Enumeration of barely set-valued tableaux and plane partitions

George Washington University Combinatorics & Algebra Seminar

Sam Hopkins

based on joint works with Melody Chan, Colin Defant, Shahrzad Haddadan,

Alexander Lazar, Svante Linusson, Luca Moci, Svetlana Poznanović, and James Propp

Howard University

January 21st, 2022

Section 1

Tableaux and plane partitions

Standard Young tableaux

The **Young diagram** of a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is left-justified array of boxes with λ_i boxes in ith row:

$$(4, 3, 1) = $$

We will care most about the **rectangle shape** $a \times b := (\overbrace{b, b, \ldots, b}^{a \text{ times}})$.
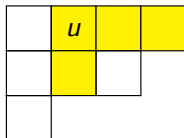
A **standard Young tableau** of shape λ is a filling of the Young diagram with numbers $1, 2, \ldots, n := |\lambda|$, each appearing once, which is increasing along rows and down columns.

Let $\mathcal{SYT}(\lambda) := \{\text{SYTs of shape } \lambda\}$.

$$\mathcal{SYT}(2 \times 2) = \left\{ \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \right\}$$

The Hook Length Formula

The **hook** of box u of a Young diagram is all boxes weakly left or below u:

Hook length $h(u) :=$ number of boxes in hook.

Theorem (Hook Length Formula; Frame–Robinson–Thrall, 1954)

$$\#\mathcal{SYT}(\lambda) = \frac{n!}{\prod_{u \in \lambda} h(u)}$$

For example, $\#\mathcal{SYT}(2 \times 2) = \frac{4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 2 \cdot 1} = 2$.

Set-valued tableaux

A **standard set-valued tableau** of shape λ is a filling of Young diagram with numbers $1, 2, \ldots, n + k$ for some $k \geq 0$, each appearing once, but where multiple numbers can be in the same box.

(Each box must get at least one number, and still needs to be increasing.)

Let $\mathcal{SYT}^{+k}(\lambda)$ be the set of these tableaux. So $\mathcal{SYT}^{+0}(\lambda) = \mathcal{SYT}(\lambda)$.

Our focus is on **barely set-valued tableaux** $\mathcal{SYT}^{+1}(\lambda)$.

For example, there are 10 tableaux in $\mathcal{SYT}^{+1}(2 \times 2)$:

1	2
3	4, 5

1	3
2	4, 5

1	2
3, 4	5

1	3
2, 4	5

1	4
2, 3	5

1	2, 3
4	5

1	2, 4
3	5

1	3, 4
2	5

1, 2	3
4	5

1, 2	4
3	5

Aside: Schur & Grothendieck polynomials

The **Schur function**

$$s_\lambda(x_1, x_2, \ldots) = \sum_{\substack{\text{SSYT } T, \\ \text{shape}(T) = \lambda}} \mathbf{x}^{\text{content}(T)}$$

is the generating function for **semistandard tableaux** (I won't define). Schur functions have many algebraic/geometric guises; one is that they represent Schubert cycles in the cohomology of the Grassmannian.

Similarly, the **(stable) Grothendieck polynomials**

$$G_\lambda(x_1, x_2, \ldots) = \sum_{\substack{\text{set-valued SSYT } T, \\ \text{shape}(T) = \lambda}} (-1)^{|T| - |\lambda|} \mathbf{x}^{\text{content}(T)}$$

represent Schubert cycles in K-theory of the Grassmannian (Buch, 2002).

Plane partitions

An **plane partition** of shape λ is filling of the Young diagram with nonnegative integers, weakly decreasing in rows and columns.

Let $\mathcal{PP}_m(\lambda) :=$ plane partitions of shape λ with entries in $\{0, 1, \ldots, m\}$.

There is a beautiful 3D representation of plane partitions:

5	3	3	2	2	1
4	2	1	1	1	0
4	1	1	1	0	0
3	1	1	0	0	0

$\Leftrightarrow$

$\in \mathcal{PP}_5(4 \times 6)$
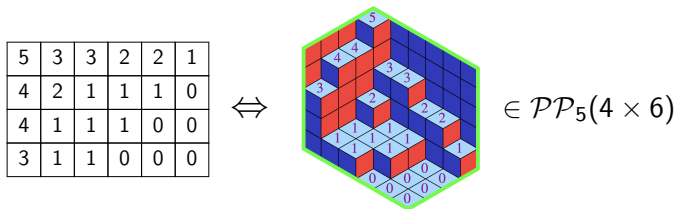
Theorem (MacMahon, c. 1915)

$$\#\mathcal{PP}_m(a \times b) = \prod_{i=1}^{a}\prod_{j=1}^{b} \frac{m + i + j - 1}{i + j - 1}$$

Section 2

Motivation from algebraic geometry

Brill–Noether theory

Let C be a "general" curve of genus g. The **Brill–Noether space** $G_d^r(C)$ is moduli space of maps from C to r-dim'l projective space $\mathbb{P}^r$ of degree d:

$$G_d^r(C) = \left\{ \; \mathbf{8} \; \xrightarrow[\text{degree}(\phi)=d]{\phi} \mathbb{P}^r \right\}$$

Define the **Brill–Noether number** $\rho = \rho(g, d, r)$ as

$$\rho := g - (r+1)(g - d + r)$$

Theorem (Brill–Noether Theorem; Griffiths–Harris, 1980)

$G_d^r(C)$ is nonempty iff $\rho \geq 0$, and in that case $\dim(G_d^r(C)) = \rho$.

Finer invariants of moduli spaces

We could ask for finer information about $G_d^r(C)$ than just its dimension.

For example, when $\rho = 0$, $G_d^r(C)$ is a finite set of points, and the number of points is known to be

$$\#G_d^r(C) = g! \cdot \prod_{i=0}^{r} \frac{i!}{(g-d+r+i)!}$$

Or when $\rho = 1$, $G_d^r(C)$ is itself a smooth curve, and the genus of this curve is known to be

$$\mathrm{genus}(G_d^r(C)) = 1 + \frac{(r+1)(g-d+r)}{g-d+2r+1} \cdot g! \cdot \prod_{i=0}^{r} \frac{i!}{(g-d+r+i)!}$$

Interesting product formulas...

Euler characteristics via tableaux

Comparing to the Hook Length Formula we see that when $\rho = 0$,

$$\#G_d^r(C) = \#\mathcal{SYT}((r+1) \times (g-d+r))$$

Chan–López-Martín–Pflueger–Teixidor i Bigas (2018) showed when $\rho = 1$,

$$\mathrm{genus}(G_d^r(C)) = 1 + \#\mathcal{SYT}^{+1}((r+1) \times (g-d+r))$$

> **Corollary (Chan–López-Martín–Pflueger–Teixidor i Bigas, 2018)**
> $$\#\mathcal{SYT}^{+1}(a \times b) = (ab+1) \cdot \frac{ab}{a+b} \cdot \#\mathcal{SYT}(a \times b)$$

For example, $\#\mathcal{SYT}^{+1}(2 \times 2) = 5 \cdot \frac{4}{4} \cdot \#\mathcal{SYT}(2 \times 2) = 5 \cdot 1 \cdot 2 = 10$.

Chan–Pflueger (2021) showed more generally that for any $\rho \geq 0$, the Euler characteristic of $G_d^r(C)$ is $(-1)^\rho$ times $\#\mathcal{SYT}^{+\rho}((r+1) \times (g-d+r))$. But apparently no product formulas for $\rho > 1$!

Section 3

Down-degree expectations

Decomposing barely set-valued tableaux

A barely set-valued tableau $T' \in \mathcal{SYT}^{+1}(\lambda)$ has a rather simple structure: one special box has two numbers, while all others have a single number.

This leads to a decomposition of T' into a triple (T, i, u) where:

- $T \in \mathcal{SYT}(\lambda)$ is a usual standard tableau;
- $i \in \{0, 1, \ldots, n\}$ is some number;
- u is a **removable box** of the **subshape** $T^{-1}(\{1, 2, \ldots, i\})$.

*(A **subshape** of λ is a Young diagram σ with $\sigma \subseteq \lambda$. A **removable box** of a subshape $\sigma \subseteq \lambda$ is a box whose removal gives another subshape.)*

$$T' = \begin{array}{|c|c|c|} \hline 1 & 2 & 4,7 \\ \hline 3 & 5 & 8 \\ \hline 6 & 9 & 10 \\ \hline \end{array} \qquad \Leftrightarrow \qquad \begin{array}{|c|c|c|} \hline 1 & 2 & ④ \\ \hline 3 & 5 & 7 \\ \hline 6 & 8 & 9 \\ \hline \end{array} = T$$

$i = 6$, $u =$ circled box,
$T^{-1}(\{1 \ldots, i\}) =$ yellow

Distributions on subshapes

The decomposition of barely set-valued tableaux T' motivates us to consider the following probability distribution on subshapes of λ:

- choose $T \in \mathcal{SYT}(\lambda)$ uniformly at random;
- choose $i \in \{0, 1, \ldots, n\}$ uniformly at random;
- select the subshape $T^{-1}(\{1, 2, \ldots, i\})$.

Call this distrubtion on subshapes μ_{SYT}. Also, denote the number of removable boxes of a subshape σ by $\mathrm{ddeg}(\sigma)$, the **down-degree** of σ.

For example, with $\lambda = 2 \times 2$:

σ						
$\mu_{\mathrm{SYT}}(\sigma)$	1/5	1/5	1/10	1/10	1/5	1/5
$\mathrm{ddeg}(\sigma)$	0	1	1	1	2	1

Down-degree expectations

The decomposition of barely set-valued T' can be restated in terms of **expected down-degrees** as follows:

Proposition

$$\mathbb{E}_{\mu_{\mathrm{SYT}}}(\mathrm{ddeg}) = \frac{\#\mathcal{SYT}^{+1}(\lambda)}{(n+1) \cdot \#\mathcal{SYT}(\lambda)}$$

For example, with $\lambda = 2 \times 2$:

$$\mathbb{E}_{\mu_{\mathrm{SYT}}}(\mathrm{ddeg}) = (0 \cdot \frac{1}{5} + 1 \cdot \frac{1}{5} + 1 \cdot \frac{1}{10} + 1 \cdot \frac{1}{10} + 2 \cdot \frac{1}{5} + 1 \cdot \frac{1}{5}) = 1 = \frac{10}{5 \cdot 2}.$$

"Expected down-degrees" terminology due to Reiner–Tenner–Yong (2018).

Barely set-valued plane partitions

For any $m \geq 1$, we can define distribution μ_{PP_m} on subshapes by:

- choose $\pi \in \mathcal{PP}_m(\lambda)$ uniformly at random;
- choose $i \in \{0, 1, \ldots, m-1\}$ uniformly at random;
- select the subshape $\pi^{-1}(\{0, 1, \ldots, i\})$.

Note: $\mu_{\mathrm{SYT}} = \lim_{m \to \infty} \mu_{\mathrm{PP}_m}$ and $\mu_{\mathrm{PP}_1} = $ **uniform** distribution.

Proposition

$$\mathbb{E}_{\mu_{\mathrm{PP}_m}}(\mathrm{ddeg}) = \frac{\#\mathcal{PP}_m^{+1}(\lambda)}{m \cdot \#\mathcal{PP}_m(\lambda)}$$

Here $\mathcal{PP}_m^{+1}(\lambda)$ is the set of **"barely set-valued plane partitions"** which look like what you'd expect:

$$\begin{array}{|c|c|c|} \hline 2 & 2 & 2,0 \\ \hline 2 & 1 & 0 \\ \hline \end{array} \in \mathcal{PP}_2^{+1}(2 \times 3).$$

Section 4

Toggles, toggle-symmetry, and rooks

Toggling subshapes

Let $u \in \lambda$ be a box & $\sigma \subseteq \lambda$ a subshape. Define the **toggle** $\tau_u(\sigma)$ to be

$$\tau_u(\sigma) := \begin{cases} \sigma \setminus u & \text{if } u \text{ is a removable from } \sigma; \\ \sigma \cup u & \text{if } u \text{ is addable to } \sigma; \\ \sigma & \text{otherwise.} \end{cases}$$

*(Here u being **addable** means we can add u to σ and get a subshape.)*
For example,

$$\tau_{(1,2)} \left(\begin{array}{c} \text{(figure)} \end{array} \right) = \begin{array}{c} \text{(figure)} \end{array}$$

$$\tau_{(1,2)} \left(\begin{array}{c} \text{(figure)} \end{array} \right) = \begin{array}{c} \text{(figure)} \end{array}$$

$$\tau_{(1,2)} \left(\begin{array}{c} \text{(figure)} \end{array} \right) = \begin{array}{c} \text{(figure)} \end{array}$$

Toggle-symmetric distributions

For box $u \in \lambda$, define toggleability statistics $\mathcal{T}_u^+, \mathcal{T}_u^-, \mathcal{T}_u$ on subshapes by

$$\mathcal{T}_u^+(\sigma) := \begin{cases} 1 & \text{if } u \text{ is addable to } \sigma; \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{T}_u^-(\sigma) := \begin{cases} 1 & \text{if } u \text{ is removable from } \sigma; \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{T}_u(\sigma) := \mathcal{T}_u^+(\sigma) - \mathcal{T}_u^-(\sigma).$$

Definition

A probability distribution μ on subshapes is called **toggle-symmetric** if we have $\mathbb{E}_\mu(\mathcal{T}_u) = 0$ for all boxes $u \in \lambda$.

In other words, we are as likely to be able to toggle u in as toggle it out.

SYT & plane partition distributions are toggle-symmetric

Lemma (Chan–Haddadan–H.–Moci, 2017)

- *The distribution μ_{SYT} is toggle-symmetric.*
- *For any $m \geq 1$, the distribution μ_{PP_m} is toggle-symmetric.*

Proof sketch: For μ_{SYT}: use $\mu_{\mathrm{SYT}} = \lim_{m \to \infty} \mu_{\mathrm{PP}_m}$.

For μ_{PP_m}: for any $\pi \in \mathcal{PP}_m(\lambda)$, the contribution of π to $\mathbb{E}_{\mu_{\mathrm{PP}_m}}(\mathcal{T}_u)$ is negative the contribution of $\tau_u(\pi)$, where the **(piecewise-linear) plane partition toggle** $\tau_u(\pi)$ is defined by the formula

$$\pi = \begin{array}{|c|c|c|} \hline & b & \\ \hline a & u & d \\ \hline & c & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline & b & \\ \hline a & u' & d \\ \hline & c & \\ \hline \end{array} = \tau_u(\pi)$$

with $u' := \min(a, b) + \max(c, d) - u$. $\square$

Down-degree as sum of toggleability statistics

What's the point? We can sometimes write down-degree in a clever way...

Theorem (Chan–Haddadan–H.–Moci, 2017)

For the rectangle $\lambda = a \times b$, there are coefficients $c_u \in \mathbb{Q}$, $u \in \lambda$ for which

$$\mathrm{ddeg} = \frac{ab}{a+b} + \sum_{u \in \lambda} c_u \mathcal{T}_u$$

By linearity of expectation we obtain enumerative corollaries:

Corollary

$$\frac{\#\mathcal{SYT}^{+1}(a \times b)}{(n+1) \cdot \#\mathcal{SYT}(a \times b)} = \mathbb{E}_{\mu_{\mathrm{SYT}}}(\mathrm{ddeg}) = \frac{ab}{a+b}$$

$$\frac{\#\mathcal{PP}_m^{+1}(a \times b)}{m \cdot \#\mathcal{PP}_m(a \times b)} = \mathbb{E}_{\mu_{\mathrm{PP}_m}}(\mathrm{ddeg}) = \frac{ab}{a+b}$$

Key technical tool: "rooks"

How to write down-degree as a sum of the $\mathcal{T}_u$? Note $\mathrm{ddeg} = \sum_{u \in \lambda} \mathcal{T}_u^-$.
So the key is to find <u>relations</u> among the toggleability statistics.

The "building block" of toggleability statistics relations is the **rook** R_{ij}:

Lemma

We have $R_{ij}(\sigma) = 1$ for any subshape $\sigma \subseteq a \times b$.

Section 5

q-analogs

Comajor index for SYTs

Let $T \in \mathcal{SYT}(\lambda)$ be a standard tableau. A **descent*** of T is an entry i such that $i + 1$ is in a higher row than i. Denote set of descents by $D(T)$.
The **comajor index** of T is $\mathrm{comaj}(T) := \sum_{i \in D(T)} (n - i)$.

Theorem (q-Hook-Length-Theorem; Stanley, c. 1970?)

$$\sum_{T \in \mathcal{SYT}(\lambda)} q^{\mathrm{comaj}(T)} = \frac{[n]_q \cdot [n-1]_q \cdots [2]_q \cdot [1]_q}{\prod_{u \in \lambda} [h(u)]_q}$$

We use standard q-notation $[n]_q := 1 + q + \cdots + q^{n-1} = (1 - q^n)/(1 - q)$.

T	$\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}$	$\begin{array}{cc} 1 & 3 \\ 2 & 4 \end{array}$
$D(T)$	$\varnothing$	$\{2\}$
$\mathrm{comaj}(T)$	0	2

$$\sum_{T \in \mathcal{SYT}(2 \times 2)} q^{\mathrm{comaj}(T)} = q^2 + 1$$

$$= \frac{[4]_q [3]_q [2]_q [1]_q}{[3]_q [2]_q [2]_q [1]_q}$$

Comajor index for barely set-valued SYTs

Let $T \in \mathcal{SYT}^{+1}(\lambda)$ be a barely set-valued tableau. Let $i_*(T)$ denote the bigger number in the special box that has two numbers. A **descent** of T is an entry i such that $i+1$ is in a higher row than i, <u>except that</u>:

- $i_*(T) - 1$ is <u>never</u> a descent;
- $i_*(T)$ is <u>always</u> a descent.

Denote set of descents by $D(T)$. Let $\mathrm{comaj}(T) := \sum_{i \in D(T)} (n + 1 - i)$.

Theorem (H.–Lazar–Linusson, 2021)

$$\sum_{T \in \mathcal{SYT}^{+1}(\lambda)} q^{\mathrm{comaj}(T)} = [ab+1]_q \cdot \frac{[a]_q [b]_q}{[a+b]_q} \cdot \sum_{T \in \mathcal{SYT}(\lambda)} q^{\mathrm{comaj}(T)}$$

Comajor index generating functions: example

T	$\begin{array}{cc} 1 & 2 \\ 3 & 4,5 \end{array}$	$\begin{array}{cc} 1 & 3 \\ 2 & 4,5 \end{array}$	$\begin{array}{cc} 1 & 2 \\ 3,4 & 5 \end{array}$	$\begin{array}{cc} 1 & 3 \\ 2,4 & 5 \end{array}$	$\begin{array}{cc} 1 & 4 \\ 2,3 & 5 \end{array}$
$D(T)$	$\{5\}$	$\{2,5\}$	$\{4\}$	$\{2,4\}$	$\{3\}$
$\mathrm{comaj}(T)$	0	3	1	4	2
T	$\begin{array}{cc} 1 & 2,3 \\ 4 & 5 \end{array}$	$\begin{array}{cc} 1 & 2,4 \\ 3 & 5 \end{array}$	$\begin{array}{cc} 1 & 3,4 \\ 2 & 5 \end{array}$	$\begin{array}{cc} 1,2 & 3 \\ 4 & 5 \end{array}$	$\begin{array}{cc} 1,2 & 4 \\ 3 & 5 \end{array}$
$D(T)$	$\{3\}$	$\{4\}$	$\{2,4\}$	$\{2\}$	$\{2,3\}$
$\mathrm{comaj}(T)$	2	1	4	3	5

$$\sum_{T \in \mathcal{SYT}^{+1}(2\times 2)} q^{\mathrm{comaj}(T)} = q^5 + 2q^4 + 2q^3 + 2q^2 + 2q + 1$$

$$= [5]_q \cdot \frac{[2]_q [2]_q}{[4]_q} \cdot (q^2 + 1)$$

Size generating functions for plane partitions

The **size** $|\pi|$ of a plane partition $\pi \in \mathcal{PP}_m(\lambda)$ is the sum of its entries.

Theorem (MacMahon, c. 1915)

$$\sum_{\pi \in \mathcal{PP}_m(a \times b)} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{[m + i + j - 1]_q}{[i + j - 1]_q}$$

Define size for barely set-valued plane partitions similarly.

Theorem (H.–Lazar–Linusson, 2021)

$$\sum_{\pi \in \mathcal{PP}_m^{+1}(a \times b)} q^{|\pi| - 1} = [m]_q \cdot \frac{[a]_q [b]_q}{[a + b]_q} \cdot \sum_{\pi \in \mathcal{PP}_m(a \times b)} q^{|\pi|}$$

Size generating functions: example

$$\sum_{\pi\in\mathcal{PP}_1(2\times 2)} q^{|\pi|} = q^4 + q^3 + 2q^2 + q + 1 = \frac{[4]_q[3]_q[3]_q[2]_q}{[3]_q[2]_q[2]_q[1]_q}$$

$$\sum_{\pi\in\mathcal{PP}_1^{+1}(2\times 2)} q^{|\pi|-1} = q^3 + 2q^2 + 2q + 1 = [1]_q \cdot \frac{[2]_q[2]_q}{[4]_q} \cdot (q^4 + q^3 + 2q^2 + q + 1)$$

Proofs of q-analogs: q-toggle-symmetry

The basic outline of proofs for q-analogs is same as in case $q = 1$.

For a box u of λ, set $\mathcal{T}_u^q := \mathcal{T}_u^+ - q\mathcal{T}_u^-$. Call a probability distribution μ on subshapes q-**toggle-symmetric** if $\mathbb{E}_\mu(\mathcal{T}_u^q) = 0$ for all $u \in \lambda$.

We define appropriate q-analogs of distributions μ_{SYT}^q and $\mu_{\mathrm{PP}_m}^q$ and show:

Lemma (H.–Lazar–Linusson, 2021)

The distributions μ_{SYT}^q and $\mu_{\mathrm{PP}_m}^q$ are q-toggle-symmetric.

The other ingredient of the proof is:

Theorem (Defant–H.–Poznanović–Propp, 2021)

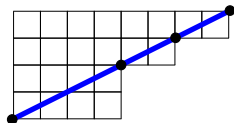*For $\lambda = a \times b$, there are coefficients $c_u(q) \in \mathbb{Q}(q), u \in \lambda$ for which*

$$\mathrm{ddeg} = \frac{[a]_q[b]_q}{[a+b]_q} + \sum_{u \in \lambda} c_u(q)\mathcal{T}_u^q$$

Section 6

Concluding remarks

Concluding remarks

- Not all shapes λ have product formulas for $\#\mathcal{SYT}^{+1}(\lambda)$, but the rook technique does work for a broader class of **"balanced" shapes**:

- Can also look at **shifted shapes** (see Kim–Schlosser–Yoo (2020)), other posets, etc. In fact the q-analogs hold for all **minuscule posets**.
- There are interesting toggle-symmetric distributions not coming from tableaux/plane partitions. For instance, some come from **dynamics** on subshapes. Related to study of **homomesy** for these dynamics.

Thank you!

these slides are available on my website
and papers are on the arXiv:

- Chan, Haddadan, Hopkins, and Moci. "The expected jaggedness of order ideals." `arXiv:1507.00249`
- Reiner, Tenner, and Yong. "Poset edge densities, nearly reduced words, and barely set-valued tableaux." `arXiv:1603.09589`.
- Hopkins, Lazar, and Linusson. "On the q-enumeration of barely set-valued tableaux and plane partitions." `arXiv:2106.07418`.
- Defant, Hopkins, Poznanović, and Propp. "Homomesy via toggleability statistics." `arXiv:2108.13227`.