

Rank and characteristic generating functions of upper homogeneous posets

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based on
arXiv: 2202.12103

these slides are on my website:

SamuelHopkins.com

A finite poset P is graded of rank n if we can write $P = P_0 \sqcup P_1 \sqcup \dots \sqcup P_n$ such that every maximal chain looks like $p_0 < p_1 < \dots < p_n$, $p_i \in P_i$.

Rank function $\text{rk}: P \rightarrow \mathbb{N}$ is $\text{rk}(p) = i \Leftrightarrow p \in P_i$.

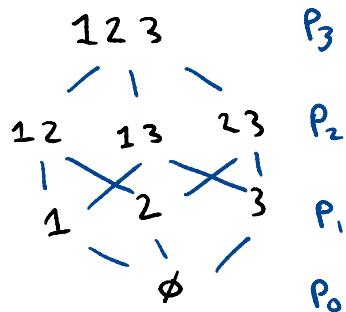
Rank generating polynomial is

$$F_P(x) = \sum_{i \geq 0} \#P_i \cdot x^i = \sum_{p \in P} x^{\text{rk}(p)}$$

e.g.

P = Boolean lattice B_n of subsets of $[n] = \{1, 2, \dots, n\}$

$$F_{B_n}(x) = \sum_{k=0}^n \binom{n}{k} x^k = (1+x)^n$$



Suppose further that P has minimum $\hat{0}$.

Then the characteristic polynomial is

$$\chi_p(x) = \sum_{p \in P} \mu(\hat{0}, p) x^{\text{rk}(p)} \quad \text{M\"obius fn.!}$$

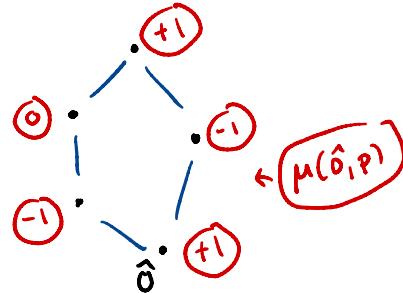
Rmk: Often reciprocal polynomial $\sum_p \mu(\hat{0}, p) x^{n-\text{rk}(p)}$ is studied instead, since it better matches e.g. char. poly. of a matroid.

Review of Möbius functions:

The Möbius function $\mu(p, q)$ for $p \leq q \in P$ is the inverse of the zeta function $\zeta(p, q) = 1$ in $\mathcal{I}(P)$.

It can be computed recursively by:

↑
incidence
algebra!



$$\cdot \mu(p, p) = 1$$

$$\cdot \mu(p, q) = - \sum_{p \leq r < q} \mu(p, r) \text{ for } p < q.$$

Recall Möbius inversion:

$$f, g : P \rightarrow R \quad \begin{matrix} \text{(abelian} \\ \text{gp.)} \end{matrix} \quad f(p) = \sum_{q \geq p} g(q) \Leftrightarrow g(p) = \sum_{q \geq p} \mu(p, q) f(q)$$

Also "Philip Hall's Thm.":

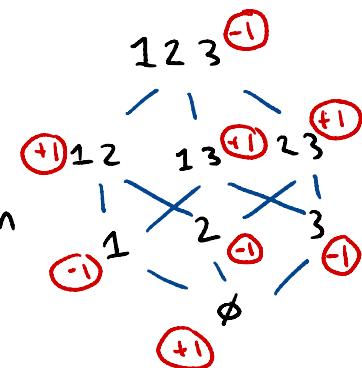
$$\mu(p, q) = C_0 - C_1 + C_2 - C_3 + \dots$$

← Euler characteristic!

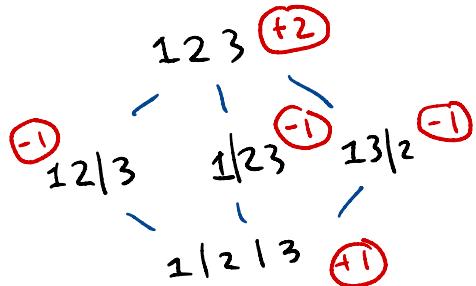
where $C_i = \# \text{ chains } p = p_0 < p_1 < \dots < p_i = q$.

e.g. In B_n , $\mu(s, T) = (-1)^{\# T \setminus s}$

so $\chi_{B_n}(x) = \sum_{k=0}^n (-1)^k \binom{n}{k} x^k = (1-x)^n$



e.g. $P = \text{lattice } \mathbb{T}\Pi_n$ of
set partitions of $[n]$
ordered by refinement



$$F_{\mathbb{T}\Pi_n}(x) = \sum_{k=0}^{n-1} S(n, n-k) x^k$$

↑ Stirling # of 2nd kind

$$\chi_{\mathbb{T}\Pi_n}(x) = (1-x)(1-2x)\cdots(1-(n-1)x) = \sum_{k=0}^{n-1} \Delta(n, n-k) x^k$$

↑ Stirling # of 1st kind

Now we try to do same for certain infinite posets.

(Infinite) poset P is \mathbb{N} -graded if $P = P_0 \sqcup P_1 \sqcup P_2 \sqcup \dots$
s.t. every max'l chain is $p_0 < p_1 < p_2 < \dots$, $p_i \in P_i$.

As before, rank fn. $\text{rk}: P \rightarrow \mathbb{N}$ is $\text{rk}(p) = i \Leftrightarrow p \in P_i$.

Say P is finite type if $\# P_i < \infty \forall i$.

Then we can form the rank generating function

$$F_P(x) = \sum_{i \geq 0} \# P_i x^i = \sum_{p \in P} x^{\text{rk}(p)}$$

Suppose further P has minimum $\hat{0}$.

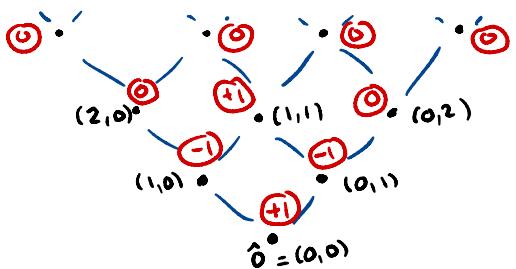
Then define the characteristic generating fn.

$$\chi_p(x) = \sum_{p \in P} \mu(\hat{0}, p) x^{\text{rk}(p)}$$

e.g. $P = \mathbb{N}^2$

$$F_P(x) = \sum_{k \geq 0} (k+1) x^k$$

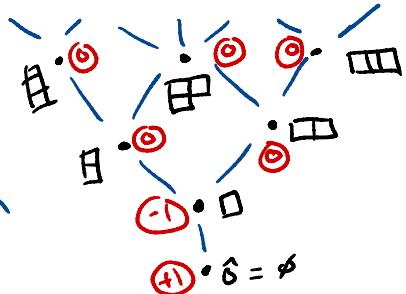
$$X_P(x) = 1 - 2x + x^2$$



e.g. $P = \mathcal{Y}$, Young's lattice of partitions

$$F_P(x) = \sum_{n \geq 0} P(n) x^n$$

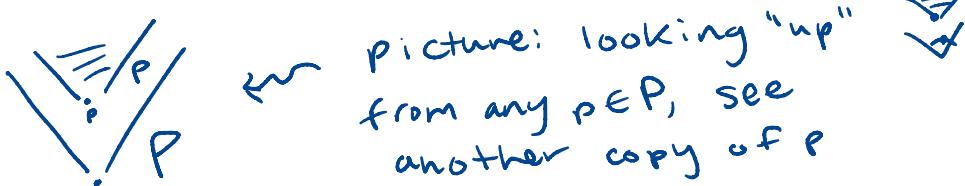
$$X_P(x) = 1 - x$$



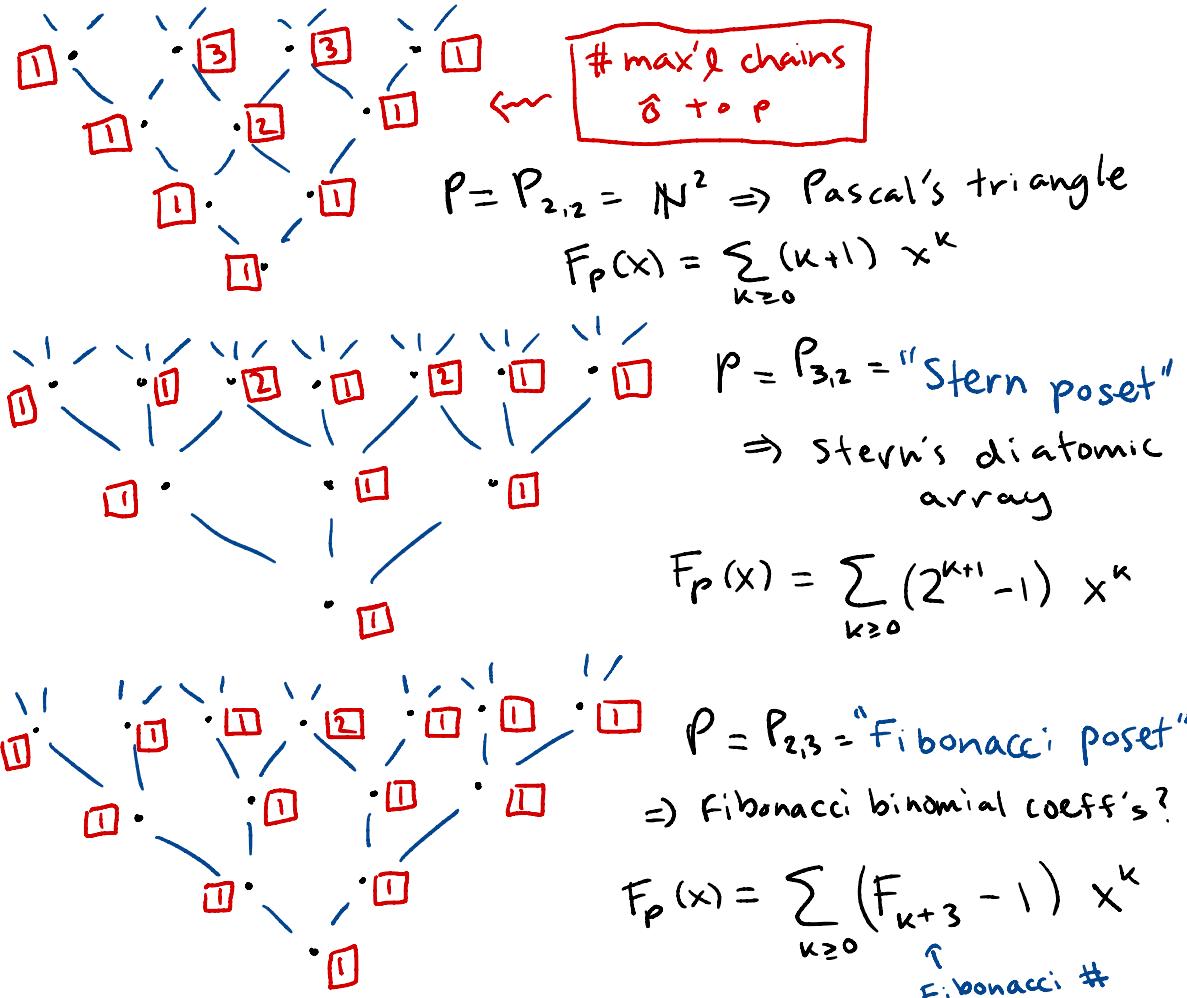
Stanley recently introduced following class of posets:

Say P is **upper homogeneous** or "**upho**" if

$V_p \cong P \forall p \in P$, where $V_p = \{q : q \geq p\}$ is the **principal order filter** generated by p .



Stanley was interested in certain analogs of **Pascal's triangle** coming from some **planar upho posets**:
(= planar Hasse diagram)



An MIT PRIMES gp. looked at rank g.f.'s of upho posets:

Thm (Gao - Guo - Seetharaman - Seidel, 2022)

For a planar upho poset P , have $F_P(x) = \frac{1}{Q(x)}$

where $Q(x) = 1 - bx + c_2 x^2 + c_3 x^3 + \dots + c_n x^n$

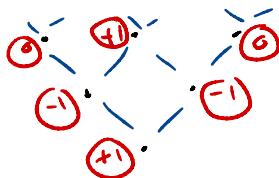
is a polynomial w/ $b, c_2, \dots, c_n \in \mathbb{N}$, $c_2 + \dots + c_n - b \leq 0$.

Thm (GGSS, 2022)

There are uncountably many $F_p(x)$'s among all upho posets P. \curvearrowleft read: very bad!

I recently noticed that for upho P , there's a very simple relationship between $F_p(x)$ and $X_p(x)$:

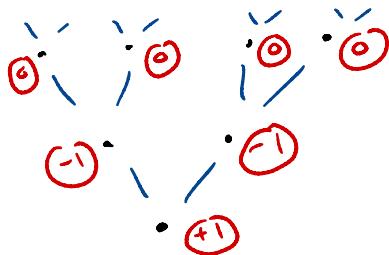
e.g.



$$P = \mathbb{N}^2$$

$$F_p(x) = \sum_{k \geq 0} (k+1) x^k = \frac{1}{(1-x)^2}$$

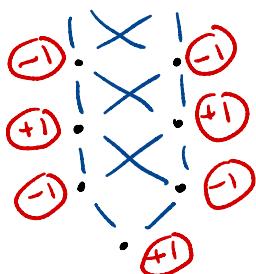
$$X_p(x) = 1 - 2x + x^2 = (1-x)^2$$



$$P = \text{"binary tree"}$$

$$F_p(x) = \sum_{k \geq 0} 2^k x^k = \frac{1}{1-2x}$$

$$X_p(x) = 1 - 2x$$



$$P = \text{"bowtie poset"}$$

$$F_p(x) = 1 + \sum_{k \geq 1} 2 x^k = \frac{1+x}{1-x}$$

$$X_p(x) = 1 + \sum_{k \geq 1} (-1)^k 2 x^k = \frac{1-x}{1+x}$$

Thm (H., 2022)

For P upho, we have $F_p(x) = X_p(x)^{-1}$.

Pf: Set $f(p) = x^{\text{rk}(p)}$ and $g(p) = \sum_{q \geq p} f(q) = \sum_{q \geq p} x^{\text{rk}(q)}$ if $p \in P$.

By Möbius inversion,

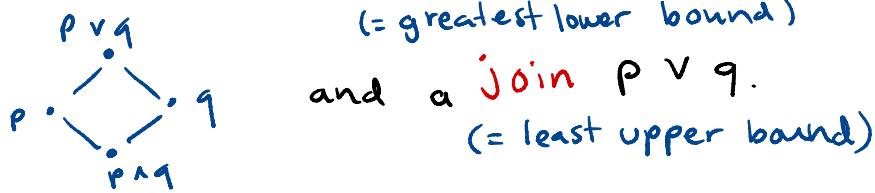
$$1 = x^0 = f(\hat{0}) = \sum_{q \geq \hat{0}} \mu(\hat{0}, q) g(q),$$

but since $\vee_q \simeq P \setminus q$, $g(q) = x^{\text{rk}(q)} \cdot F_p(x)$, so

$$1 = \sum_{q \in P} \mu(\hat{0}, q) x^{\text{rk}(q)} F_p(x) = F_p(x) \cdot \chi_p(x)$$



Recall that a **lattice** is a poset P where every pair $p, q \in P$ has a **meet** $p \wedge q$,



Lattices have good Möbius fn.'s, e.g.:

Cor (to Rota's crosscut thm)

If P is a finite lattice, w/ minimum $\hat{0}$,

maximum $\hat{1}$, and **atoms** a_1, \dots, a_m

(= covers of $\hat{0}$: $a_1 \swarrow \searrow \dots \swarrow \searrow a_m$)

then $\mu(\hat{0}, \hat{1}) = 0$ if $\hat{1} \neq a_1 \wedge \dots \wedge a_m$.

↑
join of all atoms

Hence... If P is an upho lattice, then

$$F_P(x) = X_P(x)^{-1} = X_{P'}(x)^{-1}, \text{ where}$$

P' = Subposet of P below joins of atoms

is a finite graded lattice ...

so $F_P(x)^{-1} = X_{P'}(x)$ is a polynomial.

(In fact, can replace "lattice" by "meet semilattice" above. And all planar upho P are meet semilattices.)

Note Upgo lattice P is not determined by this finite graded lattice P' :

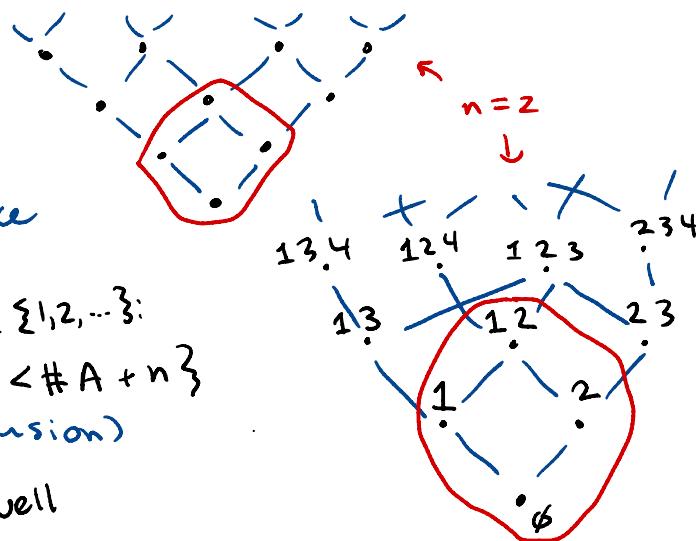
e.g. $P = \mathbb{N}^n$

$$\Rightarrow P' = B_n$$

Boolean lattice

but... $P = \left\{ \text{finite } A \subseteq \{1, 2, \dots\} : \max(A) < \#A + n \right\}$
(order = inclusion)

$$\Rightarrow P' = B_n \text{ as well}$$



Nevertheless, still interesting to ask...

maybe at
an REU...

Q: When can we extend a finite graded lattice P' to an upho lattice P ?

e.g. Just saw 2 ways for $P' = B_n \dots$

Note Since $F_p(x) = X_{P'}(x)^{-1}$, we need

$X_{P'}(x)^{-1}$ to have nonnegative coeff.'s,

so a "random" finite lattice P' does not extend.

e.g. fix $n \geq 1$ and a prime p and set

$P =$ subgroups of \mathbb{Z}^n of index p^k ($k \geq 0$)
ordered by reverse inclusion.

Then P is an upho lattice

and $P' = B_n(p)$

= lattice of subspaces

of $(\mathbb{Z}/p\mathbb{Z})^n$, with

$$X_{P'}(x) = (1-x)(1-px)\cdots(1-p^{n-1}x) = F_p(x)^{-1}.$$



Rmk: More generally, R is a local P.I.D. with finite residue field, then submodules of R^d of finite colength (ordered by rev. inclusion) give an upho lattice.

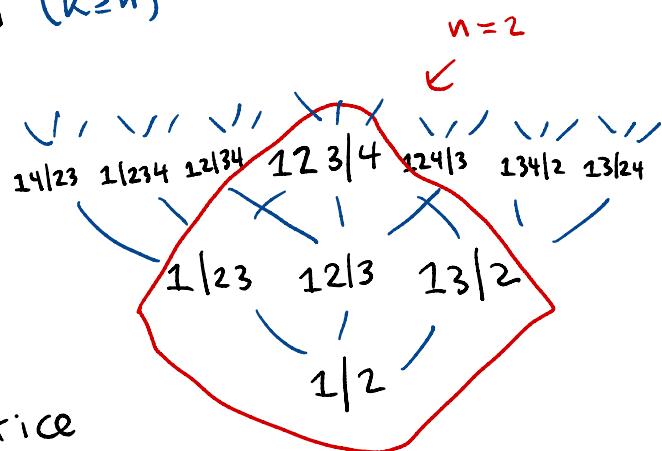
Conjecture (Stanley)

All* **modular** upho lattices are of this form.

$$\begin{aligned} & \stackrel{\leftarrow}{\text{rk}} (\text{rk}(p) + \text{rk}(q)) \\ & = \text{rk}(p \cap q) + \text{rk}(p \vee q) \quad \forall p, q \in P \end{aligned}$$

e.g. Fix $n \geq 1$ and set
 $P = \text{partitions of } [k] \quad (k \geq n)$
 into n blocks

where $\pi_1 \leq \pi_2$ if
 $\forall B_1 \in \pi_1, \exists B_2 \in \pi_2$
 s.t. $B_1 \subseteq B_2$.



Then P is an upho lattice

and $P' = \prod_{n+1}$, so that
 set partition lattice

$$F_p(x) = \sum_{k \geq n} S(k, n) x^{k-n} = \frac{1}{(1-x)(1-2x)\cdots(1-nx)} = x_{\prod_{n+1}}(x)^{-1}.$$

Finite poset P , graded of rank n , w/ min. $\hat{0}$ & max. $\hat{1}$
 is called **uniform** if for all $i = 0, 1, \dots, n$
 there exists a single poset Q_i such that
 $V_p = [\rho, \hat{1}] \cong Q_i \quad \forall p \in P \text{ with } \text{rk}(p) = n-i.$

e.g. $P = B_n, B_n(q), \Pi_{n+1}$ are all uniform
 (with $Q_i = B_i, B_i(q), \Pi_{i+1}$ respectively).

For such uniform P , define the **Whitney #'**s
 of the 1st and 2nd kind $v(i, j)$ and $V(i, j)$

by $X_{Q_i}(x) = \sum_{j=0}^i v(i, i-j) x^j$

and $F_{Q_i}(x) = \sum_{j=0}^i V(i, i-j) x^j$

Thm (Dowling, 1971) \leftarrow See Stanley EC1
 exercise 3.130

For P uniform, the matrices

$$\left(v(i, j) \right)_{\substack{i=0, \dots, n \\ j=0, \dots, n}} \text{ and } \left(V(i, j) \right)_{\substack{i=0, \dots, n \\ j=0, \dots, n}}$$

are inverses of one another.

e.g. For $P = \text{Th}_{n+1}$, says the Stirling # matrices $(S(i,j))$ and $(S(i,j))$ are inverse:

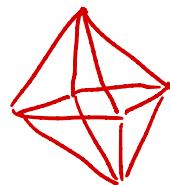
$$n=4 \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -6 & 11 & -6 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{bmatrix} \quad \text{well-known!}$$

Uniform posets are "as close" to being upho as finite posets can be, and the Whitney # thm. "feels similar" to the thm. that $F_p(x)$ and $X_p(x)$ are inverse for upho P .

So trying to extend uniform lattices P' to upho lattices P might make sense...

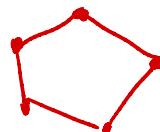
e.g. Can you extend ...

P' = face lattice of ?
the cross polytope



a Type B analog of B_n . Can also do Type B analog of Th_n .

P' = bond lattice of ?
odd cycle graph



or more generally... certain uniform matroids?

Thank You!

And can't wait to see
all of you **in person**
at **OPAC 2022** in
just a couple weeks ...



SECRET BONUS SLIDE:

Call a sequence P_0, P_1, P_2, \dots **ultra uniform** if

- each P_n is a finite n -graded poset,
- for any atom $a \in P_n$, have $[a, \hat{1}_{P_n}] \cong P_{n-1}$,
(this recursively implies all P_n are uniform)
- for each $n \geq 1$, have fixed coatom $t_n \in P_n$
and isomorphism $\varphi_n : P_{n-1} \cong [\hat{0}_{P_n}, t_n]$
(e.g. $\varphi_6(13|2|45) = 13|2|45|6$ for Π_n)

Let P_0, P_1, \dots be an ultra uniform seq. of lattices,

and let $P_\infty = \bigcup_{n \geq 0} P_n \quad / \quad p = \varphi_n(p)$

(e.g. $\Pi_\infty =$ set partitions of $\{1, 2, 3, \dots\}$
s.t. all but ∞ -many blocks are singletons)

Then I think P_∞ should be upho lattice...
but will not be finite type (e.g. #atoms = ∞)

So for $k \geq 1$, let $P_\infty^{(k)} = \{p \in P_\infty : p \in P_{k+1}(p) + k\}$

Then I think $P_\infty^{(k)}$ should be finite type upho lattice
w/ finite sublattice below join of atoms = P_{k+1}
↑ get students to (dis)prove this! //