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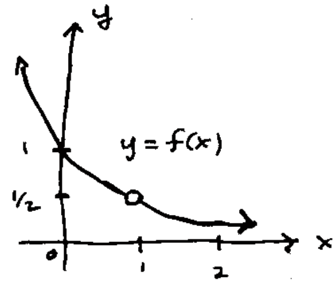
Intro to limits and derivatives § 2.1 + 2.2

So far we have reviewed functions, and hopefully you had seen most of that material before in algebra/pre-calculus. Today, we will introduce calculus in earnest.

The first important notion in calculus is a limit.

Consider the function

$$f(x) = \frac{x-1}{x^2-1}$$



If we graph it near $x=1$, it looks something like this ⇒

Notice the "O" at $x=1$: this shows that $x=1$ is not in the domain of f (because we would divide by zero at $x=1$).

However, it looks like there is a value $f(x)$ "should" take at $x=1$: the value $1/2$.

At x values near 1, $f(x)$ gets close to $1/2$, and it gets closer to $1/2$ the nearer to $x=1$ we get.

We express this by $\displaystyle\lim_{x\to 1} \frac{x-1}{x^2-1} = \frac{1}{2}$

or in words, "the limit of $f(x)$ as x goes to 1 is $1/2$."

Def'n (Intuitive definition of a limit)

The limit of $f(x)$ at $x=a$ is L, written

$$\lim_{x\to a} f(x) = L$$

if we can force $f(x)$ to be as close to L as we want by requiring the input to be sufficiently close, but not equal, to a.

Notice how the definition of the limit does not require $f(x)$ to be defined at $x=a$, or for $f(a)$ to equal the limit $\lim_{x \to a} f(x)$ if it is defined. But... if this is the case we say $f(x)$ is continuous at a.

<u>Def'n</u> $f(x)$ is continuous at a point $x=a$ in its domain if $f(a) = \lim_{x \to a} f(x)$.

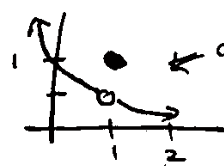
Most of the functions we've looked at so far, like x^n, $\sqrt[n]{x}$, $\sin(x)$, $\cos(x)$, e^x, $\ln(x)$, etc. are continuous at all points in their domain.

<u>Very roughly</u>, this means we can "draw the graph without lifting our pencil."

For an example of a function that is <u>not</u> continuous (i.e., <u>discontinuous</u>) at a point in its domain:

<u>E.g.</u> Let $f(x) = \begin{cases} \frac{x-1}{x^2-1} & \text{if } x \neq 1, -1 \\ 1 & \text{if } x = 1 \end{cases}$

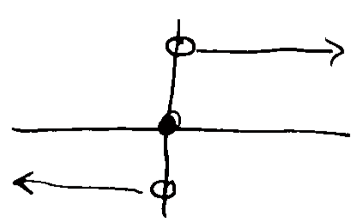
The graph of $f(x)$ near $x=1$ is



← discontinuity at $x=1$

and since $\lim_{x \to 1} f(x) = 1/2 \neq 1 = f(1)$, it's <u>discontinuous</u> at $x=1$.

<u>E.g.</u> Let $f(x) = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$



Then $\lim_{x \to 0} f(x)$ <u>does not exist</u>, because for values of x slightly more than 0, $f(x) = 1$, while for values of x slightly less than 0, $f(x) = -1$.

Does not get close to a <u>single</u> value near $x=0$!

This last example is related to <u>one-sided limits</u>:

<u>Def'n</u> We write $\lim_{x\to a^-} f(x) = L$ and say the <u>left-hand limit</u> of $f(x)$ at $x=a$ is L (or "limit as x approaches a from the left") if we can make $f(x)$ as close to L as we want by requiring x to be sufficiently close to <u>and less than</u> a.

We write $\lim_{x\to a^+} = L$ and say the <u>right-hand limit</u> is L for analogous thing but with values <u>greater than</u> a.

<u>E.g.</u> With $f(x)$ as in previous example, we have

$$\lim_{x\to 0^-} f(x) = -1 \text{ and } \lim_{x\to 0^+} f(x) = 1.$$

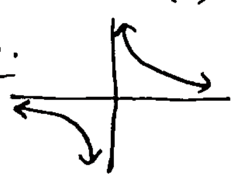
<u>Note</u> $\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x).$

Related to one-sided limits are <u>limits at infinity</u>.

<u>Def'n</u> We write $\lim_{x\to\infty} f(x) = L$ if we can make $f(x)$ arbitrarily close to L by requiring x to be <u>big enough</u>.

We write $\lim_{x\to-\infty} f(x) = L$ if same but for x <u>small enough</u>.

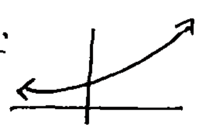
<u>E.g.</u>



For $f(x) = 1/x$, we have

$$\lim_{x\to\infty} f(x) = 0 = \lim_{x\to-\infty} f(x)$$

<u>E.g.</u>



For $f(x) = e^x$, we have

$$\lim_{x\to-\infty} f(x) = 0 \quad (\text{but not as } x\to\infty).$$

<u>E.g.</u> When we defined $e = \lim_{n\to\infty}(1+1/n)^n$, we were using limit at infinity of $f(n) = (1+1/n)^n$.

We can check
$$f(1) = (1+1)^1 = 2$$
$$f(2) = (1+1/2)^2 = 2.25$$
$$\vdots$$
$$f(100) = 2.7048\ldots$$

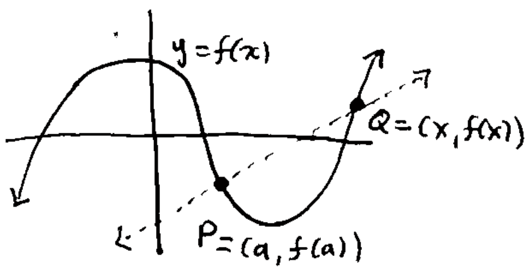
and it gets closer to $e = 2.71\ldots$ as $n\to\infty$. //

Derivative as a limit §2.1, 2.7

If most "normal" functions we work with are continuous at all points in their domain, you might wonder why we define limits at all, especially for points not in domain.

Reason is we want to define the derivative as a limit, and this naturally involves a limit that is "0/0" (so not computable just by "plugging in values").

Recall our discussion from 1st day of class:



We have a point P on a curve, i.e. graph of function $f(x)$. Assume $P=(a,f(a))$ is fixed. For another point Q on the curve, with $Q=(x,f(x))$,

what is the slope of the secant line from P to Q?

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f(x)-f(a)}{x-a}$$

Recall that the tangent line of the curve at P is the limit of the secant line as we send Q to P.

So what is the slope of the tangent line at P?

$$\text{slope of tangent} = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

This is the derivative of $f(x)$ at $x=a$!

Def'n The derivative of $f(x)$ at a point $x = a$ in its domain is $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

E.g. Let's compute the derivative of $f(x) = x^2$ at point $x = 1$. We need to compute

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

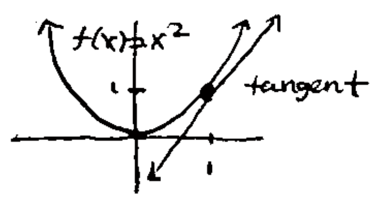
To do this, we use the algebraic trick:

$$(x^2 - 1) = (x+1)(x-1)$$

So $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}} = \lim_{x \to 1} (x+1) = \underline{2}$.

We will justify all these steps later when we talk about rules for computing limits (but it should match $\lim_{x \to 1} \frac{x-1}{x^2-1} = 1/2$ from before...)

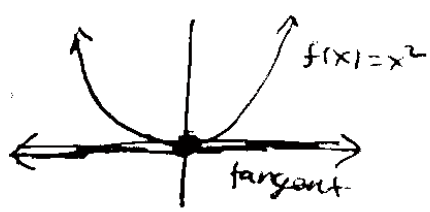
And it looks reasonable that the slope of the tangent at $x = 1$ is 2:



E.g. If instead we compute the derivative of $f(x) = x^2$ at point $x = 0$, we get

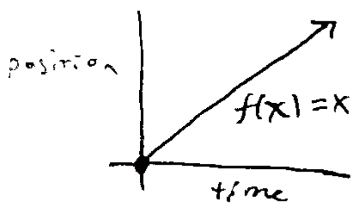
$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 - 0}{x - 0} = \lim_{x \to 0} \frac{x^2}{x} = \lim_{x \to 0} x = \underline{0}.$$

and again it looks like the slope of tangent at $x = 0$ is zero (horizontal):

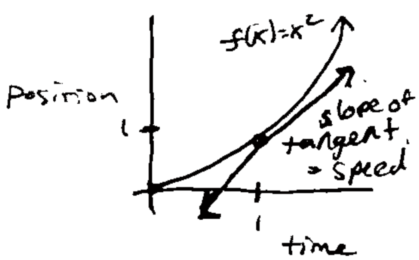


But why do we care about derivatives?
They tell us "instantaneous rate of change."

E.g. Suppose a car's position in meters (away from some initial point) after x seconds is given by function $f(x)$.
How can we find the speed of the car at time $x = a$?



If $f(x) = x$, so that the car is moving at a constant rate of 1 m/s, then clearly at any time its speed is this value of 1 m/s.



But what if instead $f(x) = x^2$ (which represents an accelerating car).
To find the speed at time $x = 1$, we could measure its position at time $x=1$ and $x=b$ for b a little more than 1.
We then compute:

$$\underset{\text{at } x=1}{\text{speed}} \approx \frac{f(b) - f(1)}{b - 1} \leftarrow \text{rate of growth} : \frac{\text{rise}}{\text{run}}$$

To be super accurate, we want b to be very close to 1, so the best definition of speed at time 1 is:

$$\lim_{b \to 1} \frac{f(b) - f(1)}{b - 1}$$, i.e., the derivative of $f(x)$ at $x = 1$!

We saw before that the derivative of $f(x) = x^2$ at $x = 1$ is 2, so the accelerating car is moving faster than the constant speed car at time $x = 1$. However, at time $x = 0$, the derivative is 0, because car is just starting to move!