Howard Math 274, HW# 2,

Spring 2022; Instructor: Sam Hopkins; Due: Friday, March 25th

1. Let $\lambda = (\lambda_1, \ldots), \mu = (\mu_1, \ldots) \vdash n$ be partitions of n. Recall that the *lexicographic order* $\prec$ on partitions of n is given by $\mu \prec \lambda$ iff there is some j such that $\mu_i = \lambda_i$ for all $i < j$ and $\mu_j < \lambda_j$. It is a total order: we either have $\mu \prec \lambda$ or $\lambda \prec \mu$ or $\lambda = \mu$.

   A different order on partitions of n is the dominance order. The *dominance order* $\leq$ is defined by $\mu \leq \lambda$ iff $\mu_1 + \mu_2 + \cdots + \mu_j \leq \lambda_1 + \lambda_2 + \cdots + \lambda_j$ for all j. The dominance order is only partial order: we might have neither $\mu \leq \lambda$ nor $\lambda \leq \mu$.

   Show that the lexicographic order *extends* the dominance order in the sense that if $\mu \leq \lambda$ and $\mu \neq \lambda$ then necessarily $\mu \prec \lambda$.

   Bonus problem, just to think about: Recall from the previous semester that a *lattice* is a partial order where every pair of elements has a least upper bound and a greatest lower bound. Show that dominance order on partitions of n is a lattice.

2. Show that we could've used dominance order instead of lexicographic order in our arguments about the triangularity of the transition matrices from p_λ or e_λ to m_μ. That is, show that

   $$p_\lambda = \sum_{\lambda \leq \mu} \alpha_\mu m_\mu \qquad \text{and} \qquad e_\lambda = \sum_{\mu \leq \lambda^t} \beta_\mu m_\mu \quad \text{for coefficients } \alpha_\mu, \beta_\mu \in \mathbb{C}$$

   for any $\lambda \vdash n$, where $\leq$ denotes dominance order and λ^t denotes the transpose of λ.

3. Let $\lambda \vdash n$ and define f^λ to be the coefficient of $x_1 x_2 \cdots x_n$ in the Schur function $s_\lambda(x_1, x_2, \ldots)$. Explain why $f^\lambda = f^{\lambda^t}$. Give an example showing that this is not true for other coefficients of Schur functions, i.e., that $s_\lambda \neq s_{\lambda^t}$ in general.

4. The Cauchy–Binet formula says that if $A = (A_{i,j})$ is an $m \times n$ matrix and $B = (B_{i,j})$ is an $n \times m$ matrix, then the determinant of the $m \times m$ matrix AB can be computed by

   $$\det(AB) = \sum_{I \subseteq [n],\ \#I = m} \det(A \mid_{\text{cols}=I}) \det(B \mid_{\text{rows}=I}).$$

   Here, as always, $[n] := \{1, 2, \ldots, n\}$, and $A \mid_{\text{cols}=I}$ (resp., $B \mid_{\text{rows}=I}$) means the $m \times m$ matrix we get by restricting A to the columns in I (resp., by restricting B to the rows in I).

   Deduce the Cauchy–Binet formula from the Lindström–Gessel–Viennot formula.

   Hint: Consider the network with source vertices $s_1, \ldots, s_m$, target vertices $t_1, \ldots, t_m$, and internal vertices $k_1, \ldots, k_n$, and edges $s_i \to k_j$ with weight $A_{i,j}$ and $k_i \to t_j$ with weight $B_{i,j}$.

5. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition and k an integer. Give a formula for $m_\lambda(\overbrace{1, 1, \ldots, 1}^{k \text{ 1's}})$.

   Hint: Your formula can use the *length* $\ell(\lambda) := \max\{i \colon \lambda_i > 0\}$ of the partition, as well as the *multiplicities* $m_i(\lambda) := \{j \colon \lambda_j = i\}$ for $i \geq 1$.