Total: 25/50 + 5 bonus points for presenting = 30/50

Q.:

(a) Show that the lexicographic order extends the dominance order in the sense that if we have partitions $\lambda, \mu \vdash n$ with $\mu \leq \lambda$ and $\mu \neq \lambda$ then $\Rightarrow \mu < \lambda$.

8/10.

Suppose that $\mu \neq \lambda$. let j be smallest number s.t $\mu_j \neq \lambda_j$. then $\lambda_i = \mu_i \ \forall\, i < j$

so μ j^{th}, has $< \lambda_j$ j's.

$\Rightarrow \mu < \lambda$.

(b) Give an example of partitions $\lambda, \mu \vdash n$ with $\mu < \lambda$ but $\mu \leq \lambda$.

$(3,2,2,1) > (3,2,1,1,1)$

Text

Not quite, since these *are* comparable in dominance order. [-2pts]

CamScanner

Answer of Q2:

10/10

to show $e_\lambda = \sum_{\mu \le \lambda^t} B_\mu^\lambda m_\mu$

Assume $M_{\lambda\mu} \neq 0$ then $\exists$ $(0,1)$-matrix A with $row(A) = \lambda$ and $col(A) = \mu$.

Now let A' be obtained from A by left-justifying all of the 1's in each row (i.e. move all of the 1's in row i to the 1^{st} λ_i positions).

note that $col(A') = \lambda^t$. Also, the number of 1's in the 1^{st} i columns of A' is at least as many as the number of 1's in the first i columns of A, so $\lambda_1^t + \cdots + \lambda_i^t \geq \mu_1 + \cdots + \mu_i$, i.e. $\lambda^t \geq \mu$. Moreover, if $\mu = \lambda^t$, A' is the only $(0,1)$-matrix with $row(A') = \lambda$ and $col(A') = \lambda^t$.

So via the transpose $\Rightarrow$ $e_\lambda = (x_1 x_2 \cdots x_{\lambda_1} + \cdots)(x_1 x_2 \cdots x_{\lambda_2} + \cdots)$
$\cdots (x_1 x_2 \cdots x_{\lambda_k} + \cdots)$

to make the biggest monomial, we should take all terms ~~of~~ the form $x_1 \cdots x_{\lambda_i}$ that product gives $x_1^{\lambda_1^t} x_2^{\lambda_2^t} \cdots$

$$\Rightarrow e_\lambda = \sum_{\mu \le \lambda^t} B_\mu^\lambda m_\mu \quad \text{w/} \quad B_{\lambda^t}^\lambda M_{\lambda^t} = ?$$

Nice argument.

* to show $P_\lambda = \sum_{\lambda \le \mu} d_\mu^\lambda m_\mu$

Every monomial in P_λ is ~~of~~ the form

$x_{i_1}^{\lambda_1} \cdots x_{i_k}^{\lambda_k}$ for some choice of $i_1, \dots, i_k$.

Suppose this is Equal to x^μ for a partition μ.

So we get μ by reordering the i_j and merging together equal indices. In other words, there is a decomposition $B_1 \sqcup \dots \sqcup B_r = \{1, \dots, k\}$ So $\mu_j = \sum_{i \in B_j} \lambda_i$ $\Rightarrow$ we use this to show that

$\mu_1 + \mu_2 + \dots + \mu_i \ge \lambda_1 + \dots + \lambda_i$ for each i.

So fix a value of i. It's immediate from the def of the B_j that $\sum_{s \in S} \mu_s \ge \sum_{s \in S} \lambda_s$, where $S = B_1 \cup \dots \cup B_i$. If $j \le i$ and $j \notin B_1 \cup \dots \cup B_i$ then $j \in B_{i'}$ for some $i' > i$, and then

Good.

$\mu_j \ge \mu_{i'} \ge \lambda_j$. Add all of these inequalities up to get $\mu_1 + \dots + \mu_i \ge \lambda_1 + \dots + \lambda_i$.

Expanding $P_\lambda = (x_1^{\lambda_1} + x_2^{\lambda_1} + \dots)(x_1^{\lambda_2} + x_2^{\lambda_2} + \dots)(x_1^{\lambda_k} + x_2^{\lambda_k} + \dots)$

So to Find the smallest m_μ that appears in P_λ, we need to find the smallest monomial of the form $x_1^{\mu_1} x_2^{\mu_2} \cdots x_\ell^{\mu_\ell}$, So we show the matrix w/ rows of coeffients expressing P_λ in the basis m_μ is the transpose of the upper triangular w/ nonzero entries on digonal.

thus, M is invertible.

so we can write m_μ as a sum of the P_λ

are a basis so

$$P_\lambda = \sum_{\lambda \leq \mu} \alpha_\mu^\lambda m_\mu, \quad w/ \alpha_\lambda^\lambda \mu_\lambda = 0$$

Q4

$$\det N = \sum_{P} \operatorname{sign} P \, w(P) \qquad (*)$$

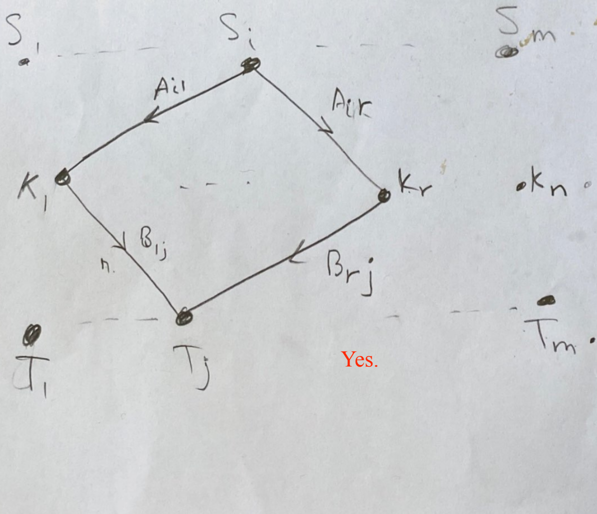
. Rewrite $(*)$ in terms of permutations of matrix elements

$$\det N = \sum_{\sigma} \operatorname{Sign} \sigma \; n_{1\sigma(1)} \, n_{2\sigma(2)} \cdots n_{m\sigma(m)}$$

. Now, let's set $N = AB$, where $A = (A_{ij})$ is an $m \times n$ path matrix of the bipartite graph on S and K, relating to a path sys A. with source vertices $\{S_1 \ldots S_m\}$ and ~~target~~ target vertices $\{K_1 - K_n\}$

And $B = (B_{ij})$ is an $n \times m$ path matrix of the bipartite graph on K and T, relating to path sys B. with source ~~with~~ $\{K_1 - K_n\}$ and with target verts $\{T_1 - T_m\}$ with

. Now, linking these together to create a new graph between S and T



Yes.

we will consider a weighted directed (bipartite) ...

$$\det N = \sum_{P \text{ vertex-disjoint path system}} \operatorname{sign} P \, w(P)$$

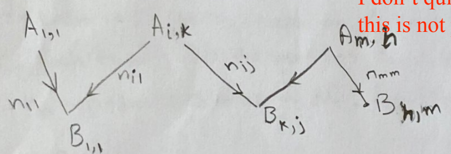
N is an $m \times m$ matrix, $N = (n_{ij})$ and writing the $\det N$ it terms of the permutations of the matrix elements.

So we have $\det N = \sum_{\sigma} \operatorname{sign} \sigma \; n_{1\sigma(1)} \cdots n_{m\sigma(m)}$ where the sign can be -1 or 1 depending on the nuber of the transpositions of it, even or odd.

let the vertices $A_{1,1}, \ldots, A_{m,n}$ represent rows of N, and $B_{1,1}, \ldots, B_{m,m}$ represent columns of N.

For some $A_{i,k} \to B_{k,j}$ the weight will be n_{ij}

I don't quite understand this picture:
this is not part of the graph you defined earlier.



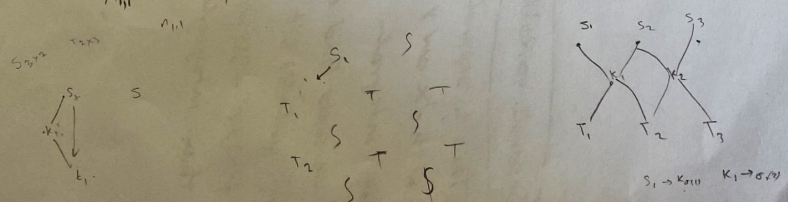
let $A = \{A_{1,1}, \ldots, A_{m,n}\}$ an $B = \{B_{1,1} \ldots B_{n,m}\}$. Let's assume that the weighted (signed) sum over all vertex-disjoint path systems $A \to B$ is given by

~~For~~ a given system, P_σ, paths

$A_{1,1} \to B_{\sigma(1)}, \ldots, A_{m,n} \to B_{\sigma}$.

$(S_1 \to K_{\sigma(1)}) \to K_1 \to t_{\sigma(1)}$ $\quad (S_m \to K_{\sigma(n)}) \to K_n \to t_{\sigma(m)}$

$A_{1,1} \xrightarrow{n_{1,1}} B_{1,1}$



...tite graph on S and K,

Answer of Q4:

The question asked to deduce the Cauchy-Binet Formula From Lindstrom-Gessel-viennot Formula.

So, the Lindstrom-Gessel-viennot Formula : (*)

let M be the path matrix from A to B. Then

$$\det N = \sum_{P \text{ vertex-disjoint path system}} \operatorname{sign} P\, w(P)$$

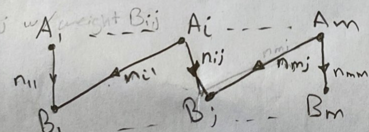
• Now, we will deduce the Cauchy-Binet Formula. From (*).

First, we start with the $m \times m$ matrix, $N = (n_{ij})$, and the determinant of N given in terms of the permutations of the matrix elements

So, we have $\boxed{\det N = \sum_{\sigma} \operatorname{sign}\sigma\, n_{1\sigma(1)} n_{2\sigma(2)} \cdots n_{m\sigma(m)}.}$

and that the sign of σ is -1 or 1, depending on whether the number of transpositions is even or odd.

• we will consider a weighted directed (bipartite) graph. let the vertices $A_1, \dots, A_m$ represent rows of N, and $B_1, \dots, B_m$ represent columns of N. For some $A_i \to B_j$, the weight will be represented by n_{ij}.



• let $A = \{A_1, \dots, A_m\}$ and $B = \{B_1 \dots B_m\}$. For a given system, P_σ, the weighted (signed) sum over all vertex-disjoint path systems $A \to B$ is given by paths

$$A_1 \to B_{\sigma(1)}, \dots, A_n \to B_{\sigma(n)}$$

The product of each individual weight represents the weight on the system so we have

$$w(P_\sigma) = w(A_1 \to B_{\sigma(1)}) \cdots w(A_n \to B_{\sigma(n)})$$

Thus we find for the matrix N

$$\det N = \sum_{\sigma} \operatorname{sign}\sigma\, w(P_\sigma)$$

You seem to be missing one key thing: the subset I of [n] which appears in the Cauchy-Binet formula corresponds to the subset of the "k" vertices in your network (the "middle" ones) which a given tuple of non-interseting lattice paths goes through. This is the key connection between the Cauchy-Binet and LGV formulas. [-3pts]