Howard Math 273, HW# 2,

Fall 2021; Instructor: Sam Hopkins; Due: Friday, November 5th

1. Fix a positive integer k. We showed the ordinary generating function $F_k(x) := \sum_{n\geq 0} S(n,k)x^n$ of the Stirling numbers of the 2nd kind satisfies $F_k(x) = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$. Find the partial fraction decomposition of $F_k(x)$, i..e., find the coefficients $a_j \in \mathbb{R}$, $j = 1,2,\ldots,k$, for which $F_k(x) = \frac{a_1}{(1-x)} + \frac{a_2}{(1-2x)} + \cdots + \frac{a_k}{(1-kx)}$. Conclude $S(n,k) = \sum_{j=1}^{k} a_j \cdot j^n$.
   Hint: clear denominators, and then plug in $x = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{k}$.
   Bonus just to think about, not do: prove $S(n,k) = \sum_{j=1}^{k} a_j \cdot j^n$ using (i) the *exponential* g.f. $\sum_{n\geq 0} S(n,k)\frac{x^n}{n!} = \frac{1}{k!}(e^x-1)^k$; or (ii) the Principle of Inclusion-Exclusion (P.I.E.).

2. *(Stanley, EC1, #2.2)* Let A be some finite set of objects, and suppose these objects potentially posses n different *properties* $p_1, p_2, \ldots, p_n$: e.g., $p_1 =$ "is green"; $p_2 =$ "is solid"; et cetera. For $X \subseteq [n]$, let $f_=(X)$ denote the number of elements in A possessing *exactly* the properties p_i for $i \in X$ (and not possessing any of the properties p_j for $j \notin X$); and let $f_\geq(X)$ denote the number of elements in A possessing *at least* the properties p_i for $i \in X$ (but potentially also some properties p_j for $j \notin X$). Give a bijective proof of the P.I.E. identity
$$\sum_{X\subseteq[n]} f_=(X)(1+y)^{\#X} = \sum_{Y\subseteq[n]} f_\geq(Y)y^{\#Y},$$
   i.e., give a bijective proof, for each k, that the coefficients of y^k on the L- and RHS are equal.

3. *(Stanley, EC1, #2.25(a))* Let $f_i(m,n)$ be the number of $m \times n$ matrices of 0's and 1's, with a total of i 1's, and with at least one 1 in each row and column. Use the P.I.E. to show
$$\sum_{i\geq 0} f_i(m,n)t^i = \sum_{k=0}^{n}(-1)^k\binom{n}{k}((1+t)^{n-k}-1)^m.$$

4. *(Stanley, EC1, #2.25(b))* With $f_i(m,n)$ as in the previous problem, show that
$$\sum_{m,n\geq 0}\left(\sum_{i\geq 0} f_i(m,n)t^i\right)\frac{x^m\,y^n}{m!\,n!} = e^{-x-y}\cdot\sum_{m,n\geq 0}(1+t)^{mn}\frac{x^m\,y^n}{m!\,n!}.$$
   Hint: use the formula from the previous problem, and do some algebraic manipulations.

5. The q-binomial coefficient satisfies $\left[{n \atop k}\right]_q = \sum_{w\in\mathcal{W}_{n,k}} q^{\mathrm{inv}(w)}$, where $\mathcal{W}_{n,k}$ is the set of words that are rearrangements of $(n-k)$ 0's, and k 1's, and $\mathrm{inv}(w)$ is the number of inversions of w.
   Suppose $n = 2m$ is even. Prove that $\left[{n \atop k}\right]_{q:=-1}$ (the evaluation of the q-binomial at $q=-1$) is equal to $\#\mathcal{P}_{n,k}$, where $\mathcal{P}_{n,k}$ is the subset of words $w = w_1w_2\ldots w_n \in \mathcal{W}_{n,k}$ that are *palindromes* (i.e., which satisfy $w_i = w_{n+1-i}$ for all i). Do this by defining a **sign-reversing involution**. That is, define an involution $\tau\colon \mathcal{W}_{n,k} \to \mathcal{W}_{n,k}$ satisfying:
   - $\mathrm{inv}(w)$ and $\mathrm{inv}(\tau(w))$ have opposite parity for all $w \in \mathcal{W}_{n,k}$ with $\tau(w) \neq w$;
   - $\mathrm{inv}(w)$ is even for all $w \in \mathcal{W}_{n,k}$ with $\tau(w) = w$;
   - $\#\{w \in \mathcal{W}_{n,k}\colon \tau(w) = w\} = \#\mathcal{P}_{n,k}$.