = 2, I0 = 1.5, Y0 = 1$

$$\mathcal{R}_0 = \frac{K\beta}{c} = \frac{6 \times 0.25}{0.1} = 15$$

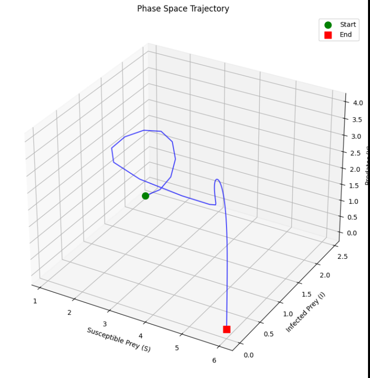

Figure: Plot of one orbit for $b = 0.6$.

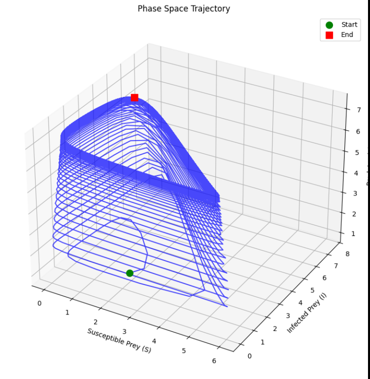

Figure: Plot of one orbit for $b = 0.2$.

$0.15 < k < 0.34$ Hopf bifurcation. $1 > k \geq 0.34$ S, I increasing and Y decreasing.

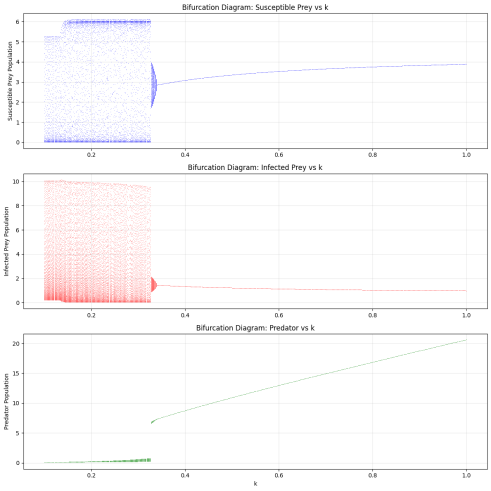

Figure: $b = 0.3, c = 0.1, d = 0.04, m = 0.3, r = 1.2, \beta = 0.25, K = 6, k \in [0.15, 1], S0 = 2, I0 = 1.5, Y0 = 1$

$$\mathcal{R}_0 = \frac{K\beta}{c} = \frac{6 \times 0.25}{0.1} = 15$$

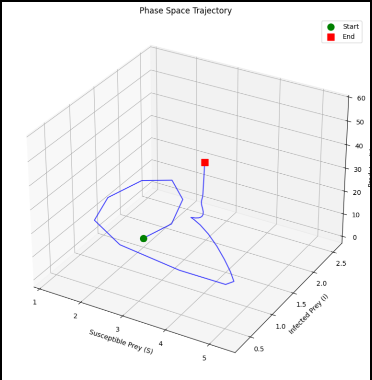

Figure: Plot of one orbit for $k = 0.3$.

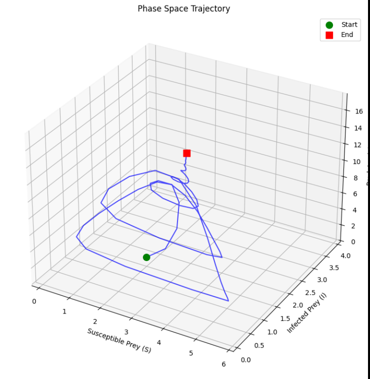

Figure: Plot of one orbit for $k = 0.8$.

Influence of the parameters b and k on populations.

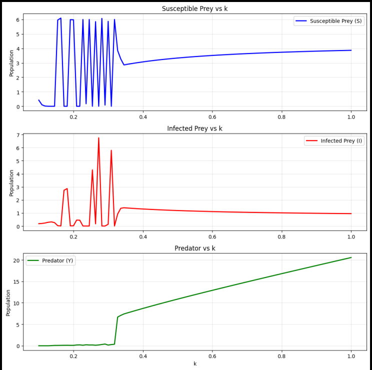

Figure: b fixed and varying k.

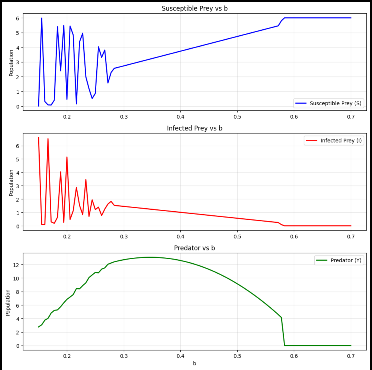

Figure: k fixed and varying b.

Influence of β on the endemic equilibrium

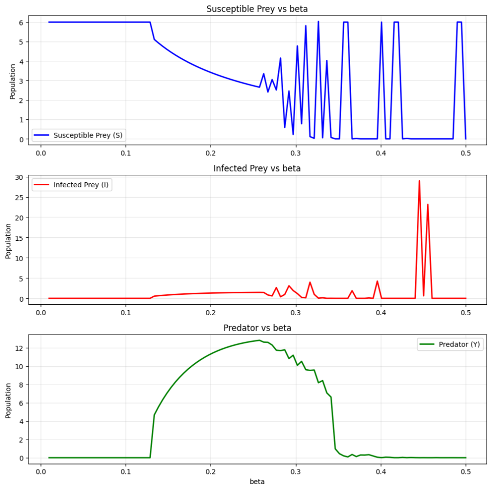

Figure: $b = 0.3, c = 0.1, d = 0.04, k = 0.34, m = 0.4, r = 1.2, K = 6, \beta \in [0.01, 0.5], S0 = 2, I0 = 1.5, Y0 = 1$

Summary

The discrete-time model exhibits a richer dynamical behavior compared to its continuous-time counterpart.

- When $\mathcal{R}_0 > 1$ and $bk < d$: S and I coexist while Y becomes extinct.
- r, K, β, c directly influence the dynamical behaviors of the system. By varying these parameters, we observe local stability, period-doubling bifurcations, Hopf bifurcations, and even chaotic dynamics.
- When b and k increase, Y increase and I decrease. If b is large this can lead to the extinction of I and ultimately Y (since predators can only eat infected prey).

Bibliography I

Zengyun Hu, Zhidong Teng, Chaojun Jia, Long Zhang and Xi Chen
Complex dynamical behaviors in a discrete eco-epidemiological model with disease in prey.
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