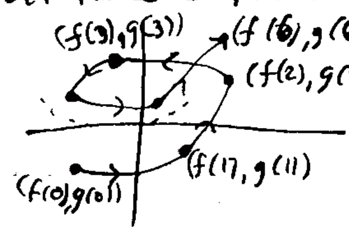
3/13 **Parametric Equations § 10.1**

The 1st half of the semester was focused on integration.
In 2nd half, we will study other topics. We start with a short chapter (Ch. 10) on parametric equations & polar coordinates.

Up until now we have looked at curves of the form $y = f(x)$ (or, more rarely, $f(x,y) = 0$).
A parameterized curve is defined by two equations:

$$x = f(t) \quad \text{and} \quad y = g(t)$$

where t is an auxilliary variable. Often think of t as time, so the curve describes motion of a particle where at time t particle is at position $(f(t), g(t))$:

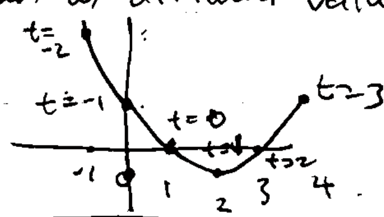


In this picture the arrows → show movement of particle over time.

Eg. Consider curve $x = t+1$, $y = t^2 - 2t$.
We can make a chart w/ different values of t:

t	x	y
-2	-1	8
-1	0	3
0	1	0
1	2	-1
2	3	0
3	4	3

⇒



⇐ plot of points $(f(t), g(t))$ for $t = -1, 0, \dots, 4$ looks like parabola

In this case, we can eliminate the variable t:

$$x = t+1 \Rightarrow t = x-1$$

$$y = t^2 - 2t \Rightarrow y = (x-1)^2 - 2(x-1)$$

$$= x^2 - 4x + 3$$

So this parametric curve is just $\boxed{y = x^2 - 4x + 3}$

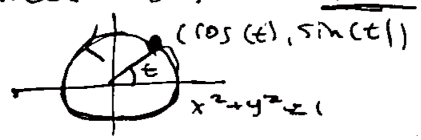
initial time
$\Rightarrow$ initial point $(f(0), g(0))$

terminal time
$\Rightarrow$ terminal point $(f(2\pi), g(2\pi))$

E.g. Consider parametric curve:
$x = \cos(t)$, $y = \sin(t)$ $\quad 0 \le t \le 2\pi$

How can we visualize this curve?

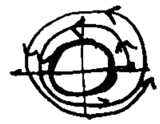
Notice that $x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$, so this parameterizes a circle $\boxed{x^2+y^2=1}$:



here t = angle (in radians) of point $(\cos(t), \sin(t))$

E.g. What about $x = \cos(2t)$, $y = \sin(2t)$, $0 \le t \le 2\pi$?
Notice we still have $x^2 + y^2 = \cos^2(2t) + \sin^2(2t) = 1$, so the parameterized curve still traces a circle.

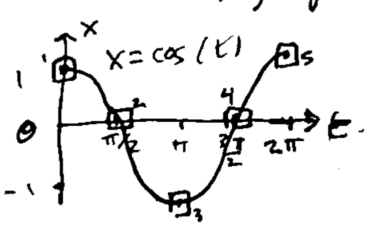
curve spirals around 2 times! $\Rightarrow$

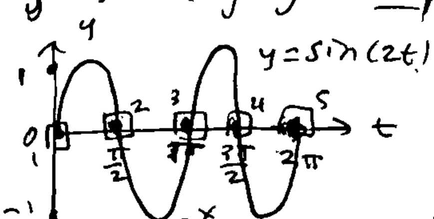


But: now traces circle twice: once for $0 \le t \le \pi$ and once for $\pi \le t \le 2\pi$

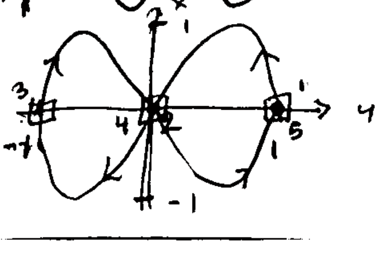
So we see that the same curve can be parameterized in different ways!
Can think of the particle as moving "faster"!

E.g. Consider the curve $x = \cos(t)$, $y = \sin(2t)$.
It is possible to eliminate t to get $y^2 = 4x^2 - 4x^4$, but that equation is hard to visualize.
Instead, graph $x = f(t)$ and $y = g(t)$ separately:





Then combine into one picture: showing $(f(t), g(t))$ $\Rightarrow$



1 ... 5 are 5 time "snapshots" of particle as it traces the curve

3/15 Calculus with parameterized curves § 10.2

Much of what we have done with curves $y=f(x)$ in Calculus can also be done for parameterized curves:

Tangent vectors: Let $(x,y)=(f(t),g(t))$ be a curve. Then, at time t, the slope of tangent vector is given by:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)} \quad (\text{if } f'(t) \neq 0).$$

chain rule

If $dy/dt=0$ and $dx/dt \neq 0 \Rightarrow$ horizontal tangent
If $dx/dt=0$ and $dy/dt \neq 0 \Rightarrow$ vertical tangent.

E.g. Consider curve $x=t^2$, $y=t^3-3t$.

First, notice when $t=\pm\sqrt{3}$ have

$$x=t^2=3 \quad \text{and} \quad y=t^3-3t=t(t^2-3)=0,$$

so curve passes through $(3,0)$ at two times $t=-\sqrt{3}$ and $t=\sqrt{3}$.

With the above formula we can compute:

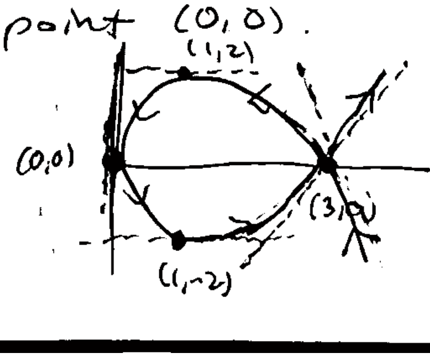
$$dy/dx = \frac{dy/dt}{dx/dt} = \frac{3t^2-3}{2t} \to = -6/2\sqrt{3} = -\sqrt{3} \text{ at } t=-\sqrt{3}$$
$$\to = 6/2\sqrt{3} = \sqrt{3} \text{ at } t=\sqrt{3}$$

So two tangent lines, of slopes $\pm\sqrt{3}$, pass through curve at $(3,0)$.

When is the tangent horizontal? When $dy/dt=3t^2-3=0$, which is for $t=\pm 1$, at points $(1,2)$ and $(1,-2)$.

When is tangent vertical? When $dx/dt=2t=0$, at $t=0$, which is point $(0,0)$.

Putting all this info together lets us give a good sketch of the curve:



(Exercise: Show circumfrence of unit circle $= 1$ using parametrization $x = \cos(t)$, $y = \sin(t)$ for $0 \le t \le 2\pi$)

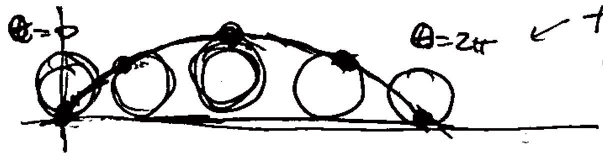
Arc lengths: We saw several times how to find lengths of curves by breaking into line segments:

$\leftarrow$ recall length of each small segment $= \sqrt{(\Delta x)^2 + (\Delta y)^2}$

For a parameterized curve $(x,y) = (f(t), g(t))$, where t is in the range $\alpha \le t \le \beta$, this gives:

$$\text{length of curve} = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_\alpha^\beta \sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

E.g. A cycloid is the path a point on circle traces as the circle rolls:



$\leftarrow$ think of this as animation of circle rolling, with point • marked where $\theta =$ "time"

The cycloid is parameterized by:

$$x = \theta - \sin\theta, \quad y = 1 - \cos\theta \quad \text{for } 0 \le \theta \le 2\pi$$

(assuming circle has radius 1; θ represents angle rolled).

Q: what is the arclength of the cycloid?

we compute: $\frac{dx}{d\theta} = 1 - \cos\theta$, $\frac{dy}{d\theta} = \sin\theta$, so that

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2} = \sqrt{2(1-\cos\theta)}$$

(using identity $\frac{1}{2}(1-\cos 2x) = \sin^2 x$) $\longrightarrow$

$$= \sqrt{4\sin^2(\theta/2)}$$

$$= 2\sin(\theta/2)$$

$$\Rightarrow \text{length of cycloid} = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

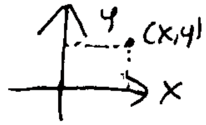
$$= \int_0^{2\pi} 2\sin\left(\frac{\theta}{2}\right) d\theta = \left[-4\cos\left(\frac{\theta}{2}\right)\right]_0^{2\pi}$$

$$= \left((-4 \cdot -1) - (-4 \cdot 1)\right) = 8.$$

3/17

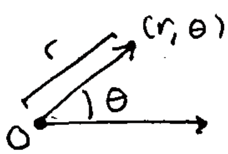
Polar Coordinates §10.3

We are used to working with the "Cartesian" coordinate system where a point on the plane is represented by (x, y)



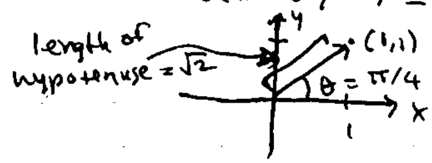
telling us how far to move along two orthogonal axes to reach that point.

The polar coordinate system is a different way to represent points on the plane by a pair (r, θ):



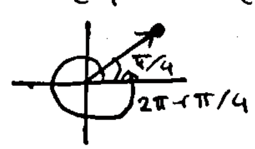
Here we have one fixed axis ray $\longrightarrow$ emanating from origin, and we reach a point (r, θ) by making an angle of θ radians and going out a distance of r.

E.g. The point $(x,y) = (1,1)$ in Cartesian coord's is the same as $(r, \theta) = (\sqrt{2}, \frac{\pi}{4})$ in polar coord's:



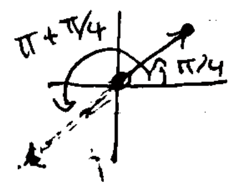
Notice: There are multiple ways to represent any point in polar coord's because we can add 2π to θ:

$(r, \theta) = (\sqrt{2}, \frac{\pi}{4})$ same as $(r, \theta) = (\sqrt{2}, 2\pi + \frac{\pi}{4})$



Also... can add π to θ and replace r by $-r$:

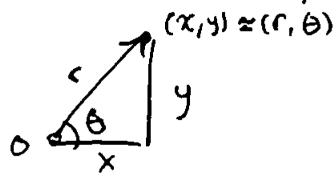
$(r, \theta) = (\sqrt{2}, \pi/4)$ same as $(r, \theta) = (-\sqrt{2}, \pi/4 + \pi)$



Negative value of r means go backwards that distance along the ray.

Question: How to convert between Cartesian & polar coord's?

Let's draw a right triangle to help us:



$\Leftarrow$ From this picture we see that

$$\boxed{x = r\cos\theta \text{ and } y = r\sin\theta}$$

which gives (x,y) in terms of (r,θ)

Also we have that:

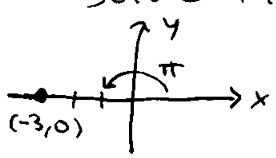
$$\boxed{r^2 = x^2 + y^2 \text{ and } \tan\theta = \frac{y}{x}}$$

which gives us (r,θ) in terms of (x,y)

(Specifically, $r = \pm\sqrt{x^2+y^2}$ and $\theta = \arctan(\frac{y}{x})$).

E.g.: Find the polar coordinates of $(x,y) = (-3,0)$.

To solve this problem easiest to draw point:



we see that this point is at angle $\theta = \pi$ and radius $r = 3$.

Check: $3^2 = r^2 = x^2 + y^2 = (-3)^2 + 0^2$

and $0 = \tan(\theta) = \frac{y}{x} = \frac{0}{-3}$

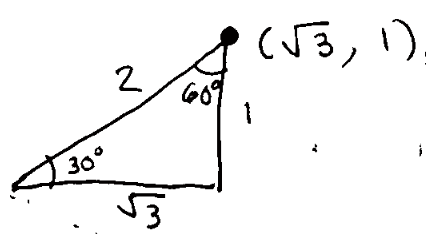
Could also choose $(r,\theta) = (-3, 0)$ ....

E.g. Find the Cartesian coordinates of $(r,\theta) = (2, \frac{\pi}{6})$.

Here we have $y = r\sin\theta = 2\sin(\frac{\pi}{6}) = 2 \cdot \frac{1}{2} = 1$

and $x = r\cos\theta = 2\cos(\frac{\pi}{6}) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$

Can also draw triangle:

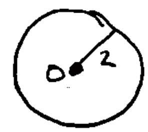


$\Leftarrow$ recall $\theta = \pi/6$ radians $= 30°$ degrees corresponds to a special right triangle

Polar equations and curves:

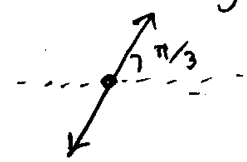
Just like we draw curves $f(x,y)=0$ in Cartesian coord's, we can draw curves $f(r,\theta)=0$ in polar coord's.

E.g. The equation $r=2$ gives circle of radius 2, centered at origin:



← circle = all points at radial distance 2 from O

E.g. The equation $\theta = \pi/3$ gives line at angle $\pi/3$ thru origin:



← line thru origin = all points at given angle...

3/20 E.g. What about equation $r = 2\cos\theta$?

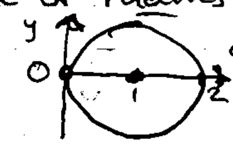
Here it's helpful to switch to Cartesian coord's:

multiplying by r gives $r^2 = 2r\cos\theta$

$$\Leftrightarrow \quad x^2+y^2 = 2x$$

$$\Leftrightarrow \quad (x-1)^2+y^2 = 1$$

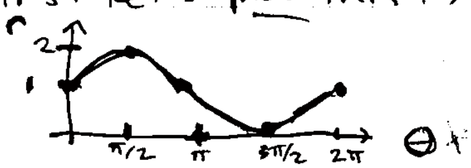
which is a circle of radius 1, centered at $(x,y)=(1,0)$:



← $(x-1)^2+y^2=1$ a.k.a. $r=2\cos\theta$.

E.g. What about $r = 1+\sin(\theta)$?

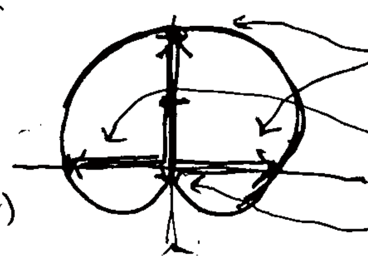
First let's plot this in Cartesian coord's:



← Shows us how radius of figure changes with angle:

cardiod

this "heart shaped" ⟹ figure is the polar curve $r = 1+\sin(\theta)$



① at angle $\theta=0$, $r=1$

② at angle $\theta=\frac{\pi}{2}$, $r=2$ so we move out to this point up top

③ at $\theta=\pi$, back to $r=1$

④ at $\theta=\frac{3\pi}{2}$, radius shrinks to $r=0$ //

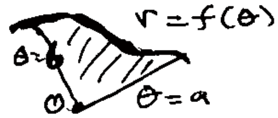
3/22

Calculus in Polar coordinates §10.4

We can do all types of calculus stuff in polar coord's too...

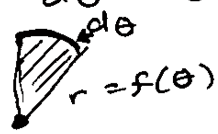
Areas: How to compute area "inside" polar curve $r = f(\theta)$?
where $a \le \theta \le b$

The polar curve looks something like this:



For a small change $d\theta$ in θ we get roughly a pie slice:

$\underline{\text{area}} = \pi r^2 \cdot \frac{d\theta}{2\pi}$
$= \frac{1}{2}(f(\theta))^2 \, d\theta$

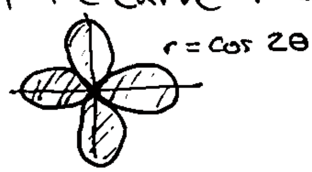


As usual, breaking up area into many pie slices and summing up areas gives an integral in limit:

$$\text{area inside polar curve} = \boxed{\int_a^b \frac{1}{2}(f(\theta))^2 \, d\theta}$$

Eg: Let's look at the curve $r = \cos 2\theta$ for $0 \le \theta \le 2\pi$:

"four-leaf clover" shape $\Rightarrow$



What is area inside this shape? Using formula...

$$\text{Area} = \int_0^{2\pi} \frac{1}{2}(f(\theta))^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} \cos^2 2\theta \, d\theta$$

We've seen before that: $\int \cos^2 x \, dx = \frac{1}{2}(x + \sin(x)\cos(x))$
(using integration by parts)

so w/ a simple u-sub: $\int \frac{1}{2}\cos^2 2\theta \, d\theta = \frac{1}{4}\theta + \frac{1}{8}\sin(2\theta)\cos(2\theta)$

Thus, area $= \int_0^{2\pi} \frac{1}{2}\cos^2 2\theta \, d\theta = \left[\frac{1}{4}\theta + \frac{1}{8}\sin(2\theta)\cos(2\theta)\right]_0^{2\pi}$

$= \left(\left(\frac{1}{4}\cdot 2\pi + \frac{1}{8}\underbrace{\sin(4\pi)}_{0}\cos(4\pi)\right) - \left(\frac{1}{4}\cdot 0 + \frac{1}{8}\underbrace{\sin(0)}_{0}\cos(0)\right)\right) = \boxed{\frac{\pi}{2}}$

3/24

Arc lengths: How to compute length of polar curve $r = f(\theta)$?

Recall $x = r\cos\theta$ and $y = r\sin\theta$ in Cartesian coord's.

So using the product rule we get:

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

So that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta$$
$$+ \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta$$
$$= \left(\frac{dr}{d\theta}\right)^2 + r^2 \quad (\text{using } \sin^2\theta + \cos^2\theta = 1)$$

If we think of (x,y) as parameterized by θ, then

$$\text{length of curve} = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

which in terms of r and θ is then

$$\text{length} = \boxed{\int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

E.g. For a circle $r = m$ centered at origin, this formula gives

$$\text{length} = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{m^2 + 0^2}\, d\theta$$
$$= \int_0^{2\pi} m\, d\theta = 2\pi m$$

circumfrence!

E.g. We saw before that $r = 2\cos\theta$, $0 \le \theta \le \pi$ gives a circle of radius 1 centered at $(x,y) = (1,0)$.

Here $\frac{dr}{d\theta} = -2\sin\theta$, so the formula gives...

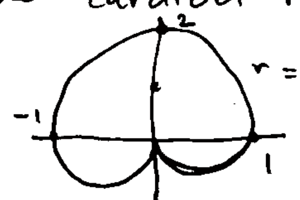
$$\text{length} = \int_0^{\pi} \sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta = \int_0^{\pi} 2\, d\theta = 2\pi.$$

Tangents: How to find slope of tangent to polar curve $r = f(\theta)$?
We again think in terms of Cartesian coord's (x, y):

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

This is pretty complicated, but easy to derive if you remember $x = r\cos\theta$ and $y = r\sin\theta$.

E.g. Consider cardiod $r = 1 + \sin\theta$:

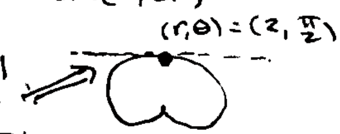


Here
$$\frac{dy}{dx} = \frac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta}$$

$$= \frac{\cos\theta(1 + 2\sin\theta)}{1 - 2\sin^2\theta - \sin\theta} = \frac{\cos\theta(1 + 2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}$$

So at $\theta = \frac{\pi}{2}$ get
$$\frac{dy}{dx} = \frac{\cos(\pi/2)(1 + 2\sin(\pi/2))}{(1+\sin(\pi/2))(1 - 2\sin(\pi/2))}$$

$$= \frac{0(1+2)}{(1+1)(1-2)} = \underline{\underline{0}} \rightarrow \text{horizontal tangent at } \theta = \pi/2$$



And at $\theta = \frac{\pi}{3}$ get
$$\frac{dy}{dx} = \frac{\cos(\pi/3)(1 + 2\sin(\pi/3))}{(1+\sin(\pi/3))(1 - 2\sin(\pi/3))}$$

$$= \frac{(1/2)(1 + 2\frac{\sqrt{3}}{2})}{(1+\frac{\sqrt{3}}{2})(1 - 2\sqrt{3}/2)} = \frac{1+\sqrt{3}}{(2+\sqrt{3})(1-\sqrt{3})} = \frac{1+\sqrt{3}}{-1-\sqrt{3}} = \underline{\underline{-1}}$$

← tangent slope $= -1$ at $\theta = \pi/3$ //