(Piecewise linear & birational) involutions on Dyck paths

Howard Mathematics Colloquium

Sam Hopkins

based on joint work with Michael Joseph (Dalton State College)

Howard University

September 24, 2021

Section 1

Catalan numbers, Dyck paths, Naryana numbers, and the Lalanne–Kreweras involution

Montserrat Mountain, Catalonia, Spain

Catalan numbers

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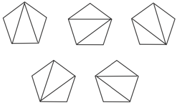
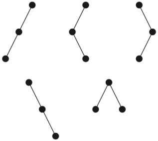


binary trees with n nodes



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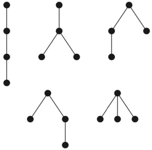

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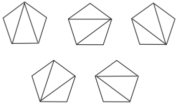

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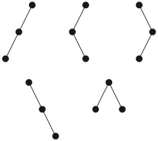

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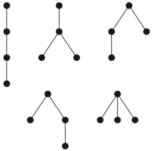

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There is a well-known product formula for the Catalan numbers:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$$

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It's a good thing the C_n are not named after Euler, since there are already

- *Euler numbers* & *Eulerian numbers*, counting certain permutations;
- *Euler's number* $e \approx 2.71$ & the *Euler–Mascheroni constant* $\gamma \approx 0.57$.

Catalan numbers: the book

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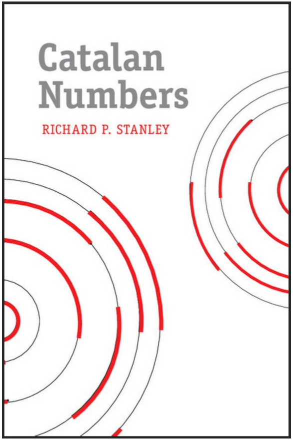

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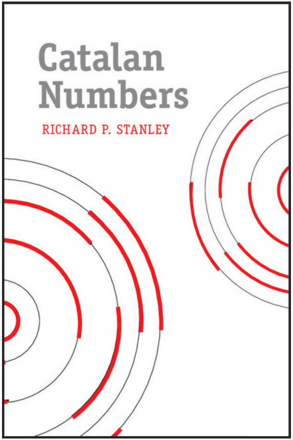

In it, he gives an astounding 214 different interpretations of C_n.

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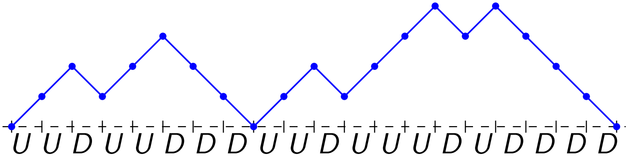

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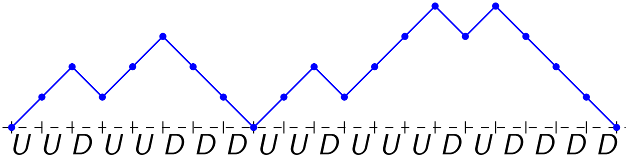

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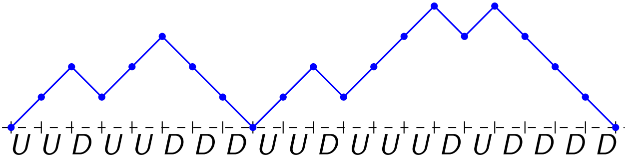

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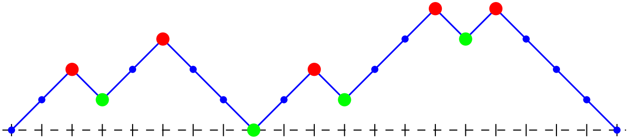

Here the peaks are marked by red circles and the valleys by green circles.

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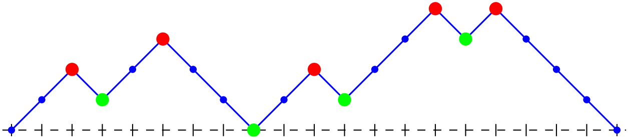

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It's easy to see that a Dyck path which has k valleys has $k+1$ peaks.

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They are named after Canadian matthematician/statistician *Tadepalli Venkata Narayana* (1930 – 1987), who in 1959 showed that

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}.$$

Symmetry of Narayana numbers

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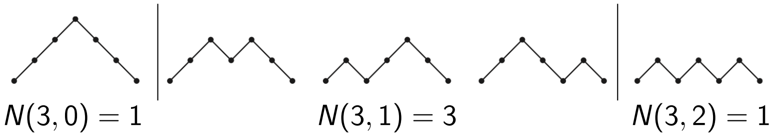

However, it is not combinatorially obvious why the number of Dyck paths with k valleys should be the same as the number with $n-1-k$ valleys.

The Lalanne–Kreweras involution

The **Lalanne–Kreweras involution** (after *G. Kreweras* and *J.-C. Lalanne*) is a map on Dyck paths which combinatorially demonstrates the symmetry of the Narayana numbers: $\#\mathrm{valleys}(\Gamma) + \#\mathrm{valleys}(\mathrm{LK}(\Gamma)) = n - 1$.

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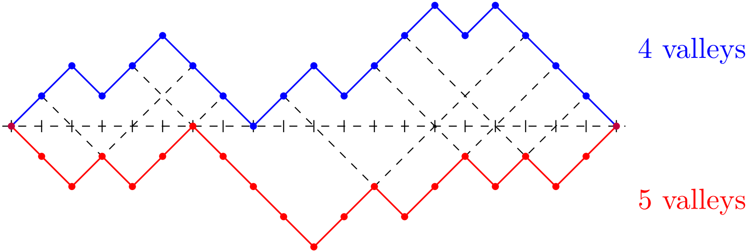

As depicted above, to compute the LK involution of a Dyck path Γ, we draw dashed lines emanating from the middle of every double up step and every double down step of Γ, at -45° and 45° respectively; these dashed lines intersect at the valleys of (an upside copy of) the Dyck path $\mathrm{LK}(\Gamma)$.

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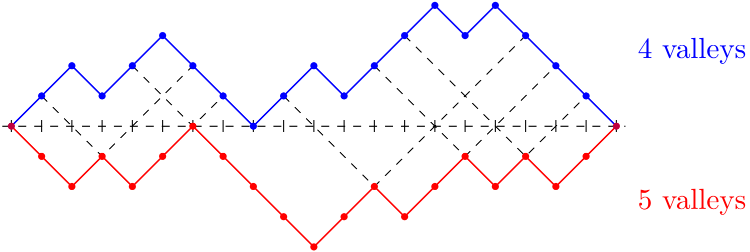

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That LK is an involution means $\mathrm{LK}^2(\Gamma) = \Gamma$ for all Dyck paths Γ.

Section 2

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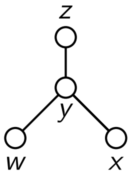

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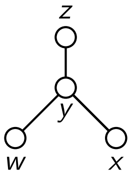

Here an edge from x (below) to y (above) represents the **cover relation** $x \lessdot y$ in P, which means $x \leq y$ and there is no $p \in P$ with $x \leq p \leq y$.

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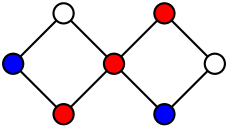

Here the red elements form a maximal chain C, and the blue elements form an antichain $A \in \mathcal{A}(P)$.

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Δ^{n-1} is the poset whose elements are **intervals** $[i,j] := \{i, i+1, \ldots, j\}$ with $1 \leq i \leq j \leq n-1$, and with the partial order given by **inclusion**:

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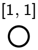

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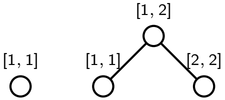

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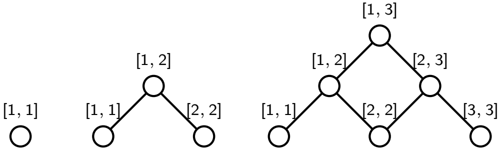

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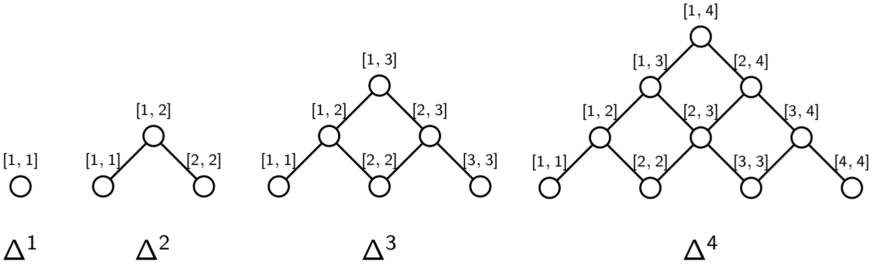

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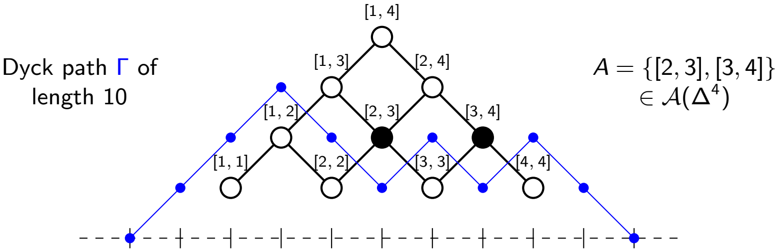

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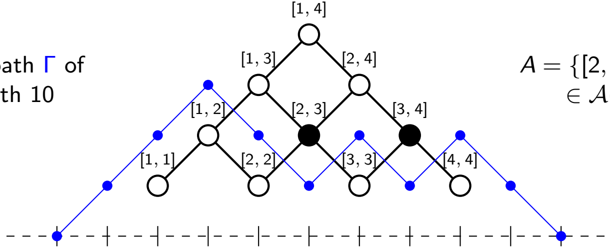
There is a natural, pictoral bijection between the Dyck paths of length $2n$ and the antichains of Δ^{n-1}:

Dyck path Γ of length 10



$A = \{[2,3],[3,4]\}$
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Observe how, under this bijection, the number of valleys of a Dyck path Γ becomes the number of elements of an antichain A.

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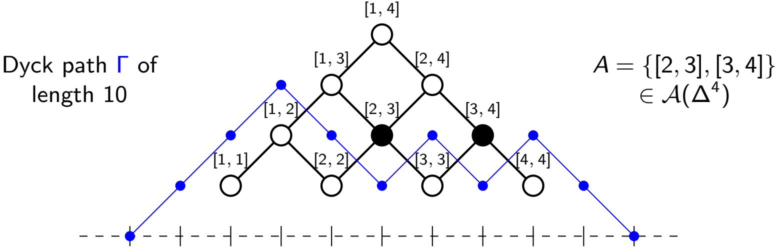

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Via this bijection, we can view the LK involution as an involution on antichains $\mathrm{LK}\colon \mathcal{A}(\Delta^{n-1}) \to \mathcal{A}(\Delta^{n-1})$ which satisfies

$$\#A + \#\mathrm{LK}(A) = n - 1.$$

The LK involution on antichains

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Theorem (Panyushev, 2004)

Let $A = \{[i_1, j_1], [i_2, j_2], \ldots, [i_k, j_k]\} \in \mathcal{A}(\Delta^{n-1})$ *with* $i_1 < i_2 < \cdots < i_k$. *Then* $\mathrm{LK}(A) = \{[i'_1, j'_1], [i'_2, j'_2], \ldots, [i'_{n-1-k}, j'_{n-1-k}]\} \in \mathcal{A}(\Delta^{n-1})$, *where*

- $\{i'_1 < i'_2 < \cdots < i'_{n-1-k}\} = \{1, 2, \ldots, n-1\} \setminus \{j_1, j_2, \ldots, j_k\}$;
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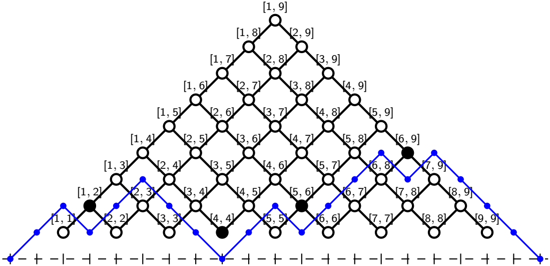
From Panyushev's description, it is immediate that this operation is an involution (i.e., $\mathrm{LK}^2(A) = A$), and that $\#A + \#\mathrm{LK}(A) = n - 1$.

The LK involution on antichains: example



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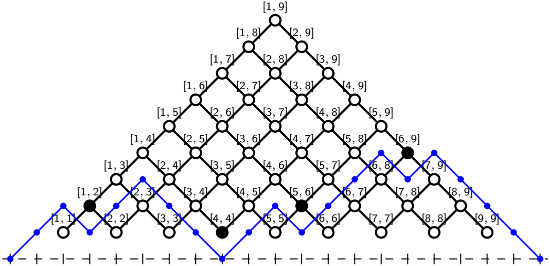

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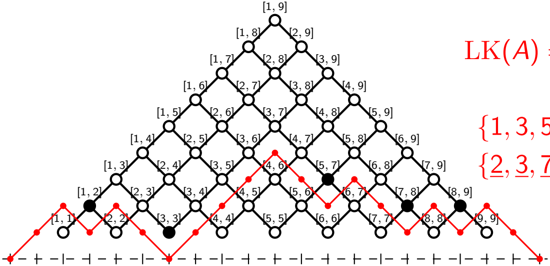

$\mathrm{LK}(A) = \{[1, \underline{2}], [3, \underline{3}], [5, \underline{7}], [7, \underline{8}], [8, \underline{9}]\}$

$\{1, 3, 5, 7, 8\} = \{1, \ldots, 9\} \setminus \{\underline{2}, \underline{4}, \underline{6}, \underline{9}\}$

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Section 3

Toggling

Toggling for antichains

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Let P be a poset and $A \in \mathcal{A}(P)$ an antichain. Let $p \in P$ be any element. The **toggle of** p **in** A is the antichain $\tau_p(A) \in \mathcal{A}(P)$, where

$$\tau_p(A) := \begin{cases} A \setminus \{p\} & \text{if } p \in A; \\ A \cup \{p\} & \text{if } p \notin A \text{ and } A \cup \{p\} \text{ remains an antichain;} \\ A & \text{otherwise.} \end{cases}$$

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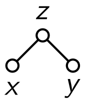
$P =$ 

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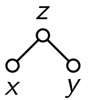
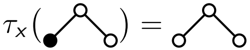
$P =$ 



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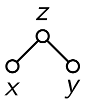
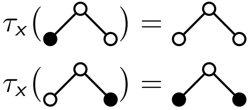
$P =$ 



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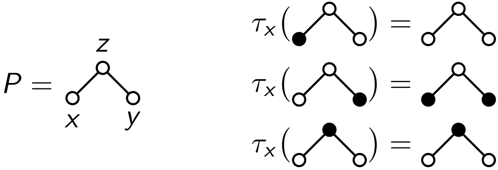
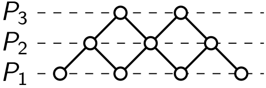


Toggling in ranked posets

A poset P is **ranked** if we can write $P = P_1 \sqcup P_2 \sqcup \cdots \sqcup P_r$ so that all the edges of the Hasse diagram of P are from P_i (below) to P_{i+1} (above):



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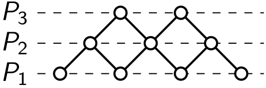

Since τ_p and τ_q commute if p and q are incomparable, and all the elements within a rank are incomparable, we can define

$$\boldsymbol{\tau}_i := \prod_{p \in P_i} \tau_p$$

to be the composition of all toggles at rank i, for $i = 1, \ldots, r$:

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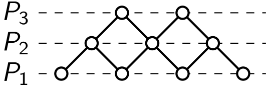

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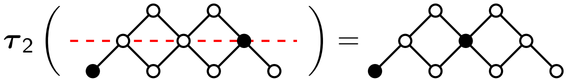

The LK involution as a composition of toggles

Theorem (H.–Joseph, 2021)

The LK involution $\mathrm{LK}\colon \mathcal{A}(\Delta^{n-1}) \to \mathcal{A}(\Delta^{n-1})$ *can be written as the following composition of toggles:*

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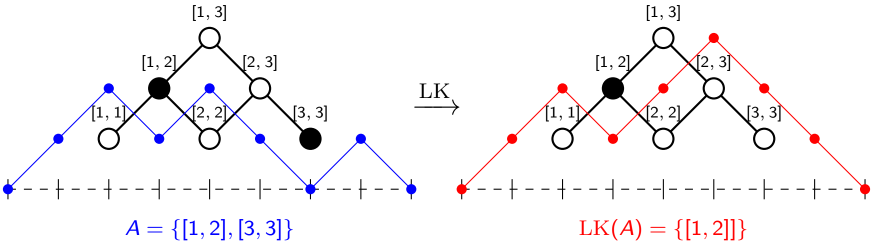
Remark: for a ranked poset P, the composition of toggles $\boldsymbol{\tau}_r\cdots\boldsymbol{\tau}_2\boldsymbol{\tau}_1$ "from bottom to top" is called **rowmotion** and has been studied by many authors (Cameron–Fon-Der-Flaass, Striker–Williams, Propp–Roby, Joseph, etc...) in the emerging subfield of **dynamical algebraic combinatorics**.

The LK involution as a composition of toggles: example



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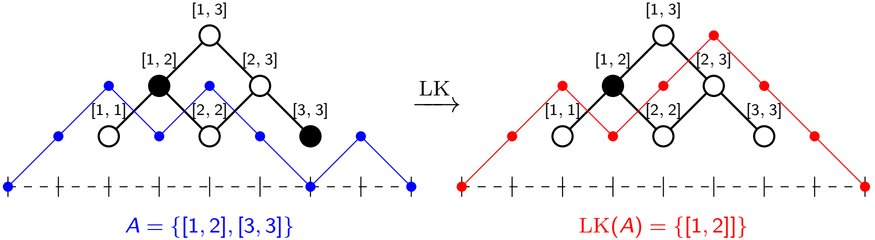

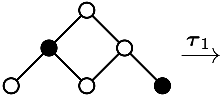

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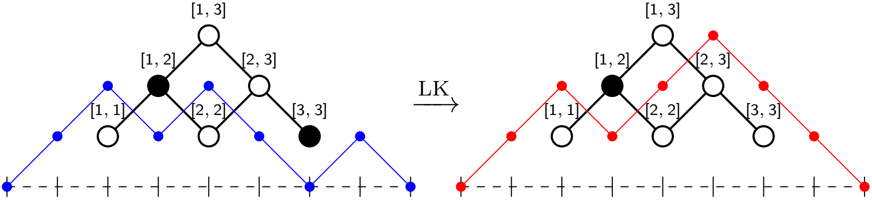

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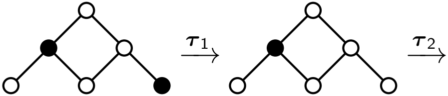

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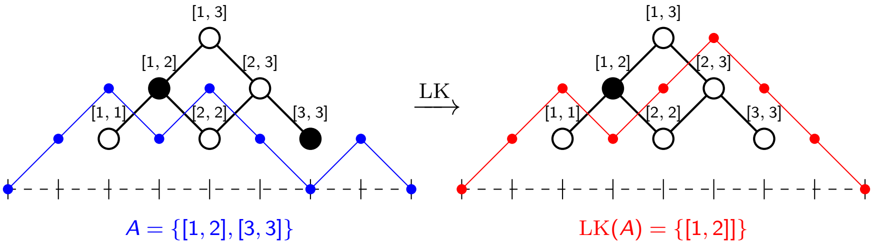

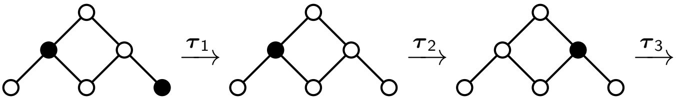

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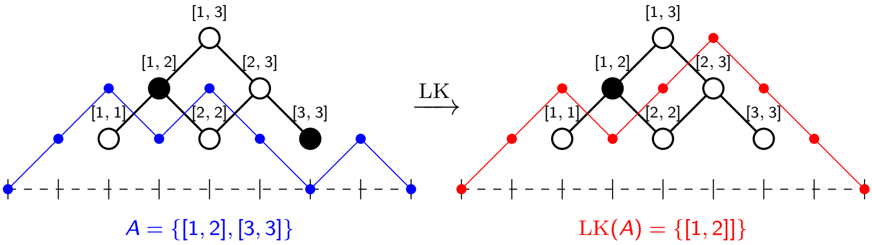

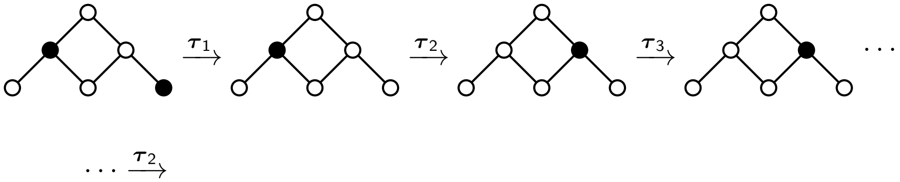

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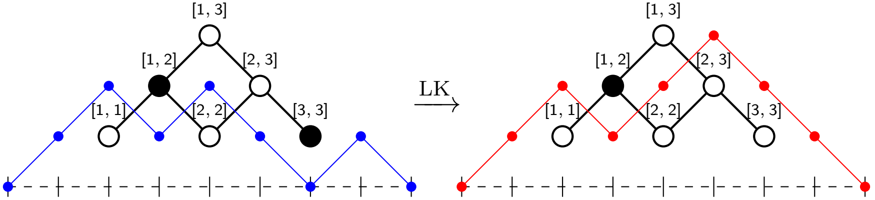

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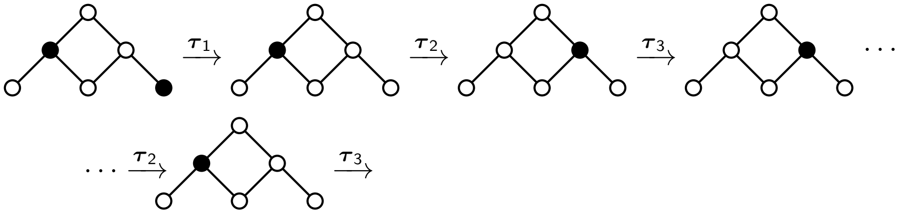

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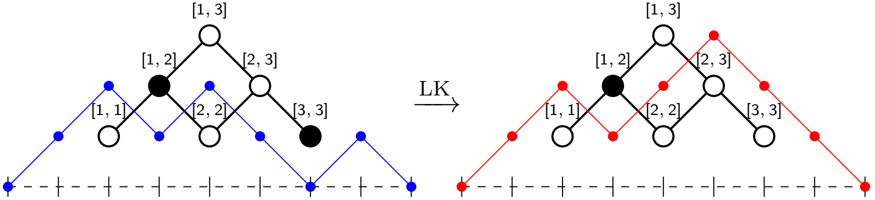

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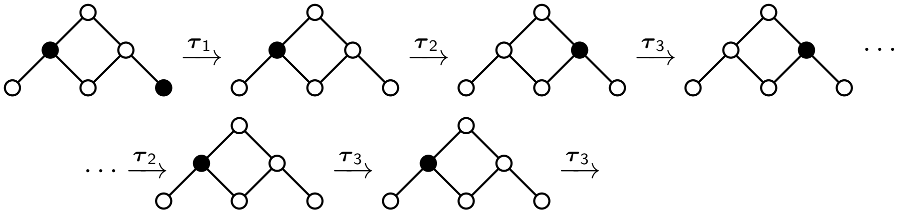

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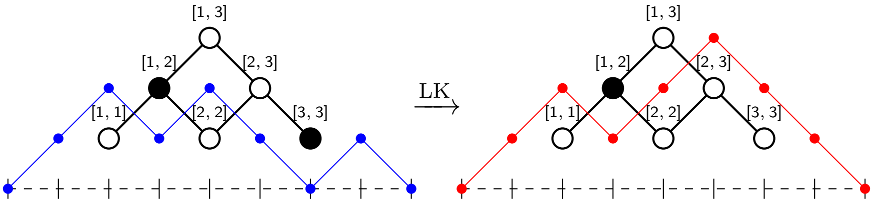

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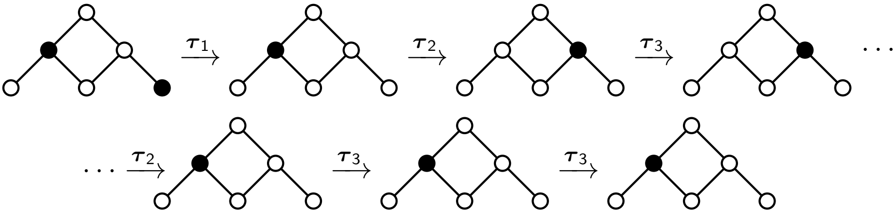

Section 4

Piecewise linear and birational lifts

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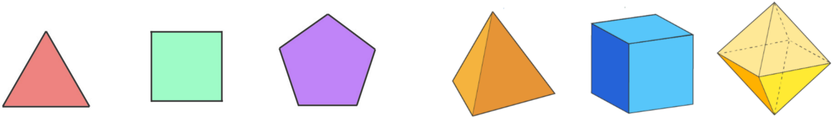

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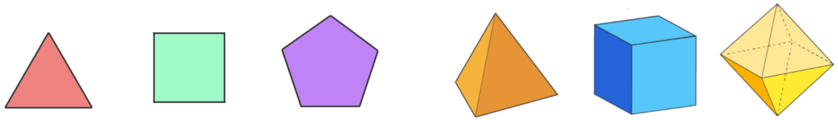

There is a rich interplay between combinatorics and convex geometry, because combinatorial objects can often be "realized" polytopally: e.g., *the subsets of* $\{1, 2, \ldots, n\}$ *correspond to the vertices of the* n*-hypercube*.

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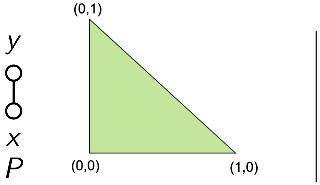

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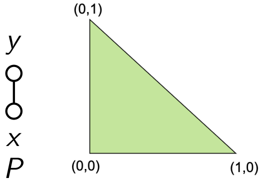

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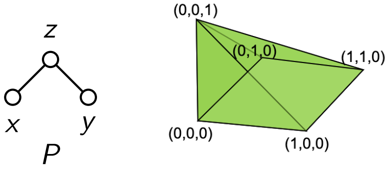

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Piecewise linear toggling

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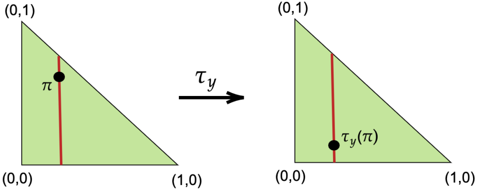
Geometrically, τ_p **reflects** π within line segment in $\mathcal{C}(P)$ in direction x_p:



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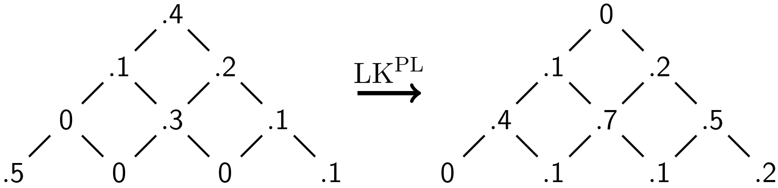
Observe that (2) is an extension of the fact that LK combinatorially exhibits the symmetry of the Narayana numbers.

The PL LK involution: example



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$$\begin{tikzcd}[column sep=small, row sep=small]
& & & .4 \arrow[dl, dash] \arrow[dr, dash] & & & \\
& & .1 \arrow[dl, dash] \arrow[dr, dash] & & .2 \arrow[dl, dash] \arrow[dr, dash] & & \\
& 0 \arrow[dl, dash] \arrow[dr, dash] & & .3 \arrow[dl, dash] \arrow[dr, dash] & & .1 \arrow[dl, dash] \arrow[dr, dash] & \\
.5 & & 0 & & 0 & & .1
\end{tikzcd}
\xrightarrow{\mathrm{LK}^{\mathrm{PL}}}
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0 & & .1 & & .1 & & .2
\end{tikzcd}$$

We can check that

$$(.5+0+0+.1+0+.3+.1+.1+.2+.4)+(0+.1+.1+.2+.4+.7+.5+.1+.2+0) =$$

$$1.7 + 2.3 = 4$$

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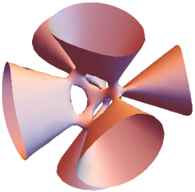

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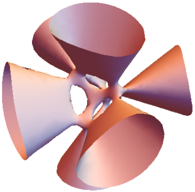

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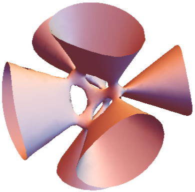

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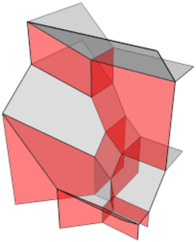

"De-tropicalization"

The process of replacing $(\times, +)$ with $(+, \max)$ in a polynomial expression is called **tropicalization**:

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It is often interesting to try to de-tropicalize PL maps, like those coming from classical combinatorial constructions.

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The birational toggle τ_p^{B} tropicalizes to the PL toggle τ_p^{PL}.

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(2) is the birational analog of the fact that LK combinatorially exhibits the symmetry of the Narayana numbers.

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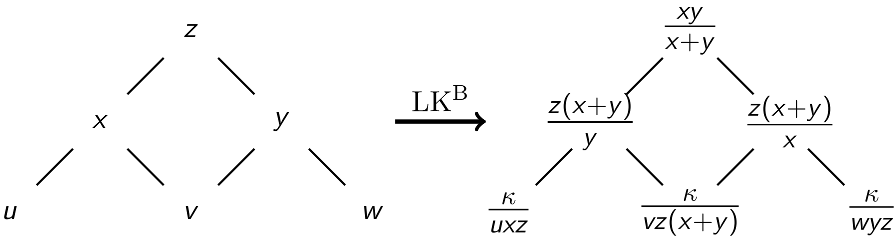

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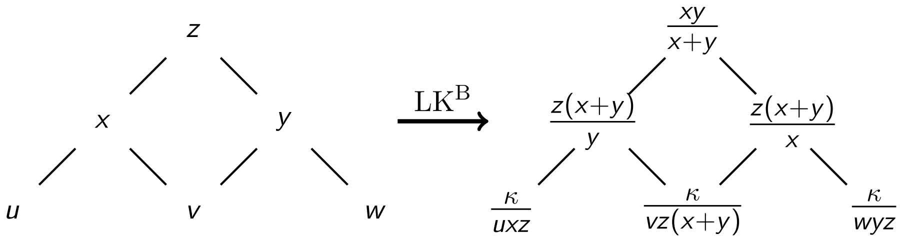

We can check that this operation really is an involution; e.g.,

$$\frac{z'(x'+y')}{y'} = \frac{\frac{xy}{x+y}\cdot\left(\frac{z(x+y)}{y}+\frac{z(x+y)}{x}\right)}{\frac{z(x+y)}{x}} = \frac{zx+zy}{\frac{z(x+y)}{x}} = \frac{z(x+y)}{\frac{z(x+y)}{x}} = x.$$

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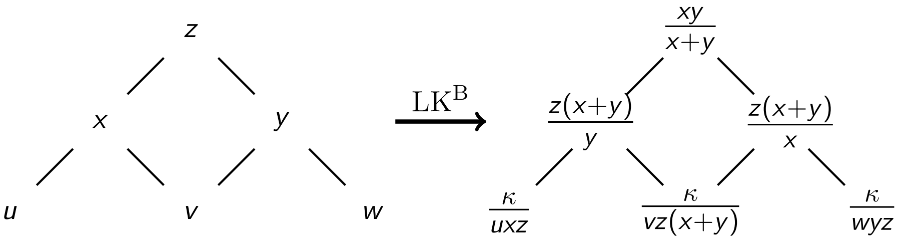

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And if we multiply together all the above values, we get κ^3.

Thank you!

these slides are available on my website
and the paper on the arXiv: `arXiv:2012.15795`

Exercises 231

6.24. [?] Explain the significance of the following sequence:

un, dos, tres, quatre, cinc, sis, set, vuit, nou, deu, . . .

R. Stanley, *Enumerative Combinatorics*, Vol. 2