Trimer covers in the triangular grid:
twenty mostly open problems

James Propp, UMass Lowell
November 10, 2022

Abstract

Physicists have studied the asymptotics of the trimer model on the triangular lattice but have not given exact results for finite subgraphs of the triangular lattice analogous to known solutions of the dimer model on finite subgraphs of various lattices. Here I introduce finite subgraphs of the triangular lattice that will be of interest to enumerative combinatorialists inasmuch as the precise number of trimer covers appears to be given by exact formulas in many cases.

1 Introduction

If $G = (V, E)$ is a finite graph, a **trimer** on G is a three-element subset of V whose induced subgraph in G is connected, and a **trimer cover** is a partition of V into trimers. Solving the trimer model for G means counting the possible trimer covers.

Physicists have studied the asymptotics of the trimer model on the triangular lattice and obtained formulas for the entropy in various regimes (see e.g. [VeNi]). These results can be seen as analogous to formulas for the entropy of dimer models on various kinds of finite graphs; the original dimer results, due to Temperley and Fisher [TeFi] and Kasteleyn [Kast], were proved using the determinant method, while other approaches were used in more recent work such as [EKLP]. These rigorously proved exact formulas have no counterpart in the literature on trimers.

Here I introduce finite subgraphs of the triangular lattice that should interest enumerative combinatorialists inasmuch as the number of trimer covers appears to be given by exact formulas in many cases. This is a companion to the talk I gave at the Open Problems in Algebraic Combinatorics conference on May 18, 2022; the slides and video can be accessed through the conference website at `http://www.samuelfhopkins.com/OPAC/opac.html`.

Consider the complex plane tiled by unit hexagonal cells centered at 1, ω, and ω^2 (here and hereafter ω always denotes a primitive 3rd root of unity);

the cell centered at α has corners at $\alpha \pm 1$, $\alpha \pm \omega$, and $\alpha \pm \omega^2$. Given positive integers a, b satisfying $2 \leq a \leq 2b$ and $2 \leq b \leq 2a$, we define the **(a, b)-benzel** as the union of the cells that lie fully inside the hexagon with vertices $a\omega + b$, $-a\omega^2 - b$, $a\omega^2 + b\omega$, $-a - b\omega$, $a + b\omega^2$, and $-a\omega - b\omega^2$ (a hexagon centered at 0 with threefold rotational symmetry whose side-lengths alternate between $2a - b$ and $2b - a$, degenerating to a triangle when $a = 2b$ or $b = 2a$), as shown in Figure 1 for $a = 4$, $b = 6$. We lose no generality in assuming $2 \leq a \leq 2b$ and $2 \leq b \leq 2a$, since the (a, b)-benzel as defined above coincides with the $(a, a - b)$-benzel when $a > 2b$ and with the $(b - a, b)$-benzel when $b > 2a$, the former satisfying $a \leq 2(a - b)$ and the latter satisfying $b \leq 2(b - a)$.

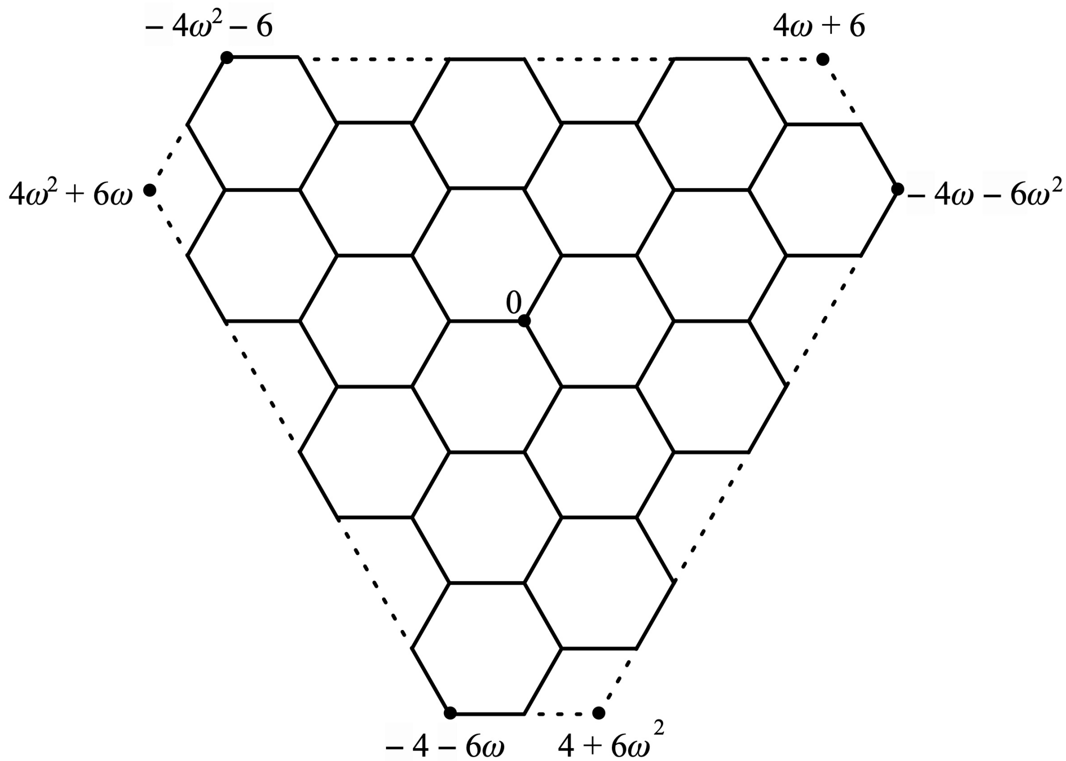

Figure 1: The (4,6)-benzel and its enclosing hexagon.

Here is an alternative description of benzels in terms of the centers of the cells using barycentric coordinates relative to the triangle with vertices 1, ω, and ω^2. Each cell's center point α belongs to $\mathbb{Z}[\omega]$ and can be represented by the unique $(i, j, k) \in \mathbb{Z}^3$ satisfying $i + j\omega + k\omega^2 = \alpha$ and $i + j + k = 1$ (here and hereafter i never denotes a primitive 4th root of unity). Then the (a, b)-benzel consists of those cells whose centers (i, j, k) satisfy $-(a-1) \leq j-i, k-j, i-k \leq b-1$. Figure 2 shows the (4,6)-benzel with its cells marked with the barycentric coordinates of their respective center points.

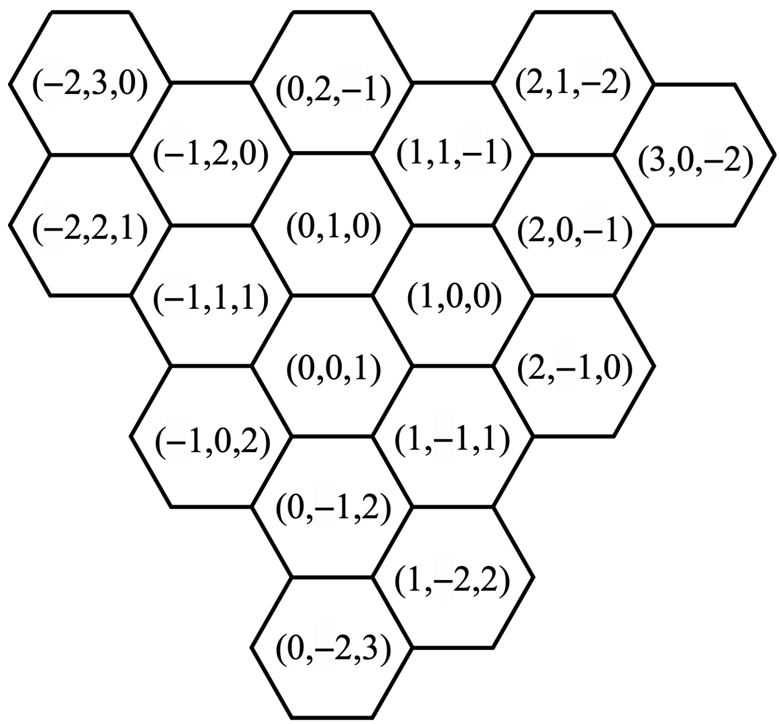

Figure 2: Barycentric coordinates for the cells of the (4,6)-benzel.

Let $V = \{(i, j, k) \in \mathbb{Z}^3 \; : \; i+j+k = 1, \; -(a-1) \leq j-i, k-j, i-k \leq b-1\}$. Given (i_1, j_1, k_1) and (i_2, j_2, k_2) in V, join (i_1, j_1, k_1) and (i_2, j_2, k_2) by an edge when $|i_1-i_2|+|j_1-j_2|+|k_1-k_2| = 2$ (that is, when the unit hexagons centered on those two vertices share an edge). This is the **(a, b)-benzel graph**.

The (a, b)-benzel has threefold rotational symmetry but for most a, b it does not have bilateral symmetry. Exchanging a and b corresponds to reflecting the benzel across a horizontal axis (or if one prefers across an axis making a 60 degree angle with the horizontal axis).

We consider tilings of the (a, b)-benzel by way of five sorts of prototiles, which we may translate but not rotate. These prototiles (shown in Figure 3) are the **right(-pointing) stone**, the **left(-pointing) stone**, the **vertical bone**, the **rising bone**, and the **falling bone**. The right stone is a benzel (specifically, the (2,2)-benzel) but the left stone is not. Dually we form spanning subgraphs of the (a, b)-benzel graph whose connected components all consist of three vertices. Figure 4 shows a tiling of the (9,9)-benzel and the associated trimer cover of the (9,9)-benzel graph.

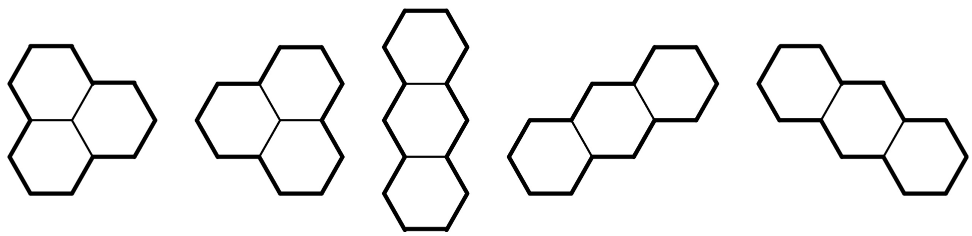

Figure 3: The five prototiles: the right stone, the left stone, the vertical bone, the rising bone, and the falling bone.

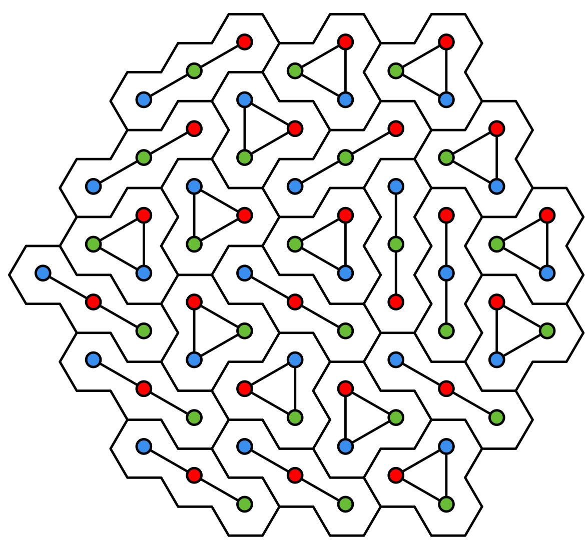

Figure 4: The (9,9)-benzel tiled by stones and bones and the associated trimer cover of the (9,9)-benzel graph.

Conway and Lagarias studied such tilings (calling stones and bones T_2 and

L_3 tiles respectively). They showed that for any simply-connected region in the hexagonal grid that can be tiled by stones and bones, the total area of the right stones minus the total area of the left stones does not depend on the specific tiling but only depends on the region being tiled. This is the Conway-Lagarias invariant of the region. It can be shown (see the companion article [KiPr]) that the area of the (a, b)-benzel (as measured by the number of tiles) is given by

$$\begin{array}{ll} (-a^2+4ab-b^2-a-b)/2 & \text{if } a+b \equiv 0 \text{ or } 2 \pmod 3, \\ (-a^2+4ab-b^2-a-b+2)/2 & \text{if } a+b \equiv 1 \pmod 3 \end{array}$$

while the value of the Conway-Lagarias invariant is

$$\begin{array}{ll} (3a^2-6ab+3b^2-a-b)/2 & \text{if } a+b \equiv 0 \pmod 3, \\ (-a^2+4ab-b^2-a-b+2)/2 & \text{if } a+b \equiv 1 \pmod 3, \\ (3a^2-6ab+3b^2+a+b-2)/2 & \text{if } a+b \equiv 2 \pmod 3. \end{array}$$

(The expressions in Theorem 2 of [KiPr] have opposite sign because the benzels in that article are mirror images of the benzels considered here.) Note that when $a+b \equiv 1 \pmod 3$, the Conway-Lagarias invariant is equal to the area of the benzel so that the tiling must consist entirely of right-pointing stones; for instance, this is the case with the (4,6)-benzel shown earlier.

The set of 5 prototiles has $2^5 - 1 = 31$ nonempty subsets, and for each, we can ask in how many ways it is possible to tile the (a, b)-benzel, that is, find translates of the prototiles whose interiors are disjoint and whose union is the benzel. There is some redundancy here. Because the benzel has threefold rotational symmetry, and because 120 degree rotations preserve the two stones' orientation (right versus left), the number of tilings depends only on (a) whether right stones are allowed, (b) whether left stones are allowed, and (c) how many of the three kinds of bones are allowed (0, 1, 2, or 3). Thus there are really only $(2)(2)(4) - 1 = 15$ tiling problems to consider. For $0 \le i, j \le 1$ and $0 \le k \le 3$ we define $T_{ijk}(a, b)$ as the number of ways to tile the (a, b)-benzel if the set of allowed prototiles contains the right stone iff $i = 0$, contains the left stone iff $j = 0$, and contains k of the bones. We say that such a tiling is an i, j, k tiling.

It is not hard to show (using the aforementioned rotational symmetry) that for each of the 15 cases, $T_{ijk}(a, b) = T_{ijk}(b, a)$. It is also not hard to show that the $(n, 2n)$-benzel is the same as the $(n, 2n-1)$-benzel and the $(n, 2n-2)$-benzel. Consequently, in the tables that follow we will often assume $2 \le a \le 2b - 2$ and $2 \le b \le 2a - 2$.

It is also easy to show that when $i = 0$ and $a+b \equiv 2 \pmod 3$, $T_{ijk}(a,b) = 0$. That is because in this case the Conway-Lagarias invariant is $(3(a-b)^2 + (a+b-2))/2 > 0$, implying that every tiling of the (a,b)-benzel must have at least one right-pointing stone.

David desJardins wrote a general purpose program `TilingCount` that I used to enumerate tilings of regions with various sets of allowed prototiles. This led to the questions and conjectures that appear below. I am happy to share the code and the data on which my conjectures are based (some of which appear in the Appendix). Here is a map of the first eighteen problems presented in this article as they relate to those fifteen cases. Rows describe which stones are allowed; columns describe how many bones are allowed.

	Two kinds of bones	Three kinds of bones
No stones	(no tilings exist)	type 003: prob. 1
Left stones	type 012: probs. 2–3	type 013: prob. 4
Right stones	type 102: prob. 5	type 103: probs. 6–7
Both kinds of stones	type 112: probs. 8–13	type 113: probs. 14–18

(This table omits cases where the number of allowed bone prototiles is zero or one; in such situations at most one tiling exists, even when both stone prototiles are allowed.)

Benzels behave differently according to whether $a + b$ is 0, 1, or 2 (mod 3), so in what follows we will often divide conjectures into these three cases.

In the cases where only two kinds of bone tiles are permitted, the allowed tilings can be viewed as ribbon tilings, as in [Pak]; indeed, this was the mode of presentation employed in [CoLa], with four of the five prototiles being depicted as ribbons. Switching over to the ribbon tilings presentation has already yielded solutions to problems 2 and 3 in [DLPY], and is likely to provide leverage on other problems in the first column of the table.

2 No stones, three kinds of bones

Prior to the conference, I was able to show that if an (a,b)-benzel can be tiled by bones, then we must have $a = k(3k-1)/2$ and $b = k(3k+1)/2$ (or vice versa) for some $k \geq 2$. Several attending students found a proof that this necessary condition is also sufficient. (Note that such benzels belongs to the case $a+b \equiv 0 \pmod 3$.) Jesse Kim found the most complete solution, providing an explicit proof that the tiling he described works for all k. It appears that the number of such tilings grows exponentially in k^4 (see Appendix 1 for the data).

Problem 1: Find an exact formula for $T_{003}(k(3k-1)/2, k(3k+1)/2)$.

Of course, even short of an exact formula, any method of determining the number of tilings that is more efficient than brute-force enumeration (e.g., a recurrence relation) would be of interest.

See [KiPr] for more discussion of no-stones tilings of benzels.

3 Left stones, two kinds of bones

Figure 5 shows the values of $T_{012}(a,b)$ for $a, b \leq 10$.

$a\backslash b$	2	3	4	5	6	7	8	9	10
2	0								
3		**2**	0						
4		0	0	**2**	0				
5			**2**	0	0	0	0		
6			0	0	**8**	0	0	0	0
7				0	0	0	**8**	0	0
8				0	0	**8**	0	0	0
9					0	0	0	**48**	0
10					0	0	0	0	0

Figure 5: Values of $T_{012}(a,b)$.

Problem 2: Is it true that $T_{012}(3n,3n) = 2^n n!$ for $n \geq 1$?

Problem 3: Is it true that $T_{012}(3n+1,3n+2) = 2^n n!$ for $n \geq 1$?

Comment: Note that by a, b symmetry the formula implies $T_{012}(3n+2,3n+1) = 2^n n!$. Henceforth we will omit such corollaries without comment.

Comment: Colin Defant, Rupert Li, Benjamin Young and I worked on problem 2 during the conference and later succeeded in solving Problems 2 and 3; see [DLPY]. We were also able to verify that the pattern of entries equal to 0 and 1 in Figures 5 and 7 persists for all larger values of a and b.

4 Left stones, three kinds of bones

Figure 6 shows the values of $T_{013}(a, b)$ for $a, b \leq 10$.

$a\backslash b$	2	3	4	5	6	7	8	9	10
2	0								
3		**3**	0						
4		0	0	**9**	0				
5			**9**	0	0	**2**	0		
6			0	0	**144**	0	0	0	0
7				**2**	0	0	**1143**	0	0
8				0	0	**1143**	0	0	**825**
9					0	0	0	**73454**	0
10					0	0	**825**	0	0

Figure 6: Values of $T_{013}(a, b)$.

When a and b are such that the Conway-Lagarias invariant is strictly positive, the (a, b)-benzel cannot be tiled by bones and left stones; the corresponding entries in the table must be zero. On the other hand, when a and b are such that the Conway-Lagarias invariant is negative or zero, the entries in the table are observed to be positive, though I see no reason for concluding that they are.

Problem 4: Is it true that when the Conway-Lagarias invariant associated with the (a, b)-benzel is negative or zero, tilings of type 013 exist?

Comment: In fact, in this regime (and in the regimes described in succeeding sections), the number of tilings appears to grow as an exponential function of the area of the region being tiled, at least when the asymptotic ratio of a to b is chosen properly.

5 Right stones, two kinds of bones

Figure 7 shows the values of $T_{102}(a, b)$ for $a, b \leq 10$.

Problem 5: Is it true that

$$T_{102}(n + 3k, 2n + 3k - 1) = \prod_{i=1}^{k} \frac{(2i)!(2i + 2n - 2))!}{(i + n - 1)!(i + n + k - 1)!}$$

$a\backslash b$	2	3	4	5	6	7	8	9	10
2	1								
3		0	1						
4		1	**2**	0	1				
5			0	1	**4**	0	1		
6			1	4	0	1	**10**	0	1
7				0	1	**8**	0	1	**28**
8				1	10	0	1	**24**	0
9					0	1	24	0	1
10					1	28	0	1	**48**

Figure 7: Values of $T_{102}(a,b)$.

for $k \geq 0$ and $n \geq 1$ (except $(k,n) = (0,1)$)?

Comment: This formula and the a, b symmetry relation together provide a conjectural enumeration of tilings of type 102 of the (a,b)-benzel for all a, b satisfying $a + b \equiv 2 \pmod 3$.

Comment: Three special cases merit special attention. When $k = 1$, the right-hand side of the equation is twice the nth Catalan number; when $n = 1$, the right-hand side of the equation is $2^k k!$; and when $n = 2$, the right-hand side of the equation is $2^k(k+1)!$.

6 Right stones, three kinds of bones

Figure 8 shows the values of $T_{103}(a,b)$ for $a, b \leq 10$.

Problem 6: Is is true that $T_{103}(n, 2n-3) = (3n+3)(3n-7)!/(n-5)!(2n-1)!$ for $n \geq 5$? (The formula works for $n = 3$ and $n = 4$ if one treats $1/(-1)!$ and $1/(-2)!$ as 0.)

Aside from the fact that Problems 2 and 6 involve different prototile sets, the two problems differ in another important way: in problem 2 we have $b/a \to 1$ as $n \to \infty$ while in problem 6 we have $b/a \to 2$ as $n \to \infty$. In the former case we say that the sequence is associated with a **central diagonal** of the table of values while in the latter case we say that the sequence is associated with a **peripheral diagonal** (recall that a/b and b/a cannot exceed 2).

$a \backslash b$	2	3	4	5	6	7	8	9	10
2	1								
3		**0**	1						
4		1	7	**0**	1				
5			0	1	33	**2**	1		
6			1	33	0	1	164	**21**	1
7				2	1	666	0	1	864
8				1	164	0	1	12430	0
9					21	1	12430	0	1
10					1	864	0	1	655721

Figure 8: Values of $T_{103}(a,b)$.

In parallel with the observations that preceded Problem 4, note that when the Conway-Lagarias invariant of a benzel is strictly negative, the benzel cannot be tiled by bones and right-pointing stones; the corresponding entries in the table must be zero. On the other hand, when a and b are such that the Conway-Lagarias invariant is positive or zero, the entries in the table are observed to be positive, though I see no reason for concluding that they are.

Problem 7: Is it true that when the Conway-Lagarias invariant associated with the (a,b)-benzel is positive or zero, tilings of type 103 exist?

7 Both stones, two kinds of bones

Figure 9 shows the values of $T_{112}(a,b)$ for $a,b \leq 10$.

Here are the first few values of $T_{112}(3n,3n)$, given in factored form:

$$\begin{gathered} 2^1, \\ 2^4\,3^1, \\ 2^{10}\,3^1\,5^1, \\ 2^{16}\,7^1\,11^1\,13^1, \\ 2^{28}\,3^2\,7^1\,13^1\,17^1, \\ 2^{38}\,3^2\,11^1\,17^2\,19^2, \\ 2^{50}\,5^1\,11^2\,13^1\,17^1\,19^2\,23^2, \\ 2^{64}\,3^3\,5^4\,11^1\,13^2\,19^1\,23^3\,29^1, \\ 2^{84}\,3^4\,5^3\,13^2\,17^1\,23^2\,29^3\,31^2, \ldots \end{gathered}$$

$a\backslash b$	2	3	4	5	6	7	8	9	10
2	1								
3		**2**	1						
4		1	**4**	**6**	1				
5			6	1	**16**	**22**	1		
6			1	16	**48**	1	**68**	**90**	1
7				22	1	**224**	**512**	1	**304**
8				1	68	512	1	**3360**	6736
9					90	1	3360	**15360**	1
10					1	304	6736	1	**168960**

Figure 9: Values of $T_{112}(a, b)$.

Problem 8: With $T(n)$ denoting $T_{112}(3n, 3n)$ is the second quotient $T(n)T(n+2)/T(n+1)^2$ equal to

$$\frac{256(2n+3)^2(4n+1)(4n+3)^2(4n+5)}{27(3n+1)(3n+2)^2(3n+4)^2(3n+5)}$$

for $n \geq 1$?

Comment: David desJardins found the pattern governing the numbers $T(n)$, with assistance from Christian Krattenthaler, Greg Kuperberg and other members of the `domino` listserv. The same is true for Problem 10.

Here are the first few values of $T_{112}(3n+1, 3n+1)$, given in factored form:

$$\begin{gathered} 2^2, \\ 2^5\,7^1, \\ 2^{10}\,3^1\,5^1\,11^1, \\ 2^{17}\,7^1\,11^1\,13^2, \\ 2^{30}\,3^1\,13^1\,17^2\,19^1, \\ 2^{38}\,3^1\,11^1\,17^2\,19^3\,23^1, \ldots \end{gathered}$$

With $T(n) = T_{112}(3n+1, 3n+1)$ with $n \geq 1$, it appears that $T(n)$ has no prime factor greater than or equal to $4n$.

Problem 9: Find a formula governing this sequence.

Here are the first few values of $T_{112}(3n+1, 3n+2)$, given in factored form:

$$\begin{gathered} 2^1\, 3^1, \\ 2^9, \\ 2^9\, 3^1\, 5^1\, 7^1\, 11^1, \\ 2^{25}\, 3^1\, 7^1\, 13^1, \\ 2^{28}\, 3^2\, 11^1\, 13^1\, 17^2\, 19^1, \\ 2^{50}\, 3^1\, 11^1\, 17^1\, 19^2\, 23^1, \\ 2^{49}\, 3^2\, 5^2\, 11^2\, 13^2\, 17^1\, 19^2\, 23^3, \\ 2^{81}\, 3^3\, 5^4\, 13^2\, 23^2\, 29^2\, 31^1, \ldots \end{gathered}$$

Problem 10: With $T(n)$ denoting $T_{112}(3n+1, 3n+2)$, is it true that $T(n)T(n+3)/T(n+1)T(n+2)$ is always equal to

$$\frac{65536(2n+3)(2n+5)^2(2n+7)(4n+3)(4n+5)^2(4n+7)^2(4n+9)^2(4n+11)}{729(3n+2)(3n+4)^3(3n+5)^2(3n+7)^2(3n+8)^3(3n+10)}$$

for $n \geq 1$?

Here are the first few values of $T_{112}(3n-1, 3n)$, given in factored form:

$$\begin{gathered} 1^1, \\ 2^4, \\ 2^5\, 3^1\, 5^1\, 7^1, \\ 2^{16}\, 11^1\, 13^1, \\ 2^{19}\, 3^1\, 7^1\, 11^1\, 13^2\, 17^1, \\ 2^{39}\, 3^1\, 17^2\, 19^2, \\ 2^{37}\, 5^1\, 11^2\, 13^1\, 17^2\, 19^3\, 23^2, \ldots \end{gathered}$$

With $T(n) = T_{112}(3n-1, 3n)$ with $n \geq 1$, it appears that $T(n)$ has no prime factor greater than or equal to $4n$.

Problem 11: Find a formula governing this sequence.

We now switch from central diagonals to peripheral diagonals.

Problem 12: Is it true that $T_{112}(n+2, 2n+1)$ is the nth large Schröder number (see sequence A006318 in the OEIS) for $n \geq 1$?

Problem 13: Is it true that $T_{112}(n+2, 2n)$ is always the number of "royal paths in a lattice of order n" (see sequence A006319 in the OEIS) for $n \geq 1$?

8 Both stones, three kinds of bones

Finally we come to the most permissive situation: all prototiles are allowed.

Figure 10 shows the values of $T_{113}(a,b)$ for $a,b \leq 10$.

$a \backslash b$	2	3	4	5	6	7	8	9	10
2	1								
3		3	1						
4		1	10	**18**	1				
5			18	1	**84**	**142**	1		
6			1	84	459	1	**724**	**1266**	1
7				142	1	5766	19057	1	**6516**
8				1	724	19057	1	380597	1077681
9					1266	1	380597	3759277	1
10					1	6516	1077681	1	185961668

Figure 10: Values of $T_{113}(a,b)$.

Problem 14: Is it true that $T_{113} = 1$ when $a+b$ is 1 (mod 3)? (Note that $a+b \equiv 1$ is the situation in which the Conway-Lagarias invariant coincides with the area of the region being tiled, so that all the tiles must be right-pointing stones.)

Comment: This problem is resolved in the affirmative by Theorem 1.1 of [DLPY].

It is disappointing that the data for $a+b \not\equiv 1$ (mod 3) do not suggest exact conjectures. On the other hand, it is intriguing that congruence phenomena occur, analogous to Cohn's 2-adic continuity theorem proved in [Cohn] and conjectural 2-adic phenomena of a similar kind discussed in [Pro].

Problem 15: Is $T_{113}(n, 2n-4)$ 2-adically continuous as a function of $n \geq 5$?

Comment: The sequence appears to be constant mod 2, constant mod 4, 2-periodic mod 8, and 8-periodic mod 16 (with repeating pattern 4, 4, 4, 4, 4, 12, 4, 12). The case $n = 4$ breaks the pattern.

Problem 16: Is $T_{113}(n, 2n-3)$ 2-adically continuous as a function of $n \geq 4$?

Comment: The sequence appears to be constant mod 2, constant mod 4, 2-periodic mod 8, and 8-periodic mod 16 (with repeating pattern 2, 14, 2, 14, 10, 6, 10, 6). The case $n = 3$ breaks the pattern.

At the OPAC 2022 meeting, David Speyer suggested that one might use a different definition of a trimer, namely, a path consisting of three vertices and two edges. Thus, each stone would correspond to three different trimers according to which 2 of the 3 possible edges one used. Equivalently, one could use the original definition of a trimer, with the proviso that each stone will count with weight 3. This changes the enumerations listed above in cases where both kinds of stones are allowed (the Conway-Lagarias invariant forces every tiling to have the same weight as every other if only one kind of stone is allowed) but the new numbers do not satisfy any nice patterns, with one exception: in the case where all prototiles are allowed, using stones of weight 3 seems to give rise to the 2-adic continuity phenomenon we saw in Problems 11 and 12.

Let $T_{ijk}(a, b; 3)$ denote the weighted sum of the i, j, k tilings of the (a, b)-benzel as defined near the end of section 1, where a tiling with m stones has weight 3^m.

Problem 17: Is $T_{113}(n, 2n - 4; 3)$ 2-adically continuous as a function of $n \geq 5$?

Comment: The sequence appears to be constant mod 2, constant mod 4, constant mod 8, and 8-periodic mod 16 (with repeating pattern 4, 4, 12, 12, 4, 12, 12, 4). The case $n = 4$ breaks the pattern.

Problem 18: Is $T_{113}(n, 2n - 3; 3)$ 2-adically continuous as a function of $n \geq 4$?

Comment: The sequence appears to be constant mod 2, constant mod 4, 2-periodic mod 8, and 8-periodic mod 16 (with repeating pattern 14, 6, 10, 2, 6, 14, 2, 10). The case $n = 3$ breaks the pattern.

9 Miscellaneous

The next problem is not enumerative; it is an old structural problem that has gone unresolved for decades. There are two natural kinds of "2-flips" that can turn one stones-and-bones tiling into another; the first trades two stones of opposite sign for two bones, and the second trades a stone and a bone for a stone of the same orientation and a bone of a different orientation; see Figure 11.

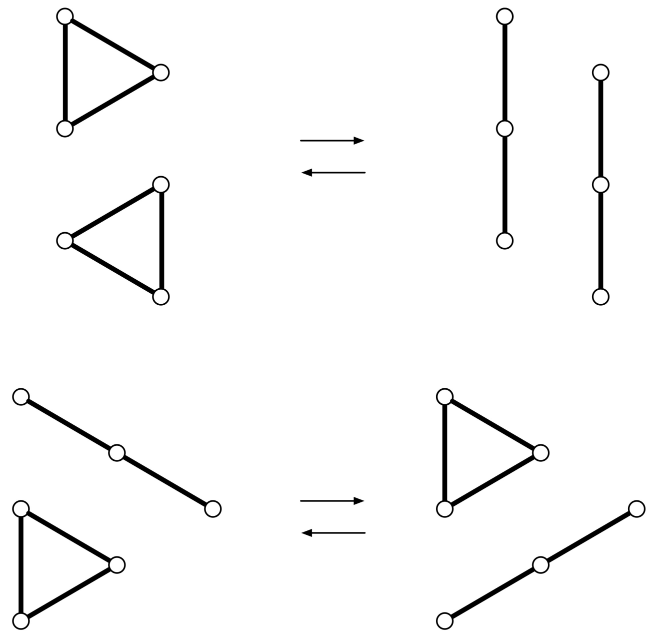

Figure 11: 2-flips for changing a trimer cover.

Problem 19: Can every tiling of a finite simply-connected region using stones and bones be mutated into every other such tiling by means of a succession of 2-flips?

Comment: It is known that the hypothesis that the region be simply-connected cannot be dropped. It is also known that if one restricts to tilings of type 112 (that is, if one prohibits one of the three orientations of bones), then the claim is true; Sheffield proved an equivalent claim in the context of ribbon tilings [Shef].

In closing, we turn to the regions that Conway, Lagarias, and Thurston originally studied: triangles of hexagonal cells, with n cells on each side ("T_n regions"). All three authors showed that if one uses stones alone, T_n can be tiled by T_2's (that is, by stones) precisely when n is congruent to 0, 2, 9, or 11 (mod 12). (In our notation, these are tilings of type 110; such tilings were not discussed above since for benzels they are not very interesting from an enumerative perspective.)

The question we ask is, how many such tiling are there? Sequence A334875 in the OEIS gives the answers for many small values of n. If we look at the prime factorizations of the answers, we notice that the exponent of the prime 2 is creeping upward. Specifically, the multiplicity of the prime 2 in the factorizations of the nonzero terms in this sequence goes 0, 0, 1, 3, 2, 3, 4, 3, 4, 3, 5, 8, 6, 8, ... This is a priori surprising, since the probability that a "random" positive integer is divisible by 2^m decreases exponentially as m increases.

Problem 20: As n goes to infinity within the set of natural numbers congruent to 0, 2, 9, or 11 (mod 12), does the number of tiling of T_n by stones converge 2-adically to 0?

10 Appendix

Here are some of the empirical data on which the conjectures stated in this article are based.

Note: This article will undergo an "appendectomy" once the relevant sequences are in the OEIS.

Problem 1: $T_{003}(k(3k-1)/2, k(3k+1)/2)$ for $2 \leq k \leq 4$: 2, 42705, 7501790059160666750.

Problem 2: $T_{012}(3n, 3n)$ for $1 \leq n \leq 6$: 2, 8, 48, 384, 3840, 46080.

Problem 3: $T_{012}(3n+1, 3n+2)$ for $1 \leq n \leq 6$: 2, 8, 48, 384, 3840, 46080.

Problem 5: The conjecture is true for $0 \leq k \leq 5$, $1 \leq n \leq 5$.

Problem 6: The conjecture is true for $5 \leq n \leq 16$.

Problem 8: $T_{112}(3n, 3n)$ for $1 \leq n \leq 9$:

$$\begin{aligned}&2, 48, 15360, 65601536, 3737426853888,\\&2839095978202497024, 2874817669362069482242048 0,\\&387952004963238149100725600256 0000,\\&6976271067658190025590579601863413334016000.\end{aligned}$$

The conjecture is true for $1 \leq n \leq 7$.

Problem 9: $T_{112}(3n+1, 3n+1)$ for $1 \leq n \leq 7$: 4, 224, 168960, 1705639936, 229940737867776, 41356164749149706496, 991812095929913971373506560 0.

Problem 10: $T_{112}(3n+1, 3n+2)$ for $1 \leq n \leq 8$:

$$6, 512, 591360, 9160359936, 1897011087409152,\\
5244422625774526267392,\\
1934033587063332244178337779200,\\
950984627208089329318875493721702400000.$$

Problem 11: $T_{112}(3n-1, 3n)$ for $1 \leq n \leq 7$: 1, 16, 3360, 9371648, 347950546944, 172066422921363456, 1133503548832944876421120.

Problem 12: The conjecture is true for $1 \leq n \leq 15$.

Problem 13: The conjecture is true for $1 \leq n \leq 15$.

Problem 14: The conjecture is true for $a + b \leq 28$.

Problem 15: $T_{113}(n, 2n-4)$ for $3 \leq n \leq 14$: 1, 10, 84, 724, 6516, 60900, 586404, 5777916, 57952212, 589381020, 6060195316, 62863155972.

Problem 16: $T_{113}(n, 2n-3)$ for $3 \leq n \leq 14$: 3, 18, 142, 1266, 12030, 118650, 1198230, 12296202, 127633590, 1336133730, 14079114270, 149124688482.

Problem 17: $T_{113}(n, 2n-4; 3)$ for $3 \leq n \leq 14$:

$$3, 102, 10260, 3267540, 3272495580, 10170919805580,\\
97112573496153540, 2829427113881208115260,\\
250440846963119234063024220,\\
6714319716839273852162816812240,\\
5441161364761844583846480805250817906 0,\\
13308556095374126636077976363771602176718554 0.$$

Problem 18: $T_{113}(n, 2n-3; 3)$ for $3 \leq n \leq 14$:

$$9, 270, 27110, 8798490, 8980383330, 28344705113430,\\
273927748387623390, 8057418594145673168610,\\
718650987298253553656580570,\\
193874673319110717570773876192670,\\
157927323459469084048485672225266775510,\\
3879624319582472677735278022720806271273188 90.$$

Acknowledgments: I am grateful to the organizers of OPAC 2022 for inviting me to give this presentation. I also think Colin Defant, Rupert Li, and Ben Young for useful conversations.

References

[Cohn] Henry Cohn, 2-adic behavior of numbers of domino tilings, Electronic Journal of Combinatorics **6** (1999), #R14.

[CoLa] John Conway and Jeffrey Lagarias, Tiling with polyominoes and combinatorial group theory, J. Combin. Theory (Ser. A) **53** (1990), no. 2, 183–208.

[DLPY] Colin Defant, Rupert Li, James Propp, and Benjamin Young, Tilings of benzels via the abacus bijection, submitted to Combinatorial Theory; https://arxiv.org/abs/2209.05717.

[EKLP] Noam Elkies, Greg Kuperberg, Michael Larsen, and James Propp, Alternating sign matrices and domino tilings, J. Algebraic Combin. **1** (1992), no. 2, 111–132; J. Algebraic Combin. **1** (1992), no. 3, 219–234.

[Kast] Pieter Kasteleyn, The statistics of dimers on a lattice, I: The number of dimer arrangements on a quadratic lattice, Physica **27** (1961), 1209–1225.

[KiPr] Jesse Kim and James Propp, A pentagonal number theorem for tribone tilings, in preparation. http://arxiv.org/abs/2206.04223

[Pak] Igor Pak, Ribbon tile invariants, Trans. Amer. Math. Soc. **352** (2000), no. 12, 5525–5561.

[Pro] James Propp, Some 2-adic conjectures concerning polyomino tilings of Aztec diamonds, in preparation.

[Shef] Scott Sheffield, Ribbon tilings and multidimensional height functions, Trans. Amer. Math. Soc. **354** (2002), no. 12, 4789–4813.

[TeFi] Harold Neville Temperley and Michael Fisher, Dimer problem in statistical mechanics – an exact result, Phil. Mag. J. Theor. Exp. Appl. Phys. **6** (1961) no. 68, 1061–1063.

[Thur] William Thurston, Conway's tiling groups, Amer. Math. Monthly **97** (1990), no. 8, 757–773.

[VeNi] Alain Verberkmoes and Bernard Nienhuis, Bethe Ansatz solution of triangular trimers on the triangular lattice, Phys. Rev. E **63** (2001), 066122.