

Howard Math 273, HW# 1,

Fall 2023; Instructor: Sam Hopkins; Due: Friday, September 29th

1. (Stanley, EC1, #1.66) Let $p_k(n)$ denote the number of partitions of n into exactly k parts. Give a **bijective** proof that

$$p_0(n) + p_1(n) + p_2(n) + \cdots + p_k(n) = p_k(n+k).$$

Hint: Think about Young diagrams.

2. (Stanley, EC1, #1.5) Show that

$$\sum_{n_1, \dots, n_k \geq 0} \min(n_1, \dots, n_k) x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} = \frac{x_1 x_2 \cdots x_k}{(1-x_1)(1-x_2) \cdots (1-x_k) \cdot (1-x_1 x_2 \cdots x_k)},$$

where $\min(n_1, \dots, n_k)$ means the minimum of the integers n_1, \dots, n_k .

3. (Stanley, EC1, #1.26) Let $\bar{c}(n, m)$ denote the number of compositions of n into parts of size at most m . Show that

$$\sum_{n \geq 0} \bar{c}(n, m) x^n = \frac{1-x}{1-2x+x^{m+1}}.$$

4. Prove that, for any $n \geq 0$,

$$4^n = \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k}.$$

Hint: We discussed the generating function $\sum_{n=0}^{\infty} \binom{2n}{n} x^n$ of the central binomial coefficients. How can you use what we proved about this generating function to deduce this result?

5. Let $n \geq 1$, and let $\text{ODD}(n)$ denote the subset of permutations in the symmetric group \mathfrak{S}_n with no cycles of even size. Prove that

$$\sum_{\sigma \in \text{ODD}(n)} 2^{\#\text{cycles}(\sigma)} = 2 \cdot n!.$$

Hint: Recall *Touchard's theorem*, which says that

$$\sum_{n \geq 0} \frac{1}{n!} \left(\sum_{\sigma \in \mathfrak{S}_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots t_n^{c_n(\sigma)} \right) x^n = e^{t_1 \frac{x}{1} + t_2 \frac{x^2}{2} + t_3 \frac{x^3}{3} + \cdots} = e^{\sum_{j=1}^{\infty} t_j \frac{x^j}{j}},$$

where $c_k(\sigma)$ is the number of cycles of σ of size k . How can you use Touchard's theorem to deduce this result?