Howard Math 157: Calculus II Spring 2023
Instructor: Sam Hopkins (sam.hopkins@howard.edu)
(call me "Sam")

1/9 Logistics:

Classes: MWRF 10:10 - 11 am ASB-B #100
Office HRs: R 12-2 pm Annex III - #220
or by appointment - email me!

Website: samuelfhopkins.com/classes/157.html

Text: Calculus, Early Transcendentals by Stewart, 9e

Grading: 40% (inperson) quizzes
40% two (in-person) midterms
20% final exam

There will be 12 in-person quizzes taken on Thursdays (about 20 mins, we will go over answers in class).
Your lowest 2 scores will be dropped (so 10/12 count).

The 2 midterms will happen in-class, also on Thursdays.

The final will take place during finals week.

Beyond that, I may assign additional problems for practice (not graded)
and I expect you to SHOW UP TO CLASS
+ PARTICIPATE! :)

that means... Interrupt me by
ASKING QUESTIONS

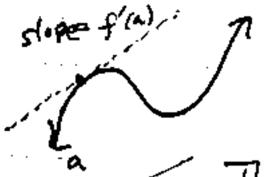
(and please say your names when you ask a question so I learn to put names to faces)

Overview of the course:

In Calculus I we learned two important and related operations on functions $f(x): \mathbb{R} \to \mathbb{R}$:

- differentiation and
- integration

The derivative $f'(a)$ of $f(x)$ at a point $x=a$ is the slope of the tangent to $y=f(x)$ at $(a, f(a))$:

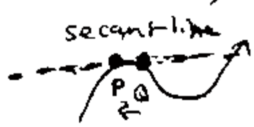


It is also the "instantaneous rate of change" of the function $f(x)$ at $x=a$.

The integral $\int_a^b f(x)\,dx$ is the area under the curve $y=f(x)$ from $x=a$ to $x=b$:

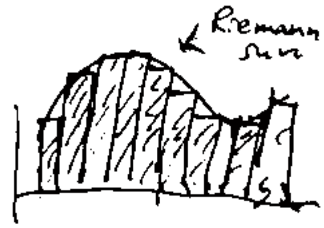
Both the derivative and integral are formally defined as limits:

- the derivative is the limit of slopes of secant lines approximating the tangent



$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

- the integral is the limit of Riemann sums (= rectangles) approximating the area under curve:



$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

The Fundamental Theorem of Calculus says that differentiation and integration are inverse operations:

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F'(x) = f(x)$.

In Calculus II we will continue to study derivatives & integrals.
Some of the things we will learn are:

- Applications of integration:

In Calc I we learned many applications of derivatives (minimums & maximums, concavity, etc.)
In Calc II we will learn more things we can compute using integrals (beyond area under curve) like:

- volumes (3D version of area)
- lengths (1D version of area).

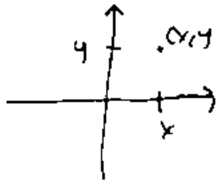
Also, FTC says that integral represents net change, so we will study some physical applications of integrals like to work (in the sense of force).

- Techniques for integration:

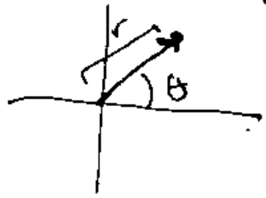
Using rules for differentiation like the product and chain rules, we know how to take the derivative of "any" function, e.g. $d/dx\ (x\sin(e^{x^2}+5x-6))$

But... integrating a "random" function like this can be really hard or not even possible. We will learn more techniques for computing integrals, when possible.
[Recall we already learned one technique: u-substitution.]

- Polar coordinates: We are used to working with (x,y) a.k.a. "Cartesian coordinates"



A different, also useful coordinate system is called polar coordinates (r,θ).



Calculus can also be done using polar coordinates as we will see.

• Taylor series:

How do we evaluate a function $f(x)$ at a particular value, e.g. compute $f(1.5)$?

If $f(x)$ is a polynomial like $f(x) = 6x^2 - 2x + 3$ we can use arithmetic: $f(1.5) = 6(1.5)^2 - 2(1.5) + 3, \ldots$

If it is a rational function like $f(x) = \frac{x+1}{x^2-1}$ we can use division similarly: $f(1.5) = \frac{1.5+1}{(1.5)^2-1}$

But what about something like $f(x) = \sin(x)$ or $f(x) = e^x$? How to compute $e^{1.5}$? what does your calculator even do?

Even though e^x is not a polynomial, it has a representation as a kind of "infinite" polynomial:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$$

This is called a Taylor series, and lets us compute $e^{1.5}$ (at least approximately).

We will learn how to deal with these kind of infinite sums called series (specifically, power series) and related mathematical constructions called sequences.

We will also learn Taylor's theorem, telling us that the coefficients of the Taylor series can be computed using the derivative of the function (which is where calculus comes in...)

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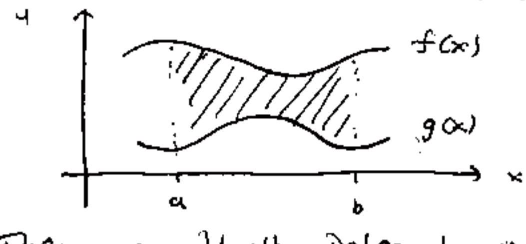
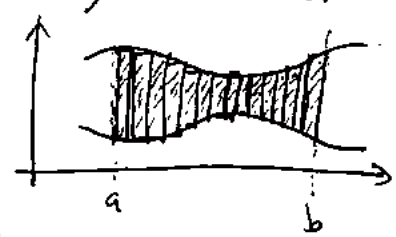
Area between curves (§6.1 of textbook)

The integral computes the area under a curve. What if we have two curves, $y=f(x)$ and $y=g(x)$, and we want to know the area between the curves?



Specifically, suppose that $f(x) \geq g(x)$ for all x in some closed interval from $x=a$ to $x=b$.

Then, as with the integral, we can define the area between the curves on $[a,b]$ by approximating it with a large number of thin rectangles:



Let $\Delta x = \frac{b-a}{n}$ (for some $n \geq 1$) and let $x_i = a + i \cdot \Delta x$ for $i=0,1,\ldots,n$ so that $[a,b]$ is divided into n sub-intervals $[x_0,x_1],[x_1,x_2]\cdots[x_{n-1},x_n]$

for each sub-interval, choose a $x_i^* \in [x_{i-1}, x_i]$, and consider the thin rectangles of width Δx and height $= f(x_i^*) - g(x_i^*)$ ← difference in hts of two curves at $x = x_i^*$

Then area between curves from $x=a$ to $x=b$ $\approx \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*))\,\Delta x$

and is exactly → $= \lim_{n\to\infty} \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*))\,\Delta x$

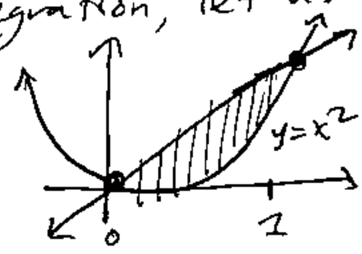
$$\boxed{= \int_a^b f(x) - g(x)\, dx}$$

So... area between two curves can be computed as integral of difference function.

Note: If we let $g(x) = 0$ be the function corresponding to the x-axis $y=0$, then we recover area under curve as $\int_a^b f(x)\,dx$ from

E.g. Let's compute the area bounded by the curves
$y = x$ and $y = x^2$.

Since the problem does not tell us the bounds of integration, let us sketch the curves:



Let $f(x) = x$ and $g(x) = x^2$

We can find out where the curves intersect by setting: $f(x) = g(x)$

$$\Rightarrow x = x^2 \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0 \Rightarrow x = 0 \text{ or } x = 1.$$

Also, choosing $x = \frac{1}{2}$, we see that between $x = 0$ and $x = 1$
$f(x) = \frac{1}{2}$, $g(x) = \frac{1}{4}$
we have that curve $y = f(x) = x$ is above curve $y = g(x) = x^2$.

Thus, the area bounded by the two curves is

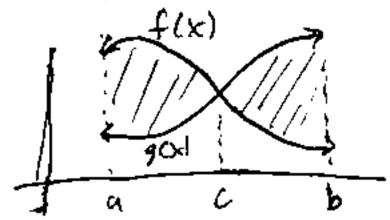
$$\int_a^b f(x) - g(x)\,dx = \int_0^1 x - x^2\,dx = \frac{x^2}{2} - \frac{x^3}{3}\Big]_0^1$$

$$= \frac{1^2}{2} - \frac{1^3}{3} - \left(\frac{0^2}{2} - \frac{0^3}{3}\right) = \frac{1}{2} - \frac{1}{3} = \boxed{\frac{1}{6}}.$$

If on the interval $[a,b]$, sometimes $f(x) \geq g(x)$ and sometimes $g(x) > f(x)$, then to correctly find the area, we need to take absolute value of difference.

$$\text{area between curves} = \int_a^b |f(x) - g(x)|\,dx.$$

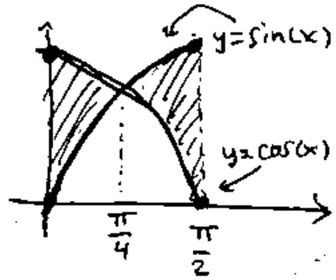
In practice, break up this integral into parts where $f(x) \geq g(x)$ and parts where $g(x) \geq f(x)$;



$$\int_a^c f(x) - g(x)\,dx + \int_c^b g(x) - f(x)\,dx$$

E.g. Compute the area between $y = f(x) = \cos(x)$ and $y = g(x) = \sin(x)$ for $x = 0$ to $x = \pi/2$

Again, good idea to sketch curves to see what's going on:



$\cos(0) = 1 > 0 = \sin(0)$

but $\sin(\frac{\pi}{2}) = 1 > 0 = \cos(\frac{\pi}{2})$

so which curve is on top changes:

in fact have $\cos(\theta) = \sin(\theta)$ at $\theta = \frac{\pi}{4}$

(think about isosceles right triangle ...)

Thus, area between $y = \cos(x)$ and $y = \sin(x)$ from $x = 0$ to $x = \frac{\pi}{2}$

$$= \int_0^{\frac{\pi}{4}} \cos(x) - \sin(x)\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x) - \cos(x)\,dx$$

$$= \sin(x) + \cos(x)\Big]_0^{\frac{\pi}{4}} + -\cos(x) - \sin(x)\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

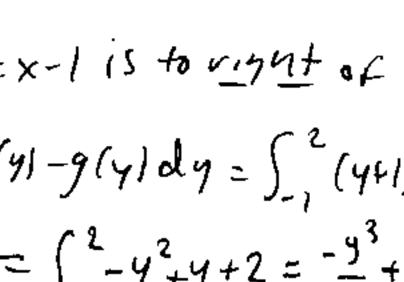
$$= \left(\sin(\tfrac{\pi}{4}) + \cos(\tfrac{\pi}{4}) - \sin(0) - \cos(0)\right) + \left(-\cos(\tfrac{\pi}{2}) - \sin(\tfrac{\pi}{2}) + \cos(\tfrac{\pi}{4}) + \sin(\tfrac{\pi}{4})\right)$$

$$= (1/\sqrt{2} + 1/\sqrt{2} - 0 - 1) + (-0 - 1 + 1/\sqrt{2} + 1/\sqrt{2}) = 2\sqrt{2} - 2$$ //

E.g. Sometimes it is easier to integrate w.r.t. y variable

Let's find area between $y = x - 1$ and $y^2 = x + 1$;

We sketch the curves:



they intersect at $y = -1$ and $y = 2$

$x = y^2 - 1 = g(y)$

and $x = y + 1 = f(y)$

set equal

$\Rightarrow y^2 - 1 = y + 1$

$\Rightarrow y^2 - y - 2 = 0$

$\Rightarrow (y-2)(y+1) = 0$

$\Rightarrow y = 2$ or $y = -1$

Then since $y = x - 1$ is to right of $y^2 = x + 1$ for $y = -1$ to $y = 2$

$$\text{area} = \int_{-1}^{2} f(y) - g(y)\,dy = \int_{-1}^{2} (y+1) - (y^2 - 1)\,dy$$

$$= \int_{-1}^{2} -y^2 + y + 2 = \frac{-y^3}{3} + \frac{y^2}{2} + 2y\Big]_{-1}^{2} = \frac{-8}{3} + 2 + 4 - \left(\frac{+1}{3} + \frac{1}{2} - 2\right)$$ //

$\boxed{4.5}$

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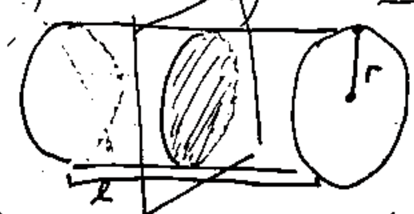
Volumes (§6.2)

Volumes are the 3-dimensional version of areas.

Let's start by considering a circular cylinder:



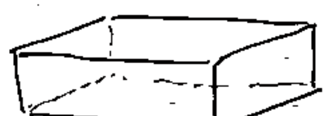
The cross-section (= intersection w/ y, z-plane) of this cylinder at any x-coordinate is a circle (of radius r).

We thus define the volume of the cylinder

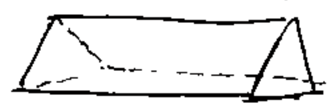
to be = area of cross-section × length of cylinder

$= \pi r^2 \cdot l$

We can also consider cylinder whose cross-sections are other shapes, e.g., rectangles or triangles:



rectangular cylinder (or 'rectangular prism')



triangular cylinder ('Toblerone' bar)

The important thing is that the cylinder has a certain length and across the whole length cross-sections are same.

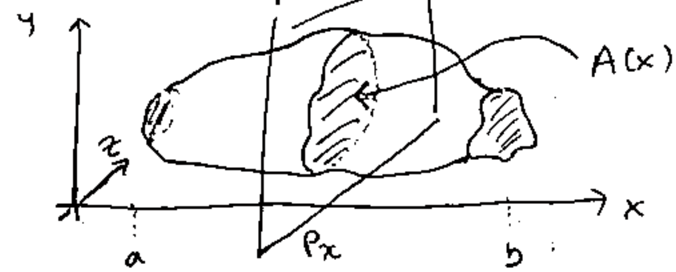
Thus, for any cylinder we set

volume of cylinder = area of cross section × length

E.g. volume of rectangular prism = width × height × length.

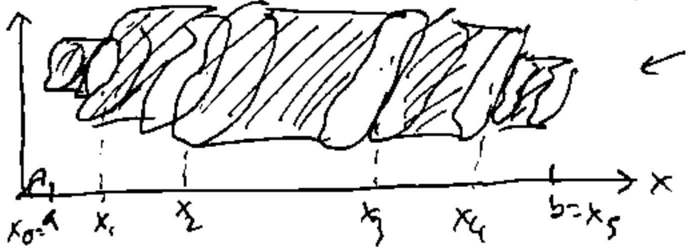
Q: What if the cross-section of our solid is not constant?

Let's draw a picture of our solid:



Suppose the solid extends between $x=a$ and $x=b$, and let $A(x)$ for $a \le x \le b$ be the area of the cross-section obtained by intersecting with plane P_x perpendicular to x-axis at that point.

We can approximate the volume by dividing the solid into several short cylinders:



← sliced into 5 cylinders

As w/ integral, we break up interval $[a,b]$ into n subintervals $[x_{i-1}, x_i]$ $i=1,\dots,n$, $x_i = x_{i-1}+\Delta x$

Then the volume of the solid $\approx \sum_{i=1}^{n}$ area of cross-section of i^{th} short cylinder $\times \Delta x$

$$= \sum_{i=1}^{n} A(x_i^*) \, \Delta x$$

and is exactly $= \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*) \, \Delta x$

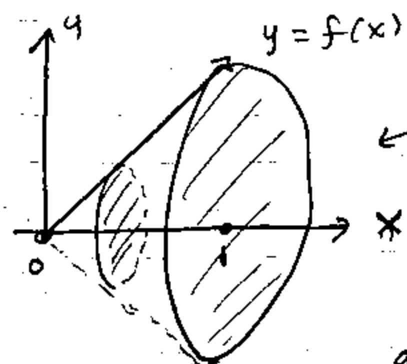
$$\boxed{= \int_a^b A(x) \, dx}$$

letting us compute volumes as integrals!

An important class of solids are the <u>Solids of revolutions</u> obtained by rotating a region in x,y-plane about x-axis:

<u>E.g.</u> Find the volume of the <u>cone</u> obtained by rotating the area below $y = x$ (and above x-axis) from $x = 0$ to $x = 1$ about the x-axis.

<u>Sketch</u>:



at any x w/ $0 \le x \le 1$ cross-section of cone is a <u>circle</u> of radius $f(x) = x$

Since in this case $A(x) =$ area of circle of radius $f(x)$

$$= \pi\,(f(x))^2 = \pi x^2$$

we can use the integral formula for volume to get

$$\text{Volume of cone} = \int_0^1 \pi x^2\,dx = \frac{\pi}{3} x^3\Big]_0^1 = \boxed{\frac{\pi}{3}}$$

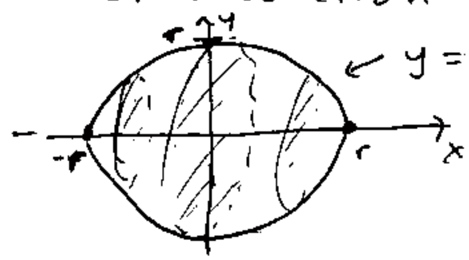
We see that in general the volume of a solid of revolution obtained by rotating the area below the curve $y = f(x)$ from $x = a$ to $x = b$ about x-axis

is $$\boxed{= \int_a^b \pi\,(f(x))^2\,dx}$$

since every cross-section is a circle of radius $= f(x)$

__Eg.__ Find the volume of a __sphere__ of radius r using an integral.

To do this, we have to realize the sphere as a solid of revolution:

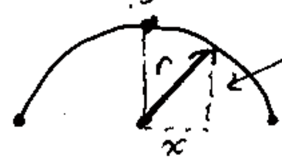


$\leftarrow y = \sqrt{r^2 - x^2}$

We see that a sphere is obtained by rotating __semicircle__ of radius r about x-axis,

and semicircle of radius r = area below curve $y = \sqrt{r^2 - x^2}$ from $x = -r$ to r

(Think:



$y = \sqrt{r^2 - x^2}$ since $x^2 + y^2 = r^2$ by Pythagorean Thm.)

Thus, according to the formula for volume of a solid of revolution, we have

$$\text{volume of sphere of radius } r = \int_{-r}^{r} \pi \left(\sqrt{r^2 - x^2}\right)^2 dx$$

$$= \pi \int_{-r}^{r} (r^2 - x^2)\, dx$$

$$= \pi \left(r^2 x - \frac{x^3}{3} \Big]_{-r}^{r} \right)$$

$$= \pi \left(\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 - \frac{-r^3}{3}\right) \right)$$

$$= \pi \left(2r^3 - \frac{2}{3} r^3 \right) = \boxed{\frac{4}{3} \pi r^3}$$

//

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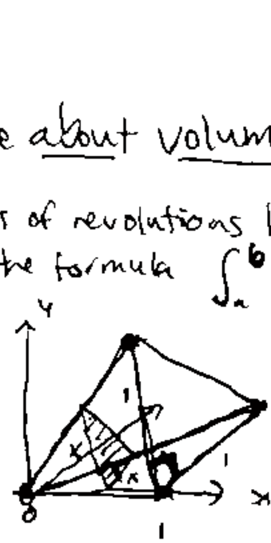
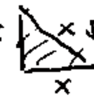
More about volumes §6.2

Solids of revolutions have cross-sections that are circles (or annuluses: see below) but the formula $\int_a^b A(x)\,dx$ for volume works w/ other shapes too..

E.g.



Let's compute the volume of the triangular cone which extends from $x=0$ to $x=1$ and whose cross-section at x is a right isosceles triangle:



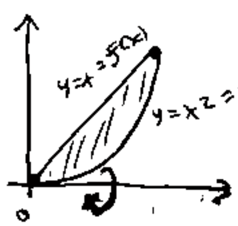
← area = $\frac{1}{2}$ base × height $= \frac{1}{2}x^2$

Then $\begin{matrix}\text{Volume of}\\ \Delta\text{-cone}\end{matrix} = \int_0^1 A(x)\,dx$

$$= \int_0^1 \frac{1}{2}x^2\,dx = \frac{1}{2}\,\frac{1}{3}\,x^3\Big]_0^1 = \boxed{\frac{1}{6}}$$

Returning to solids of revolution ... we can also rotate the region between two curves over an axis.

E.g.



Let's rotate the region between the curves $y=x$ and $y=x^2$ from $x=0$ to $x=1$ about the x-axis to make a solid.

The cross-section of this solid is an annulus: the region between two circles

"annulus" a.k.a. "washer" shape →

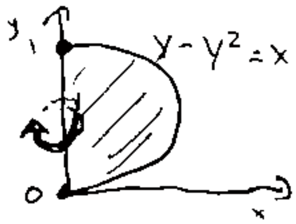
Area of annulus ← is $\pi(r_2^2 - r_1^2)$.

In the case of region between two curves, the area of this cross-section is $A(x) = \pi(f(x)^2 - g(x)^2)$.

For our $f(x)=x$ and $g(x)=x^2$ example, this gives:

$$\begin{matrix}\text{Volume of}\\ \text{solid of}\\ \text{revolution}\end{matrix} = \int_0^1 \pi(x^2 - (x^2)^2)\,dx = \pi\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \pi\left(\frac{1}{3}-\frac{1}{5}\right) = \boxed{\frac{\pi 2}{15}}$$

Sometimes we want to rotate across y-axis instead of x-axis.

E.g. 

How can we compute the volume of the solid obtained by rotating the region between y-axis and curve $y-y^2=x$ about the y-axis?

We just do the same thing we've been doing, but with respect to y.

$$\text{Volume of solid} = \int_a^b A(y)\,dy \quad A(y) = \text{area of y cross-section}$$

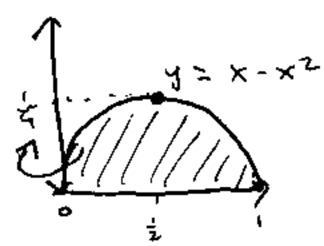
$$= \int_0^1 \pi (y-y^2)^2\,dy$$

since y-cross-section is a circle of radius $f(y) = y-y^2$

$$= \int_0^1 \pi (y^2 - 2y^3 + y^4)\,dy$$

$$= \pi\left(\tfrac{1}{3}y^3 - \tfrac{2}{4}y^4 + \tfrac{1}{5}y^5\right]_0^1 = \pi\left(\tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5}\right) = \boxed{\tfrac{\pi}{30}}$$ //

1/20 What about the following solid of revolution problem?

E.g. 

Compute the volume of solid obtained by rotating region below $y=x-x^2$ (and above x-axis) about the y-axis.

To do this following the method above, we have to realize this region as between two curves

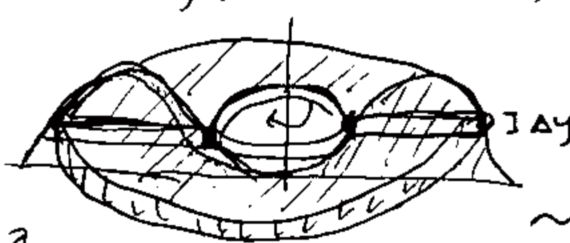
$x=f(y)$ and $x=g(y)$ and integrate w.r.t. y.

(To find $f(y)$ and $g(y)$ we need to "invert" $y=x-x^2$ using quadratic formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.)

But... there is a better approach using integration w.r.t. x ...

The method of cylindrical shells §6.3

To compute the volume of solid of revolution obtained by rotating region below $y = f(x)$ about the y-axis using previous method, we break into "thin washers":



thickness of "washer" $= \Delta y$

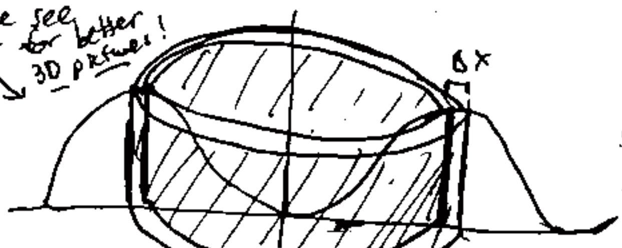
volume of washer $= \Delta y \times$ area of annulus

$= \Delta y \cdot \pi(r_2^2 - r_1^2)$

$\rightsquigarrow \int \text{—} \, dy$ (integrate w.r.t. y.)

But we can also break this solid into hollow cylindrical shells.

please see text for better 3D pictures!



these shells look like

gap in radii $= \Delta x$

height $= f(x_i^*)$

very thin tubes of toilet paper

By breaking the solid into many cylindrical shells, we obtain:

$$\text{Volume of solid} \approx \sum_{i=1}^{n} \text{volume of } i^{\text{th}} \text{ shell}$$

$$= \sum_{i=1}^{n} \text{height of } i^{\text{th}} \text{ shell} \cdot \text{area of annulus base}$$

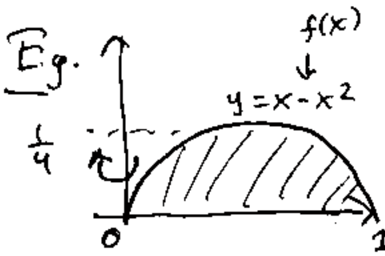
$$= \sum_{i=1}^{n} f(x_i^*) \cdot \pi\left((x_i^* + \Delta x)^2 - (x_i^*)^2\right)$$

$$= \sum_{i=1}^{n} f(x_i^*) \cdot 2\pi x_i^* \Delta x + \sum_{i=1}^{n} f(x_i^*) \cdot \pi (\Delta x)^2$$

and in the limit . . .

$$\text{Volume} = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(x_i^*) x_i^* \Delta x + \underbrace{\lim_{n\to\infty} \sum_{i=1}^{n} \pi f(x_i^*)(\Delta x)^2}_{\to 0 \text{ as } n\to\infty}$$

$$= \boxed{\int_a^b 2\pi f(x) \cdot x \, dx}$$

E.g. 

Returning to the example of solid obtained by rotating region below $y = x - x^2$ about y-axis, its

$$\text{Volume} = \int_a^b 2\pi f(x)\cdot x \, dx = \int_0^1 2\pi (x-x^2)\, x \, dx$$

$$= 2\pi \int_0^1 x^2 - x^3 dx = 2\pi \left[\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right]_0^1 = \boxed{\frac{\pi}{6}}$$

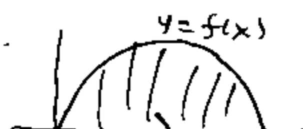
Using the "washer" method instead, we would have to compute:

$$\text{volume} = \int_0^{\frac{1}{4}} \pi \cdot \left(\left(\frac{1+\sqrt{1-4y}}{2}\right)^2 - \left(\frac{1-\sqrt{1-4y}}{2}\right)^2 \right) dy$$

$$= \cdots = \frac{\pi}{6}$$

which is much harder algebra!

Upshot: Both the "disks/washers" method and the "cylindrical shells" method will work to compute the volume of a solid of revolution, but sometimes one will lead to an easier integral:

E.g. 

for region below curve $y = f(x)$ rotated about x-axis, use "disk/washer" method to get formula $\text{volume} = \int_a^b \pi (f(x))^2 dx$

E.g. 

for region below curve $y = f(x)$ rotated about y-axis, use "cylindrical shells" method to get formula $\text{volume} = \int_a^b 2\pi f(x)\cdot x \;\; dx$

For other regions: Guess, or try both methods! //