Combinatorial reciprocity for non-intersecting paths

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based on joint work with Gjergji Zaimi

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Combinatorial reciprocity

A *combinatorial reciprocity theorem* asserts $f(-n) = \pm g(n)$, where $f(n)$ and $g(n)$ are two related counting functions. It's a "hidden duality."

For example, the most basic combinatorial reciprocity theorem is

$$\binom{-n}{k} = -1^k \left(\!\!\binom{n}{k}\!\!\right)$$

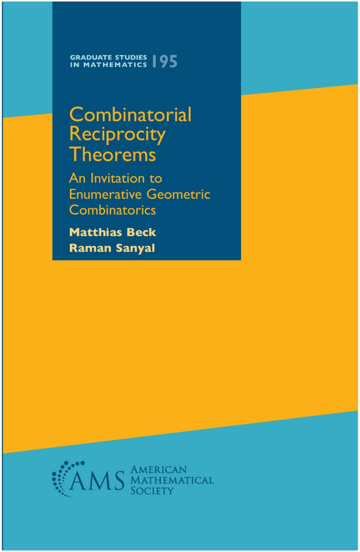
where $\binom{n}{k}$ of course counts the number of k-subsets of $[n] = \{1, 2, \ldots, n\}$, and $\left(\!\binom{n}{k}\!\right)$ counts the number of k-multisets on $[n]$.

In order to make sense of $\binom{-n}{k}$, we observe that

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-(k-1))}{k!}$$

is a polynomial in n, which can then be evaluated at negative numbers.

Combinatorial reciprocity for polynomials



There are many combinatorial reciprocity theorems for polynomial counting functions, including:

- for the order polynomial $\Omega_P(n)$ of a poset P;
- for the chromatic polynomial $\chi_G(n)$ of a graph G;
- for the Ehrhart polynomial $L_{\mathcal{P}}(n)$ of a lattice polytope $\mathcal{P}$.

Combinatorial reciprocity beyond polynomials

But sometimes we can make sense of $f(-n)$, and prove combinatorial reciprocity theorems, for counting functions $f(n)$ that are not polynomials.

We say that $f\colon \mathbb{N} \to \mathbb{C}$ satisfies a *linear recurrence* if there are $d \geq 0$ and $\alpha_1, \ldots, \alpha_d \in \mathbb{C}$ for which

$$f(n+d) + \alpha_1\, f(n+d-1) + \alpha_2\, f(n+d-2) + \cdots + \alpha_d\, f(n) = 0$$

for all $n \geq 0$.

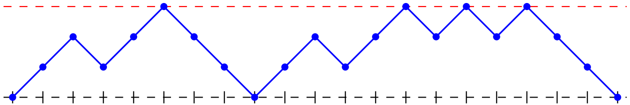
For such an f, we define $f(-n)$ by "running the recurrence backwards." That is, we set

$$f(-n) = \frac{-1}{\alpha_d}\left(f(-n+d) + \alpha_1\, f(-n+d-1) + \cdots + \alpha_{d-1}\, f(-n+1)\right)$$

for all $n \geq 1$.

Bounded Dyck paths

Recall that a *Dyck path* is a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to $(2n,0)$, whose steps are $(1,1)$ or $(1,-1)$, and which never goes below the x-axis. We say a Dyck path is *r-bounded* if it never goes above the line $y = r$.



Example

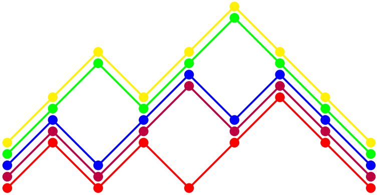
Let $f(n)$ be the number of 3-bounded Dyck paths of length $2n$.

Exercise: Show that $f(n) = F_{2n-1}$, where F_n are the *Fibonacci numbers* defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$.

Therefore, $f(n) = \frac{1}{\sqrt{5}}(\varphi^{2n-1} + \varphi^{-2n+1})$, and thus $f(-n) = f(n+1)$.

Reciprocity for fans of bounded Dyck paths

For two Dyck paths D and D', we write $D \leq D'$ if D is weakly below D'. An *m-fan* of Dyck paths is a tuple $D_1 \leq \cdots \leq D_m$ of nested Dyck paths.



Let $d(m,k;n) = \#$ m-fans of $(2k+1)$-bounded Dyck paths of length $2n$.

Theorem (Cigler–Krattenthaler, 2020)

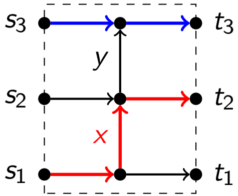
$d(m,k;n)$ satisfies a linear recurrence, and $d(m,k;-n) = d(k,m;n+1)$.

See also follow up work of Jang–Kim–Kim–Song–Song, 2022 on reciprocity for other kinds of bounded lattice paths (Motzkin, Schröder, et cetera).

Acyclic planar networks

An *acyclic planar network* is an acyclic directed graph $G = (V, E)$ embedded in a disk, with boundary vertices $s_1, \ldots, s_m$ (*sources*) and $t_m, \ldots, t_1$ (*sinks*) in clockwise order, and with *edge weights* $w\colon E \to \mathbb{C}$.

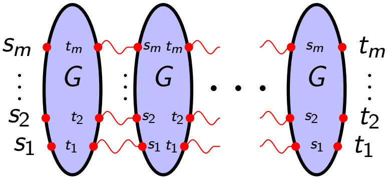
We write $\pi\colon s_i \to t_j$ to mean π is a path in G connecting s_i to t_j, and we write $\Pi = (\pi_1, \ldots, \pi_k)\colon (s_{i_1}, \ldots, s_{i_k}) \to (t_{j_1}, \ldots, t_{j_k})$ to mean Π is a tuple of paths $\pi_\ell\colon s_{i_\ell} \to t_{j_\ell}$. The tuple Π is *non-intersecting* if no two of its paths share any vertices. We set $w(\pi) = \prod_{e\in\pi} w(e)$ and $w(\Pi) = \prod_{\pi\in\Pi} w(\pi)$.

$G =$ 

The above non-intersecting tuple $\Pi\colon (s_1, s_3) \to (t_2, t_3)$ has $w(\Pi) = x$, because by convention edges without labels have weight one.

Reciprocity for non-intersecting paths

Let G be an acyclic planar network for which there is a unique, weight one non-intersecting tuple of paths connecting all the sinks to all the sources. Let G^n denote n copies of G glued together like this (red lines = identify):



For $I = \{i_1 < \cdots < i_k\}, J = \{j_1 < \cdots < j_k\} \subseteq [m]$ let $f(I, J; n) = \sum w(\Pi)$ a sum over non-intersecting tuples $\Pi\colon (s_{i_1}, \ldots, s_{i_k}) \to (t_{i_1}, \ldots, t_{i_k})$ in G^n.

Theorem (H.–Zaimi, 2023)

$f(I, J; n)$ satisfies a linear recurrence. $f(I, J; -n) = -1^{\sigma(I)+\sigma(J)} f(I^c, J^c; n)$ where for $K \subseteq [m]$ we use $\sigma(K) = \sum_{i \in K} i$ and $K^c = [m] \setminus K$.

Proof ingredients I: LGV lemma

Unsurprisingly, the LGV lemma is a major ingredient in our proof.

For network G, let $\mathrm{P}_G = (p_{i,j})$ be *path matrix* of G: $p_{i,j} = \sum_{\pi\colon s_i \to s_j} w(\pi)$.

For an $m \times m$ matrix M and k-subsets $I, J \subseteq [m]$, let $M[I, J]$ denote the square submatrix of M with column indices in I and row indices in J.

Lemma (Lindström–Gessel–Viennot)

For $I = \{i_1 < \cdots < i_k\}, J = \{j_1 < \cdots < j_k\} \subseteq [m]$,

$$\det(\mathrm{P}_G[I, J]) = \sum w(\Pi)$$

a sum over non-intersecting tuples $\Pi\colon (s_{i_1}, \ldots, s_{i_k}) \to (t_{i_1}, \ldots, t_{i_k})$ in G.

Proof ingredients II: compound and adjugate matrices

The other ingredient in our proof is a result from elementary linear algebra.

For an $m \times m$ matrix M, let $\mathrm{com}_k(\mathsf{M})$ and $\mathrm{adj}_k(\mathsf{M})$ be the kth *compound* and *adjugate* matrices of M. These are $\binom{m}{k} \times \binom{m}{k}$ matrices whose rows & columns are indexed by k-subsets $I, J \subseteq [m]$. Specifically, the entries are:

$$\mathrm{com}_k(\mathsf{M})_{I,J} = \det(\mathsf{M}[I,J]) \text{ and } \mathrm{adj}_k(\mathsf{M})_{I,J} = -1^{\sigma(I)+\sigma(J)} \det(\mathsf{M}[I^c, J^c])$$

Lemma (Generalized Laplace expansion of determinant)

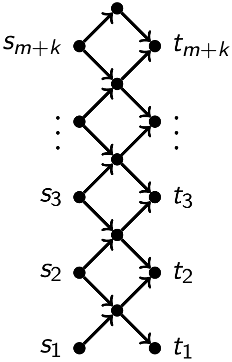
For any $k \geq 0$,

$$\mathrm{com}_k(\mathsf{M}) \times \mathrm{adj}_k(\mathsf{M}) = \mathrm{adj}_k(\mathsf{M}) \times \mathrm{com}_k(\mathsf{M}) = \det(\mathsf{M}) \cdot \mathsf{I},$$

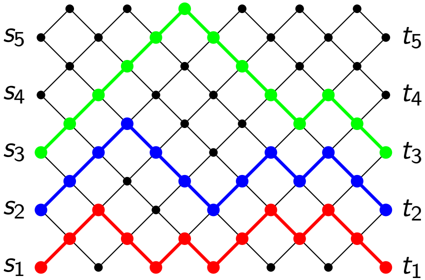
where I *is the* $\binom{m}{k} \times \binom{m}{k}$ *identity matrix.*

Recovering reciprocity for fans of bounded Dyck paths

To recover the fans of bounded Dyck paths reciprocity from our result, we use this network G:

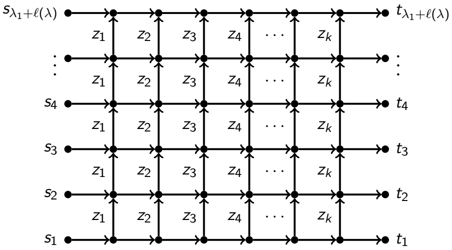


It's easy to see that non-intersecting tuples of paths in G^n correspond to fans of bounded Dyck paths.

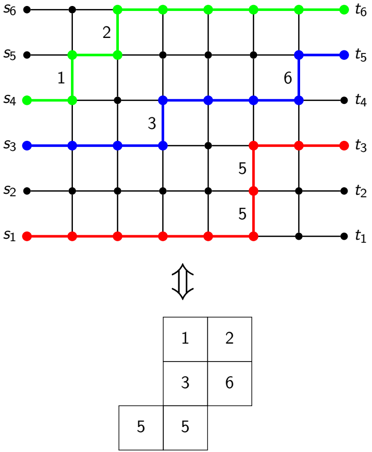


Another important network: Schur polynomials

Consider the following network G:



For appropriate I, J depending on λ and μ, non-intersecting tuples in G correspond to SSYT of shape λ/μ.



Thus, the Schur polynomial $s_{\lambda/\mu}(z_1, \ldots, z_n)$ is the generating function of these non-intersecting tuples.

Reciprocity for Schur functions with repeated entries

What does our reciprocity result say for this Schur polynomial network G?

Fix $\mathbf{z} = (z_1, \ldots, z_k) \in \mathbb{C}^k$. Let $\mathbf{z}^n = (z_1, \ldots, z_k, z_1, \ldots, z_k, \ldots, z_1, \ldots, z_k)$, with each value repeated n times. Then, our result yields the following:

Theorem

$s_{\lambda/\mu}(\mathbf{z}^n)$ satisfies a linear recurrence, and $s_{\lambda/\mu}(\mathbf{z}^{-n}) = -1^{|\lambda/\mu|} \, s_{\lambda^t/\mu^t}(\mathbf{z}^n)$.

More generally, for any homogeneous symmetric function f of degree m, we have that $f(\mathbf{z}^n)$ is a polynomial in n and $f(\mathbf{z}^{-n}) = -1^m \, \omega f(\mathbf{z}^n)$, where $\omega \colon \Lambda \to \Lambda$ is the canonical involution on the ring of symmetric functions Λ. (This will appear as an exercise in the *new edition* of Stanley's EC2).

Extends to quasi-symmetric functions, combinatorial Hopf algebras, etc.

How this project happened: MathOverflow

mathoverflow

- MO:372642
- MO:372811
- MO:373030
- MO:430249

J. Cigler asked a series of questions on MathOverflow about bounded Dyck paths of "negative length." These attracted comments and answers, including from R. Stanley. Subsequently, J. Cigler and C. Krattenthaler wrote their paper.

I noticed the non-intersecting paths interpretation of Cigler's inquiries, and asked a follow-up MO question. G. Zaimi answered, explaining the argument with compound and adjugate matrices. I later asked an MO question about the symmetric function reciprocity, and again R. Stanley and G. Zaimi provided interesting answers. Then, G. Zaimi and I wrote our joint paper.

Open problems

- Find more **interesting networks** to which we can apply the non-intersecting paths reciprocity theorem. For example, can we recover the Motzkin, Schröder, ... reciprocity of Jang et al. this way?
- In an unpublished manuscript from his days as a Harvard undergrad, D. Speyer proved a combinatorial reciprocity theorem for counting **perfect matchings** in a linearly growing sequence of graphs:

  `http://www-personal.umich.edu/~speyer/TransferMatrices.pdf`

  Generalizes an earlier reciprocity result of J. Propp for domino tilings. Is there a connection to the non-intersecting paths reciprocity?
- Find a **bijective** proof of the relationship between compound and adjugate matrices, even in the special case of a path matrix P_G.

Thank you!

these slides are on my website:

`https://www.samuelfhopkins.com/docs/reciprocity_talk.pdf`

and the relevant papers are:

- J. Cigler and C. Krattenthaler. "Bounded Dyck paths, bounded alternating sequences, orthogonal polynomials, and reciprocity." Forthcoming, *European J. Combin.*, 2024. `arXiv:2012.03878`
- S. Hopkins, G. Zaimi. "Combinatorial reciprocity for non-intersecting paths." *Enumer. Comb. Appl.* 3, no. 2, 2023. `arXiv:2301.00405`
- J. Jang, D. Kim, J. S. Kim, M. Song, and U.-K. Song. "Negative moments of orthogonal polynomials." *Forum Math. Sigma* 11, 2023. `arXiv:2201.11344`