# Cyclic Symmetry in Combinatorics

Howard University Colloquium

Sam Hopkins

University of Minnesota

February 9th, 2021

Section 1

# Introductory example: subsets under rotation

# Subsets and binomial coefficients

One of the first sequences of numbers you learn about in enumerative combinatorics are the **binomial coefficients**

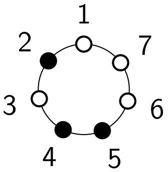
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

which of course count the number of $k$-element subsets of $\{1, 2, \ldots, n\}$.

There's a nice way to represent these $k$-subsets via circular diagrams:

$n = 7,$
$k = 3$

$\{2, 4, 5\} \Leftrightarrow$

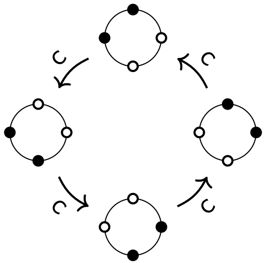


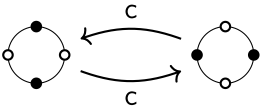

This suggests a natural **cyclic symmetry** of the collection of $k$-subsets.

# Cyclic action on subsets

Indeed, the cyclic action $c\colon i \mapsto i+1 \mod n$ on subsets corresponds to **rotation** of the circular diagrams:

$n = 4,$
$k = 2$




We might want to know the orbit structure of this action. Turns out there is a very nice answer in terms of **$q$-binomial coefficients**.

**Note:** there are two perspectives here, both useful. One is that we have a cyclic symmetry. Other is that we're studying **dynamics**.

# $q$-binomial coefficients

The **$q$-binomial coefficient** (or **Gaussian binomial coefficient**) is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}$$

where we use the **$q$-factorial** notation $[n]_q! := [n]_q[n-1]_q \cdots [2]_q[1]_q$ and **$q$-number** notation $[n]_q := \frac{(1-q^n)}{(1-q)} = (1 + q + q^2 + \cdots + q^{n-1})$.

**Example**

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{[4]_q!}{[2]_q![2]_q!} = \frac{(1+q+q^2+q^3)(1+q+q^2)}{(1+q)} = 1+q+2q^2+q^3+q^4$$

In fact, $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is always a **polynomial** in $q$:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{-\binom{k+1}{2}} \cdot \sum_{\{i_1,i_2,\ldots,i_k\} \subseteq \{1,2,\ldots,n\}} q^{i_1+i_2+\cdots+i_k}$$

# The cyclic sieving phenomenon for subsets

## Theorem (Reiner–Stanton–White)

*The number of $k$-subsets of $\{1, 2, \ldots, n\}$ fixed by the $m$th power $c^m$ of rotation is given by setting $q := \zeta^m$ in $\begin{bmatrix} n \\ k \end{bmatrix}_q$, where $\zeta := e^{2\pi i/n}$ is a primitive $n$th root of unity.*

This is a prototypical **cyclic sieving phenomenon (CSP)**. It says the whole orbit structure of subset rotation is recorded in $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

## Example ($n = 4, k = 2$)

$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$, so $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_1 = 6$, $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_{\pm i} = 0$, and $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_{-1} = 2$.

This means all 6 subsets are fixed by $c^0$, no subsets are fixed by $c^1$ or $c^3$, and two subsets are fixed by $c^2$, agreeing with what we saw earlier.

# Algebra behind CSP: exterior powers of vector spaces

There's a very nice algebraic explanation of this CSP.

Namely, let $V \simeq \mathbb{C}^n$ be an $n$-dimensional complex vector space. Then

$$\dim(\Lambda^k V) = \binom{n}{k},$$

where $\Lambda^k V$ is the **$k$th exterior power** of $V$. In fact, if $\{e_1, \ldots, e_n\}$ is a basis of $V$ then

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \colon \{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}\}$$

is a basis of $\Lambda^k V$.

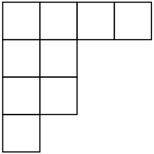
Moreover, the **symmetric group** $S_n$ of permutations of $\{1, \ldots, n\}$ acts naturally on $V$, hence on $\Lambda^k V$, and the action of $c = (1, 2, \ldots, n) \in S_n$ corresponds to rotation of subsets. The CSP can be proved by computing the **character** (i.e., **trace**) of this element $c$.

Section 2

# Promotion of Standard Young Tableaux

# Young diagrams and Standard Young Tableaux

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be an **integer partition**. We represent $\lambda$ via its **Young diagram**, which has $\lambda_i$ left-justified boxes in row $i$:

$(4, 2, 2, 1) \Leftrightarrow$ 

A **Standard Young Tableau (SYT)** of shape $\lambda$ is a bijective filling of its boxes with the numbers $1, 2, \ldots, n = |\lambda|$, increasing in rows and columns:

| | | | |
|---|---|---|---|
| 1 | 3 | 4 | 8 |
| 2 | 5 | | |
| 6 | 7 | | |
| 9 | | | |

SYTs are a central object in modern algebraic combinatorics.

# SYTs in algebra

SYTs appear in many parts of algebra and geometry.

For example, the partitions of $n$ index the **irreducible representations** of the symmetric group $S_n$. And the **dimension** of the representation $S^\lambda$ indexed by $\lambda$ is given by the number of SYTs of shape $\lambda$.

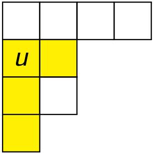
There's a beautiful formula for this quantity:

**Theorem (Hook-length formula)**

*The number of SYTs of shape $\lambda$ is*

$$n! \cdot \prod_{u \in \lambda} \frac{1}{h(u)}$$

*where $h(u)$ is* **hook-length** *of box $u$.*




$h(u) = 4$

# Promotion of SYTs

**Promotion** is the following invertible operation on the SYTs of shape $\lambda$:

- Delete the entry 1.
- Slide boxes into the resulting hole.
- Decrement all entries.
- Fill the hole with $n$.

## Example

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} \xrightarrow{\text{del. } 1} \begin{array}{|c|c|c|}\hline \bullet & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} \xrightarrow{\text{slide}} \begin{array}{|c|c|c|}\hline \bullet & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} \xrightarrow{\text{decr.}} \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & \bullet \\ \hline \end{array} \xrightarrow{\text{fill } 6} \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array} = \mathrm{Pro}(T)$$

Together with a partner operation called **evacuation**, promotion was first defined by Schützenberger in his study of combinatorics of representation theory of the symmetric group and related theory of **symmetric functions**.

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# Promotion of rectangular SYTs

For most $\lambda$, promotion behaves **chaotically**: e.g., for $\lambda = (4, 2, 2, 1)$, the order of promotion is 32760.

But for a **rectangular** shape $\lambda = a \times b$, Schützenberger proved that we have $\mathrm{Pro}^{ab}(T) = T$ for all SYTs $T$ of shape $\lambda$:

$$\cdots \xrightarrow{\mathrm{Pro}} \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline\end{array} \xrightarrow{\mathrm{Pro}} \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline\end{array} \xrightarrow{\mathrm{Pro}} \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline\end{array} \cdots$$

Moreover, Rhoades proved a remarkable CSP for promotion of rectangular SYTs, involving the polynomial

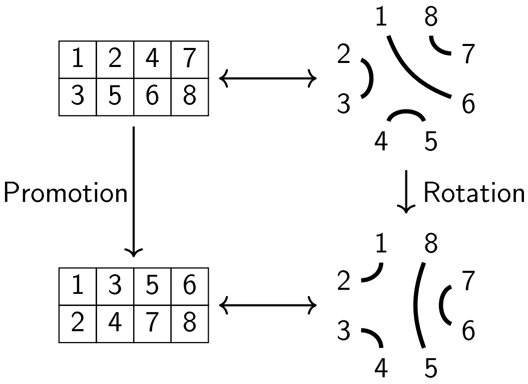
$$f(q) = [n]_q! \cdot \prod_{u \in \lambda} \frac{1}{[h(u)]_q}.$$

To do this, he employed some rather advanced machinery from algebra: the **Kazhdan–Lusztig cellular basis** for representations of $S_n$.

# $2 \times b$ SYTs and noncrossing matchings

For certain short rectangles, can avoid advanced algebra machinery using **diagrammatic models** of promotion.

White observed that promotion of $2 \times b$ SYTs is the same as rotation of **noncrossing matchings** of $1, 2, \ldots, 2b$ (counted by **Catalan numbers**):

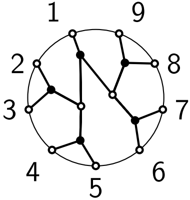


# $3 \times b$ SYTs and webs

**Webs** are certain planar diagrams that Kuperberg introduced to study the invariant theory of simple Lie algebras. Petersen, Pyylavskyy, and Rhoades showed that promotion of $3 \times b$ SYTs is the same as rotation of $\mathfrak{sl}_3$-webs:

| | | |
|---|---|---|
| 1 | 2 | 6 |
| 3 | 4 | 8 |
| 5 | 7 | 9 |

$\longleftrightarrow$

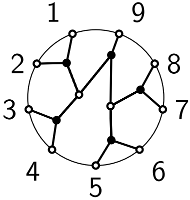



Promotion $\downarrow$

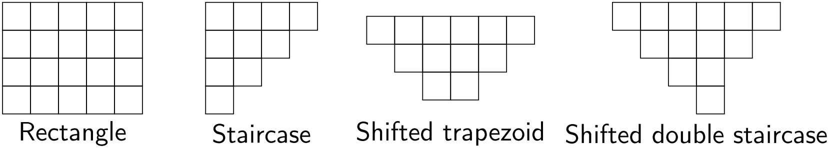
$\downarrow$ Rotation

| | | |
|---|---|---|
| 1 | 2 | 6 |
| 3 | 4 | 8 |
| 5 | 7 | 9 |

$\longleftrightarrow$



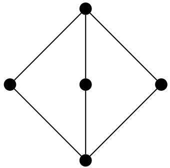
# Promotion of other shapes



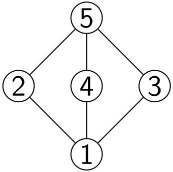

- (Schützenberger) For $\lambda$ a rectangle, $\mathrm{Pro}^{|\lambda|}$ is the identity.
- (Edelman–Greene) For $\lambda$ a **staircase**, $\mathrm{Pro}^{|\lambda|}$ is **transposition**.
- (Haiman) For $\lambda$ a **shifted trapezoid** or **shifted double staircase**, $\mathrm{Pro}^{|\lambda|}$ is the identity.
- (Haiman–Kim) These are the **only** four families of shapes with good promotion behavior.

# Promotion for linear extensions of posets

Young diagram shapes $\lambda$ and their SYTs are a special case of **posets (partially ordered sets)** and their **linear extensions**:



Poset $P$




A linear extension of $P$

Promotion makes sense more generally acting on the linear extensions of any finite poset, with essentially the same definition.

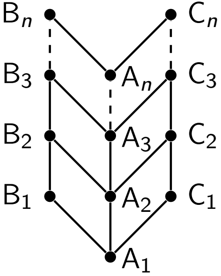
**Stanley's question**: Are there any other posets, beyond the four families of shapes classified by Haiman, for which promotion behaves well?

Section 3

# New results on promotion

# The chain of V's

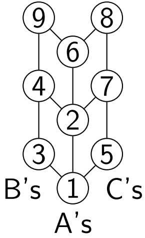
Let $V(n)$ be the following poset:




We will see that $V(n)$ addresses Stanley's question.

# Linear extensions of the chain of V's

Linear extensions of $V(n)$ correspond to **words** with $n$ A's, $n$ B's, and $n$ C's such that every prefix has as many A's as B's and as many as A's as C's:




$\Leftrightarrow$ AABBCACCB

Kreweras showed that the number of such words is

$$\frac{4^n}{(n+1)(2n+1)}\binom{3n}{n}.$$

Compare to **Dyck words**, which are counted by Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$.

# Promotion for the chain of V's

**Example**

$w = \text{AABBCACCB}$

$\mathrm{Pro}(w) = \text{ABACACCBB}$

$\mathrm{Pro}^2(w) = \text{AACACCBBB}$

$\mathrm{Pro}^3(w) = \text{ACACABBBC}$

$\mathrm{Pro}^4(w) = \text{AACABBBCC}$

$\mathrm{Pro}^5(w) = \text{ACABBACCB}$

$\mathrm{Pro}^6(w) = \text{AABBACCBC}$

$\mathrm{Pro}^7(w) = \text{ABAACCBCB}$

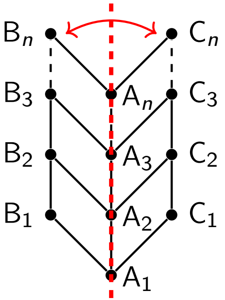
$\mathrm{Pro}^8(w) = \text{AAACCBCBB}$

$\mathrm{Pro}^9(w) = \text{AACCBABBC}$

With Martin Rubey, we showed that $V(n)$ has good promotion behavior:

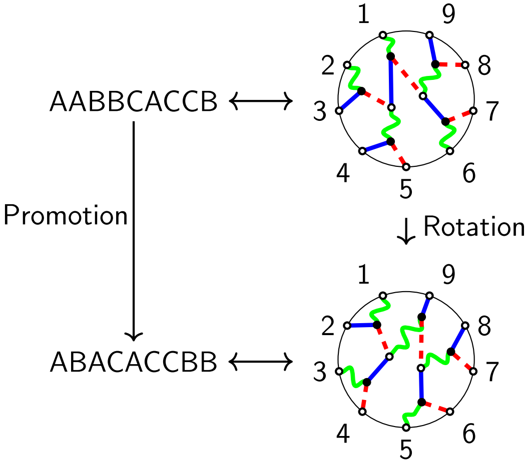
**Theorem (Hopkins–Rubey, 2020)**

*For $P = V(n)$, $\mathrm{Pro}^{\#P}$ is reflection across the vertical axis of symmetry.*



# Promotion and webs, again

We showed that Kreweras words can be encoded as certain **3-edge-colored** $\mathfrak{sl}_3$-webs so that once again promotion corresponds to rotation:




(**Note**: the web rotates, but the edge-coloring does slightly more.)

Section 4

# Where does this come from, and where is it going?

# Order polynomials of posets

For $P$ a poset, its **order polynomial** is

$$\Omega_P(m) = \#\text{ weakly order-preserving maps } P \to \{0, 1, \ldots, m\}.$$

In other words, we count fillings of $P$ with numbers 0 to $m$ that weakly increase as we move up in the poset.

Basic facts:

- $\Omega_P(m)$ is a polynomial in $m$ of degree $\#P$.
- Its leading coefficient is $\frac{1}{\#P!}$ times $\#$ of linear extensions of $P$.

**Note**: $\Omega_P(m)$ is intimately connected to **polyhedral combinatorics**.

# Plane partitions & product formulas

An $a \times b$ **plane partition** is an $a \times b$ array of nonnegative integers that are weakly decreasing in rows and columns.

Let $\mathcal{PP}^m(a \times b) := \{a \times b \text{ plane partitions with entries } \leq m\}$:

$$\begin{array}{|c|c|c|c|} \hline 5 & 2 & 1 & 0 \\ \hline 5 & 1 & 0 & 0 \\ \hline \end{array} \in \mathcal{PP}^5(2 \times 4)$$

Evidently, $\Omega_{a \times b}(m) = \#\mathcal{PP}^m(a \times b)$.

**Theorem (MacMahon's formula for plane partitions in a box)**

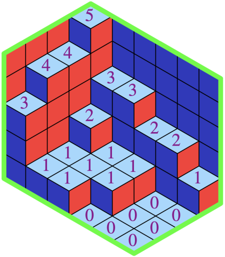
$$\Omega_{a \times b}(m) = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{m+i+j-1}{i+j-1}$$

# Visualization of plane partitions

Plane partitions have a beautiful $3D$ representation as a stacking of cubes:

| | | | | | |
|---|---|---|---|---|---|
| 5 | 3 | 3 | 2 | 2 | 1 |
| 4 | 2 | 1 | 1 | 1 | 0 |
| 4 | 1 | 1 | 1 | 0 | 0 |
| 3 | 1 | 1 | 0 | 0 | 0 |

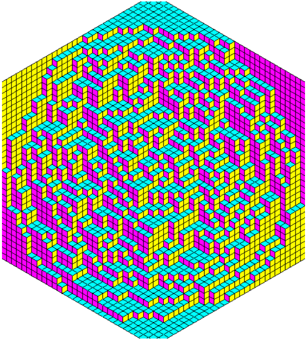
$\Leftrightarrow$




In this way they correspond to **lozenge tilings** of regions of the triangular lattice, and are a special case of the **dimer model** in statistical mechanics.

# Aside: limit shapes for plane partitions

A popular topic in the past 30 years has been looking at **limit shapes** of plane partitions:



**Product formulas**, like MacMahon's formula, are the starting point of any analysis of limit shapes.

# Main heuristic: order polynomials and dynamics

Over the past couple years I've had success developing and applying the following heuristic for finding special posets:

**posets with good dynamical properties**
**= posets with order polynomial product formulas**

Here **"good dynamical properties"** includes good behavior of promotion.

For example, it was shown by Kreweras and Niederhausen that $\Omega_{V(n)}(m)$ has a product formula, which is why I looked at promotion of $V(n)$.

# Applications of heuristic: more product formulas

I've also successfully applied the heuristic "in the other direction," i.e., to find new posets with order polynomial product formulas.

It was already known that staircases and shifted trapezoids have order polynomial product formulas. With Tri Lai, we showed:

**Theorem (Hopkins–Lai 2020)**

$$\Omega_P(m) = \prod_{1 \leq i \leq j \leq n} \frac{m+i+j-1}{i+j-1} \prod_{1 \leq i \leq j \leq k} \frac{m+i+j}{i+j}$$

*for $P$ a shifted double staircase shape $(n, n-1, \ldots, 1) + (k, k-1, \ldots, 1)$.*

We proved this theorem by using recursive techniques from the theory of lozenge tilings. Subsequently, Okada gave a different, algebraic proof.

# Future directions

I hope to explore my heuristic for finding special posets more in the future. There are many problems remaining, including:

- studying other dynamical actions on other objects associated to the posets (e.g. **rowmotion** acting on plane partitions);
- proving CSPs for the cyclic actions with good behavior;
- finding more examples of posets satisfying either side of the heuristic;
- finding formal relations between the two sides of the heuristic;
- finding algebraic "explanations" for the remarkable facts about these special posets.

# Thank you!

a survey paper with many
more details about my heuristic
and the conjectures it produces
is available at `arXiv:2006.01568`