Pre-tutorial questions

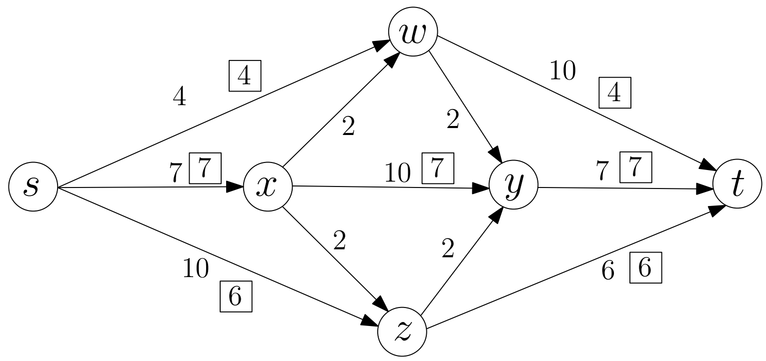
Do you know the basic concepts of this week's lecture content? These questions are only to test yourself. They will not be explicitly discussed in the tutorial, and no solutions will be given to them.

1. Flow network $G = (V, E)$.
   (a) Is G directed or undirected?
   (b) What does it mean that an edge has a capacity?
   (c) What is a flow f in G?
   (d) Why is the flow of an edge bounded by the capacity of an edge?
   (e) What are the capacity and conservation constraints?
2. Min Cut (A, B) of G.
   (a) What is a cut (A, B) of G?
   (b) What is the capacity of a cut?
3. Min cut vs. max flow
   (a) Why is the value of a cut an upper bound on the maximum flow?
   (b) What does the Min Cut - Max Flow theorem state?
4. Ford-Fulkerson
   (a) The Ford-Fulkerson algorithm iteratively increases the flow. How is this done in each iteration?
   (b) Can you upper bound the number of iterations performed by the Ford-Fulkerson algorithm?

Tutorial

Problem 1
The figure below shows a flow network on which an $s - t$ flow is shown. The capacity of each edge appears as a label next to the edge, and the numbers in boxes give the amount of flow sent on each edge. (Edges without boxed numbers have no flow being sent on them.)

1. What is the value of this flow?
2. Is this a maximum $s - t$ flow in this graph? If not, find a maximum $s - t$ flow.
3. Find a minimum $s - t$ cut. (Specify which vertices belong to the sets of the cut.)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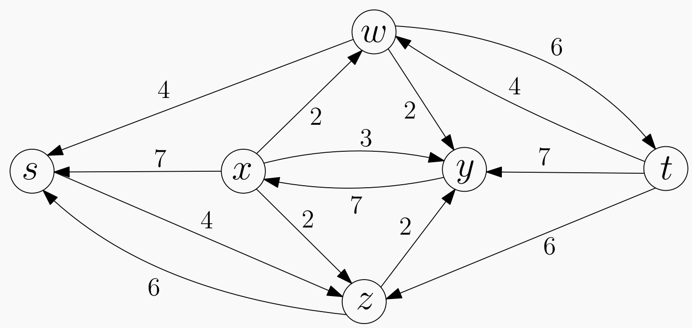
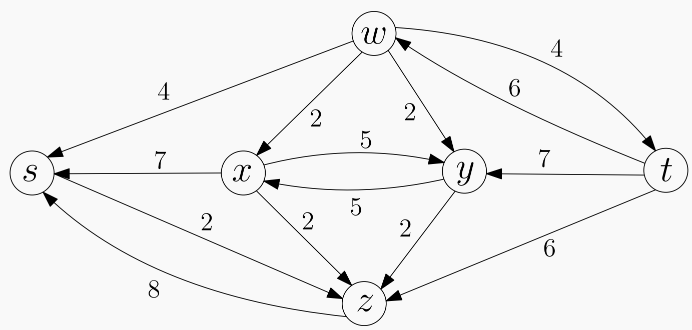
Solution:

1. The value of this flow is 17. (The total amount of flow passing from s to t.)

2. No, this is not a maximum flow. We can find a maximum flow by looking at the residual network. In this case, from the given flow, we can get a max flow with only a single augmenting path, as shown in the picture below. So the maximum flow has value 19. The residual network for the given network is as shown in the figure:

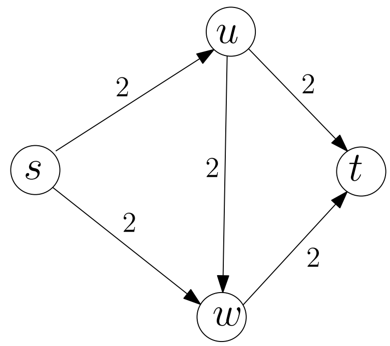


Augment flow with path s, z, y, x, w, t. Capacity of this path is 2 so send 2 new units of flow on this path. The new residual flow shows that no augmenting paths exists, hence, the maximum flow is 19.



3. As explained in class, we can find a minimum cut using the residual network for the final maximum flow. Starting at s, find all vertices that are reachable from s in the residual network. This forms one set in the minimum cut. The other vertices form the other set part of the cut. So a minimum cut in this case are the two sets $\{s, z\}$ and $\{x, w, y, t\}$. You can check that the capacity of this cut (i.e. the total capacity of the edges from $\{s, z\}$ to $\{x, w, y, t\}$, consisting of the directed edges $(s, w), (s, x), (z, y)$, and (z, t)) is 19, as the Max Flow/Min Cut Theorem guarantees.

Problem 2
Find all minimum $s - t$ cuts in the following graph. The capacity of each edge appears as a label next to the edge.



Solution: First, there are exactly four possible cuts in the graph. These are (recalling that the cut must separate s and t):

(a) $\{s\}, \{u, w, t\}$,

(b) $\{s, u\}, \{w, t\}$,

(c) $\{s, w\}, \{u, t\}$, and

(d) $\{s, u, w\}, \{t\}$.

It is a simple matter to determine the capacity of each of these cuts, and find those that are minimum cuts. In the order given above, the capacities of the cuts are 4, 6, 4, and 4. So we see that there are three minimum cuts in this graph, namely

(a) $\{s\}, \{u, w, t\}$,

(b) $\{s, w\}, \{u, t\}$, and

(c) $\{s, u, w\}, \{t\}$.

Problem 3
Let $G = (V, E)$ be an arbitrary flow network, with source s, sink t, and edge capacities $c : E \to \mathbb{Z}^+$. Decide whether each of the following statements are true of false. If it is true, give a short explanation. If it is false, give a counterexample.

1. If f is a maximum s-t flow in G, then f saturates every edge out of s; that is, $f(e) = c(e)$ for each edge e coming out of s.

2. Let (A, B) be a minimum s-t cut with respect to c. Define new capacities $\hat{c}(e) = c(e) + 1$ for each $e \in E$. Then (A, B) is a minimum s-t cut with respect $\hat{c}$.

Problem 4
Design a linear time algorithm that given a flow f verifies that f is maximum. If the flow is maximum your algorithm should output "yes", otherwise, it should output "no". No other action is required.

Problem 5
Let G be a flow network such that, for every edge e in the network, $c(e)$ is an even integer.

1. Argue that the maximum value of a flow is an even integer.

2. Show that there is a maximum flow f such that, for every edge e, the flow over e is an even integer.

Problem 6

Consider a generalization of the minimum cut problem where we want to separate two sets of vertices. An instance is defined by a graph $G = (V, E)$, a pair of disjoint subsets of vertices $S, T \subseteq V$, and edge capacities $c : E \to \mathbb{Z}^+$. The objective is to find a minimum capacity cut (A, B) such that $S \subseteq A$ and $T \subseteq B$.

Design an efficient algorithm for this problem.

Problem 7

Given an **undirected** graph $G = (V, E)$ and edge capacities $c : E \to \mathbb{R}_+$ and two vertices s and t, design an algorithm for computing a minimum capacity s-t cut.

Problem 8

[**Think on your own, will be proved in next lecture**] Consider a directed graph $G = (V, E)$ with capacities c and a source $s \in V$ and sink $t \in V$. Suppose f is a flow such that the residual graph G_f has no augmenting path. First, show that (A, B) is a min cut where A is the set of vertices reachable from s in G_f. Second, show that (A', B') is also a min cut where B' is the set of vertices that can reach t in G_f. *[Hint: use weak duality]*

Problem 9

[**COMP3927 only**] Consider the following simplest-possible randomized algorithm for global min cut: Given an undirected and unweighted graph $G = (V, E)$, choose a random subset A of vertices and if $\emptyset \subsetneq A \subsetneq V$, return the cut $(A, V \setminus A)$; otherwise resample A until we get an A such that $\emptyset \subsetneq A \subsetneq V$. Your task is to give an example of an n-vertex graph such that the probability that the algorithm returns a global min cut is at most $c/2^n$ for some constant c.