# Structure constants: complexity and asymptotics

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$S_n$ – irreducible representations: the Specht modules $\mathbb{S}_\lambda$ for $\lambda \vdash n$.

$GL_N$–irreducible [polynomial] representations: the Weyl modules $V_\alpha$ for $\ell(\lambda) \leq N$.

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**Plethysm:** Compositions of $GL$-representations.

$$S^d(S^n V) = \oplus_{\lambda \vdash dn} V_\lambda^{a_\lambda(d[n])}$$

# Via Schur functions and $S_n$ characters

$$s_\lambda(x,y) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(x) s_\nu(y) \quad \Longleftrightarrow \quad s_{\lambda/\mu}(x) = \sum_\nu c^\lambda_{\mu,\nu} s_\nu(x)$$

$$s_\lambda[x.y] = \sum_{\mu,\nu} g(\lambda,\mu,\nu) s_\mu(x) s_\nu(y) \quad \Longleftrightarrow \quad \sum_{\lambda,\mu,\nu} g(\lambda,\mu,\nu) s_\lambda(x) s_\mu(y) s_\nu(z) = \prod_{i,j,k} \frac{1}{1 - x_i y_j z_k}$$

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| | | | | | |
|---|---|---|---|---|---|
| | | | 1 | 1 | 1 |
| | 1 | 2 | 2 | | |
| 2 | 3 | 3 | | | |

| | | | | | |
|---|---|---|---|---|---|
| | | | 1 | 1 | 1 |
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( $c^{(6,4,3)}_{(3,1)(4,3,2)} = 2$)

# Combinatorics I

## Problem (Murnaghan 1938, Stanley)

*Find a positive combinatorial interpretation for $g(\lambda, \mu, \nu)$, i.e. a family of combinatorial objects $\mathcal{O}_{\lambda,\mu,\nu}$, s.t. $g(\lambda, \mu, \nu) = \#\mathcal{O}_{\lambda,\mu,\nu}$.*

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**Theorem [Murnaghan]** If $|\lambda| + |\mu| = |\nu|$ and $n > |\nu|$, then

$$g((n + |\mu|, \lambda), (n + |\lambda|, \mu), (n, \nu)) = c^{\nu}_{\lambda\mu}.$$

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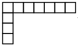
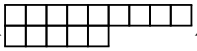
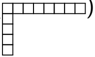
Combinatorial formulas for $g(\lambda, \mu, \nu)$, when:

- $\mu$ and $\nu$ are hooks ( ), [*Remmel, 1989*]
- $\nu = (n - k, k)$ () and $\lambda_1 \geq 2k - 1$, [*Ballantine–Orellana, 2006*]
- $\nu = (n - k, k)$, $\lambda = (n - r, r)$ [*Remmel–Whitehead*, 1994; *Blasiak–Mulmuley–Sohoni*,2013]
- $\nu = (n - k, 1^k)$ (), [*Blasiak 2012, Blasiak-Liu 2014*]
- Other special cases [Colmenarejo-Rosas, Ikenmeyer-Mulmuley-Walter, Pak-Panova, Mishna-Rosas-Sundaram].

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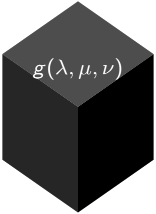
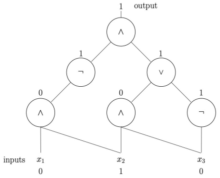


# Computational Complexity in a Nutshell

`Input:` $I$, $\text{size}(I) = n$ (bits)
`Problem:` $C(I)$



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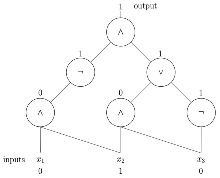


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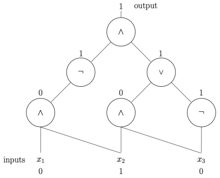


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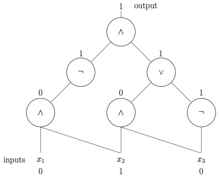


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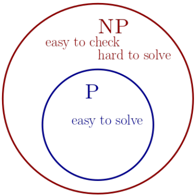


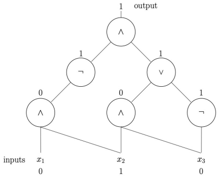


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**Millennium Problem: Is P = NP ?**

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## Combinatorial interpretations vs Computational Complexity

**Littlewood-Richardson**:

$\mathrm{LR}$: Input: $\lambda, \mu, \nu$ Output: $c^{\lambda}_{\mu\nu}$

$\mathrm{LR}\textsc{pos}$: Input: $\lambda, \mu, \nu$ Output: Is $c^{\lambda}_{\mu\nu} > 0$?

LR rule $\Longrightarrow \mathrm{LR} \in \#\mathsf{P}$

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**Characters**:

$\textsc{Char}$: Input: $n, \lambda, \alpha \vdash n$ (unary) Output: Is $\chi^{\lambda}[\alpha] \neq 0$?

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**Kronecker**:

$\mathrm{K{\small RON}}$: Input: $\lambda, \mu, \nu$ Output: $g(\lambda, \mu, \nu)$

$\mathrm{K{\small RON}P{\small OS}}$: Input: $\lambda, \mu, \nu$ Output: Is $g(\lambda, \mu, \nu) > 0$?

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$\textsc{KronPos}$: Input: $\lambda, \mu, \nu$ Output: Is $g(\lambda, \mu, \nu) > 0$?

[Bürgisser-Ikenmeyer, Pak-P]: $\textsc{Kron} \in \mathtt{GapP}$.

[Ikenmeyer-Mulmuley-Walter]: $\textsc{KronPos}$ is [strongly] NP-hard.

**Question**[Pak-P]: is $\textsc{Kron} \in \#\mathrm{P}$?

# Positivity

## Conjecture (Tensor square, Saxl'12)

*For every $n \geq 9$ there is an irreducible $S_n$ representations, $\mathbb{S}_\lambda$, such that $\mathbb{S}_\lambda \otimes \mathbb{S}_\lambda$ contains every irreducible representation. I.e. $g(\lambda, \lambda, \mu) > 0$ for every $\mu \vdash n$. Saxl conjecture: for $n = \binom{k}{2}$ such partition is $\lambda = \delta_k = (k-1, \ldots, 1)$* 

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**Partial results:**

[Pak-P-Vallejo'13]: for $\mu$ - 2-row, hook, hook + boxes etc

$$\text{[PPV'13],[PP'16]} \qquad g(\lambda, \lambda, \mu) \geq |\chi^\mu(2\lambda_1 - 1, 2\lambda_2 - 3, \dots)| \qquad \text{for } \lambda = \lambda'$$

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**Other positivity results:**

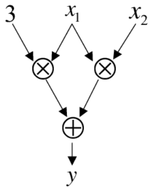
[Ikenmeyer-P,'16]:
$g((N - ab, a^b), (N - ab, a^b), (N - |\gamma|, \gamma)) > 0$ for large $N$ and almost all $\gamma, a, b$ (with some restrictions), related to Geometric Complexity Theory.

$\rightarrow$ NEXT

# Algebraic P vs NP: VP vs VNP

**Arithmetic Circuits:**

$y = 3x_1 + x_1 x_2$




**Input:** $X_1, \ldots, X_n$ and constants from $\mathbb{F}$.

**Circuit:** nodes are $+, -, \times, \div$ gates.

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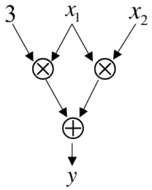


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polynomials that can be computed with circuits with $poly(n)$ nodes

**Class** VNP (Valliant's NP ):
polynomials $f_n$, s.t. $\exists g_n \in$ VP with

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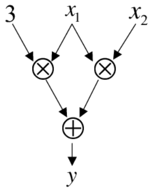


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**Theorem**[Bürgisser]:
If VP = VNP over finite $\mathbb{F}$ or Generalized Riemann Hypothesis holds, then P = NP.

# Geometric Complexity Theory in a Nutshell

VP vs VNP : permanent vs determinant

$$\mathsf{det}_n := \sum_{\sigma \in S_n} \mathsf{sgn}(\sigma) \prod_{i=1}^{n} x_{i,\sigma(i)} \qquad \mathrm{per}_m := \sum_{\sigma \in S_m} \prod_{i=1}^{m} x_{i,\sigma(i)}$$

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The (normalized) permanent $x_{11}^{n-m}\mathrm{per}_m \neq \det_n[A\mathbf{x}^T]$ ($n \times n$ determinant of affine linear forms in $\{x_{ij}\}_{i,j=1}^{m}$) for $n = poly(m)$. (and thus VP $\neq$ VNP)

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$$x_{11}^{n-m}\mathrm{per}_m = \det_n[A\mathbf{x}^T] \Longrightarrow \overline{GL_{n^2} x_{11}^{n-m}\mathrm{per}_m} \subset \overline{GL_{n^2}\det_n}$$

GCT program (Mulmuley and Sohoni): If $\mathbb{C}[\overline{GL_{n^2}\mathrm{per}_m^n}]_d \subset \mathbb{C}[\overline{GL_{n^2}\det_n}]_d$, show that $n > poly(m)$.

# Geometric Complexity Theory in a Nutshell

VP vs VNP : permanent vs determinant

$$\det_n := \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} x_{i,\sigma(i)} \qquad \mathrm{per}_m := \sum_{\sigma \in S_m} \prod_{i=1}^{m} x_{i,\sigma(i)}$$

**Conjecture [Valiant'78]**:
The (normalized) permanent $x_{11}^{n-m}\mathrm{per}_m \neq \det_n[A\mathbf{x}^T]$ ($n \times n$ determinant of affine linear forms in $\{x_{ij}\}_{i,j=1}^{m}$) for $n = poly(m)$. (and thus VP $\neq$ VNP)

$$x_{11}^{n-m}\mathrm{per}_m = \det_n[A\mathbf{x}^T] \Longrightarrow \overline{GL_{n^2} x_{11}^{n-m}\mathrm{per}_m} \subset \overline{GL_{n^2}\det_n}$$

GCT program (Mulmuley and Sohoni): If $\mathbb{C}[\overline{GL_{n^2}\mathrm{per}_m^n}]_d \subset \mathbb{C}[\overline{GL_{n^2}\det_n}]_d$, show that $n > poly(m)$.

$$\mathbb{C}[\overline{GL_{n^2}\det_n}]_d \simeq \bigoplus_{\lambda \vdash nd} V_\lambda^{\oplus \delta_{\lambda,d,n}}, \qquad \mathbb{C}[\overline{GL_{n^2}\mathrm{per}_m^n}]_d \simeq \bigoplus_{\lambda \vdash nd} V_\lambda^{\oplus \gamma_{\lambda,d,n,m}},$$

**Obstructions** $\lambda$: if $\delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m}$ for $n > poly(m) \Longrightarrow$ VP $\neq$ VNP.

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*There exist occurrence obstructions that show $n > poly(m)$.*

## Theorem (Bürgisser-Ikenmeyer-P)

*This Conjecture is false. There are no such occurrence obstructions for $n > m^{25}$.*

# Kronecker coefficients and GCT

VP vs VNP

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$$\delta_{\lambda,d,n} \leq g(\lambda, n^d, n^d) \qquad \gamma_{\lambda,d,n,m} \leq a_\lambda(d[n])$$

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$$\delta_{\lambda,d,n} \leq g(\lambda, n^d, n^d) \qquad \gamma_{\lambda,d,n,m} \leq a_\lambda(d[n])$$

## Conjecture (GCT, Mulmuley and Sohoni)

*There exist $\lambda$, s.t. $g(\lambda, n^d, n^d) = 0$ and $\gamma_{\lambda,d,n,m} > 0$ for some $n > poly(m)$.*

## Theorem (Ikenmeyer-P )

*Let $n > 3m^4$, $\lambda \vdash nd$. If $g(\lambda, n^d, n^d) = 0$(so $mult_\lambda \mathbb{C}[GL_{n^2}\mathrm{det}_n] = 0$), then $mult_\lambda(\mathbb{C}[\overline{GL_{n^2}\mathrm{per}^n_m}] = 0$.*

## Theorem (Ikenmeyer-P)

*For every partition $\rho$, let $n \geq |\rho|$, $d \geq 2$, $\lambda := (nd - |\rho|, \rho)$. Then $g(\lambda, n^d, n^d) \geq a_\lambda(d[n])$.*

# No occurrence obstructions: positive Kroneckers

## Theorem (Ikenmeyer-Panova)

*Let $n > 3m^4$, $\lambda \vdash nd$. If $g(\lambda, n \times d, n \times d) = 0$ (so $\mathrm{mult}_\lambda \mathbb{C}[\overline{GL_{n^2}\det_n}]_d = 0$), then $\mathrm{mult}_\lambda(\mathbb{C}[\overline{GL_{n^2} per_m^n}]_d = 0$.*

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**Proof ingredients:**

## Theorem (Kadish-Landsberg)

*If $\mathrm{mult}_\lambda \mathbb{C}[\overline{GL_{n^2}per_m^n}]_d > 0$, then $\lambda_1 \geq nd - md$ and $\ell(\lambda) \leq m^2$.*

## Theorem (Degree lower bound, [IP] )

*If $\lambda_1 \geq nd - md$ with $\gamma_{\lambda,d,n,m} > g(\lambda, n \times d, n \times d)$, then $d > \frac{n}{m}$.*

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### Theorem (Kronecker positivity, [IP] )

*If $\ell(\lambda) \leq m^2$, $\lambda_1 \geq nd - md$, $d > 3m^3$, and $n > 3m^4$, then $g(\lambda, n \times d, n \times d) > 0$, except for 6 special cases.*

Proof uses semigroup property, symmetries, positivity for squares.

## Multiplicity obstructions in GCT

$$\mathbb{C}[\overline{GL_{n^2}\det_n}]_d \simeq \bigoplus_{\lambda \vdash nd} V_\lambda^{\oplus \delta_{\lambda,d,n}}, \qquad \mathbb{C}[\overline{GL_{n^2}\mathrm{per}_m^n}]_d \simeq \bigoplus_{\lambda \vdash nd} V_\lambda^{\oplus \gamma_{\lambda,d,n,m}},$$

**[GCT paradigm]** : There exist multiplicity obstructions that show $n > poly(m)$, so VP $\neq$ VNP, i.e. there is some $\lambda$ and $n, m$ with $n > poly(m)$, s.t. $\delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m}$

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**Toy problem:** Factor $\ell_1^n + \cdots + \ell_k^n$ into linear forms? $(k > 2)$

$$\mathrm{Ch}^n_m := \{\ell_1 \cdots \ell_n \mid \ell_i \in V\} \qquad \text{vs} \qquad \mathrm{Ps}^n_{m,k} := \overline{\{\ell_1^n + \cdots + \ell_k^n \mid \ell_i \in V\}},$$

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### Theorem (Dörfler–Ikenmeyer-P'20)

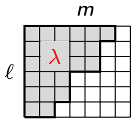
*Let $m \geq 3$, $n \geq 2$. We have* $\mathrm{mult}_\lambda(\mathbb{C}[\mathrm{Ch}_m^n]_{n+1}) < \mathrm{mult}_\lambda(\mathbb{C}[\mathrm{Ps}_{m,n+1}^n]_{n+1})$ *for* $\lambda = (n^2-2, n, 2)$, *i.e.,* $\lambda$ **is a multiplicity obstruction** *that shows* $P_{m,n+1}^n \not\subseteq \mathrm{Ch}_m^n$.
**No occurrence obstructions**, *for explicit values of* $k, n, m$.

[BIP'16] $\mathrm{mult}_\lambda(\mathbb{C}[\mathrm{Ps}_{m,k}^n]_d) = a_\lambda(d[n])$ for $k \geq d$.
[Landsberg] $\mathrm{mult}_\lambda(\mathbb{C}[\mathrm{Ch}_m^n]_d) \leq a_\lambda(n[d])$
Explicit plethysm formula: $a_{(n^2-2,n,2)}((n+1)[n]) = 1 + a_{(n^2-2,n,2)}(n[n+1])$

# Tight asymptotics

$p_n(\ell, m) := \#\{\lambda \vdash n;\ \lambda \subset (m^\ell)\}$

$$\sum_{k \geq 0} p_n(\ell, m) q^n \; = \; \begin{bmatrix} m+\ell \\ m \end{bmatrix}_q$$



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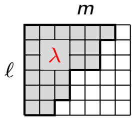


## Theorem (Pak-P'15)

*For all* $m \geq \ell \geq 8$ *and* $2 \leq k \leq \ell m/2$, *let* $s = \min\{2k, \ell^2\}$. *We have:*

$$g(m^\ell, m^\ell, (m\ell - k, k)) = p_k(\ell, m) \, - \, p_{k-1}(\ell, m) \, > \, 0.004 \, \frac{2^{\sqrt{s}}}{s^{9/4}} \, .$$

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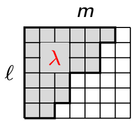


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## Theorem (Melczer-P-Pemantle'19)

*Let* $A := \frac{\ell}{m}$ $B := \frac{n-1}{m^2}$. *Let* $c, d$ *be solutions of [a system of integral equations]*

$$p_n(\ell, m) - p_{n-1}(\ell, m) \sim \frac{d}{m} p_{n-1}(\ell, m) \sim \frac{d \ e^{m\left[cA + 2dB - \log(1 - e^{-c-d})\right]}}{2\pi m^3 \sqrt{D}}.$$

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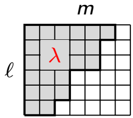


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# Maximal multiplicities

Theorem [Stanley]

$$\max_{\lambda\vdash n}\max_{\mu\vdash n}\max_{\nu\vdash n}\; g(\lambda,\mu,\nu) \;=\; \sqrt{n!}\, e^{-O(\sqrt{n})}\,,$$

$$\max_{0\le k\le n}\max_{\lambda\vdash n}\max_{\mu\vdash k}\max_{\nu\vdash n-k}\; c^{\lambda}_{\mu,\nu} \;=\; 2^{n/2-O(\sqrt{n})}.$$

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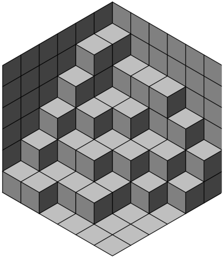
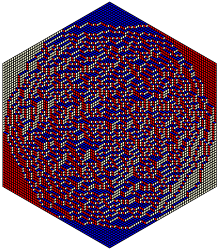
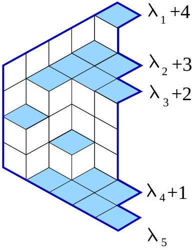
**Question:** [Stanley] For which $\lambda, \mu, \nu$ are these maxima achieved?

## Stat mech motivation: lozenge tilings








$$\lim_{n\to\infty} \frac{s_{\lambda^n}(x_1,\ldots,x_k,1^{n-k})}{s_{\lambda^n}(1^n)}$$

[Gorin-P'15] effective asymptotics giving GUE near boundary, also in [Novak, Petrov] etc, subsequently used for LLN and CLT for trapezoidal domains [Bufetov-Gorin, Aggarwal-Gorin] etc

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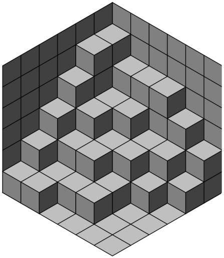

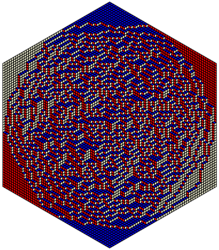

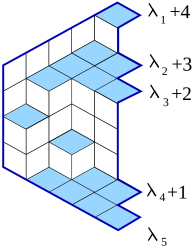


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**Question:** What about nontrapezoidal domains, can we analyze asymptotically $\dfrac{s_{\lambda/\mu}(x_1,\ldots,x_k,1^{n-k})}{s_{\lambda/\mu}(1^n)}$?

**Question:** Asymptotics of $K_{\lambda/\mu,\nu}$, $c^{\lambda}_{\mu\nu}$ etc as $\lambda,\mu,\nu$ grow..?

# Largest Kroneckers

Inequalities

$$\sum_{\lambda,\mu,\nu\vdash n} g(\lambda,\mu,\nu)^2 = \sum_{\alpha\vdash n} z_\alpha \geq z_{1^n} = n!,$$

where $z_\alpha = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$ when $\alpha = (1^{m_1} 2^{m_2} \ldots)$,

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## Theorem (Pak-Panova-Yeliussizov'18)

*Let $\{\lambda^{(n)} \vdash n\}$, $\{\mu^{(n)} \vdash n\}$, $\{\nu^{(n)} \vdash n\}$ be three partition sequences, such that*

$$(*) \qquad g\left(\lambda^{(n)}, \mu^{(n)}, \nu^{(n)}\right) = \sqrt{n!}\, e^{-O(\sqrt{n})}.$$

*Then $\lambda^{(n)}, \mu^{(n)}, \nu^{(n)}$ are Plancherel (i.e. VKLS shape). Conversely, for every two Plancherel sequences $\{\lambda^{(n)} \vdash n\}$ and $\{\mu^{(n)} \vdash n\}$, there exists a Plancherel partition sequence $\{\nu^{(n)} \vdash n\}$, s.t. (*) holds.*

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$$\mathbf{D}(n) := \max_{\lambda\vdash n} f^\lambda$$

**Theorem**[PPY]: Let $\mu, \nu \vdash n$, s.t. $f^\mu, f^\nu \geq \mathbf{D}(n)/a$ for some $a \geq 1$. Then there exist $\lambda \vdash n$, s.t.

$$f^\lambda \geq \frac{\mathbf{D}(n)}{a\sqrt{p(n)}} \quad \text{and} \quad g(\lambda,\mu,\nu) \geq \frac{\mathbf{D}(n)}{a^2\, p(n)}.$$

# Littlewood-Richardson

## Theorem (PPY'18)

*There exists a constant $d > 0$, s.t. for all $n > k \geq 1$:*

$$\sqrt{\binom{n}{k}}\, e^{-d\sqrt{n}} \leq \max_{\lambda\vdash n} \max_{\mu\vdash k} \max_{\nu\vdash n-k} c^{\lambda}_{\mu,\nu} \leq \sqrt{\binom{n}{k}}.$$

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## Theorem (PPY'18)

*Fix $0 < \theta < 1$ and let $k_n := \lfloor \theta n \rfloor$. Then:*

*1. for every Plancherel partition sequence $\{\lambda^{(n)} \vdash n\}$, there exist Plancherel partition sequences $\{\mu^{(n)} \vdash k_n\}$ and $\{\nu^{(n)} \vdash n - k_n\}$, s.t.*

$$(**) \qquad c^{\lambda^{(n)}}_{\mu^{(n)},\, \nu^{(n)}} = \binom{n}{k_n}^{1/2} e^{-O(\sqrt{n})},$$

*2. for all Plancherel partition sequences $\{\mu^{(n)} \vdash k_n\}$ and $\{\nu^{(n)} \vdash n - k_n\}$, there exists a Plancherel partition sequence $\{\lambda^{(n)} \vdash n\}$, s.t. $(**)$ holds,*
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$$f^{\nu^{(n)}} = \sqrt{n!}\, e^{-O(n^{2/3}\log n)} \quad \text{and} \quad c^{\lambda^{(n)}}_{\mu^{(n)},\, \nu^{(n)}} = \binom{n}{k_n}^{1/2} e^{-O(n^{2/3}\log n)}.$$

# Littlewood-Richardson

## Theorem (PPY'18)

*There exists a constant $d > 0$, s.t. for all $n > k \geq 1$:*

$$\sqrt{\binom{n}{k}}\, e^{-d\sqrt{n}} \leq \max_{\lambda\vdash n} \max_{\mu\vdash k} \max_{\nu\vdash n-k} c^{\lambda}_{\mu,\nu} \leq \sqrt{\binom{n}{k}}.$$

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[Belinschi-Guionnet-Huang'20+]: General upper bounds on $c^{\lambda}_{\mu\nu}$ for "nice measures" via elliptical [random matrix] integrals.

# Small number of rows

**Theorem (Pak-P'20)**

*Let $\lambda, \mu, \nu \vdash n$ such that $\ell(\lambda) = \ell$, $\ell(\mu) = m$, and $\ell(\nu) = r$. Then:*

$$g(\lambda, \mu, \nu) \le \left(1 + \frac{\ell m r}{n}\right)^n \left(1 + \frac{n}{\ell m r}\right)^{\ell m r}.$$

**Corollary:** Let $\lambda = (\ell^2)^\ell$, where $\ell = \sqrt[3]{n}$, then

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$$\sum_{\lambda,\mu,\nu} g(\lambda,\mu,\nu)\, s_\lambda(x)\, s_\mu(y)\, s_\nu(z) = \sum_{\alpha,\beta,\gamma} T(\alpha,\beta,\gamma)\, x^\alpha\, y^\beta\, z^\gamma.$$

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[Barvinok]: The number of 3d contingency tables with marginals $(\alpha, \beta, \gamma)$ is

$$\le \exp\left( \max_{Z \in P(\alpha,\beta,\gamma)} \sum_{i,j,k} (Z_{ijk} + 1) \log(Z_{ijk} + 1) - Z_{ijk} \log(Z_{ijk}) \right)$$

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# The reduced (but not simpler!) Kronecker coefficients

$$\overline{g}(\alpha,\beta,\gamma) := \lim_{n\to\infty} g\big(\alpha[n],\beta[n],\gamma[n]\big), \qquad \alpha[n] := (n-|\alpha|,\alpha_1,\alpha_2,\ldots),\ n \geq |\alpha|+\alpha_1,$$

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*The reduced Kronecker coefficients satisfy the saturation property:*

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*For all $k\geq 3$, the triple of partitions $\big(1^{k^2-1},1^{k^2-1},k^{k-1}\big)$ is a counterexample to the Conjecture. For every partition $\gamma$ s.t. $\gamma_2\geq 3$, there are infinitely many pairs $(a,b)\in\mathbb{N}^2$ s.t. $(a^b,a^b,\gamma)$ is a counterexample.*

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**Example:**
$\overline{g}(1^5, 1^5, (3,3)) = 0$, but $\overline{g}(2^5, 2^5, (6,6)) > 0$.

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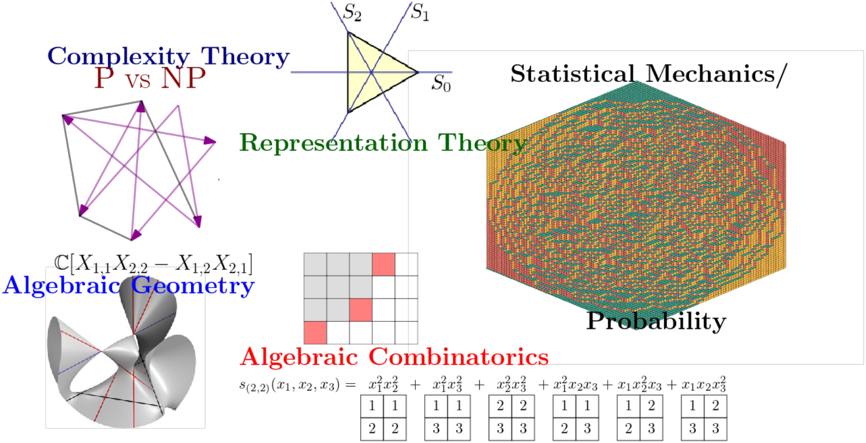
## Theorem (Pak-P'20)

*Computing the reduced Kronecker coefficients $\overline{g}(\alpha,\beta,\gamma)$ is strongly #P-hard.*

# Thank you!


Complexity Theory
P vs NP
$S_2$
$S_1$
$S_0$
Representation Theory
Statistical Mechanics/
Probability
$\mathbb{C}[X_{1,1}X_{2,2} - X_{1,2}X_{2,1}]$
Algebraic Geometry
Algebraic Combinatorics
$s_{(2,2)}(x_1, x_2, x_3) = x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2$
1 1 / 2 2
1 1 / 3 3
2 2 / 3 3
1 1 / 2 3
1 2 / 2 3
1 2 / 3 3