Howard Math 273, HW# 1,

Fall 2023; Instructor: Sam Hopkins; Due: Friday, September 29th

1. *(Stanley, EC1, #1.66)* Let $p_k(n)$ denote the number of partitions of n into exactly k parts. Give a **bijective** proof that

$$p_0(n) + p_1(n) + p_2(n) + \cdots + p_k(n) = p_k(n + k).$$

2. *(Stanley, EC1, #1.5)* Show that

$$\sum_{n_1,\ldots,n_k \geq 0} \min(n_1, \ldots, n_k) x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} = \frac{x_1 x_2 \cdots x_k}{(1 - x_1)(1 - x_2) \cdots (1 - x_k) \cdot (1 - x_1 x_2 \cdots x_k)},$$

where $\min(n_1, \ldots, n_k)$ means the minimum of the integers $n_1, \ldots, n_k$.

3. *(Stanley, EC1, #1.26)* Let $\overline{c}(n, m)$ denote the number of compositions of n into parts of size at most m. Show that

$$\sum_{n \geq 0} \overline{c}(n, m) x^n = \frac{1 - x}{1 - 2x + x^{m+1}}.$$

4. Prove that, for any $n \geq 0$,

$$4^n = \sum_{k=0}^{n} \binom{2k}{k} \binom{2(n-k)}{n-k}.$$

Hint: We discussed the generating function $\sum_{n=0}^{\infty} \binom{2n}{n} x^n$ of the central binomial coefficients. How can you use what we proved about this generating function to deduce this result?

5. Let $n \geq 1$, and let $\mathrm{ODD}(n)$ denote the subset of permutations in the symmetric group $\mathfrak{S}_n$ with no cycles of even size. Prove that

$$\sum_{\sigma \in \mathrm{ODD}(n)} 2^{\#\mathrm{cycles}(\sigma)} = 2 \cdot n!.$$

Hint: Recall *Touchard's theorem*, which says that

$$\sum_{n \geq 0} \frac{1}{n!} \left(\sum_{\sigma \in \mathfrak{S}_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots t_n^{c_n(\sigma)} \right) x^n = e^{t_1 \frac{x}{1} + t_2 \frac{x^2}{2} + t_3 \frac{x^3}{3} + \cdots} = e^{\sum_{j=1}^{\infty} t_j \frac{x^j}{j}},$$

where $c_k(\sigma)$ is the number of cycles of σ of size k. How can you use Touchard's theorem to deduce this result?