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Rules for differentiation §3.1

Now we will spend a lot of time learning rules for derivatives.
The simplest derivative is for a constant function:

Thm If $f(x) = c$ for some constant $c \in \mathbb{R}$,
then $f'(x) = \underline{\underline{0}}$.

Pf: We could write a limit, but it's easier to just remember the tangent line definition of the derivative.
If $y = f(x)$ is a line, then the tangent line at any point is $y = f(x)$. In this case, the slope $= 0$ (horizontal) since $f(x) = c$. ☒

$\longleftrightarrow$ $y = f(x) = c$

Actually, the same argument works for any linear function $f(x)$.

Thm If $f(x) = mx + b$ is a linear function,
then $f'(x) = m$ (slope of line).

Some other simple rules for derivatives are:

Thm
- (sum) $(f+g)'(x) = f'(x) + g'(x)$
- (difference) $(f-g)'(x) = f'(x) - g'(x)$
- (scaling) $(c \cdot f)'(x) = c\, f'(x)$ for $c \in \mathbb{R}$.

Pf: These all follow from the corresponding limit laws.
E.g., for sum rule have

$$(f+g)'(x) = \lim_{h \to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x).$$ ☒

The first really interesting derivative is for $f(x) = x^n$, a power function. We've seen:

$d/dx\,(x^0) = 0$, $d/dx\,(x^1) = 1$, $d/dx\,(x^2) = 2x$.

Do you see a pattern?

Thm for any nonnegative integer n, if $f(x) = x^n$

then $\boxed{f'(x) = n \cdot x^{n-1}}$

"bring n down" from the exponent

Pf: We can use an algebra trick:

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^n - a^n}{x-a}$$

check that this multiplies correctly!

$$= \lim_{x\to a} \frac{(x-a)(x^{n-1} + a x^{n-2} + a^2 x^{n-3} + \cdots + a^{n-2}x + a^{n-1})}{x-a}$$

$$= \lim_{x\to a} (x^{n-1} + a x^{n-2} + \cdots + a^{n-1}) = \underbrace{a^{n-1} + a^{n-1} + \cdots + a^{n-1}}_{n \text{ times}}$$

$$= n \cdot a^{n-1}. \checkmark \blacksquare$$

This is one of the most important formulas in calculus! Please memorize it.

E.g. If $f(x) = 3x^4 - 2x^3 + 6x^2 + 5x - 9$ then

$f'(x) = 12x^3 - 6x^2 + 12x + 5$.

Can easily take derivative of any polynomial!

E.g. If $f(x) = x^3$, what is $f''(x)$?

Well, $f'(x) = 3x^2$, so $f''(x) = 3 \cdot 2x$

$= 6x$.

All derivatives of x^n easy to compute this way! //

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Derivatives for more kinds of functions §3.1

Thm For any real number n, if $f(x) = x^n$

then $\boxed{f'(x) = n \cdot x^{n-1}}$

Exactly same formula as for positive integers n.
Proof is similar, and we will skip it...

E.g. Q: If $f(x) = \sqrt{x}$, what is $f'(x)$?
A: $f(x) = x^{1/2}$, so $f'(x) = 1/2\, x^{1/2 - 1} = 1/2\, x^{-1/2}$
$= 1/2 \frac{1}{x^{1/2}} = \frac{1}{2\sqrt{x}}$.

Q: If $f(x) = \frac{1}{x}$, what is $f'(x)$?
A: $f(x) = x^{-1}$, so $f'(x) = -1 \cdot x^{-1-1} = -x^{-2} = \frac{-1}{x^2}$.

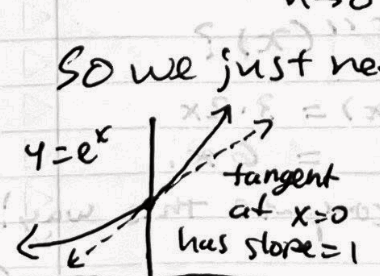
The exponential fn. e^x has a surprisingly simple derivative:

Thm If $f(x) = e^x$, then $f'(x) = e^x = (f(x))$.

Taking derivative of e^x does not change it!
So also $f''(x) = e^x$, $f'''(x) = e^x$, etc...

Pf: We write
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$
$$= \lim_{h\to 0} \frac{e^x \cdot e^h - e^x \cdot e^0}{h} = e^x \cdot \lim_{h\to 0} \frac{e^h - e^0}{h} = e^x \cdot f'(0)$$

So we just need to show $f'(0) = 1$.



But remember, we defined e as the unique $b > 1$ for which slope of tangent of b^x at $x=0$ is one.
So $f'(0) = 1$ by definition of e! ☒

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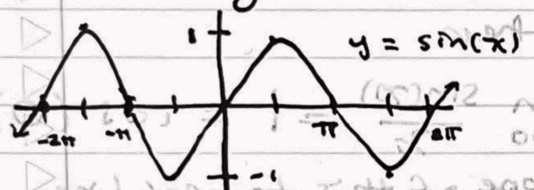
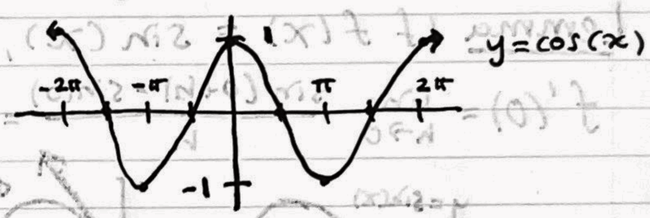
Derivatives of trigonometric functions §3.3

Looking at the graphs of sin(x) and cos(x):





We notice that:

- sin(x) is increasing $\Leftrightarrow$ cos(x) > 0
- sin(x) is decreasing $\Leftrightarrow$ cos(x) < 0
- sin(x) has min./max. $\Leftrightarrow$ cos(x) = 0

- cos(x) is increasing $\Leftrightarrow$ sin(x) < 0
- cos(x) is decreasing $\Leftrightarrow$ sin(x) > 0
- cos(x) has min./max. $\Leftrightarrow$ sin(x) = 0

From these qualitative properties, reasonable to guess:

Thm $\boxed{\begin{aligned} d/dx(\sin(x)) &= \cos(x) \\ \text{and } d/dx(\cos(x)) &= -\sin(x) \end{aligned}}$

E.g. If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$,

so $f''(x) = -\sin(x)$, and $f'''(x) = -\cos(x)$,

and $f^{(4)}(x) = -(-\sin(x)) = \sin(x) = f(x)$.

After 4 derivatives, we get back what we started with!

Can also check that if $f(x) = \cos(x)$, then $f^{(4)}(x) = \cos(x) = f(x)$.

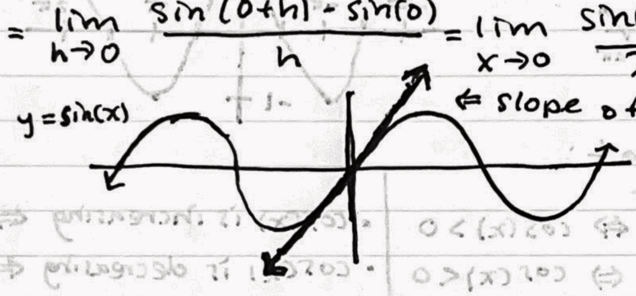
In this way, the trig functions sin(x) and cos(x) behave like e^x, where taking enough derivatives gives us back the original function we started with.

Whereas with a polynomial function like $f(x) = 5x^4 - 3x^3 + 6x^2 + 10x - 9$, taking enough derivatives always gives us zero!

The key step for proving $d/dx(\sin(x)) = \cos(x)$ is this:

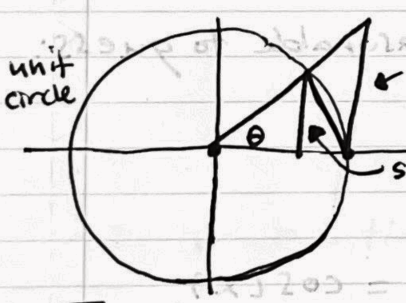
Lemma If $f(x) = \sin(x)$, then

$$f'(0) = \lim_{h\to 0} \frac{\sin(0+h) - \sin(0)}{h} = \lim_{x\to 0} \frac{\sin(x)}{x} = 1 = \cos(0).$$



⇐ slope of this tangent line at $x=0$ should be $= \underline{\underline{1}}$.

There is a nice geometric proof of this Lemma:



⇐ idea of proof is to compare areas of triangles in this drawing. See book for details!

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For our purposes, we will just use the formulas.

To summarize, it is worth memorizing the following important derivatives:

$$\boxed{\begin{array}{ll} d/dx(x^n) = n\cdot x^{n-1} & d/dx(\sin(x)) = \cos(x) \\ d/dx(e^x) = e^x & d/dx(\cos(x)) = -\sin(x) \end{array}}$$

don't forget this <u>negative</u> sign: it's important!