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More ways limits can fail to exist: § 2.2

So far we only saw one example of a limit not existing, and it was when ^{the} two one-sided limits disagreed.

But limits can fail to exist for many reasons.

E.g. Consider $f(x) = \frac{1}{x^2}$ for x near zero,



$f(x)$ will be a big positive #, and gets bigger + bigger as x gets closer + closer to 0.

so $\lim_{x \rightarrow 0} \frac{1}{x^2}$ does not exist.

In this case we write $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$ to mean that

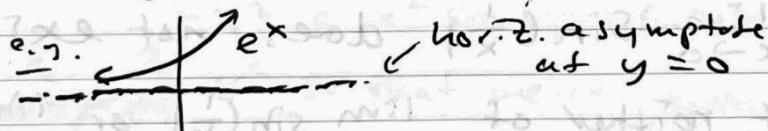
as ~~x~~ gets closer to zero (on either side), $f(x)$ can be made arbitrarily big.

Note: $\lim_{x \rightarrow a} f(x) = \infty$ (or $\lim_{x \rightarrow a^-} f(x) = -\infty$)

counts as the limit not existing.

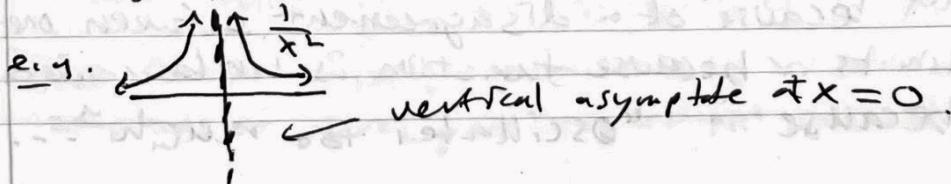
Compare: if $\lim_{x \rightarrow \infty} f(x) = L$ or $\lim_{x \rightarrow -\infty} f(x) = L$

then $f(x)$ has a "horizontal asymptote at $y = L$ ".



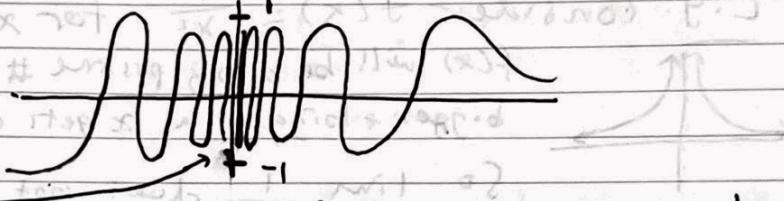
(if $\lim_{x \rightarrow a} f(x) = \infty$ or $\lim_{x \rightarrow a} f(x) = -\infty$)

then $f(x)$ has a "vertical asymptote at $x = a$ ".



Limits can fail to exist for even more "complicated" reasons:

E.g. Let $f(x) = \sin(\frac{1}{x})$. We saw a while ago that its graph looks like:



As we bring x closer and closer to zero, $\frac{1}{x}$ passes through many values, so $\sin(\frac{1}{x})$ passes through many periods. In each period, it attains a max. value of 1 and a min. value of -1.

Thus, near zero, there are many x for which $\sin(\frac{1}{x}) = 1$ and many for which $\sin(\frac{1}{x}) = -1$.

Since it oscillates rapidly between these values, there is no single value that $f(x)$ approaches as x gets closer to zero. Thus,

$$\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right) \text{ does not exist.}$$

In fact, neither of $\lim_{x \rightarrow 0^-} \sin(\frac{1}{x})$ or $\lim_{x \rightarrow 0^+} \sin(\frac{1}{x})$ exists either. (So this limit fails to exist not because of a disagreement between one-sided limits or because function is unbounded, but because it "oscillates too much"...) //

The Squeeze Theorem § 2.3

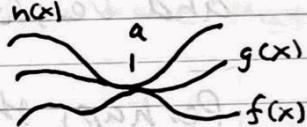
Sometimes we can calculate a limit for a function $f(x)$ by comparing it to other functions in size.

Thm If $f(x) \leq g(x)$ for x near a (except possibly at a) and the limits of f & g at a both exist, then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x).$$

Thm (Squeeze Theorem) If $f(x) \leq g(x) \leq h(x)$ for x near a (except possibly at a) and $\lim_{x \rightarrow a} f(x) = L = \lim_{x \rightarrow a} h(x)$

then $\lim_{x \rightarrow a} g(x) = L$. "picture":



Eg. Let's use the Squeeze theorem to compute

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right).$$

Note we cannot use product law for limits here

Since $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$ does not exist. But...

Since $\sin\left(\frac{1}{x}\right)$ is always between -1 and 1, hence

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2 \quad \forall x$$

so that we can apply squeeze theorem

$$\text{with } \lim_{x \rightarrow 0} -x^2 = \lim_{x \rightarrow 0} x^2 = 0$$

$$\text{and thus } \lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

(Even though $x^2 \sin\left(\frac{1}{x}\right)$ "oscillates" a lot as $x \rightarrow 0$, amplitude of waves gets smaller and smaller...)

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More about one-sided limits + limits at ∞ §2.6

Basically all of the laws/theorems for limits also hold for one-sided limits and limits at infinity.

E.g. we have $\lim_{x \rightarrow a^-} f(x) \cdot g(x) = \lim_{x \rightarrow a^-} f(x) + \lim_{x \rightarrow a^-} g(x)$

$$\lim_{x \rightarrow \infty} f(x) \cdot g(x) = \lim_{x \rightarrow \infty} f(x) \cdot \lim_{x \rightarrow \infty} g(x), \text{ etc.}$$

(when these limits exist) and even

Thm If $f(x) \leq g(x)$ then $\lim_{x \rightarrow \infty} f(x) \leq \lim_{x \rightarrow \infty} g(x)$

and versions of the squeeze thm, etc.

Perhaps the one additional law for limits at ∞ is:

Prop. For any integer $r > 0$, have

$$\lim_{x \rightarrow \infty} \frac{1}{x^r} = \lim_{x \rightarrow -\infty} \frac{1}{x^r} = 0.$$

(For $r > 0$ have $\lim_{x \rightarrow \infty} x^r = \infty$ and $\lim_{x \rightarrow -\infty} x^r = \begin{cases} +\infty & r \text{ even} \\ -\infty & r \text{ odd} \end{cases}$)

Let's see one example how to use this:

$$\text{E.g. } \lim_{x \rightarrow \infty} \frac{3x^2 - 2x + 4}{5x^2 + x - 7} \xrightarrow{\text{divide top + bottom by } x^2} \lim_{x \rightarrow \infty} \frac{3 - \frac{2}{x} + \frac{4}{x^2}}{5 + \frac{1}{x} - \frac{7}{x^2}}$$

quotient law

$$= \lim_{x \rightarrow \infty} 3 - \frac{2}{x} + \frac{4}{x^2} = \dots = \frac{3 - 2 \cdot 0 + 4 \cdot 0}{5 + 1 \cdot 0 - 7 \cdot 0}$$

$$\lim_{x \rightarrow \infty} 5 + \frac{1}{x} - \frac{7}{x^2} = \dots = \frac{5 + 1 \cdot 0 - 7 \cdot 0}{5 + 1 \cdot 0 - 7 \cdot 0}$$

upshot: only "leading term" matters at ∞ . $\rightarrow = \frac{3}{5}$.

Precise definition of limit. § 2.4

The way we defined a limit so far has been a little vague because of imprecise terms like "near" and "close to". The precise definition of a limit is:

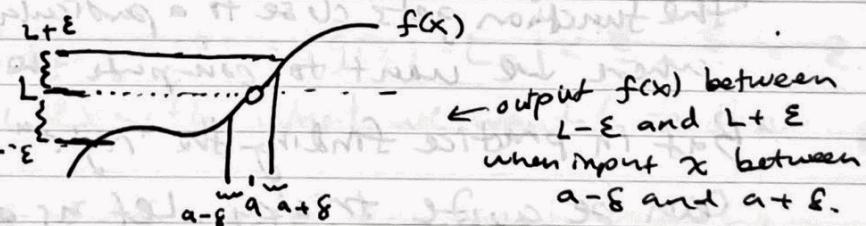
Def'n Let $f(x)$ be a function defined on an open interval containing $a \in \mathbb{R}$, except possibly at a itself.

We say $\lim_{x \rightarrow a} f(x) = L$ for a number $L \in \mathbb{R}$ if:

- for every $\epsilon > 0$ there is a $\delta > 0$ such that for all x with $0 < |x - a| < \delta$, we have that $|f(x) - L| < \epsilon$.

Think: However close ($\epsilon > 0$) we desire the output ($f(x)$) to be to the ~~limit value~~ (L), we can get that close by requiring the input (x) to be close ($\delta > 0$) to ~~limit point (a)~~.

Graphically:

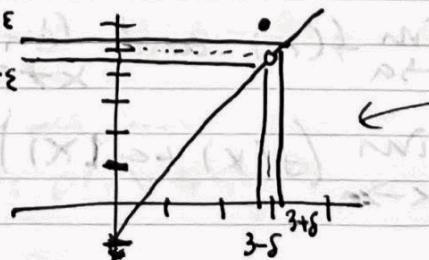


← output $f(x)$ between $L - \epsilon$ and $L + \epsilon$ when input x between $a - \delta$ and $a + \delta$.

Let's see an example of showing that

$$\lim_{x \rightarrow 3} f(x) = 5 \text{ when } f(x) = \begin{cases} 2x-1 & \text{if } x \neq 3 \\ 6 & \text{if } x=3 \end{cases}$$

Graphically:



looks like we can find a narrow band of inputs to fall into a narrow band of outputs.

Think: my "enemy" gives me $\epsilon > 0$. I need to find a $\delta > 0$ so that $|f(x) - 5| \leq \epsilon$,
 i.e. $5 - \epsilon \leq f(x) \leq 5 + \epsilon$,
 for all x with $0 < |x - 3| < \delta$,
 i.e. $3 - \delta < x < 3 + \delta$ and $x \neq 3$.

A good choice for this $f(x)$ is $\delta = \frac{\epsilon}{2}$.

Indeed, if $3 - \delta < x < 3 + \delta$ (and $x \neq 3$)

that means $3 - \frac{\epsilon}{2} < x < 3 + \frac{\epsilon}{2}$

so that $6 - \epsilon < 2x < 6 + \epsilon$

i.e., $5 - \epsilon < 2x - 1 < 5 + \epsilon$

which is $5 - \epsilon < f(x) < 5 + \epsilon$,

what we wanted to show!

We see how this definition captures the idea of "the function gets close to a particular value a where we want to compute the limit" precisely.

But in practice finding the "right" δ in terms of ϵ can be quite tricky. Let us give one example of what formally proving the limit laws looks like using the $\epsilon - \delta$ definition of limit.

Thm If $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist,

then $\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$.

Pf: Let $L_1 = \lim_{x \rightarrow a} f(x)$ and $L_2 = \lim_{x \rightarrow a} g(x)$.

We want to show that $\lim_{x \rightarrow a} (f(x) + g(x)) = L_1 + L_2$.

So let $\epsilon > 0$ be given.

By supposition that $\lim_{x \rightarrow a} f(x) = L_1$, there

is a $\delta_1 > 0$ s.t. $|f(x) - L_1| < \frac{\epsilon}{2}$ for all $0 < |x - a| < \delta_1$.

Simil. w.r.t. g , there is a $\delta_2 > 0$ s.t. $|g(x) - L_2| < \frac{\epsilon}{2}$ for all $0 < |x - a| < \delta_2$.

Set $\delta = \min(\delta_1, \delta_2)$. Then for all

$0 < |x - a| < \delta \leq \delta_1, \delta_2$ have

$$|f(x) - L_1| < \frac{\epsilon}{2} \text{ and } |g(x) - L_2| < \frac{\epsilon}{2}$$

$$\text{So that } |(f(x) + g(x)) - (L_1 + L_2)|$$

$$= |(f(x) - L_1) + (g(x) - L_2)|$$

$$\leq |f(x) - L_1| + |g(x) - L_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

which is exactly what we wanted to show. \square

See that $\epsilon-\delta$ style arguments can be rather tricky, and also sometimes tedious.

Exercise: Prove another limit law using $\epsilon-\delta$ definition of limit.

from now on, we will compute limits (and derivatives, etc.) using the laws, and not $\epsilon-\delta$ defn.

9/23 Continuity § 2.5

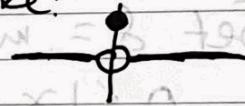
Recall that we say $f(x)$ is continuous at a if $f(a) = \lim_{x \rightarrow a} f(x)$. This requires 3 things:

- $f(x)$ is defined at $x=a$, i.e., $a \in$ domain of f ,
- $\lim_{x \rightarrow a} f(x)$ exists,
- and $f(a)$ and $\lim_{x \rightarrow a} f(x)$ are the same number.

If $f(x)$ is not continuous at a we say it is discontinuous there.

Most of the examples of discontinuity we've seen were piecewise functions like:

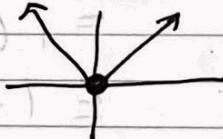
$$f(x) = \begin{cases} 1 & \text{if } x=0 \\ 0 & \text{otherwise} \end{cases}$$



where the function "jumps" suddenly at a point.

But note that not all piecewise functions are discontinuous, e.g. the absolute value function

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



is continuous even at $x=0$.

The reason examples of discontinuity we've seen look "contrived" is because

thus the following kinds of functions are continuous at all points in their domain:

- polynomials
- rational functions
- root functions like \sqrt{x}
- trig functions like $\sin(x)$ and $\cos(x)$
- exponentials like e^x
- logarithms like $\ln(x)$

Furthermore...

Thm If f and g are continuous at a , then so are
 $\circ f+g \circ f-g \circ f \cdot g \circ \frac{f}{g}$ if $g(a) \neq 0 \circ c \cdot f$ for any constant $c \in \mathbb{R}$.

And we can even say the following about composition:

Thm If $\lim_{x \rightarrow a} g(x) = b$ and f is continuous at b ,
then $\lim_{x \rightarrow a} f(g(x)) = f(b)$ ($= f(\lim_{x \rightarrow a} g(x))$)

"Can push the limit thru continuous functions"

Cor If g is continuous at a and f is continuous at $g(a)$
then composite $f \circ g$ is continuous at a .

Upshot: All the ways of combining all the "normal" functions
we've considered give functions continuous at all pts in their domain.
So... to compute a limit of a function like this
try plugging in!

E.g. $\lim_{x \rightarrow 0} \sin\left(\frac{\pi}{2} \cdot e^x\right) = \sin\left(\frac{\pi}{2} \cdot e^0\right) = \sin\left(\frac{\pi}{2}\right) = 1.$

Warning: Remember $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$ D.N.E.
But 0 is not in the domain of $\sin\left(\frac{1}{x}\right)$.

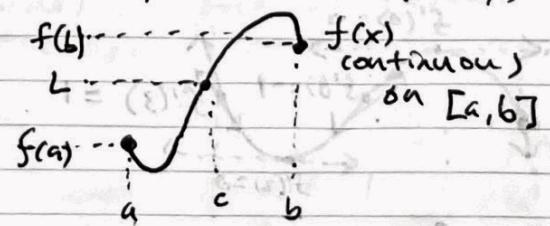
One more important property of continuous functions:

Thm Let $f(x)$ be continuous on some closed interval $[a, b]$.

Then for every L w/ $f(a) \leq L \leq f(b)$, there is a $c \in [a, b]$ w/ $f(c) = L$.

Called the "intermediate value theorem"

It says f takes on all values "intermediate" between $f(a)$ and $f(b)$.



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The derivative as a function

§ 2.8

Recall that we defined the derivative of $f(x)$ at a in ^{2 ways}

- the slope of the tangent to the curve $y = f(x)$ at $(a, f(a))$
- the limit $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$

We were thinking of the point a as fixed. But now, let us consider the point we're taking the derivative at to vary. Thus, we define the derivative at x :

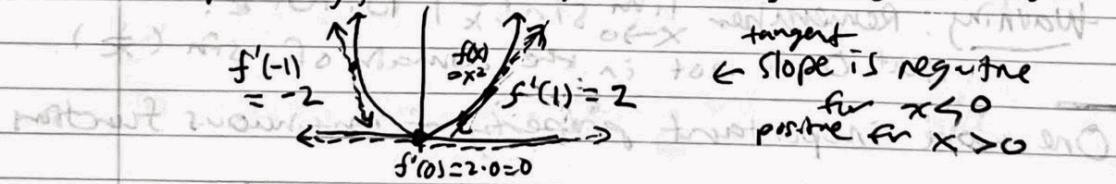
$$f'(x) := \lim_{k \rightarrow x} \frac{f(k) - f(x)}{k - x}$$

We think of $f'(x)$ as a new function defined from $f(x)$.

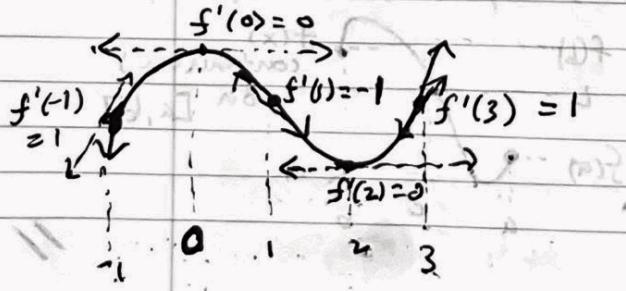
E.g. Let's compute $f'(x)$ for $f(x) = x^2$.

$$\begin{aligned} f'(x) &= \lim_{k \rightarrow x} \frac{f(k) - f(x)}{k - x} = \lim_{k \rightarrow x} \frac{k^2 - x^2}{k - x} \\ &\stackrel{\text{"difference of squares"}}{=} \lim_{k \rightarrow x} \frac{(k-x)(k+x)}{k-x} = \lim_{k \rightarrow x} k+x = 2x \end{aligned}$$

Graphically, this answer makes sense in terms of tangent lines:



E.g. We can estimate $f'(x)$ from a graph of $f(x)$ using tangents!



We see that

- $f'(x) > 0 \Leftrightarrow f$ is increasing at x
- $f'(x) < 0 \Leftrightarrow f$ is decreasing at x
- $f'(x) = 0 \Leftrightarrow f$ has a local min./max. at x

Def'n We say $f(x)$ is differentiable at x if $f'(x)$ exists.

Since it's a limit, it doesn't have to exist!

In fact, we have the following important theorem

Theorem If $f(x)$ is differentiable at x , then
it is continuous at x .

E.g. Let $f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$



Then, since $f(x)$ is not continuous at $x = 0$

$f'(0)$ does not exist ("it's not defined").

$$\bullet \quad \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h - 0} = \lim_{h \rightarrow 0} \frac{0 - 1}{h} = \lim_{h \rightarrow 0} \frac{-1}{h} \text{ D.N.E.}$$

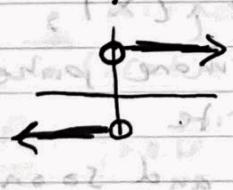
But... there are other ways $f(x)$ can fail to be differentiable.

E.g. Let $f(x) = |x|$. We mentioned before that



$f(x)$ is continuous at $x = 0$.

But it is not differentiable at $x = 0$.



Indeed, for $x > 0$, have $f'(x) = 1$

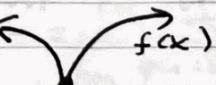
since slope is clearly = 1.

for $x < 0$, have $f'(x) = -1$

since slope is = -1.

But at $x = 0$, slopes on left- and right-hand sides disagree, so cannot define derivative as a single number.

In general, a major way differentiability can fail is if there is a "crease" (or "cusp")



"cusp" where $f'(x)$ D.N.E.