Howard Math 273, HW# 1,

Fall 2023; Instructor: Sam Hopkins; Due: Friday, September 29th

1. *(Stanley, EC1, #1.66)* Let $p_k(n)$ denote the number of partitions of n into exactly k parts. Give a **bijective** proof that

$$p_0(n) + p_1(n) + p_2(n) + \cdots + p_k(n) = p_k(n+k).$$

   Hint: Think about Young diagrams.

2. *(Stanley, EC1, #1.5)* Show that

$$\sum_{n_1,\ldots,n_k \geq 0} \min(n_1,\ldots,n_k) x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} = \frac{x_1 x_2 \cdots x_k}{(1-x_1)(1-x_2)\cdots(1-x_k)\cdot(1-x_1x_2\cdots x_k)},$$

   where $\min(n_1,\ldots,n_k)$ means the minimum of the integers $n_1,\ldots,n_k$.

3. *(Stanley, EC1, #1.26)* Let $\overline{c}(n,m)$ denote the number of compositions of n into parts of size at most m. Show that

$$\sum_{n\geq 0} \overline{c}(n,m)x^n = \frac{1-x}{1-2x+x^{m+1}}.$$

4. Prove that, for any $n \geq 0$,

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2(n-k)}{n-k} = 4^n.$$

   Hint: We discussed the generating function $\sum_{n\geq 0}\binom{2n}{n}x^n$ of the central binomial coefficients. How can you use what we proved about this generating function to deduce the desired result?

5. Let $n \geq 1$, and let $\mathrm{ODD}(n)$ denote the subset of permutations in the symmetric group S_n with no cycles of even size. Prove that

$$\sum_{\sigma\in \mathrm{ODD}(n)} 2^{\#\mathrm{cycles}(\sigma)} = 2\cdot n!.$$

   Hint: Recall that we showed

$$\sum_{n\geq 0}\left(\sum_{\sigma\in S_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots t_n^{c_n(\sigma)}\right)\frac{x^n}{n!} = e^{t_1\frac{x}{1}+t_2\frac{x^2}{2}+t_3\frac{x^3}{3}+t_4\frac{x^4}{4}+\cdots},$$

   where $c_k(\sigma)$ is the number of cycles of σ of size k. How can you use this generating function identity to deduce the desired result?