Total: 47/50 + 5 bonus points for presenting = 52/50

Homework 2 – Combinatorics 2

1. Let $\lambda = (\lambda_1, \lambda_2, \ldots), \mu = (\mu_1, \mu_2, \ldots) \vdash n$ be partitions of n. Recall that the *lexicographic order* $\prec$ on partitions of n is given by $\mu \prec \lambda$ iff there is some j such that $\mu_i = \lambda_i$ for all $i < j$ and $\mu_j < \lambda_j$. It is a total order: we either have $\mu \prec \lambda$ or $\lambda \prec \mu$ or $\lambda = \mu$.

   A different order on partitions of n is the dominance order. The *dominance order* $\leq$ is defined by $\mu \leq \lambda$ iff $\mu_1 + \mu_2 + \cdots + \mu_j \leq \lambda_1 + \lambda_2 + \cdots + \lambda_j$ for all j. The dominance order is only partial order: we might have neither $\mu \leq \lambda$ nor $\lambda \leq \mu$.

   (a) Show that the lexicographic order *extends* the dominance order in the sense that if we have partitions $\lambda, \mu \vdash n$ with $\mu \leq \lambda$ and $\mu \neq \lambda$ then necessarily $\mu \prec \lambda$.

   (b) Give an example of partitions $\lambda, \mu \vdash n$ with $\mu \prec \lambda$ but $\mu \not\leq \lambda$.

a) Consider the partitions $\lambda, \mu \vdash n$ where $\mu \leq \lambda$ and $\mu \neq \lambda$.

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WLOG note that if the length of $\mu = s > t =$ length of λ, then $\mu < \lambda$ or μ isn't a partition of the same number since there is $\mu_{t+1}, \ldots, \mu_s$.

Now we know that $\mu_1 \leq \lambda_1$, $\mu_1 + \mu_2 \leq \lambda_1 + \lambda_2$, $\mu_1 + \mu_2 + \mu_3 \leq \lambda_1 + \lambda_2 + \lambda_3$, etc.

Observe that if $\mu_1 = \lambda_1$ and $\mu_1 + \mu_2 = \lambda_1 + \lambda_2$, then $\mu_1 = \lambda_1$, and if also $\mu_1 + \mu_2 + \mu_3 = \lambda_1 + \lambda_2 + \lambda_3$, then $\mu_3 = \lambda_3$, and so on.

If we continue this pattern, then we'll get $\mu_1 + \mu_2 + \ldots + \mu_{n-1} = \lambda_1 + \lambda_2 + \ldots + \lambda_{n-1}$ and $\mu_1 + \mu_2 + \ldots + \mu_n \leq \lambda_1 + \lambda_2 + \ldots + \lambda_n$. But recall that $\mu \neq \lambda$, so $\mu_1 + \mu_2 + \ldots + \mu_n = \lambda_1 + \lambda_2 + \ldots + \lambda_n$ can't be true because then $\forall i$, $\mu_i = \lambda_i \Rightarrow \mu = \lambda$. Thus, $\mu_1 + \mu_2 + \ldots + \mu_n < \lambda_1 + \lambda_2 + \ldots + \lambda_n$.

And $\mu_1 + \mu_2 + \ldots + \mu_{n-1} = \lambda_1 + \lambda_2 + \ldots + \lambda_{n-1} \Rightarrow \mu_n < \lambda_n \Rightarrow \mu \prec \lambda$.

Good!

b) Let $\mu = 67333$ and $\lambda = 682222$, then $\mu, \lambda \vdash n$ since $6+7+3+3+3 = 6+8+2+2+2+2 = 22$.

The partial sums for μ and λ are as follows:

μ	λ
6	6
13	14
16	16
19	18
22	20
22	22

Observe that for the second one, $\mu < \lambda$ and for the fourth one $\mu > \lambda \Rightarrow \mu \not\leq \lambda$. Also, $6 = 6$ and

Good.

$7 < 8 \implies \mu \prec \lambda$ $\square$

2. Show that we could've used dominance order instead of lexicographic order in our arguments about the triangularity of the transition matrices from p_λ or e_λ to m_μ. That is, show that

$$p_\lambda = \sum_{\lambda \le \mu} \alpha^\lambda_\mu \, m_\mu \qquad \text{and} \qquad e_\lambda = \sum_{\mu \le \lambda^t} \beta^\lambda_\mu \, m_\mu \quad \text{for coefficients } \alpha^\lambda_\mu, \beta^\lambda_\mu \in \mathbb{C}$$

for any $\lambda \vdash n$, where $\le$ is dominance order and λ^t is the transpose (a.k.a. conjugate) of λ.

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Show $p_\lambda = \alpha^\lambda_\lambda m_\lambda + \sum_{\mu \geq \lambda} \alpha^\lambda_\mu m_\mu$ for $\alpha^\lambda_\mu \in \mathbb{C}$.

By expanding p_λ, we get $(x_1^{\lambda_1} + x_2^{\lambda_1} + \ldots)(x_1^{\lambda_2} + x_2^{\lambda_2} + \ldots) \ldots (x_1^{\lambda_k} + x_2^{\lambda_k} + \ldots)$

$\implies$ there will be a term of $x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_k^{\lambda_k} \implies \alpha^\lambda_\lambda \neq 0$. The exponent vectors for $\mu \neq \lambda$ can all be obtained by summing up the parts of λ,

Yes, that's a good way of putting it.
Could also say mu is obtained by "combining the parts" of lambda.

so $\mu \geq \lambda \implies p_\lambda = \sum_{\lambda \le \mu} \alpha^\lambda_\mu m_\mu$.

Show $e_\lambda = \sum_{\mu \le \lambda^t} \beta^\lambda_\mu m_\mu$ for $\beta^\lambda_\mu \in \mathbb{C}$.

By expanding e_λ, we get $(x_1 x_2 \ldots x_{\lambda_1} \ldots)(x_1 x_2 \ldots \lambda_2 \ldots) \ldots \implies$ the biggest monomial in dominance order is $x_1^{\lambda_1^t} x_2^{\lambda_2^t} \ldots \implies \beta^\lambda_\mu \neq 0$. And the exponent vectors for $\mu \neq \lambda$ can all be obtained by summing up parts of μ, so $\mu \le \lambda^t \implies e_\lambda = \sum_{\mu \le \lambda^t} \beta^t_\mu m_\mu$.

Good.

3. Let $\lambda \vdash n$ and define f^λ to be the coefficient of $x_1 x_2 \cdots x_n$ in the Schur function $s_\lambda(x_1, x_2, \ldots)$. Explain why $f^\lambda = f^{\lambda^t}$. Give an example showing that this is not true for other coefficients of Schur functions (i.e., that $s_\lambda \neq s_{\lambda^t}$ in general).

8/10

Let $\lambda \vdash n$ and define f^λ to be the coefficient of $x_1 x_2 \ldots x_n$ in the Schur function $s_\lambda(x_1, x_2, \ldots)$.

By definition, the Semistandard Young Tableau is a filling of the Young Tableau where the rows are weakly increasing and the columns are strictly increasing. And since f^λ is the coefficient of $x_1 x_2 \ldots x_n$ where $x_1, x_2, \ldots, x_n$ all have order 1, each element appears in the Semistandard Young Tableau only once. This means that the rows are strictly increasing.

Now observe that filling the rows in λ equates to filling the columns in λ^t and filling the columns in λ equates to filling the rows in λ^t. So since every element appears once, making it so that the rows of λ and λ^t are strictly increasing, giving λ and λ^t the same restrictions $\Rightarrow$ λ and λ^t can be filled the same way $\Rightarrow f^\lambda = f^{\lambda^t}$ $\square$ Good. But what about an example of unequal coefficients? [-2pts]

4. The Cauchy–Binet formula says that if $A = (A_{i,j})$ is an $m \times n$ matrix and $B = (B_{i,j})$ is an $n \times m$ matrix, then the determinant of the $m \times m$ matrix AB can be computed by

$$\det(AB) = \sum_{I \subseteq [n],\ \#I = m} \det(A\mid_{\text{cols}=I}) \det(B\mid_{\text{rows}=I}).$$

Here, as always, $[n] := \{1, 2, \ldots, n\}$, and $A\mid_{\text{cols}=I}$ (resp., $B\mid_{\text{rows}=I}$) means the $m \times m$ matrix we get by restricting A to the columns in I (resp., by restricting B to the rows in I).

Deduce the Cauchy–Binet formula from the Lindström–Gessel–Viennot formula.

Hint: Consider the network with source vertices $s_1, \ldots, s_m$, target vertices $t_1, \ldots, t_m$, and internal vertices $k_1, \ldots, k_n$, and edges $s_i \to k_j$ with weight $A_{i,j}$ and $k_i \to t_j$ with weight $B_{i,j}$.

9/10 Let $A = (A_{i,j})$ be an $m \times n$ matrix and $B = (B_{i,j})$ be an $n \times m$ matrix.

Consider the network with source vertices $s_1, \ldots, s_m$, target vertices $t_1, \ldots, t_m$, and internal vertices $k_1, \ldots, k_n$, and edges $s_i \to k_j$ with weight $A_{i,j}$ and $k_i \to t_j$ with weight $B_{i,j}$. Then $AB_{i,j} = \sum_x^n A_{i,x} B_{j,x}$.

Let $I \subset [n]$.

Let P_{SI} be the set of all disjoint paths from the source vertices to exclusively internal vertices with indices in I, and let P_{IT} be the set of all disjoint paths from internal vertices with indices in I to any target vertices.

Using the Lindström-Gessel-Viennot formula, we get that $\det(A) = \sum_{R \in P_{SI}} \text{sign}(R)\, \omega(R)$ and $\det(B) = \sum_{S \in P_{IT}} \text{sign}(S)\, \omega(S) \Rightarrow \det(A)\det(B) = \sum_{R \in P_{SI}} \text{sign}(R)\, \omega(R) \sum_{S \in P_{IT}} \text{sign}(S)\, \omega(S)$ $= \sum_{P \in P_{SI} \times P_{IT}} \text{sign}(P)\, \omega(P)$. $P_{SI} \times P_{IT}$ contains all of the non-intersecting tuples of

Good. But to be complete, there is one subtle thing that needs to be checked in this proof: that sign(R) * sign(S) = sign(P). [-1pt]

paths passing through the internal vertices that have indices I. Thus,

$$\sum_{I \subseteq [n], \#I = m} \det(A|_{cols=I}) \det(B|_{rows=I}) = \sum_{I \subseteq [n], \#I=m} \left[\sum_{P \in P_{SI} \times P_{IT}} \operatorname{sign}(P)\, \omega(P) \right] = \det(AB).$$

$$\therefore \det(AB) = \sum_{I \subseteq [n], \#I=m} \det(A|_{cols=I}) \det(B|_{rows=I}) \quad \square$$

5. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition and k a positive integer. Give a formula for $m_\lambda(\overbrace{1, 1, \ldots, 1}^{k \text{ 1's}})$.

Hint: Your formula can use the *length* $\ell(\lambda) := \max\{i \colon \lambda_i > 0\}$ of the partition λ, as well as the *multiplicities* $m_i(\lambda) := \{j \colon \lambda_j = i\}$ for $i \geq 1$.

10/10 There are k variables and $\ell(\lambda)$ exponents. Now if we want to find the number of ways to put $\ell(\lambda)$ exponents onto k variables, we would have $\frac{k!}{(k-\ell(\lambda))!}$ ways of doing that. But some of the parts of λ could be the same, and since it doesn't matter what order they're in, we have to divide by the multiplicity of each part factorial (the number of ways to permute $m_i(\lambda)$), i.e., $\prod_{i \geq 1} [(m_i(\lambda)!]$.

$\therefore$ The formula for $m_\lambda(\overbrace{1, 1, \ldots, 1}^{k \text{ 1's}})$ is $\dfrac{k!}{(k-\ell(\lambda))! \prod_{i \geq 1} [(m_i(\lambda)!]}$ $\square$

Good, and nice explanation.

(Just for your info: this can also be written as a certain mulitnomial coefficient.)