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Work §6.4

Intuitively **work** is the amount of energy spent accomplishing some task. The formal definition in physics depends on the notion of force.

You can think of force as the push/pull on an object. Formal definition comes from Newton's 2nd Law:

$$F = ma$$

$$\text{Force} = \text{mass} \times \text{acceleration}$$

E.g. The acceleration due to gravity of an object on Earth is $9.8\ m/s^2$ (meters per second squared). So the amount of force that gravity applies to a 10kg object is $10\ kg \times 9.8\ m/s^2 = 98\ kg\ m/s^2 = 98$ Newtons $= 98\ N$

This is called weight — SI unit of force — mass

Work is force applied over a distance. Specifically, if an object moves a distance d while experiencing a constant force F (i.e., constant acceleration) we define

$$\underset{\text{done}}{\text{work}} = F\,d = \text{Force} \times \text{distance}.$$

E.g. What is the work done lifting a 10kg object 100 m in the air? We use the formula:

$$\text{Work} = \text{Force} \times \text{distance} = 9.8\ kg\ m/s^2 \times 100\ m$$

to lift an object we must counteract gravity

$$= 980\ kg\,m^2/s^2$$

$$= 980\ \text{Joules}$$

SI unit of energy

$$= 980\ J.$$

But what if the object experiences a non-constant force? That's where calculus comes in! ...

Suppose our object moves from $x=a$ to $x=b$ and at each point x in between experiences force $f(x)$.
As usual, we can <u>approximate</u> the work done by selecting points $x_0, x_1, \dots, x_n$ in $[a,b]$ at distance $\Delta x = \frac{b-a}{n}$ and within each interval $[x_{i-1}, x_i]$ picking a point x_i^*.
The work done moving the object across the i^{th} interval is

$$W_i \approx \underset{\text{force}}{f(x_i^*)} \cdot \underset{\text{distance}}{\Delta x}$$

So the total work is approximately:

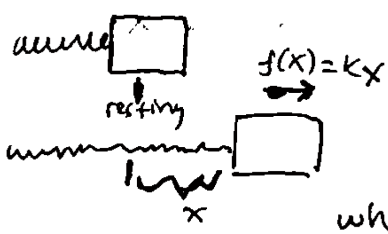
$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

We get an exact value for work as an <u>integral</u>:

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \int_a^b f(x)\,dx$$

$\boxed{\text{Work = integral of force over distance}}$

<u>E.g.</u>:



<u>Hooke</u>'s <u>Law</u> says that the force needed to maintain a <u>spring</u> stretched a distance x from its resting state is given by

$$f(x) = kx$$

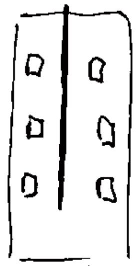
where k is the "<u>spring</u> <u>constant</u>".

<u>Q</u>: Suppose a spring has a spring constant of $10\frac{N}{m}$.
How much work is done stretching this spring 0.5 m?

<u>A</u>: At a stretch distance of x, we need to apply force $f(x) = kx = 0.1\text{ N}$ by Hooke's Law. So

$$\text{Work} = \underset{\text{distance}}{\underset{\text{force over}}{\text{integral of}}} = \int_0^{0.5} f(x)\,dx = \int_0^{0.5} 10x\,dx = 10\,\tfrac{1}{2}x^2\Big]_0^{0.5}$$

$$= 10 \cdot \tfrac{1}{2} \cdot 0.25 = \boxed{1.25\text{J}}$$

E.g.  a 100 meter cable hangs off a building. Its mass is 250 Kilograms. How much work is done lifting the rope to the top of the building?

Let's show 2 (related) approaches to this problem;

(1) Break the cable into many intervals of length Δx.
Let x_i^* be a point in the i^{th} interval.
All the points in the i^{th} interval must be raised
$\approx x_i^*$ meters up to bring them to the top.
Since the density of the cable is $\frac{250 \text{ kg}}{100 \text{ m}} = 2.5 \text{ kg/m}$,
the mass of the i^{th} segment is $2.5 \text{ kg} \cdot \Delta x$,
and weight (force from gravity) is $9.8 \cdot 2.5 \cdot \Delta x$ N.

So total work $\approx \sum_{i=1}^{n} 9.8 \times 2.5 \times x_i^* \, \Delta x$

and taking limit gives Work $= \int_0^{100} 9.8 \times 2.5 \times x \, dx$

$$= 9.8 \cdot 2.5 \cdot \tfrac{1}{2} x^2 \Big]_0^{100} = 9.8 \cdot 2.5 \cdot \tfrac{1}{2} \cdot (100)^2$$

$$= 122{,}500 \text{ J}.$$

(2) After we have pulled up x meters of the cable, there is $(100-x)$ meters left, and this weighs

$$f(x) = 9.8 \cdot (100 - x) \cdot \underbrace{2.5}_{\text{density}} \text{ N}.$$

Integrating this force over the distance gives;

$$\int_0^{100} 9.8\,(100-x) \cdot 2.5 \, dx = -\tfrac{1}{2} \cdot 9.8 \cdot 2.5 \cdot (100-x)^2 \Big]_0^{100}$$

simple u-sub. to antidifferentiate

$$= 122{,}500 \text{ J}$$

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Average value of a function §6.5

To compute the average of a finite list $y_1, y_2, \dots, y_n \in \mathbb{R}$ of real numbers, we add them up and then divide by the number of items in the list:

$$\text{average value} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

E.g. To compute the average height of person in the room, we sum heights of all people and divide by # of people.

But what about computing: the average temperature during a day. A day has ∞-many times, so we cannot just add all the temperatures and divide.
Instead, we approximate by choosing n times to measure temperature at, then let $n \to \infty$.

Def'n If $f(x)$ is a continuous function on $[a,b]$, pick some n and let $x_0 = a$, $x_i = x_{i-1} + \Delta x$ for $i=1,\dots,n$ where $\Delta x = \frac{b-a}{n}$ as usual. To approximate the average of $f(x)$ on $[a,b]$, we sample f at the points $x_1, x_2, \dots, x_n$ and average them:

$$\text{avg. value of } f(x) \text{ on } [a,b] \approx \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$$

And to define average exactly, we let $n \to \infty$:

$$\text{avg. value of } f(x) \text{ on } [a,b] = \lim_{n\to\infty} \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$$

$$= \lim_{n\to\infty} \frac{\sum_{i=1}^{n} f(x_i)}{n} = \frac{1}{b-a} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

since $\Delta x = \frac{b-a}{n}$

$$\boxed{= \frac{1}{b-a}\int_a^b f(x)\,dx}$$

$\Rightarrow$ Average of function on interval =

$$\frac{\text{integral of function on interval}}{\text{length of interval}}$$

E.g. Let's compute the average of $f(x) = 1+x^2$ on $[-1,2]$.

$$\text{avg.} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2-(-1)}\int_{-1}^{2} 1+x^2\,dx$$

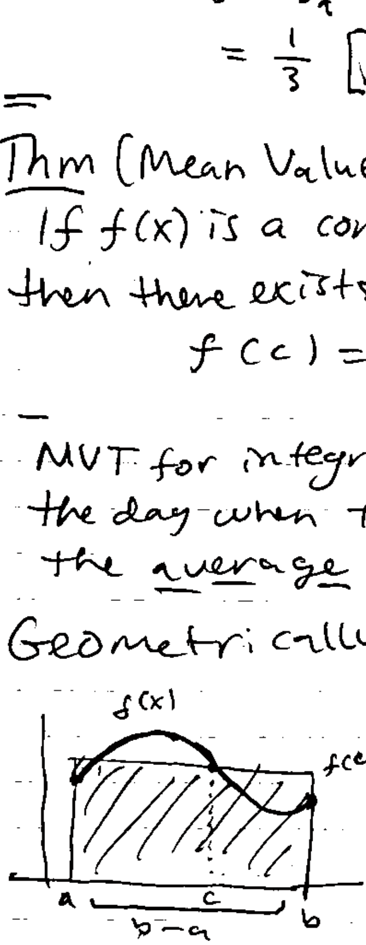
$$= \frac{1}{3}\left[x + \frac{1}{3}x^3\right]_{-1}^{2} = \frac{1}{3}\left(\left(2+\frac{8}{3}\right) - \left(-1-\frac{1}{3}\right)\right) = \frac{6}{3} = \boxed{2.}$$

Thm (Mean Value Theorem for Integrals)
If $f(x)$ is a continuous function defined on $[a,b]$, then there exists a point c with $a \le c \le b$ s.t.

$$f(c) = f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

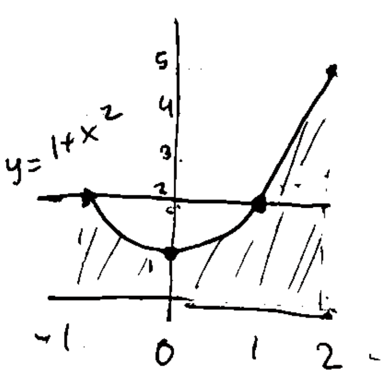
MVT for integrals says there is some time during the day when the temperature is exactly the average temperature for that day.

Geometrically:



MVT for integrals says that there is a $c \in [a,b]$ s.t. $\Leftarrow$ area under curve $y=f(x)$ from a to b is same as area of rectangle of ht. $f(c)$ and width $b-a$.

E.g. Since the average of $f(x) = 1+x^2$ on $[-1,2]$ is 2, MVT for integrals says $\exists\, c \in [-1,2]$ s.t. $f(c) = 2$. Actually, there are two such c's : $c=1$ and $c=-1$ (Since $1+(-1)^2 = 1+(1)^2 = 2$).



Could solve for c by using

$$2 = 1+c^2 \Rightarrow c = \pm 1.$$

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1/27 # Techniques for integration (Chapter 7)

Now that we've seen many applications of (definite) integrals, we will return to the problem of: how to compute integrals, which by Fund. Thm. of Calc. means anti-derivatives (aka "indefinite" integrals)

From Calc I we already know the following integrals:

$\int x^n dx = \frac{1}{n+1} x^{n+1}$ $(n \neq -1)$

$\int 1/x \, dx = \ln(x)$

$\int e^x dx = e^x$

$\int \sin(x) dx = -\cos(x)$

$\int \cos(x) dx = \sin(x)$

We also know that the integral is linear in sense that

$$\int \alpha f(x) + \beta g(x) dx = \alpha \int f(x) dx + \beta \int g(x) dx \quad \text{for } \alpha, \beta \in \mathbb{R}$$

This lets us compute many integrals, but far from all.

At end of Calc I we learned u-substitution technique for computing integrals:

$$\int g(f(x)) \cdot f'(x) \, dx = \int g(u) \, du$$

where $u = f(x)$ and $du = f'(x) dx$

The u-substitution technique lets us compute

e.g. $\int x \sin(x^2) dx = -\frac{1}{2} \cos(x^2)$

(take $u = x^2$ so that $du = 2x dx$)

The u-substitution technique was the "opposite" of the chain rule for derivatives.

We can come up with more integration techniques by doing the "opposite" of other derivative rules, like the product rule ...

Integration by parts §7.1

Recall the product rule says that

$$\frac{d}{dx}\left(f(x)g(x) \right) = f(x)g'(x) + g(x)f'(x)$$

Integrating both sides gives

$$f(x)g(x) = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx$$

Rearranging, this gives

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx}$$

This formula is called integration by parts.

It is more often written in the form:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

where $u = f(x)$ and $v = g(x)$
so that $du = f'(x)dx$ and $dv = g'(x)\,dx$.

In the u-sub. technique we had to make good choice of u.
Integration by parts is similar, but now we have to make good choices for both u and v!
It's easiest to see how this works in examples...

E.g. Compute $\int x \cdot \sin(x)\,dx$.

How to choose u?: General rule of thumb:
choose a u such that du is simpler than u.
In this case, let's therefore choose

$u = x$ which leaves $dv = \sin(x)dx$

$(\Rightarrow du = dx)$ $\Rightarrow$ $v = -\cos(x)$
(by integrating ...)

So the integration by parts formula gives:

$$\int \underbrace{x}_{u} \underbrace{\sin(x)\,dx}_{dv} = \underbrace{x}_{u}\underbrace{(-\cos(x))}_{v} - \int \underbrace{(-\cos(x))}_{v}\underbrace{dx}_{du}$$

This is useful because $\int \cos(x)\,dx$ is something we already know!

$$\Rightarrow \int x\sin(x)\,dx = -x\cos(x) + \int \cos(x)\,dx$$
$$= -x\cos(x) + \sin(x) + C$$

(always remember $+C$!!)

E.g: Compute $\int \ln(x)\,dx$.

Since $d/dx(\ln(x)) = 1/x$ is "simpler" than $\ln(x)$, makes sense to choose $u = \ln(x)$, $dv = dx$
$\Rightarrow du = 1/x\,dx$, $v = x$

$$\Rightarrow \int \underbrace{\ln(x)}_{u}\underbrace{dx}_{dv} = \underbrace{\ln(x)}_{u}\underbrace{x}_{v} - \int \underbrace{x}_{v}\underbrace{1/x\,dx}_{du}$$
$$= x\ln(x) - \int dx = x\ln(x) - x + C$$

A good rule of thumb when picking u is to follow the order:

L - logarithm ($\ln(x)$) ← we didn't really talk about these...
I - inverse trig (like $\arcsin(x)$)
A - algebraic (like polynomials $x^2 + 5x$)
T - trig functions (like $\sin(x)$)
E - exponentials (like e^x)

Remember LIATE: so pick $u = \ln(x)$ over $u = x^2$, $u = x^2$ over $u = \sin(x)$, $u = \sin(x)$ over $u = e^x$, et cetera... (this will make du "simple")

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E.g. Compute $\int x^2 e^x dx$. Following LIATE,
we pick $u = x^2$, $dv = e^x dx$
$\Rightarrow du = 2x\,dx$, $v = e^x$

$$\Rightarrow \int x^2 e^x dx = x^2 e^x - \int e^x \cdot 2x\,dx = x^2 e^x - 2\int x e^x dx$$

But how do we finish? We need to compute $\int x e^x dx$.
To do this, let's use integration by parts again

$$\int \underbrace{x}_{u} \underbrace{e^x\,dx}_{dv} = \underbrace{x}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v} \underbrace{dx}_{du} = xe^x - e^x$$

$$\Rightarrow \int x^2 e^x dx = x^2 e^x - 2\int x e^x dx = x^2 e^x - 2(xe^x - e^x)$$
$$= x^2 e^x - 2xe^x + 2e^x$$

E.g. Compute $\int \sin(x) e^x dx$. Following LIATE,
we pick $u = \sin(x)$, $dv = e^x dx$
$\Rightarrow du = \cos(x)dx$, $v = e^x$

$$\Rightarrow \int \sin(x) e^x dx = \sin(x)e^x - \int e^x \cos(x)\,dx.$$

We need to integrate by parts again for this!

$$\int \underbrace{e^x}_{dv} \underbrace{\cos(x)}_{u} dx = \underbrace{\cos(x)}_{u} \underbrace{e^x}_{v} - \int \underbrace{e^x}_{v} \underbrace{(-\sin(x))\,dx}_{du}$$

Looks like we didn't make any progress! But...

$$\int \sin(x)e^x dx = \sin(x)e^x - \int e^x\cos(x)dx = \sin(x)e^x - \left(\cos(x)e^x + \int \sin(x)e^x dx\right)$$

i.e., $\int \sin(x)e^x dx = \sin(x)e^x - \cos(x)e^x - \int \sin(x)e^x dx$

move to other side!

$$\Rightarrow 2\int \sin(x)e^x dx = \sin(x)e^x - \cos(x)e^x$$

$$\Rightarrow \int \sin(x) e^x dx = \frac{1}{2} e^x(\sin(x) - \cos(x)) + C$$

Definite Integrals: To compute definite integral, always:
(1) First fully compute the indefinite integral.
(2) Then plug in bounds at very end, using Fund. Thm. Calc.

E.g.: To compute $\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx$ we
(1) use u-sub. to get $\int x\sin(x^2)dx = -\frac{1}{2}\cos(x^2)+C$
(2) use FTC: $\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx = \left[-\frac{1}{2}\cos(x^2)\right]_0^{\sqrt{\pi}} = -\frac{1}{2}\cos(\pi)+\frac{1}{2}\cos(0)$
$= -\frac{1}{2}\cdot -1+\frac{1}{2}\cdot 1 = \boxed{1}$

E.g. To compute $\int_0^{\pi} x\sin(x)\,dx$ we
(1) use int. by parts to get $\int x\sin(x)dx = -x\cos(x)+\sin(x)+C$
(2) use FTC: $\int_0^{\pi} x\sin(x)dx = \left[-x\cos(x)+\sin(x)\right]_0^{\pi}$
$= (-\pi\cdot\cos(\pi)+\sin(\pi)) - (-0\cdot\cos(0)+\sin(0)) = -\pi\cdot -1 = \boxed{\pi}$

2/1 Trigonometric Integrals: §7.2

Sometimes we can get a recurrence formula using int. by parts.

E.g.: Prove $\int \sin^n(x)dx = \frac{-1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$.
(Here $n\geq 1$ is an integer and recall $\sin^n(x) = (\sin(x))^n$.)

Pf: We let $u = \sin^{n-1}(x)$ $\quad dv = \sin(x)dx$
$du = (n-1)\sin^{n-2}(x)\cos(x)dx \quad v = -\cos(x)$

so that $\int \sin^n x\,dx = -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\cos^2(x)\,dx$

But recall Pythagorean Identity: $\boxed{\sin^2\theta+\cos^2\theta = 1}$

$\cos^2 x = 1-\sin^2 x$
$\Rightarrow \int \sin^n x\,dx = -\cos x\sin^{n-1}x + (n-1)\int\sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$

move to LHS $\Rightarrow n\int \sin^n x\,dx = -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx$

divide by n $\Rightarrow \int \sin^n x\,dx = -\frac{1}{n}\cos x\sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$

$\blacksquare$

So we can compute iteratively... $\int \sin^0 x\,dx = \int dx = x$

$\int \sin^1 x\,dx = -\cos x$

$\int \sin^2 x\,dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}\int \sin^0 x\,dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x$

$\int \sin^3 x\,dx = -\frac{1}{3}\cos x \sin^2 x + \frac{2}{3}\int \sin^1 x\,dx = -\frac{1}{3}\cos x \sin^2 x - \frac{2}{3}\cos x.$

et cetera...

Another approach to integrating powers of trig fn's:

E.g. Compute $\int \cos^3 x\,dx$. We'll use u-substitution: $\boxed{u = \sin(x) \Rightarrow du = \cos(x)\,dx}$. Trick is to again use Pythagorean identity $\cos^2 x = 1 - \sin^2 x$:

$$\int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx = \int (1-\sin^2 x)\cos x\,dx$$

$$= \int (1-u^2)\,du = u - \frac{1}{3}u^3 + C = \boxed{\sin x - \frac{1}{3}\sin^3 x + C}.$$

E.g. Compute $\int \sin^5 x \cos^2 x\,dx$. This time we write

$$\sin^5 x \cos^2 x = (\sin^2 x)^2 \cos^2 x \sin x$$
$$= (1-\cos^2 x)^2 \cos^2 x \sin x$$

So letting $\boxed{u = \cos(x) \Rightarrow du = -\sin x\,dx \text{ get}}$

$$\int \sin^5 x \cos^2 x\,dx = \int (1-\cos^2 x)^2 \cos^2 x \sin x\,dx$$
$$= \int (1-u^2)^2 u^2 (-du) = -\int (u^2 - 2u^4 + u^6)\,du$$
$$= -\left(\frac{u^3}{3} + 2\frac{u^5}{2} + \frac{u^7}{7}\right) + C$$
$$= \boxed{-\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C.}$$

See from these two examples that you want to make:

1. exactly one factor of $\sin x$ / $\cos x$ next to the dx
2. everything else in terms of "opposite" $\cos x$ / $\sin x$
3. so you set $u = \cos x$ or $\sin x$ and get $du = -\sin x\,dx$ or $\cos x\,dx$.

Using $\cos^2 x = 1 - \sin^2 x$ and $\sin^2 x = 1 - \cos^2 x$,

This strategy will let you compute $\int \sin^m x \cos^n x \, dx$ whenever at least one of m or n is odd.

If both are even, have to use int. by parts recurrence, or "half-angle identities" $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

I don't want to make you memorize these

Recall the other trig functions $\tan\theta$ and $\sec\theta$:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

We have $\boxed{\sec^2\theta = 1 + \tan^2\theta}$ (divide Pythagorean id. by $\cos^2\theta$)

and last semester we saw:

$$d/dx\,(\tan x) = \frac{1}{\cos^2 x} = \sec^2 x, \quad d/dx(\sec x) = \frac{\sin x}{\cos^2 x} = \tan x \sec x.$$

We can thus compute $\int \tan^m x \sec^n x \, dx$ using same u-sub. strategy, factoring out $\sec^2 x \, dx$ or $\tan x \sec x \, dx$:

E.g. Compute $\int \tan^6 x \sec^4 x \, dx$. We can write:

$\tan^6 x \sec^4 x = \tan^6 x \sec^2 x \sec^2 x$ so with

$u = \tan x \Rightarrow du = \sec^2 x$ we have

$$\int \tan^6 x \sec^4 x = \int \tan^6 x \sec^2 x \sec^2 x = \int \tan^6 x (1 + \tan^2 x) \sec^2 x \, dx$$

$$= \int u^6 (1 + u^2) \, du = \int u^6 + u^8 \, du = \frac{u^7}{7} + \frac{u^9}{9} + C$$

$$= \boxed{1/7 \tan^7 x + 1/9 \tan^9 x + C}$$

Exercise: Compute $\int \tan^5 x \sec^7 x \, dx$ using this strategy.

Hint: $\tan^5 x \sec^7 x = \tan^4 x \sec^4 x \tan x \sec x$

$$= (\sec^2 x - 1)^2 \sec^4 x \underbrace{\tan x \sec x}_{d/dx(\sec x)}$$

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