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Techniques for Integration (Chapter 7)

Now that we've seen many applications of (definite) integrals, we will return to the problem of: how to compute integrals, which by Fund. Thm. Calculus means anti-derivatives (a.k.a. "indefinite integrals")

From Calc I we already know the following integrals:

$\int x^n \, dx = \frac{1}{n+1} x^{n+1} \quad (n \neq -1)$ $\qquad \int e^x \, dx = e^x$

$\int \frac{1}{x} \, dx = \ln(x)$ $\qquad \int \sin(x)\, dx = -\cos(x)$ & $\int \cos(x)\, dx = \sin(x)$

We also know that the integral is linear in sense that

$$\int \alpha \cdot f(x) + \beta \cdot g(x)\, dx = \alpha \int f(x)\, dx + \beta \int g(x)\, dx \quad \text{for } \alpha, \beta \in \mathbb{R}.$$

This lets us compute many integrals, but far from all.

At end of Calc I we learned u-substitution technique for computing integrals:

$$\int g(f(x)) \cdot f'(x)\, dx = \int g(u)\, du$$

where $u = f(x)$ and $du = f'(x)\, dx$.

The u-substitution technique lets us compute

e.g. $\int x \sin(x^2)\, dx = -\frac{1}{2}\cos(x^2) + C$

(take $u = x^2$ so $du = 2x\, dx$)

The u-substitution technique was the "opposite" of the chain rule for derivatives.

We can find more integration techniques by doing the "opposite" of other derivative rules, like the product rule..

Integration by parts §7.1

Recall the product rule says that

$$\frac{d}{dx}\left(f(x)\,g(x)\right) = f(x)\,g'(x) + g(x)\,f'(x)$$

Integrating both sides of this equation gives

$$f(x)\,g(x) = \int f(x)\,g'(x)\,dx + \int g(x)\,f'(x)\,dx$$

Rearranging this gives:

$$\boxed{\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int g(x)\,f'(x)\,dx}$$

This formula is called integration by parts.

It is more often written in the form:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

where $u = f(x)$ and $v = g(x)$, so that

$du = f'(x)dx$ and $dv = g'(x)\,dx$.

In the u-sub. technique, we had to make good choice of u.

Integration by parts is similar, but now we have to make good choices for u and v!

It's easiest to see how this works in examples.

E.g. Compute $\int x \cdot \sin(x)\,dx$.

How to choose u? General rule of thumb:

choose a u such that du is simpler than u.

In this case, let's therefore choose

$u = x$ which leaves $dv = \sin(x)\,dx$

$\Rightarrow du = dx$ $\qquad \Rightarrow v = -\cos(x)$

(by integrating...)

So the integration by parts formula gives

$$\int \underbrace{x}_{u} \underbrace{\sin(x)\,dx}_{dv} = \underbrace{x}_{u} \underbrace{(-\cos(x))}_{v} - \int \underbrace{(-\cos(x))}_{v} \underbrace{dx}_{du}$$

This is useful because $\int \cos(x)\,dx$ is something we already know!

$$\Rightarrow \int x \sin(x)\,dx = -x\cos(x) + \int \cos(x)\,dx$$

$$= \boxed{-x\cos(x) + \sin(x) + C.}$$

(good to remember the +C)

E.g. Compute $\int \ln(x)\,dx$.

Since $d/dx(\ln(x)) = 1/x$ is "simpler" than $\ln(x)$, makes sense to choose $u = \ln(x)$, $dv = dx$

$\Rightarrow du = 1/x\,dx$ $v = x$

$$\Rightarrow \int \underbrace{\ln(x)}_{u} \underbrace{dx}_{dv} = \underbrace{\ln(x)}_{u} \underbrace{x}_{v} - \int \underbrace{x}_{v} \underbrace{1/x\,dx}_{du}$$

$$= x\ln(x) - \int dx = \boxed{x\ln(x) - x + C}$$

A good rule of thumb when picking u in integration by parts is to follow the order:

- L - logarithm ($\ln(x)$)
- I - inverse trig (like $\arcsin(x)$)
- A - algebraic (like polynomials x^2+5x)
- T - trig functions (like $\sin(x)$, $\cos(x)$, ...)
- E - exponentials (e^x)

we haven't talked much about these, but we will soon...!

The earlier letters in LIATE are better choices of u:

so pick $u = \ln(x)$ over $u = x^2$,
but $u = x^2$ over $u = \sin(x)$,
and $u = \sin(x)$ over $u = e^x$, etc...

(these choices will make du "simpler")

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Let's see some more examples of integration by parts:

E.g. Compute $\int x^2 e^x dx$.

Following LIATE, we pick $u = x^2$, $dv = e^x dx$

$$\Rightarrow du = 2x dx, \; v = e^x$$

$$\Rightarrow \int x^2 e^x dx = x^2 e^x - \int e^x 2x\, dx = x^2 e^x - 2\int x e^x dx.$$

But how do we finish? We need to find $\int x e^x dx \ldots$

To do this, let's use integration by parts *again*:

$$\int \underbrace{x}_{u} \underbrace{e^x dx}_{dv} = \underbrace{x}_{u} \underbrace{e^x}_{v} - \int \underbrace{e^x}_{v} \underbrace{dx}_{du} = x e^x - e^x$$

$$\Rightarrow \int x^2 e^x dx = x^2 e^x - 2\int x e^x dx = x^2 e^x - 2(x e^x - e^x)$$

$$= \boxed{x^2 e^x - 2x e^x + 2e^x + C} \checkmark$$

E.g. Compute $\int \sin(x)\, e^x dx$.

Following LIATE, choose $u = \sin(x)$, $dv = e^x dx$

$$\Rightarrow du = \cos(x) dx, \; v = e^x$$

$$\Rightarrow \int \sin(x) e^x dx = \sin(x) e^x - \underbrace{\int e^x \cos(x) dx}$$

We need to integrate by parts again for this!

$$\int \underbrace{\cos(x)}_{u} \underbrace{e^x dx}_{dv} = \underbrace{\cos(x)}_{u} \underbrace{e^x}_{v} - \int \underbrace{e^x}_{v} \underbrace{(-\sin(x))\, dx}_{du}$$

$$= \cos(x) e^x + \int e^x \sin(x)\, dx$$

$$\Rightarrow \int \sin(x) e^x dx = \sin(x) e^x - \int \cos(x) e^x dx$$

$$= \sin(x) e^x - \cos(x) e^x - \underbrace{\int e^x \sin(x)\, dx}.$$

Looks like we didn't make progress, because of this term.

However... what if we move all the $\int \sin(x) e^x dx$ to one side:

$$\Rightarrow 2 \int \sin(x) e^x \, dx = \sin(x) e^x - \cos(x) e^x$$

$$\Rightarrow \int \sin(x) e^x \, dx = \boxed{\tfrac{1}{2} e^x (\sin(x) - \cos(x)) + C} \checkmark$$

This trick is often useful for integrating things with sin/cos.

Definite Integrals

To compute definite integrals, always:

1. First fully compute the indefinite integral.
2. Then plug in bounds at end, using Fund. Thm. Calculus.

Doing it in this order ensures you get right answer!

E.g. Compute $\int_0^{\sqrt{\pi}} x \sin(x^2) \, dx$.

1. Using u-substitution, we get
$$\int x \sin(x^2) \, dx = -\tfrac{1}{2} \cos(x^2) + C$$
2. Then using FTC, we get
$$\int_0^{\sqrt{\pi}} x \sin(x^2) \, dx = \left[-\tfrac{1}{2} \cos(x^2)\right]_0^{\sqrt{\pi}} = -\tfrac{1}{2} \cos(\pi) + \tfrac{1}{2} \cos(0)$$
$$= -\tfrac{1}{2} \cdot -1 + \tfrac{1}{2} \cdot 1 = \boxed{1}$$

E.g. Compute $\int_0^{\pi} x \sin(x) \, dx$.

1. Using integration by parts, we get
$$\int x \sin(x) \, dx = -x \cos(x) + \sin(x) + C$$
2. Then using FTC, we get
$$\int_0^{\pi} x \sin(x) \, dx = \left[-x \cos(x) + \sin(x)\right]_0^{\pi}$$
$$= (-\pi \cdot \cos(\pi) + \sin(\pi)) - (-0 \cdot \cos(0) + \sin(0)) = -\pi \cdot -1 = \boxed{\pi}$$

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Trigonometric Integrals §7.2

Integration by parts can let us compute integrals of powers of trig functions, like $\cos^2(x)$. recall: this means $(\cos(x))^2$

E.g. Compute $\int \cos^2(x)\,dx$.

Our only real choice is $u = \cos(x)$, $dv = \cos(x)\,dx$
$du = -\sin(x)\,dx$, $v = \sin(x)$

$$\Rightarrow \int \cos^2(x)\,dx = \cos(x)\sin(x) - \int \sin(x)(-\sin(x))\,dx$$
$$= \cos(x)\sin(x) + \int \sin^2(x)\,dx.$$

How do we deal with this term? We could try integration by parts again, but won't help...

Instead, recall Pythagorean Identity: $\boxed{\cos^2(x)+\sin^2(x)=1}$,

which can also be written $\sin^2(x) = 1-\cos^2(x)$.

$$\Rightarrow \int \cos^2(x)\,dx = \cos(x)\sin(x) + \int \sin^2(x)\,dx$$
$$= \cos(x)\sin(x) + \int (1-\cos^2(x))\,dx$$
$$= \cos(x)\sin(x) + \int 1\,dx - \int \cos^2(x)\,dx$$
$$= \cos(x)\sin(x) + x - \int \cos^2(x)\,dx$$

Now we do same trick of moving $\int \cos^2(x)dx$ terms to one side:

$$\Rightarrow 2\int \cos^2(x)\,dx = \cos(x)\sin(x) + x$$
$$\Rightarrow \int \cos^2(x)\,dx = \boxed{\tfrac{1}{2}(\cos(x)\sin(x)+x) + C} \checkmark$$

Exercise: Compute $\int \sin^2(x)\,dx$ similarly.

A different approach to integrating powers of trig functions is using u-substitution instead...

E.g. Compute $\int \cos^3(x)\,dx$.

We use u-sub., with $u = \sin(x) \Rightarrow du = \cos(x)\,dx$.

The trick is to again use Pyth. Identity $\cos^2(x) = 1 - \sin^2(x)$.

$$\Rightarrow \int \cos^3(x)\,dx = \int \cos^2(x)\cdot\cos(x)\,dx = \int (1-\sin^2(x))\cdot\cos(x)\,dx$$

sub. in u and du $\rightsquigarrow$ $= \int (1-u^2)\,du = u - \frac{1}{3}u^3 + C$

$$= \boxed{\sin(x) - \tfrac{1}{3}\sin^3(x) + C} \checkmark$$

Can even mix powers of sin & cos this way:

E.g. Compute $\int \sin^5(x)\cos^2(x)\,dx$.

We have $\sin^5(x)\cos^2(x) = (\sin^2(x))^2\cos(x)\sin(x)$

$= (1-\cos^2(x))^2\cos(x)\sin(x)$

so letting $u = \cos(x) \Rightarrow du = -\sin(x)\,dx$ we get

$$\int \sin^5(x)\cos^2(x)\,dx = \int (1-\cos^2(x))^2\cos^2(x)\sin(x)\,dx$$

$$= \int (1-u^2)^2u^2(-du) = -\int u^2 - 2u^4 + u^6\,du$$

$$= -\left(\frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7}\right) + C$$

$$= \boxed{-\tfrac{1}{3}\cos^3(x) + \tfrac{2}{5}\cos^5(x) - \tfrac{1}{7}\cos^7(x) + C} \checkmark$$

From these examples we see the goal is to make

1. exactly one factor of $\sin(x)$ or $\cos(x)$ next to dx
2. everything else in terms of "opposite" $\cos(x)$ or $\sin(x)$ using Pyth. Identity $\cos^2(x) + \sin^2(x) = 1$
3. so you set $u = \cos(x)$ or $\sin(x)$ and $du = -\sin(x)\,dx$ or $\cos(x)\,dx$.

This strategy will let you compute $\int \sin^m(x)\cos^n(x)\,dx$ whenever at least one of m or n is odd. //

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Recall the two other trig functions $\tan(x)$ and $\sec(x)$:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \sec(x) = \frac{1}{\cos(x)}$$

Last semester we saw, using quotient rule, that

$$\boxed{d/dx(\tan(x)) = \frac{1}{\cos^2(x)} = \sec^2(x)} \quad \boxed{d/dx(\sec(x)) = \frac{\sin(x)}{\cos^2(x)} = \tan(x)\sec(x)}$$

We also can divide the Py. Identity by $\cos^2(x)$ to get:

$$\boxed{\sec^2(x) = 1 + \tan^2(x)}$$

We can then compute $\int \tan^m(x)\sec^n(x)\,dx$ using a similar u-sub. strategy:

E.g: Compute $\int \tan^6(x)\sec^4(x)\,dx$.

We have $\tan^6(x)\sec^4(x) = \tan^6(x)\sec^2(x)\sec^2(x)$

So that with $u = \tan(x)$ $= \tan^6(x)(1+\tan^2(x))\sec^2(x)$

$\Rightarrow du = \sec^2(x)\,dx$

We get $\int \tan^6(x)\sec^4(x)\,dx = \int \tan^6(x)(1+\tan^2(x))\sec^2(x)\,dx$

$$= \int u^6(1+u^2)\,du = \int u^6 + u^8\,du$$

$$= \frac{u^7}{7} + \frac{u^9}{9} + C = \boxed{1/7\tan^7(x) + 1/9\tan^9(x) + C} \checkmark$$

Exercise: Compute $\int \tan^5(x)\sec^7(x)\,dx$ using this strategy.

Hint: $\tan^5(x)\sec^7(x) = \tan^4(x)\sec^4(x)\tan(x)\sec(x)$

$$= (\sec^2(x)-1)^2\sec^4(x)\underbrace{\tan(x)\sec(x)}_{d/dx(\sec(x))}.$$

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Trigonometric Substitution §7.3

It is often possible to compute integrals involving $(a^2 - x^2)$ where $a \in \mathbb{R}$, by writing $x = a \cdot \sin(u)$ so that

$$(a^2 - x^2) = (a^2 - a^2 \sin^2(u))$$
$$= a^2 (1 - \sin^2(u)) = a^2 \cos^2(u).$$

E.g: Let's compute $\int \frac{1}{\sqrt{1-x^2}} dx$ this way.

Write $x = \sin(u) \Rightarrow dx = \cos(u)\, du$ so that

$$\int \frac{1}{\sqrt{1-x^2}} dx = \int \frac{1}{\sqrt{1-\sin^2(u)}} \cos(u)\, du = \int \frac{1}{\sqrt{\cos^2(u)}} \cos(u)\, du$$
$$= \int \frac{1}{\cos(u)} \cos(u)\, du = \int du = u + C$$

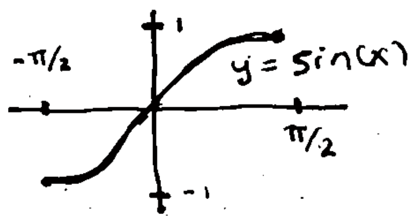
This is the answer in terms of u, but we want the x answer.

Since $x = \sin(u) \Rightarrow u = \arcsin(x)$ (also written $\sin^{-1}(x)$)

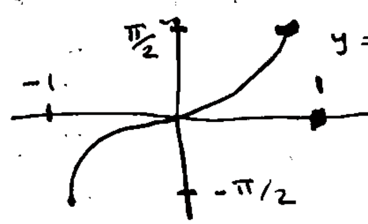
Thus, $\boxed{\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + C.}$

Recall: arcsin is the inverse of the sin function:

$y = \arcsin(x) \Leftrightarrow \sin(y) = x$ for $-\pi/2 \leq x \leq \frac{\pi}{2}$



$\Rightarrow$
(flip over $y=x$)



e.g. since $\sin(\pi/2) = 1$ we have $\arcsin(1) = \pi/2$
since $\sin(\pi/6) = 1/2$ we have $\arcsin(1/2) = \pi/6$, etc...

Note: with this technique of "trig substitution" we do a u-substitution, but it is a "reverse" u-substitution where we write $x = f(u)$ instead of $u = f(x)$.
This is okay as long as you do $dx = f'(u)\, du$.
Also sometimes we use θ instead of u.

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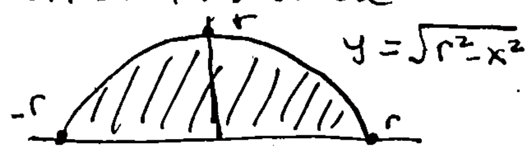
Trig substitution is useful when working with circles:

E.g. Let's compute the area of circle of radius r with an integral.
The equation of this circle is $x^2+y^2=r^2$.



$\Leftarrow$ area under curve = semi-circle

So area of circle of radius r $= 2\cdot\int_{-r}^{r}\sqrt{r^2-x^2}\,dx$, which we solve using trig sub.

Since we see r^2-x^2 we set $x=r\cdot\sin(\theta) \Rightarrow dx = r\cos(\theta)\,d\theta$.

$$\Rightarrow \int\sqrt{r^2-x^2}\,dx = \int\sqrt{r^2-r^2\sin^2(\theta)}\;r\cos(\theta)\,d\theta$$

$$= \int r\sqrt{1-\sin^2(\theta)}\;r\cos(\theta)\,d\theta = r^2\int\cos(\theta)\cdot\cos(\theta)\,d\theta$$

$$= r^2\int\cos^2(\theta)\,d\theta = r^2\cdot\tfrac{1}{2}(\cos(\theta)\sin(\theta)+\theta)$$

recall: we found $\int\cos^2(x)\,dx$ before!

Picture of relationship between x & θ:

$\sin(\theta)=\frac{x}{r}$ $\quad$ $\theta=\arcsin(\frac{x}{r})$

$\cos(\theta)=\frac{\sqrt{r^2-x^2}}{r}$

$$\Rightarrow \int\sqrt{r^2-x^2}\,dx = \frac{r^2}{2}\left(\frac{\sqrt{x^2-r^2}}{r}\cdot\frac{x}{r}+\arcsin(\tfrac{x}{r})\right)$$

$$= \frac{x}{2}\sqrt{r^2-x^2}+\frac{r^2}{2}\arcsin(\tfrac{x}{r})$$

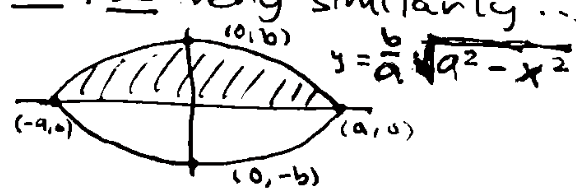
$$\Rightarrow \tfrac{1}{2}\text{ area of circle} = \int_{-r}^{r}\sqrt{r^2-x^2}\,dx = \left[\frac{x}{2}\sqrt{r^2-x^2}+\frac{r^2}{2}\arcsin(\tfrac{x}{r})\right]_{-r}^{r}$$

$$= \left(0+\frac{r^2}{2}\arcsin(1)\right)-\left(0+\frac{r^2}{2}\arcsin(-1)\right) = \frac{r^2}{2}\left(\frac{\pi}{2}-\frac{-\pi}{2}\right) = \boxed{\frac{r^2\pi}{2}}$$

E.g. We can find area of an ellipse very similarly...

Ellipse equation:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



$$\Rightarrow \tfrac{1}{2}\text{ area of ellipse} = \int_{-a}^{a}\frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$= \frac{b}{a}\left(\int_{-a}^{a}\sqrt{a^2-x^2}\,dx\right) = \frac{b}{a}\left(\frac{a^2\pi}{2}\right) = \boxed{\frac{ab\pi}{2}}$$

take $x=a\sin\theta$
$dx = a\cos\theta\,d\theta$
and do same steps as in circle example.

Sometimes we see expressions of the form (a^2+x^2) in our integral. In that case, we take $x = a\cdot\tan(\theta) \Rightarrow dx = a\sec^2(\theta)\,d\theta$ because of identity $\boxed{1+\tan^2(\theta) = \sec^2(\theta)}$

E.g. Let's compute $\int \frac{1}{1+x^2}\,dx$ this way.

We let $x = \tan(\theta) \Rightarrow dx = \sec^2(\theta)\,d\theta$ so that

$$\int \frac{1}{1+x^2}\,dx = \int \frac{1}{1+\tan^2(\theta)}\sec^2(\theta)\,d\theta$$

$$= \int \frac{1}{\sec^2(\theta)}\sec^2(\theta)\,d\theta = \int d\theta = \theta + C$$

and since $x = \tan(\theta) \Rightarrow \theta = \arctan(x)$ (inverse function for tan)

$$\Rightarrow \boxed{\int \frac{1}{1+x^2}\,dx = \arctan(x) + C.}$$

E.g. Now let's compute $\int \frac{1}{(1+x^2)^2}\,dx$ with a trig sub.

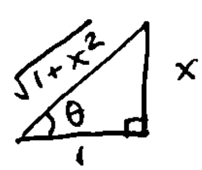
Again, let $x = \tan(\theta) \Rightarrow dx = \sec^2(\theta)\,d\theta$ so that

$$\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{1}{(1+\tan^2(\theta))^2}\sec^2(\theta)\,d\theta = \int \frac{1}{(\sec^2(\theta))^2}\sec^2(\theta)\,d\theta$$

$$= \int \frac{1}{\sec^2(\theta)}\,d\theta = \int \cos^2(\theta)\,d\theta = \frac{1}{2}\left(\cos(\theta)\sin(\theta) + \theta\right) + C$$

as we just saw...

Picture of relationship between x & θ:



$\tan(\theta) = x$

$\sin(\theta) = \frac{x}{\sqrt{1+x^2}}$ $\quad \theta = \arctan(x)$

$\cos(\theta) = \frac{1}{\sqrt{1+x^2}}$

$$\Rightarrow \int \frac{1}{(1+x^2)^2}\,dx = \frac{1}{2}\left(\frac{x}{\sqrt{1+x^2}}\,\frac{1}{\sqrt{1+x^2}} + \arctan(x)\right) + C$$

$$= \boxed{\frac{1}{2}\left(\frac{x}{1+x^2} + \arctan(x)\right) + C.}$$

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