11/9 Anti-derivatives §4.9

Whenever we have some "operation" in mathematics, it is useful to think about "undoing" this operation: e.g. we discussed how inverse functions (like $\ln(x)$) undo the original functions (like e^x).

Differentiation is an important operation, and its "inverse" is called anti-differentiation:

Def'n We say that $F(x)$ is an anti-derivative of $f(x)$ if $F'(x) = f(x)$ (on some interval).

E.g. $F(x) = x^2$ is an antiderivative of $f(x) = 2x$ since $d/dx(x^2) = 2x$.

Note: There are multiple anti-derivatives of $f(x)$:

E.g. x^2+1 is another anti-derivative of $2x$.

But... Theorem If $F(x)$ is one particular anti-derivative of $f(x)$, then the general anti-derivative is $F(x)+C$ for any constant $C \in \mathbb{R}$.

Pf: We explained this before, using the Mean Value Thm. ∎

The $+C$ part is important, but this theorem tells us it is enough to know one anti-derivative of $f(x)$ in order to understand all of them.

Unfortunately, it can be pretty hard to find antiderivatives, e.g. for $f(x) = e^{x^2}$ we know how to compute its derivative, but there is no simple way to describe its anti-derivative.

But... we will still learn how to compute certain anti-derivatives.
Let's start with something easy:

Theorem • If $F(x)$ is anti-deriv. of $f(x)$, then $c \cdot F(x)$ is a.d. of $c \cdot f(x)$ for any $c \in \mathbb{R}$.

• If $F(x)$ is a.d. of $f(x)$ and $G(x)$ is a.-d. of $g(x)$, then $F(x) + G(x)$ is a.d. of $f(x) + g(x)$.

Pf: These follow from linearity of derivative:
$$\frac{d}{dx}\left(c \cdot F(x) + d \cdot G(x)\right) = c \cdot F'(x) + d \cdot G'(x). \quad \blacksquare$$

But what about something like $f(x) = x^n$?
How do we find an anti-deriv. of x^n?

Notice that $\frac{d}{dx}(x^{n+1}) = (n+1) \cdot x^n$, almost what we want, just need to divide by $\frac{1}{n+1}$. (But with $n = -1$, this doesn't work!)

Let us record some common anti-derivatives in a table:

$f(x)$	particular anti-derivative $F(x)$
x^n $(n \neq -1)$	$\frac{1}{n+1} \cdot x^{n+1}$
$1/x$	$\ln(x)$
e^x	e^x
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$

← notice the − sign is "backwards" from derivative.

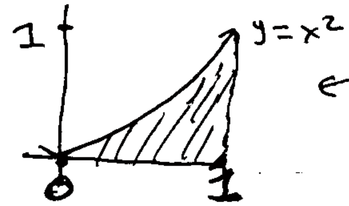
This already gives us a lot of anti-derivatives, but to deal with more complicated things, like $\cos^2(x)$, we will have to learn more anti-differentiation techniques! //.

11/16 **Area under a curve §5.1**

At the beginning of the semester we briefly discussed two problems that calculus solves: the tangent to a curve, and the area under a curve.

We've spent many weeks discussing the tangent and its relation to the derivative. We end the semester discussing the area under a curve, and the integral.

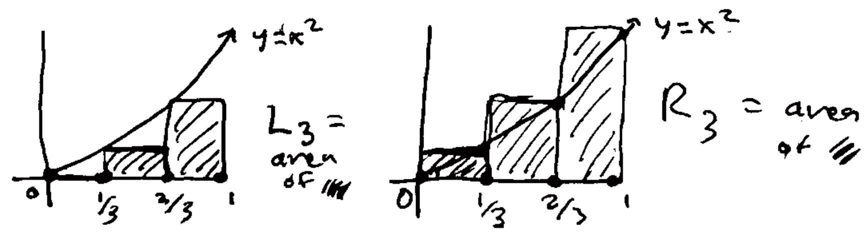
Consider curve $y = f(x)$. What is the area between this curve and the x-axis, between $x=0$ and $x=1$?



← A = area of shaded region ///

In geometry we learn formulas for area of shapes like triangles, rectangles, and circles, but this is not one of those.

However, ... we could approximate the area by using shapes like rectangles which are easy to work with:



On the left we drew 3 rectangles of width 1/3 where the left vertex of the top of the rectangle touches $y=f(x)$, on the right we drew 3 rectangles of width 1/3 where the right vertex of top touches curve $y=f(x)$.

We see that $L_3 < A < R_3$

width of 1st rectangle ↓ · height of 1st rectangle ↓ · width of 2nd rect. ↓ · height of 2nd rect. ↓ · hgt of 3rd rect. ↓

We can compute $L_3 = \left(\frac{1}{3}\right)\cdot 0^2 + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 = \frac{11}{81}$

and $R_3 = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)1^2 = \frac{42}{81}$ so

$$0.1358\ldots \approx \frac{11}{81} < A < \frac{42}{81} \approx 0.5185\ldots$$

If we let L_n and R_n denote the analogous areas of rectangles but where we use n rectangles of width $1/n$ (touching curve at left and right vertices, resp.), then we always have $L_n < A < R_n$ and bigger values of n give better approximations.

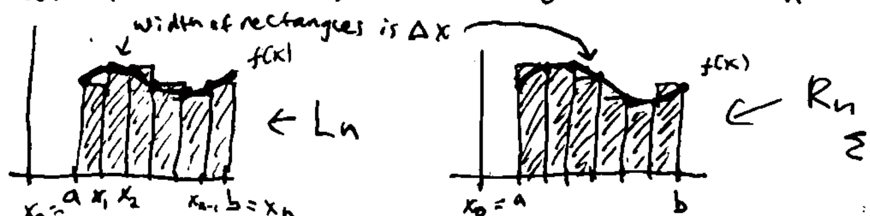
e.g. $n=10 \Rightarrow 0.285\ldots < A < 0.385\ldots$

$n=100 \Rightarrow 0.328\ldots < A < 0.338\ldots$

$n=1000 \Rightarrow 0.332\ldots < A < 0.333\ldots$

It looks like the bounds are converging to $1/3 = 0.333\ldots$ This is right, and suggests we can define area under the curve as a limit.

Def'n Let $f(x)$ be defined on a closed interval $[a,b]$. Fix n, and let $\Delta x = \frac{b-a}{n}$, and let $x_i = a + i\cdot\Delta x$ for all $i = 0,1,2,\ldots,n$ (so $x_0 = a$ and $x_n = b$).



Σ shorthand for sum

Let $L_n = \Delta x\cdot f(x_0) + \Delta x\cdot f(x_1) + \cdots + \Delta x\cdot f(x_{n-1}) = \sum_{i=0}^{n-1} \Delta x \cdot f(x_i)$

and $R_n = \Delta x\cdot f(x_1) + \Delta x\cdot f(x_2) + \cdots + \Delta x\cdot f(x_n) = \sum_{i=1}^{n} \Delta x\cdot f(x_i)$.

Then, as long as $f(x)$ is continuous, the limits of these areas $\lim_{n\to\infty} L_n$ and $\lim_{n\to\infty} R_n$ exist and are equal, so we define $A =$ area under curve of $f(x) = \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n$.

E.g. Let us return to $f(x)=x^2$ defined on $[0,1]$.

Then $R_n = \frac{1}{n}\cdot f(\frac{1}{n}) + \frac{1}{n}\cdot f(\frac{2}{n}) + \cdots + \frac{1}{n} f(\frac{n}{n})$

$$= \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$

$$= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right).$$

Prop. $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

E.g. $1^2 = 1 = \frac{1(1+1)(2+1)}{6}$, $1^2+2^2 = 5 = \frac{2(2+1)(4+1)}{6}$, ...

pf: This can be proved using mathematical induction.

Maybe you have seen the simpler formula:

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

This is slightly more complicated, but basically the same. ☒

So $R_n = \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6n^3}$

Thus $A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{2n^3+3n^2+n}{6n^3} = \frac{2}{6} = \frac{1}{3}.$ //

This definition of area under the curve is conceptually clear, but difficult to work with: we have to come up with formulas like $1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$.

We will give a way to compute areas under the curve using anti-derivatives, which is much more straightforward, and connects the problem to calculus! ...

11/18 <u>The Definite Integral</u> §5.2.

Area under the curve is so important that we give it a special name and notation.

<u>Def'n</u> Let $f(x)$ be a continuous function defined on $[a, b]$.
The (<u>definite</u>) <u>integral</u> of $f(x)$ from a to b is

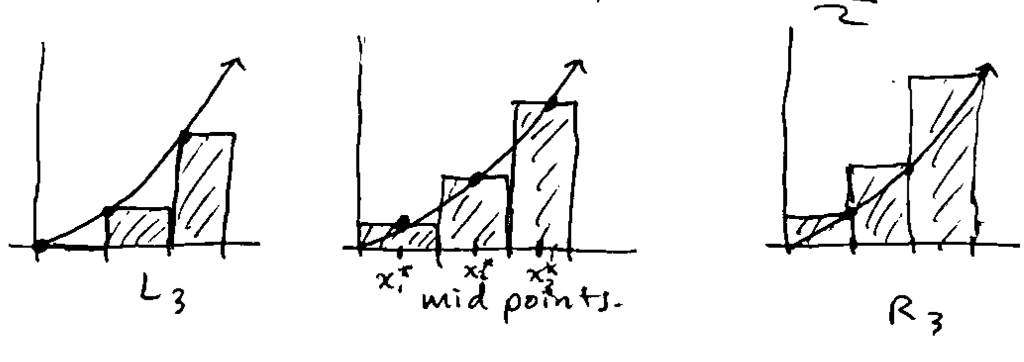
$$\int_a^b f(x)\,dx = \text{area under curve of } y=f(x) \text{ from } a \text{ to } b.$$

More precisely, ... let $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\cdot\Delta x$ for $i = 0, 1, \ldots, n$. Choose a point $x_i^* \in [x_{i-1}, x_i]$ for each $i = 1, \ldots, n$ and set:

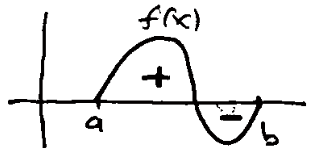
$$A_n = \sum_{i=1}^{n} \Delta x \cdot f(x_i^*).$$

Then $\int_a^b f(x)\,dx = \lim_{n\to\infty} A_n.$

<u>Note</u>: If we choose $x_i^* = x_{i-1}$ for all i, gives L_n; if we choose $x_i^* = x_i$ for all, gives R_n. No matter what point we choose to make the <u>height</u> of the <u>thin rectangles</u> in our approximation of area under curve, in <u>limit</u> all give same value. <u>However</u>, for <u>fixed</u> value of n, approximations are different, and often the <u>best choice</u> is <u>midpoints</u> $x_i^* = \frac{x_{i-1}+x_i}{2}$



For $f(x)$ that are always above x-axis, $\int_a^b f(x)\,dx$ really is area under the curve, but for $f(x)$ that might go below the x-axis, we subtract that area:



$$\int_a^b f(x)\,dx = +(\text{area above x-axis and below } y=f(x)) - (\text{area below x-axis and above } y=f(x))$$

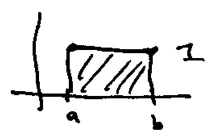
Some more properties of the integral:

For constants $c, d \in \mathbb{R}$ → **Theorem** $\int_a^b \big(c\cdot f(x) + d\cdot g(x)\big)dx = c\cdot\int_a^b f(x)\,dx + d\cdot\int_a^b g(x)\,dx$.

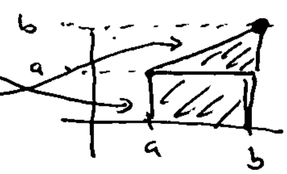
In other words the integral is linear, just like the derivative.

Pf. $\sum_{i=1}^n \Delta x\,(c\cdot f(x) + d\cdot g(x)) = c\cdot\sum_{i=1}^n \Delta x\cdot f(x) + d\cdot\sum_{i=1}^n \Delta x\, g(x)$. ∎

E.g. $\int_a^b 1\,dx = (b-a)$ since just get a rectangle



$$\int_a^b x\,dx = \underset{\text{rectangle}}{a\cdot(b-a)} + \underset{\text{triangle}}{\tfrac{1}{2}(b-a)(b-a)}$$
$$= \tfrac{1}{2}(a+b)(b-a) = \tfrac{1}{2}(b^2-a^2)$$



So that $\int_a^b (mx+c)\,dx = \frac{m}{2}(b^2-a^2) + c(b-a)$

and now we can integrate any linear function...

Even though we only defined $\int_a^b f(x)\,dx$ when $a \le b$

it also makes sense to set $\boxed{\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx}$

(swapping endpoints of integral negates it).

Notice $\int_a^a f(x)\,dx = 0$ by this definition.

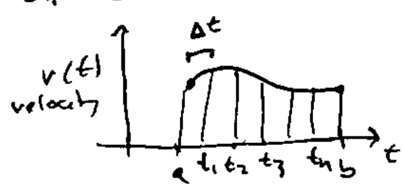
Also--

Prop. For $c \in [a,b]$ have
$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

Pf: Picture:

Position from velocity: We explained how the derivative (slope of tangent) lets us compute the velocity $v(t)$ of a car at time t if all we know is its position function $p(t)$. The integral lets us do the opposite thing!

Specifically, suppose we know $v(t)$ velocity of car as function of time on some interval $[a,b]$:



If $v(t)$ were constantly $= v$ then the distance the car goes from time a to b would just be $v \cdot (a-b)$.

But since velocity is changing, we need to measure it at multiple times. We could approximate the distance traveled by setting
$$\Delta t = \frac{b-a}{n} \text{ and } t_i = a + i \cdot \Delta t \text{ for } i = 0, 1, \ldots, n.$$

Then distance traveled $\approx \sum_{i=1}^{n} \Delta t \cdot v(t_i)$

Since on each short time interval $[t_{i-1}, t_i]$ the velocity is approximately constant.

This means that in the limit we have exactly:

$$\text{distance car travels in time a to b} = \int_a^b v(t)\,dt \leftarrow \text{the integral}$$

11/21 **The Fundamental Theorem of Calculus** § 5.3

The following theorem gives us a way to compute integrals:

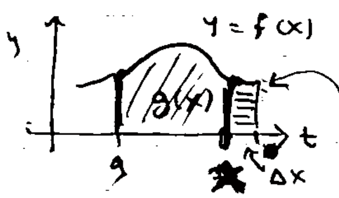
Theorem Let $f(x)$ be a continuous function.

1) Define the function $g(x) = \int_a^x f(t)\,dt$ (for some fixed $a \in \mathbb{R}$).
Then $g'(x) = f(x)$.

2) Suppose that $F(x)$ is an anti-derivative of $f(x)$.
Then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Pf: We only give a sketch of the proof, see book for details...

1) The function $g(x)$ computes the area under the curve $y = f(t)$ for $t = a$ to $t = x$:



If we increase x by Δx, how does $g(x)$ change? Well, since $f(x)$ is continuous, we roughly add $f(x) \cdot \Delta x$ to $g(x)$

Thus $\Delta g \approx \Delta x \cdot f(x)$, i.e., $f(x) \approx \frac{\Delta g}{\Delta x}$. As $\Delta x \to 0$, get that $dg/dx = f(x)$ ✓

For 2): We know from 1 that $g(x)$ is one anti-derivative of $f(x)$ (since $g'(x) = f(x)$),
So there is some constant C s.t. $g(x) = F(x) + C$
Now, $g(a) = \int_a^a f(x)\,dx = 0$, so $C = -F(a)$. Thus,
$\int_a^b f(x)\,dx = g(b) = F(b) - F(a)$ ✓ ☒

For us the point of the Fund. Thm. Calculus is that it lets us evaluate integrals by computing anti-derivatives.

E.g. We saw before that $\int_0^1 x^2\,dx = 1/3$.
Let's do this again, faster. Recall that $F(x) = \frac{1}{3}x^3$ is an anti-derivative of $f(x) = x^2$ since $F'(x) = f(x)$.
Thus by FTC, $\int_0^1 x^2\,dx = F(1) - F(0) = \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 = \frac{1}{3}$.

Since we so often want to compute $F(b) - F(a)$, we use shorthand notation $F(x)\Big]_a^b = F(b) - F(a)$.
Then FTC says $\int_a^b f(x)\,dx = F(x)\Big]_a^b$.

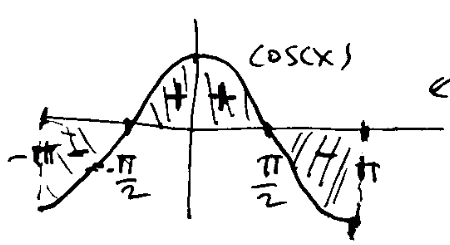
E.g. To compute $\int_1^2 e^x\,dx$, we recall that e^x is the anti-derivative of e^x so that

$$\int_1^2 e^x\,dx = e^x\Big]_1^2 = e^2 - e^1 = e(e-1)$$

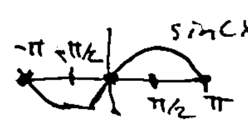
E.g. $\sin(x)$ is an anti derivative of $\cos(x)$, so

$$\int_{-\pi}^{\pi} \cos(x)\,dx = \sin(x)\Big]_{-\pi}^{\pi} = \sin(\pi) - \sin(-\pi) = 0 - 0 = 0$$

This makes sense since:



positive and negative areas cancel by symmetry: 0 result



11/28 Indefinite Integrals §5.4

We want a better notation for anti-derivatives. This will come from the so-called indefinite integral.

Def'n We write $\boxed{\int f(x)\,dx = F(x)}$ to mean that $F'(x) = f(x)$. The expression "$\int f(x)\,dx$" is called an indefinite integral.

Note: Do not confuse definite and indefinite integrals. The definite integral $\int_a^b f(x)\,dx$ is a number: it is the area under the curve $y=f(x)$ from $x=a$ to b. The indefinite integral $\int f(x)\,dx$ is a function: it is the anti-derivative of $f(x)$.

E.g. $\int_0^1 x^2\,dx = 1/3$ as we have seen.

But $\int x^2\,dx = 1/3\,x^3 + C$ (for any $C \in \mathbb{R}$).

Table of indefinite integrals we know so far:

- $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$ (for $n \neq -1$)
- $\int 1/x\,dx = \ln(x) + C$
- $\int \cos(x)\,dx = \sin(x) + C$
- $\int e^x\,dx = e^x + C$
- $\int \sin(x)\,dx = -\cos(x) + C$

(here $C \in \mathbb{R}$ is any constant)

With this indefinite integral notation, we can restate the Fundamental Theorem of Calculus as:

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big]_a^b$$

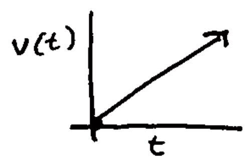
definite integral . indefinite integral $F(x)$ evaluated: $F(b)-F(a)$

Net Change: Another way to think of FTC:

$$\int_a^b F'(x)\,dx = F(b) - F(a) \quad \text{is}$$

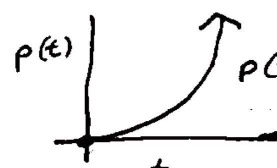
"the integral of the (instantaneous) rate of change is the net change (over some time interval)."

E.g. 1) If $p(t)$ is the position of a car (from some point on the road) at time t, we have seen that $v(t) = p'(t)$ is the velocity a.k.a. speed. Thus $\int_a^b v(t)\,dt = \int_a^b p'(t)\,dt = p(b) - p(a)$ means that the integral of velocity (from time a to b) is the net displacement of the car from time a to b (distance traveled).



Velocity of car experiencing constant acceleration

$\Longrightarrow$



$p(t) = \int v(t)\,dt$

position function is integral of velocity.

2) In biology, if $n(t)$ is the number of organisms of some population, then dn/dt is the rate of growth of the population.

at time t

Hence $\int_a^b dn/dt\,dt = n(b) - n(a)$ is the net population growth from time a to time b.

3) In economics, if $p(x)$ is the profit from selling x units of some product, then dp/dx is the marginal profit. The FTC says the integral of marginal profit = total profit.

//

11/29 Integration by Substitution § 5.5

There are many integrals like $\int x \cdot \cos(x^2+1)\,dx$ where our rules for integration don't apply.
One technique for integration is called integration by substitution or "u-substitution" for short.

Theorem If f, g are differentiable functions then

$$\int f'(g(x)) \cdot g'(x)\,dx = f(g(x)) + C.$$

Pf: By chain rule, $d/dx\,(f(g(x))) = f'(g(x)) \cdot g'(x)$. ∎

How to use this theorem in practice? Let's see...

E.g. We want to compute $\int x \cdot \cos(x^2+1)\,dx$.
Let's set $\boxed{u = x^2+1}$ (think $u = g(x)$ is a function of x).
Then $\frac{du}{dx} = 2x$, or in differential notation $\boxed{du = 2x\,dx}$

Then
$$\int x \cdot \cos(x^2+1)\,dx = \int \underbrace{\cos(x^2+1) \cdot \tfrac{1}{2}} \cdot \underbrace{2x\,dx}$$
$$= \int \tfrac{1}{2}\cos(u) \cdot du$$
$$= \tfrac{1}{2}\int \cos(u)\,du = \tfrac{1}{2}\sin(u) + C$$
$$= \boxed{\tfrac{1}{2}\sin(x^2+1) + C}$$

This is how the u-substitution technique works.
The previous theorem says we can treat the dx (and du) in integral like the dx in $\frac{du}{dx}$, etc. But... must only integrate things of form $\int h(u)\,du$, not $\int h(u)\,dx$.

The steps to u-substitution are:

- decide what $u = g(x)$ should be
- figure out what du is in terms of dx
- Convert $\int f(x)\,dx = \int h(u)\,du$ by making the appropriate substitutions
- hopefully $\int h(u)\,du = H(u)$ is an integral you already know how to do
- Convert from u back to x: write $H(u) = F(x)$ ✓

Let's do some more examples:

E.g. $\int x^2 \cdot e^{4x^3+2}\,dx$.

We see "$4x^3+2$" inside the exponential, so a guess is that a good choice for u might be

$$u = 4x^3+2 \Rightarrow \text{"}du = 12x^2\,dx\text{"}$$

Since x^2 is there, we're in luck!:

$$\int x^2 \cdot e^{4x^3+2}\,dx = \int \underbrace{\tfrac{1}{12}}\, e^{4x^3+2} \cdot \underbrace{12\,x^2\,dx}$$

$$= \int \tfrac{1}{12} e^u \cdot du$$ we know how to integrate e^u

$$= \tfrac{1}{12} e^u + C$$ since $u = 4x^3+2$ this is how we go back to a function of x

$$= \tfrac{1}{12} e^{4x^3+2} + C$$

If we're ever in doubt we did something wrong, can double-check by differentiating:

$$d/dx\left(\tfrac{1}{12} e^{4x^3+2}\right) = \tfrac{1}{12} e^{4x^3+2} \cdot 12x^2 = x^2 e^{4x^3+2}$$ ✓

chain rule

E.g. $\int 2x\sqrt{3x^2+1}\,dx$

Good choice of u is $u = 3x^2+1 \Rightarrow du = 6x\,dx$

$$\int 2x\sqrt{3x^2+1}\,dx = \int \tfrac{1}{3}\sqrt{3x^2+1}\;6x\,dx$$
$$= \int \tfrac{1}{3}\sqrt{u}\,du = \tfrac{1}{3}\cdot\tfrac{2}{3}u^{\frac{3}{2}} + C$$
$$= \tfrac{2}{9}(3x^2+1)^{\frac{3}{2}} + C \quad ✓$$

E.g. $\int \sin(x)\cos(x)\,dx$

This one seems a little trickier... no obvious polynomial expression involving x appears.

Let's try $u = \sin(x) \Rightarrow du = \cos(x)\,dx$

This is good since both sin(x) and cos(x) appear!

$$\int \sin(x)\cos(x)\,dx = \int u \cdot du$$
$$= \tfrac{1}{2}u^2 + C$$
$$= \tfrac{1}{2}\sin(x)^2 + C$$

(Could also try $u=\cos(x)$... what would that give?)

As you can see, using the u-substitution technique is a bit of an art because you often have to guess a smart choice for what u should be!