UPHO LATTICES

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Abstract. A poset is called upper homogeneous, or "upho," if every principal order filter of the poset is isomorphic to the whole poset. We study (finite type $\mathbb{N}$-graded) upho lattices, with an eye towards their classification.

Any upho lattice has associated to it a finite graded lattice called its core, which determines its rank generating function. We investigate which finite graded lattices arise as cores of upho lattices, providing both positive and negative results. On the one hand, we show that many well-studied finite lattices do arise as cores, and we present combinatorial and algebraic constructions of the upho lattices into which they embed. On the other hand, we show there are obstructions which prevent many finite lattices from being cores.

We also focus on some important subvarieties of lattices: namely, distributive and modular lattices. It is easy to show that the only upho distributive lattices are $\mathbb{N}^n$. Upho modular lattices are more interesting. Stanley observed that the poset of full rank submodules of a free module over a discrete valuation ring with finite residue field gives an upho modular lattice, and furthermore conjectured that essentially all upho modular lattices are of this form. We refute Stanley's conjecture by producing an exceptional example of an upho modular lattice whose core is a non-Desarguesian finite projective plane.

1. Introduction

Symmetry is a fundamental theme in mathematics. A close cousin of symmetry is self-similarity, where a part resembles the whole. In this paper, we study certain partially ordered sets that are self-similar in a precise sense. Namely, a poset is called *upper homogeneous*, or "*upho*," if every principal order filter of the poset is isomorphic to the whole poset. In other words, a poset $\mathcal{P}$ is upho if, looking up from each element $p \in \mathcal{P}$, we see another copy of $\mathcal{P}$. Upho posets were introduced recently by Stanley [10, 12]. We believe they are a natural and rich class of posets which deserve further attention.

Upho posets are infinite. In order to be able to apply the tools of enumerative and algebraic combinatorics, we need to impose some finiteness condition on the posets we consider. Thus, we restrict our attention to finite type $\mathbb{N}$-graded posets. These are the infinite posets $\mathcal{P}$ that possess a rank function $\rho\colon \mathcal{P} \to \mathbb{N}$ for which we can form the rank generating function

$$F(\mathcal{P};x) := \sum_{p\in\mathcal{P}} x^{\rho(p)}.$$

Henceforth, upho posets are assumed finite type $\mathbb{N}$-graded unless otherwise specified.

The main problem we pursue in this paper is the following.

Problem 1.1. Classify upho lattices.

Problem 1.1 is likely a hard problem, perhaps even impossible. But let us explain why there is some hope of making progress on this problem. It was shown by Gao et al. [4] that there are uncountably many different rank generating functions of (finite type $\mathbb{N}$-graded) upho posets. This prevents us from being able to say much about upho posets in general. However, the situation is different for *lattices*: in [6] we showed that the rank generating function of a upho lattice is the inverse of a polynomial with integer coefficients.

More precisely, we made the following observation about the rank generating function of an upho lattice. Let $\mathcal{L}$ be an upho lattice, and let $L := [\hat{0}, a_1 \vee \cdots \vee a_r] \subseteq \mathcal{L}$ denote the interval in $\mathcal{L}$ from its minimum $\hat{0}$ to the join of its atoms $a_1, \ldots, a_r$. We refer to the finite graded lattice L as the *core* of the upho lattice $\mathcal{L}$. We showed in [6] that

$$F(\mathcal{L}; x) = \widetilde{\chi}(L; x)^{-1}, \tag{1.1}$$

where $\widetilde{\chi}(L; x) = \sum_{p \in L} \mu(\hat{0}, p) x^{\rho(p)}$ is the (reciprocal) characteristic polynomial of L. In this way, the core of an upho lattice determines its rank generating function.[1]

The core does not determine the upho lattice completely. In other words, there are different upho lattices with the same core. Nevertheless, to resolve Problem 1.1 we would certainly need to answer the following question.

Question 1.2. Which finite graded lattices are cores of upho lattices?

Question 1.2 can be thought of as a kind of tiling problem: our goal is to tile an infinite, fractal lattice $\mathcal{L}$ using copies of some fixed finite lattice L, or show that no such tiling is possible. In addressing Question 1.2 here, we provide both positive and negative results.

On the positive side, we show that many well-studied families of finite graded lattices are cores of upho lattices. Our first major result is the following.

Theorem 1.3. *Any member of a uniform sequence of supersolvable lattices is the core of some upho lattice.*

Supersolvable lattices were introduced by Stanley in [8]. They have a recursive structure which makes them amenable to inductive arguments. Examples of uniform sequences of supersolvable lattices include:

- the finite Boolean lattices B_n, i.e., the lattices of subsets of $\{1, 2, \ldots, n\}$;
- the q-analogues $B_n(q)$ of B_n, i.e., the lattices of $\mathbb{F}_q$-subspaces of $\mathbb{F}_q^n$;
- the partition lattices Π_n, i.e., the lattices of set partitions of $\{1, 2, \ldots, n\}$;
- the Type B partition lattices Π_n^B, i.e., the intersection lattices of the Type B_n Coxeter hyperplane arrangements;
- (generalizing the previous two items) the "Dowling lattices" [3, 2] $Q_n(G)$ associated to any finite group G.

Hence, these are all cores of upho lattices. We discuss these examples in detail, providing explicit descriptions of the upho lattices for which they are cores.

[1] In fact, since the flag f-vector of any upho poset is determined by its rank generating function (see [12, §3]), the core of an upho lattice determines its entire flag f-vector.

In addition to combinatorial constructions, we also explore algebraic constructions of upho lattices. Monoids provide one algebraic source of upho lattices, as the following lemma explains.

Lemma 1.4. *(c.f. [4, Lemma 5.1]) Let $M = \langle S \mid R \rangle$ be a finitely generated monoid whose defining relations are homogeneous. Let $\preceq$ denote the partial order of left-divisibility on M. If M is left cancellative and every pair of elements in M have a least common multiple (with respect to $\preceq$), then $(M, \preceq)$ is an upho lattice.*

A class of monoids satisfying the conditions of Lemma 1.4 are the (homogeneous) Garside monoids [1]. The core of a Garside monoid consists of its simple elements. Examples of lattices of simple elements in Garside monoids include:

- the weak order of a finite Coxeter group W;
- the noncrossing partition lattice of a finite Coxeter group W.

Hence, these are also cores of upho lattices. We review these examples coming from Garside and Coxeter theory in detail.

On the negative side, we show that there are various obstructions which prevent arbitrary finite graded lattices from being realized as cores of upho lattices. There are restrictions on the characteristic polynomial of the lattice coming from the equation (1.1). There are also some structural obstructions, requiring the lattice to be partly self-similar. These obstructions allow us to show, for instance, that the following plausible candidates cannot in fact be realized as cores:

- the face lattice of the n-dimensional cross polytope, for $n \geq 3$;
- the bond lattice of the cycle graph C_n, for $n \geq 4$;
- (generalizing the previous item) the lattice of flats of the uniform matroid $U(n, k)$, for $2 < k < n$.

The upshot is that Question 1.2 is quite subtle: it can be difficult to recognize when a given finite graded lattice is the core of an upho lattice. Many well-behaved finite lattices are cores of upho lattices, but many too are not.

In the absence of a definitive resolution of Problem 1.1, we might instead try to classify subvarieties of upho lattices. Two of the most important subvarities of lattices are the distributive lattices and the modular lattices. Upho distributive lattices are not very interesting. Their classification, which we state in the following theorem, is a simple consequence of Birkhoff's representation theorem for locally finite distributive lattices (see [9, Proposition 3.4.3]).

Theorem 1.5. *The only upho distributive lattices are $\mathbb{N}^n$.*

Upho modular lattices, on the other hand, are quite interesting. Stanley observed the following source of upho modular lattices, which we review in detail.

Theorem 1.6 (Stanley)**.** *Let R be a discrete valuation ring (DVR) with residue field $\mathbb{F}_q$. Then the poset of finite colength R-submodules of R^n ordered by reserve inclusion is an upho modular lattice.*

Note that the core of $\mathbb{N}^n$ is B_n, while the core of the upho modular lattice of finite colength submodules of R^n, for R a DVR with residue field $\mathbb{F}_q$, is $B_n(q)$. Stanley conjectured, moreover, that these examples give essentially all the upho modular lattices (see [4, Conjecture 1.1]). We disprove Stanley's conjecture by producing the following exceptional example of an upho modular lattice, which cannot come from a DVR because its core is different from $B_n(q)$.

Theorem 1.7. *There is an upho modular lattice whose core is (the rank three modular lattice associated to) a non-Desarguesian finite projective plane.*

Specifically, following [5], we explain a way to produce an upho modular lattice from any (sufficiently symmetric) affine building. Then we appeal to [7], where a $\widetilde{A}_2$-building whose residue planes are all non-Desarguesian projective planes is constructed. These exceptional examples can only exist in rank three, so it remains possible that Stanley's conjecture on upho modular lattices is true in higher rank.

To conclude this introduction, we remark that the following questions, which we do not pursue here, are also naturally suggested by our work.

Question 1.8.

(1) For a finite graded lattice L, let $\kappa(L)$ denote the cardinality of the collection of (isomorphism classes of) upho lattices $\mathcal{L}$ with core L. How does $\kappa(L)$ behave? For example, is $\kappa(L)$: finite for each L; infinite for some L, but always countable; or uncountably infinite for some L?

(2) For a finite modular lattice L, let $\kappa_{\mathrm{mod}}(L)$ denote the cardinality of the collection of upho modular lattices $\mathcal{L}$ with core L. How does $\kappa_{\mathrm{mod}}(L)$ behave?

In short, part (1) of Question 1.8 asks how far apart Question 1.2 and Problem 1.1 are, and part (2) asks the same but for modular lattices.

The rest of the paper is structured as follows. In Section 2, we go over some definitions and preliminary results. In Section 3, we construct upho lattices from uniform sequences of supersolvable lattices. In Section 4, we explain how monoids give rise to upho lattices. In Section 5, we discuss obstructions to realizing a finite graded lattice as the core of an upho lattice. Finally, in Section 6, we explore distributive and modular upho lattices.

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2. Preliminaries

In this section, we review some basics regarding posets and upho posets. We generally stick to standard notation for posets, as laid out of instance in [9, §3]. We use $\mathbb{N} := \{0, 1, \ldots\}$ to denote the natural numbers, $\mathbb{Z}$ to denote the integers, $\mathbb{Q}$ the rationals, and $\mathbb{R}$ the reals.

2.1. **Poset basics.** Let $P = (P, \leq)$ be a poset. When working with multiple posets we write $\leq_P$ for clarity. We use standard conventions like writing $y \geq x$ to mean $x \leq y$, writing $x < y$ to mean $x \leq y$ and $x \neq y$, and so on. We also routinely identify any subset $S \subseteq P$ with the corresponding induced subposet $S = (S, \leq)$.

2.1.1. *Basic terminology.* An *interval* of P is a subset $[x, y] := \{z \in P : x \leq z \leq y\}$ for $x \leq y \in P$. The poset P is *locally finite* if every interval of P is finite.

For $x, y \in P$, we say x is *covered* by y, written $x \lessdot y$, if $x < y$ and there is no $z \in P$ with $x < z < y$. The *Hasse diagram* of P is the directed graph whose vertices are the elements of P with an edge from x to y when $x \lessdot y$. We draw the Hasse diagram of P in the plane, with x below y if there is an edge from x to y. A locally finite poset is determined by its Hasse diagram.

A *chain* of P is a totally ordered subset, i.e., a subset $C \subseteq P$ for which any two elements in C are comparable. An *antichain* of P is a subset $A \subseteq P$ for which any two elements in A are incomparable. We say a chain is *maximal* if it is maximal by inclusion among chains, and similarly for antichains.

An *order ideal* of P is a downwards-closed subset, i.e., a subset $I \subseteq P$ such that if $y \in I$ and $x \leq y$ then $x \in I$. An order ideal is *principal* if it is of the form $\Lambda_p := \{q \in P : q \leq p\}$ for some $p \in P$. Dually, an *order filter* is an upwards-closed subset, i.e., a subset $F \subseteq P$ such that if $x \in I$ and $x \leq y$ then $y \in I$. An order filter is *principal* if it is of the form $V_p := \{q \in P : p \leq q\}$ for some $p \in P$.

A *minimum* of P, which we always denote by $\hat{0} \in P$, is an element with $\hat{0} \leq x$ for all $x \in P$. Dually, a *maximum*, denoted $\hat{1} \in P$, is an element with $x \leq \hat{1}$ for all $x \in P$. Clearly, minimums and maximums are unique if they exist. If P has a minimum $\hat{0}$, then we call $x \in P$ an *atom* if $\hat{0} \lessdot x$. Dually, if P has a maximum $\hat{1}$, then we call $x \in P$ a *coatom* if $x \lessdot \hat{1}$.

2.1.2. *New posets from old.* The *dual poset* P^* of P is the poset with the same set of elements but with the opposite order, i.e., $x \leq_{P^*} y$ if and only if $y \leq_P x$. We say P is *self-dual* if it is isomorphic to its dual poset.

Now let Q be another poset. The *direct sum* $P+Q$ of P and Q is the poset whose set of elements is the (disjoint) union $P \cup Q$, with $x \leq_{P+Q} y$ if either $x, y \in P$ and $x \leq_P y$, or $x, y \in Q$ and $x \leq_Q y$. We say P is *connected* if it cannot be written as a nontrivial direct sum. For a positive integer $n \geq 1$, we denote the direct sum of n copies of P by nP.

The *direct product* $P \times Q$ of P and Q is the poset whose set of elements is the (Cartesian) product $P \times Q$, with $(p_1, q_1) \leq_{P \times Q} (p_2, q_2)$ if $p_1 \leq_P p_2$ and $q_1 \leq_Q q_2$. We say P is *indecomposable* if it cannot be written as a nontrivial direct product. For a positive integer $n \geq 1$, we denote the direct product of n copies of P by P^n.

2.1.3. *Möbius functions.* Suppose for the moment that P is locally finite, and let $\mathrm{Int}(P)$ denote the set of intervals of P. The *Möbius function* $\mu : \mathrm{Int}(P) \to \mathbb{Z}$ of P is defined recursively by

$$\begin{aligned} \mu(x, x) &= 1 && \text{for all } x \in P; \\ \mu(x, y) &= -\sum_{x \leq z < y} \mu(x, z) && \text{for all } x < y \in P, \end{aligned}$$

where we use the standard notational shorthand $\mu(x, y) = \mu([x, y])$. The most important application of Möbius functions is the Möbius inversion formula (see [9, §3.7]), a kind of generalization of the principle of inclusion-exclusion to any poset.

The Möbius function of a product of posets decomposes as a product of Möbius functions. In other words, we have

$$\mu_{P\times Q}((p_1,q_1),(p_2,q_2)) = \mu_P(p_1,p_2)\cdot \mu_Q(q_1,q_2),$$

for all $(p_1,q_1)\leq(p_2,q_2)\in P\times Q$ (see [9, Proposition 3.8.2]).

2.1.4. *Lattices.* For $x,y\in P$, an *upper bound* of x and y is a $z\in P$ with $x\leq z$ and $y\leq z$, and the *join* (or *least upper bound*) of x and y, denoted $x\vee y$, is the minimum among all upper bounds of x and y, if such a minimum exists. Dually, a *lower bound* of x and y is a $z\in P$ with $z\leq x$ and $z\leq y$, and the *meet* (or *greatest lower bound*) of x and y, denoted $x\wedge y$, is the maximum among all lower bounds of x and y, if such a maximum exists. If $x\vee y$ exists for every $x,y\in P$, then P is called a *join semilattice*. Dually, if $x\wedge y$ exists for every $x,y\in P$, then P is called a *meet semilattice*. The poset P is a *lattice* if it is both a join and meet semilattice.

Now let L be a lattice. The operations of $\vee$ and $\wedge$ are associative and commutative, and therefore for any finite, nonempty subset $S=\{x_1,\ldots,x_n\}\subseteq L$ we can set $\bigvee S := x_1\vee\cdots\vee x_n$ and $\bigwedge S := x_1\wedge\cdots\wedge x_n$. If L has a minimum $\hat{0}$ then by convention we set $\bigvee\varnothing := \hat{0}$, and dually if L has a maximum $\hat{1}$ we set $\bigwedge\varnothing := \hat{1}$. Note that a *finite* lattice L always has a minimum (namely, $\hat{0}=\bigwedge L$) and a maximum (namely, $\hat{1}=\bigvee L$) .

The lattice L is *distributive* if the operation of meet distributes over that of join, i.e., $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$ for all $x,y,z\in L$. It is well-known that this is also equivalent to join distributing over meet, i.e., $x\vee(y\wedge z)=(x\vee y)\wedge(x\vee z)$ for all $x,y,z\in L$.

Example 2.1. For any poset P, we use $J(P)$ to denote the poset of order ideals of P, ordered by inclusion. Then, $J(P)$ is always a distributive lattice, where the operations of join and meet are union and intersection, respectively. As a variant of this construction, we use $J_{\mathrm{fin}}(P)$ to denote the *finite* order ideals of P, which again always gives a distributive lattice. We note that $J(P+Q)=J(P)\times J(Q)$ and similarly $J_{\mathrm{fin}}(P+Q)=J_{\mathrm{fin}}(P)\times J_{\mathrm{fin}}(Q)$.

The lattice L is *modular* if whenever $a\leq b\in L$, we have $a\vee(x\wedge b)=(a\vee x)\wedge b$ for all $x\in L$. Observe that modularity is a weaker condition than distributivity: all distributive lattices are modular, but most modular lattices are not distributive.

We note that the product $L_1\times L_2$ of two lattices L_1 and L_2 remains a lattice, and similarly if we append the adjectives "distributive" or "modular." Also, any interval in a lattice is a lattice, and similarly if we append the adjectives "distributive" or "modular." (These properties all follow formally from the fact that lattices, distributive lattices, and modular lattices, are varieties in the sense of universal algebra.)

2.1.5. *Convention for finite versus infinite posets.* We will routinely work with both finite and infinite posets, although the posets will always be at least locally finite. For clarity, we will from now on use the following convention: normal script letters (like P and L) will denote finite posets, while caligraphic letters (like $\mathcal{P}$ and $\mathcal{L}$) will denote infinite posets.

2.2. **Finite graded posets.** Let P be a finite poset. For a nonnegative integer $n \geq 0$, we say that P is *n-graded* if P has a minimum $\hat{0}$, a maximum $\hat{1}$, and every maximal chain of P is of the form $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$. In this case, the *rank function* $\rho\colon P \to \mathbb{N}$ is defined by setting $\rho(x_i) := i$ for each element x_i of a maximal chain $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n$. Equivalently, the rank function is determined by the requirements that $\rho(\hat{0}) = 0$ and $\rho(y) = \rho(x) + 1$ whenever $x \lessdot y \in P$.

Example 2.2. For any positive integer $n \geq 1$, we denote $[n] := \{1, 2, \ldots, n\}$. We view $[n]$ as a poset, with the usual (total) order. This chain poset $[n]$ is the most basic example of a finite $(n-1)$-graded poset.

We say that the finite poset P is *graded* if it is n-graded for some n. In this case, we say that the *rank* of P is n and, slightly abusing notation, write $\rho(P) := n$. If P and Q are two finite graded posets, then their product $P \times Q$ is also graded of rank $\rho(P \times Q) = \rho(P) + \rho(Q)$. Also, any interval in a finite graded poset is graded. Specifically, if P is a finite graded poset, then the interval $[x, y]$ will be a finite $(\rho(y) - \rho(x))$-graded poset for any $x \leq y \in P$.

2.2.1. *Generating polynomials for finite graded posets.* Now assume that P is graded. The *rank generating polynomial* of P is

$$F(P; x) := \sum_{p \in P} x^{\rho(p)}.$$

The *characteristic polynomial* of P is

$$\chi(P; x) := \sum_{p \in P} \mu(\hat{0}, p)\, x^{\rho(P) - \rho(p)}.$$

The exponent of x in each term of the characteristic polynomial $\chi(P; x)$ records the corank $\rho(P) - \rho(p)$ of the element $p \in P$. Using the corank in the characteristic polynomial is very standard, but, for reasons that will become clear soon, we need a version of the characteristic polynomial where the exponent records the usual rank instead. Hence, we define the *reciprocal characteristic polynomial* of P to be

$$\widetilde{\chi}(P; x) := \sum_{p \in P} \mu(\hat{0}, p)\, x^{\rho(p)}.$$

Observe that $\widetilde{\chi}(P; x) = x^{\rho(P)} \cdot \chi_P(x^{-1})$.

These invariants of finite graded posets all play nicely with products. Namely, $F(P \times Q; x) = F(P; x) \cdot F(Q; x)$ and $\widetilde{\chi}(P \times Q; x) = \widetilde{\chi}(P; x) \cdot \widetilde{\chi}(Q; x)$.

2.2.2. *Finite graded lattices.* Here we are most interested in finite graded lattices.

Example 2.3. The rank n *(finite) Boolean lattice* B_n is the poset of subsets of $[n]$, ordered by inclusion. B_n is a finite graded lattice, with $\rho(S) = \#S$ for all $S \in B_n$. Its Möbius function is given by $\mu(S, T) = (-1)^{\#T \setminus S}$ for all $S \leq T \in B_n$. Hence, $F(B_n; x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1 + x)^n$ and $\widetilde{\chi}(B_n; x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k = (1 - x)^n$. These formulas can also be seen from the fact that $B_n = J(n\,[1]) = [2]^n$.

Example 2.4. Birkhoff's representation theorem for finite distributive lattices says that every finite distributive lattice L has the form $L = J(P)$ for a finite poset P. (In fact, P is the subposet of join-irreducible elements of L; see [9, §3.4].) Let $L = J(P)$ be a finite distributive lattice. Then L is graded, with $\rho(I) = \#I$ for $I \in J(P)$. Its Möbius function is given by

$$\mu(I, I') = \begin{cases} (-1)^{\#I' \setminus I} & \text{if } I' \setminus I \text{ is an antichain of } P; \\ 0 & \text{otherwise,} \end{cases}$$

for $I \leq I' \in J(P)$ (see [9, Example 3.9.6]). Hence, $F(L; x) = \sum_{I \in J(P)} x^{\#I}$ and $\widetilde{\chi}(L; x) = \sum_{I \subseteq \min(P)} (-x)^{\#I} = (1-x)^{\#\min(P)}$, where $\min(P)$ is the set of minimal elements of P. Observe how this example generalizes Example 2.3.

Example 2.4 explains that all finite distributive lattices are graded. More generally, all finite modular lattices are graded. In fact, a finite lattice L is modular if and only if L is graded and $\rho(p) + \rho(q) = \rho(p \vee q) + \rho(p \wedge q)$ for all $p, q \in L$ (see, for instance, [9, §3.3]).

Example 2.5. Let q be a prime power, and $\mathbb{F}_q$ the finite field with q elements. We denote by $B_n(q)$ the poset of $\mathbb{F}_q$-subspaces of the vector space $\mathbb{F}_q^n$, ordered by inclusion. This *subspace lattice* $B_n(q)$ is a finite modular lattice. It is also known as the *q-analogue of the Boolean lattice* B_n. Its rank function is $\rho(U) = \dim(U)$, and its Möbius function is $\mu(U, V) = (-1)^k q^{\binom{k}{2}}$ for $U \subseteq V$, where $k = \dim(V) - \dim(U)$ (see [9, Example 3.10.2]). Hence, $F(B_n(q); x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k$ and $\widetilde{\chi}(B_n(q); x) = \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = (1-x)(1-qx)(1-q^2x)\cdots(1-q^{n-1}x)$, where we use the standard notation for *q-number* $[n]_q \coloneqq \frac{(1-q^n)}{(1-q)} = 1 + q + q^2 + \cdots + q^{n-1}$, *$q$-factorial* $[n]_q! \coloneqq [n]_q \cdot [n-1]_q \cdots [1]_q$, and *$q$-binomial coefficient* $\begin{bmatrix} n \\ k \end{bmatrix}_q \coloneqq \frac{[n]_q!}{[k]_q! [n-k]_q!}$.

There are various weakenings of the modular property for finite lattices that are very interesting from a combinatorial point of view. A finite lattice L is called *(upper) semimodular* if it is graded and satisfies $\rho(p) + \rho(q) \geq \rho(p \vee q) + \rho(p \wedge q)$ for all $p, q \in L$. The finite lattice L is called *atomic* if every element is a join of atoms. The finite lattice L is *geometric* if it is both semimodular and atomic. For example, the modular lattices B_n and $B_n(q)$ are geometric, although most geometric lattices are not modular. Geometric lattices are intensely studied, because they are precisely the lattices of flats of matroids (see [9, §3.3]).

Example 2.6. The *partition lattice* Π_n is the poset of set partitions of $[n]$, ordered by reverse refinement. In other words, for two set partitions π and π' of $[n]$, we have $\pi \leq \pi'$ if every block $B \in \pi$ satisfies $B \subseteq B'$ for some block $B' \in \pi'$. The partition lattice is a geometric lattice of rank $\rho(\Pi_n) = n - 1$, and $\rho(\pi) = n - \#\pi$ for $\pi \in \Pi_n$. Hence, $F(\Pi_n; x) = \sum_{k=0}^{n} S(n, n-k) x^k$, where $S(n, k)$ are the *Stirling number of the second kind*. It is also well-known that $\widetilde{\chi}(\Pi_n; x) = (1-x)(1-2x)\cdots(1-(n-1)x) = \sum_{k=0}^{n} s(n, n-k) x^k$, where $s(n, k)$ are the *(signed) Stirling number of the first kind* (see [9, Example 3.10.4]).

2.3. **Infinite graded posets.** Let $\mathcal{P}$ be an infinite poset. We say $\mathcal{P}$ is *$\mathbb{N}$-graded* if $\mathcal{P}$ has a minimum $\hat{0}$ and every maximal chain of P is of the form $\hat{0} = x_0 \lessdot x_1 \lessdot x_2 \lessdot \cdots$. In this case, the *rank function* $\rho\colon \mathcal{P} \to \mathbb{N}$ is defined by setting $\rho(x_i) := i$ for each element x_i of a maximal chain $x_0 \lessdot x_1 \lessdot x_2 \lessdot \cdots$. Equivalently, the rank function is determined by the requirements that $\rho(\hat{0}) = 0$ and $\rho(y) = \rho(x) + 1$ whenever $x \lessdot y \in \mathcal{P}$.

Example 2.7. The set of natural numbers $\mathbb{N} = \{0, 1, \ldots\}$, with the usual order, is the most basic example of an $\mathbb{N}$-graded poset.

If $\mathcal{P}$ and $\mathcal{Q}$ are $\mathbb{N}$-graded, then $\mathcal{P} \times \mathcal{Q}$ is $\mathbb{N}$-graded. Also, any interval in a *locally finite* $\mathbb{N}$-graded poset is a finite graded poset. Specifically, if $\mathcal{P}$ is a locally finite $\mathbb{N}$-graded poset, then the interval $[x, y]$ will be a finite $(\rho(y) - \rho(x))$-graded poset for any $x \le y \in \mathcal{P}$.

2.3.1. *Generating functions for infinite graded posets.* Let $\mathcal{P}$ be an $\mathbb{N}$-graded poset. In order to define sensible analogs of the rank generating and characteristic polynomials for $\mathcal{P}$, we need to make a further finiteness assumption. So let us say that $\mathcal{P}$ is *finite type* if $\{p \in \mathcal{P}\colon \rho(p) = i\}$ is finite for each $i \in \mathbb{N}$. Observe that a finite type $\mathbb{N}$-graded poset is locally finite (but finite type is stronger than locally finite).

So now assume that $\mathcal{P}$ is a finite type $\mathbb{N}$-graded poset. Then we define the *rank generating function* of $\mathcal{P}$ to be

$$F(\mathcal{P}; x) = \sum_{p \in \mathcal{P}} x^{\rho(p)},$$

a formal power series in the variable x. And we define the *characteristic generating function* of $\mathcal{P}$ to be

$$\widetilde{\chi}(\mathcal{P}; x) = \sum_{p \in \mathcal{P}} \mu(\hat{0}, p)\, x^{\rho(p)},$$

again, a formal power series. Notice how we write $\widetilde{\chi}(\mathcal{P}; x)$ to emphasize that the characteristic generating function of an infinite poset $\mathcal{P}$ uses rank in the exponent, like the *reciprocal* characteristic polynomial $\widetilde{\chi}(P; x)$ of a finite poset P.

Again, these invariants play nicely with products: $F(\mathcal{P} \times \mathcal{Q}; x) = F(\mathcal{P}; x) \cdot F(\mathcal{Q}; x)$ and $\widetilde{\chi}(\mathcal{P} \times \mathcal{Q}; x) = \widetilde{\chi}(\mathcal{P}; x) \cdot \widetilde{\chi}(\mathcal{Q}; x)$.

Example 2.8. $\mathbb{N}$ is a finite type $\mathbb{N}$-graded lattice with $F(\mathbb{N}; x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ and $\widetilde{\chi}(\mathbb{N}; x) = 1 - x$. Hence, for any positive integer $n \ge 1$, $\mathbb{N}^n$ is a finite type $\mathbb{N}$-graded lattice with $F(\mathbb{N}^n; x) = \frac{1}{(1-x)^n}$ and $\widetilde{\chi}(\mathbb{N}^n; x) = (1 - x)^n$

2.4. **Upho posets.** Let $\mathcal{P}$ be an infinite poset. $\mathcal{P}$ is *upper homogeneous*, or "*upho*," if for every $p \in \mathcal{P}$, the principal order filter $V_p = \{q \in \mathcal{P}\colon q \ge p\}$ is isomorphic to $\mathcal{P}$.

Example 2.9. The natural numbers $\mathbb{N}$ form an upho poset. Similarly, the nonnegative rational numbers $\{x \in \mathbb{Q}\colon x \ge 0\}$, with their usual order, form an upho poset. And ditto for the nonnegative real numbers $\{x \in \mathbb{R}\colon x \ge 0\}$.

Example 2.10. Let X be any infinite set. Then the poset of finite subsets of X, ordered by inclusion, is upho.

In this paper we are primarily concerned with upho posets (in fact, upho lattices). However, in order to be able to apply the tools of enumerative and algebraic combinatorics to study them, we must impose some finiteness conditions. Hence, from now on, **all upho posets are assumed finite type $\mathbb{N}$-graded** unless otherwise specified. Of the preceding examples, only $\mathbb{N}$ is finite type $\mathbb{N}$-graded.

The product $\mathcal{P} \times \mathcal{Q}$ of two upho posets $\mathcal{P}$ and $\mathcal{Q}$ remains upho. So, for instance, $\mathbb{N}^n$ is an upho lattice for any $n \geq 1$. We will see many more examples of upho lattices later, but for now $\mathbb{N}^n$ is a good prototypical example.

Remark 2.11. Upho posets were introduced by Stanley [10, 12]. Stanley was mainly interested in *planar* upho posets (i.e., those with planar Hasse diagrams). In particular, for these planar upho posets $\mathcal{P}$, he considered the number of maximal chains in $[\hat{0}, p]$ for all $p \in \mathcal{P}$. When $\mathcal{P} = \mathbb{N}^2$, these numbers form Pascal's triangle. Stanley used these chain counts for other planar upho posets $\mathcal{P}$ to produce analogues of Pascal's triangle [11, 12]. We note that planar upho posets have a rather simple structure, as described in [4]. All planar upho posets are meet semilattices, but most are not lattices. We will see that upho lattices can have a very intricate structure.

2.4.1. *Rank and characteristic generating functions of upho posets.* The following important result on rank generating functions of upho posets can be proved by a simple application of Möbius inversion.

Theorem 2.12 ([6, Theorem 1])**.** *For any upho poset $\mathcal{P}$, the rank and characteristic generating functions of $\mathcal{P}$ are multiplicative inverses as formal power series, i.e., $F(\mathcal{P}; x) = \widetilde{\chi}(\mathcal{P}; x)^{-1}$.*

Example 2.13. We have seen that $F(\mathbb{N}^n; x) = \frac{1}{(1-x)^n} = \widetilde{\chi}(\mathbb{N}^n; x)^{-1}$ for any $n \geq 1$.

2.4.2. *Upho lattices and their cores.* Now suppose that $\mathcal{L}$ is an upho lattice. Then we define the *core* of $\mathcal{L}$ to be $L := [\hat{0}, a_1 \vee a_2 \vee \cdots \vee a_r] \subseteq \mathcal{L}$, where $a_1, \ldots, a_r$ are the atoms of $\mathcal{L}$. Evidently, the core of an upho lattice is a finite graded lattice. The point of the core is the following, which can be proved for instance using Rota's cross-cut theorem (see [9, Corollary 3.9.4]).

Corollary 2.14 ([6, Corollary 6])**.** *Let $\mathcal{L}$ be an upho lattice with core L. Then $\widetilde{\chi}(\mathcal{L}; x) = \widetilde{\chi}(L; x)$. Hence, from Theorem 2.12 we conclude that $F(\mathcal{L}; x) = \widetilde{\chi}(L; x)^{-1}$.*

Note that Corollary 2.14 was stated as equation (1.1) in the introduction.

Example 2.15. For any $n \geq 1$, $\mathbb{N}^n$ is an upho lattice with core B_n, and indeed $F(\mathbb{N}^n; x) = \frac{1}{(1-x)^n} = \widetilde{\chi}(B_n; x)^{-1}$.

With all the terminology and preliminary results fully explained, we now return to Question 1.2: which finite graded lattices L arise as cores of upho lattices $\mathcal{L}$? For example, we just saw that the Boolean lattices B_n do. We will explore this question in the next three sections.

3. Upho lattices from sequences of finite lattices

In this section, we explain a method for producing upho lattices from "limits" of sequences of finite graded lattices which are appropriately embedded in one another.

3.1. Supersolvable lattices.

3.2. Limits of uniform sequences of supersolvable lattices.

3.3. Examples of uniform sequences of supersolvable lattices.

3.3.1. *Boolean lattices.*

3.3.2. *Subspace lattices.*

3.3.3. *Partition lattices.*

3.3.4. *Type B partition lattices.*

3.3.5. *Dowling lattices.*

3.4. Rank two cores: rank-by-rank construction.

4. Upho lattices from monoids

4.1. Monoids and monoid presentations.

4.2. Upho posets from cancellative monoids.

4.3. Garside monoids and finite Coxeter groups.

4.3.1. *The classical positive braid monoid and weak order.*

4.3.2. *The dual braid monoid and noncrossing partition lattice.*

5. Obstructions for cores of upho lattices

5.1. Characteristic polynomial obstructions.

5.2. Structural obstructions.

6. Distributive and modular upho lattices

6.1. Classification of upho distributive lattices.

6.2. Upho modular lattices with rank two cores.

6.3. Upho modular lattices from DVRs with finite residue fields.

6.4. Stanley's conjecture on upho modular lattices.

6.5. Upho modular lattices from affine buildings.

References

[1] P. Dehornoy, F. Digne, E. Godelle, D. Krammer, and J. Michel. *Foundations of Garside theory*, volume 22 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2015. Author name on title page: Daan Kramer.

[2] T. A. Dowling. A class of geometric lattices based on finite groups. *J. Combinatorial Theory Ser. B*, 14:61–86, 1973.

[3] T. A. Dowling. A q-analog of the partition lattice. In *A survey of combinatorial theory (Proc. Internat. Sympos., Colorado State Univ., Fort Collins, Colo., 1971)*, pages 101–115. North-Holland, Amsterdam, 1973.

[4] Y. Gao, J. Guo, K. Seetharaman, and I. Seidel. The rank-generating functions of upho posets. *Discrete Math.*, 345(1):Paper No. 112629, 14, 2022.

[5] H. Hirai. Uniform modular lattices and affine buildings. *Adv. Geom.*, 20(3):375–390, 2020.

[6] S. Hopkins. A note on Möbius functions of upho posets. *Electron. J. Combin.*, 29(2):Paper No. 2.39, 6, 2022.

[7] N. Radu. A lattice in a residually non-Desarguesian $\widetilde{A}_2$-building. *Bull. Lond. Math. Soc.*, 49(2):274–290, 2017.

[8] R. P. Stanley. Supersolvable lattices. *Algebra Universalis*, 2:197–217, 1972.

[9] R. P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.

[10] R. P. Stanley. From Stern's triangle to upper homogeneous posets. Talk transparencies, available at `https://math.mit.edu/~rstan/transparencies/stern-ml.pdf`, 2020.

[11] R. P. Stanley. Some linear recurrences motivated by Stern's diatomic array. *Amer. Math. Monthly*, 127(2):99–111, 2020.

[12] R. P. Stanley. Theorems and conjectures on some rational generating functions. `arXiv:2101.02131`, 2021.

[13] W. Stein et al. *Sage Mathematics Software (Version 9.0)*. The Sage Development Team, 2020. `http://www.sagemath.org`.

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