10/28 Relations §3.3

You can think of a relation from one set X to another set Y as a chart that lists how elements from X are "related" to elements from Y. For example, we can imagine a chart that lists for each student in a school the classes they're taking.

Student	Class
Bill	Economics
Bill	English
Alexis	English
Jordan	Chemistry ...

Notice that unlike a function, each student can take multiple classes. Also, maybe a student is taking no classes at all (e.g. they're on a leave of absence).

Def'n Formally, a (binary) relation R from set X to set Y is any subset of $X \times Y$, i.e., any set of ordered pairs (x,y) w/ $x \in X$ and $y \in Y$.
If $(x,y) \in R$ then we write $x R y$ and say that "x is related to y."

E.g. With the student/class example the relation is
$R = \{$(Bill, Econ.), (Bill, Eng.), (Alexis, Eng.), (Jordan, Chem)...$\}$.
and since Alexis is taking English we could also write Alexis R English.

Notice: A function $f: X \to Y$ is a very special kind of relation from X to Y.

The most important relations are when $X = Y$:

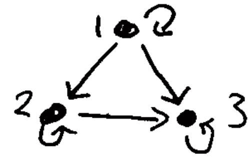
<u>Def'n</u> If R is a relation from X to X, we say it is a <u>relation</u> <u>on</u> the <u>set</u> X.

<u>E.g.</u> If $X = \{1,2,3\}$ then $\leq$ defines a relation on X (i.e. "a is related to b" if and only if "$a \leq b$").

The set of ordered pairs for this relation is:

$R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$.

We can represent this same information as a <u>digraph</u>



Here we draw a "vertex" (dot •) for each element of X, and draw an arrow $a \bullet\!\!\longrightarrow\!\bullet\, b$ whenever $a R b$.

Notice that if $a R a$ then we have a <u>loop</u>: a

<u>Def'n</u> The relation R on X is <u>reflexive</u> if $x R x$ for all $x \in X$.

<u>E.g.</u> The $\leq$ relation on $\{1,2,3\}$ is reflexive (means we have a <u>loop</u> at every vertex).

But if we did the relation given by $<$ instead:

this is <u>not</u> reflexive (no loops at all here).

Def'n The relation R on X is <u>symmetric</u> if whenever $x R y$ for $x, y \in X$ we also have $y R x$.

E.g. The relation $\leq$ on $\{1,2,3\}$ is <u>not</u> symmetric since $1 \leq 2$ but $2 \not\leq 1$. For a symmetric relation we need arrows to look like:

$$a \bullet \rightleftarrows \bullet b \qquad \Big(\text{or } \underset{\text{no arrows}}{a \bullet \qquad \bullet b}\Big)$$

E.g. An example of a symmetric relation R is
$X = \{\text{Students at Howard}\}$ and
$x R y$ if "x has a class with y" for $x, y \in X$.
This is because if Person x has a class with Person y, well then also Person y has a class with Person x (the same class that they're in!).

Relations $\leq$ are "opposite" from symmetric, so:

Def'n The relation R on X is <u>anti-symmetric</u> if whenever $x R y$ and $y R x$ for $x, y \in X$ then $x = y$.

E.g. The relation $\leq$ (on $X = \{1,2,3\}$ or on any set of numbers like $\mathbb{Z}, \mathbb{R}$, etc.) is antisymmetric since if
$x \leq y$ and $y \leq x$ then we must have $x = y$.
(The relation $<$ is also anti-symmetric since there are no x, y at all with $x < y$ and $y < x$).

<u>Antisymmetric</u>: No $\cancel{a \bullet \rightleftarrows \bullet b}$ but loops $\overset{\circlearrowleft}{\bullet}_{a}$ OK ✓

10/31

There is one more important property of relation $\leq$:

<u>Def'n</u> A relation R on X is called <u>transitive</u> if whenever $x R y$ and $y R z$ for $x, y, z \in X$, we must have that $x R z$:

must have the dotted arrow in this triangle.

<u>E.g.</u> The relation $\leq$ (or $\geq$) is transitive because if $a \leq b$ and $b \leq c$ then certainly $a \leq c$.

<u>Q</u>: Is the relation "has a class with" on students transitive?

<u>A</u>: <u>No</u>! Maybe Bill has a class with Alexis (like English) and Alexis has a class with Cole (like Biology), but Bill has no class with Cole (he's not taking Biology).

<u>Def'n</u> A relation R on X that is:

- <u>reflexive</u>
- <u>anti-symmetric</u>
- and <u>transitive</u>

is called a <u>partial order</u> on X.

<u>E.g.</u> $\leq$ is a partial order on $X = \{1,2,3\}$ (or any set of numbers)

Partial orders behave like $\leq$: they let us "<u>compare</u>" things in X.

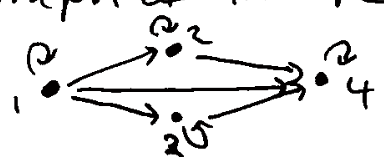
E.g. Consider a list of tasks you have to do to complete some project. Maybe the project is "make a PB&J sandwich" and so the tasks are:

1. Toast two slices of bread.
2. Spread peanut butter on one slice.
3. Spread jelly on the other slice.
4. Put the two slices together.

Some of these tasks have to be done before others (e.g. have to do 1 before 2). So define relation R on the set of tasks by:

$i R j$ if $i = j$ or task i must be done before task j.

The digraph of this relation is:



reflexive ✓
anti-symmetric ✓
transitive ✓

Notice how no arrows between 2 and 3 since spreading PB and J can be done in either order.

Also: notice we get a partial order on the tasks!

If R is a partial order on X and $x, y \in X$ we say x and y are comparable if $x R y$ or $y R x$ and say they are incomparable otherwise.

E.g. In PB&J example, the tasks of spreading PB and spreading J are incomparable (can be done in any order).

The partial order R on X is called a total order if every pair $x, y \in X$ is comparable.

E.g. Relation $\leq$ (on any set of #'s) is a total order, but "do before" relation on tasks not a total order.

Compositions of relations and inverse relations

Now let's return to discussing relations R from X to Y. Recall that a function $f: X \to Y$ is a special such relation, and we can generalize to relations the important functional notions of composition and inversion.

Def'n Let R_1 be a relation from X to Y and R_2 a relation from Y to Z. The composition $R_2 \circ R_1$ is a relation from X to Z where we have

for $x \in X$, $z \in Z$: $x\,(R_2 \circ R_1)\,z$ if and only if there is $y \in Y$ with $x \, R_1 \, y$ and $y \, R_2 \, z$.

$$X \xrightarrow{\;R_1\;} Y \xrightarrow{\;R_2\;} Z \implies X \xrightarrow{\;R_2 \circ R_1\;} Z$$

Def'n Let R be a relation from X to Y. The inverse relation R^{-1} is a relation from Y to X where

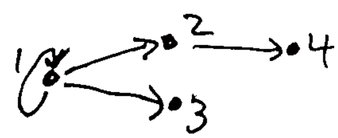
$$R^{-1} = \{(y, x) : (x, y) \in R\}.$$

"reverse" every ordered pair.

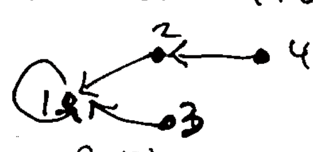
Note: For a function $f: X \to Y$, the inverse $f^{-1}: Y \to X$ is defined only when f is a bijection (1-to-1 and onto). But inverse relation R^{-1} is always defined

Note: If R is relation on X (i.e. from X to X), then digraph of R^{-1} obtained from digraph of R by reversing direction of all arrows.

e.g.



R



R^{-1}

11/2

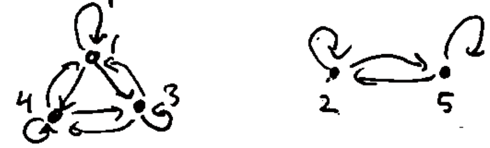
Equivalence Relations §3.4

Let X be a set. Recall that a partition of X is a collection S of nonempty subsets of X such that every element of X belongs to exactly one of the subsets in S.

E.g. If $X = \{1,2,3,4,5\}$ then one partition of X is $S = \{\{1,3,4\}, \{2,5\}\}$

A partition of X is a way of "breaking X into groups" and we can use a partition S to define a relation on X: we have xRy if and only if x and y are in same subset in S.

E.g. with previous set partition, the digraph of R is:



Theorem The relation R on set X defined from a partition S of X is:
- reflexive
- symmetric
- and transitive.

PS: These are all easy to check directly. Reflexive: x is in same subset as x. Symmetric: if x is in same subset as y, then y is same as x. Transitive: if x and y in same subset, and y and z, then same for x and z. ∎

Def'n A relation R on X that is:
- reflexive
- symmetric ← (compare to def. of partial order)
- and transitive

is called an equivalence relation on X.

An equivalence relation on X is a way elements of X can be "the same."

E.g. Relation R on $\mathbb{R}$ where xRy if $x^2=y^2$ is an equiv. relation.

E.g. Let n be any positive integer. We define relation R on $\mathbb{Z}$ by xRy if $x-y$ is a multiple of n.

Exercise: This is an equivalence relation on $\mathbb{Z}$.

We've seen that partitions give equiv. relations. Converse is also true.

Theorem Let R be an equiv. relation on X. Let $a \in X$ be any element and define $[a] := \{x \in X : x R a\}$ (things that relate to a). Then $S = \{[a] : a \in X\}$ is a partition of X.

Pf: We need to show that every $x \in X$ belongs to exactly one subset in S. It clearly belongs to $[x]$ (by reflex. of R). So suppose it also belonged to $[y]$. We want to show that $[x]=[y]$. So let $z \in [x]$. Then zRx, and since xRy, we have zRy, i.e. $z \in [y]$ (using trans.). By symmetry, also have yRx, so if $z \in [y]$ then by same argument $z \in [x]$. Thus $[x]=[y]$, as claimed. ∎

Def'n The sets $[a]$ for $a \in X$ from the previous theorem are called the equivalence classes of the equiv. relation R.

E.g. With R being equiv. relation on $\mathbb{R}$ w/ xRy if $x^2=y^2$, equivalence classes are $\{a, -a\}$ for $a \in \mathbb{R}$, i.e. each number is grouped with its negative.

E.g. Exercise what are the equivalence classes for the "$x-y$ is a multiple of n" equiv. relation on the integers $\mathbb{Z}$?

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11/4

Combinatorics: Basic Counting Principles §6.1

We are now starting a new chapter (our last of the semester): Chapter 6 on combinatorics, which is just a fancy word for "counting!" We will learn many techniques for counting the elements of a finite set.

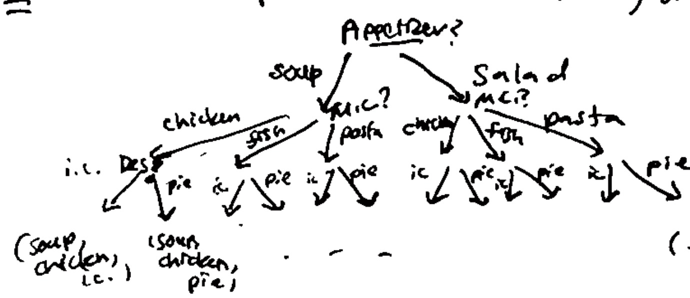
We start with some basic counting principles:

E.g. Suppose for a meal you get to choose

- an appetizer: either soup or salad,
- a main course: chicken, fish, or pasta,
- a dessert: either ice cream or pie.

How many total different meals are possible with these choices?

A: We can represent all the choices by drawing a "decision tree":



we talked about these a while ago

We see that there are $12 = 2 \times 3 \times 2$ total meals: We multiply the choices at each step to get total.

Thm (Multiplication Principle for Counting)

Suppose we make an object via a series of steps, where we have k_1 choices for step 1, k_2 choices for step 2, etc. down to k_m choices for step m. Then the total # of objects we can make is $k_1 \times k_2 \times \cdots \times k_m$.

Remark: We saw before that for Cartesian product $X_1 \times X_2 \times \cdots \times X_m$ of sets we have $\#(X_1 \times \cdots \times X_m) = \#X_1 \times \#X_2 \times \cdots \times \#X_m$. This is basically the same as multiplication principle.

Let's see some more examples of multiplication principle,

<u>Q</u>: A US telephone # is 10 digits long, and the 1st digit cannot be a zero. How many telephone #'s are there?

<u>A</u>: We have 9 digits to choose for the 1st digit and 10 for each of the 9 other digits, so by multiplication principle:

$$9 \times \overbrace{10 \times 10 \times \cdots \times 10}^{9 \text{ times}} = 9 \times 10^9 = 9 \text{ billion}.$$

<u>Recall</u>: We've seen that the total # of subsets of $\{1, 2, \ldots, n\}$ is 2^n.

This is easy to see with the multiplication principle:

to make a subset we decide:
- include 1 or not? (2 choices)
- include 2 or not? (2 choices)
- ... include n or not? (2 choices)

This is n steps, with 2 choices at each step, so in total, # possibilities $= \overbrace{2 \times 2 \times \cdots \times 2}^{n \text{ times}} = 2^n$. ✓

Sometimes it takes a little more thought to see how to apply the multiplication principle to a counting problem:

<u>Q</u>: How many reflexive relations on $X = \{1, 2, \ldots, n\}$ are there?

<u>A</u>: Remember that a relation is a subset of $X \times X$. We know that (x, x) must be in our relation R for each $x \in X$ so that it will be reflexive. How many choices do we have ~~for~~ other elements of R?

Well, for each $(x, y) \in X \times X$ with $x \neq y$, we can include that (x, y) or not ($=$ 2 choices). The # of (x, y) w/ $x \neq y$ is $n \times (n-1)$ since we can choose any $x \in X$ for 1st coordinate, and then one of the

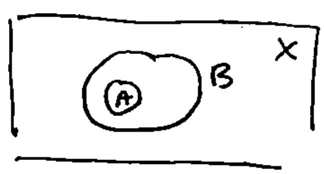
$(n-1)$ other elements of X for the 2nd coordinate y.
Thus, we have $n \times (n-1)$ binary choices to make our reflexive relation R, so # of such relations is:

$$\underbrace{2 \times 2 \times \cdots \times 2}_{n \times (n-1)} = 2^{n(n-1)} = 2^{n^2 - n}.$$

(A similar, but simpler, argument shows that there are 2^{n^2} total relations on $X = \{1, 2, \ldots, n\}$.)

Q: Let $X = \{1, 2, \cdots, n\}$. How many ordered pairs (A, B) of subsets of X satisfying $A \subseteq B \subseteq X$ are there?

A: It is helpful to draw a Venn diagram of our situation:



← We see that there are 3 "regions" in this Venn diagram:

- things in A, • things in $B - A$, • things in $X - B$.

To make an ordered pair (A, B) of this form, we can therefore choose for each $i = 1, 2, \cdots, n$ where to place i:
- Place 1 in A, $B - A$, or $X - B$? (3 choices)
- Place 2 in A, $B - A$, or $X - B$? (3 choices)
- Place n in A, $B - A$, or $X - B$? (3 choices).

Thus, we have n steps with 3 choices at each step, so total # of possibilities $= 3 \times 3 \times \cdots \times 3 = \boxed{3^n}$.

Exercise: What about (A, B, C) with $A \subseteq B \subseteq C \subseteq \{1, 2, \ldots, n\}$?

11/7 Addition Principle + Principle of Inclusion - Exclusion §6.1

Sometimes we are trying to count objects that have multiple "kinds":

E.g. Q: Let $X = \{a, b\}$. How many strings in X^* are there which have length 3 or length 4?

A: The # of strings of length 3 in $X^* = 2 \times 2 \times 2 = 2^3$ by mult. principle.

# of strings of length 4 $= 2 \times 2 \times 2 \times 2 = 2^4$

# of strings of length 3 or 4 $= 2^3 + 2^4 = 8 + 16 = 24$.

We see another counting principle in action here:

Theorem (Addition Principle for Counting)

If $X_1, X_2, \ldots, X_m$ are disjoint sets (meaning $X_i \cap X_j = \emptyset$ for all $i \neq j$, i.e. sets have no common elements)

then $\#(X_1 \cup X_2 \cup \cdots \cup X_m) = \#X_1 + \#X_2 + \cdots + \#X_m$.

We see that, as long as the sets are disjoint, we can count any grouping of sets just by adding together:

E.g. Q: # of strings in $\{a, b\}^*$ of length 3 or 4 or 5?

A: $2^3 + 2^4 + 2^5$, by the addition principle.

E.g. Alexis, Ben, Cole, David, and Erica are a 5 person group. They have to choose a: President, Vice President, & Treasurer.

a) How many ways are there to do this?

b) How many ways are there if we require that either Alexis or Ben is the President?

a): We can choose any of the 5 people for prez. Then for VP we can choose any of the remaining 4, and similarly for treasurer we can choose any of the remaining 3.

By mult. principle: $5 \times 4 \times 3 = 60$.

b) If Alexis is President, we have $4 \times 3 = 12$ choices for VP + treasurer.
If Ben is prez, we also have $4 \times 3 = 12$ choices for VP + treasurer.
By addition principle # of choices is $12 + 12 = 24$.

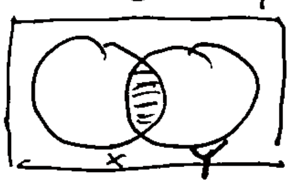
But what if the sets are not disjoint? Then we use:

Theorem (Principle of Inclusion - Exclusion)

$$\#(X \cup Y) = \#X + \#Y - \#(X \cap Y)$$

notice $X \cap Y = \emptyset$ if X + Y are disjoint.

To see why P.I.E. works, look at Venn diagram



when we add $\#X$ to $\#Y$, we count things in $X \cap Y$ double, have to subtract $-\#(X \cap Y)$ to correct.

E.g. c) How many ways are there to pick prez, VP, + treasurer, where either Alexis is Prez. or Ben is VP?

c): Let X = assignments where Alexis is Prez.
Then $\#X = 4 \times 3$, # of choices of VP + treasurer.
Let Y = assignments where Ben is VP
Then $\#Y = 4 \times 3$, # of choices of Prez + treasurer.
We want to compute $\#(X \cup Y)$. By P.I.E., we also need to know the size of $X \cap Y$:
$\#X \cap Y = 3$, since if A is Prez and B is VP we can choose any of remaining 3 to be treasurer.

So... $\#(X \cup Y) = \underset{\#X}{12} + \underset{\#Y}{12} - \underset{\#(X \cap Y)}{3} = \boxed{21}$ ways to make Alexis Prez. or Ben VP.