

Promotion, webs, and plabic graphs

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[Howard University]

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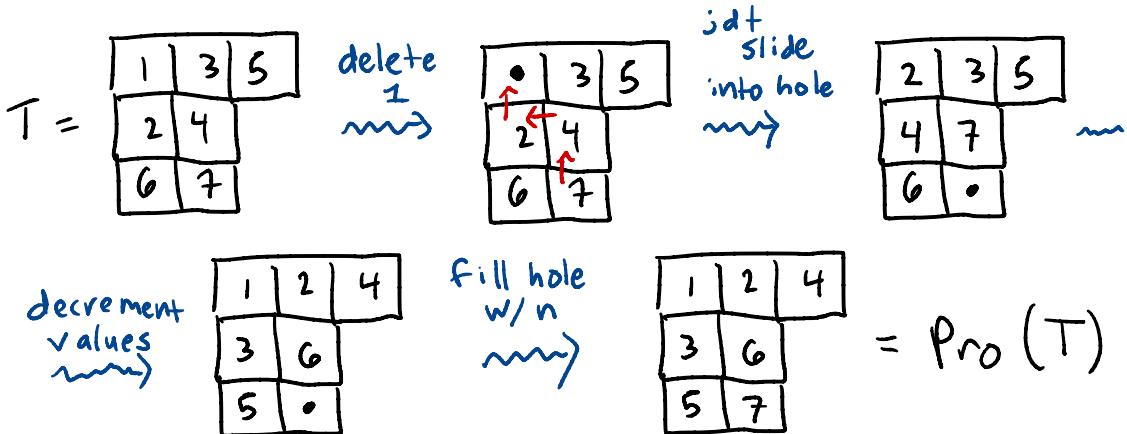
BIRS Workshop on Dynamical Algebraic
Combinatorics

(sort of) based on joint work w/ Martin Rubey
arXiv:2005.14031 [TU Wien]

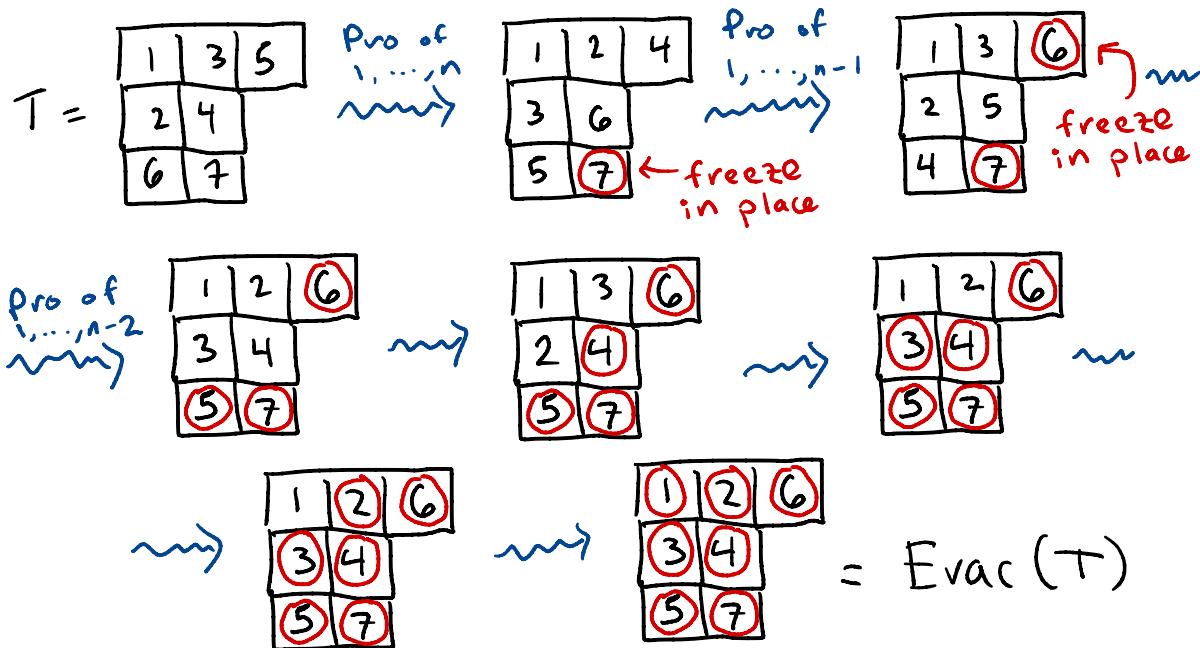
these slides are on my website:
SamuelHopkins.com

(+ this talk is being recorded)

Promotion is a cyclic action on the set of
Standard Young tableaux (SYTs) of shape $\lambda + n$:



Introduced by Schützenberger, together w/
related operation of evacuation:



While $\text{Evac}^2(T) = T$ for any T , the behavior of Pro is in general hard to predict, except for special shapes λ :

Thm (Schützenberger)

For $\lambda = a \begin{array}{|c|c|c|}\hline & & b \\ \hline & & \\ \hline & & \\ \hline \end{array}$ (the $a \times b$ rectangle),

$\boxed{\text{Pro}^{ab}(T) = T}$ for all T .

Want to study Pro (+Evac) via models:

Thm (Rhoades)

"long cycle"

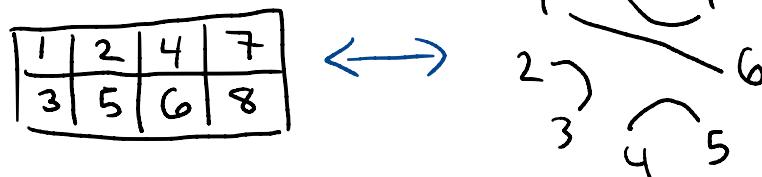
For $\lambda = a \times b + n$, the action of $C = (1, 2, \dots, n)$ on G_n -irrep S^λ corresponds* to Pro in the KL cellular basis.

Thm (Stembridge) For any $\lambda + n$, the action of $w_0 = (n \ n-1 \ \cdots \ 1)$ on S^λ corresponds* to Evac in KL cell. basis.

These algebraic models are nice,
but what about **combinatorial** models?

Prop. (D. White)

Promotion of $2 \times n$ SYTs corresponds
to rotation of noncrossing matchings of $[2n]$
via **standard bijection** between
these Catalan objects:



Pro ↓ ↓ Rot



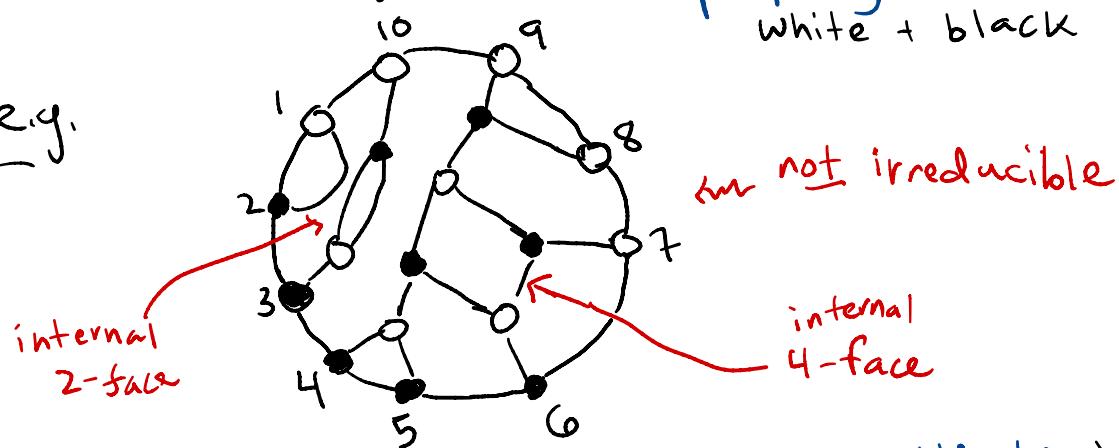
What about more than 2 rows...?

Def'n (Kuperberg)

An (sl_3 -) web is a planar graph drawn in a disc w/ boundary vertices $1, 2, \dots, m$, and some internal vertices, such that:

- bdy. vertices have degree one,
- int. vertices are trivalent,
- vertices (bdy. + int.) are properly 2-colored white + black

e.g.

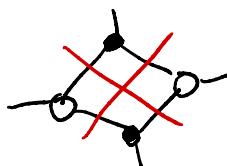


Web is called irreducible (or non-elliptic) if it has no internal (i.e., disjoint from bdy.)

2-faces



and 4-faces



Thm (Khovanov-Kuperberg,
Petersen-Polyavskyy-Rhoades, Tymoczko)

There is a **bijection**

$$\{ \text{3x}n \text{ SYTs} \} \xrightarrow{\sim} \{ \begin{array}{l} \text{irred. webs } w \\ \text{3n white bdy. vert's} \end{array} \}$$

such that $\text{Pro of } T = \text{Rot of } W$.

~~~

How does bij. work? Uses M-diagram:

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 6 \\ \hline 3 & 4 & 8 \\ \hline 5 & 7 & 9 \\ \hline \end{array} \quad \begin{array}{l} w \\ x \\ y \end{array} \quad \rightsquigarrow \quad \begin{array}{ccccccccc} w & w & x & x & y & w & y & x & y \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$

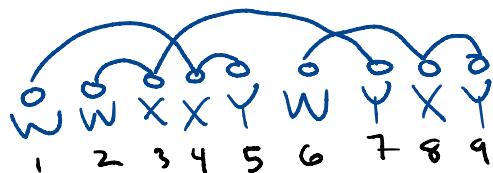
noncross. match  
of w's + x's:



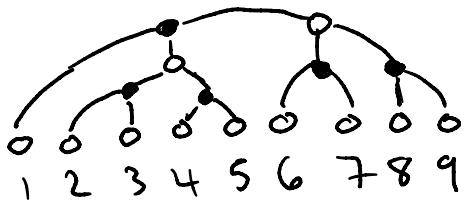
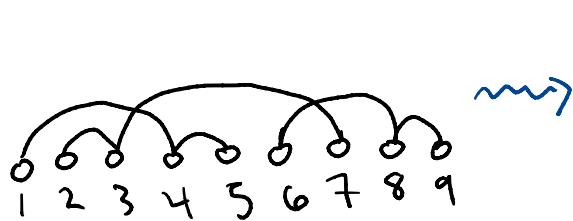
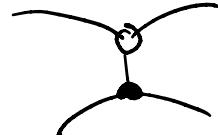
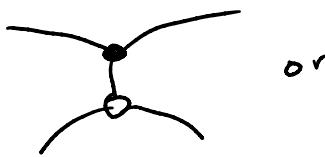
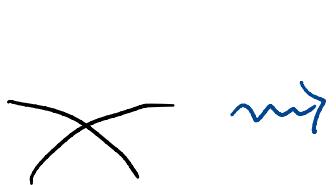
noncross. match  
of x's + y's:

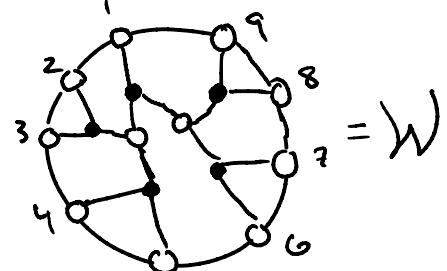


Super-impose  
to get M-diagram:



Have to resolve crossings to get web:

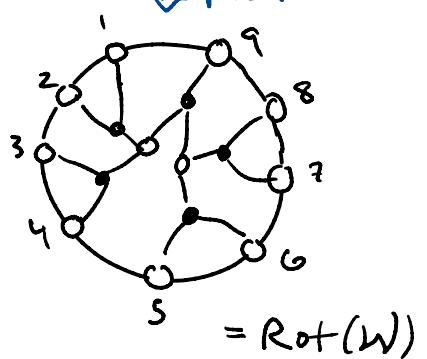


$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 6 \\ \hline 3 & 4 & 8 \\ \hline 5 & 7 & 9 \\ \hline \end{array}$$


Pro ↓

$$\text{Pro}(T) = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 6 & 7 \\ \hline 4 & 8 & 9 \\ \hline \end{array}$$


↓ Rot

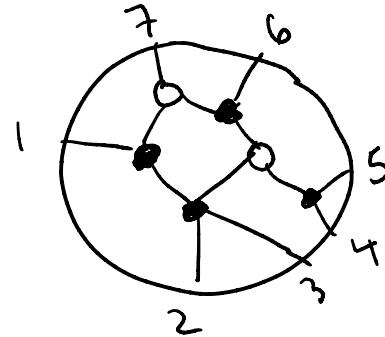


## Def'n (Postnikov)

A plabic graph is a planar graph, drawn in a disc, w/  $\deg = 1$  bdy. Vert's 1, 2, ...,  $m$ , and some int. vert's, such that int. vert's are colored black + white.

<sup>↗</sup>  
not necessarily properly!

e.g.:



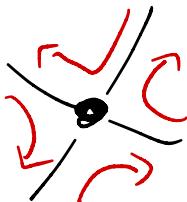
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Observation: A web is (basically) a special case of a plabic graph!

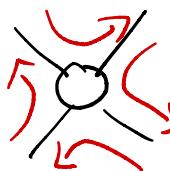
Q: What does this observation buy us?

## Def'n (Postnikov)

The trip permutation\* of a plabic graph has  $\pi(i) = \text{bdy. vertex you end at if you start a trip at } i \text{ and follow the } \underline{\text{rules of the road}}$ :

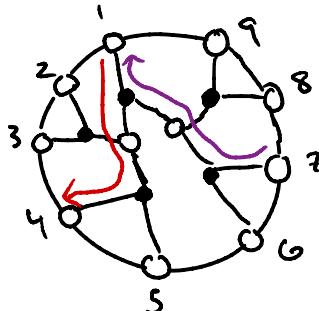


right at black



left at white

e.g.

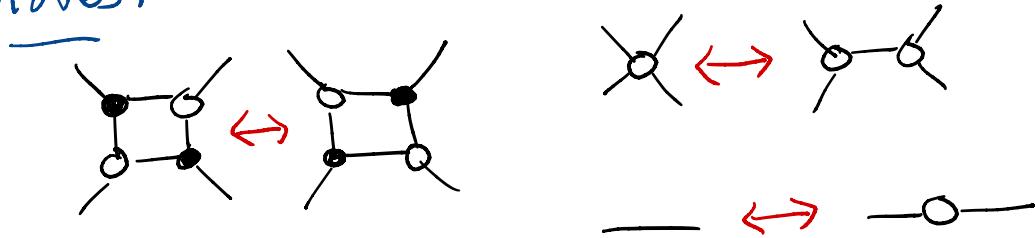


trip starting at 1 ends at 4  
trip starting at 7 ends at 1  
etc ...

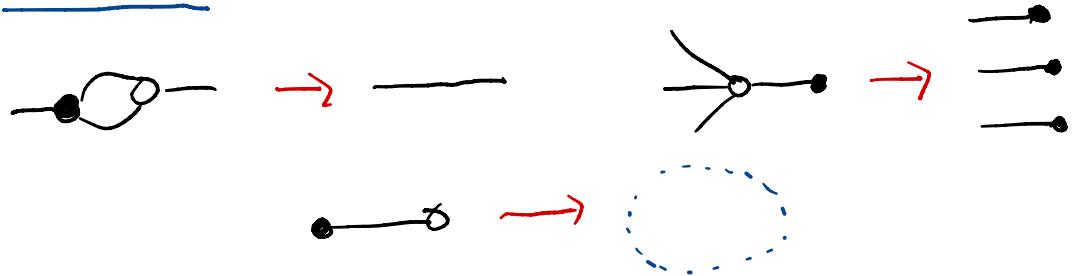
$$\pi = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)\ (4\ 3\ 8\ 5\ 2\ 7\ 1\ 9\ 6)$$

Def'n (Postnikov) Moves and reductions  
are local transformations of plabic graphs:

Moves:



Reductions:



Write  $G \sim G' :=$   $G'$  can be obtained from  $G$   
via a series of moves.

Say  $G$  is reduced if there is no  $G' \sim G$   
s.t. can apply a reduction to  $G'$ .

Thm (Postnikov)  $G, G'$  reduced plabic  
graphs  
have same trip perm.  $\Leftrightarrow G \sim G'$ .

Cor Two irreducible webs  $\mathcal{W}, \mathcal{W}'$  have same trip perm.  $\Leftrightarrow \mathcal{W} = \mathcal{W}'$ .

Pf: Only moves/reductions we can

do are trivial: —  $\leftrightarrow$  —



~

For  $3 \times n$  SYT  $T$ , let  $\pi_{\text{trip}}$  be the trip perm. of associated web  $\mathcal{W}$ .

So  $T$  is determined by  $\pi_{\text{trip}}$ .

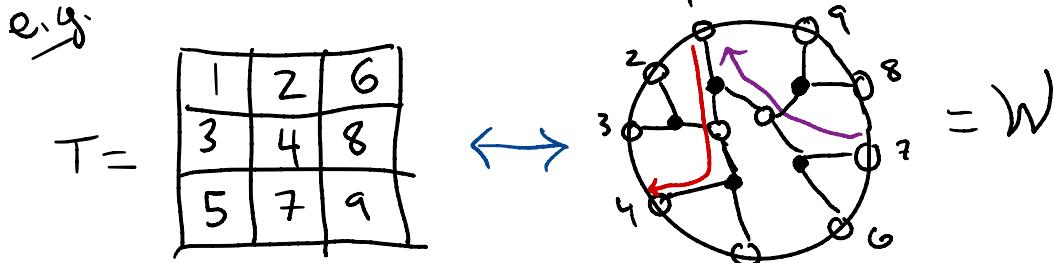
Q: Another way to find  $\pi_{\text{trip}}$ ?

A: YES! Using promotion:

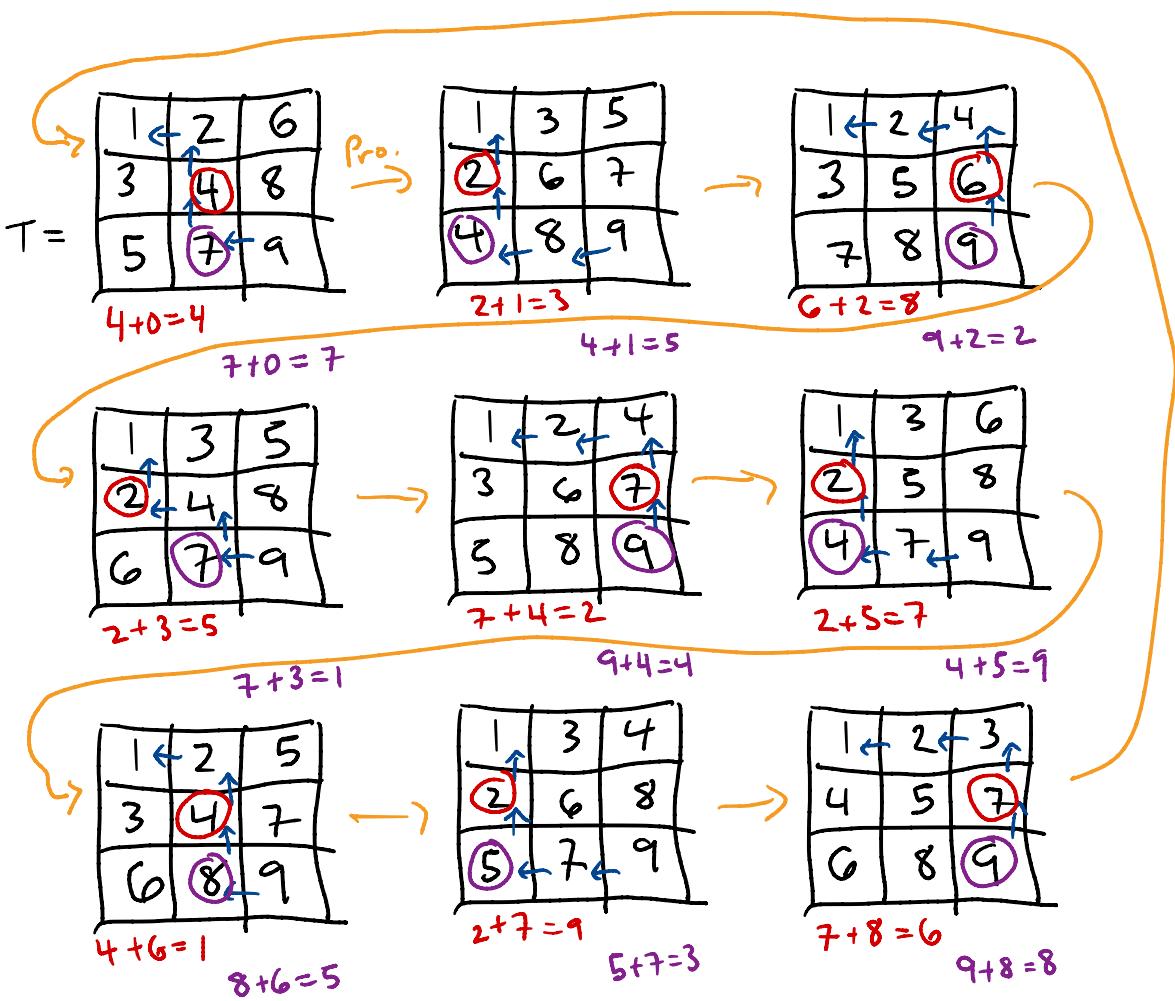
the # that moves up

$\pi_x(i) \equiv$  from 2<sup>nd</sup> row when  $+ (i-1) \pmod{3n}$   
applying Pro to  
 $\text{Pro}^{i-1}(T)$

$\pi_y$  = Same but w/ 3<sup>rd</sup> row instead of 2<sup>nd</sup>



$$\pi_{trip} = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9)$$



$$\pi_x = 438527196 = \pi_{trip} \quad \pi_y = 752149538 = \pi_{trip}^{-1}$$

Thm For any  $3 \times n$  SYT  $T$ ,

(a)  $\pi_X = \pi_{\text{trip}}$

(b)  $\pi_Y = \pi_{\text{trip}}^{-1}$ .

Pf! For (a), first note that  $\pi_{\text{trip}}, \pi_X$  both rotate  $\xrightarrow{\text{w} \mapsto \text{cwc}^{-1}}$  when we apply Pro to  $T$ .

So suffices to prove  $\pi_{\text{trip}}(1) = \pi_X(1)$ .

But it's easy to see both  
= what 1 is matched with in M-diagram.

For (b), follows by considering dual SYT.  $\downarrow 180^\circ \text{ rot.}$  □

Rmk: Representing  $T$  via a chain of partitions  $x^0 \xrightarrow{} x^1 \xrightarrow{} x^2 \dots \xrightarrow{} x^{3n-1} \xrightarrow{} x^{3n}$  in usual way,

can read off  $\pi_X, \pi_Y$  in a beautiful way from the growth diagram of  $T$ .

Rule for growth diagram: (see Fomin, Roby, Stanley...)

$$(a, b, c) + e_i \quad (a, b, c) + e_i + e_j$$

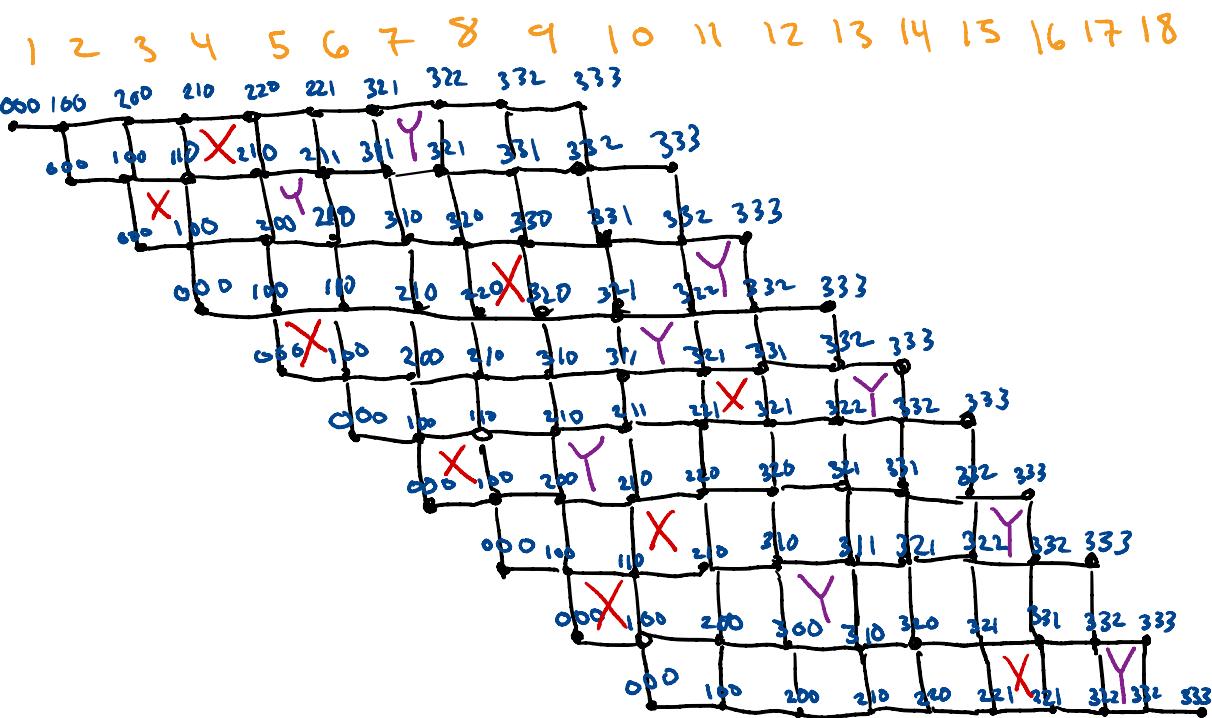
$\left\{ \begin{array}{l} (a, b, c) + e_j \text{ if this is a partition} \\ (a, b, c) + e_i \text{ otherwise} \end{array} \right.$

↳ In this 2<sup>nd</sup> case mark box w/

X if  $j=2$

Y if  $j=3$

Growth diagram for full promotion orbit of T:



$$\pi_X = 4, 3, 8, 5, 11, 7, 10, 9, 15 = 4, 3, 8, 5, 2, 7, 1, 9, 6$$

$$\pi_Y = 7, 5, 11, 10, 13, 9, 15, 12, 17 = 7, 5, 2, 1, 4, 9, 6, 3, 8$$

So... what? What can we do w/ this?

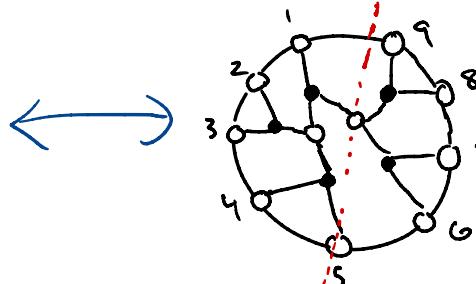
Well... I'd love you to tell me!

But here is one thing I know we can do:

Cor (c.f. Patrias-Pechenik)  $\text{Evac}(T) \leftrightarrow \text{Flip}(W)$ ,  
where  $\text{Flip}(W) = \text{refl. of } W \text{ across a diam.}$

e.g.

|   |   |   |
|---|---|---|
| 1 | 2 | 6 |
| 3 | 4 | 8 |
| 5 | 7 | 9 |

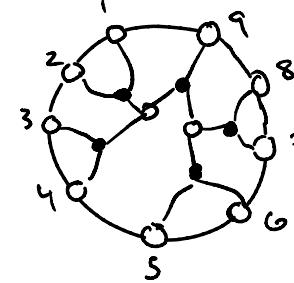


$\text{Evac} \downarrow$

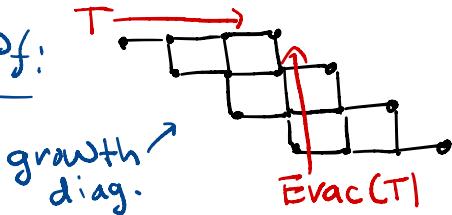
|   |   |   |
|---|---|---|
| 1 | 3 | 5 |
| 2 | 6 | 7 |
| 4 | 8 | 9 |



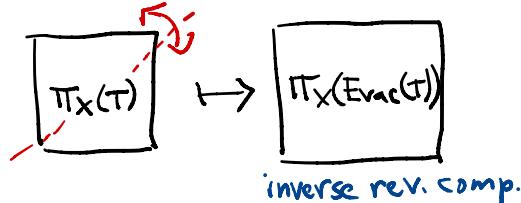
$\downarrow \text{Flip}$



Pf:



hence



But since trip perm. of  $\text{Flip}(W) = \text{inv. r.c. of trip perm. of } W$ ,  
 $\text{Flip}(W)$  is only possibility for  $\text{Evac}(T)$ . □

But... what else? :

- do this in other, similar contexts:  
e.g., semistandard  $3 \times n$  tableaux (Russell)  
or linear extensions of  $V \times [n]$  (H.-Rubey)  
or  $a \times b$  SYT w/  $a > 3$  (ambitious!)
- Can we classify perm's  $\pi$  trip,  
say for  $3 \times n$  SYT ?
  - Well-known that we can either make a  
(red.) plabic graph trivalent or bipartite  
Via moves. Webs are when we can do both.  
Any significance to that?
- Plabic graphs + Webs both related to the  
Grassmannian  $\text{Gr}(k, n)$ :
  - i) Plabic graphs parameterize cells in the  
totally nonnegative Grass.  $\text{Gr}_{\geq 0}(k, n)$ ,
  - ii) Webs yield tensor invariants, which can  
be viewed as elements of  $\mathbb{C}[\text{Gr}(k, n)]$ .

Is there a direct connection (c.f. Fraser-Lam-Le)?  
Where does promotion fit in?

Thank you!

(+ hope to see you all in person  
Sometime soon...  
maybe in DC?)