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## Rules for differentiation §3.1

Now we will spend a lot of time learning rules for derivatives.
The simplest derivative is for a constant function:

**Thm** If $f(x) = c$ for some constant $c \in \mathbb{R}$,
then $f'(x) = 0$.

Pf: We could write a limit, but it's easier to just remember the tangent line definition of the derivative.
If $y = f(x)$ is a line, then the tangent line at any point is $y = f(x)$. In this case, the slope $= 0$ (horizontal) since $f(x) = c$. $\blacksquare$

$\longleftrightarrow \; y = f(x) = c$

Actually, the same argument works for any linear function $f(x)$.

**Thm** If $f(x) = mx + b$ is a linear function,
then $f'(x) = m$ (slope of line).

Some other simple rules for derivatives are:

**Thm**
- (sum) $(f+g)'(x) = f'(x) + g'(x)$
- (difference) $(f-g)'(x) = f'(x) - g'(x)$
- (scaling) $(c \cdot f)'(x) = c\, f'(x)$ for $c \in \mathbb{R}$.

Pf: These all follow from the corresponding limit laws.
E.g., for sum rule have

$$(f+g)'(x) = \lim_{h \to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x). \quad \blacksquare$$

The first really interesting derivative is for $f(x) = x^n$, a power function. We've seen:

$d/dx\,(x^0) = 0$, $d/dx\,(x^1) = 1$, $d/dx\,(x^2) = 2x$

Do you see a pattern?

Thm for any nonnegative integer $n$, if $f(x) = x^n$ then

$$\boxed{f'(x) = n \cdot x^{n-1}}$$

"bring n down" from the exponent

Pf: We can use an algebra trick:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^n - a^n}{x - a}$$

$$= \lim_{x \to a} \frac{(x-a)(x^{n-1} + a x^{n-2} + a^2 x^{n-3} + \cdots + a^{n-2} x + a^{n-1})}{x - a}$$

check that this multiplies correctly!

$$= \lim_{x \to a} (x^{n-1} + a x^{n-2} + \cdots + a^{n-1}) = \underbrace{a^{n-1} + a^{n-1} + \cdots + a^{n-1}}_{n \text{ times}}$$

$$= n \cdot a^{n-1}. \checkmark \quad \blacksquare$$

This is one of the most important formulas in calculus! Please memorize it.

E.g. If $f(x) = 3x^4 - 2x^3 + 6x^2 + 5x - 9$ then

$f'(x) = 12x^3 - 6x^2 + 12x + 5$.

Can easily take derivative of any polynomial!

E.g. If $f(x) = x^3$, what is $f''(x)$?

Well, $f'(x) = 3x^2$, so $f''(x) = 3 \cdot 2x$
$= 6x$.

All derivatives of $x^n$ easy to compute this way! //

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## Derivatives for more kinds of functions §3.1

<u>Thm</u> For <u>any real number</u> $n$, if $f(x) = x^n$
then $\boxed{f'(x) = n \cdot x^{n-1}}$

Exactly same formula as for positive integers $n$.
Proof is similar, and we will skip it...

<u>E.g.</u> <u>Q</u>: If $f(x) = \sqrt{x}$, what is $f'(x)$?
<u>A</u>: $f(x) = x^{1/2}$, so $f'(x) = \frac{1}{2} x^{1/2 - 1} = \frac{1}{2} x^{-1/2}$
$= \frac{1}{2} \frac{1}{x^{1/2}} = \frac{1}{2\sqrt{x}}$.

<u>Q</u>: If $f(x) = \frac{1}{x}$, what is $f'(x)$?
<u>A</u>: $f(x) = x^{-1}$, so $f'(x) = -1 \cdot x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$.

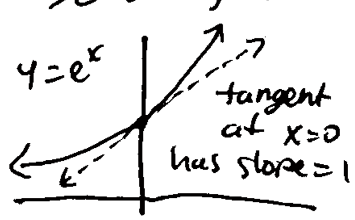
The exponential fn. $e^x$ has a surprisingly <u>simple</u> derivative:

<u>Thm</u> If $f(x) = e^x$, then $f'(x) = e^x = (f(x))$.

Taking derivative of $e^x$ does <u>not change</u> it!
So also $f''(x) = e^x$, $f'''(x) = e^x$, etc...

<u>Pf</u>: We write
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$
$$= \lim_{h \to 0} \frac{e^x \cdot e^h - e^x \cdot e^0}{h} = e^x \cdot \underbrace{\lim_{h \to 0} \frac{e^h - e^0}{h}}_{} = e^x \cdot f'(0)$$

So we just need to show $f'(0) = 1$.




But remember, we <u>defined</u> $e$ as the unique $b > 1$ for which slope of tangent of $b^x$ at $x=0$ is <u>one</u>.
So $f'(0) = 1$ by definition of $e$! ∎

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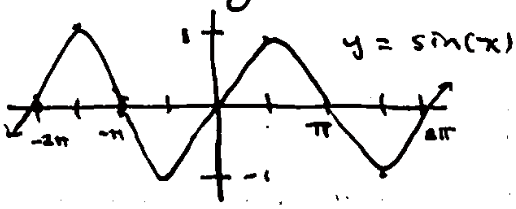
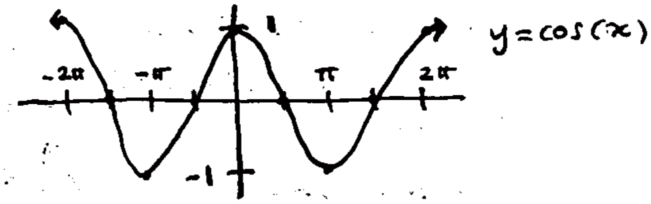
## Derivatives of trigonometric functions §3.3

Looking at the graphs of sin(x) and cos(x):




We notice that:

- $\sin(x)$ is increasing $\Leftrightarrow$ $\cos(x) > 0$
- $\sin(x)$ is decreasing $\Leftrightarrow$ $\cos(x) < 0$
- $\sin(x)$ has min./max. $\Leftrightarrow$ $\cos(x) = 0$

- $\cos(x)$ is increasing $\Leftrightarrow$ $\sin(x) < 0$
- $\cos(x)$ is decreasing $\Leftrightarrow$ $\sin(x) > 0$
- $\cos(x)$ has min./max. $\Leftrightarrow$ $\sin(x) = 0$

From these qualitative properties, reasonable to guess:

**Thm** $\boxed{\begin{aligned} d/dx(\sin(x)) &= \cos(x) \\ \text{and } d/dx(\cos(x)) &= -\sin(x) \end{aligned}}$

E.g. If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$,
so $f''(x) = -\sin(x)$, and $f'''(x) = -\cos(x)$,
and $f^{(4)}(x) = -(-\sin(x)) = \sin(x) = f(x)$.
After 4 derivatives, we get back what we started with!
Can also check that if $f(x) = \cos(x)$, then $f^{(4)}(x) = \cos(x) = f(x)$.

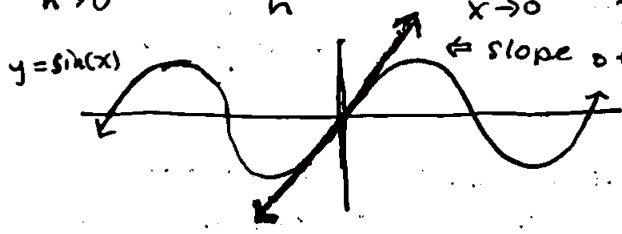
In this way the trig functions sin(x) and cos(x) behave like $e^x$, where taking enough derivatives gives us back the original function we started with.

Whereas with a polynomial function like $f(x) = 5x^4 - 3x^3 + 6x^2 + 10x - 9$, taking enough derivatives always gives us zero!

The key step for proving $d/dx(\sin(x)) = \cos(x)$ is this:

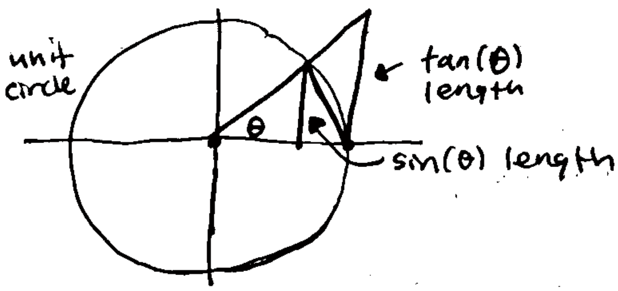
<u>Lemma</u> If $f(x) = \sin(x)$, then

$$f'(0) = \lim_{h\to 0} \frac{\sin(0+h) - \sin(0)}{h} = \lim_{x\to 0} \frac{\sin(x)}{x} = 1 = \cos(0).$$




⇐ slope of this tangent line at $x=0$ should be $=\underline{\underline{1}}$.

There is a nice geometric proof of this Lemma:




⇐ idea of proof is to compare areas of triangles in this drawing. See book for <u>details</u>!

For our purposes, we will just use the formulas. To summarize, it is worth memorizing the following important derivatives:

$$\boxed{\begin{array}{ll} d/dx\,(x^n) = n\cdot x^{n-1} & d/dx\,(\sin(x)) = \cos(x) \\ d/dx\,(e^x) = e^x & d/dx\,(\cos(x)) = -\sin(x) \end{array}}$$

don't forget this <u>negative</u> sign: it's important!

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## The product and quotient rules § 3.2

Suppose we want to take the derivative of a product $f \cdot g$ of two (differentiable) functions $f(x)$ and $g(x)$.
Might think/hope its derivative is product of derivatives, but easy to find examples where $(f \cdot g)'(x) \neq f'(x) \cdot g'(x)$.

E.g. Let $f(x) = x$, $g(x) = x^2$, then $f'(x) \cdot g'(x) = 1 \cdot 2x = 2x$, but $(f \cdot g)(x) = x^3$, so $(fg)'(x) = 3x^2$.

Instead, we have the product rule:

Thm For two (differentiable) functions $f(x), g(x)$:

$$\boxed{(f \cdot g)'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)}$$

"First times derivative of second plus second times derivative of first."

In differential notation: $\boxed{d/dx\,(f \cdot g) = f \cdot \dfrac{dg}{dx} + g \cdot \dfrac{df}{dx}}$

E.g. with $f(x) = x$ and $g(x) = x^2$, we compute

$$(f \cdot g)'(x) = f(x)\, g'(x) + g(x)\, f'(x) = x \cdot 2x + x^2 \cdot 1$$
$$= 3x^2 = d/dx\,(x^3) \checkmark$$

E.g. $d/dx\,(x e^x) = x\, d/dx\,(e^x) + e^x\, d/dx\,(x) = x e^x + e^x$

E.g. $d/dx\,(x^2 \sin(x)) = x^2\, d/dx\,(\sin(x)) + \sin(x) \cdot d/dx\,(x^2)$
$= x^2 \cos(x) + 2x \sin(x)$

Pf sketch for product rule:
Write $u = f(x)$, $v = g(x)$, $\Delta u = f(x+h) - f(x)$, $\Delta v = g(x+h) - g(x)$.

Then $\Delta(uv) = (u + \Delta u)(v + \Delta v) - uv$
$= u \Delta v + v \Delta u - \Delta u \Delta v$ this term goes away in limit!

See book for details!

The quotient rule is a bit more complicated:

Thm For two (differentiable) functions $f(x), g(x)$ (with $g(x) \neq 0$),

$$\boxed{\left(\frac{f}{g}\right)'(x) = \frac{g(x)\cdot f'(x) - f(x)\, g'(x)}{g(x)^2}}$$

, or in differential notation:

$$d/dx\left(\frac{f}{g}\right) = \left(g\frac{df}{dx} - f\frac{dg}{dx}\right)/g^2.$$

---

Looks similar in many ways to product rule, but more complicated. When we learn the chain rule, you will see that you don't need to separately memorize the quotient rule.

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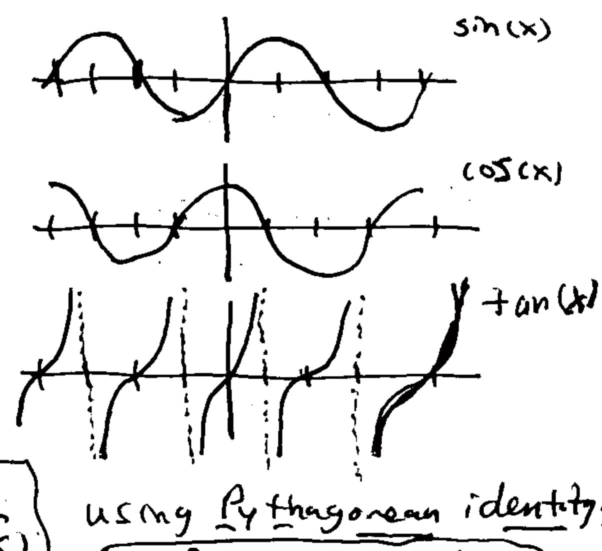
E.g. Let $f(x) = x$, $g(x) = 1-x$, so $\frac{f}{g}(x) = \frac{x}{1-x}$

Then $\left(\frac{f}{g}\right)'(x) = \frac{g(x)\cdot f'(x) - f(x)\, g'(x)}{g(x)^2} = \frac{(1-x)\cdot 1 - x\cdot(-1)}{(1-x)^2}$

$$= \frac{1-x+x}{(1-x)^2} = \frac{1}{(1-x)^2}.$$

Any rational function can be differentiated this way...

E.g. Recall $\tan(x) = \frac{\sin(x)}{\cos(x)}$




Thus, $\boxed{(\tan)'(x)} =$

$$\frac{\cos(x)\cdot(\sin)'(x) - \sin(x)\cos'(x)}{(\cos(x))^2}$$

$$= \frac{\cos(x)\cdot\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \boxed{\frac{1}{\cos^2(x)}}$$

using Pythagorean identity: $\boxed{\sin^2(x) + \cos^2(x) = 1}$ for any $x$

the one trig identity really worth knowing

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## Chain rule §3.4

Let $f(x) = \sqrt{x^2+1}$. How do we find $f'(x)$?
So far we don't know how... to do this we need the chain rule, which tells us how to take derivatives of compositions of functions:

**Theorem** For two (differentiable) fn's $f(x)$ and $g(x)$, we have $\boxed{(f \circ g)'(x) = f'(g(x)) \cdot g'(x)}$

In differential notation, this can be written

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

where $y = f(g(x))$ and $u = g(x)$. So roughly speaking the chain rule lets us "cancel" the $du$'s.

E.g. For $f(x) = \sqrt{x^2+1}$, write $f(x) = h(g(x))$ where $h(x) = \sqrt{x}$ and $g(x) = x^2+1$. Then the chain rule says $f'(x) = h'(g(x)) \cdot g'(x)$

$$= \tfrac{1}{2}(g(x)^{-\frac{1}{2}}) \cdot 2x = \frac{x}{\sqrt{x^2+1}}.$$

E.g. Let $f(x) = \sin(x^2)$. Then

$$f'(x) = \underbrace{\cos(x^2)}_{d/dx(\sin(x)),\ \text{plug in } x^2} \cdot \underbrace{2x}_{d/dx(x^2)}$$

E.g. Let $f(x) = \sin^2(x)$ (meaning $(\sin(x))^2$).

Then $$f'(x) = \underbrace{2 \cdot \sin(x)}_{d/dx(x^2),\ \text{plug in } \sin(x)} \cdot \underbrace{\cos(x)}_{d/dx(\sin(x))}.$$

E.g. with $f(x) = \frac{1}{\sin(x)} = (\sin(x))^{-1}$, we have

$$f'(x) = \underbrace{-(\sin(x))^{-2}}_{d/dx(x^{-1}),\ \text{plug in } \sin(x)} \cdot \underbrace{\cos(x)}_{d/dx(\sin(x))} = \frac{-\cos(x)}{\sin^2(x)}$$

With this last example, we see how we <u>don't</u> <u>need</u> the <u>quotient rule</u>. In fact, quotient rule can be deduced from product rule and chain rule:

Let $h(x) = \frac{f(x)}{g(x)} = f(x) \cdot (g(x))^{-1}$

Then $h'(x) = f(x) \cdot d/dx(g(x)^{-1}) + \frac{1}{g(x)} f'(x)$.

But by the chain rule,

$$d/dx\,(g(x)^{-1}) = -g(x)^{-2} \cdot g'(x) = \frac{-g'(x)}{g(x)^2}$$

So that

$$h'(x) = f(x) \cdot \frac{-g'(x)}{g(x)^2} + \frac{f'(x)}{g(x)}$$

$$= \frac{-f(x) \cdot g'(x)}{g(x)^2} + \frac{f'(x)\, g(x)}{g(x)^2}$$

$$= \frac{g(x)\, f'(x) - f(x)\, g'(x)}{g(x)^2},$$

which is exactly the quotient rule we learned. So you never need to separately memorize the quotient rule: the product rule and chain rule are enough. //