Final Exam, 12/7
Math 181 (Discrete Structures), Fall 2022

Each problem is worth 10 points, for a total of 80 points. You have 90 minutes to do the exam. Remember to *show your work* and *explain your answers* on all problems!

1. Let $A = \{1, 3, 4\}$, $B = \{2, 3, 4, 6\}$ and $C = \{1, 3, 5, 6\}$.
   (a) Draw a Venn diagram for this situation.
   (b) Let $X = (A \setminus B) \cup (B \setminus A)$. Shade the area of the Venn diagram corresponding to X.
   (c) Write the elements of $C \cap X$.

2. Convert the following argument to symbolic form and decide (with explanation) if it's valid.

   Hypotheses: If it's raining then I bring an umbrella or I wear a jacket. I bring an umbrella.
   Conclusion: It's raining.

3. Give a proof of this theorem: "For any sets X and Y, if $X \cup Y = \varnothing$ then $X = \varnothing$ and $Y = \varnothing$."

4. Prove by induction that, for all $n \geq 1$,
$$1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n! = (n+1)! - 1.$$

5. Let $X = \{0, 1, 2\}$. Define functions $f\colon X \to X$ and $g\colon X \to X$ by letting
$$f(x) = 2x \mod 3 \qquad \text{and} \qquad g(x) = x^2 \mod 3$$
for all $x \in X$.
   (a) Draw the arrow diagrams for f, for g, and for $f \circ g$.
   (b) Which of f, g, and $f \circ g$ are bijections? Explain.

6. Define a relation R on the set $\mathbb{Z} \times \mathbb{Z}$ of pairs of integers by declaring (a, b) R (c, d) if and only if $a + b \leq c + d$. For each of the following four properties, explain whether R has that property or not: (i) reflexive, (ii) symmetric, (iii) anti-symmetric, and (iv) transitive.

7. (a) How many rearrangements of the word HAWAII start with an H?
   (b) How many rearrangements of HAWAII end with an I?
   (c) How many rearrangements of HAWAII start with an H or end with an I (or both)?

8. Recall that Pascal's triangle of binomial coefficients $C(n, k)$ begins:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

   (a) Write down the next three rows of Pascal's triangle, i.e., the rows for $n = 4$, 5, and 6.
   (b) Use this to expand $(x + y)^5$.
   (c) How many three element subsets of a six element set are there?