# Enumeration of barely set-valued tableaux and plane partitions

George Washington University Combinatorics & Algebra Seminar

#### Sam Hopkins

based on joint works with Melody Chan, Colin Defant, Shahrzad Haddadan,

Alexander Lazar, Svante Linusson, Luca Moci, Svetlana Poznanović, and James Propp

Howard University

February 25th, 2022

#### Section 1

### Tableaux and plane partitions

### Standard Young tableaux

The **Young diagram** of a partition  $\lambda = (\lambda_1, \lambda_2, ...)$  is left-justified array of boxes with  $\lambda_i$  boxes in *i*th row:

We will care most about the **rectangle shape**  $a \times b := (b, b, \dots, b)$ .

A **standard Young tableau** of shape  $\lambda$  is a filling of the Young diagram with numbers  $1, 2, \ldots, n := |\lambda|$ , each appearing once, which is increasing along rows and down columns.

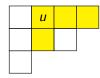
Let  $SYT(\lambda) := \{SYTs \text{ of shape } \lambda\}.$ 

$$\mathcal{SYT}(2 \times 2) = \left\{ egin{bmatrix} 1 & 2 \\ \hline 3 & 4 \end{pmatrix}, \quad egin{bmatrix} 1 & 3 \\ \hline 2 & 4 \end{bmatrix} \right\}$$

a times

### The Hook Length Formula

The **hook** of box u of a Young diagram is all boxes weakly left or below u:



**Hook length** h(u) := number of boxes in hook.

Theorem (Hook Length Formula; Frame–Robinson–Thrall, 1954)

$$\#\mathcal{SYT}(\lambda) = \frac{n!}{\prod_{u \in \lambda} h(u)}$$

For example, 
$$\#\mathcal{SYT}(2\times 2)=\frac{4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 2\cdot 1}=2.$$

#### Set-valued tableaux

A standard set-valued tableau of shape  $\lambda$  is a filling of Young diagram with numbers  $1, 2, \ldots, n+k$  for some  $k \geq 0$ , each appearing once, but where multiple numbers can be in the same box.

(Each box must get at least one number, and still needs to be increasing.)

Let 
$$\mathcal{SYT}^{+k}(\lambda)$$
 be the set of these tableaux. So  $\mathcal{SYT}^{+0}(\lambda) = \mathcal{SYT}(\lambda)$ .

Our focus is on barely set-valued tableaux  $SYT^{+1}(\lambda)$ .

For example, there are 10 tableaux in  $\mathcal{SYT}^{+1}(2 \times 2)$ :

1	2	1	3	1		2	1	3	1	4
3	4,5	2	4,5	3,	4	5	2, 4	5	2, 3	5
1	2,3	1	2, 4	1	3	, 4	1, 2	3	1, 2	4
4	5	3	5	2		5	4	5	3	5

### Aside: Schur & Grothendieck polynomials

#### The Schur function

$$s_{\lambda}(x_1, x_2, \ldots) = \sum_{\substack{\text{SSYT}T, \\ \text{shape}(T) = \lambda}} \mathbf{x}^{\text{content}(T)}$$

is the generating function for **semistandard tableaux** (I won't define). Schur functions have many algebraic/geometric guises; one is that they represent Schubert cycles in the cohomology of the Grassmannian.

#### Similarly, the (stable) Grothendieck polynomials

$$G_{\lambda}(x_1, x_2, \ldots) = \sum_{\substack{\text{set-valued SSYT}\mathcal{T}, \\ \text{shape}(\mathcal{T}) = \lambda}} (-1)^{|\mathcal{T}| - |\lambda|} \mathbf{x}^{\text{content}(\mathcal{T})}$$

represent Schubert cycles in K-theory of the Grassmannian (Buch, 2002).

### Plane partitions

An **plane partition** of shape  $\lambda$  is filling of the Young diagram with nonnegative integers, weakly decreasing in rows and columns.

Let  $\mathcal{PP}_m(\lambda) :=$  plane partitions of shape  $\lambda$  with entries in  $\{0, 1, \dots, m\}$ . There is a beautiful 3D representation of plane partitions:

5

5	3	3	2	2	1	44 3	
4	2	1	1	1	0	3	$\in \mathcal{PP}_5(4 \times 6)$
4	1	1	1	0	0	1 1 2	$\in PP_5(4 \times 0)$
3	1	1	0	0	0		
						00000	

#### Theorem (MacMahon, c. 1915)

$$\#\mathcal{PP}_{m}(a \times b) = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{m+i+j-1}{i+j-1}$$

#### Section 2

Motivation from algebraic geometry

### Brill-Noether theory

Let C be a "general" curve of genus g. The **Brill–Noether space**  $G_d^r(C)$  is moduli space of maps from C to r-dim'l projective space  $\mathbb{P}^r$  of degree d:

$$G_d^r(C) = \left\{ \bigcap_{\deg \operatorname{rec}(\phi) = d}^{\phi} \mathbb{P}^r \right\}$$

Define the **Brill–Noether number**  $\rho = \rho(g, d, r)$  as

$$\rho := g - (r+1)(g-d+r)$$

#### Theorem (Brill-Noether Theorem; Griffiths-Harris, 1980)

 $G_d^r(C)$  is nonempty iff  $\rho \geq 0$ , and in that case  $\dim(G_d^r(C)) = \rho$ .

### Finer invariants of moduli spaces

We could ask for finer information about  $G_d^r(C)$  than just its dimension.

For example, when  $\rho=0$ ,  $G_d^r(\mathcal{C})$  is a finite set of points, and the number of points is known to be

$$\#G_d^r(C) = g! \cdot \prod_{i=0}^r \frac{i!}{(g-d+r+i)!}$$

Or when  $\rho = 1$ ,  $G_d^r(C)$  is itself a smooth curve, and the genus of this curve is known to be

$$\operatorname{genus}(G_d^r(C)) = 1 + \frac{(r+1)(g-d+r)}{g-d+2r+1} \cdot g! \cdot \prod_{i=0}^r \frac{i!}{(g-d+r+i)!}$$

Interesting product formulas...

#### Euler characteristics via tableaux

Comparing to the Hook Length Formula we see that when ho= 0,

$$\#G_d^r(C) = \#\mathcal{SYT}((r+1) \times (g-d+r))$$

Chan–López-Martín–Pflueger–Teixidor i Bigas (2018) showed when ho=1,

$$\operatorname{genus}(G_d^r(C)) = 1 + \#\mathcal{SYT}^{+1}((r+1) \times (g-d+r))$$

### Corollary (Chan-López-Martín-Pflueger-Teixidor i Bigas, 2018)

$$\#\mathcal{SYT}^{+1}(a \times b) = (ab+1) \cdot \frac{ab}{a+b} \cdot \#\mathcal{SYT}(a \times b)$$

For example,  $\#\mathcal{SYT}^{+1}(2\times 2) = 5 \cdot \frac{4}{4} \cdot \#\mathcal{SYT}(2\times 2) = 5 \cdot 1 \cdot 2 = 10$ .

Chan–Pflueger (2021) showed more generally that for any  $\rho \geq 0$ , the Euler characteristic of  $G_d^r(C)$  is  $(-1)^\rho$  times  $\#\mathcal{SYT}^{+\rho}((r+1)\times(g-d+r))$ . But apparently no product formulas for  $\rho > 1$ !

#### Section 3

Down-degree expectations

### Decomposing barely set-valued tableaux

A barely set-valued tableau  $T' \in \mathcal{SYT}^{+1}(\lambda)$  has a rather simple structure: one special box has two numbers, while all others have a single number.

This leads to a decomposition of T' into a triple (T, i, u) where:

- $T \in \mathcal{SYT}(\lambda)$  is a usual standard tableau;
- $i \in \{0, 1, \dots, n\}$  is some number;
- u is a **removable box** of the **subshape**  $T^{-1}(\{1,2,\ldots,i\})$ .

(A subshape of  $\lambda$  is a Young diagram  $\sigma$  with  $\sigma \subseteq \lambda$ . A removable box of a subshape  $\sigma \subseteq \lambda$  is a box whose removal gives another subshape.)

$$T' = \begin{array}{|c|c|c|}\hline 1 & 2 & 4,7\\\hline 3 & 5 & 8\\\hline 6 & 9 & 10\\\hline \end{array} \iff$$

$$i = 6$$
,  $u = \text{circled box}$ ,  
 $T^{-1}(\{1...,i\}) = \text{yellow}$ 

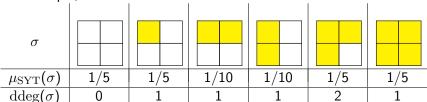
### Distributions on subshapes

The decomposition of barely set-valued tableaux T' motivates us to consider the following probability distribution on subshapes of  $\lambda$ :

- choose  $T \in \mathcal{SYT}(\lambda)$  uniformly at random;
- choose  $i \in \{0, 1, ..., n\}$  uniformly at random;
- select the subshape  $T^{-1}(\{1,2,\ldots,i\})$ .

Call this distribution on subshapes  $\mu_{SYT}$ . Also, denote the number of removable boxes of a subshape  $\sigma$  by  $ddeg(\sigma)$ , the **down-degree** of  $\sigma$ .

For example, with  $\lambda = 2 \times 2$ :



### Down-degree expectations

The decomposition of barely set-valued T' can be restated in terms of **expected down-degrees** as follows:

#### Proposition

$$\mathbb{E}_{\mu_{\text{SYT}}}(\text{ddeg}) = \frac{\#\mathcal{SYT}^{+1}(\lambda)}{(n+1) \cdot \#\mathcal{SYT}(\lambda)}$$

For example, with  $\lambda = 2 \times 2$ :

$$\mathbb{E}_{\mu_{\mathrm{SYT}}}(\mathrm{ddeg}) = (0 \cdot \frac{1}{5} + 1 \cdot \frac{1}{5} + 1 \cdot \frac{1}{10} + 1 \cdot \frac{1}{10} + 2 \cdot \frac{1}{5} + 1 \cdot \frac{1}{5}) = 1 = \frac{10}{5 \cdot 2}.$$

"Expected down-degrees" terminology due to Reiner-Tenner-Yong (2018).

### Barely set-valued plane partitions

For any  $m \ge 1$ , we can define distribution  $\mu_{\text{PP}_m}$  on subshapes by:

- choose  $\pi \in \mathcal{PP}_m(\lambda)$  uniformly at random;
- choose  $i \in \{1, ..., m\}$  uniformly at random;
- select the subshape  $\pi^{-1}(\{i, i+1, \ldots, m\})$ .

**Note**:  $\mu_{\mathrm{SYT}} = \lim_{m \to \infty} \mu_{\mathrm{PP}_m}$  and  $\mu_{\mathrm{PP}_1} = \text{uniform}$  distribution.

#### **Proposition**

$$\mathbb{E}_{\mu_{\mathrm{PP}_m}}(\mathrm{ddeg}) = \frac{\#\mathcal{PP}_m^{+1}(\lambda)}{m \cdot \#\mathcal{PP}_m(\lambda)}$$

Here  $\mathcal{PP}_m^{+1}(\lambda)$  is the set of "barely set-valued plane partitions" which look like what you'd expect:

#### Section 4

Toggles, toggle-symmetry, and rooks

### Toggling subshapes

Let  $u \in \lambda$  be a box &  $\sigma \subseteq \lambda$  a subshape. Define the **toggle**  $\tau_u(\sigma)$  to be

$$\tau_u(\sigma) := \begin{cases} \sigma \setminus u & \text{if } u \text{ is a removable from } \sigma; \\ \sigma \cup u & \text{if } u \text{ is addable to } \sigma; \\ \sigma & \text{otherwise.} \end{cases}$$

(Here u being addable means we can add u to  $\sigma$  and get a subshape.) For example,

$$\tau_{(1,2)}\left(\begin{array}{c} \\ \\ \\ \end{array}\right) = \begin{array}{c} \\ \\ \\ \end{array}$$

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### Toggle-symmetric distributions

For box  $u \in \lambda$ , define toggleability statistics  $\mathcal{T}_u^+, \mathcal{T}_u^-, \mathcal{T}_u$  on subshapes by

$$\mathcal{T}_{u}^{+}(\sigma) := \begin{cases} 1 & \text{if } u \text{ is addable to } \sigma; \\ 0 & \text{otherwise,} \end{cases} 
\mathcal{T}_{u}^{-}(\sigma) := \begin{cases} 1 & \text{if } u \text{ is removable from } \sigma; \\ 0 & \text{otherwise,} \end{cases} 
\mathcal{T}_{u}(\sigma) := \mathcal{T}_{u}^{+}(\sigma) - \mathcal{T}_{u}^{-}(\sigma).$$

#### Definition

A probability distribution  $\mu$  on subshapes is called **toggle-symmetric** if we have  $\mathbb{E}_{\mu}(\mathcal{T}_u) = 0$  for all boxes  $u \in \lambda$ .

In other words, we are as likely to be able to toggle u in as toggle it out.

### SYT & plane partition distributions are toggle-symmetric

### Lemma (Chan-Haddadan-H.-Moci, 2017)

- The distribution  $\mu_{SYT}$  is toggle-symmetric.
- For any  $m \ge 1$ , the distribution  $\mu_{PP_m}$  is toggle-symmetric.

Proof sketch: For  $\mu_{SYT}$ : use  $\mu_{SYT} = \lim_{m \to \infty} \mu_{PP_m}$ .

For  $\mu_{\mathrm{PP}_m}$ : for any  $\pi \in \mathcal{PP}_m(\lambda)$ , the contribution of  $\pi$  to  $\mathbb{E}_{\mu_{\mathrm{PP}_m}}(\mathcal{T}_u)$  is negative the contribution of  $\tau_u(\pi)$ , where the **(piecewise-linear) plane** partition toggle  $\tau_u(\pi)$  is defined by the formula

with  $u' := \min(w, x) + \max(y, z) - u$ .  $\square$ 

### Down-degree as sum of toggleability statistics

What's the point? We can sometimes write down-degree in a clever way...

#### Theorem (Chan-Haddadan-H.-Moci, 2017)

For the rectangle  $\lambda = a \times b$ , there are coefficients  $c_u \in \mathbb{Q}$ ,  $u \in \lambda$  for which

$$ddeg = \frac{ab}{a+b} + \sum_{u \in \lambda} c_u \mathcal{T}_u$$

By linearity of expectation we obtain enumerative corollaries:

#### Corollary

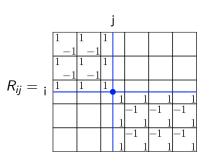
$$\frac{\#\mathcal{SYT}^{+1}(\mathsf{a}\times \mathsf{b})}{(\mathsf{n}+1)\cdot\#\mathcal{SYT}(\mathsf{a}\times \mathsf{b})} = \mathbb{E}_{\mu_{\mathrm{SYT}}}(\mathrm{ddeg}) = \frac{\mathsf{ab}}{\mathsf{a}+\mathsf{b}}$$

$$\frac{\#\mathcal{PP}_m^{+1}(a \times b)}{m \cdot \#\mathcal{PP}_m(a \times b)} = \mathbb{E}_{\mu_{\mathrm{PP}_m}}(\mathrm{ddeg}) = \frac{ab}{a+b}$$

### Key technical tool: "rooks"

How to write down-degree as a sum of the  $\mathcal{T}_u$ ? Note  $\mathrm{ddeg} = \sum_{u \in \lambda} \mathcal{T}_u^-$ . So the key is to find <u>relations</u> among the toggleability statistics.

The "building block" of toggleability statistics relations is the **rook**  $R_{ij}$ :



1	1	1						
-1	-1							
1	1	1						
-1	-1							
1	1	1						
			1	1		1		1
				-1	-1		-1	
			1	1		1		1
				-1	-1		-1	
			1	1		1		1

#### Lemma

We have  $R_{ii}(\sigma) = 1$  for any subshape  $\sigma \subseteq a \times b$ .

#### Section 5

q-analogs

### Comajor index for SYTs

Let  $T \in \mathcal{SYT}(\lambda)$  be a standard tableau. A **descent**\* of T is an entry i such that i+1 is in a higher row than i. Denote set of descents by D(T). The **comajor index** of T is  $\operatorname{comaj}(T) := \sum_{i \in D(T)} (n-i)$ .

### Theorem (g-Hook-Length-Theorem; Stanley, c. 1970?)

$$\sum_{T \in \mathcal{SYT}(\lambda)} q^{\operatorname{comaj}(T)} = \frac{[n]_q \cdot [n-1]_q \cdots [2]_q \cdot [1]_q}{\prod_{u \in \lambda} [h(u)]_q}$$

We use standard q-notation  $[n]_q:=1+q+\cdots+q^{n-1}=(1-q^n)/(1-q)$ .

Т	1 2 3 4	1 3 2 4	$\sum_{T \in \mathcal{SYT}(2 imes2)} q^{ ext{comaj}(T)} = q^2 + 1$
D(T)	Ø	{2}	$= \frac{[4]_q[3]_q[2]_q[1]}{[3]_q[2]_q[1]}$
comaj(T)	0	2	$[3]_q[2]_q[2]_q[1]$

### Comajor index for barely set-valued SYTs

Let  $T \in \mathcal{SYT}^{+1}(\lambda)$  be a barely set-valued tableau. Let  $i_*(T)$  denote the bigger number in the special box that has two numbers. A **descent** of T is an entry i such that i+1 is in a higher row than i, except that:

- $i_*(T) 1$  is <u>never</u> a descent;
- $i_*(T)$  is always a descent.

Denote set of descents by D(T). Let  $\operatorname{comaj}(T) := \sum_{i \in D(T)} (n+1-i)$ .

### Theorem (H.-Lazar-Linusson, 2021)

$$\sum_{T \in \mathcal{SYT}^{+1}(a \times b)} q^{\operatorname{comaj}(T)} = [ab+1]_q \cdot \frac{[a]_q[b]_q}{[a+b]_q} \cdot \sum_{T \in \mathcal{SYT}(\lambda)} q^{\operatorname{comaj}(T)}$$

### Comajor index generating functions: example

Т	1 2 3 4,5	1 3 2 4,5	1 2 3,4 5	1 3 2,4 5	1 4 2,3 5
D(T)	{5}	$\{2, 5\}$	{4}	{2,4}	{3}
$\operatorname{comaj}(T)$	0	3	1	4	2
Т	1 2,3	1 2, 4 3 5	1 3, 4 2 5	1, 2 3 4 5	1, 2 4 3 5
D(T)	{3}	{4}	{2,4}	{2}	{2,3}
comaj(T)	2	1	4	3	5

$$\sum_{T \in \mathcal{SYT}^{+1}(2 \times 2)} q^{\operatorname{comaj}(T)} = q^5 + 2q^4 + 2q^3 + 2q^2 + 2q + 1$$

$$= [5]_q \cdot \frac{[2]_q [2]_q}{[4]_q} \cdot (q^2 + 1)$$

### Size generating functions for plane partitions

The **size**  $|\pi|$  of a plane partition  $\pi \in \mathcal{PP}_m(\lambda)$  is the sum of its entries.

#### Theorem (MacMahon, c. 1915)

$$\sum_{\pi \in \mathcal{PP}_m(a \times b)} q^{|\pi|} = \prod_{i=1}^{\mathsf{a}} \prod_{j=1}^{b} \frac{[m+i+j-1]_q}{[i+j-1]_q}$$

Define size for barely set-valued plane partitions similarly.

#### Theorem (H.-Lazar-Linusson, 2021)

$$\sum_{\pi \in \mathcal{PP}_m^{+1}(\mathsf{a} imes b)} q^{|\pi|-1} = [m]_q \cdot rac{[\mathsf{a}]_q [b]_q}{[\mathsf{a} + b]_q} \cdot \sum_{\pi \in \mathcal{PP}_m(\mathsf{a} imes b)} q^{|\pi|}$$

### Size generating functions: example

$$\sum_{\pi \in \mathcal{PP}_1(2 \times 2)} q^{|\pi|} = q^4 + q^3 + 2q^2 + q + 1 = \frac{[4]_q[3]_q[3]_q[2]_q}{[3]_q[2]_q[2]_q[1]_q}$$

$$\sum_{\pi \in \mathcal{PP}_1^{+1}(2 \times 2)} q^{|\pi|-1} = q^3 + 2q^2 + 2q + 1 = [1]_q \cdot \frac{[2]_q [2]_q}{[4]_q} \cdot (q^4 + q^3 + 2q^2 + q + 1)$$

### Proofs of q-analogs: q-toggle-symmetry

The basic outline of proofs for q-analogs is same as in case q = 1.

For a box u of  $\lambda$ , set  $\mathcal{T}_u^q := \mathcal{T}_u^+ - q \mathcal{T}_u^-$ . Call a probability distribution  $\mu$  on subshapes q-toggle-symmetric if  $\mathbb{E}_{\mu}(\mathcal{T}_u^q) = 0$  for all  $u \in \lambda$ .

We define appropriate q-analogs of distributions  $\mu_{\mathrm{SYT}}^q$  and  $\mu_{\mathrm{PP}_m}^q$  and show:

#### Lemma (H.-Lazar-Linusson, 2021)

The distributions  $\mu_{\mathrm{SYT}}^{q}$  and  $\mu_{\mathrm{PP}_{m}}^{q}$  are q-toggle-symmetric.

The other ingredient of the proof is:

#### Theorem (Defant-H.-Poznanović-Propp, 2021)

For  $\lambda = \mathsf{a} \times \mathsf{b}$ , there are coefficients  $c_u(q) \in \mathbb{Q}(q), u \in \lambda$  for which

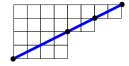
$$ddeg = \frac{[a]_q[b]_q}{[a+b]_q} + \sum_{u \in \lambda} c_u(q) \mathcal{T}_u^q$$

#### Section 6

### Concluding remarks

### Concluding remarks

• Not all shapes  $\lambda$  have product formulas for  $\#\mathcal{SYT}^{+1}(\lambda)$ , but the rook technique does work for a broader class of "balanced" shapes:



- Can also look at shifted shapes (see Kim–Schlosser–Yoo (2020)), other posets, etc. In fact the q-analogs hold for all minuscule posets.
- Beyond distributive lattices: "nearly reduced words," ... (see RTY).
- There are interesting toggle-symmetric distributions not coming from tableaux/plane partitions. For instance, some come from dynamics on subshapes. Related to study of homomesy for these dynamics.

## Thank you!

these slides are available on my website and papers are on the arXiv:

- Chan, Haddadan, Hopkins, and Moci. "The expected jaggedness of order ideals." arXiv:1507.00249
- Reiner, Tenner, and Yong. "Poset edge densities, nearly reduced words, and barely set-valued tableaux." arXiv:1603.09589.
- Hopkins, Lazar, and Linusson. "On the *q*-enumeration of barely set-valued tableaux and plane partitions." arXiv:2106.07418.
- Defant, Hopkins, Poznanović, and Propp. "Homomesy via toggleability statistics." arXiv:2108.13227.