4/12 Power series § 11.8

A power series is a series of the form
$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$
Here the c_n are a sequence of numbers we call coefficients, while "x" is a variable, which we can specialize to any number.

For example, if $c_n = 1$ for all $n \geq 1$, then we get
$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots,$$
the geometric series with ratio x, which converges $\Leftrightarrow |x| < 1$.
We can think of the power series as defining a function
$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$
which gives a value when x converges. E.g., $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x|<1$.

More generally, for a number a, we can consider a power series centered at a, which is a series of form
$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

E.g. Find the values of x for which the power series
$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$$
converges. Idea: use ratio test.

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x-3|^{n+1}}{n+1} \cdot \frac{n}{|x-3|^n} = \lim_{n\to\infty} |x-3| \left(\frac{n}{n+1}\right) = |x-3|$$

So when $|x-3| < 1$, series converges & when $|x-3|>1$, series diverges.
Notice $|x-3| < 1 \Leftrightarrow 2 < x < 4$. For $x=2$ and $x=4$, ratio test inconclusive.
But $x=2 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ alt. harmonic series $\Rightarrow$ converges, + $x=4 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$ harmonic series $\Rightarrow$ diverges.
Altogether, series converges exactly for $2 \leq x < 4$.

Thm For power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, only three things can happen:

i) The series converges only when $x = a$.

ii) The series converges for all x.

iii) There is a positive number R such that the series converges when $|x-a| < R$ and diverges when $|x-a| > R$.

Pf idea: Ratio test, like last example. ☒

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The number R in the above thm called radius of convergence, and we declare $R = 0$ in case i) and $R = \infty$ in case ii).

The interval $a - R \overset{?}{\leq} x \overset{?}{\leq} a + R$ is called the interval of convergence of the series.

WARNING: whether the series converges at end points $a-R, a+R$ is tricky, usually have to use something beyond ratio test.

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4/14 E.g. For n a positive integer, the number n factorial is $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$ (and $0! = 1$).

Consider the power series centered at 0 with coeff's $c_n = \frac{1}{n!}$:

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = \frac{1}{0!} + \frac{1}{1!} x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \cdots$$

Let's find the radius of convergence of this series.

$$L = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \lim_{n \to \infty} \frac{|x|}{(n+1)}$$

For any fixed x, $(n+1)$ is eventually much bigger than $|x|$,

So $L = \lim_{n \to \infty} \frac{|x|}{(n+1)} = 0$ for every x.

Thus, Ratio Test says $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converges for all x, i.e., radius of convergence is $R = \infty$.

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Exercise

E.g. Show radius of convergence of $\sum_{n=0}^{\infty} n! \, x^n$ is $\underline{0}$. //

Representing functions as power series §11.9

We have seen that $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots = \frac{1}{1-x}$ for $|x|<1$

So we can represent the function $f(x) = \frac{1}{1-x}$ as a power series $f(x) = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$.

Another way to think about this: we have the partial sums $S_n(x) = 1 + x + x^2 + \cdots + x^n$, which are polynomials in x.

And $f(x) = \sum_{n=0}^{\infty} x^n$ means $f(x) = \lim_{n\to\infty} S_n(x)$ for $|x|<1$.

We can represent many other functions (especially rational functions) as power series (especially geometric series);

E.g. How to write $f(x) = \frac{1}{1+x^2}$ as a power series?

Write $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

important: substitution technique!

$$= 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

This geometric series converges for $|(-x^2)|<1$, i.e., $|x|<1$.

E.g. How to find power series representation of $f(x) = \frac{1}{x+2}$?

Write $\frac{1}{2+x} = \frac{1}{2(1+\frac{x}{2})} = \frac{1}{2}\cdot\frac{1}{1+\frac{x}{2}} = \frac{1}{2}\,\frac{1}{1-(\frac{-x}{2})}$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}\, x^n$$

This geo. series converges for $|\frac{-x}{2}|<1$, i.e. $|x|<2$ meaning $x \in (-2,2)$.

E.g. What about $\frac{x^3}{x+2}$? Here we write:

$$\frac{x^3}{x+2} = x^3\cdot\frac{1}{x+2} = x^3\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{n+3}$$

$$= \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 - \cdots = \sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{2^{n-2}}x^n$$

As in the previous example, the interval of convergence is $(-2,2)$.

4/17 Differentiating and Integrating Power Series §11.9

Thm If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ is a power series at $x=a$ with nonzero radius of convergence $R>0$, then

(i) $f'(x) = \sum_{n=0}^{\infty} n \cdot c_n (x-a)^{n-1}$ is the derivative,

(ii) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1}$ is the integral (where C is any constant)

and these power series also have radius of convergence $R>0$.

Note: This is saying we can differentiate / integrate power series "as though they were polynomials":

$$d/dx\,(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots$$

$$\int c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots\,dx = C + c_0 x + \frac{c_1}{2} x^2 + \frac{c_2}{3} x^3 + \cdots$$

E.g. We know that $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$.

$$d/dx\left(\frac{1}{1-x}\right) = d/dx\left((1-x)^{-1}\right) = -(1-x)^{-2} \cdot -1 = \frac{1}{(1-x)^2}$$

so the rule for differentiating power series says

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + \cdots = \sum_{n=0}^{\infty} (n+1) x^n.$$

E.g. How to find power series representation of $\ln(1+x)$?

Notice that $\int \ln(1+x)\,dx = \frac{1}{1+x}$ and we know

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n,$$

so by the rule for integrating power series we get

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}\right) + C$$

$$= C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = C + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

At $x=0$ have $\ln(1+0)=0$, so the integration constant is $C=0$

$$\Rightarrow \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

In both above examples, radius of convergence is $R=1$

Taylor Series § 11.10

Let $f(x)$ be infinitely differentiable in an interval containing $x=a$.

Use $f^{(n)}(x)$ to mean the n^{th} derivative of $f(x)$;

e.g. $f^{(0)}(x) = f(x)$, $f^{(1)}(x) = f'(x)$, $f^{(2)}(x) = f''(x)$, etc.

Def'n The Taylor series of $f(x)$ at $x=a$ is the power series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$

Most important case is when $a=0$, and then is called Taylor-Maclaurin (or just Maclaurin) series;

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Why do we define Taylor series like this? Look what happens when we take n derivatives:

$$d^n/dx^n \left(\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \right) = \frac{f^{(n)}(0)}{n!} \cdot n \cdot (n-1) \cdot \cdots \cdot 3 \cdot 2 \cdot 1 \cdot x^0 + \text{higher powers of } x$$

which means that the n^{th} derivative of Taylor series at $x = a (=0)$ is $f^{(n)}(a)$ ($= f^{(n)}(0)$ for Maclaurin series).

This means if $f(x)$ has a power series representation (at $x=a$) it must be the Taylor series!

E.g. Exercise Show Maclaurin series of $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$.

E.g. Let's find the Maclaurin series of $f(x) = e^x$.

We know $d/dx(e^x) = e^x$, so in fact $f^{(n)}(x) = e^x$ for all $n \geq 0$, and thus $f^{(n)}(0) = e^0 = 1$ for all $n \geq 0$.

This means the Taylor-Maclaurin series of e^x is $\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$

Recall: We saw this power series had radius of convergence $R = \infty$.

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WARNING: There is no reason the Taylor series has to converge (i.e., have positive radius of convergence $R > 0$), and even if it does, it doesn't necessarily converge to same function as $f(x)$ itself.

E.g. Try $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ as an exercise.

So how to show in practice that $f(x)$ equals its Taylor series?

Let us define the degree n Taylor polynomial $T_n(x)$ (centered at $x=a$) of $f(x)$ to be n^{th} partial sum of Taylor series:

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = (x-a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

E.g. For $f(x) = e^x$ and $a=0$, $T_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$.

By definition, the Taylor series is $T(x) = \lim_{n\to\infty} T_n(x)$.

So in order to show that $T(x) = f(x)$, in some open interval $|x-a| < d$, we need to look at the remainder

$$R_n(x) = f(x) - T_n(x)$$

and show that $\lim_{n\to\infty} R_n(x) = 0$. To do that...

Theorem (Taylor's Inequality)

Suppose that $|f^{(n+1)}(x)| \leq M$ for all $|x-a| \leq d$.

Then the remainder for n^{th} Taylor polynomial satisfies

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \text{ for all } |x-a| \leq d.$$

Note: Notice how we bound the error for $T_n(x)$ in terms of $f^{(n+1)}(x)$, i.e., the next derivative after those appearing in $T_n(x)$.

Let's use Taylor's inequality to show $f(x) = e^x$ is equal to its Taylor-Maclaurin series for all x.

We need to show that for $T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k$ and remainder $R_n(x) = f(x) - T_n(x)$, have $\lim_{n\to\infty} R_n(x)$.

Fix an arbitrary d and focus on x where $|x| \le d$.

By Taylor's Inequality, have

$$|R_n(x)| \le \frac{M}{(n+1)!} \cdot |x|^{n+1}, \text{ where}$$

$|f^{(n+1)}(x)| \le M$ is a bound on the $(n+1)^{st}$ derivative.

But notice that for any n, $f^{(n+1)}(x) = e^x$, so a bound on $|f^{(n+1)}(x)|$ is e^d if $|x| \le d$. Thus,

$$|R_n(x)| \le \frac{e^d}{(n+1)!} |x|^{n+1} \quad \text{for all } |x| \le d.$$

Hence,

$$\lim_{n\to\infty} |R_n(x)| \le \lim_{n\to\infty} \frac{e^d}{(n+1)!} |x|^{n+1} = e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

where we use the important fact $\boxed{\lim_{n\to\infty} \frac{r^n}{n!} = 0}$ for any fixed r (factorial is "super-exponential").

Since the d we fixed was arbitrary, we get

$$\lim_{n\to\infty} R_n(x) = 0 \quad \text{for all } x,$$

and thus $\boxed{e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots$ for all x.

<u>Key point</u>: This worked because the derivative $f^{(n)}(x)$ of $f(x) = e^x$ does not increase as $n \to \infty$. Same idea works for other similar $f(x)$...

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More important Taylor series § 11.10

Let's find the Taylor-Maclaurin series for $f(x) = \sin(x)$.
To do this, we need to take derivatives of $\sin(x)$:

$$f^{(0)}(x) = \sin(x) \Rightarrow f^{(0)}(0) = 0$$
$$f^{(1)}(x) = \cos(x) \Rightarrow f^{(1)}(0) = 1$$
$$f^{(2)}(x) = -\sin(x) \Rightarrow f^{(2)}(0) = 0$$
$$f^{(3)}(x) = -\cos(x) \Rightarrow f^{(3)}(0) = -1$$

and then $f^{(4)}(x) = \sin(x) = f^{(0)}(x)$ so this pattern repeats

This means $f^{(n)}(0) = \begin{cases} 0 & n \text{ even} \\ (-1)^m & \text{if } n = 2m+1 \text{ is odd} \end{cases}$

So the Taylor-Maclaurin series of $\sin(x)$ is

$$\sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Moreover, because $|f^{(n)}(x)|$ is bounded for all x, the same technique using Taylor's inequality we employed to show that e^x equals its Taylor series for all x works for $\sin(x)$:

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \text{ for all } x.$$

Something very similar happens for $f(x) = \cos(x)$.
This time the pattern of $f^{(n)}(x)$ is $1, 0, -1, 0, \ldots$,
and again $\cos(x)$ equals its Taylor series for all x, so:

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Note: Can also find this by taking derivative of Taylor series for $\sin(x)$.

Another interesting example is $f(x) = (1+x)^k$, for which:

$$f^{(0)}(x) = (1+x)^k \Rightarrow f^{(0)}(0) = 1$$
$$f^{(1)}(x) = k\cdot(1+x)^{k-1} \Rightarrow f^{(1)}(0) = k$$
$$f^{(2)}(x) = k\cdot(k-1)(1+x)^{k-2} \Rightarrow f^{(2)}(0) = k\cdot(k-1)$$
$$f^{(3)}(x) = k(k-1)(k-2)(1+x)^{k-3} \Rightarrow f^{(3)}(0) = k(k-1)(k-2)$$
$$\vdots \qquad\qquad \vdots$$
$$f^{(n)}(x) = k(k-1)(k-2)\cdots(k-n+1)(1+x)^{k-n} \Rightarrow f^{(n)}(0) = k(k-1)\cdots(k-n+1)$$

This gives us the Taylor series:

$$\sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n = 1 + kx + \frac{k(k-1)}{2}x^2 + \frac{k(k-1)(k-2)}{6}x^3 + \cdots$$

This series is called the <u>binomial series</u>, and its coefficients are called <u>binomial coefficients</u> which also have notation $\binom{k}{n} = \dfrac{k(k-1)\cdots(k-n+1)}{n!}$

It can be shown that this Taylor series has radius of convergence $R=1$, and ~~...~~ where it converges it equals $f(x) = (1+x)^k$:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n \text{ for } |x|<1$$

<u>Notice</u>: Case $k=-1$ of the above gives:

$$\frac{1}{1+x} = (1+x)^{-1} = \sum_{n=0}^{\infty} \frac{-1(-1-1)\cdots(-1-n+1)}{n!} x^n = \sum_{n=0}^{\infty} \frac{-1\cdot -2 \cdot \cdots \cdot -n}{1\cdot 2\cdot\cdots\cdot n} x^n$$
$$= \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$$

We <u>know</u> this since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$

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4/24 Multiplying power series §11.10

We have already seen how to get new power series from old using substitution:

E.g. Since $\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$, (for $|x|<1$)

$$\frac{1}{1+2x} = \frac{1}{1-(-2x)} = \sum_{n=0}^{\infty} (-2x)^n = \sum_{n=0}^{\infty} (-1)^n 2^n x^n \quad (\text{for } |x| < \tfrac{1}{2})$$

E.g. Use $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ to write power series for e^{-x^2}.

Using this technique is much faster than re-deriving the Taylor series by taking derivatives...

We can do something similar for multiplication:

E.g. Let's write the first ~~three~~ four terms of the (Maclaurin-) Taylor series of $f(x) = e^x \cdot \sin(x)$.

We could do this by taking derivatives of $f(x)$, but instead let's use what we already know:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

The trick is that we can multiply these series like they're polynomials:

$$e^x \cdot \sin(x) = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{2} + \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{5!} + \cdots\right)$$

$$= x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right) - \frac{x^3}{6}\left(1 + x + \frac{x^2}{2} + \cdots\right) + \frac{x^5}{5!}(1 + \cdots) + \cdots$$

$$= \left(x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} \cdots\right) - \left(\frac{x^3}{6} + \frac{x^4}{6} + \frac{x^5}{12} + \cdots\right) + \left(\frac{x^5}{5!} + \cdots\right) + \cdots$$

$$= x + x^2 + \left(\frac{1}{2} - \frac{1}{6}\right)x^3 + \left(\frac{1}{6} - \frac{1}{6}\right)x^4 + \left(\frac{1}{24} - \frac{1}{12} + \frac{1}{5!}\right)x^5 + \cdots$$

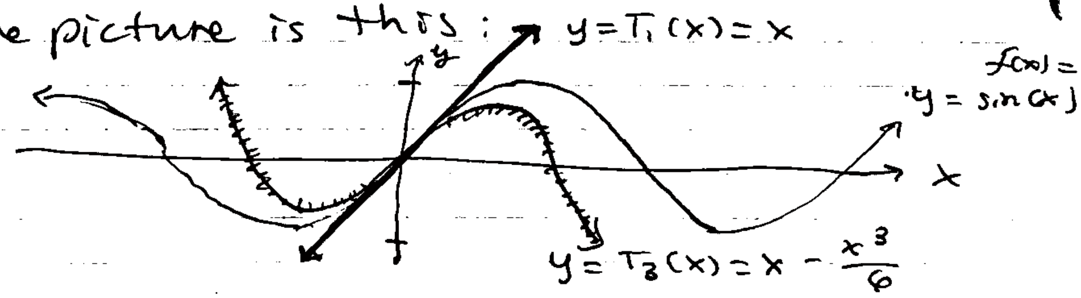
$$= x + x^2 + x^3/3 - x^5/30 + \cdots \leftarrow \text{1st four nonzero terms!}$$ //

Applications of Taylor series: approximation

The main application of Taylor series/polynomials is approximation.
If $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is Taylor series of $f(x)$, and
$T(x) = f(x)$ for all $|x-a| < R$ (radius of convergence)
Then we can expect that $f(x) \approx T_n(x)$ for $x \approx a$,
where $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ is a Taylor polynomial.

E.g. To evaluate $\sin(\frac{1}{2})$ we can use the degree 3 Taylor polynomial approximation: $\sin(x) \approx x - \frac{x^3}{6}$
So $\sin(\frac{1}{2}) \approx \frac{1}{2} - \frac{1}{6}(\frac{1}{2})^3 = \frac{1}{2} - \frac{1}{48} \approx 0.48\ldots$

To get a better approximation, use a higher value of n.
The picture is this:



Each $T_n(x)$ does a better and better job of approximating $f(x)$ for x near the center a of the Taylor series.
Notice that $y = T_1(x) = f(a) + f'(a)(x-a)$ is tangent line to curve $y = f(x)$ at $x = a$, the best linear approximation.

To bound the error of our approximations of $f(x)$ by the Taylor polynomial $T_n(x)$, we can use Taylor's inequality, or sometimes other tools like the alternating series inequality!