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Modules over a ring §4.1

We now begin the last chapter of the semester, on modules. When we studied groups, we saw that looking at their actions on sets was very useful. A module is something that a ring acts on; but it is more than just a set: it's an abelian group.

Def'n Let R be a ring (possibly noncommutative, but with 1). A (left) R-module is an abelian group A together with a map $R \times A \to A$ (we denote $(r,a) \mapsto ra$) such that

- $r(a+b) = ra + rb \quad \forall r \in R,\ a,b \in A$
- $(r+s)a = ra + sa \quad \forall r,s \in R,\ a \in A$
- $r(sa) = (rs)a \quad \forall r,s \in R,\ a \in A$
- $1 \cdot a = a \quad \forall a \in A.$

Def'n If A and B are R-modules, a (R-module) homomorphism is a map $\varphi: A \to B$ such that $\varphi(x+y) = \varphi(x) + \varphi(y) \ \forall x,y \in A$ and $\varphi(rx) = r\varphi(x) \ \forall x \in A,\ r \in R$.

E.g. If $R = \mathbb{Z}$, then an R-module is the same thing as an abelian group: indeed $\mathbb{Z}$ acts on any abelian group G by $n \cdot g = \underbrace{g + g + \cdots + g}_{n \text{ times}}$ for $g \in G$ and $n \in \mathbb{Z}$ (where $(-1)\cdot g = g^{-1}$, etc.). And a $\mathbb{Z}$-module homo. $A \to B$ is the same as a group homo.

So modules generalize abelian groups. They also generalize vector spaces:

E.g. If $R = K$ is a field, then an R-module is the same thing as a vector space V over K, and a R-module homo. $V \to W$ is the same as a linear transformation.

So the study of modules is like a version of linear algebra for rings (but we have to be careful since linear independence does not hold.)

E.g. If $R = M_n(K)$, matrix algebra over a field K, then one R-module is K^n, where Mv for $M \in M_n(K)$ and $v \in K^n$ is given by usual matrix multiplication, viewing v as a column vector.

E.g. Consider $R = K[G]$, the group algebra of a group G over a field K. Then an R-module is the same thing as a vector space V over K together with a homomorphism $\varphi: G \to GL(V)$, where $GL(V)$ is the general linear group of V, the ~~set~~ group of all invertible linear transformations $V \to V$.

This is also called a representation of group G over field K, and the study of group representations is a huge subject!

We see that modules over noncommutative rings are very interesting, but we will mostly consider commutative rings from now on.

E.g. If R is a commutative ring and $I \subseteq R$ is an ideal, then I is an R-module (w/ the natural multiplication by elts of R) but also R/I is an R-module. In commutative algebra, quotients by ideals are a major source of modules.

E.g. Let's do a particular example. Let $R = \mathbb{C}[x]$ be the poly. ring. And let $I = \langle x^2 + 2x - 1 \rangle \subseteq R$ and $M = R/I$, as an R-module. Note that $M = \{a \cdot 1 + bx : a, b \in \mathbb{C}\} \simeq \mathbb{C}^2$ as an abelian gp., but we have also the action of R on M to understand. Of course $1 \cdot m = m$ for all $m \in M$, but what about $x \in R$? Note that $x \cdot 1 = x$, while

$$x \cdot x = x^2 = -2x + 1 \in M \quad (\text{since } x^2 + 2x - 1 = 0)$$

From this we can deduce the action of any $f \in \mathbb{C}[x]$ on M.

Just like in linear algebra, where even more important than vector spaces are linear transformations (a.k.a. matrices), we care about module homomorphisms.

Def'n Let $\varphi: A \to B$ be an R-module homomorphism. We define its image $im(\varphi) = \{\varphi(a) : a \in A\} \subseteq B$ and kernel $ker(\varphi) = \{a \in A : \varphi(a) = 0\} \subseteq A$ as usual, and we say φ is an epimorphism if it's surjective ($im(\varphi) = B$) and a monomorphism if it's injective ($ker(\varphi) = 0$), isomorphism if both.

Def'n Let $A \xrightarrow{\varphi_1} B \xrightarrow{\varphi_2} C$ be a sequence of R-module homomorphisms. We say this sequence is exact if $im(\varphi_1) = ker(\varphi_2)$. Similarly if $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} A_4 \cdots$ is a sequence of R-mod. hom's we say it is exact if $im(\varphi_i) = ker(\varphi_{i+1})$ for all i.

Exact sequences are extremely important in the study of modules, but it can be a bit hard to understand their significance at first...

Def'n A short exact sequence is a sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ that is exact, where 0 is the trivial R-module (trivial group). What does this mean? Well since $ker(\alpha) = im(0 \to A) = 0$, we must have that α is a monomorphism, and since $im(\beta) = ker(C \to 0) = C$, must have that β is an epimorphism. Together with $im(\alpha) = ker(\beta)$, this is all we need.

Def'n Let A and B be two R-modules. The direct sum $A \oplus B$ is the direct sum as an abelian group, with $r \cdot (a, b) = (ra, rb)$ for all $r \in R$, $(a,b) \in A \oplus B$.

E.g. Given two R-modules A and B, there is a SES

$$0 \to A \xrightarrow{i} A \oplus B \xrightarrow{\pi} B \to 0$$

where $A \xrightarrow{i} A \oplus B$ is the canonical inclusion, and $A \oplus B \xrightarrow{\pi} B$ is the canonical projection. Are all SES like this? ...

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Def'n We say that two SES: $0 \to A \to B \to C \to 0$, $0 \to A' \to B' \to C' \to 0$ are isomorphic if there are iso's $f: A \to A'$, $g: B \to B'$, $h: C \to C'$ s.t.

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \to & A' & \to & B' & \to & C' & \to & 0 \end{array}$$

making the diagram commute (going two ways around square gives the same map).

Rmk: "Homological algebra" studies commutative diagrams ("diagram chasing").

Def'n A SES $0 \to A \to B \to C \to 0$ is split if it is isomorphic to one of the form $0 \to X \xrightarrow{i} X \oplus Y \xrightarrow{\pi} Y \to 0$

Thm If $R = K$ is a field, then any SES of vector spaces $0 \to A \to B \to C \to 0$ is split.

We will discuss the proof of this thm later, but it amounts to the fact that any set of linearly independent vectors extends to a basis.

So is every SES split? No!

E.g. Let $R = \mathbb{Z}$, so that R-modules are just abelian groups. Let $n \geq 1$. Consider the sequence $0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$.

Here $\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}$ is the "multiplication by n" map $a \mapsto n \cdot a$. This is injective, so $0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}$ is exact.

And $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is the quotient map $a \mapsto a \bmod n$, which is surjective, so $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ is exact.

Finally, notice that $\operatorname{im}(\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}) = n\mathbb{Z} = \ker(\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z})$, so we indeed have a short exact sequence of abelian groups.

But it is not split!: $\mathbb{Z}$ is not isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ because it has no torsion elements!

Free Modules and Vector Spaces §4.2

Def'n For M an R-module, a submodule $N \subseteq M$ is a subset that is a sub-abelian group and is closed under the action of R: i.e., $r \cdot n \in N$ for all $n \in N$, $r \in R$.

Given a subset $X \subseteq M$, the submodule generated by X, $\langle X \rangle$, is the smallest submodule containing X; concretely

$$\langle X \rangle = \{ r_1 a_1 + r_2 a_2 + \cdots + r_n a_n : a_1, \ldots, a_n \in X, r_1, \ldots, r_n \in R \}.$$

We say M is finitely generated if $M = \langle X \rangle$ for a finite $X \subseteq M$, and say M is cyclic if it is generated by a single element, i.e. $M = \langle x \rangle$ for some $x \in M$.

If $\langle X \rangle = M$ for some $X \subseteq M$, then we say the subset X spans M (like in linear algebra).

Def'n A subset $X \subseteq M$ is linearly independent if whenever

$$r_1 a_1 + r_2 a_2 + \cdots + r_n a_n = 0 \text{ for } a_1, \ldots, a_n \in X, r_1, \ldots, r_n \in R$$

then we must have $r_i = 0$ for all i. (Just like linear algebra!)

We say X is a basis of M if it spans M and is linearly independent. We say the R-module M is free if it has a basis.

E.g. For any ring R, R is naturally a (left) R-module, and in fact it is a free R-module since $1 \in R$ is a basis.

More generally $R^n = R \oplus R \oplus \cdots \oplus R$ is a free R-module with basis $\{(1,0,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,0,0,\ldots,0,1)\}$.

E.g. Let $R = \mathbb{Z}/6\mathbb{Z}$. Then $\mathbb{Z}/3\mathbb{Z}$ is naturally an R-module (viewing $\mathbb{Z}/3\mathbb{Z} = (\mathbb{Z}/6\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})$), but it is not a free R-module because $\pm 1 \in \mathbb{Z}/3\mathbb{Z}$ would need to be in a basis, but $3 \cdot (\pm 1) = 0 \in \mathbb{Z}/3\mathbb{Z}$, so it is not linearly independent.

Thm For any ring R (with 1), the following are equivalent for M an R-mod.:

1) M is a free R-module
2) M is isomorphic to $\bigoplus_{i \in I} R$, direct sum of copies of R indexed by some (possibly infinite) set I.

Moreover, if M is a finitely generated free R-module, then $M \simeq R^n$ for some $n \geq 1$. Pf: Skipped, see book. $\boxtimes$

Free R-modules behave like vector spaces over a field. Now we will recall some facts from linear algebra about v.s.'s.

Thm If K is a field, then every K-module is free, since it is a vector space and every vector space has a basis.

Thm Let V be a vector space over a field K.
Then: • any linearly independent subset of V can be extended to a maximal linearly independent subset, which spans V, i.e., is a basis.
Moreover, all bases of V have same cardinality.

Rmk: All of this remains true for a skew field K like the quaternions $\mathbb{H}$: see the book.

Def'n The dimension $\dim_K(V)$ of a vector space V over a field K is the cardinality of any K-basis of V.
If $\dim_K(V) < \infty$ we say V is finite dimensional, and in this case we will have $V \simeq K^{\dim_K(V)}$

E.g. For $K = \mathbb{Z}/p\mathbb{Z}$ (p prime) a finite field with p elements, and V a finite dimensional vector space over K with $\dim_K(V) = n$, we have $(\mathbb{Z}/p\mathbb{Z})^n \simeq V$, so in particular $|V| = |(\mathbb{Z}/p\mathbb{Z})|^n = p^n$.

We would like to define an analog of dimension, which we will call the <u>rank</u>, for any free R-module M for any ring R.

E.g. For $R = \mathbb{Z}$, we know every finitely generated free abelian group (i.e. free $\mathbb{Z}$-module) is isomorphic to $\mathbb{Z}^n$, where n is the <u>rank</u> we are talking about.

However, it is a bizarre fact that there are some <u>noncommutative</u> rings R which have $R \cong R \oplus R$ as R-modules, meaning there cannot be a coherent notion of rank for free modules over such R (See Exercise 13 in § 4.2 of book - example is complicated.)

Nevertheless, this cannot happen for <u>commutative</u> R:

<u>Thm</u> Let R be a commutative ring, and let M be a free R-module. Then every basis of M has the same cardinality, which we call the <u>rank</u> of M.

<u>Pf sketch</u>: The idea is to view M as a vector space over some field and then use its dimension over that field as the rank over R. More precisely, choose any maximal ideal I of R. Then we know $K = R/I$ is a field. And also,

$M \otimes_R K$ is a K-module, i.e., a vector space over K, where $\otimes_R$ denotes <u>tensor product</u> of R-modules, a concept we will learn about soon. Any R-basis of M becomes a K-basis of $M \otimes_R K$, so indeed the <u>rank</u> of M is well defined as $\dim_K (M \otimes_R K)$. ∎

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Hom and duality §4.4

Def'n For R a ring, and A and B R-modules, we use $Hom_R(A,B)$ to denote the set of R-mod. homo's $\varphi: A \to B$.

Note that $Hom_R(A,B)$ has the structure of an abelian group, where $(\varphi_1+\varphi_2)(a) = \varphi_1(a)+\varphi_2(a)$ for all $\varphi_1, \varphi_2 \in Hom_R(A,B)$.

E.g. Let's compute $Hom_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/6\mathbb{Z})$. $1 \in \mathbb{Z}/3\mathbb{Z}$ is a generator, so any $\varphi: \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ is determined by $\varphi(1)$.
And where can we send 1? We can send it to any $x \in \mathbb{Z}/6\mathbb{Z}$ satisfying $3x = 0$ (since $3 \cdot 1 = 0 \in \mathbb{Z}/3\mathbb{Z}$).
So $Hom_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}) \simeq \{x \in \mathbb{Z}/6\mathbb{Z} : 3x = 0\}$
$= \{0,2,4\} \simeq \mathbb{Z}/3\mathbb{Z}$.

We want to view $Hom_R(A,B)$ as not just an abelian group, but as an R-module itself. However, we will have to restrict to commutative R for this to work...

Def'n Let R and S be two rings. An (R,S)-module A is an abelian group that is simultaneously a left R-module and a right S-module, s.t. those actions of R and S commute in sense that $(ra)s = r(as)$ $\forall r \in R, s \in S, a \in A$.

E.g. If R is a commutative ring, then any R-module A is an (R,R)-module if we set $a \cdot r = ra$ for all $r \in R, a \in A$.

Prop: Let R be a ring and A, B R-modules. Suppose that A is an (R,S)-module for some ring S. Then $Hom_R(A,B)$ is a left S-module by $s \cdot \varphi(a) = \varphi(as)$ $\forall s \in S, \varphi \in Hom_R(A,B)$.
Similarly, if B is an (R,S)-module then $Hom_R(A,B)$ is a left S-module by $s \cdot \varphi(a) = \varphi(a)s$ $\forall s \in S, \varphi \in Hom_R(A,B)$.

E.g. Let $R = M_2(\mathbb{C})$. Then $M = \mathbb{C}^2$ is an R-module as we saw and $\mathrm{Hom}_R(\mathbb{C}^2, \mathbb{C}^2) = \{$linear maps $f: \mathbb{C}^2 \to \mathbb{C}^2$: f commutes w/ all matrices in $M_2(\mathbb{C})\}$ $=$ Center of $M_2(\mathbb{C}) = \left\{\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix} : \lambda \in \mathbb{C}\right\}$. There is no natural action of $M_2(\mathbb{C})$ on this set of diagonal matrices, but since $\mathbb{C}^2$ has a right action of $\mathbb{C}$ commuting w/ left action of $M_2(\mathbb{C})$, $\mathrm{Hom}_R(\mathbb{C}^2, \mathbb{C}^2)$ is at least a $\mathbb{C}$-vector space.

Cor If R is a commutative ring, then $\mathrm{Hom}_R(A, B)$ is naturally an R-module for any R-modules A and B. We have $r \cdot \varphi(a) = r\varphi(a) = \varphi(ra)$ $\forall\, r \in R$, $\varphi \in \mathrm{Hom}_R(A, B)$.

E.g. If $R = \mathbb{Z}$, then for any abelian groups A, B, $\mathrm{Hom}_{\mathbb{Z}}(A, B)$ is an abelian group, a.k.a. $\mathbb{Z}$-module.

E.g. If $R = k$ is a field and V and W are two k-vector spaces then $\mathrm{Hom}_k(V, W) = \{k\text{-linear maps } f: V \to W\}$ is a k-vector space. If $V \simeq k^n$ and $W \simeq k^m$ then $\mathrm{Hom}_k(V, W) \simeq \{n \times m$ matrices with ~~coeffs~~ entries in $k\}$, so $\dim_k(\mathrm{Hom}_k(V, W)) = n \cdot m = \dim_k(V) \cdot \dim_k(W)$.

E.g. If R is any commutative ring, then $\mathrm{Hom}_R(R^n, R^m)$ can be viewed as set of $n \times m$ matrices w/ ~~coeffs~~ entries in R.
We'll discuss this more (especially when R is a PID) later.

Prop. Let R be a commutative ring. Then for any R-mod. A, have canonical isomorphism $\mathrm{Hom}_R(R, A) \simeq A$.
Pf: The isomorphism is given by $\varphi \mapsto \varphi(1)$ for $\varphi \in \mathrm{Hom}_R(R, A)$. This works since 1 generates R as an R-module. $\blacksquare$

So $\mathrm{Hom}_R(R,A) = A$. What about other direction, i.e., $\mathrm{Hom}_R(A,R)$?

Def'n For R a com. ring and A an R-mod., its dual module is
$A^* = \mathrm{Hom}_R(A,R)$.

E.g. If $R=K$ is a field, and V is a K-vector space, then $V^* = \{$linear functions $f: V \to K\}$ is the dual space, also often called the space of linear functionals on V. You might know that if V is finite dimensional then $\dim_K(V) = \dim_K(V^*)$. However, there is no canonical isomorphism $V \to V^*$. But...

Thm For any R-mod. A, there is a canonical map $A \to A^{**}$ to the double dual given by $a \mapsto (f \mapsto f(a))$ for $a \in A$, $f \in A^*$.

Def'n A module A is reflexive if the canonical homo. $A \to A^{**}$ is an isomorphism.

E.g. For any field K and finite dimensional vector space V over K, V is reflexive, i.e. canonically isomorphic to V^{**}.

E.g. On next HW you will show $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$, hence the dual of $\mathbb{Z}/n\mathbb{Z}$, and also the double dual, as a $\mathbb{Z}$-mod. $=0$.
Notice how these non-reflexive come from torsion in the module (where we recall torsion element means an $m \in M$ with $rm = 0$ for some non zero divisor $r \in R$).

One last thing about duality is how it interacts with module homo's:

Thm Let $A \xrightarrow{f} B$ be two R-mod. with a homomorphism between them. Then we have a hom. $B^* \xrightarrow{f^*} A^*$ given by $(f^*\varphi)(a) = \varphi(f(a))$ for all $\varphi \in B^* = \mathrm{Hom}_R(B,R)$.

Rmk: This means duality is a "contravariant functor", i.e., it reverses direction of arrows in category of R-modules. //

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Tensor Product of modules §4.5

We now discuss an operation on R-modules called tensor product that produces a new R-mod. $A \otimes B$ from two R-mod.'s A and B. It is related to "multilinear algebra" and also intimately related to the Hom construction we discussed last class.

For convenience today we assume R is a commutative ring, although this mostly all works the same for noncom. R.

Def'n Let A and B be two R-mod.'s. Let F be the free abelian group on set $A \times B$. So the elements of F are (finite) formal sums of form $\sum_i n_i (a_i, b_i)$ with $a_i \in A$, $b_i \in B$, $n_i \in \mathbb{Z}$.
Let S be the subgroup of F generated by elements:

$$(a+a', b) - (a,b) - (a',b) \quad \forall a,a' \in A, b \in B,$$
$$(a, b+b') - (a,b) - (a,b') \quad \forall a \in A, b, b' \in B$$
$$(ra, b) - (a, rb) \quad \forall a \in A, b \in B, r \in R.$$

The quotient F/S is called the tensor product of A and B, and is denoted $A \otimes_R B$. The image of (a,b) in $A \otimes_R B$ is denoted $a \otimes b$ and is called a pure tensor.

Note: Not every element of $A \otimes_R B$ is a pure tensor. In general an element of $A \otimes_R B$ is a (formal) sum of pure tensors: $\sum_i n_i \, a_i \otimes b_i$ $n_i \in \mathbb{Z}$, $a_i \in A$, $b_i \in B$.

Remark: The pure tensors in $A \otimes_R B$ satisfy these relations:

$$(a+a') \otimes b = a \otimes b + a' \otimes b$$
$$a \otimes (b+b') = a \otimes b + a \otimes b'$$
$$ra \otimes b = a \otimes rb \qquad \forall r \in R$$

This is the sense in which the tensor product is "multilinear," i.e., linear in both components.

Prop. $A \otimes_R B$ has the structure of an R-mod. where
$r\left(\sum_i n_i a_i \otimes b_i\right) = \sum_i n_i \, ra_i \otimes b_i = \sum_i n_i \, a_i \otimes rb_i$

Prop. The $\otimes$ operation is associative and commutative in sense that $(A \otimes_R B) \otimes_R C \cong A \otimes_R (B \otimes_R C)$
and $A \otimes_R B \cong B \otimes_R A$.

Prop. We have $A \otimes_R R \cong A \cong R \otimes_R A$ for any R-mod A.

Pf: All of these propositions are relatively straightforward. Let's prove the last one about $A \otimes_R R \cong A$. First note that $a \otimes r = ra \otimes 1$ for any pure tensor, hence every pure tensor is of form $a \otimes 1$ for $a \in A$. Then any element of $A \otimes_R R$ is of form $\sum_i n_i a_i \otimes 1$ but this is $= (\sum_i n_i a_i) \otimes 1 = a' \otimes 1$ for some $a' \in A$. So $A \otimes_R R = \{a \otimes 1 : a \in A\} \cong A$, as claimed. ☒

E.g.: Let's do an example of tensor products for $R = \mathbb{Z}/2\mathbb{Z}$. Let $A = R^2 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} = B$. What does $(\mathbb{Z}/2\mathbb{Z})^2 \otimes_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})^2$ look like? Let's consider some elements:

$$(0,0) \otimes (1,1) = 0 \cdot (0,0) \otimes (1,1) = (0,0) \otimes 0 \cdot (1,1) = (0,0) \otimes (0,0)$$

So in fact $(0,0) \otimes a = 0$ for any $a \in A$. Also we have

$$(1,1) \otimes (1,0) = ((1,0) + (0,1)) \otimes (1,0) = (1,0) \otimes (1,0) + (0,1) \otimes (1,0).$$

In fact we can see that a basis over $\mathbb{Z}/2\mathbb{Z}$ of $A \otimes_R A$ is $\{(1,0) \otimes (1,0), (1,0) \otimes (0,1), (0,1) \otimes (1,0), (0,1) \otimes (0,1)\}$.

Note that $\dim_R(A \otimes_R A) = 4 = 2 \cdot 2 = \dim_R(A) \cdot \dim_R(A)$. Also note that $(1,0) \otimes (1,0) + (0,1) \otimes (0,1)$ is an element in $A \otimes_R A$ which is <u>not</u> a pure tensor. //

Thm Let $R=K$ be a field and V and W two K-vector spaces. Suppose $\{e_i : i\in I\}$ and $\{f_j : j\in J\}$ are bases of V and W, then $\{e_i\otimes f_j : i\in I, j\in J\}$ is a basis of $V\otimes_K W$. In particular if V and W are finite dimensional with $n=\dim_K(V)$ and $m=\dim_K(W)$ then $\dim_K(V\otimes_K W)=n\cdot m=\dim_K(V)\cdot\dim_K(W)$.

Pf: Exercise for you, similar to the example we saw. ☒

However when R is not a field, $\otimes_R$ behaves differently, especially if there are torsion elements in the modules.

E.g. Let's consider $R=\mathbb{Z}$, so R-mod.'s are just abelian gp.s. In particular let's consider $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Q}$ is the (additive) group of the rational numbers. Notice that for any pure tensor $x\otimes b$ for $x\in\mathbb{Q}$, $b\in\mathbb{Z}/2\mathbb{Z}$ we have $x\otimes b = 2(\frac{x}{2})\otimes b = (\frac{x}{2})\otimes 2b = \frac{x}{2}\otimes 0 = 0$ since $2b\in\mathbb{Z}/2\mathbb{Z}=0$ and since $\frac{x}{2}$ exists for any $x\in\mathbb{Q}$. Since any pure tensor $=0$, $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}=0$. Exercise: what is different with $\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}$?

(On HW#6 you will take this example further...)

Thm Let A, B, C be R-mod's and $A\xrightarrow{f}B$ a $\overset{R\text{-mod.}}{\text{homo.}}$ Then $\exists$ a $\overset{R\text{-mod.}}{\text{homo.}}$ $A\otimes_R C\xrightarrow{f\otimes 1_C}B\otimes_R C$ where
$f\otimes 1_C\left(\sum_i n_i\, a_i\otimes c_i\right)=\sum_i n_i\, f(a_i)\otimes c_i$.

Rmk: Since $-\otimes_R C$ preserves the direction of the arrows we say it is a "covariant functor" on category of R-mod's

Finally, let's discuss the relationship between tensor and hom.

<u>Theorem</u> (Tensor-Hom Adjunction) For A, B, C R-mod's, have $\mathrm{Hom}_R(A \otimes_R B, C) \simeq \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C))$.

This says $\otimes_R$ and $\mathrm{Hom}_R(-,-)$ are "opposite" or "dual" in a certain sense... Let's focus on one special case.

<u>Cor</u> $(A \otimes_R B)^* \simeq \mathrm{Hom}_R(A, B^*)$

<u>Pf</u>: Take $C = R$ in the tensor-hom adjunction. ☒

<u>Cor</u> Suppose B is reflexive, i.e., $B^{**} = B$.
Then $\mathrm{Hom}_R(A, B) \simeq (A \otimes_R B^*)^*$.

<u>Rmk</u>: Recall that every finite-dimensional v.s. V over a field K is reflexive. Hence $\mathrm{Hom}_K(V, W) \simeq (V \otimes_K W^*)^* \simeq V^* \otimes_K W$ for any two fin.-dim'l v.s.'s. This shows that we can build up all hom spaces between fin.-dim'l vector spaces just using $\otimes$ and duality.

<u>Pf sketch</u> for <u>tensor-hom</u> <u>adjunction</u>:
See the book for details, but key point is that
$$\mathrm{Hom}_R(A \otimes_R B, C) \simeq \mathrm{Bil}_R(A \times B, C)$$
where $\mathrm{Bil}_R(A \times B, C)$ is the set of "<u>bilinear</u> <u>maps</u>" $A \times B \to C$. Then $\mathrm{Bil}_R(A \times B, C) \simeq \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C))$ via the map $f \mapsto (a \mapsto (b \mapsto f(a, b)))$ ☒