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Sequences §11.1

We now start a new chapter, Ch. 11, on sequences, series, and power series. This is the final topic of the semester.

Def'n An (infinite) sequence is an infinite list $a_1, a_2, a_3, \ldots, a_n, \ldots$ of real numbers. We also use $\{a_n\}$ and $\{a_n\}_{n=1}^{\infty}$ to denote this sequence.

E.g. We can let $a_n = \frac{1}{2^n}$ for $n \geq 1$, which gives the sequence $1/2, 1/4, 1/8, 1/16, 1/32, \ldots$

E.g. $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$

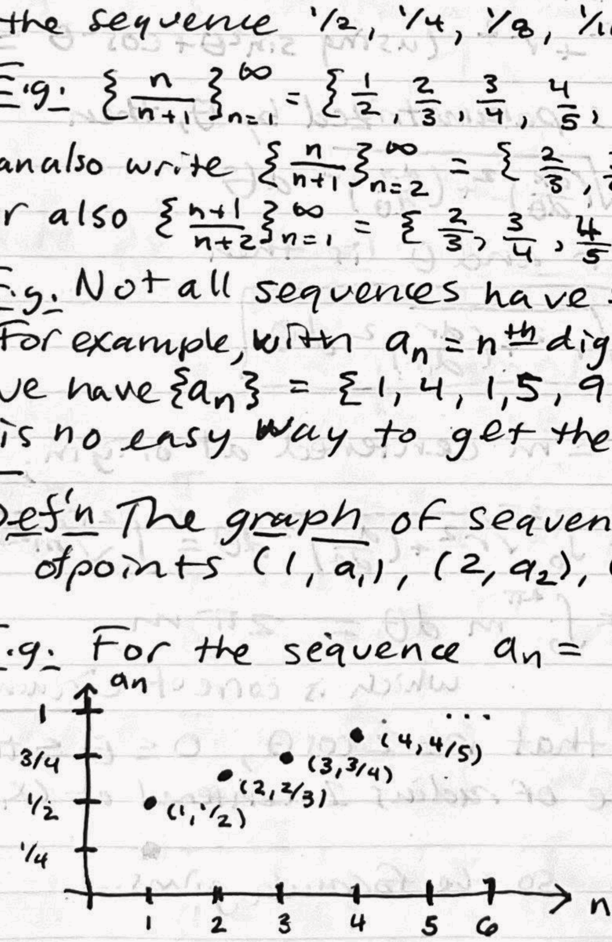
Can also write $\left\{\frac{n}{n+1}\right\}_{n=2}^{\infty} = \left\{\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$ to start at $n=2$,

or also $\left\{\frac{n+1}{n+2}\right\}_{n=1}^{\infty} = \left\{\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$.

E.g. Not all sequences have simple formulas for the n^{th} term. For example, with $a_n = n^{\text{th}}$ digit of π after the decimal point, we have $\{a_n\} = \{1, 4, 1, 5, 9, 2, 6, 5, \ldots\}$ but there is no easy way to get the n^{th} term here...

Def'n The graph of sequence $\{a_n\}_{n=1}^{\infty}$ is the collection of points $(1, a_1), (2, a_2), (3, a_3), \ldots$ in the plane.

E.g. For the sequence $a_n = \frac{n}{n+1}$, its graph is

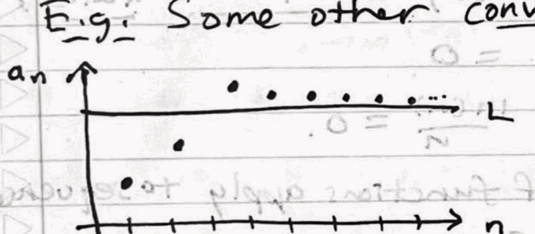


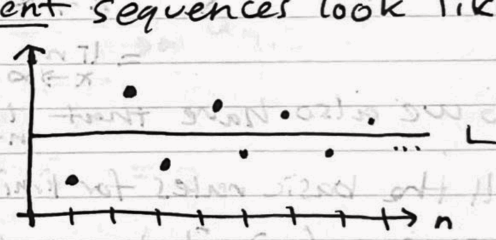
The graph of a sequence is like the graph of a function, but we get discrete points instead of a continuous curve. Notice how for this graph, points approach line $y=1$ ...

Def'n We say the <u>limit</u> of sequence $\{a_n\}$ is L, written "$\lim_{n\to\infty} a_n = L$" or "$a_n \to L$ as $n \to \infty$" if, intuitively, we can make the <u>terms</u> <u>a_n</u> <u>as close to L</u> as we'd like by taking <u>n</u> <u>sufficiently large</u>.
(Precise definition uses ε, like limits in Calc I...)
If $\lim_{n\to\infty} a_n$ exists, we say the sequence <u>converges</u>.
Otherwise, we say the sequence <u>diverges</u>.

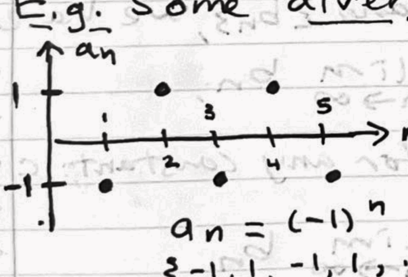
E.g. The sequence $a_n = \frac{n}{n+1}$ has $\lim_{n\to\infty} a_n = 1$ (we'll prove this later...)

E.g. Some other con<u>vergent</u> sequences look like:

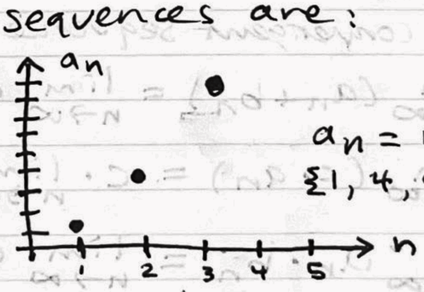




E.g. Some <u>divergent</u> sequences are:



$a_n = (-1)^n$
$\{-1, 1, -1, 1, \ldots\}$



$a_n = n^2$
$\{1, 4, 9, 16, \ldots\}$

Notice how this 2^{nd} example $a_n = n^2$ "goes off to ∞."

Def'n The notation "$\lim_{n\to\infty} a_n = \infty$" means that for every M there is an N such that $a_n > M$ for all $n > N$.
We define "$\lim_{n\to\infty} a_n = -\infty$" similarly.

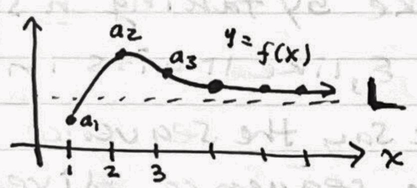
E.g. $\lim_{n\to\infty} n^2 = \infty$ and $\lim_{n\to\infty} -n = -\infty$.

Having an <u>infinite</u> <u>limit</u> is one way a sequence can <u>diverge</u>.

Limits of sequences are very similar to limits of functions:

<u>Theorem</u> If $f(x)$ is a function with $f(n) = a_n$ for all positive integers n, then if $\lim_{x \to \infty} f(x) = L$ also $\lim_{n \to \infty} a_n = L$.

<u>Picture</u>:



<u>E.g.</u> How to find $\lim_{n \to \infty} \frac{\ln(n)}{n}$? Instead, let $f(x) = \frac{\ln(x)}{x}$,

then $\lim_{x \to \infty} \frac{\ln(x)}{x} = \lim_{x \to \infty} \frac{1/x}{1}$ (by <u>L'Hôpital's Rule</u>)

$= \lim_{x \to \infty} 1/x = 0$

So we also have that $\lim_{n \to \infty} \frac{\ln(n)}{n} = 0$.

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All the basic rules for limits of functions apply to sequences:

<u>Theorem</u> (Limit Laws for Sequences)

For convergent sequences $\{a_n\}$ and $\{b_n\}$, we have:

- $\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n$.
- $\lim_{n \to \infty} (c \cdot a_n) = c \cdot \lim_{n \to \infty} a_n$ for any constant $c \in \mathbb{R}$.
- $\lim_{n \to \infty} a_n \cdot b_n = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n$
- $\lim_{n \to \infty} \frac{a_n}{b_n} = (\lim_{n \to \infty} a_n) / (\lim_{n \to \infty} b_n)$ if $\lim_{n \to \infty} b_n \neq 0$.

<u>E.g.</u> To compute $\lim_{n \to \infty} \frac{n}{n+1}$, we can use these rules:

$$\lim_{n \to \infty} \frac{n}{n+1} = \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n}} = \frac{\lim_{n \to \infty} 1}{\lim_{n \to \infty} 1 + \lim_{n \to \infty} \frac{1}{n}} = \frac{1}{1+0} = 1$$

multiply top and bottom by $1/n$

as claimed! ✓

Another very useful lemma for computing limits of sequences:

Lemma If $\lim_{n\to\infty} a_n = L$ and $f(x)$ is continuous at $x=L$, then $\lim_{n\to\infty} f(a_n) = f(L)$.

E.g. Q: What is $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right)$?

A: Notice $\lim_{n\to\infty} \frac{\pi}{n} = 0$ and cos is continuous at 0, so that $\lim_{n\to\infty} \cos(\pi/n) = \cos(0) = 1$.

Another useful lemma for limits of sequences with signs:

Lemma If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.

E.g. How to compute $\lim_{n\to\infty} \frac{(-1)^n}{n}$? Since $\lim_{n\to\infty} \frac{1}{n} = 0$, we also have that $\lim_{n\to\infty} (-1)^n/n = 0$.

Compare this to $a_n = (-1)^n$, which diverges!

One of the most important kind of sequences are the sequences $a_n = r^n$ for some fixed number $r \in \mathbb{R}$.

When does this sequence converge?

We have seen in Calc I that for $0 < r < 1$,

$$\lim_{x\to\infty} (r^x) = 0 \quad \left(\text{think: } \lim_{x\to\infty} \left(\tfrac{1}{2}\right)^x = 0\right)$$

So $\lim_{n\to\infty} r^n = 0$ for $0<r<1$ too.

By the absolute value lemma, $\lim_{n\to\infty} r^n = 0$ when $-1 < r < 0$ as well.

Also, clearly $\lim_{n\to\infty} 1^n = \lim_{n\to\infty} 1 = 1$. But other r diverge!

Theorem

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \text{does not exist} & \text{for all other } r. \end{cases}$$

Monotone and bounded sequences §11.1

Def'n The sequence $\{a_n\}$ is increasing if $a_n < a_{n+1}$ for all $n \geq 1$, and decreasing if $a_n > a_{n+1}$ for all $n \geq 1$. It is called monotone if it is either increasing or decreasing.

E.g. The sequence $a_n = n$ is increasing (hence monotone).
The sequence $a_n = (-1)^n$ is neither increasing nor decreasing.

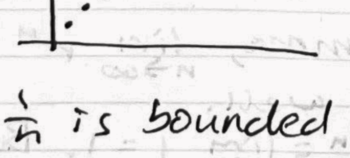
Def'n $\{a_n\}$ is bounded above if there is some M such that $a_n < M$ for all $n \geq 1$, it is bounded below if there is M such that $a_n > M$ for all $n \geq 1$, and it is bounded if it is both bounded above and below.

E.g. $a_n = (-1)^n$ is bounded (above by 2 and below by -2).
but $a_n = n$ is unbounded since it goes off to ∞.

Clearly a sequence which is unbounded (like $a_n = n$) cannot be convergent. Some bounded sequences, (like $a_n = (-1)^n$), are also divergent. But if our sequence is both bounded and monotone, then it must converge:

Thm (Monotone Sequence Theorem) Every bounded, monotone (either increasing or decreasing) sequence converges.

Picture proof.



$\Leftarrow$ an increasing sequence bounded by M will converge to an L with $L \leq M$.

E.g. $a_n = \frac{1}{n}$ is bounded and monotone (decreasing) so it converges, as we were already aware.

Exercise Use the Monotone Convergence Theorem to explain why $a_n = \frac{n}{n+1}$ converges (which we also knew...)

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Series §11.2

A series is basically an "infinite sum."
If we have an (infinite) sequence $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \dots\}$
the corresponding series is

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \cdots$$

An infinite sum like this does not always make sense:

$$\sum_{n=1}^{\infty} n = 1+2+3+4+5+\dots = \text{"}\infty\text{"}$$

But sometimes we can sum ∞-many terms & get a finite number:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = ???$$

Well, $\frac{1}{2} = 0.5$, $\frac{1}{2}+\frac{1}{4} = 0.75$, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8} = 0.875$,
and it seems that if we add up more and more terms,
we don't go off to ∞, but instead get closer and closer to $\underline{\underline{1}}$.

Def'n For series $\sum_{n=1}^{\infty} a_n$, the associated partial sums
are $S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \dots + a_n$ for $n \geq 1$.
If $\lim_{n\to\infty} S_n = L$ then we write $\sum_{n=1}^{\infty} a_n = L$ and we
say the series converges. Otherwise, it diverges.

key idea: $\boxed{\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} (a_1 + a_2 + \dots + a_n)}$

E.g.: Let $a_n = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$. What is $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$?

Well, $S_n = \underbrace{\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right)}_{a_1} + \underbrace{\left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right)}_{a_2} + \cancel{\cdots} + \underbrace{\left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right)}_{a_{n-1}} + \underbrace{\left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)}_{a_n}$

$= 1 - \frac{1}{n+1}$, so that $\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = \underline{\underline{1}}$.

Thus, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \underline{\underline{1}}$.

One of the most important kind of series are the geometric series:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots, \quad \text{for real numbers } a \text{ and } r \neq 0.$$

Notice that $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$

and $r \cdot S_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$

$\Rightarrow (1-r) \cdot S_n = a - ar^n$

$$\Rightarrow S_n = \frac{a - ar^n}{(1-r)}$$

Since $\lim_{n\to\infty} r^n = 0$ for $|r|<1$, we have:

$$\sum_{n=1}^{\infty} ar^{n-1} = \lim_{n\to\infty} \frac{a-ar^n}{1-r} = \frac{a}{1-r} \text{ for } |r|<1.$$

important formula to remember: value of geo. series when $|r|<1$.

E.g.: $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ is geo. series with $a = 1/2$ and $r = 1/2$. So $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1/2}{1-1/2} = \underline{1}$.

This is what we expected above!

For $|r| \geq 1$, geo. series $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

Consider in particular the case $a = r = 1$.

Then $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} 1 = 1+1+1+\cdots$, so the partial sums are $S_n = 1+1+\cdots+1 = n$, and $\lim_{n\to\infty} S_n = \infty$.

In general, in order to converge, the terms in a series must approach zero:

Theorem (Divergence Test) If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. So if $\lim_{n\to\infty} a_n \neq 0$, $\sum_{n=1}^{\infty} a_n$ diverges.

WARNING: The divergence test says that if terms do not go to zero, the series diverges.

But converse does not hold: the terms a_n can go to 0, while the series $\sum_{n=1}^{\infty} a_n$ still diverges.

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The most important (counter)example is the harmonic series:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

Of course, $\lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ still diverges.

How to see this? Ignore the 1 at the start, and consider

$$\underbrace{\frac{1}{2}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq 2\cdot\frac{1}{4} = \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq 4\cdot\frac{1}{8} = \frac{1}{2}} + \cdots \quad \cdots$$

The trick, as shown above, is to break the series into chunks consisting of 1, 2, 4, 8, ... terms.

If we add up the terms in each chunk, we get a sum bigger than $\frac{1}{2}$. So overall sum is $\geq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$

But a sum of ∞-many $\frac{1}{2}$'s must diverge!

So the harmonic series, which is bigger than that, diverges too.

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Theorem (Laws for series)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge.

Then • $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

and • $\sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n$ for any $c \in \mathbb{R}$.

WARNING: $\sum_{n=1}^{\infty} a_n \cdot b_n \neq \left(\sum_{n=1}^{\infty} a_n\right) \cdot \left(\sum_{n=1}^{\infty} b_n\right)$.