

# (Piecewise linear & birational) involutions on Dyck paths

Howard Mathematics Colloquium

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# Section 1

## Catalan numbers, Dyck paths, Naryana numbers, and the Lalanne–Kreweras involution



Montserrat Mountain, Catalonia, Spain

# Catalan numbers

The **Catalan numbers**  $C_n$  are a famous sequence of numbers

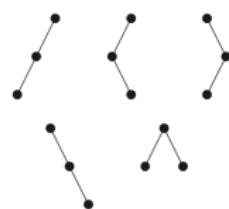
$$1, 2, 5, 14, 42, 132, 429, 1430, \dots,$$

which count numerous combinatorial collections including:

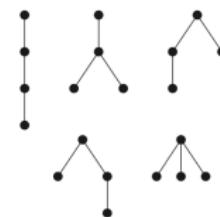
triangulations  
of an  $n + 2$ -gon



binary trees  
with  $n$  nodes



plane trees with  
 $n + 1$  nodes



bracketings of  
 $n + 1$  terms

$$\begin{aligned} &a(b(cd)) \quad a((bc)d) \\ &(ab)(cd) \quad (a(bc))d \\ &((ab)c)d \end{aligned}$$

There is a well-known product formula for the Catalan numbers:

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$$

# History of Catalan numbers

The Catalan numbers are named after Belgian mathematician *Eugène Catalan* (1814 – 1894), who studied them in conjunction with bracketings.

But they were studied combinatorially much earlier by *Leonhard Euler* (1707 – 1783), who showed they count triangulations of convex polygons.

In fact, even earlier, Mongolian mathematician/scientist *Minggatu* (c.1692 – c.1763) used Catalan numbers in certain trigonometric identities.



E. Catalan



L. Euler



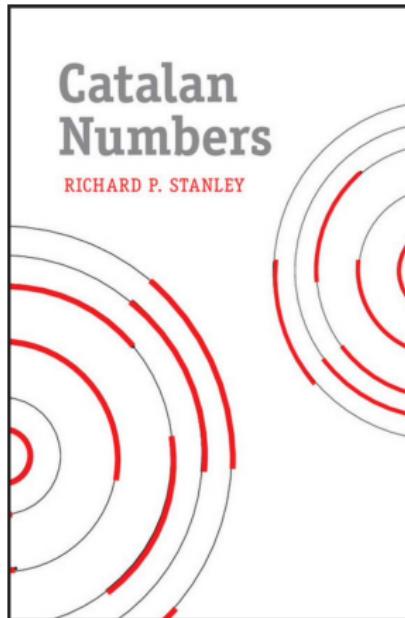
Minggatu

It's a good thing the  $C_n$  are not named after Euler, since there are already

- *Euler numbers & Eulerian numbers*, counting certain permutations;
- *Euler's number*  $e \approx 2.71$  & the *Euler–Mascheroni constant*  $\gamma \approx 0.57$ .

# Catalan numbers: the book

Richard Stanley has a whole book devoted to the Catalan numbers.

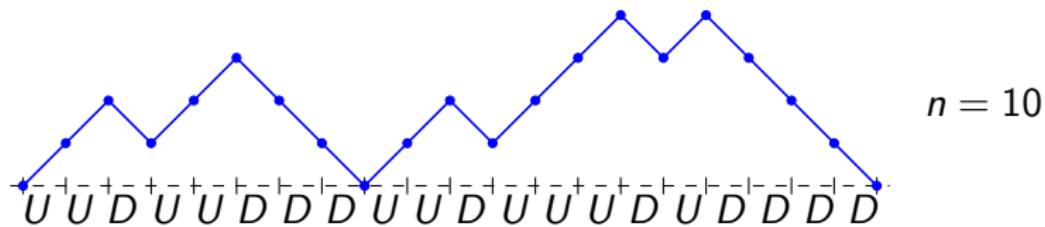


In it, he gives an astounding 214 different interpretations of  $C_n$ .

# Dyck paths

The interpretation of  $C_n$  I want to focus on is in terms of Dyck paths.

A **Dyck path** of length  $2n$  is a lattice path in  $\mathbb{Z}^2$  from  $(0, 0)$  to  $(2n, 0)$  consisting of  $n$  up steps  $U = (1, 1)$  and  $n$  down steps  $D = (1, -1)$  that never goes below the  $x$ -axis:



The number of Dyck paths of length  $2n$  is  $C_n$ :

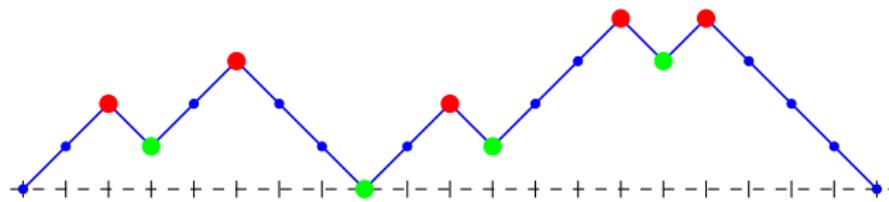


They are named after German algebraist *Walther von Dyck* (1856 – 1934).

# Peaks and valleys in Dyck paths

Dyck paths look like mountain ranges. So we use some topographic terminology when working with Dyck paths.

A **peak** in a Dyck path is an up step that is immediately followed by a down step; a **valley** is a down step immediately followed by an up step.



Here the peaks are marked by red circles and the valleys by green circles.  
It's easy to see that a Dyck path which has  $k$  valleys has  $k + 1$  peaks.

# Narayana numbers

The **Narayana number**  $N(n, k)$  is the number of Dyck paths of length  $2n$  with exactly  $k$  valleys.

$n \setminus k$	0	1	2	3
1	1			
2	1	1		
3	1	3	1	
4	1	6	6	1

$\leftarrow$  array of  $N(n, k)$

Evidently, the Narayana numbers  $N(n, k)$  refine the Catalan number  $C_n$ :

$$C_n = \sum_{k=0}^{n-1} N(n, k).$$

They are named after Canadian mathematician/statistician *Tadepalli Venkata Narayana* (1930 – 1987), who in 1959 showed that

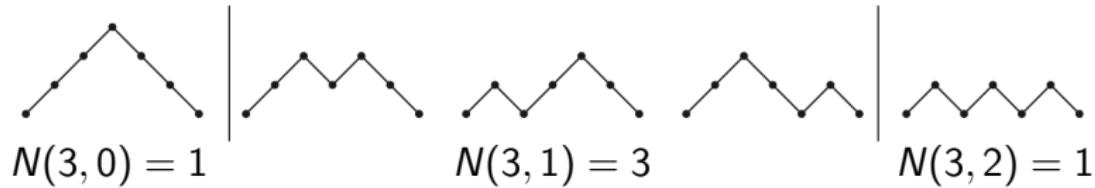
$$N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k+1}.$$

# Symmetry of Narayana numbers

From Narayana's formula, it follows immediately that

$$N(n, k) = N(n, n - 1 - k)$$

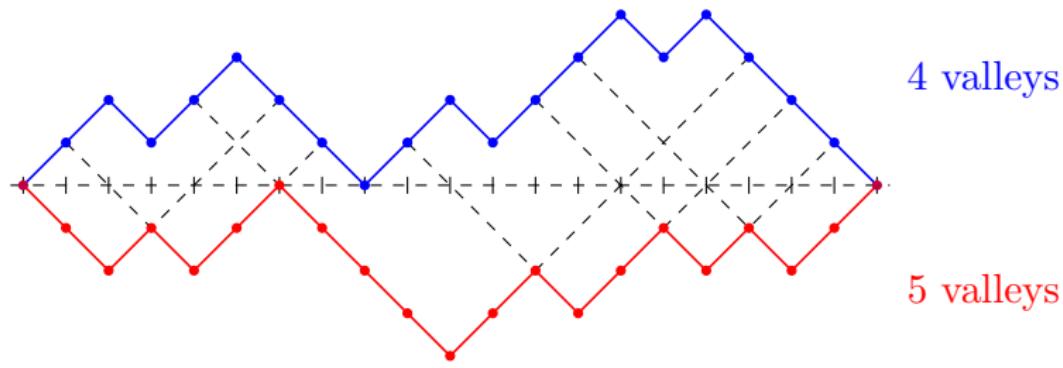
for all  $k$ . That is, the sequence of Narayana numbers is *symmetric*.



However, it is not combinatorially obvious why the number of Dyck paths with  $k$  valleys should be the same as the number with  $n - 1 - k$  valleys.

# The Lalanne–Kreweras involution

The **Lalanne–Kreweras involution** (after G. Kreweras and J.-C. Lalanne) is a map on Dyck paths which combinatorially demonstrates the symmetry of the Narayana numbers:  $\#\text{valleys}(\Gamma) + \#\text{valleys}(\text{LK}(\Gamma)) = n - 1$ .



As depicted above, to compute the LK involution of a Dyck path  $\Gamma$ , we draw dashed lines emanating from the middle of every double up step and every double down step of  $\Gamma$ , at  $-45^\circ$  and  $45^\circ$  respectively; these dashed lines intersect at the valleys of (an upside copy of) the Dyck path  $\text{LK}(\Gamma)$ . That LK is an involution means  $\text{LK}^2(\Gamma) = \Gamma$  for all Dyck paths  $\Gamma$ .

## Section 2

### Posets

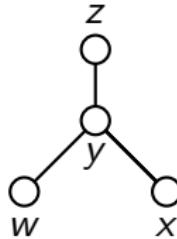
# Posets

We will now reinterpret the LK involution using the theory of finite posets.

A (finite) **poset**, or *partially ordered set*, is a (finite) set  $P$  together with a relation  $\leq$  satisfying the usual axioms of a partial order:

- *transitivity* ( $x \leq y, y \leq z \Rightarrow x \leq z$ );
- *anti-symmetry* ( $x \leq y, y \leq x \Rightarrow x = y$ );
- *reflexivity* ( $x \leq x$ ).

We represent posets via their **Hasse diagrams**:

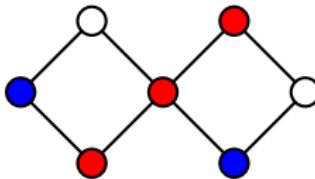


Here an edge from  $x$  (below) to  $y$  (above) represents the **cover relation**  $x \lessdot y$  in  $P$ , which means  $x < y$  and there is no  $p \in P$  with  $x < p < y$ .

# Chains and antichains

Two elements  $x, y$  in a poset  $P$  are **comparable** if either  $x \leq y$  or  $y \leq x$ . A **chain**  $C \subseteq P$  of  $P$  is a subset of pairwise comparable elements (i.e., a chain is a *totally ordered* subset  $C = \{x_1 < x_2 < \dots < x_k\}$ ). A chain  $C$  is **maximal** if it is not strictly contained in another chain.

Two elements  $x, y \in P$  are **incomparable** if they are not comparable. An **antichain**  $A \subseteq P$  of  $P$  is a subset of pairwise incomparable elements. We use  $\mathcal{A}(P)$  to denote the set of antichains of  $P$ .



Here the red elements form a maximal chain  $C$ , and the blue elements form an antichain  $A \in \mathcal{A}(P)$ .

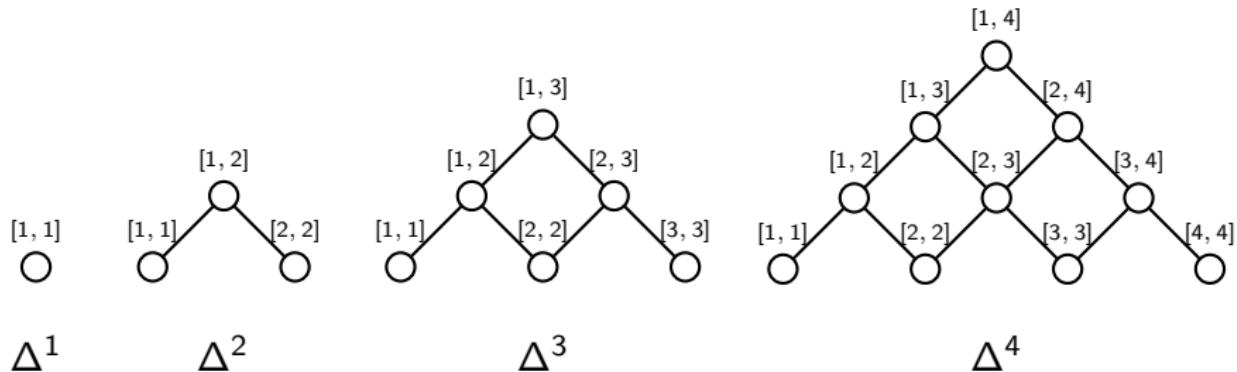
# The poset $\Delta^{n-1}$

One particular family of posets  $\Delta^{n-1}$  is relevant to the LK involution.

$\Delta^{n-1}$  is the poset whose elements are **intervals**  $[i, j] := \{i, i + 1, \dots, j\}$  with  $1 \leq i \leq j \leq n - 1$ , and with the partial order given by **inclusion**:

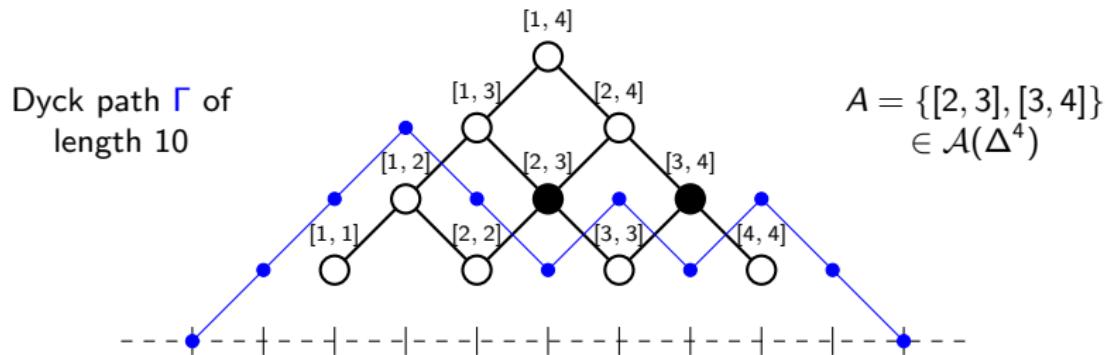
$$[i, j] \leq [i', j'] \iff [i, j] \subseteq [i', j'] \iff i \leq i' \leq j' \leq j$$

$\Delta^{n-1}$  has a “triangular” Hasse diagram:



# Dyck paths are antichains in $\Delta^{n-1}$

There is a natural, pictorial bijection between the Dyck paths of length  $2n$  and the antichains of  $\Delta^{n-1}$ :



Observe how, under this bijection, the number of valleys of a Dyck path  $\Gamma$  becomes the number of elements of an antichain  $A$ .

Via this bijection, we can view the LK involution as an involution on antichains  $\text{LK}: \mathcal{A}(\Delta^{n-1}) \rightarrow \mathcal{A}(\Delta^{n-1})$  which satisfies

$$\#A + \#\text{LK}(A) = n - 1.$$

# The LK involution on antichains

D. Panyushev gave a simple description of the LK involution on  $\mathcal{A}(\Delta^{n-1})$ :

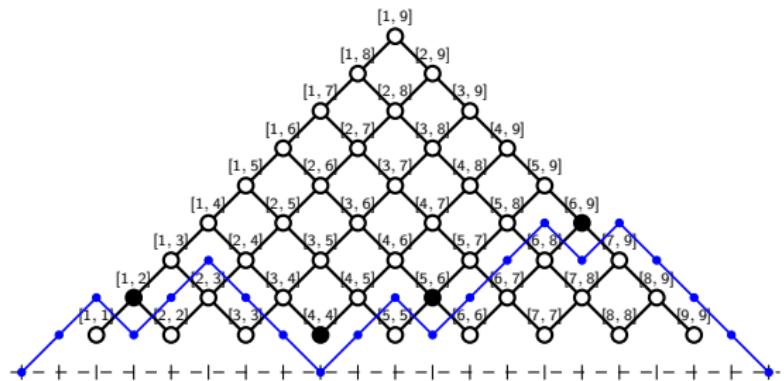
## Theorem (Panyushev, 2004)

Let  $A = \{[i_1, j_1], [i_2, j_2], \dots, [i_k, j_k]\} \in \mathcal{A}(\Delta^{n-1})$  with  $i_1 < i_2 < \dots < i_k$ .  
 Then  $\text{LK}(A) = \{[i'_1, j'_1], [i'_2, j'_2], \dots, [i'_{n-1-k}, j'_{n-1-k}]\} \in \mathcal{A}(\Delta^{n-1})$ , where

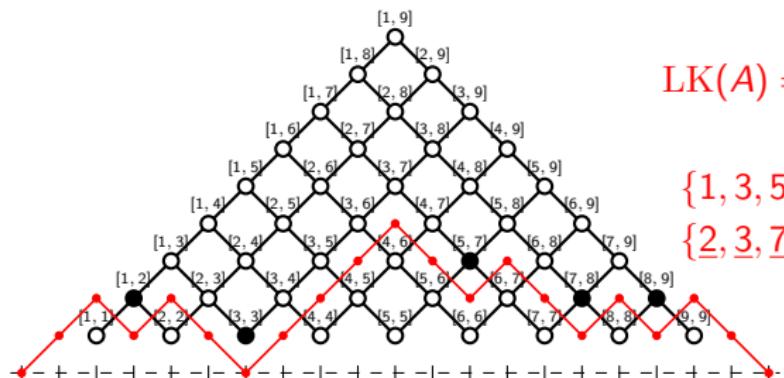
- $\{i'_1 < i'_2 < \dots < i'_{n-1-k}\} = \{1, 2, \dots, n-1\} \setminus \{j_1, j_2, \dots, j_k\}$ ;
- $\{j'_1 < j'_2 < \dots < j'_{n-1-k}\} = \{1, 2, \dots, n-1\} \setminus \{i_1, i_2, \dots, i_k\}$ .

From Panyushev's description, it is immediate that this operation is an involution (i.e.,  $\text{LK}^2(A) = A$ ), and that  $\#A + \#\text{LK}(A) = n - 1$ .

# The LK involution on antichains: example



$$A = \{[1, \underline{2}], [4, \underline{4}], [5, \underline{6}], [6, \underline{9}]\}$$



$$\text{LK}(A) = \{[1, \underline{2}], [\underline{3}, \underline{3}], [5, \underline{7}], [7, \underline{8}], [8, \underline{9}]\}$$

$$\begin{aligned} \{1, 3, 5, 7, 8\} &= \{1, \dots, 9\} \setminus \{\underline{2}, \underline{4}, \underline{6}, \underline{9}\} \\ \{\underline{2}, \underline{3}, \underline{7}, \underline{8}, \underline{9}\} &= \{1, \dots, 9\} \setminus \{1, 4, 5, 6\} \end{aligned}$$

## Section 3

Toggling

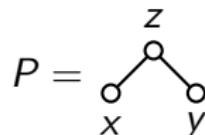
# Toggling for antichains

Our first new result gives another expression for the LK involution in terms of certain “local” involutions called **toggles**.

Let  $P$  be a poset and  $A \in \mathcal{A}(P)$  an antichain. Let  $p \in P$  be any element. The **toggle of  $p$  in  $A$**  is the antichain  $\tau_p(A) \in \mathcal{A}(P)$ , where

$$\tau_p(A) := \begin{cases} A \setminus \{p\} & \text{if } p \in A; \\ A \cup \{p\} & \text{if } p \notin A \text{ and } A \cup \{p\} \text{ remains an antichain;} \\ A & \text{otherwise.} \end{cases}$$

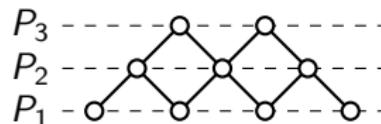
In other words, we “toggle” the status of  $p$  in  $A$ , if possible:



$$\begin{aligned} \tau_x(\bullet \diagup \circ \diagdown \circ) &= \circ \diagup \circ \diagdown \circ \\ \tau_x(\circ \diagup \bullet \diagdown \bullet) &= \bullet \diagup \circ \diagdown \bullet \\ \tau_x(\circ \diagup \bullet \diagdown \circ) &= \circ \diagup \bullet \diagdown \circ \end{aligned}$$

# Toggling in ranked posets

A poset  $P$  is **ranked** if we can write  $P = P_1 \sqcup P_2 \sqcup \cdots \sqcup P_r$  so that all the edges of the Hasse diagram of  $P$  are from  $P_i$  (below) to  $P_{i+1}$  (above):



Since  $\tau_p$  and  $\tau_q$  commute if  $p$  and  $q$  are incomparable, and all the elements within a rank are incomparable, we can define

$$\tau_i := \prod_{p \in P_i} \tau_p$$

to be the composition of all toggles at rank  $i$ , for  $i = 1, \dots, r$ :



# The LK involution as a composition of toggles

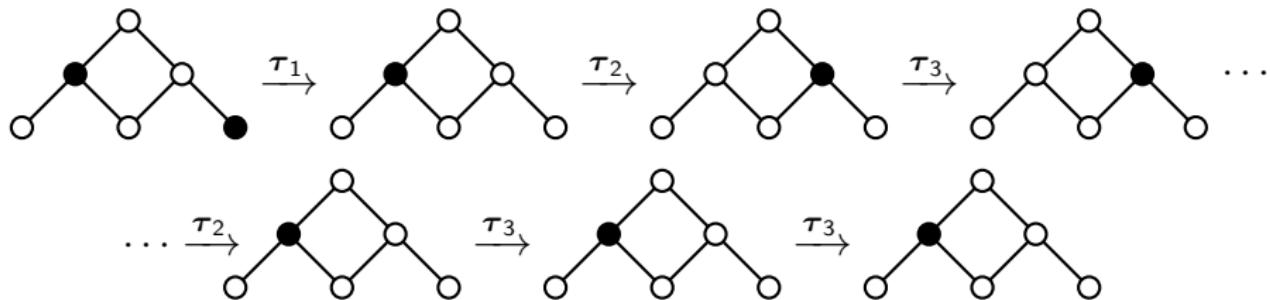
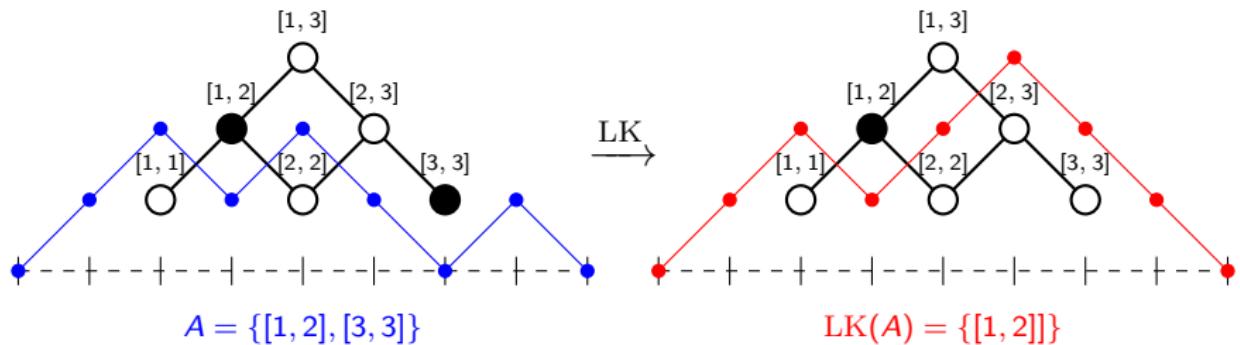
## Theorem (H.-Joseph, 2021)

*The LK involution  $\text{LK}: \mathcal{A}(\Delta^{n-1}) \rightarrow \mathcal{A}(\Delta^{n-1})$  can be written as the following composition of toggles:*

$$\text{LK} = (\tau_{n-1})(\tau_{n-1}\tau_{n-2}) \cdots (\tau_{n-1} \cdots \tau_3\tau_2)(\tau_{n-1} \cdots \tau_2\tau_1)$$

**Remark:** for a ranked poset  $P$ , the composition of toggles  $\tau_r \cdots \tau_2\tau_1$  “from bottom to top” is called **rowmotion** and has been studied by many authors (Cameron–Fon-Der-Flaass, Striker–Williams, Propp–Roby, Joseph, etc...) in the emerging subfield of **dynamical algebraic combinatorics**.

# The LK involution as a composition of toggles: example



## Section 4

Piecewise linear and birational lifts

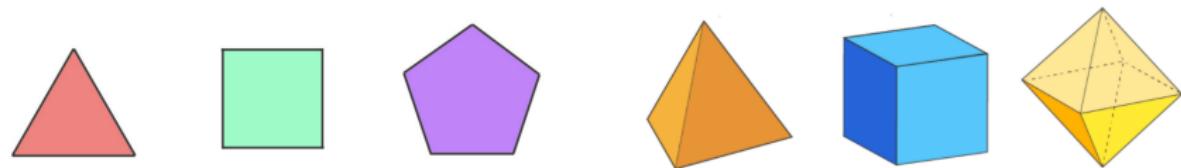
# Convex polytopes

Why did we want to write the LK involution as a composition of toggles?  
In order to **extend** it to the **piecewise linear** realm...

A **convex polytope** in  $\mathbb{R}^n$  can be defined either as

- a convex hull of finitely many points (**vertices**);
- a bounded intersection of finitely many linear inequalities (**facets**).

In dimensions 2 and 3, these are familiar shapes:



There is a rich interplay between combinatorics and convex geometry,  
because combinatorial objects can often be “realized” polytopally: e.g.,  
*the subsets of  $\{1, 2, \dots, n\}$  correspond to the vertices of the  $n$ -hypercube.*

# The chain polytope of a poset

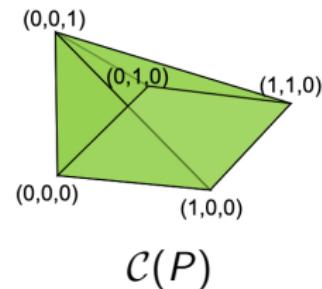
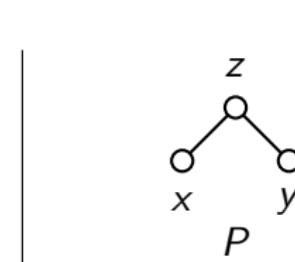
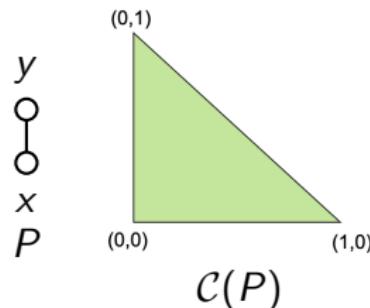
In 1986, Richard Stanley associated to any poset  $P$  two polytopes in  $\mathbb{R}^P$ , the **order polytope**  $\mathcal{O}(P)$  and the **chain polytope**  $\mathcal{C}(P)$ .

The **chain polytope**  $\mathcal{C}(P)$  has facets

$$0 \leq x_p, \quad \forall p \in P$$

$$\sum_{p \in C} x_p \leq 1, \quad \forall C \subseteq P \text{ a maximal chain.}$$

Stanley proved that the **vertices** of  $\mathcal{C}(P)$  are precisely the **indicator functions of antichains**  $A \in \mathcal{A}(P)$ :



# Piecewise linear toggling

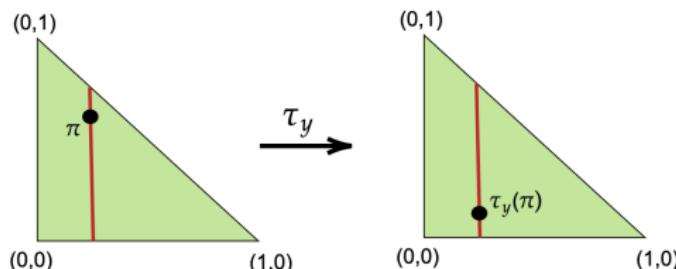
In 2013, D. Einstein and J. Propp (c.f. Joseph) introduced a (continuous) **piecewise linear extension** of the toggles  $\tau_p$ .

For  $p \in P$ , the **PL toggle**  $\tau_p^{\text{PL}}: \mathcal{C}(P) \rightarrow \mathcal{C}(P)$  is defined by

$$\tau_p^{\text{PL}}(\pi)(q) := \begin{cases} \pi(q) & \text{if } q \neq p; \\ 1 - \max \left\{ \sum_{r \in C} \pi(r) : \begin{array}{l} C \subseteq P \text{ a maximal} \\ \text{chain with } p \in C \end{array} \right\} & \text{if } p = q. \end{cases}$$

Restricted to the vertices of the chain polytope  $\mathcal{C}(P)$ , it is the same as  $\tau_p$ .

Geometrically,  $\tau_p$  **reflects**  $\pi$  within line segment in  $\mathcal{C}(P)$  in direction  $x_p$ :



# The PL LK involution

As before, for a ranked poset  $P$  we use  $\tau_i^{\text{PL}} := \prod_{p \in P_i} \tau_p^{\text{PL}}$  to denote the composition of all toggles at rank  $i$ .

We define the **PL LK involution**  $\text{LK}^{\text{PL}} : \mathcal{C}(\Delta^{n-1}) \rightarrow \mathcal{C}(\Delta^{n-1})$  to be

$$\text{LK}^{\text{PL}} := (\tau_{n-1}^{\text{PL}})(\tau_{n-1}^{\text{PL}} \tau_{n-2}^{\text{PL}}) \cdots (\tau_{n-1}^{\text{PL}} \cdots \tau_3^{\text{PL}} \tau_2^{\text{PL}})(\tau_{n-1}^{\text{PL}} \cdots \tau_2^{\text{PL}} \tau_1^{\text{PL}})$$

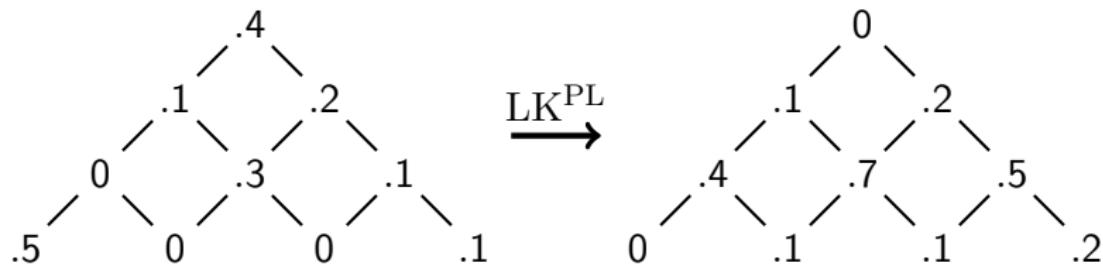
By prior theorem, it's same as LK when restricted to the vertices of  $\mathcal{C}(P)$ .

**Theorem (H.-Joseph, 2021)**

- (1)  $\text{LK}^{\text{PL}}$  is an involution.
- (2) For any  $\pi \in \mathcal{C}(\Delta^{n-1})$ ,  $\sum_{p \in P} \pi(p) + \sum_{p \in P} \text{LK}^{\text{PL}}(\pi)(p) = n - 1$ .

Observe that (2) is an extension of the fact that LK combinatorially exhibits the symmetry of the Narayana numbers.

# The PL LK involution: example



We can check that

$$(0.5+0+0+0.1+0+0.3+0.1+0.1+0.2+0.4)+(0+0.1+0.1+0.2+0.4+0.7+0.5+0.1+0.2+0) =$$

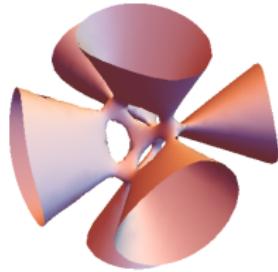
$$1.7 + 2.3 = 4$$

# Tropical geometry

**Algebraic geometry** studies  
**polynomial** expressions like

$$x^3y + y^3z + z^3x$$

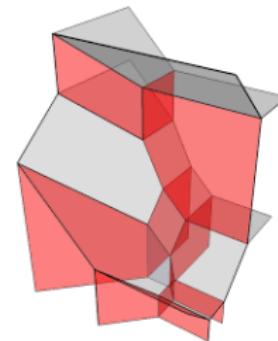
which lead to “curvy” hypersurfaces  
like



**Tropical geometry** studies  
**piecewise linear** expressions like

$$\max(3x + y, 3y + z, 3z + x)$$

which lead to “flat” polytopal  
complexes like



# “De-tropicalization”

The process of replacing  $(\times, +)$  with  $(+, \max)$  in a polynomial expression is called **tropicalization**:

$$x^3y + y^3z + z^3x \mapsto \max(3x + y, 3y + z, 3z + x)$$

$\Rightarrow$  important interactions between algebraic & convex geometry

(Adjective “tropical” comes from fact that computer scientist & pioneer of tropical geometry Imre Simon worked at University of São Paulo,  

The process of replacing  $(+, \max)$  with  $(\times, +)$  in a piecewise linear expression is called **de-tropicalization\***:

$$\max(3x + y, 3y + z, 3z + x) \mapsto x^3y + y^3z + z^3x$$

It is often interesting to try to de-tropicalize PL maps, like those coming from classical combinatorial constructions.

# Birational toggling

Einstein–Propp (c.f. Joseph–Roby) also introduced a **birational extension** of the toggles  $\tau_p$ , via de-tropicalization.

For  $p \in P$ , the **birational toggle**  $\tau_p^B : \mathbb{C}^P \dashrightarrow \mathbb{C}^P$  is

$$\tau_p^B(\pi)(q) := \begin{cases} \pi(q) & \text{if } q \neq p; \\ \kappa \cdot \left( \prod_{\substack{C \subseteq P \\ \text{max. chain,} \\ p \in C}} \sum_{r \in C} \pi(r) \right)^{-1} & \text{if } p = q, \end{cases}$$

where  $\kappa \in \mathbb{C}$  is some fixed constant.

The birational toggle  $\tau_p^B$  tropicalizes to the PL toggle  $\tau_p^{\text{PL}}$ .

# The birational LK involution

As before, if  $P$  is ranked we set  $\tau_i^B := \prod_{p \in P_i} \tau_p^B$ .

We define the birational LK involution  $\text{LK}^B : \mathbb{C}^{\Delta^{n-1}} \dashrightarrow \mathbb{C}^{\Delta^{n-1}}$  by

$$\text{LK}^B := (\tau_{n-1}^B)(\tau_{n-1}^B \tau_{n-2}^B) \cdots (\tau_{n-1}^B \cdots \tau_3^B \tau_2^B)(\tau_{n-1}^B \cdots \tau_2^B \tau_1^B)$$

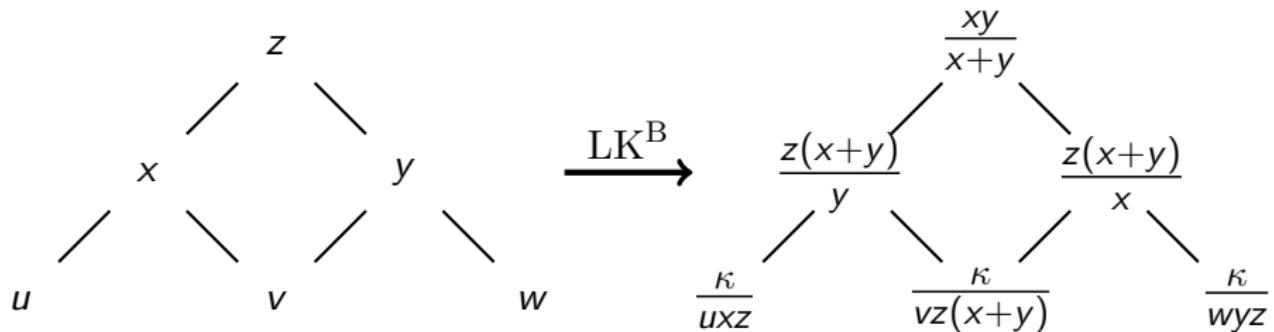
It tropicalizes to  $\text{LK}^{\text{PL}}$ .

**Theorem (H.-Joseph, 2021)**

- (1)  $\text{LK}^B$  is an involution.
- (2) For any  $\pi \in \mathbb{C}^{\Delta^{n-1}}$ ,  $\prod_{p \in P} \pi(p) \cdot \prod_{p \in P} \text{LK}^B(\pi)(p) = \kappa^{n-1}$ .

(2) is the birational analog of the fact that LK combinatorially exhibits the symmetry of the Narayana numbers.

# The birational LK involution: example



We can check that this operation really is an involution; e.g.,

$$\frac{z'(x' + y')}{y'} = \frac{\frac{xy}{x+y} \cdot \left( \frac{z(x+y)}{y} + \frac{z(x+y)}{x} \right)}{\frac{z(x+y)}{x}} = \frac{zx + zy}{\frac{z(x+y)}{x}} = \frac{z(x+y)}{\frac{z(x+y)}{x}} = x.$$

And if we multiply together all the above values, we get  $\kappa^3$ .

# Thank you!

these slides are available on my website  
and the paper on the arXiv: arXiv:2012.15795

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## *Exercises*

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- 6.24.** [?] Explain the significance of the following sequence:

un, dos, tres, quatre, cinc, sis, set, vuit, nou, deu, . . .

R. Stanley, *Enumerative Combinatorics*, Vol. 2