Rank and characteristic generating functions of upper homogeneous posets

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(these slides are on my website
samuelfhopkins.com)

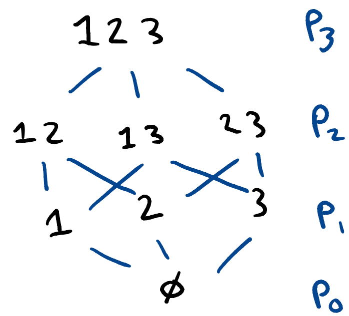
A finite poset P is graded of rank n if we can write $P = P_0 \sqcup P_1 \sqcup \cdots \sqcup P_n$ such that every maximal chain looks like $p_0 \lessdot p_1 \lessdot \cdots \lessdot p_n$, $p_i \in P_i$.

Rank function $rk: P \to \mathbb{N}$ is $rk(p) = i \Leftrightarrow p \in P_i$.

Rank generating polynomial is

$$F_P(x) = \sum_{i \geq 0} \# P_i \; x^i = \sum_{p \in P} x^{rk(p)}$$

e.g. P = Boolean lattice B_n of subsets of $[n] = \{1, 2, \cdots, n\}$



$$F_{B_n}(x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$$

—

Suppose further that P has minimum $\hat{0}$.

Then the characteristic polynomial is

$$\chi_P(x) = \sum_{p \in P} \mu(\hat{0}, p) \, x^{rk(p)}$$

Möbius fn.!

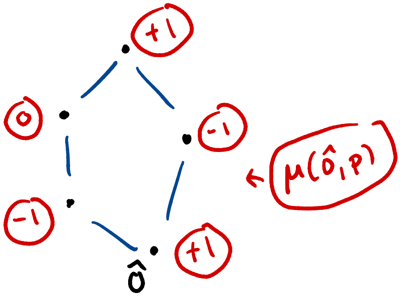
Rmk: Often reciprocal polynomial $\sum_p \mu(\hat{0}, p) \, x^{n - rk(p)}$ is studied instead, since it better matches e.g. char. poly. of a matroid.

Review of Möbius functions:

The **Möbius function** $\mu(p,q)$ for $p \le q \in P$ is the inverse of the **zeta function** $\zeta(p,q) = 1$ in $\mathcal{I}(P)$.

incidence algebra!

It can be computed recursively by:



- $\mu(p,p) = 1$
- $\mu(p,q) = -\sum_{p \le r < q} \mu(p,r)$ for $p<q$.

Recall **Möbius inversion**:

$f,g : P \to R$ (abelian gp.)

$$f(p) = \sum_{q \ge p} g(q) \iff g(p) = \sum_{q \ge p} \mu(p,q) f(q)$$

Also **"Philip Hall's Thm."**:

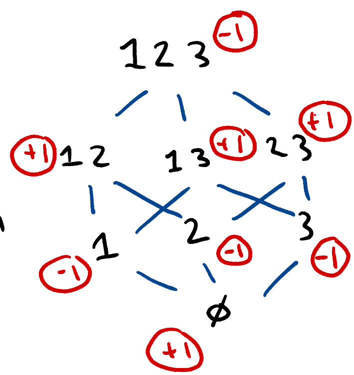
$$\mu(p,q) = c_0 - c_1 + c_2 - c_3 + \cdots$$

Euler characteristic!

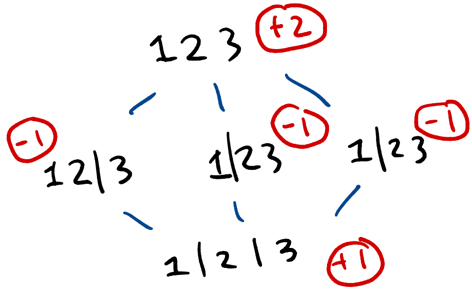
where $c_i = \#$ chains $p = p_0 < p_1 < \cdots < p_i = q$.

e.g. In B_n, $\mu(S,T) = (-1)^{\# T \setminus S}$,

so
$$\chi_{B_n}(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k = (1-x)^n$$



e.g. P = lattice Π_n of set partitions of $[n]$ ordered by refinement



$$F_{\Pi_n}(x) = \sum_{k=0}^{n-1} S(n, n-k)\, x^k$$

Stirling # of 2nd kind

$$\chi_{\Pi_n}(x) = (1-x)(1-2x)\cdots(1-(n-1)x) = \sum_{k=0}^{n-1} s(n, n-k)\, x^k$$

Stirling # of 1st kind

Now we try to do same for certain infinite posets.

(Infinite) poset P is **$\mathbb{N}$-graded** if $P = P_0 \sqcup P_1 \sqcup P_2 \sqcup \cdots$ s.t. every max'l chain is $p_0 \lessdot p_1 \lessdot p_2 \lessdot \cdots$, $p_i \in P_i$.

As before, **rank fn.** $rk : P \to \mathbb{N}$ is $rk(p) = i \iff p \in P_i$.

Say P is **finite type** if $\# P_i < \infty$ $\forall i$.

Then we can form the **rank generating function**

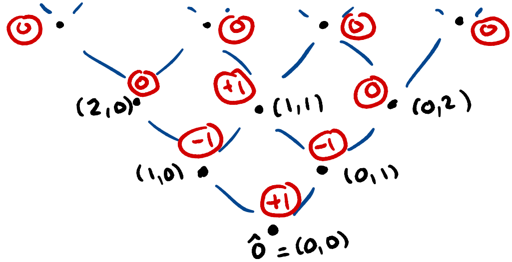
$$F_P(x) = \sum_{i \geq 0} \# P_i\, x^i = \sum_{p \in P} x^{rk(p)}$$

Suppose further P has minimum $\hat{0}$.

Then define the **characteristic generating fn.**

$$\chi_P(x) = \sum_{p \in P} \mu(\hat{0}, p)\, x^{rk(p)}$$

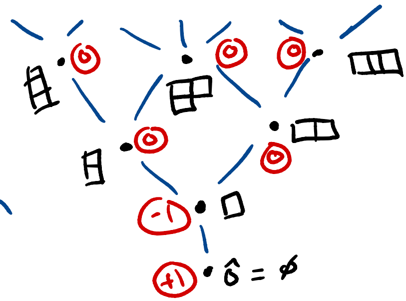
e.g. $P = \mathbb{N}^2$



$$F_P(x) = \sum_{k \geq 0} (k+1)\, x^k$$

$$\chi_P(x) = 1 - 2x + x^2$$

e.g. $P = \mathbb{Y}$, Young's lattice of partitions



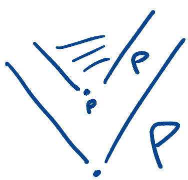
$$F_P(x) = \sum_{n \geq 0} p(n)\, x^n$$

$p(n)$ ← # partitions $\lambda \vdash n$

$$\chi_P(x) = 1 - x$$

Stanley recently introduced following class of posets:

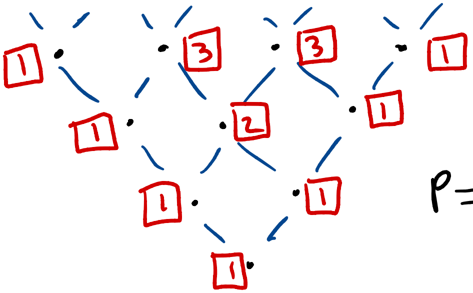
Say P is upper homogeneous or "upho" if $V_p \simeq P$ $\forall\, p \in P$, where $V_p = \{q : q \geq p\}$ is the principal order filter generated by p.



← picture: looking "up" from any $p \in P$, see another copy of P

Stanley was interested in certain analogs of Pascal's triangle coming from some planar upho posets:

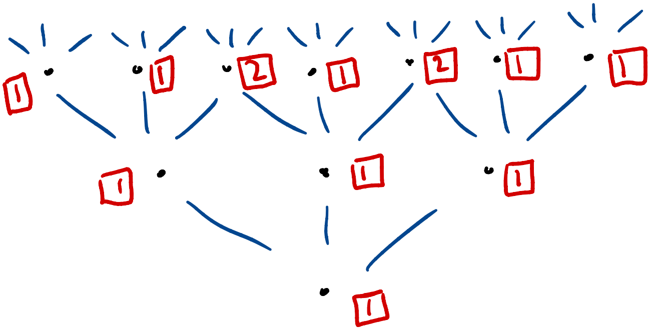
(= planar Hasse diagram)



$\boxed{\begin{array}{c}\#\text{ max'l chains}\\ \hat{0}\text{ to }p\end{array}}$

$P = P_{2,2} = \mathbb{N}^2 \Rightarrow$ Pascal's triangle

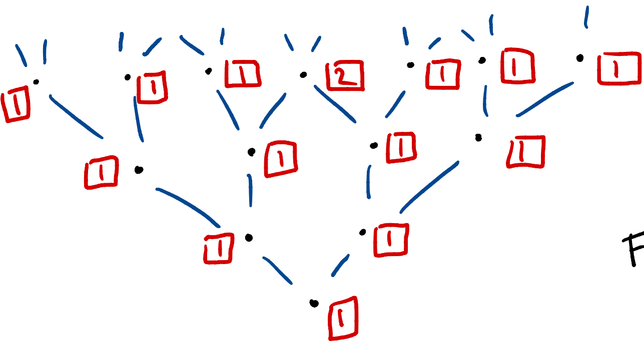
$$F_P(x) = \sum_{k \geq 0} (k+1)\, x^k$$



$P = P_{3,2} =$ "Stern poset"

$\Rightarrow$ Stern's diatomic array

$$F_P(x) = \sum_{k \geq 0} (2^{k+1} - 1)\, x^k$$



$P = P_{2,3} =$ "Fibonacci poset"

$\Rightarrow$ Fibonacci binomial coeff's?

$$F_P(x) = \sum_{k \geq 0} (F_{k+3} - 1)\, x^k$$

Fibonacci #

An MIT PRIMES gp. looked at rank g.f.'s of upho posets:

Thm (Gao-Guo-Seetharaman-Seidel, 2022)

For a planar upho poset P, have $F_P(x) = \dfrac{1}{Q(x)}$

where $Q(x) = 1 - bx + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n$

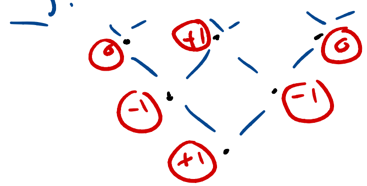
is a polynomial w/ $b, c_2, \ldots, c_n \in \mathbb{N}$, $c_2 + \cdots + c_n - b \leq 0$.

Thm (GGSS, 2022)

There are uncountably many $F_P(x)$'s among all upho posets P. ← read: very bad!

I recently noticed that for upho P, there is a very simple relationship between $F_P(x)$ and $\chi_P(x)$:

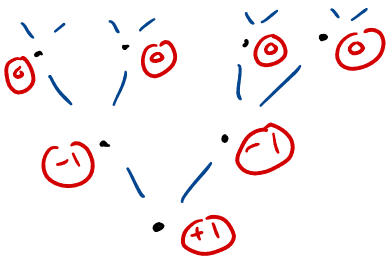
e.g.



$$P = \mathbb{N}^2$$

$$F_P(x) = \sum_{k \geq 0} (k+1)\, x^k = \frac{1}{(1-x)^2}$$

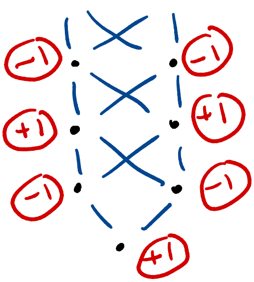
$$\chi_P(x) = 1 - 2x + x^2 = (1-x)^2$$



P = "binary tree"

$$F_P(x) = \sum_{k \geq 0} 2^k x^k = \frac{1}{1-2x}$$

$$\chi_P(x) = 1 - 2x$$



P = "bowtie poset"

$$F_P(x) = 1 + \sum_{k \geq 1} 2\, x^k = \frac{1+x}{1-x}$$

$$\chi_P(x) = 1 + \sum_{k \geq 1} (-1)^k 2x^k = \frac{1-x}{1+x}$$

Thm (H., 2022)

For P upho, we have $F_P(x) = \chi_P(x)^{-1}$.

Pf: Set $f(p) = x^{rk(p)}$ and $g(p) = \sum_{q \geq p} f(q) = \sum_{q \geq p} x^{rk(q)}$ $\forall p \in P$.

By Möbius inversion,

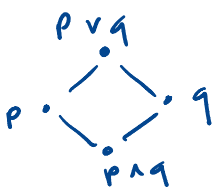
$$1 = x^0 = f(\hat{0}) = \sum_{q \geq \hat{0}} \mu(\hat{0}, q)\, g(q),$$

but since $V_q \simeq P$ $\forall q$, $g(q) = x^{rk(q)} \cdot F_P(x)$, so

$$1 = \sum_{q \in P} \mu(\hat{0}, q)\, x^{rk(q)}\, F_P(x) = F_P(x) \cdot \chi_P(x)$$

$\boxed{}$

Recall that a **lattice** is a poset P where every pair $p, q \in P$ has a **meet** $p \wedge q$,
(= greatest lower bound)



and a **join** $p \vee q$.
(= least upper bound)

Lattices have good Möbius fn.'s, e.g.:

Cor (to Rota's crosscut thm)

If P is a finite lattice, w/ minimum $\hat{0}$, **maximum** $\hat{1}$, and **atoms** $a_1, \ldots, a_m$
(= covers of $\hat{0}$:

)

then $\mu(\hat{0}, \hat{1}) = 0$ if $\hat{1} \neq a_1 \wedge \cdots \wedge a_m$.

join of all atoms

Hence... If P is an upho lattice, then

$$F_P(x) = \chi_P(x)^{-1} = \chi_{P'}(x)^{-1}, \text{ where}$$

P' = subposet of P below joins of atoms

is a finite graded lattice ...

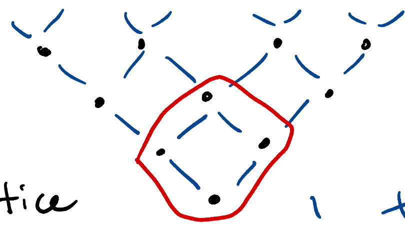
so $F_P(x)^{-1} = \chi_{P'}(x)$ is a polynomial.

(In fact, can replace "lattice" by "meet semilattice" above. And all planar upho P are meet semilattices.)

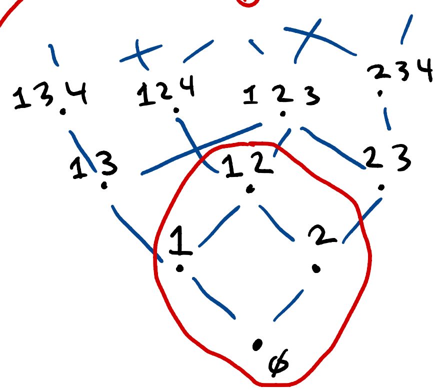
Note Upho lattice P is not determined by this finite graded lattice P':

e.g. $P = \mathbb{N}^n$

$\Rightarrow P' = B_n$
Boolean lattice



$n = 2$



but... $P = \{\text{finite } A \subseteq \{1,2,\ldots\} : \max(A) < \#A + n\}$
(order = inclusion)

$\Rightarrow P' = B_n$ as well

Nevertheless, still interesting to ask... ← maybe at an REU...

Q: When can we **extend** a finite graded lattice P' to an upho lattice P?

e.g. Just saw 2 ways for $P' = B_n$...

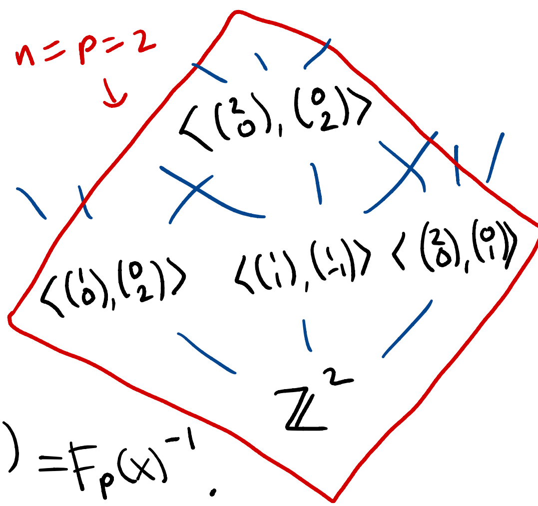
Note Since $F_P(x) = \chi_{P'}(x)^{-1}$, we need $\chi_{P'}(x)^{-1}$ to have nonnegative coeff.'s, so a "random" finite lattice P' does not extend.

e.g. fix $n \geq 1$ and a prime p and set

$P =$ subgroups of $\mathbb{Z}^n$ of index p^k $(k \geq 0)$ ordered by reverse inclusion.

Then P is an upho lattice and $P' = B_n(p)$ = lattice of subspaces of $(\mathbb{Z}/p\mathbb{Z})^n$, with

$n=p=2$ ↓



$$\chi_{P'}(x) = (1-x)(1-px)\cdots(1-p^{n-1}x) = F_P(x)^{-1}.$$

Rmk: More generally, R is a local P.I.D. with finite residue field, then submodules of R^d of finite colength (ordered by rev. inclusion) give an upho lattice.

Conjecture (Stanley)

All* modular upho lattices are of this form.

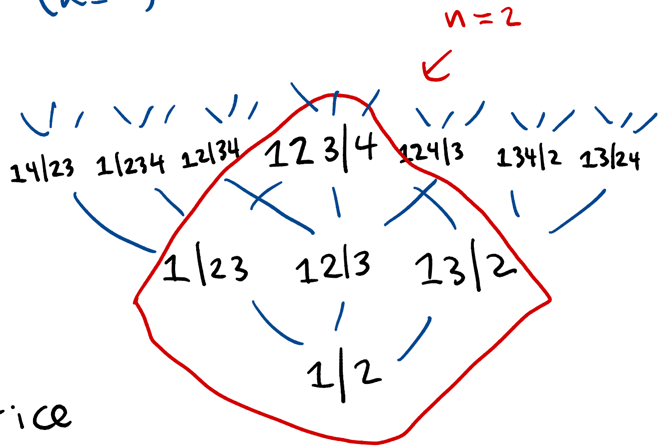
$rk(p) + rk(q) = rk(p \wedge p) + rk(p \vee q) \quad \forall p,q \in P$

---

e.g. Fix $n \geq 1$ and set

$P =$ partitions of $[k]$ $(k \geq n)$ into n blocks

where $\pi_1 \leq \pi_2$ if $\forall B_1 \in \pi_1, \exists B_2 \in \pi_2$ s.t. $B_1 \subseteq B_2$.



Then P is an upho lattice and $P' = \Pi_{n+1}$, so that
(set partition lattice)

$$F_P(x) = \sum_{k \geq n} S(k,n)\, x^{k-n} = \frac{1}{(1-x)(1-2x)\cdots(1-nx)} = \chi_{\Pi_{n+1}}(x)^{-1}.$$

Finite poset P, graded of rank n, w/ min. $\hat{0}$ & max. $\hat{1}$ is called uniform if for all $i = 0, 1, \cdots, n$ there exists a single poset Q_i such that $V_p = [p, \hat{1}] \simeq Q_i$ $\forall p \in P$ with $rk(p) = n - i$.

e.g. $P = B_n$, $B_n(q)$, Π_{n+1} are all uniform (with $Q_i = B_i$, $B_i(q)$, Π_{i+1} respectively).

For such uniform P, define the Whitney #'s of the 1st and 2nd kind $\nu(i,j)$ and $V(i,j)$ by

$$\chi_{Q_i}(x) = \sum_{j=0}^{i} \nu(i, i-j)\, x^j$$

and

$$F_{Q_i}(x) = \sum_{j=0}^{i} V(i, i-j)\, x^j$$

Thm (Dowling, 1971) $\leftarrow$ See Stanley EC1 exercise 3.130

For P uniform, the matrices

$$\left(\nu(i,j)\right)_{j=0,\ldots,n}^{i=0,\ldots,n} \quad \text{and} \quad \left(V(i,j)\right)_{j=0,\ldots,n}^{i=0,\ldots,n}$$

are inverses of one another.

e.g. For $P = \Pi_{n+1}$, says the Stirling # matrices $(s(i,j))$ and $(S(i,j))$ are inverse:

$$n=4 \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -6 & 11 & -6 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{bmatrix}$$

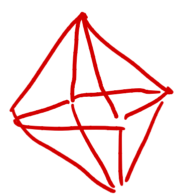
well-known!

Uniform posets are "as close" to being upho as finite posets can be, and the Whitney # thm. "feels similar" to the thm. that $F_P(x)$ and $\chi_P(x)$ are inverse for upho P.

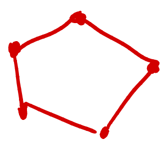
So trying to extend uniform lattices P' to upho lattices P might make sense...

e.g. Can you extend...

$P' =$ face lattice of cross polytope ?



$P' =$ bond lattice of odd cycle graph ?



Thank You!

And can't wait to see all of you in person at OPAC 2022 in just a couple weeks...