Upho lattices and their cores

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Finite & infinite graded posets

A finite poset P is **n-graded** if $P = \sqcup_{i=0}^{n} P_i$ where all maximal chains are of form $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n$ with $x_i \in P_i$ for all i. Its **rank generating** and **(reciprocal) characteristic polynomials** are

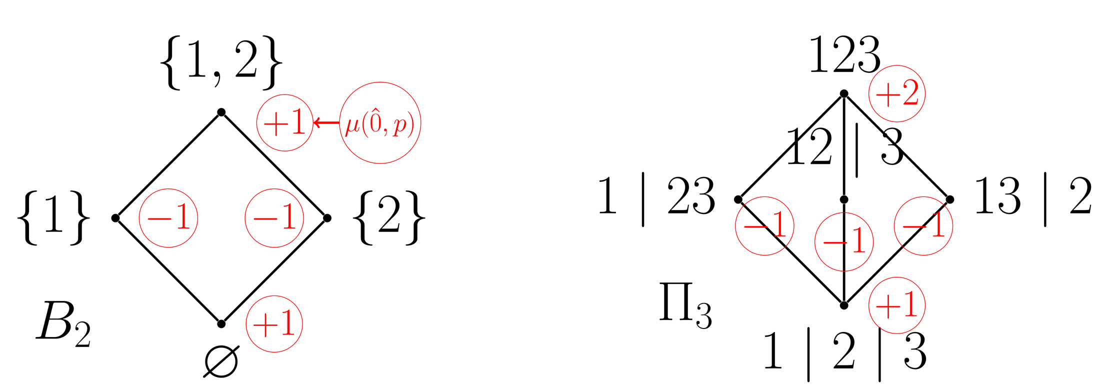
$$F(P;x) = \sum_{i=0}^{n} \#P_i\, x^i = \sum_{p \in P} x^{\rho(p)}$$
$$\chi(P;x) = \sum_{p \in P} \mu(\hat{0},p)\, x^{\rho(p)}$$

For $B_n =$ **Boolean lattice** of subsets of $[n]$:

$$F(B_n;x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$$
$$\chi(B_n;x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k = (1-x)^n$$

For $\Pi_n =$ **partition lattice** of set partitions of $[n]$:

$$F(\Pi_n;x) = \sum_{k=0}^{n} S(n,n-k)\, x^k$$
$$\chi(\Pi_n;x) = \sum_{k=0}^{n} s(n,n-k)\, x^k = \prod_{i=1}^{n-1} (1-ix)$$

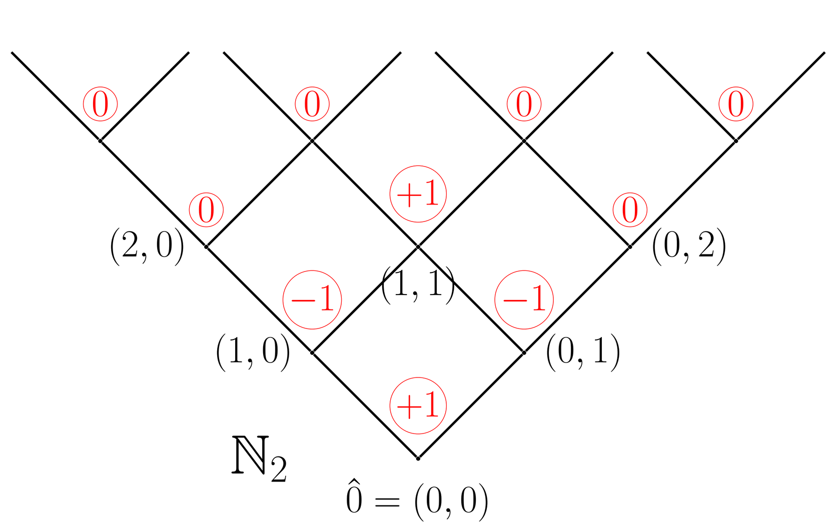


An infinite poset $\mathcal{P}$ is **finite type $\mathbb{N}$-graded** if $\mathcal{P} = \sqcup_{i=0}^{\infty} P_i$ where all maximal chains are of form $x_0 \lessdot x_1 \lessdot \cdots$ with $x_i \in P_i$ for all i and where $\#P_i < \infty$ for all i. We again define

$$F(\mathcal{P};x) = \sum_{i=0}^{\infty} \#P_i\, x^i = \sum_{p \in \mathcal{P}} x^{\rho(p)}$$
$$\chi(\mathcal{P};x) = \sum_{p \in \mathcal{P}} \mu(\hat{0},p)\, x^{\rho(p)}$$

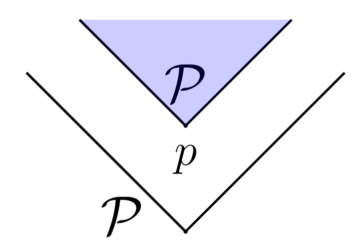
For $\mathcal{P} = \mathbb{N}^n$:

$$F(\mathbb{N}^n;x) = \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} x^k = \frac{1}{(1-x)^n}$$
$$\chi(\mathbb{N}^n;x) = (1-x)^n$$



Upho posets

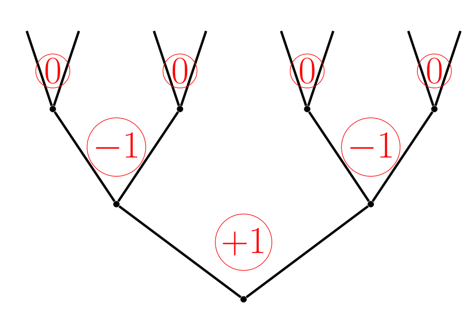
A poset $\mathcal{P}$ is **upper homogeneous**, or "**upho**," if for every $p \in \mathcal{P}$ the **principal order filter** $V_p = \{q\colon q \geq p\}$ is isomorphic to whole poset $\mathcal{P}$. Looking up from each $p \in \mathcal{P}$, we see a copy of $\mathcal{P}$:



We consider only **finite type $\mathbb{N}$-graded** posets $\mathcal{P}$.

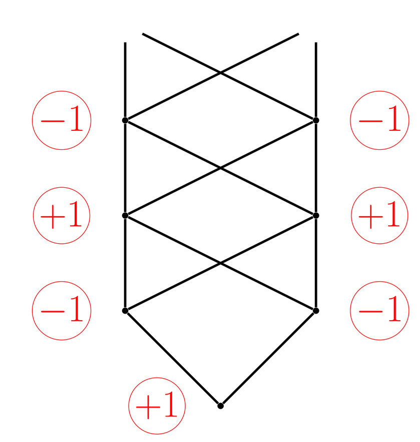
Since $\mathbb{N}$ is upho, and upho-ness is preserved by direct product, $\mathbb{N}^n$ is upho for all $n \geq 1$. Other examples...

$\mathcal{P} =$ the infinite binary tree poset:



$$F(\mathcal{P};x) = \sum_{n \geq 0} 2^n x^n = \frac{1}{1-2x}$$
$$\chi(\mathcal{P};x) = 1-2x$$

$\mathcal{P} =$ the upho poset with $\#P_i = 2$ for all $i \geq 1$:



$$F(\mathcal{P};x) = 1 + \sum_{n \geq 1} 2x^n = \frac{1+x}{1-x}$$
$$\chi(\mathcal{P};x) = 1 + \sum_{n \geq 1} (-1)^n 2x^n = \frac{1-x}{1+x}$$

These examples with two **atoms** have obvious generalizations to any number $r \geq 1$ of atoms.

From the above examples, it is not hard to guess:

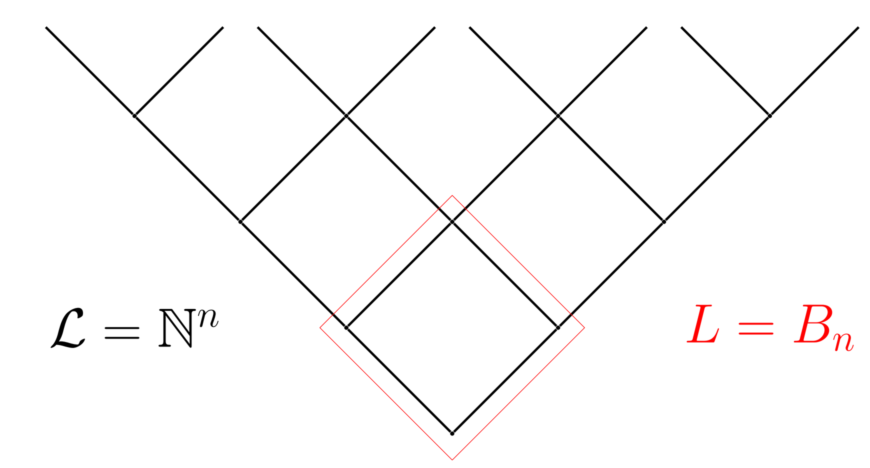
Theroem (H. 2022)

For $\mathcal{P}$ an upho poset, $F(\mathcal{P};x) = \chi(\mathcal{P};x)^{-1}$.

Note: Gao–Guo–Seetharaman–Seidel 2022 showed there are **uncountably many** rank generating functions $F(\mathcal{P};x)$ among all upho posets $\mathcal{P}$.

Upho lattices and cores

For an upho **lattice** $\mathcal{L}$ we let $L = [\hat{0}, s_1 \vee \cdots \vee s_r]$ be the interval from its minimum to the join of its atoms $s_1, \ldots, s_r$, which we call its **core**.



Corollary (from cross-cut thm.)

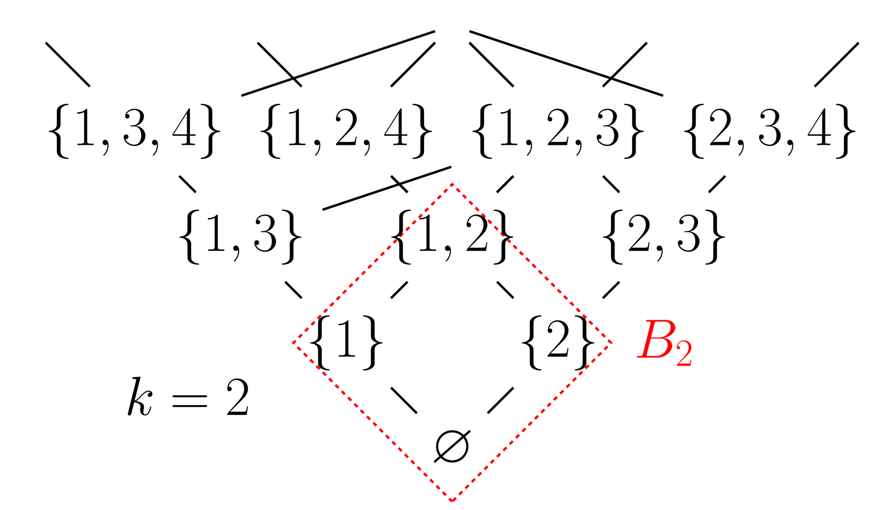
$\mathcal{L}$ upho lattice, core $L \Rightarrow F(\mathcal{L};x) = \chi(L;x)^{-1}$.

Note: the core does not determine the upho lattice, i.e., a given L can be a core of more than one $\mathcal{L}$.

For example, fix $k \geq 1$ and let

$\mathcal{L} = \{\text{finite } A \subseteq \{1,2,\ldots\}\colon \max(A) < \#A + k\}$,

ordered by inclusion.



This $\mathcal{L}$ is an upho lattice with core $L = B_k$, but it is **not** isomorphic to $\mathbb{N}^k$ (for $k \geq 2$).

Nevertheless, we are still interested in knowing:

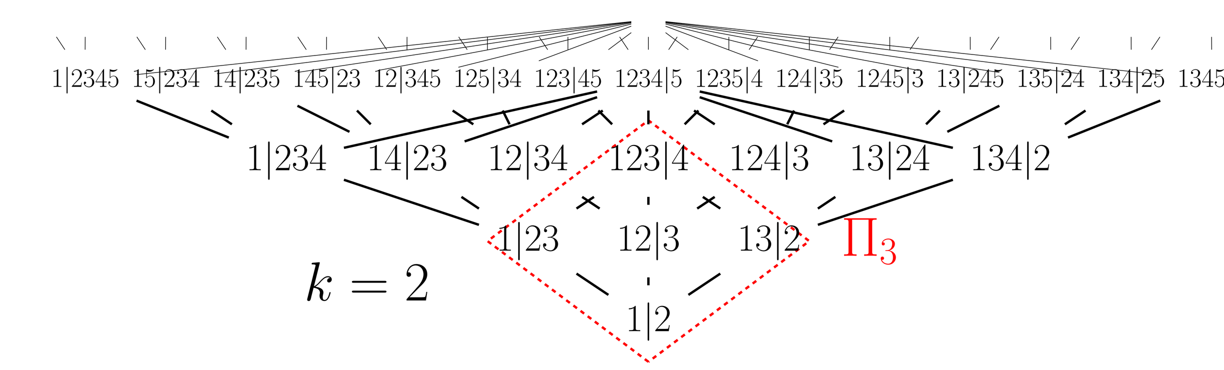
Main question

Which finite lattices L are cores of upho lattices?

For example, we know the Boolean lattice B_n is a core, for any $n \geq 1$. We cannot fully answer this question, but we can provide **positive** and **negative examples**, showing it is subtle.

Combinatorial examples of cores

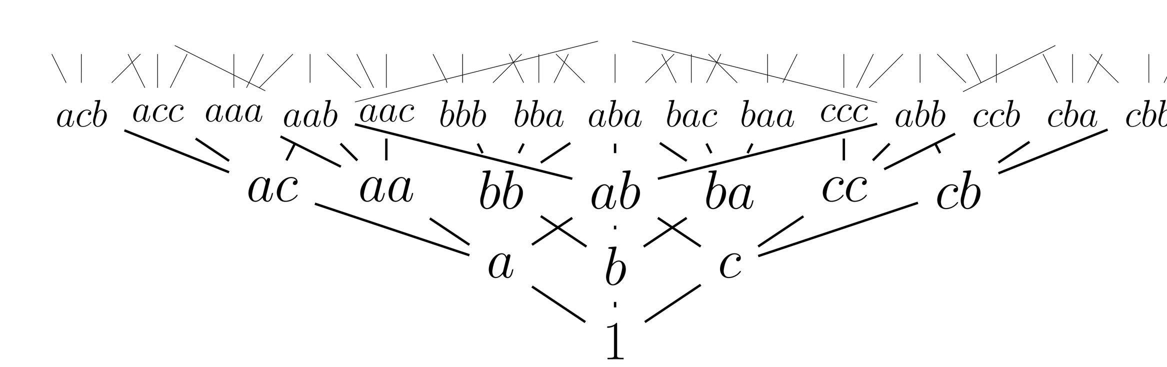
Fix $k \geq 1$ and let $\mathcal{L}$ be the set partitions of $[n]$ (for any $n \geq k$) into k blocks, ordered by refinement:



This $\mathcal{L}$ is an upho lattice with core $L = \Pi_{k+1}$. And a similar construction exists for any **uniform sequence** of **supersolvable geometric lattices**.

Algebraic examples of cores

Consider the monoid $M = \langle a,b,c \mid ab = bc = ca \rangle$, ordered by left divisibility:



This is an upho lattice. The same is true for any (homogeneous) **Garside monoid**. Hence, the **weak order** and **noncrossing partition lattice** of any **finite Coxeter group** are cores.

Non-examples of cores

Lemma

If L is a core of an upho lattice, then the power series $\chi(L;x)^{-1}$ has all positive coefficients.

If L is the **face lattice** of an **octahedron**, then $[x^{13}]\chi(L,x)^{-1} = -123704$, so L is not a core. More generally, face lattices of n-dimensional **cross polytopes** and **hypercubes** aren't cores ($n \geq 3$).

If L is the **lattice of flats** of the **uniform matroid** $U(3,4)$, then $[x^7]\chi(L;x)^{-1} = -80$, so L is not a core. More generally, the lattice of flats of $U(k,n)$ is not a core for $2 < k < n$.