Howard Math 157: Calculus II Spring 2024
Instructor: Sam Hopkins (sam.hopkins@howard.edu)
(call me "Sam")

1/8 Logistics:

Classes: MWRF 10:10 - 11 am ASB-B #100
Office HRS: R 9-10 am Annex III - #220
or by appointment - email me!

Website: samuelfhopkins.com/classes/157.html

Text: Calculus, Early Transcendentals by Stewart, 9e

Grading: 35% (in-person) quizzes
45% three (in-person) midterms
20% (in-person) final exam

There will be 11 in-person quizzes taken on Thursdays (about 20 mins, we will go over answers in class).
Your lowest 2 scores will be dropped (so 9/11 count).
The 3 midterms will happen in-class, also on Thursdays.
The final will take place during finals week.
This is an in-person class; all assessments must be taken in-person!

Beyond that, I will assign additional practice problems from the book.

and I expect you to SHOW UP TO CLASS + PARTICIPATE! :)

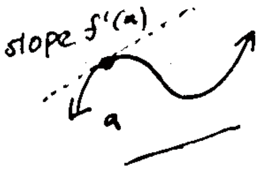
which means ASK QUESTIONS!

Overview of the course:

In Calculus I we learned two important and related operations on functions $f(x): \mathbb{R} \to \mathbb{R}$:

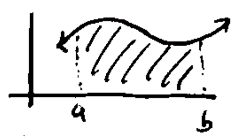
- differentiation and • integration

The derivative $f'(a)$ of $f(x)$ at a point $x=a$ is the slope of the tangent to $y=f(x)$ at $(a, f(a))$.



It is also the "instantaneous rate of change" of the function $f(x)$ at $x=a$.

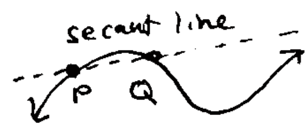
The integral $\int_a^b f(x)\,dx$ is the area under the curve $y=f(x)$ from $x=a$ to $x=b$:



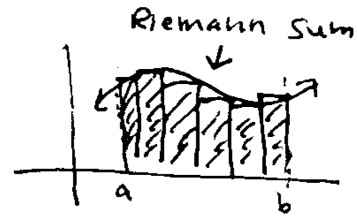
area (▨) $= \int_a^b f(x)\,dx$

Both the derivative and integral are formally defined as limits:

- the derivative is the limit of slopes of secant lines approximating the tangent:
  $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$



- the integral is the limit of Riemann sums (= rectangles) approximating area under curve:
  $$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$



The Fundamental Theorem of Calculus says that differentiation and integration are inverse operations:

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F'(x) = f(x)$.

In Calculus II we will continue to study derivatives & integrals.

Some of the things we will learn are:

- Applications of integration:

In Calc I we learned many applications of derivatives (minimums & maximums, concavity, etc.)

In Calc II we will learn more things we can compute using integrals (beyond area under curve) like

  - volumes (3D version of area)
  - lengths (1D version of area)

Also, FTC says that integral represents net change, so we will study some physical applications of integrals like to work (in the sense of force).

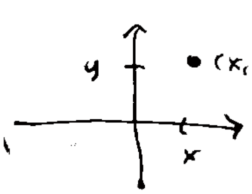
- Techniques for integration:

Using rules for differentiation like product and chain rules, we know how to take the derivative of "any" function,

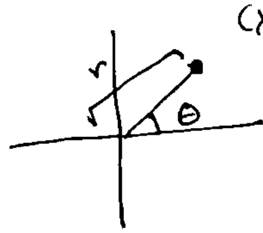
e.g. $d/dx\,(x \sin(e^{x^2} + 5x - 6))$

But... integrating a "random" function like this can be really hard or not even possible. We will learn more techniques for computing integrals, when possible.
[Recall that we already learned one technique: u-substitution]

- Polar coordinates: We are used to working with (x, y) a.k.a. "Cartesian coordinates"



vs.



Polar coordinates (r, θ) are a different system where we can also do calculus.

• Taylor series:

How do we evaluate a function $f(x)$ at a particular value, e.g. compute $f(1.5)$?

If $f(x)$ is a polynomial like $f(x) = 6x^2 - 2x + 3$ we can use arithmetic: $f(1.5) = 6(1.5)^2 - 2(1.5) + 3 = \cdots$

If it is a rational function like $f(x) = \dfrac{x+1}{x^2-2}$ we can use division similarly: $f(1.5) = \dfrac{1.5+1}{(1.5)^2-2} = \cdots$

But what about something like $f(x) = \sin(x)$ or $f(x) = e^x$? How to compute $e^{1.5}$? What does your calculator even do?

Even though e^x is not a polynomial, it has a representation as a kind of "infinite" polynomial:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$$

This is called a Taylor series, and it lets us compute things like $e^{1.5}$ (at least approximately).

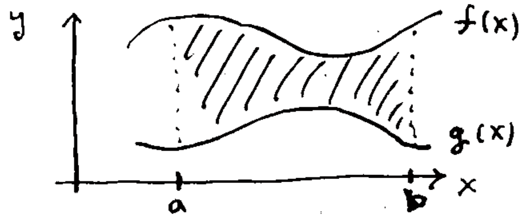
We will learn how to deal with these kind of infinite sums called series (specifically, power series) and related mathematical constructions called sequences.

We will also learn Taylor's theorem, telling us that the coefficients of the Taylor series can be computed from the derivative of the function (which is where calculus comes in!).

1/11

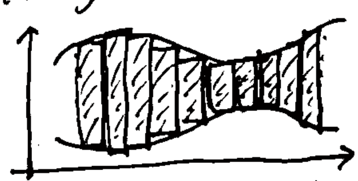
Area between curves (§6.1 of textbook)

The integral computes the area under a curve. What if we have two curves, $y=f(x)$ and $y=g(x)$, and we want to know the area between the curves?



Specifically, suppose that $f(x) \geq g(x)$ for all x in some closed interval from $x=a$ to $x=b$.

Then, as with the integral, we can define the area between the curves on $[a,b]$ by approximating it with a large number of thin rectangles:



Let $\Delta x = \frac{b-a}{n}$ (for some $n \geq 1$)
and let $x_i = a + i \cdot \Delta x$ for $i = 0, 1, \ldots, n$
So that $[a,b]$ is divided into n sub-intervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$

For each sub-interval, choose a $x_i^* \in [x_{i-1}, x_i]$, and consider the thin rectangles of width Δx and height $= f(x_i^*) - g(x_i^*)$ ← difference in hts of two curves at $x = x_i^*$

Then area between curves from $x=a$ to $x=b$ $\approx \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*))\Delta x$

and is exactly → $= \lim_{n\to\infty} \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*))\Delta x$

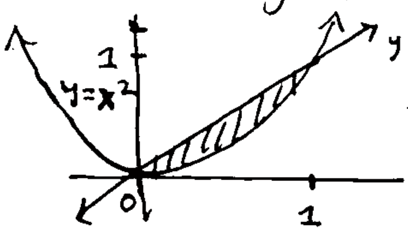
$$\boxed{= \int_a^b f(x) - g(x)\, dx}$$

So... area between two curves can be computed as integral of difference function

Note: If we let $g(x) = 0$ be the function corresponding to the x-axis $y=0$, then we recover the area under the curve as $\int_a^b f(x)\,dx$ from

E.g.: Let's compute the area bounded by the curves $y = x$ and $y = x^2$.

Since the problem does not tell us the bounds of integration, let us sketch the curves:



Letting $f(x) = x$ and $g(x) = x^2$, we can find where the curves intersect by setting $f(x) = g(x)$

$$\Rightarrow x = x^2 \Rightarrow x^2 - x = 0$$
$$\Rightarrow x(x-1) = 0$$
$$\Rightarrow x = 0 \text{ or } x = 1$$

Also, choosing $x = \frac{1}{2}$, we see that between $x=0$ and $x=1$, $f(x) = \frac{1}{2} \geq g(x) = \frac{1}{4}$, so the curve $y = f(x)$ is above $y = g(x)$ on $[0,1]$.

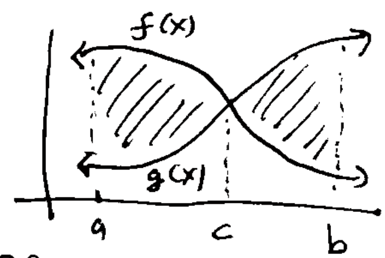
Thus, the area bounded by the curves is

$$\int_a^b f(x) - g(x)\,dx = \int_0^1 x - x^2\,dx = \frac{x^2}{2} - \frac{x^3}{3}\Big]_0^1$$
$$= \left(\frac{1^2}{2} - \frac{1^3}{3}\right) - \left(\frac{0^2}{2} - \frac{0^3}{3}\right) = \frac{1}{2} - \frac{1}{3} = \boxed{\frac{1}{6}}.$$

If on the interval $[a,b]$, sometimes $f(x) > g(x)$ and sometimes $g(x) > f(x)$, then to correctly find area between them, we need to take absolute value of difference:

$$\text{area between curves} = \int_a^b |f(x) - g(x)|\,dx.$$

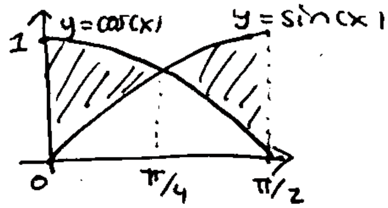
In practice, we break up this integral into the parts where $f(x) \geq g(x)$ and where $g(x) \geq f(x)$



$$\Rightarrow \int_a^c f(x) - g(x)\,dx + \int_c^b g(x) - f(x)\,dx$$

E.g.: Compute the area between $y = f(x) = \cos(x)$ and $y = g(x) = \sin(x)$ for $x = 0$ to $x = \pi/2$.

Again, good idea to sketch curves to see what's going on:



$\cos(0) = 1 > 0 = \sin(0)$, but $\sin(\pi/2) = 1 > 0 = \cos(\pi/2)$, so which curve is on top changes from $x=0$ to $x=\pi/2$. In fact, have $\cos(\pi/4) = \sin(\pi/4)$ (by symmetry, or isosceles right triangle...)

Thus...

area between $y=\cos(x)$ and $y=\sin(x)$ from $x=0$ to $x=\pi/2$ $= \int_0^{\pi/4} \cos(x) - \sin(x)\,dx + \int_{\pi/4}^{\pi/2} \sin(x) - \cos(x)\,dx$

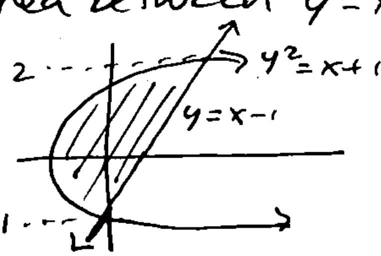
$= \sin(x) + \cos(x)\Big]_0^{\pi/4} + -\cos(x) - \sin(x)\Big]_{\pi/4}^{\pi/2}$

$= (\sin(\pi/4) + \cos(\pi/4) - \sin(0) - \cos(0)) + (-\cos(\pi/2) - \sin(\pi/2) + \cos(\pi/4) + \sin(\pi/4))$

$= (1/\sqrt{2} + 1/\sqrt{2} - 0 - 1) + (-0 - 1 + 1/\sqrt{2} + 1/\sqrt{2}) = \boxed{2\sqrt{2} - 2}$ //

E.g.: Sometimes it is easier to integrate w.r.t. y variable. Let's find area between $y = x-1$ and $y^2 = x+1$:

We sketch the curves: they intersect at $y=-1$ and $y=2$



$x = y^2 - 1 = g(y)$
and $x = y+1 = f(y)$
set equal $y^2 - 1 = y + 1$
$\Rightarrow y^2 - y - 2 = 0$
$\Rightarrow (y-2)(y+1) = 0$
$\Rightarrow y = 2$ or $y = -1$

Then, since $y = x-1$ is to right of $y^2 = x+1$ for $y=-1$ to $y=2$:

area between curves $= \int_{-1}^{2} f(y) - g(y)\,dy = \int_{-1}^{2} (y+1) - (y^2 - 1)\,dy$

$= \int_{-1}^{2} -y^2 + y + 2\,dy = \frac{-y^3}{3} + \frac{y^2}{2} + 2y\Big]_{-1}^{2} = \left(\frac{-8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right)$

$= \boxed{4.5}$ //

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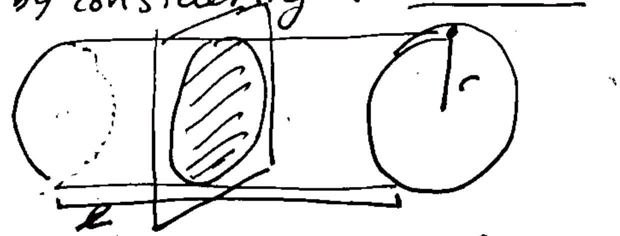
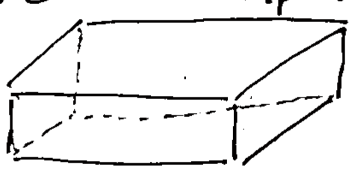
Volumes (§6.2)

Volumes are the 3-dimensional version of areas.
Let's start by considering a circular cylinder:

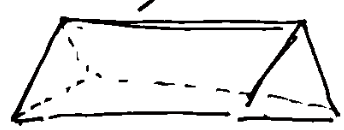


The cross-section (= intersection w/ y,z-plane) of this cylinder at any x-coordinate is a circle (of radius r). We thus define the volume of the cylinder to be = area of cross-section × length of cylinder
$$= \pi r^2 \cdot \ell.$$

We can also consider cylinders whose cross-sections are other shapes, e.g., rectangles or triangles:



rectangular prism
(or rectangular cylinder)



triangular cylinder
('Toblerone' bar)

The important thing is that the cylinder has a certain length and across the whole length cross-sections are same.

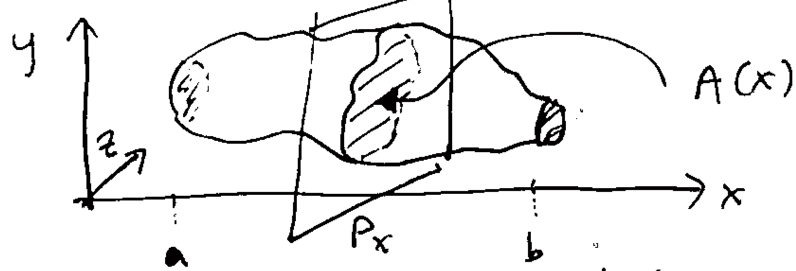
Thus, for any cylinder we define

Volume of cylinder = area of cross-section × length.

E.g.: Volume of rectangular prism = width × height × length.

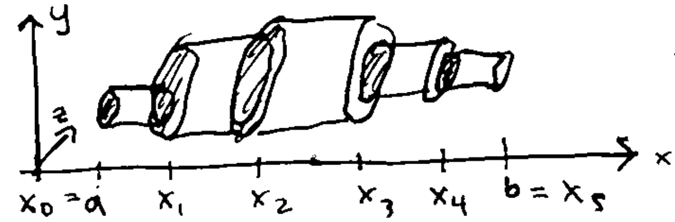
Q: What if the cross-section of our solid is not constant?

Let's draw a picture of our solid:



Suppose the solid extends between $x = a$ and $x = b$, and let $A(x)$ for $a \le x \le b$ be the area of the cross-section obtained by intersecting with plane P_x perpendicular to x-axis at that point.

We can approximate the volume by dividing the solid into several short cylinders:



As w/ integral, we break up the interval $[a,b]$ into n sub-intervals $[x_{i-1}, x_i]$, $i = 1, \dots, n$, $x_i = x_{i-1} + \Delta x$

Then the volume of the solid is $\approx \sum_{i=1}^{n}$ area of cross-section of i^{th} short cylinder $\times \Delta x$

$$= \sum_{i=1}^{n} A(x_i^*) \Delta x$$

and is exactly $= \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*) \Delta x$

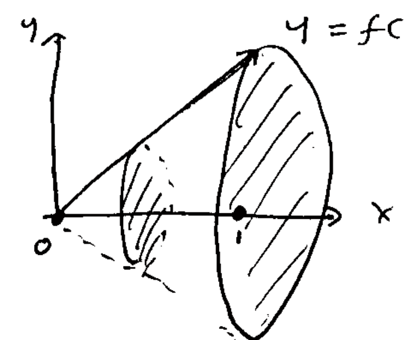
$$\boxed{= \int_a^b A(x)\, dx}$$

This lets us compute volume as an integral!

An important class of solids are the <u>solids of revolution</u> obtained by rotating a region in x,y-plane about x-axis;

<u>E.g.</u> Find the volume of the <u>cone</u> obtained by rotating the area below $y=x$ (and above x-axis) from $x=0$ to $x=1$ about the x-axis.

<u>Sketch</u>:



$\Leftarrow$ at any x with $0 \le x \le 1$ cross-section of cone is a <u>circle</u> of radius $f(x)=x$

Since in this case $A(x) =$ area of circle of radius $f(x)$

$$= \pi(f(x))^2 = \pi x^2$$

we can use the integral formula for volume to get

$$\text{Volume of cone} = \int_0^1 \pi x^2\,dx = \frac{\pi}{3}x^3\Big]_0^1 = \boxed{\frac{\pi}{3}}$$

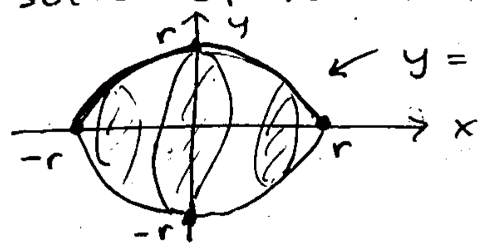
We see that in general the volume of a solid of revolution obtained by rotating the area below the curve $y=f(x)$ from $x=a$ to $x=b$ about the x-axis is

$$\boxed{= \int_a^b \pi\,(f(x))^2\,dx}$$

since every cross-section is a circle of radius $= f(x)$

E.g.: Find the volume of a sphere of radius r using an integral.

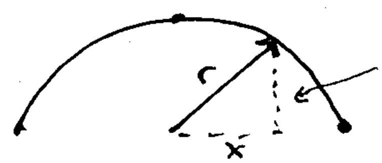
To do this, we have to realize the sphere as a solid of revolution:



We see that a sphere is obtained by rotating a semicircle of radius r about x-axis, and

semicircle of radius r = area below curve $y = \sqrt{r^2 - x^2}$ from $x = -r$ to $x = r$

(Think:



$y = \sqrt{r^2 - x^2}$ since $x^2 + y^2 = r^2$ by Pythagorean Thm.)

Thus, according to the formula for volume of a solid of revolution, we have:

$$\text{volume of sphere of radius } r = \int_{-r}^{r} \pi \left(\sqrt{r^2 - x^2}\right)^2 dx$$

$$= \pi \int_{-r}^{r} (r^2 - x^2)\, dx$$

$$= \pi \left(r^2 x - \frac{x^3}{3} \Big]_{-r}^{r} \right)$$

$$= \pi \left(\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 - \frac{-r^3}{3}\right) \right)$$

$$= \pi \left(2r^3 - \frac{2}{3} r^3\right) = \boxed{\frac{4}{3} \pi r^3}.$$

//

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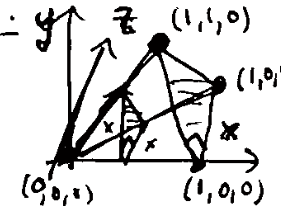
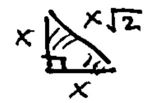
More about volumes §6.2

Solids of revolutions have cross-sections that are circles (or annulus, see below...)
but the formula $\int_a^b A(x)\,dx$ for volume works for other shapes too...

E.g.:



Let's consider the triangular cone which extends from $x=0$ to $x=1$ and whose cross-section at x is a right isosceles triangle:

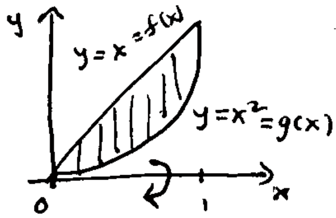


⇐ area $= A(x) = \frac{1}{2}$ base × height $= \frac{1}{2}x^2$

Then the volume of this triangular cone $= \int_0^1 A(x)\,dx = \int_0^1 \frac{1}{2}x^2\,dx = \frac{1}{2}\,\frac{1}{3}\,x^3\Big]_0^1 = \boxed{\frac{1}{6}}$.

Returning to solids of revolution ... we can also rotate the region between two curves about an axis.

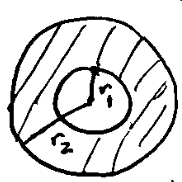
E.g.



Let's rotate the region between the curves $y=x$ and $y=x^2$ from $x=0$ to $x=1$ about the x-axis to make a solid.

The cross-section of this solid is an annulus: the region between two circles

"annulus" a.k.a. "washer" shape →



← area of annulus is $\pi(r_2^2 - r_1^2)$

In the case of region between two curves $y=f(x)$ and $y=g(x)$, the area of this cross-section is $A(x) = \pi(f(x)^2 - g(x)^2)$.

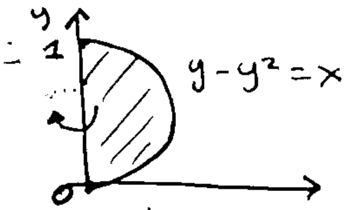
So the volume of the solid is $= \int_a^b \pi\left(f(x)^2 - g(x)^2\right)$.

In above example with $f(x) = x$ and $g(x) = x^2$, we get volume $= \int_0^1 \pi(x^2 - (x^2)^2)\,dx = \int_0^1 \pi\,(x^2 - x^4)\,dx$

$$= \pi\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \boxed{\pi\,\frac{2}{15}}$$

Sometimes we want to rotate across y-axis instead of x-axis.

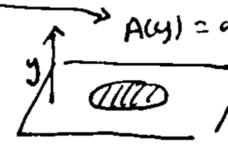
E.g.



How can we compute the volume of the solid obtained by rotating the region between y-axis and curve $y-y^2=x$ about the y-axis?

We just do same thing we've been doing, but with respect to y!

$$\text{Volume of solid} = \int_a^b A(y)\,dy \longrightarrow A(y) = \text{area of } y \text{ cross-section}$$



$$= \int_0^1 \pi(y-y^2)^2\,dy$$

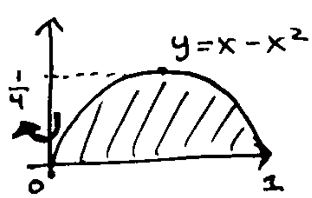
since y-cross-section is circle of radius $f(y)=y-y^2$

$$= \int_0^1 \pi(y^2-2y^3+y^4)\,dy$$

$$= \pi\left[\tfrac{1}{3}y^3-\tfrac{2}{4}y^4+\tfrac{1}{5}y^5\right]_0^1 = \pi\left(\tfrac{1}{3}-\tfrac{1}{2}+\tfrac{1}{5}\right) = \boxed{\tfrac{\pi}{30}}.$$

What about the following solid of revolution problem?

E.g.



Compute the volume of solid obtained by rotating region below $y=x-x^2$ (and above x-axis) about the y-axis.

To do this following the method above, we would have to realize this region as the region between two curves

- $x=f(y)$ and $x=g(y)$ and integrate w.r.t. y.

(To find $f(y)$ and $g(y)$ we need to "invert" $y=x-x^2$ using the quadratic formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\Rightarrow f(y)=\frac{1+\sqrt{1-4y}}{2}$ and $g(y)=\frac{1-\sqrt{1-4y}}{2}$.)

But... there is a better approach using integration w.r.t. x ...