

Final Exam Study Guide

Math 157 (Calculus II), Spring 2024

1. Geometric applications of integrals [§6.1, 6.2, 6.3, 8.1, 8.2]

- (a) Area between curves [§6.1]: area between $y = f(x)$ and $y = g(x)$ is $\int_a^b |f(x) - g(x)| dx$.
- (b) Volume of general solid [§6.2]: if $A(x)$ = area of cross-section, then volume is $\int_a^b A(x) dx$.
- (c) Volume of solid of revolution [§6.2, 6.3]: “disks/washers” & “cylindrical shells” methods.
For region below curve $y = f(x)$ from $x = a$ to $x = b$:
 - i. rotated around x -axis, “disks method” gives volume $= \int_a^b \pi f(x)^2 dx$;
 - ii. rotated around y -axis, “shells method” gives volume $= \int_a^b 2\pi f(x) x dx$.
- (d) Arc lengths of curves [§8.1]: length of $y = f(x)$ from $x = a$ to $x = b$ is $\int_a^b \sqrt{1 + (f'(x))^2} dx$.
- (e) Area of surface of revolution [§8.2]:
 - i. for $y = f(x)$ from $x = a$ to $x = b$ rotated about x -axis, area is $\int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx$;
 - ii. for $x = g(y)$ from $y = c$ to $y = d$ rotated about x -axis, area is $\int_c^d 2\pi y \sqrt{1 + (g'(y))^2} dy$.

2. Other applications of integrals [§6.4, 6.5]

- (a) Work [§6.4]: if $F(x)$ = force as function of distance, then work done is $W = \int_a^b F(x) dx$.
- (b) Average of function [§6.5]: the average of $f(x)$ from $x = a$ to $x = b$ is $\frac{1}{b-a} \int_a^b f(x) dx$.

3. Techniques for computing integrals [§7.1, 7.2, 7.3, 7.4, 7.5]

- (a) Integration by parts [§7.1]: $\int u dv = uv - \int v du$; choose u using “LIATE” rule
- (b) Trigonometric integrals [§7.2]: for $\int \sin^n(x) \cos^m(x) dx$, use the Pythagorean identity $\sin^2(x) + \cos^2(x) = 1$ to isolate single factor of $\cos(x) dx$ or $\sin(x) dx$, then do a u -sub.
- (c) Trigonometric substitution [§7.3]:
 - i. for $a^2 - x^2 \Rightarrow$ sub $x = a \sin(\theta)$, $dx = a \cos(\theta) d\theta$, and use $1 - \sin^2(\theta) = \cos^2(\theta)$;
 - ii. for $a^2 + x^2 \Rightarrow$ sub $x = a \tan(\theta)$, $dx = a \sec^2(\theta) d\theta$, and use $1 + \tan^2(\theta) = \sec^2(\theta)$.
- (d) Integrating rational functions by partial fractions [§7.4]: find roots of denominator $Q(x)$ and solve system of equations to write $P(x)/Q(x) = A/(x-a) + B/(x-b) + \dots + Z/(x-z)$ and use $\int A/(x-a) dx = A \ln(x-a)$; for repeated roots do $A_1/(x-a) + A_2/(x-a)^2 + \dots$.

4. Other concepts related to integration [§7.7, 7.8]

- (a) Approximating definite integrals [§7.7]: two good approximations of $\int_a^b f(x) dx$ are
 - i. midpoint approximation $M_n = \sum_{i=1}^n f(\bar{x}_i) \Delta x$ where $\bar{x}_i = \frac{x_{i-1} + x_i}{2}$;
 - ii. trapezoid approximation $T_n = \frac{\Delta x}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n))$.
- (b) Improper integrals [§7.8]: $\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$, et cetera.

5. Parametrized curves [§10.1, 10.2]

- (a) Curve of form $x = f(t)$ and $y = g(t)$ for some auxiliary variable t (“time”) [§10.1]
- (b) Slope of tangent [§10.2] to curve given by chain rule: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$
- (c) Arc length [§10.2] is $\int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} dt = \int_a^b \sqrt{g'(t)^2 + f'(t)^2} dt$

6. Polar coordinates and polar curves [§10.3, 10.4]

- (a) Cartesian vs. polar [§10.3]: $(x, y) = (r \cos \theta, r \sin \theta)$ and $(r, \theta) = (\sqrt{x^2 + y^2}, \arctan(\frac{y}{x}))$
- (b) Area inside [§10.4] polar curve $r = f(\theta)$ for $\alpha \leq \theta \leq \beta$ is $\int_\alpha^\beta \frac{1}{2} r^2 d\theta = \int_\alpha^\beta \frac{1}{2} f(\theta)^2 d\theta$
- (c) Slope of tangent [§10.4] to polar curve $r = f(\theta)$ given by chain and product rules:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(r \sin \theta)}{\frac{d}{d\theta}(r \cos \theta)} = \frac{f(\theta) \cos \theta + f'(\theta) \sin \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}$$

- (d) Arc length [§10.4] of polar curve $r = f(\theta)$ is $\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta$

7. Sequences and series [§11.1, 11.2, 11.3, 11.4, 11.5, 11.6, 11.7]

- (a) Sequence $\{a_n\}_{n=1}^\infty = a_1, a_2, \dots$ is list of numbers, $\lim_{n \rightarrow \infty} a_n$ defined like $\lim_{x \rightarrow \infty} f(x)$ [§11.1]
- (b) Series $\sum_n^\infty a_n$ is “infinite sum” $a_1 + a_2 + \dots$ of terms a_n ; its value is $s = \lim_{n \rightarrow \infty} s_n$ where $s_n = a_1 + a_2 + \dots + a_n$ is the n th partial sum [§11.2]
- (c) Important series: geometric series [§11.2] $\sum_{n=1}^\infty ar^{n-1}$ converges if and only if $|r| < 1$ (and $= \frac{a}{1-r}$ if it converges); p -series [§11.3] $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if and only if $p > 1$
- (d) Many tests for convergence / divergence of series:
 - i. (Divergence test [§11.2]) If $\lim_{n \rightarrow \infty} a_n \neq 0$, series $\sum_n^\infty a_n$ diverges.
 - ii. (Integral test [§11.3]) If $f(x)$ continuous, decreasing, and positive, with $a_n = f(n)$, then $\sum_n^\infty a_n$ converges if and only if $\int_1^\infty f(x) dx$ converges. In this case, have error bounds for remainder $R_n = s - s_n$: $\int_{n+1}^\infty f(x) dx \leq R_n \leq \int_n^\infty f(x) dx$.
 - iii. (Comparison tests [§11.4]) If $\sum_n^\infty b_n$ converges & $a_n \leq b_n$, then $\sum_n^\infty a_n$ converges. If $\sum_n^\infty b_n$ diverges & $a_n \geq b_n$, then $\sum_n^\infty a_n$ diverges. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ exists and is $\neq 0$, then $\sum_n^\infty a_n$ converges if and only if $\sum_n^\infty b_n$ converges.
 - iv. (Alternating series test [§11.5]) Alternating series $\sum_{n=1}^\infty (-1)^{n-1} b_n$ converges as long as $b_{n+1} \leq b_n$ and $\lim_{n \rightarrow \infty} b_n = 0$. In this case, have error bound: $|R_n| \leq b_{n+1}$.
 - v. (Ratio test [§11.6]) For series $\sum_{n=1}^\infty a_n$, let $L = \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|}$. If $L < 1$, series converges. If $L > 1$ (including ∞), series diverges. If $L = 1$, test is inconclusive.

8. Power series and Taylor series [§11.8, 11.9, 11.10, 11.11]

- (a) The ratio test tells us that any power series $\sum_{n=0}^\infty c_n(x-a)^n$ has a radius of convergence R such that it converges when $|x-a| < R$ and diverges when $|x-a| > R$ [§11.8]
- (b) Differentiate, integrate, and multiply power series like they are polynomials [§11.9, 11.10]
- (c) Taylor series of $f(x)$ at $x = a$ is $\sum_{n=0}^\infty \frac{f^{(n)}(a)}{n!} (x-a)^n$, where $f^{(n)}$ is n th derivative [§11.10]
- (d) Important Taylor series [§11.10]: $\frac{1}{1-x} = \sum_{n=0}^\infty x^n$ ($R = 1$); $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$ ($R = \infty$); $\sin(x) = \sum_{n=0}^\infty \frac{(-1)^{n-1} x^{2n+1}}{(2n+1)!}$ ($R = \infty$); $\cos(x) = \sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{(2n)!}$ ($R = \infty$)
- (e) Taylor polynomial $T_n(x)$: n th partial sum of series; $f(x) \approx T_n(x)$ if $x \approx a$ [§11.10, 11.11]