Order polynomial product formulas and poset dynamics

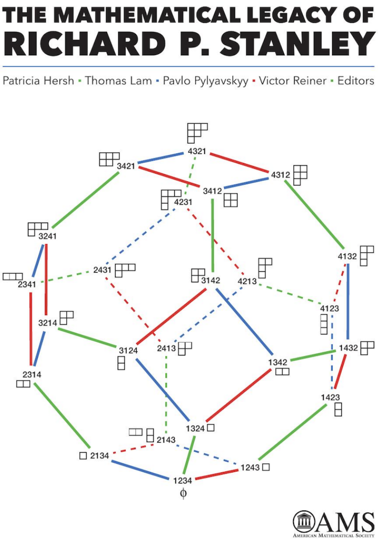
Statistical and Dynamical Combinatorics:
Jim Propp's $2^{\binom{4}{2}}$th Birthday Conference
Massachusetts Institute of Technology

Sam Hopkins

Howard University

June 26th, 2024

Alternate talk title: "Lessons I learned from Jim Propp"



Lessons I Learned from Richard Stanley

James Propp

to Richard Stanley, on the occasion of his 70th birthday

Abstract. I will share with the reader what I have learned from Richard Stanley and the ways in which he has contributed to research in combinatorics conducted by me and my collaborators.

1. Two big ideas

The biggest lesson I learned from Richard Stanley's work is, *combinatorial objects want to be partially ordered!* By which I mean: if you are trying to understand some class of combinatorial objects, you should look at ways of putting a partial order on the class, in hopes of finding one that has especially nice properties. You won't always succeed, but when you do, the gains are likely to more than justify the effort.

A related lesson that Stanley has taught me is, *combinatorial objects want to belong to polytopes!* That is: If you can find a way to view the objects you're interested in as the vertices or facets of a polytope, or as the faces (of all dimensions) of a polytope, or as the lattice points inside a polytope, then geometrical methods will give you a lot of combinatorial insight.

2. Tilings and perfect matchings

The two articles of Stanley's that had the greatest impact on my research were [**23**] and [**21**], which deal respectively with rhombus tilings of hexagons (Stanley calls them plane partitions whose three-dimensional diagram fits inside a box) and domino tilings of rectangles (Stanley, taking the dual point of view, calls them dimer covers).

Figure 1(a) shows one of the 20 ways to tile a regular hexagon of side-length 2 using twelve unit-rhombus tiles; Figure 1(b) shows the associated perfect matching of the graph whose edges correspond to allowed positions of the tiles, with vertices corresponding to triangular "half-tiles".

Figure 2(a) shows one of the 36 ways to tile a square of side-length 4 using eight 1-by-2 rectangular tiles (dominos); Figure 2(b) shows the associated dimer cover (or perfect matching) of the graph whose edges correspond to allowed positions of the tiles, with vertices corresponding to square half-tiles.

Lessons we all learned from Jim Propp

- Mathematical mentoring is important

  REACH, Tilings Research Group, ...

- Mathematical collaboration is important

  email forums (domino, robbins, DAC), workshops (AIM, BIRS), ...

- Mathematical outreach is important

  Mathematical Enchantments, National Museum of Mathematics, ...

- Mathematical names are important

  Aztec diamond, domino shuffling, rotor-routing, ...

- And above all...

  mathematics is about exploration!

One big idea

1. Two big ideas

The biggest lesson I learned from Richard Stanley's work is, *combinatorial objects want to be partially ordered!* By which I mean: if you are trying to understand some class of combinatorial objects, you should look at ways of putting a partial order on the class, in hopes of finding one that has especially nice properties. You won't always succeed, but when you do, the gains are likely to more than justify the effort.

A related lesson that Stanley has taught me is, *combinatorial objects want to belong to polytopes!* That is: If you can find a way to view the objects you're interested in as the vertices or facets of a polytope, or as the faces (of all dimensions) of a polytope, or as the lattice points inside a polytope, then geometrical methods will give you a lot of combinatorial insight.

The biggest lesson I learned from Jim Propp's work is,
combinatorial objects want to be acted on by dynamical operators!

Plane partitions

A *$a \times b$ plane partition* is an $a \times b$ array of nonnegative integers that are weakly decreasing in rows and columns.

Let $\mathcal{PP}^m(a \times b) := \{a \times b \text{ plane partitions with entries} \leq m\}$:

$$\begin{array}{|c|c|c|c|c|c|} \hline 5 & 3 & 3 & 2 & 2 & 1 \\ \hline 4 & 2 & 1 & 1 & 1 & 0 \\ \hline 4 & 1 & 1 & 1 & 0 & 0 \\ \hline 3 & 1 & 1 & 0 & 0 & 0 \\ \hline \end{array} \in \mathcal{PP}^5(4 \times 6)$$

Theorem (MacMahon's formula (c.1915) for plane partitions in a box)

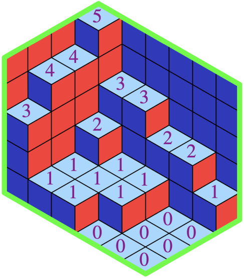
$$\#\mathcal{PP}^m(a \times b) = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{m+i+j-1}{i+j-1}$$

Other guises of plane partitions

Plane partitions have a beautiful $3D$ representation as stacks of boxes:

5	3	3	2	2	1
4	2	1	1	1	0
4	1	1	1	0	0
3	1	1	0	0	0

$\Leftrightarrow$



In this way they correspond to *lozenge tilings* of hexagonal regions of the triangular lattice (c.f. David–Tomei 1989).

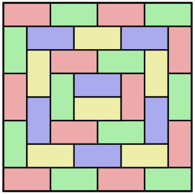
Plane partitions are also intimately related to the *representation theory of classical groups*, because $\mathcal{PP}^m(a \times b)$ indexes a basis of the irreducible representation V^λ of $\mathfrak{sl}(a+b)$ with highest weight $\lambda = m^a$.

An aside on tilings, another central theme of Jim's work

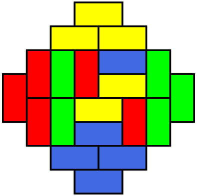
Many other tiling problems also have beautiful enumerative formulas.

Theorem (Fisher–Temperley 1961, Kasteleyn 1961)

$$\#\textit{domino tilings of } m \times n \textit{ rectangle} = \prod_{j=1}^{\lceil m/2 \rceil} \prod_{k=1}^{\lceil n/2 \rceil} \left(4\cos^2 \frac{\pi j}{m+1} + 4\cos^2 \frac{\pi k}{n+1} \right)$$



See Jim's blog post:
"My Life with Aztec Diamonds"



Theorem (Elkies–Kuperberg–Larsen–Propp 1992)

$$\#\textit{domino tilings of order } n \textit{ Aztec diamond} = 2^{n(n+1)/2}$$

Toggling plane partitions

(Piecewise-linear) toggling of an entry of a plane partition $\pi \in \mathcal{PP}^m(a \times b)$ does the following:

$$\begin{array}{|c|c|c|}\hline & v & \\ \hline w & \boxed{x} & y \\ \hline & z & \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|c|c|c|}\hline & v & \\ \hline w & x' & y \\ \hline & z & \\ \hline \end{array} \qquad \max(y,z) \_ \, x \, \_\_ \, x' \, \_ \min(v,w)$$

with $x' = \max(y,z) + \min(v,w) - x$.[1] Toggling an entry is an involution.

Let $t_{i,j} \colon \mathcal{PP}^m(a \times b) \to \mathcal{PP}^m(a \times b)$ be toggling at entry (i,j).

Example

$$t_{2,2}\left(\begin{array}{|c|c|c|}\hline 5 & 4 & 1 \\ \hline 3 & \boxed{1} & 1 \\ \hline \end{array}\right) \in \mathcal{PP}^5(2 \times 3) = \begin{array}{|c|c|c|}\hline 5 & 4 & 1 \\ \hline 3 & 3 & 1 \\ \hline \end{array}$$

[1] If v or w don't exist, treat them as m; if y or z don't exist, treat them as 0.

Rowmotion on rectangular plane partitions

(Piecewise-linear) rowmotion $\mathrm{Row}\colon \mathcal{PP}^m(a \times b) \to \mathcal{PP}^m(a \times b)$ consists of toggling *all* the entries, in sequence, from bottom-right to top-left.

Example

Let's compute rowmotion of a plane partition $\pi \in \mathcal{PP}^5(2 \times 3)$:

$$\pi = \begin{array}{|c|c|c|}\hline 5 & 4 & 1 \\ \hline 3 & 1 & 1 \\ \hline \end{array} \xrightarrow{t_{2,3}} \begin{array}{|c|c|c|}\hline 5 & 4 & 1 \\ \hline 3 & 1 & 0 \\ \hline \end{array} \xrightarrow{t_{2,2}} \begin{array}{|c|c|c|}\hline 5 & 4 & 1 \\ \hline 3 & 2 & 0 \\ \hline \end{array} \xrightarrow{t_{1,3}} \begin{array}{|c|c|c|}\hline 5 & 4 & 3 \\ \hline 3 & 2 & 0 \\ \hline \end{array}$$

$$\xrightarrow{t_{2,1}} \begin{array}{|c|c|c|}\hline 5 & 4 & 3 \\ \hline 4 & 2 & 0 \\ \hline \end{array} \xrightarrow{t_{1,2}} \begin{array}{|c|c|c|}\hline 5 & 4 & 3 \\ \hline 4 & 2 & 0 \\ \hline \end{array} \xrightarrow{t_{1,1}} \begin{array}{|c|c|c|}\hline 4 & 4 & 3 \\ \hline 4 & 2 & 0 \\ \hline \end{array} = \mathrm{Row}(\pi)$$

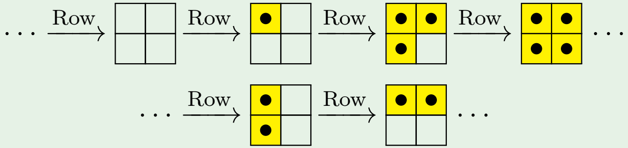
Piecewise-linear rowmotion was introduced by Einstein–Propp, 2013.

Combinatorial rowmotion

Let $\mathcal{J}(a \times b)$ be the set of sub-Young diagrams inside the $a \times b$ rectangle. We have $\mathcal{J}(a \times b) \simeq \mathcal{PP}^1(a \times b)$ via the indicator function.

(Combinatorial) rowmotion $\mathrm{Row}\colon \mathcal{J}(a \times b) \to \mathcal{J}(a \times b)$ sends $\mu \subseteq a \times b$ to the smallest Young diagram containing the minimal elements of $(a \times b) \setminus \mu$.

Example



Combinatorial rowmotion was introduced by Brouwer–Schrijver, 1974. Cameron–Fon-der-Flaass, 1995 showed how to write combinatorial rowmotion as a composition of combinatorial toggles, and Einstein–Propp's contribution was generalizing their description to the piecewise-linear realm.

Periodicity for rowmotion on rectangular plane partitions

Example

One rowmotion orbit in $\mathcal{PP}^5(2 \times 3)$ is:

$$\cdots \xrightarrow{\text{Row}} \begin{array}{|c|c|c|}\hline 5&4&1\\\hline 3&1&1\\\hline\end{array} \xrightarrow{\text{Row}} \begin{array}{|c|c|c|}\hline 4&4&3\\\hline 4&2&0\\\hline\end{array} \xrightarrow{\text{Row}} \begin{array}{|c|c|c|}\hline 5&4&3\\\hline 4&4&2\\\hline\end{array} \xrightarrow{\text{Row}} \begin{array}{|c|c|c|}\hline 3&3&2\\\hline 2&1&1\\\hline\end{array} \xrightarrow{\text{Row}} \begin{array}{|c|c|c|}\hline 4&1&1\\\hline 2&1&0\\\hline\end{array} \cdots$$

Theorem (Grinberg–Roby 2015; conjectured by Einstein–Propp)

The order of $\mathrm{Row}\colon \mathcal{PP}^m(a \times b) \to \mathcal{PP}^m(a \times b)$ *is* $a + b$.

Note: Case $m = 1$ (combinatorial rowmotion) due to Brouwer–Schrijver.

From Kirillov–Berenstein, 1995 and Striker–Williams, 2009 it follows that dynamics are same as *rectangular semistandard Young tableaux promotion*, for which order $a + b$ is known from Schützenberger, Haiman, Rhoades, ...

Rowmotion on plane partitions of other shapes

For a Young diagram λ, a *plane partition of shape λ* is a filling of its boxes with nonnegative integers that are weakly decreasing in rows and columns.

All of the prior constructions make sense for arbitrary shapes λ. But for a "random" λ, rowmotion will not behave well like it does for rectangles.

Example

For $\lambda = (4, 2, 2)$ and for

$$\pi = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 0 & 0 \\ \hline 1 & 1 & \multicolumn{2}{c}{} \\ \cline{1-2} 1 & 0 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \in \mathcal{PP}^1(\lambda)$$

the rowmotion orbit of π has 17 elements. Things get worse from there.

But Grinberg–Roby showed that rowmotion behaves well also for *staircases* and *shifted staircases*, and Johnson–Liu, 2023 showed same for *trapezoids*.

When does rowmotion behave well? The order polynomial...

What distinguishes the shapes with good rowmotion behavior?

For any shape λ, the function $\Omega_\lambda(m) = \#\mathcal{PP}^m(\lambda)$ is a *polynomial* in m, called the *order polynomial* of λ. It was introduced by Richard Stanley.

For example, MacMahon's formula says $\Omega_{a\times b}(m) = \prod_{i=1}^{a}\prod_{j=1}^{b}\frac{m+i+j-1}{i+j-1}$; in particular, all roots of $\Omega_{a\times b}(m)$ are *integers*!

Example

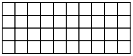
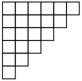
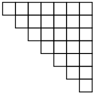
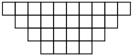
For $\lambda = (4,2,2)$,

$$\Omega_\lambda(m) = \frac{1}{720}(m+1)(m+2)^2(m+3)^2(m+4)(m^2+5m+5),$$

which has an irreducible quadratic factor.

Empirically, shapes λ with good rowmotion behavior are those with *order polynomial product formulas*, i.e., with all roots of $\Omega_\lambda(m)$ in $\mathbb{Z}$ (or $\frac{1}{2}\mathbb{Z}$).

Shapes with order polynomial product formulas

Rectangle 	$\prod_{i=1}^{a}\prod_{j=1}^{b}\frac{m+i+j-1}{i+j-1}$	$\mathfrak{sl}(n)$	MacMahon c. 1915
Staircase 	$\prod_{1\leq i\leq j\leq n}\frac{2m+i+j}{i+j}$	$\mathfrak{sp}(2n)$	Proctor 1988 *"symmetric, self-complementary plane partitions"*
Shifted staircase 	$\prod_{1\leq i\leq j\leq n}\frac{m+i+j-1}{i+j-1}$	$\mathfrak{so}(2n+1)$	Conj. MacMahon 1896, Andrews/Macdonald c. 1977 *"symmetric plane partitions"*
Shifted Trapezoid 	$\prod_{i=1}^{k}\prod_{j=1}^{2n-k+1}\frac{m+i+j-1}{i+j-1}$	$\mathfrak{sp}(2n)$	Proctor 1983 *"transpose-complementary plane partitions"*

More dynamics: promotion of standard Young tableaux

Standard Young Tableaux (SYTs) of a shape λ with n boxes are bijective fillings of the boxes with $1, \ldots, n$, increasing in rows and columns.

Promotion, $\mathrm{Pro}\colon \mathcal{SYT}(\lambda) \to \mathcal{SYT}(\lambda)$, is the following invertible operation on these SYTs:

- Delete the entry 1.
- Slide boxes into the resulting hole.
- Decrement all entries.
- Fill the hole with n.

Example

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} \xrightarrow{\text{del. } 1} \begin{array}{|c|c|c|}\hline \bullet & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} \xrightarrow{\text{slide}} \begin{array}{|c|c|c|}\hline 2 & 4 & 5 \\ \hline 3 & 6 & \bullet \\ \hline \end{array} \xrightarrow{\text{decr.}} \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & \bullet \\ \hline \end{array} \xrightarrow{\text{fill } 6} \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array} = \mathrm{Pro}(T)$$

Along with *evacuation*, defined by Schützenberger to study *RSK algorithm*.

When does promotion of SYT behave well? Same shapes!

Promotion behaves chaotically for most shapes, but:

Theorem

- *(Schützenberger 1977) For λ a rectangle, order of Pro is n*
- *(Edelman–Greene 1987) For λ a staircase, order of Pro is $2n$.*
- *(Haiman 1992) For λ a shifted trapezoid or shifted double staircase, order of Pro is n.*
- *(Haiman–Kim 1992) These are the* **only** *four families of shapes with good promotion behavior.*

Remarkably, these are (basically) the same families of shapes that have good plane partition rowmotion behavior!

The main heuristic

To summarize, we have seen that:

shapes with good dynamical properties
$=$ **shapes with order polynomial product formulas**

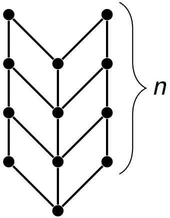
All of the constructions (order polynomial, plane partitions, rowmotion, SYTs, promotion, ...) we discussed make sense for arbitrary *finite posets*. We then put forward the following heuristic:

posets with good dynamical properties
$=$ **posets with order polynomial product formulas**

What's really cool about this heuristic is that it seems like a powerful tool for mathematical exploration *in both directions!*

Using the heuristic to find good dynamics

Let $V(n)$ be the following poset:



$V(n)$ is *not* a shape, but it has an order polynomial product formula:

Theorem (Kreweras–Niederhausen '81)

$$\Omega_{V(n)}(m) = \frac{\prod_{i=1}^{n}(m+1+i)\prod_{i=1}^{2n}(2m+i+1)}{(n+1)!(2n+1)!}$$

The heuristic lead us to:

Theorem (H.–Rubey 2022)

$\mathrm{Pro}\colon \mathcal{L}(V(n)) \to \mathcal{L}(V(n))$ *has order* $2n$*.*

Here $\mathcal{L}(P)$ is set of *linear extensions* of a poset P, the analog of SYTs.

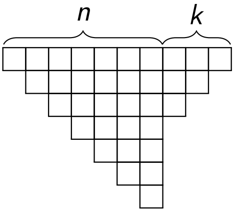
Theorem (Adenbaum 2023)

$\mathrm{Row}\colon \mathcal{PP}^m(V(n)) \to \mathcal{PP}^m(V(n))$ *has order* $2(n+2)$*.*

Note: for $m = 1$ (combinatorial rowmotion) see Plante–Roby, 2024.

Using the heuristic to find good enumeration

The *shifted double staircase* shape is the following λ:

$\lambda =$ 

Recall that this was one of the families Haiman showed has good behavior of promotion of SYTs.

The heuristic lead us to:

Theorem (H.–Lai 2021, Okada 2021)

$$\Omega_\lambda(m) = \prod_{1\leq i\leq j\leq n} \frac{m+i+j-1}{i+j-1} \cdot \prod_{1\leq i\leq j\leq k} \frac{m+i+j}{i+j}$$

Our proof with Lai is based on tilings and *Kuo condensation*.

Okada's proof is algebraic and uses Proctor's *"intermediate" symplectic group* characters.

The cyclic sieving phenomenon

Is there any connection between enumeration and dynamics? Yes, the CSP!

We can ask for even more refined information about a cyclic action than its period, such as its *orbit structure*. A compact way to record orbit structure of a cyclic action is via the *cyclic sieving phenomenon (CSP)*:

Definition (Reiner–Stanton–White 2004)

For $C = \langle c \rangle$ a $\mathbb{Z}/n$-action on a finite set X, and $f(q) \in \mathbb{N}[q]$ a polynomial, we say (X, C, f) *exhibits CSP* if for all k,

$$\# X^{c^k} = f(\zeta^k)$$

with $\zeta := e^{2\pi i/n}$ a primitive nth root of unity.

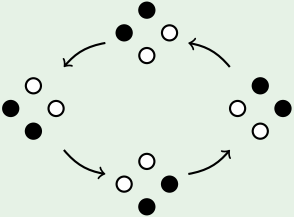
When the sieving polynomial $f(q)$ has a product formula, a CSP result implies that *every* symmetry class has a product formula.

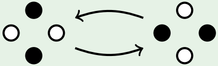
Cyclic sieving example: rotation of subsets

Theorem (Reiner–Stanton–White 2004)

$(\{k\text{-subsets of } \{1,\ldots,n\}\}, \langle i \mapsto i+1 \mod n\rangle \simeq \mathbb{Z}/n, f)$ exhibits CSP, where $f(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^{k} \frac{(1-q^{m+1-i})}{(1-q^i)}$ is the q-binomial coefficient.

Example ($n = 4$, $k = 2$)





$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4 \Rightarrow \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{q:=1} = 6$, $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_{q:=\pm i} = 0$, $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_{q:=-1} = 2$

Cyclic sieving for rectangular rowmotion and promotion

Theorem (Rhoades 2010)

$(\mathcal{PP}^m(a \times b), \langle \mathrm{Row} \rangle \simeq \mathbb{Z}/(a+b), f)$ *exhibits CSP, where*

$$f(q) = \sum_{\pi \in \mathcal{PP}^m(a \times b)} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{(1-q^{i+j+m-1})}{(1-q^{i+j-1})},$$

is MacMahon's size generating function of plane partitions in a box.

Note: case $m = 1$ recovers the subset rotation CSP.

Theorem (Rhoades 2010)

$(\mathcal{SYT}(a \times b), \langle \mathrm{Pro} \rangle \simeq \mathbb{Z}/ab, f)$ *exhibits CSP, where*

$$f(q) = \sum_{T \in \mathcal{SYT}(a \times b)} q^{\mathrm{maj}(T)} = \prod_{i=1}^{ab} (1-q^i) \cdot \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{1}{(1-q^{i+j-1})},$$

is a q-analog of the hook length formula for these SYTs.

General cyclic sieving conjecture from order polynomial

Let P be one of these posets whose order polynomial $\Omega_P(m)$ has a product formula. Define

$$\Omega_P(m;q) = \prod_{\alpha \text{ root of } \Omega_P(m)} \frac{(1-q^{\kappa(m-\alpha)})}{(1-q^{-\kappa\alpha})}, \qquad (\kappa{:=}\min\{k{>}0\colon k\alpha{\in}\mathbb{Z}\forall\alpha\})$$

the natural q-analog of $\Omega_P(m)$. *(Not obviously a polynomial!)*

Conjecture (H. 2020)

$(\mathcal{PP}^m(P), \langle \mathrm{Row} \rangle \simeq \mathbb{Z}/\kappa(\mathrm{rk}(P)+2), \Omega_P(m;q))$ *exhibits CSP (if P graded).*

Define

$$e(P;q) = (1-q^{\kappa})(1-q^{2\kappa})\cdots(1-q^{\#P\cdot\kappa}) \lim_{m\to\infty} \Omega_P(m;q),$$

the natural q-analog of $e(P) = \#\mathcal{L}(P)$, the number of linear extensions.

Conjecture (H. 2020)

$(\mathcal{L}(P), \langle \mathrm{Pro} \rangle \simeq \mathbb{Z}/\kappa \cdot \#P, e(P;q))$ *exhibits CSP.*

What's behind all the good behavior? Algebra!

Often sophisticated tools from algebra are used to prove these CSP results.

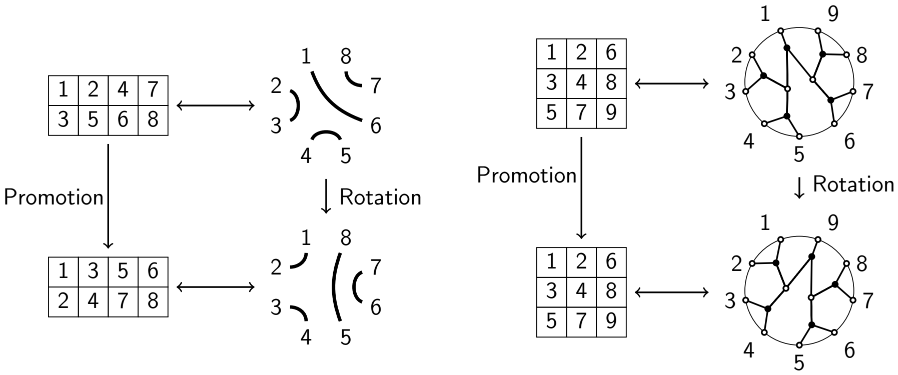
For example, Rhoades used *canonical bases* from Kazhdan–Lusztig theory to prove the rectangular pro/rowmotion CSPs. Subsequent work has connected promotion to *crystals* and tensor invariants, the *monodromy* action on the Wronski map, canonical bases from *cluster algebras*, etc.

The posets themselves often have direct connection to Lie algebras, being either *root posets* or *minuscule posets*. The Weyl dimension formula often provides the product formula for $\Omega_P(m)$.

Still, we are far from a unified algebraic explanation for all known examples.

Another perspective: pro/rowmotion as rotation

In the best situations, we can find a *diagrammatic model* (like *noncrossing matchings*, *webs*, ...) where pro/rowmotion corresponds to rotation:



Again, we are far from a unified "rotation model" for all known examples.

Further questions

- Can we find a unified algebraic explanation for all the known examples of posets with good behavior? What about a unified rotation model?
- Can we find *direct implications* between the properties in the heuristic (pro/rowmotion dynamics & order polynomial product formula)? This would upgrade the heuristic to an actual theorem!
- How do other aspects of poset dynamics come into play here? For example, the *homomesy* phenomenon, where natural statistics have constant orbit averages. Or, further lifts of the actions to the *birational* and *noncommutative* realms.

Happy Birthday Jim!

- These slides are on my website at:

  `https://www.samuelfhopkins.com/docs/jim_talk.pdf`.
- See my survey `arXiv:2006.01568` for references.