

A GEOMETRIC DYNAMICAL SYSTEM WITH RELATION TO POLYGONAL BILLIARDS

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ABSTRACT. We introduce a geometric dynamical system where iteration is defined as a cycling composition of a finite collection of geometric maps, which act on a space composed of three or more nonparallel, nonconcurrent lines in \mathbb{R}^2 . This system is motivated by the dynamics of iterated function systems, as well as billiards with modified reflection laws. We provide conditions under which this dynamical system generates periodic orbits, and use this result to prove the existence of closed nonsmooth curves over \mathbb{R}^2 which satisfy particular structural constraints with respect to a space of intersecting lines in the plane.

1. INTRODUCTION

The theory of mathematical billiards in polygons concerns the uniform motion of a point mass (billiard) in a polygonal plane domain, with elastic reflections off the boundary according to the mirror law of reflection: the angle of incidence equals the angle of reflection. In addition to billiards obeying the mirror law of reflection, well studied areas include billiards with modified reflection laws, so that the angle of reflection is some function of the angle of incidence (see e.g., [AMS09, AMS12, MDD⁺14, MDD⁺12, GT18] and the references therein), and tiling billiards, where trajectories refract through planar tilings (see [DDRL18, BSDFI18]).

The purpose of this paper is to elaborate a geometric dynamical system that serves as a generalization to the system studied in [Eve20b], and is motivated by polygonal billiard dynamics. The dynamical system studied here can be analogized to an iterated function system (see [Hut01] for review) where the defining collection of contraction mappings are geometrically defined over lines in the plane, and composed in a fixed, cycling order. The system can also be thought of as billiards where trajectories reflect off or refract through lines as a function of the line they are incident to.

As an application of this dynamical system, we prove Theorem 1.1, which asserts existence of nonsmooth closed curves satisfying particular geometric constraints with respect to a space of lines on the plane. In fact, such closed curves can coincide with periodic billiard trajectories. We state this result after giving some notation.

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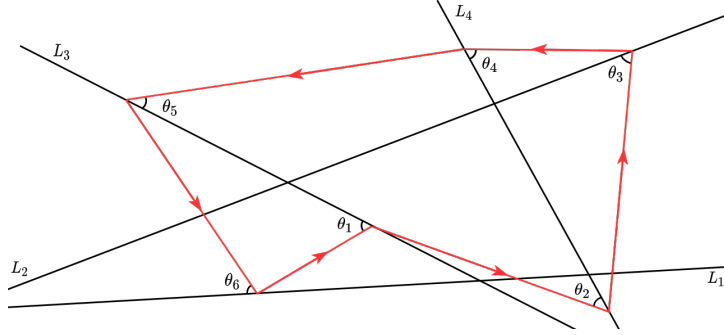


FIGURE 1. The values $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$ compose an incidence angle sequence with respect to the line sequence $L_3, L_4, L_2, L_4, L_3, L_1$.

Let X_m denote the space of $m \geq 3$ lines in \mathbb{R}^2 that are not all parallel and nonconcurrent, and assign each line in X_m a unique label L_i , $i \in \{1, 2, \dots, m\}$. Let p_1, p_2, \dots, p_n , be a sequence of $n \geq m$ points in X_m such that consecutive points, including p_n and p_1 , are distinct, and if $p_k \in L_i$, then $p_{k+1} \in L_j$, $i \neq j$ (with the convention that $p_{n+1} = p_1$). Join consecutive pairs of such points, including p_n and p_1 , with line segments to construct a *closed curve* Γ over X_m . Traversal of a closed curve in a fixed direction allows for construction of an *incidence angle sequence* $\theta_1, \theta_2, \dots, \theta_n$ with respect to a line sequence $L_{a_1}, L_{a_2}, \dots, L_{a_n}$, by taking the acute or right angle θ_i between each segment of the closed curve and a line with label L_{a_i} it is incident to, with respect to the traversal direction. Refer to Figure 1 for visual demonstration. A result obtained as a consequence of the dynamical system introduced in this paper is the following.

Theorem 1.1. *For any space X_m with labeled lines, let $\theta_1, \theta_2, \dots, \theta_n$, $n \geq m \geq 3$, be any sequence of acute or right angles, and let $L_{a_1}, L_{a_2}, \dots, L_{a_n}$ be a sequence of line labels such that no two consecutive labels are the same, including L_{a_n} and L_{a_1} , and each of the m possible labels occur at least once in the sequence.*

Then there exists a closed curve Γ over X_m that admits an incidence angle sequence $\theta_1, \theta_2, \dots, \theta_n$ with respect to the line sequence $L_{a_1}, L_{a_2}, \dots, L_{a_n}$ when traversed in a fixed direction. Moreover, there are either uncountably many such curves, or no more than 2^n .

In the case where the closed curves are strictly contained within polygons formed by the intersecting lines in X_m , the angles of incidence implicitly define angles of reflection, and the connection with billiards in polygons is made apparent.

A basic question one can ask about polygonal billiard systems is whether there exists a periodic billiard trajectory. Indeed, a long-standing question in polygonal billiards with standard reflection laws is whether every polygon contains a periodic billiard orbit (see *Problem 10* in [Gut12], and [Gut96] for a survey); in fact, the question remains unsolved for particular obtuse triangles. Intense study on this problem has led to progress (see, e.g. [Mas86] for results on rational polygons, and [Sch09, GSV92, Tro05, HH00] for results on triangles), and many deep theorems have been obtained, but the problem remains open. The dynamical system introduced in this paper lends itself to the study of periodic billiard orbits in polygons in a natural way, by way of techniques used in proving Theorem 1.1.

This paper is organized into two main parts, separated by study of two related dynamical systems. In the first, from Sections 2 through 3, we define a dynamical system that provides controllable and predictable behavior, which we use to prove Theorem 1.1. In the second part, from Sections 4 through 5, we redefine components of the introduced dynamical system in a way that introduces discontinuities. The introduction of such discontinuities leads to far more complex dynamics that can share characteristics with piecewise isometries (see [Goe00, Goe03] for review) and generalizes [Eve20b]. We prove a theorem that asserts orbits of this system are asymptotically stable when particular geometric conditions are satisfied.

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2. PRELIMINARIES

Let L_i, L_j label distinct lines in the space X_m . Then, for every $x \in L_i$ we may determine two lines, $\mathcal{L}_0, \mathcal{L}_1$ such that $\{x\} = \mathcal{L}_0 \cap \mathcal{L}_1$ and $\mathcal{L}_0, \mathcal{L}_1$ intersect with line L_j at an angle $\theta \in (0, \pi/2)$, with intersection points z_0 and z_1 in L_j , respectively. We call z_0, z_1 the *orientation 0 and 1, angle θ -projections of x onto L_j* . If $\theta = \pi/2$, then we call the line intersection point z the *perpendicular projection of x onto L_j* . In the case when $x \in L_i \cap L_j$, the projection of x onto L_i or L_j is simply x itself.

Definition 2.1. Let $\theta \in (0, \pi/2]$, $o \in \{0, 1\}$, and $i \in \{1, \dots, m\}$. We call a mapping $r : X_m \rightarrow X_m$ a *rule*, if $r(x)$ is an angle θ , orientation o projection of $x \in X_m$ onto a line L_i in X_m . We may also notate rules as $r(x; \theta, o, L_i)$ to make the parameters explicit.

When the rule projection angle θ is equal to $\pi/2$, we simply write $r(x; \theta, L_i)$ when notating rules as there is only one possible orientation. Figure 2 provides visual demonstration of the composition of two rules, $r_1(x) := r(x; \theta_1, 1, L_2)$ and $r_2(x) := r(x; \theta_2, 0, L_3)$ over a point $x_0 \in X_3$, so that

$$r_1(x_0) = x_1 \text{ and } r_2(r_1(x_0)) = x_2.$$

We require rule orientation to be defined in a predictable and consistent way, so that it is never ambiguous which projection points correspond to which orientation value. For this paper, we choose a natural and mathematically convenient convention where the orientation 0 and 1 projection points under a rule $r(x)$ are the “left” and “right” points, “from the perspective of x ”. Figures 2 and 3 demonstrate this convention under point translation.

We define a *rule sequence* associated to a space X_m to be a sequence of $n \geq m \geq 3$ rules, denoted $\{r_i\}_{i=1}^n$, with the restriction that consecutive rules in the rule sequence, including r_1 and r_n , cannot map onto the same line in X_m . Further, we require each line in X_m be mapped onto by at least one of the rules in a rule sequence $\{r_i\}_{i=1}^n$ associated to a space X_m .

Definition 2.2. Let $T_n : X_m \rightarrow X_m$ denote an n -rule map, where iteration of T_n is defined to be a cycling composition of the $n \geq 3$ rules in an associated defining rule sequence $\{r_i\}_{i=1}^n$. That is, if $\{r_i\}_{i=1}^n$ is the defining rule sequence for n -rule map T_n , then for $x \in X_m$, we define iteration of T_n so that

$$T_n(T_n^{n+1}(x)) = T_n^{n+2}(x) = r_2(r_1(r_n(\dots r_2(r_1(x)))))$$

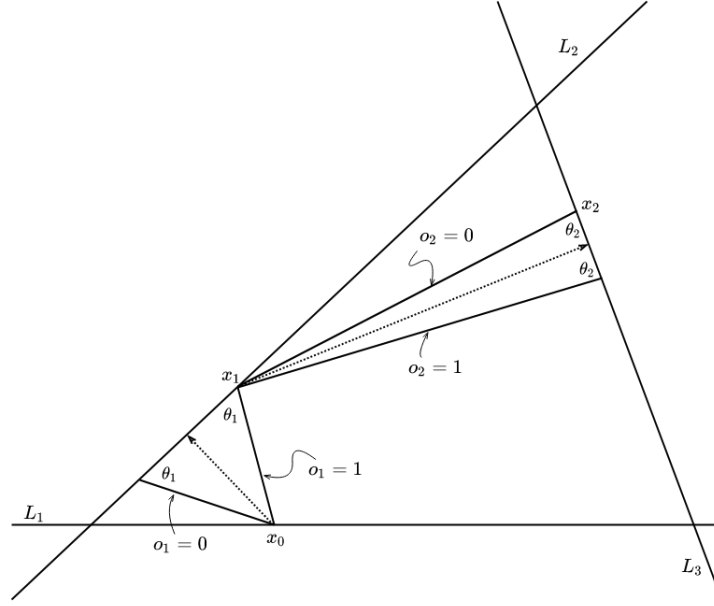


FIGURE 2. An illustration of the mapping action of two rules composed over an initial point x_0 in a space Y_3 .

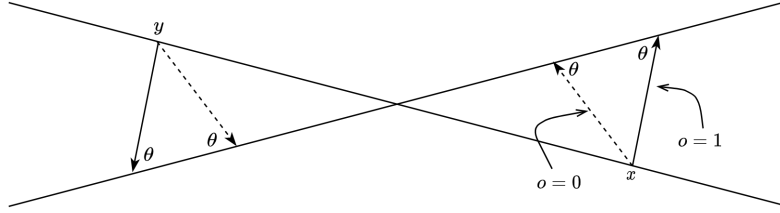


FIGURE 3. How orientation is preserved under point translation (from x to y) under a rule with projection angle θ . Note how the projection lines corresponding with a fixed orientation are antiparallel across line intersection points, and always map opposite the same line intersection angle.

Unless otherwise stated, the pair (X_m, T_n) denotes a dynamical system. For a point $x \in X_m$, we let $\mathcal{O}(x)$ denote the *orbit of x* under n -rule map T_n , so that

$$\mathcal{O}(x) := \{x, T_n(x), T_n^2(x), \dots\}.$$

We call an n -rule map *redundant* if there exists a length n' rule sequence with $m \leq n' < n$, such that for all $x \in X_m$, the orbit of x under the n' -rule map is equal to the orbit of x under the n -rule map. For the purpose of this paper we assume all n -rule maps are not redundant.

2.1. n-Rule maps and rules as similarities. Let L_1, L_2 denote lines in \mathbb{R}^2 , intersecting at point z with acute or right angle δ . Let $x, y \in L_1$ lie on the same side of the line intersection point z , and take a rule r that projects onto line L_2

with projection angle $\theta \in (0, \pi/2]$. Further, let the orientation value of r be $o = 1$, so that it maps farthest from z when δ is acute (see Figure 3).

Let d be the Euclidean metric. Assume $d(x, y) = \epsilon > 0$, and $a = d(z, x)$, $a + \epsilon = d(z, y)$. Let $\gamma = \pi - \delta - \theta$, so that $\gamma = \angle zxr(x) = \angle zyr(y)$. Then it follows by use of the law of sines that

$$d(r(x), r(y)) = \left\| \frac{a \sin(\gamma)}{\sin(\theta)} - \frac{(a + \epsilon) \sin(\gamma)}{\sin(\theta)} \right\|_2 = \frac{\epsilon \sin(\gamma)}{\sin(\theta)}$$

Let $c = \sin(\gamma)/\sin(\theta)$, and hence $d(r(x), r(y)) = cd(x, y)$. We see that if $0 < \theta < (\pi - \delta)/2$ and δ is acute, then

$$c = \frac{\sin(\gamma)}{\sin(\theta)} > 1$$

Furthermore, if $\theta = (\pi - \delta)/2$, then $c = 1$, and when

$$\frac{\pi - \delta}{2} < \theta \leq \frac{\pi}{2}$$

then $0 \leq c < 1$.

By similar argument, when the rule r has orientation parameter of $o = 0$, we see that $d(r(x), r(y)) = cd(x, y)$ for some constant $c = c(\theta, \delta)$ computable using the law of sines, where $0 \leq c < 1$ when $\delta/2 < \theta \leq \pi/2$, and $c = 1$ when $\theta = \delta/2$, and $c > 1$ when $0 < \theta < \delta/2$.

Indeed, when the points x and y are on opposite sides of the line intersection point z , it still holds that $d(r(x), r(y)) = cd(x, y)$. To see this, let $x, y \in L_1$ lie on opposite sides of line intersection point z . Then $d(r(x), z) = cd(x, z)$ and $d(r(y), z) = cd(y, z)$, and hence

$$d(r(x), r(y)) = d(r(x), z) + d(r(y), z) = cd(x, z) + cd(y, z) = cd(x, y)$$

As such, by fixing the angle projection and orientation parameters θ and o of a rule r , and restricting the mapping of a rule between two lines in X_m , then r becomes a similarity transformation. That is

$$d(r(x), r(y)) = cd(x, y), \quad c \geq 0$$

We call the constant c a *similarity coefficient*.

Iteration of n -rule maps is defined to be a cycling composition of rules in an associated rule sequence, and as a consequence, after the first iteration of a n -rule map over an point in X_m , each rule in the rule sequence will always map between the same pair of lines since each rule in the sequence always projects onto the same line. As such, by way of the above analysis, iteration of a fixed n -rule map can be thought of as a cycling composition of similarity transformations, after the first iteration of the map.

Let $\hat{T}_n := T_n^n$ denote an induced map of n -rule map T_n , so that $\hat{T}_n^l = T_n^{ln}$ for $l \in \mathbb{N}$, and $\hat{T}_n : L_{a_n} \rightarrow L_{a_n}$, where L_{a_n} is the line the n th rule in the defining rule sequence of T_n maps onto. If the rules defining T_n have similarity coefficients c_1, \dots, c_n , then let $C = c_1 \cdot c_2 \cdots c_n$ label the similarity coefficient for the induced map \hat{T}_n .

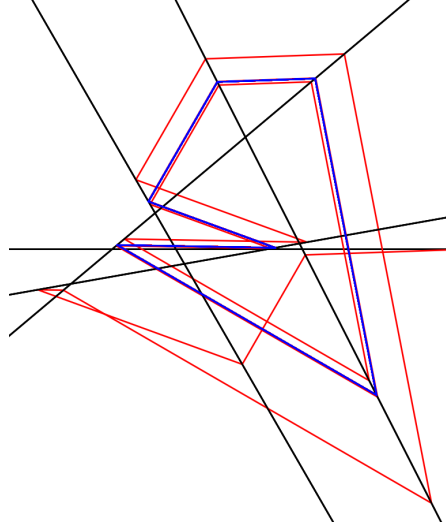


FIGURE 4. Iteration of a 6-rule map in X_4 , with the orbit converging to a six-cycle.

3. N-RULE MAPS AND CLOSED CURVES

In this section we prove Theorem 1.1. We begin by establishing the following.

Theorem 3.1. *Let (X_m, T_n) be a dynamical system, and let \hat{T}_n be the induced map of n -rule map T_n , with similarity coefficient $C = c_1 c_2 \cdots c_n$. If $0 \leq C < 1$, then T_n admits a unique periodic orbit of period n .*

Proof. Let $\hat{\mathcal{O}}(x)$ denote the orbit of $x \in X_m$ under the map \hat{T}_n . It follows from definition of n -rule maps that for any $x \in X_m$, the orbit $\hat{\mathcal{O}}(x) \setminus \{x\}$ of x under \hat{T}_n , must be a subset of some line $L_{a_n} \subset X_m$ determined by the n th rule in the defining rule sequence of T_n . Further, by hypothesis $0 \leq C < 1$, and hence

$$d(\hat{T}_n(x), \hat{T}_n(y)) \leq C d(x, y)$$

for any $x, y \in L_{a_n}$, so \hat{T}_n is a contraction mapping. But the line L_{a_n} is a closed subset of \mathbb{R}^2 and necessarily complete. Hence, by the contraction mapping theorem there exists a unique $x^* \in L_i$ such that $\hat{T}_n(x^*) = x^*$. Then T_n admits a unique periodic orbit of period n . \square

Refer to Figure 4 for visual demonstration of the type of dynamics Theorem 3.1 provides.

Assume two lines L_i, L_j in X_m intersect at angle $\delta \leq \pi/2$. Then if the defining rule sequence for an n -rule map T_n over X_m contains a rule mapping from L_i to L_j (or vice-versa) with projection angle $\theta = \delta$ and orientation value $o = 0$ (or $o = 1$ if $\theta = \delta = \pi/2$), then iteration of this rule projects onto the intersection point of L_i and L_j every n iterations, and hence the system collapses to a periodic orbit. In such a case we say the n -rule map T_n is *collapsing*. Figure 5 gives a visual example of a collapsing map.

Remark 1. *If an n -rule map is not collapsing, then it is invertible.*

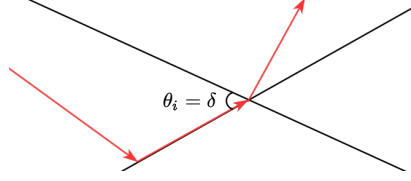


FIGURE 5. The red arrows indicate iterates of a collapsing map, projecting onto a line intersection point.

If $2 \leq m' < m$ lines in X_m intersect at a common point, then a rule sequence $\{r_i\}_{i=1}^n$ may contain a subsequence of consecutive rules which map strictly between the m' lines intersecting at the same point. As a consequence, this same subsequence of consecutive rules would map the line intersection point to itself. In such a case, we call the subsequence of rules an *absorbed* subsequence. Further, if the line intersection point is a periodic point under the n -rule map, we say it is an *absorbing* periodic point.

An absorbed subsequence may be composed of $1 \leq k \leq n - 2$ rules. That $k \leq n - 2$ is given by the nonconcurrency assumption of the lines composing X_m , and that every line in X_m must be mapped onto by at least one rule in every rule sequence associated to an X_m . Hence, there are always at least two rules in a rule sequence that cannot be absorbed.

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Given a sequence $\theta_1, \theta_2, \dots, \theta_n$ of acute or right angles, as well as a sequence $L_{a_1}, L_{a_2}, \dots, L_{a_n}$ of line labels over a space X_m with no two consecutive labels the same, we may construct a sequence of n rules $\{r_i\}_{i=1}^n$ such that $r_i = r(x; \theta_i, o_i, L_{a_i})$ with $i = 1, \dots, n$ and leave each o_i arbitrary. Use this rule sequence to define an n -rule map T_n over the fixed X_m , and the induced map \hat{T}_n with similarity coefficient $C = c_1 c_2 \cdots c_n$.

Consider the case when the angles $\theta_1, \dots, \theta_n$ and other parameters of the n -rule map are such that the similarity coefficient for the induced map is $C < 1$. Then by Theorem 3.1, T_n admits a periodic orbit, and by joining consecutive periodic points of this orbit with line segments we obtain a closed curve Γ over X_m that admits an incidence angle sequence $\theta_1, \theta_2, \dots, \theta_n$ with respect to the line sequence $L_{a_1}, L_{a_2}, \dots, L_{a_n}$, by the definition of T_n . Similarly, if T_n is collapsing, then it has a periodic orbit, and this orbit corresponds to a closed curve Γ admitting an incidence angle sequence $\theta_1, \theta_2, \dots, \theta_n$ with respect to the line sequence $L_{a_1}, L_{a_2}, \dots, L_{a_n}$.

If $C > 1$ and T_n is not collapsing, then T_n has an inverse n -rule map T_n^{-1} , with corresponding induced map \hat{T}_n^{-1} and similarity coefficient $C' = 1/C$. Then by Theorem 3.1 T_n^{-1} has a periodic orbit. But T_n^{-1} is the inverse of T_n , and hence this is also a periodic orbit for T_n . Constructing a closed curve Γ from this periodic orbit as above, we see that Γ admits an incidence angle sequence $\theta_1, \theta_2, \dots, \theta_n$ with respect to the line sequence $L_{a_1}, L_{a_2}, \dots, L_{a_n}$.

If $C = 1$, then there is a rule in the rule sequence $\{r_i\}_{i=1}^n$ defining T_n such that $\theta_i < \pi/2$. Let $\epsilon > 0$, and replace the projection angle θ_i in rule r_i with $\theta_i + \epsilon \leq \pi/2$, so that $C < 1$. Then by the above argument we obtain a closed curve Γ over X_m with incidence angle sequence $\theta_1, \dots, \theta_i + \epsilon, \dots, \theta_n$ with respect to the line sequence $L_{a_1}, L_{a_2}, \dots, L_{a_n}$. But as $\epsilon \rightarrow 0$, we obtain a sequence of closed curves that

approach a closed curve with incidence angle sequence $\theta_1, \dots, \theta_n$ with respect to the line sequence $L_{a_1}, L_{a_2}, \dots, L_{a_n}$, and this limiting closed curve exists by continuity.

When an n -rule map associated with a closed curve Γ has one or more absorbing periodic points, then one or more subsequences of rules are absorbed over a line intersection point in X_m , and as a consequence Γ will be degenerate, not admitting a full incidence angle sequence $\theta_1, \dots, \theta_n$. To construct a nondegenerate closed curve apply the following procedure: When an n -rule map contains an absorbed subsequence, flip the orientation of the rule immediately before the absorbed subsequence. Then at least one of the following rules in the rule sequence will be removed from the absorbed subsequence. Continue this process until there are no more absorbed subsequences.

This algorithm must terminate because there cannot be more than $\binom{m}{2}$ unique line intersection points, and hence no more than $\binom{m}{2}$ absorbed subsequences. Furthermore, there are 2^n distinct n -rule maps associated with the distinct rule orientation configurations, and $\binom{m}{2} < 2^n$ when $3 \leq m \leq n$, implying there exists at least one rule orientation configuration such that the associated n -rule map has no absorbed subsequences.

Finally, for the $C < 1$ and $C > 1$ cases it is clear that there cannot be more than 2^n distinct closed curves corresponding to the 2^n different orientation configurations. For the $C = 1$ case there are many examples where there are uncountably many such closed curves. The author chose to leave out the argument for the $C = 1$ case that demonstrates there cannot be countably many such curves or a finite number greater than 2^n for the sake of brevity. \square

n -Rule maps lend themselves to determining periodic billiard orbits in polygons in a natural way. If a closed curve generated by an n -rule map is strictly contained within a polygon cut out by the intersecting lines composing X_m , then there are cases in which the parameters of the n -rule map can be adjusted so that the angles of incidence equal the angles of reflection, and the closed curve corresponds with a periodic billiard orbit in the polygon. Conditions that assert when this is possible can be determined, however further discussion is out of the scope of this paper.

4. N-RULE MAPS DEFINED USING PIECEWISE CONTINUOUS RULES

This section begins the second part of our analysis, in which we redefine n -rule maps so that rules map onto lines not on a basis of some fixed line label, but rather on a basis of a distance between points and lines. Rules then become piecewise continuous, and this redefinition introduces discontinuities that complicate the dynamics. In this section we give basic results concerning the redefined n -rule maps, and then in Section 5 we prove Theorem 5.1, which shows when their orbits are asymptotically periodic.

This redefined dynamical system generalizes the system studied in [Eve20b], and shares characteristics with piecewise isometric dynamical systems. Furthermore, the interest in this dynamical system is partially given by potential applications, and the fact that such a simple geometric dynamical system can lead to rich phenomena which provide characteristics analogous to a number of well studied areas in dynamics, such as billiards.

4.1. Redefining n -rule maps. Let Y_m label the space of $m \geq 3$ pairwise nonparallel, nonconcurrent line in \mathbb{R}^2 . If L_i, L_j are two distinct lines in Y_m where $x \in L_i$,

we define

$$d(x, L_j) := \inf\{d(x, y) | y \in L_j\}$$

to be the distance between point x and line L_j , where d is the Euclidean metric and $d(x, L_i) = 0$.

For every point $x \in Y_m$, we construct the set

$$D(x) = \{d(x, L_i) | 1 \leq i \leq m\}$$

and define the partially ordered set $\mathcal{D}(x) := (D(x), \leq)$. We let l denote an index on $\mathcal{D}(x)$, so that $l = i$, $1 \leq i \leq m$ corresponds with the i th farthest line from point x . Note that if there exist m' lines in Y_m , $m' < m$, that are all the same distance from point x , then there are m' values of l that do not correspond with a unique distance value in $\mathcal{D}(x)$, and thus do not correspond with a unique line in Y_m .

Definition 4.1. Let $\theta \in (0, \pi/2]$, $o \in \{0, 1\}$, and $l \in \{2, 3, \dots, m\}$. We call a mapping $r : Y_m \rightarrow Y_m$ a piecewise rule, if $r(x)$ is an angle θ , orientation o projection $x \in Y_m$ onto the l th farthest line from x . If the index l corresponds with a distance value in $\mathcal{D}(x)$ that is not unique in $\mathcal{D}(x)$, put $r(x) = x$.

As previously, we may notate piecewise rules as $r(x; \theta, o, l)$ to make the parameters of the rule explicit. Furthermore, for the rest of the paper, we will refer to “piecewise rules” as “rules” for convenience, and when needed refer to the rules used in Sections 2 and 3 as “symbolic rules.”

In the case when the index l corresponds to a distance value in $\mathcal{D}(x)$ that is not unique so that $r(x) = x$, then we say x is an *invariant* point under rule r . Figure 2 provides visual demonstration of the composition of two rules,

$$r_1(x) := r(x; \theta_1, 1, 2) \text{ and } r_2(x) := r(x; \theta_2, 0, 3)$$

over a point $x_0 \in Y_3$, so that $r_1(x_0) = x_1$ and $r_2(r_1(x_0)) = x_2$. Intuitively, rule r_1 maps to the *closest* line from a point x , and rule r_2 maps to the *farthest* line from a point x when applied to a space Y_3 . We leave rule orientation to be defined as previously, with the convention made visually explicit for this new class of rules in Figures 2 and 3.

We leave rule sequences and n -rule maps defined as before, except for noting that n -rule maps defined by piecewise rules may only have one rule in the defining rule sequence, unlike those defined with symbolic rules which required at least three. We let K_n denote an n -rule map where the defining rules in the rule sequence are piecewise rules. We may call such maps *piecewise n -rule maps* for clarity, although for the remainder of this paper we will only work with piecewise n -rule maps, and hence we may refer to them simply as n -rule maps when the context is clear.

For any piecewise n -rule map K_n , it is required that at least one of the rules in the associated rule sequence has l index value greater than $l = 2$; such a restriction ensures the dynamics of a piecewise n -rule map are nontrivial. If all rules of the rule sequence have l index value of $l = 2$, then each iteration maps to the “closest” line, and orbits approach a line intersection point of Y_m , failing to exhibit behavior of interest.

Unless otherwise stated, the pair (Y_m, K_n) denotes a dynamical system. If a point $x^* \in Y_m$ is invariant for $n' < n$ of the rules in the n -rule sequence defining K_n , then we say x^* is *sometimes invariant* under K_n . If x^* is invariant under all rules defining K_n , we say x^* is *strictly invariant* under K_n . As such, any 1-rule map has only strictly invariant points.

Remark 2. For any (Y_m, K_n) dynamical system, the set of strictly invariant and sometimes invariant points is finite.

As before, we call an n -rule map *redundant* if there exists a length n' rule sequence with $1 \leq n' < n$, such that for all $x \in Y_m$, the orbit of x under the n' -rule map is equal to the orbit of x under the n -rule map. We assume all piecewise n -rule maps are not redundant.

4.2. Convergence and contraction of piecewise n -rule maps. We now give results pertaining to piecewise n -rule maps that are used in determining asymptotic behavior of orbits.

The space Y_m is composed of m pairwise nonparallel, nonconcurrent lines, so any given space Y_m has $\binom{m}{2}$ pairwise line intersection points. Then, for each pairwise intersection point, let η_i label the i th pairwise line intersection angle, where $0 < \eta_i \leq \pi/2$. Let

$$\delta = \min \left\{ \eta_i \mid 1 \leq i \leq \binom{m}{2} \right\}$$

label the least pairwise intersection angle between any two lines in Y_m . Note δ must be acute by definition of Y_m .

Definition 4.2 (Average Contraction Condition). For piecewise n -rule map K_n , let $\bar{\theta}$ label the average of all projection angles in the n -rule sequence defining K_n . Then if

$$(1) \quad \frac{\pi - \delta}{2} < \bar{\theta} \leq \frac{\pi}{2}$$

for least angle δ in Y_m , we say K_n satisfies the average contraction condition with respect to Y_m .

We motivate the introduction of the average contraction condition through the following observations, which are similar to those given in Section 2.1.

Let L_1, L_2 denote lines in \mathbb{R}^2 , intersecting at point z with acute angle δ . Without loss of generality, let $x, y \in L_1$, and take a rule r , such that $r(x), r(y) \in L_2$, and $x, y, r(x), r(y)$ are on the same side of intersection point z . Further, let the orientation value of r be the choice that maps farthest from z . For example, in Figure 3 rule orientation value $o = 1$ maps farther away from the line intersection point when mapping from the particular line.

Assume $d(x, y) = \epsilon > 0$, and $a = d(z, x)$, $a + \epsilon = d(z, y)$. Let θ denote the projection angle of rule r , and let $\gamma = \pi - \delta - \theta$, so that $\gamma = \angle zxr(x) = \angle zyr(y)$. Then if

$$0 < \theta < \frac{\pi - \delta}{2}$$

it follows by use of the law of sines that

$$d(r(x), r(y)) = \left\| \frac{a \sin(\gamma)}{\sin(\theta)} - \frac{(a + \epsilon) \sin(\gamma)}{\sin(\theta)} \right\|_2 = \frac{\epsilon \sin(\gamma)}{\sin(\theta)}$$

but $\theta < (\pi - \delta)/2$ and δ is acute, so under our choice of rule orientation value

$$\frac{\sin(\gamma)}{\sin(\theta)} > 1$$

and then $d(r(x), r(y)) > d(x, y)$: iteration of r over L_1 and L_2 in such a way is then expansive. By similar argument, we see that if $\theta = (\pi - \delta)/2$, then the rule defines an isometry and $d(r(x), r(y)) = d(x, y)$. When

$$\frac{\pi - \delta}{2} < \theta \leq \frac{\pi}{2}$$

then $d(r(x), r(y)) \leq cd(x, y)$, $0 \leq c < 1$. Further, δ is acute, so in the case when $r(x)$ maps opposite the angle $\pi - \delta$, we have $\pi - \delta > \delta$, so if r is contractive when mapping opposite δ , it must also be contractive when mapping opposite $\pi - \delta$.

From the above example, we see that for any rule r , with colinear x, y and colinear $r(x), r(y)$ all on the same side of the line intersection point, we have

$$d(r(x), r(y)) \leq cd(x, y), \quad c \geq 0$$

where c can be computed directly via the law of sines, as a function of the rule projection angle and the opposite line intersection angle. In this case, we call such values c , *separation coefficients*.

Lemma 4.1. *Let lines $L_1, L_2 \subset Y_m$ intersect at a point z with acute angle δ , and let $x, y \in L_1$ lie on the same side of z . Let r_1, r_2 label rules with distinct orientation values, which are chosen so that the rules map farthest from z , and let the points $r_i(x), r_i(y) \in L_2$ and $r_i(r_j(x)), r_i(r_j(y)) \in L_1$, $i, j = 1, 2$, $i \neq j$ all lie on the same side of z . Then for corresponding rule separation constants c_1 and c_2 , $0 \leq c_1 c_2 < 1$ if and only if*

$$(2) \quad \frac{\pi - \delta}{2} < \frac{\theta_1 + \theta_2}{2} \leq \frac{\pi}{2}$$

for rule projection angles θ_1, θ_2 corresponding with rules r_1, r_2 .

Note that $c_1 c_2 < 1$ implies composition of the two rules defines a contraction:

$$d(r_i(r_j(x)), r_i(r_j(y))) \leq c_1 c_2 d(x, y), \quad c_1 c_2 < 1$$

For distinct i, j . Further, the orientation values of the rules are chosen so that the rules map farthest from the line intersection point in each case, and thus the corresponding separation constants are maximized.

Proof of Lemma 4.1. Let $\gamma_1 = \pi - \theta_1 - \delta$ and $\gamma_2 = \pi - \theta_2 - \delta$. We assume that

$$\frac{\pi - \delta}{2} < \theta_1 \leq \frac{\pi}{2}$$

so by Equation 2, we require that

$$(3) \quad \pi - \theta_1 - \delta < \theta_2 \leq \pi - \theta_1$$

By Equation 3 we see that $\sin(\theta_2) > \sin(\pi - \theta_1 - \delta)$, and that

$$\begin{aligned} \sin(\gamma_2) &= \sin(\pi - \theta_2 - \delta) \\ &= \sin(\theta_2 + \delta) \\ &\leq \sin(\pi - \theta_1 + \delta) \end{aligned}$$

Then, through substitution we obtain

$$c_1 c_2 = \frac{\sin(\gamma_1) \sin(\gamma_2)}{\sin(\theta_1) \sin(\theta_2)} < \frac{\sin(\pi - \theta_1 - \delta) \sin(\pi - \theta_1 + \delta)}{\sin(\theta_1) \sin(\pi - \theta_1 - \delta)}$$

but $\sin(\pi - \theta_1 + \delta) = \sin(\theta_1 - \delta) < \sin(\theta_1)$, so

$$\frac{\sin(\pi - \theta_1 - \delta)}{\sin(\theta_1)} \frac{\sin(\pi - \theta_1 + \delta)}{\sin(\pi - \theta_1 - \delta)} = \frac{\sin(\pi - \theta_1 + \delta)}{\sin(\theta_1)} < 1$$

Going the other direction, let $\gamma_1 = \pi - \theta_1 - \delta$ and $\gamma_2 = \pi - \theta_2 - \delta$. Then from

$$c_1 c_2 = \frac{\sin(\gamma_1) \sin(\gamma_2)}{\sin(\theta_1) \sin(\theta_2)} < 1$$

with substitution we obtain

$$\sin(\pi - \theta_1 - \delta) \sin(\pi - \theta_2 - \delta) < \sin(\theta_1) \sin(\theta_2)$$

By the product identity for sine, we have

$$\frac{\cos(-\theta_1 + \theta_2) - \cos(2\pi - \theta_1 - \theta_2 - 2\delta)}{2} < \frac{\cos(\theta_1 - \theta_2) - \cos(\theta_1 + \theta_2)}{2}$$

but $\cos(-1(\theta_1 - \theta_2)) = \cos(\theta_1 - \theta_2)$ so upon simplifying we have

$$\cos(2\pi - \theta_1 - \theta_2 - 2\delta) > \cos(\theta_1 + \theta_2)$$

and by removing cosine we obtain

$$2\pi - \theta_1 - \theta_2 - 2\delta < \theta_1 + \theta_2$$

Note that although cosine is not monotone, by the restrictions on the angles we can remove cosine in such a way. This gives us

$$\pi - \delta < \theta_1 + \theta_2 \implies \frac{\pi - \delta}{2} < \frac{\theta_1 + \theta_2}{2}$$

The case for the upper bound $(\theta_1 + \theta_2)/2 \leq \pi/2$ is clear. \square

In the above lemma, we took the rule orientation values to be chosen in a way that ensures the rules map farthest from the line intersection point. By instead using rule orientation values that force the rules to map closer to the line intersection points, then so long as the two projection angles θ_1, θ_2 satisfy Equation 2, *both rules* must provide contraction (i.e. $c_1 < 1$ and $c_2 < 1$). That is, if $\gamma_2 = \pi - \theta_2 - \delta$, and

$$\frac{\sin(\gamma_2)}{\sin(\theta_2)} > 1$$

under a rule orientation value mapping farther from a line intersection point, then under opposite rule orientation value, we have $\gamma_2' = \pi - (\pi - \theta_2) - \delta$, so

$$\frac{\sin(\gamma_2')}{\sin(\pi - \theta_2)} = \frac{\sin(\theta_2 - \delta)}{\sin(\theta_2)} < 1$$

As an immediate consequence of the above remark and Lemma 4.1, we obtain the following corollary.

Corollary 4.2. *Let $L_1, L_2 \subset Y_m$ intersect at point z with acute angle δ . Further, let K_n be a piecewise n -rule map so that iterates of K_n map between L_1 and L_2 , opposite angle δ , and for initial points $x, y \in L_1$, let the first n points of $\mathcal{O}(x), \mathcal{O}(y)$ remain on the same side of z . Then if K_n satisfies the average contraction condition for least angle δ ,*

$$d(K_n^n(x), K_n^n(y)) \leq C d(x, y), \quad C \in [0, 1)$$

We remark that here $C = c_1 c_2 \cdots c_n$, is the product of the n separation constants coming from the piecewise rule sequence.

Lemma 4.3. *For all lines $L_i \subset Y_m$ and $x, y \in L_i$, if each closed interval*

$$[K_n^i(x), K_n^i(y)] \subset Y_m, \quad 0 \leq i \leq n$$

contains no line intersection points or invariant points, then if K_n satisfies the average contraction condition in Y_m ,

$$d(K_n^n(x), K_n^n(y)) \leq Cd(x, y), \quad C \in [0, 1)$$

Proof. Let δ label the least pairwise intersection angle in Y_m . Then by Corollary 4.2, if K_n satisfies the average contraction condition over Y_m , and iteration of K_n is strictly opposite angle δ , then $d(K_n^n(x), K_n^n(y)) \leq Cd(x, y)$ for $C \in [0, 1)$. But δ is the least angle in Y_m , so if iteration of K_n contracts opposite angle δ on average, then it must also contract opposite every other angle in Y_m on average: if η_i is a distinct line intersection angle, then $\eta_i \geq \delta$, and

$$\frac{\pi - \eta_i}{2} \leq \frac{\pi - \delta}{2}$$

As such, assuming the conditions of the statement, it follows that

$$d(K_n^n(x), K_n^n(y)) \leq Cd(x, y)$$

for $C \in [0, 1)$. □

We note the average contraction condition ensures contraction regardless of rule orientation. The average contraction condition provides sufficient but not necessary conditions for an n -rule map to define a contraction on average.

Lemma 4.4. *If K_n satisfies the average contraction condition over Y_m , then there exists bounded regions $R, R' \subset Y_m$ such that for all $x \in R$, $\overline{\mathcal{O}(x)} \subset R'$.*

Proof. By definition of n -rule maps and the average contraction condition, iteration of an n -rule map K_n in Y_m must, on average, map closer to line intersection points. The lines composing Y_m are pairwise nonparallel, so all lines must intersect, and there must exist a bounded region R containing all such line intersection points. As such, if iteration of K_n maps closer to line intersection points on average, then iteration of the map must remain in a bounded region R' . □

Immediate from proof of Lemma 4.4, we obtain the following corollary.

Corollary 4.5. *If K_n satisfies the average contraction condition over Y_m , then any sequence of points taken from successive preimages of K_n over noninvariant points $x \in Y_m$ diverges in Y_m .*

5. ASYMPTOTIC BEHAVIOR OF PIECEWISE N -RULE MAPS

In this section, we study the asymptotic properties of piecewise n -rule maps satisfying the average contraction condition over Y_m . For piecewise n -rule map K_n and point $x \in Y_m$, we call a *cycle* of K_n over x the application of K_n to x , n times; the cycle of $x_0 \in Y_m$ under K_n is the sequence of points x_0, x_1, \dots, x_n , where $K_n^n(x_0) = x_0$. We let $\mathcal{K}_n := K_n^n$ label the *cycle map* of K_n , so that for $x_0 \in Y_m$, $\mathcal{K}_n(x_0) = x_n$, and $\mathcal{K}_n^t(x_0) = K_n^{tn}(x_0) = x_{tn}$.

If rule r_i in the rule sequence of n -rule map K_n has sometimes invariant point q in Y_m , then for every $h \in Y_m$ such that $K_n^i(h) = q$ for $1 \leq i \leq n$, we call h a *pre-invariant* point of rule r_i . Associated with the (Y_m, K_n) dynamical system, we let Ω denote the set of invariant points of all types, as well as preimages of the

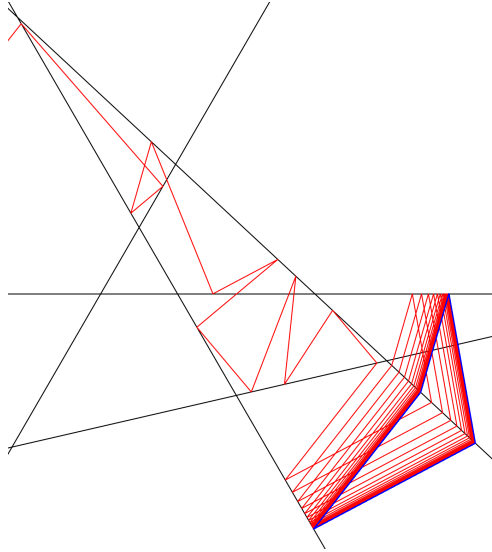


FIGURE 6. Iteration of a piecewise 4-rule map in Y_5 , with the orbit converging to a four-cycle.

cycle map \mathcal{K}_n from all pre-invariant points. Further, if p is a strictly invariant point under K_n , then all points $a \in Y_m$ such that $\mathcal{K}_n^j(a) = p$, $j \in \mathbb{Z}^+$, are also contained in Ω .

Put $Y'_m = Y_m \setminus \Omega$. We call the dynamical system (Y_m, K_n) *degenerate* when iteration of K_n eventually maps to an invariant point of any type; it follows that for the dynamical system (Y'_m, \mathcal{K}_n) to be well defined, (Y_m, K_n) must be a non-degenerate dynamical system. Such degenerate systems arise at bifurcation points, and the remainder of this section focuses on the study of non-degenerate systems. The main result of this section is as follows.

Theorem 5.1. *Let (Y_m, K_n) be a non-degenerate system, with piecewise n -rule map K_n satisfying the average contraction condition over Y_m . Then for all $x \in Y'_m$, the orbit $\mathcal{O}(x)$ converges to a periodic orbit of period kn , $k \in \mathbb{Z}^+$.*

Figure 6 illustrates the kind of dynamics Theorem 5.1 provides, showing the periodic orbit iteration of a 4-rule map converged to in a space Y_5 .

We need some preparatory lemmas to prove Theorem 5.1. First, note that as consequence of Corollary 4.5, if K_n satisfies the average contraction condition, then sequences of points taken from preimages of invariant points diverge in Y_m , so Ω is guaranteed not to be dense in Y_m since the collection of sometimes invariant and strictly invariant points is finite. If, however, K_n fails to satisfy the average contraction condition then such a guarantee may not be made.

If K_n satisfies the average contraction conditions in Y_m , then let U_m denote the set of open intervals $I_a \subset Y_m$ such that the boundary values of each I_a are given by elements in Ω ; no element in Ω is contained within an open interval I_a . Let

$$\hat{\mathcal{O}}(x) := \{x, \mathcal{K}_n(x), \mathcal{K}_n^2(x), \dots\}$$

denote the orbit of x under cycle map \mathcal{K}_n .

Lemma 5.2. *For non-degenerate dynamical system (Y_m, K_n) and piecewise n -rule map K_n satisfying the average contraction condition over Y_m , if $I_a \in U_m$, then there exists an $I_b \in U_m$ such that $\mathcal{K}_n[I_a] \subset I_b$, where I_a, I_b need not be distinct.*

Proof. We proceed by contradiction and assume $\mathcal{K}_n[I_a] \subset I_b \cup I_c$. By definition, the boundary values of each $I_a \in U_m$ are invariant points or preimages of invariant points under \mathcal{K}_n . It follows that if $\mathcal{K}_n[I_a] \subset I_b \cup I_c$, then $I_a = I_d \cup I_e$, as I_a would contain preimages of such boundary values, a contradiction. \square

Lemma 5.3. *If (Y_m, K_n) is non-degenerate with K_n satisfying the average contraction condition and $x \in Y'_m$, then $\hat{O}(x) \subset \bigcup_{i=1}^s I_i$ for finite s and $I_i \in U_m$.*

Proof. We first note that taking $x \in Y'_m$ is, by definition, equivalent to taking $x \in I_a$, for $I_a \in U_m$. For any (Y_m, K_n) dynamical system, there may only be a finite number of invariant points of any type under K_n , and by Corollary 4.5, preimages of n -rule maps satisfying the average contraction condition diverge from points in Y_m . It then follows by definition of the set Ω and corresponding construction of intervals in U_m , that for any bounded region $R \subset Y_m$, there may only be a finite number of such intervals I_a in R . Further, by Lemma 4.4, orbits of n -rule maps satisfying the average contraction condition must remain in a bounded region. Finally, by Lemma 5.2, for every $I_a \in U_m$, $\mathcal{K}_n[I_a] \subset I_b$, and it thus follows that the orbit of x under \mathcal{K}_n is contained in a finite number of intervals. \square

We call an interval $I_c \in U_m$ *confining* if there is a $t \in \mathbb{Z}^+$, $t = t(I_c, K_n)$, such that $\mathcal{K}_n^t[I_c] \subset I_c$.

Lemma 5.4. *If (Y_m, K_n) is a non-degenerate dynamical system with n -rule map K_n satisfying the average contraction conditions over Y_m , then there exists a confining interval I_c in Y_m , and iteration of \mathcal{K}_n over any $x \in Y'_m$ maps into a confining interval in a finite number of iterations.*

Proof. By Lemma 5.3, the orbit of any $x \in Y'_m$ under \mathcal{K}_n is restricted to a finite number of intervals. Thus, by way of the pigeon hole principle, iteration of \mathcal{K}_n is forced to map to an interval it has already visited in a finite number of iterations: a confining interval. And because the orbit is restricted to a finite number of intervals, it must map into a confining interval after a finite number of iterations. \square

Definition 5.1. *Let $I_c \in U_m$ be a confining interval in Y_m , and let $\hat{\mathcal{K}}_n : I_c \rightarrow I_c$ be the induced map of \mathcal{K}_n over the interval of continuity I_c , defined so that if $x \in I_c$ and $\mathcal{K}_n^k(x) \in I_c$ for minimal k , we put $\hat{\mathcal{K}}_n(x) = \mathcal{K}_n^k(x) = K_n^{kn}(x)$.*

Lemma 5.5. *If (Y_m, K_n) is a non-degenerate dynamical system with n -rule map K_n satisfying the average contraction condition over Y_m , and let I_c be a confining interval in Y_m . Then the induced map $\hat{\mathcal{K}}_n$ has a unique fixed point in I_c .*

Proof. By hypothesis, K_n satisfies the average contraction condition over Y_m , so $\hat{\mathcal{K}}_n$ is a contraction mapping over confining interval I_c as consequence of Lemma 4.3. Further, we take the system (Y_m, K_n) to be non-degenerate, so by Lemma 5.2, $\hat{\mathcal{K}}_n[I_c] \subset I_c$ (strict subset). As such, for any $x \in I_c$, the sequence of points $x, \hat{\mathcal{K}}_n(x), \hat{\mathcal{K}}_n^2(x), \dots$ is a Cauchy sequence, and must converge to a unique point in the interval of continuity I_c . It follows that there is a point $x^* \in I_c$ such that $\hat{\mathcal{K}}_n(x^*) = x^*$. \square

We now prove Theorem 5.1.

Proof of Theorem 5.1. By hypothesis, (Y_m, K_n) is a non-degenerate dynamical system, K_n satisfies the average contraction condition, and we take $x \in Y'_m$ so iteration of K_n over x does not map to an invariant point of any type. It then follows as a consequence of Lemma 5.4 that iteration of K_n over $x \in Y'_m$ maps into a confining interval $I_c \in U_m$ in a finite number of iterations. And by consequence of Lemma 5.5 and Definition 5.1, iteration of K_n in a confining interval must converge to a periodic orbit of period kn , $k \in \mathbb{Z}^+$. \square

We remark that for particular periodic orbits generated by an n -rule map in Y_m , we cannot claim that the corresponding basin of attraction is all of Y' , as the periodic orbit is also dependent on initial condition $x_0 \in Y'$. Indeed, work established in [NP15] for example, which concerns piecewise contractions of the interval, motivates questions regarding upper bounds for the number of distinct periodic orbits a fixed (Y_m, K_n) dynamical system can admit. One other question that arises from our analysis is whether there are conditions that can be used to tell whether a dynamical system (Y_m, K_m) is degenerate or not.

Software that can be used to simulate both types of n -rule maps is publicly available at [Eve20].

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