Back to the Fuss-er: Catalan objects, lattices, and dynamics

BIRS Workshop on Lattice Theory
Banff, Canada

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Section 1

Lattices on Catalan objects & their dynamics

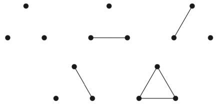
Catalan numbers

The **Catalan numbers** C_n, for $n \geq 1$, are a famous sequence of numbers

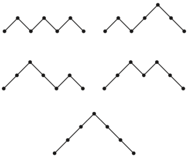
$$1, 2, 5, 14, 42, 132, 429, 1430, \ldots,$$

which count numerous sets of combinatorial objects including:

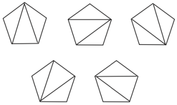
noncrossing partitions of $[n] = \{1, \ldots, n\}$



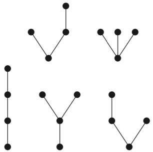
Dyck paths of length $2n$



triangulations of an $(n+2)$-gon



plane trees with $n+1$ nodes



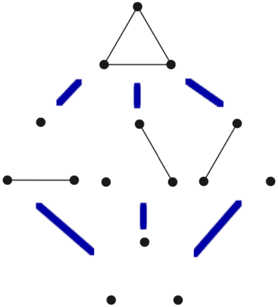
There is a well-known product formula for the Catalan numbers:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

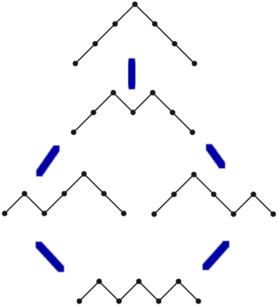
Catalan lattices

There are three important **lattices** on Catalan objects:

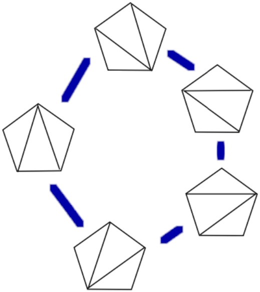
Kreweras's **lattice of noncrossing partitions**



Stanley's **lattice of Dyck path**



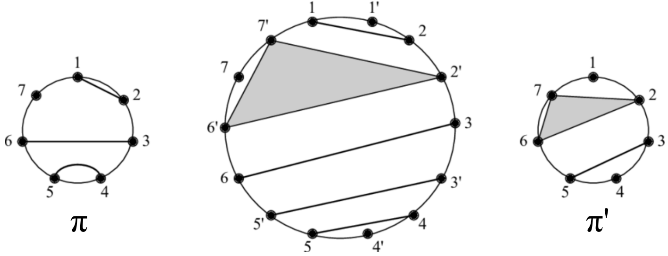
the **Tamari lattice** of triangulations



Each lattice gives us (the same!) **dynamics** on the Catalan objects...

Kreweras complement of noncrossing partitions

The **Kreweras complement** of a noncrossing partition π of $[n]$ is another noncrossing partition π' with $\pi \vee \pi' = \hat{1}$ and $\pi \wedge \pi' = \hat{0}$, defined as follows:

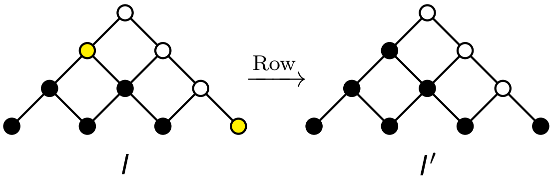


The square of $\mathrm{Krew}\colon \pi \mapsto \pi'$ is **rotation**, hence Krew itself has order $2n$.

(In fact, Krew is the same as rotation of **noncrossing matchings** of $[2n]$.)

Rowmotion of order ideals

Rowmotion acts on the **distributive lattice** $J(P)$ of **order ideals** (downwards-closed sets) of a finite poset P. It sends an order ideal I to the order ideal I' generated by the minimal elements of $P \setminus I$:

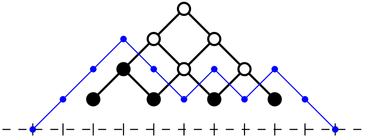


Row is an invertible operator. In fact, Cameron and Fon-der-Flaass showed how to write rowmotion as a composition of **toggles** (local involutions).

For most distributive lattices $J(P)$, rowmotion behaves chaotically, but...

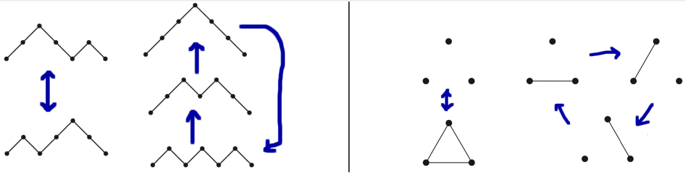
Rowmotion for Dyck paths

Dyck paths corresponds to order ideals in a **triangle poset**:



Theorem (Armstrong–Stump–Thomas)

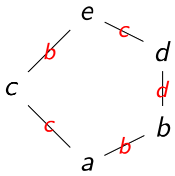
There is a bijection between Dyck paths of length $2n$ and noncrossing partitions of $[n]$ that takes Row *to* Krew.



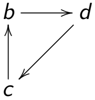
(Adenbaum and Elizalde showed that their bijection is essentially RSK.)

Tamari lattice rowmotion

Barnard explained how any **semidistributive** lattice L has a canonical labeling of its edges by **join irreducible** elements. **Rowmotion** on L sends element with up label set S to unique element with down label set S:



$a \leftrightarrow e$



Moreover, Barnard showed how for the **Tamari lattice**, which is semidistributive, this rowmotion again is the Kreweras complement!

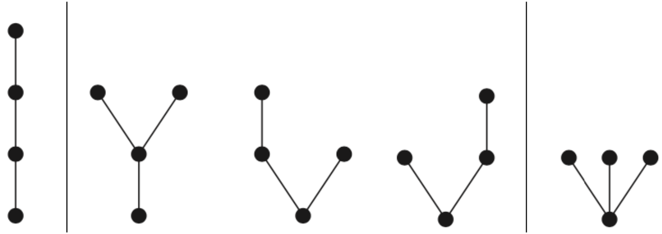
(Thomas and Williams expressed Tamari lattice rowmotion via toggles.)

Section 2

Plane tree statistics & averages

Plane trees by leaves: Narayana numbers

For $1 \leq k \leq n$, the **Narayana number** $N(n,k)$ is the number of (rooted) **plane trees** on $n+1$ nodes with k **leaves**:



$N(3,1) = 1$ $\quad$ $N(3,2) = 3$ $\quad$ $N(3,3) = 1$

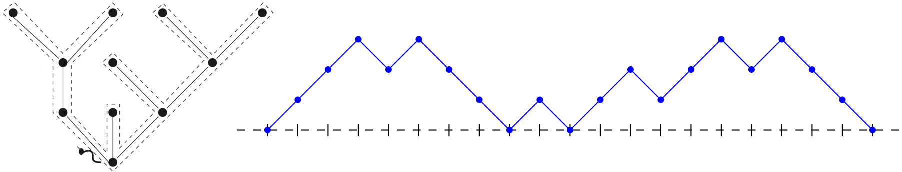
Of course, $C_n = \sum_{k=1}^{n} N(n,k)$. In fact, we have explicit product formula:

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

Notice the **symmetry** that $N(n,k) = N(n, n+1-k)$.

Reinterpretation: Dyck paths by peaks

There's a classic bijection from plane trees on $n+1$ nodes to Dyck paths of length $2n$ where we traverse the exterior of the tree, taking up steps when we move away from the root and down steps when we move towards it:



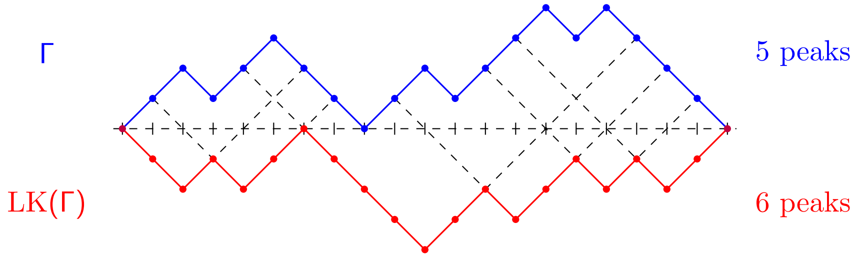
Under this bijection, **leaves** of the tree become **peaks** of the Dyck path.

But "why" is the peak statistic on Dyck paths symmetrically distributed?

Lalanne–Kreweras involution

The **Lalanne–Kreweras involution** is a map on Dyck paths which combinatorially demonstrates the symmetry of the Narayana numbers:

$$\#\mathrm{peaks}(\Gamma) + \#\mathrm{peaks}(\mathrm{LK}(\Gamma)) = n + 1.$$

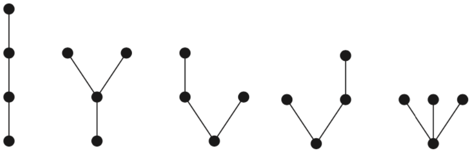


(Panyushev, and H. and Joseph, related the LK involution to rowmotion.)

Number of leaves at height one

From the symmetry of the Narayana numbers we get that the **average** number of leaves of a plane tree on $n+1$ nodes is $\frac{n+1}{2}$.

Another interesting statistical average: among plane trees on $n+1$ nodes, the average number of leaves **at height one** is 1.



$$(0+0+1+1+3)/5 = 1$$

(I learned about these leaf statistic averages from Lou Shapiro, who showed me how they can be proved by generating function arguments.)

Can we "explain" this average using an operator, like the LK involution?

Section 3

The homomesy phenomenon

Homomesy

Let X be a collection of combinatorial objects, $\varphi\colon X \to X$ an invertible operator, and f a statistic on X. We say f is **homomesic** with respect to the action of φ on X if every φ-orbit has the same f average.

For example, if X is the collection of binary strings with a 0's and b 1's, and φ is rotation, then the statistic f which is the first letter of the string is homomesic w.r.t. φ with average $\frac{b}{a+b}$:

$$\cdots \to 0011 \to 0110 \to 1100 \to 1001 \to \cdots \qquad (0+0+1+1)/4 = 1/2$$
$$\cdots \to 0101 \to 1010 \to \cdots \qquad (0+1)/2 = 1/2$$

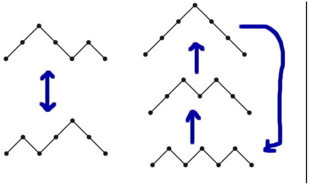
The homomesy paradigm was introduced by Propp and Roby and has since been applied throughout algebraic combinatorics.

Rowmotion homomesy for number of peaks

Panyushev conjectured, and Armstrong–Stump–Thomas proved, that the **"antichain cardinality"** statistic is homomesic for rowmotion acting on the order ideals of the triangle poset. Rephrased:

Theorem (Armstrong–Stump–Thomas)

The number of peaks statistic is homomesic with respect to the action of rowmotion on Dyck paths of length $2n$, with average $\frac{n+1}{2}$.



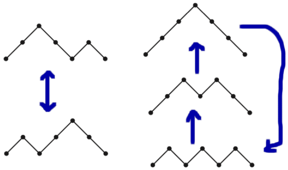
$$(2+2)/2 = 2$$
$$(3+2+1)/3 = 2$$

Homomesy for number of peaks at height one

By interpreting Dyck paths as **321-avoding permutations**, where the number of peaks at height one becomes the number of **fixed points**, Adenbaum and Elizalde established the following.

Theorem (Adenbaum–Elizalde)

The number of peaks at height one statistic is homomesic with respect to the action of rowmotion on Dyck paths of length $2n$, with average 1.



$$(1+1)/2 = 1$$
$$(3+0+0)/3 = 1$$

Section 4

The Fuss-er

Fuss–Catalan numbers

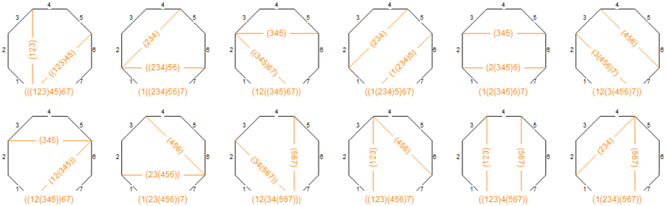
Let $m \geq 1$ be another integer parameter. The **Fuss–Catalan number** is

$$C_n^m = \frac{1}{(m+1)n+1}\binom{(m+1)n+1}{n}.$$

For $m = 1$ we have $C_n^1 = C_n$, the usual Catalan number.

And just like the Catalan number C_n counts many combinatorial objects, the Fuss–Catalan number C_n^m also counts many combinatorial objects.

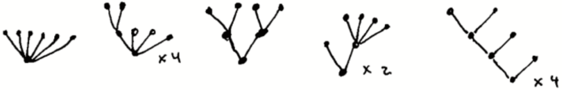
For example, C_n^m is the number of $(m+2)$-angulations of an $(mn+2)$-gon:



$$C_3^2 = \frac{1}{10}\binom{10}{3} = 12$$

m-plane trees & statistical averages

An m-**plane tree** is a plane tree all of whose nodes have a multiple of m children. The number of m-plane trees on $(mn+1)$ nodes is C_n^m:



Among m-plane trees with $(mn+1)$ nodes, the average number of leaves is $\frac{m}{m+1}(mn+1)$:

$$(6+4\times 5+4+2\times 5+4\times 4)/12 = 56/12 = \frac{2}{3}(7)$$

And the average number of leaves at height one is m:

$$(6+4\times 3+0+2\times 1+4\times 1)/12 = 24/12 = 2$$

(These averages can again be proved easily via generating functions.)

The proposed problem

The proposed problem is to extend the Dyck path rowmotion homomesies to the Fuss–Catalan setting:

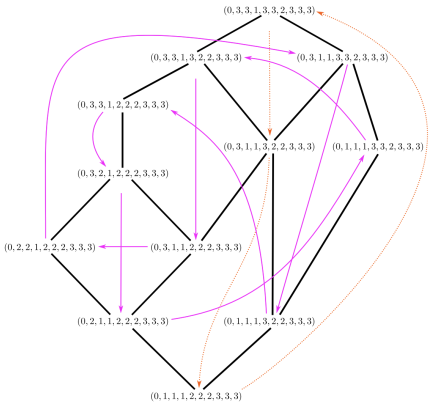
Problem

Define an operator acting on m-plane trees with $mn + 1$ *nodes for which the number of leaves and number at height one statistics are homomesic.*

Of course, we want the operator to be "natural" in some sense.
For starters, it should have a small, predictable order like $(m+1)n$.

In fact, we have a guess as to what the operator should be...

The m-Tamari lattice



The m-**Tamari lattice** is a semidistributive lattice which has C_n^m many elements.

Thomas and Williams conjecutred, and Defant and Lin proved, that m-Tamari lattice rowmotion has order $(m+1)n$.

It is very likely that this m-Tamari lattice rowmotion is the right operator for our problem.

Thank you!

see the problem description for more details and references:

https://drive.google.com/file/d/1EfNCJNIzDdzaIMPkNgnywAgI6jP_gs9I/