Total: 57/60

1 Let $p_k(n)$ denote the number of partitions of n into k parts. Prove bijectively that $p_0(n)+p_1(n)+p_2(n)+\ldots+p_k(n) = p_k(n+k)$. Caleb DeMere

Diagrammatic Argument: To form all partitions $p_k(n+k)$, begin with a column of size k. We then have n pieces remaining to form a partition of maximum size k with. We then attach this partition to the column, forming a partition of $n+k$ with k parts. Thus, $p_k(n+k) = p_0(n)+p_1(n)+p_2(n)+\ldots+p_k(n)$. Good.

Alternatively, all partitions of n into k parts can be written in the form $(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k)$. To form a partition of $n+k$ into k parts, begin with the partition $p_k(k)$, i.e. $\lambda=(\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_k)$ where $\lambda_1=\lambda_2=\ldots=\lambda_k=1$. Then form any partition of the remaining n into up to k parts, i.e. $\mu=(\mu_1,\mu_2,\mu_3,\ldots,\mu_k)$. Then add them together to form partition $\sigma=(\lambda_1+\mu_1, \lambda_2+\mu_2,\ldots,\lambda_k+\mu_k)$. Thus σ is a partition of $n+k$ into k parts. The number of permutations of n into at most k parts is given by $p_0(n)+p_1(n)+\ldots+p_k(n)$, so $p_k(n+k)=p_0(n)+p_1(n)+\ldots+p_k(n)$.

Okay, though same basic argument as previous paragraph. 10/10

n	$p(n)$	$p_0(n)$	$p_1(n)$	$p_2(n)$	$p_3(n)$	$p_4(n)$	$p_5(n)$	$p_6(n)$
1	1	0	1	x	x	x	x	x
2	2	0	1	1	x	x	x	x
3	3	0	1	1	1	x	x	x
4	5	0	1	2	1	1	x	x
5	7	0	1	2	2	1	1	x
6	11	0	1	3	3	2	1	1

2 We can consider this question as counting how many ways one can put n balls into k boxes, with boxes allowed to be empty. For each of the n balls there are k boxes we can put it in, so the number of ways we can organize the balls is k^n. Yes. 10/10

3 Show that $\sum_{n_1,\ldots,n_k \geq 0} \min(n_1,\ldots,n_k) x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k} = \frac{x_1 x_2 \ldots x_k}{(1-x_1)(1-x_2)\ldots(1-x_k)(1-x_1 x_2 \ldots x_k)}$

$$\frac{x_1 x_2 \ldots x_k}{(1-x_1)(1-x_2)\ldots(1-x_k)(1-x_1 x_2 \ldots x_k)} = (x_1 x_2 \ldots x_k)\left(\frac{1}{1-x_1}\right)\left(\frac{1}{1-x_2}\right)\ldots\left(\frac{1}{1-x_1 x_2 \ldots x_k}\right)$$

$$(*) \quad = (x_1 x_2 \ldots x_k)(1+x_1+x_1^2+\ldots)(1+x_2+x_2^2+\ldots)\ldots(1+(x_1 x_2 \ldots x_k)+(x_1 x_2 \ldots x_k)^2+\ldots)$$

This is equivalent to $\sum_{x_1,\ldots,x_k \geq 0} \alpha(n_1,n_2,\ldots,n_k) x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}$, with $\alpha(n_1,n_2,\ldots,n_k)$ representing the number of ways we can create a particular $x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}$.

Note that when constructing $x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}$ via $(*)$, selecting a power of $(x_1 x_2 \ldots x_k)$ immediately fixes all selections from the other expansions. Also note that the maximum power of $(x_1 x_2 \ldots x_k)$ that can be selected is $\min(n_1,n_2,\ldots,n_k)-1$. Thus the potential powers of $(x_1 x_2 \ldots x_k) \in [0,1,2,\ldots,\min(n_1,n_2,\ldots,n_k)-1]$, which contains a total of $\min(n_1,n_2,\ldots,n_k)$ options. Therefore $\alpha(n_1,n_2,\ldots,n_k) = \min(n_1,n_2,\ldots,n_k)$.

$\therefore \frac{x_1 x_2 \ldots x_k}{(1-x_1)(1-x_2)\ldots(1-x_k)(1-x_1 x_2 \ldots x_k)} = \sum_{x_1,\ldots,x_k \geq 0} \min(n_1,n_2,\ldots,n_k) x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}$.

Good! 10/10

4 Let $\bar{c}(n,m)$ denote the number of compositions of n into parts of size at most m. Show that $\sum_{n\geq 0} \bar{c}(n,m)x^n = \frac{1-x}{1-2x+x^{m+1}}$.

Consider $\bar{c}_k(n,m)$, the number of compositions of n into k parts of size at most m. Given $\sum_{n\geq 0}^{\infty} \bar{c}_k(n) = (x+x^2+x^3+\dots)^k$, then $\sum_{n\geq 0}^{\infty} \bar{c}_k(n,m) = (x+x^2+x^3+\dots x^m)^k$, since each part has maximum size m.

$(x+x^2+x^3+\dots x^m)^k = \left(\frac{1}{1-x} - 1 - \frac{x^{m+1}}{1-x}\right)^k$ [delete the 1 & every term after x^m.]

$\left(\frac{1}{1-x} - 1 - \frac{x^{m+1}}{1-x}\right)^k = \left(\frac{1}{1-x} - \frac{1-x}{1-x} - \frac{x^{m+1}}{1-x}\right)^k = \left(\frac{x-x^{m+1}}{1-x}\right)^k$

$$\sum_{n\geq 0} \bar{c}(n,m)x^n = \sum_{n\geq 0}\left(\sum_{k\geq 0} \bar{c}_k(n,m)x^n\right) = \sum_{k\geq 0}\left(\sum_{n\geq 0} \bar{c}_k(n,m)x^n\right) = \sum_{k\geq 0}\left(\frac{x-x^{m+1}}{1-x}\right)^k$$

$$= \frac{1}{1-\left(\frac{x-x^{m+1}}{1-x}\right)} = \frac{1-x}{(1-x)-(x-x^{m+1})} = \frac{1-x}{1-2x+x^{m+1}}.$$

Yep! 10/10

5 Prove that, for any $n\geq 0$, $4^n = \sum_{k\geq 0}^{n} \binom{2k}{k}\binom{2(n-k)}{n-k}$.

$\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}^2$

$\sum_{n\geq 0}\binom{2n}{n}x^n = (1-4x)^{-1/2}$

Note that 4^n is generated by $\frac{1}{1-4x}$.

Yes.

$$\frac{1}{\sqrt{1-4x}} = (1-4x)^{-1/2} = \sum_{k\geq 0}\binom{-1/2}{k}(-4)^k x^k = \sum_{k\geq 0} \frac{(-1/2)(-3/2)(-5/2)\dots(-(2k-1)/2)}{k!}\cdot(-4)^k x^k$$

Yes.

$$= \sum_{k\geq 0} \frac{2^k(1)(3)(5)\dots(2k-1)}{k!}x^k = \sum_{k\geq 0}\left(\frac{2^k(1)(3)(5)\dots(2k-1)}{k!}\right)\left(\frac{(2)(4)(6)\dots(2k)}{2^k k!}\right)x^k$$

$$= \sum_{k\geq 0}\frac{(2k)!}{k!k!}x^k = \sum_{k\geq 0}\binom{2k}{k}x^k.$$

Note that $\binom{2(n-k)}{(n-k)}$ is also generated by $\frac{1}{\sqrt{1-4x}}$. Thus,

$$\sum_{n\geq 0}\binom{2k}{k}\binom{2(n-k)}{(n-k)}x^n = \frac{1}{\sqrt{1-4x}}\cdot\frac{1}{\sqrt{1-4x}} = \frac{1}{1-4x} = \sum_{n\geq 0}4^n x^n,$$

$\therefore 4^n = \binom{2k}{k}\binom{2(n-k)}{(n-k)}$.

No.. this last sentence is not the way to think about it: the point is that multiplication of power series is convolution of the coefficients, and the expression with products of central binomials is exactly a convolution.

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6 $\sum_{\sigma \in ODD(n)} 2^{\#cycles(\sigma)}$ = Touchard's with $t_1 = t_3 = t_5 = \ldots = 2$
$t_2 = t_4 = t_6 = \ldots = 0$

$$= e^{2\left(\frac{x}{1}\right) + 2\left(\frac{x^3}{3}\right) + 2\left(\frac{x^5}{5}\right) + \ldots} = e^{\log(1+x) - \log(1-x)} = e^{\log\left(\frac{1+x}{1-x}\right)}$$

$$= \left(\frac{1+x}{1-x}\right) = \left(\frac{1}{1-x} + \frac{x}{1-x}\right) = (1 + x + x^2 + x^3 + \ldots) + (x + x^2 + x^3 + \ldots)$$

$$= 1 + 2(x + x^2 + x^3 + \ldots) = 1 + \left(\frac{2}{1-x}\right) = 1 + 2(1 + (-x))^{-1}$$

$$= 1 + 2 \sum_{n \geq 0} \frac{(-1)(-2)(-3)\ldots(-n)}{n!} (-x)^n$$

$$= 1 + 2 \sum_{n \geq 0} n! \left(\frac{x^n}{n!}\right)$$

$$= 1 + \sum_{n \geq 0} 2 \cdot n! \left(\frac{x^n}{n!}\right)$$

$$\left[x^n/n!\right] = 2 \cdot n! \Rightarrow \sum_{\sigma \in ODD(n)} 2^{\#cycles(\sigma)} = 2 \cdot n!$$

Strictly speaking in the first line you should write
sum_{n >= 0} (x^n/n!) * sum_{sigma in ODD(n)} 2^{#cycles(sigma)}
instead, so that it is a function of x, but right basic idea.

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Note: $\sum_{\sigma \in S_n} t^{\#of\,cycles(\sigma)} = [\text{Touchard's}]_{t_1 = t_2 = \ldots = t}$

Yes. This is what we used to get the generating function of the
Stirling numbers of the 1st kind.