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Schur functions: the "most important" basis of Sym

We now define the Schur functions $S_\lambda(x_1, x_2, \dots)$, which are another basis of Sym - and the most important one.

It's a bit hard to motivate what makes them so important:
i) W.r.t. the inner product $\langle \cdot, \cdot \rangle : \text{Sym} \times \text{Sym} \rightarrow \mathbb{R}$; just mentioned

They are orthonormal: $\langle S_\lambda, S_\mu \rangle = \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{otherwise} \end{cases}$

ii) In the representation theory interpretations of Sym,
they correspond to "irreducible representations".

The definition of $S_\lambda(x_1, x_2, \dots)$ will be very different from other bases:

DEF'N Let $\lambda \vdash n$ be a partition. Recall that λ 's

Young diagram has λ_i boxes in the i^{th} row.

$$\lambda = (3, 3, 1) \Leftrightarrow \lambda = \begin{array}{|c|c|c|} \hline & \boxed{} & \boxed{} \\ \hline & \boxed{} & \boxed{} \\ \hline & \boxed{} & \\ \hline \end{array} \quad (\text{rows are left-justified})$$

A semistandard Young tableau of shape λ is a filling of the boxes of its Young diagram w/ positive integers such that:

- entries are weakly increasing along rows
- entries are strictly increasing down columns

e.g. $T = \begin{array}{|c|c|c|} \hline & 1 & 1 \\ \hline & 2 & 2 \\ \hline & 4 & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline & 1 & 2 \\ \hline & 2 & 3 \\ \hline & 4 & \\ \hline \end{array}$ is a SSYT of shape $(3, 3, 1)$

For T a SSYT, content (T) is vector $\text{co}(T) := (c_1, c_2, \dots) \in \mathbb{N}^\infty$ where $c_i = \# \text{ boxes w/ entry } i$

e.g. $\text{co}(T) = (2, 3, 1, 1, 0, 0, \dots)$

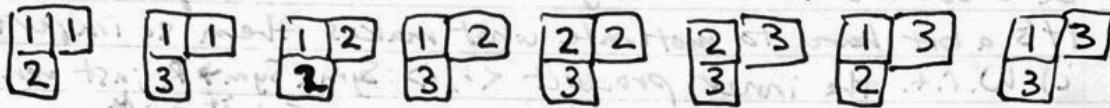
The Schur function $S_\lambda(x_1, x_2, \dots)$ is then

$$S_\lambda := \sum_{\substack{T \text{ SSYT} \\ \text{of sh.} = \lambda}} \overrightarrow{x}^{\text{co}(T)} = \sum_{\substack{T \text{ SSYT} \\ \text{sh}(T) = \lambda}} \prod_{i=1}^{c_i(T)} x_i^{c_i(T)} \in \mathbb{C}[[x_1, x_2, \dots]]$$

E.g. Let $\lambda = (2, 1)$. Let's compute the Schur polynomial

$$S_\lambda(x_1, x_2, x_3) = S_\lambda(x_1, x_2, x_3, 0, 0, \dots)$$

The SSYT of sh. = (2,1) and entries in {1, 2, 3} are:



$$\begin{aligned} S_\lambda(x_1, x_2, x_3) &= x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2 + 2x_1 x_2 x_3 \\ &= 2m_{(1,1,1)}(x_1, x_2, x_3) + m_{(2,1)}(x_1, x_2, x_3) \end{aligned}$$

A Symmetric polynomial is ↪ not a priori obvious

In fact, $S_{(2,1)}(x_1, x_2, x_3) = 2m_{(1,1,1)} + m_{(2,1)} \in \text{Sym.}$

Schur functions generalize elementary + complete homo. sym. fn's;

Prop. • $S_{(1^n)}(x_1, \dots) = e_n(x_1, \dots)$

• $S_{(n)}(x_1, \dots) = h_n(x_1, \dots)$

Pf:

$$S_{(1^n)}(x_1, \dots) \models S_{\boxed{\square}}(x_1, \dots) = \sum_{T \text{ SSYT sh. } \square} \vec{x}^{c_0(T)}$$

But an SSYT of sh. = $\boxed{\square}$ is just $\boxed{\substack{i_1 \\ i_2 \\ \vdots \\ i_n}}$ w/ $i_1 < i_2 < \dots < i_n$

$$\text{So indeed } S_{(1^n)} = \sum_{i_1 < \dots < i_n} x_{i_1} \dots x_{i_n} = e_n.$$

Similarly $S_{(n)} = \sum_{T \text{ SSYT sh. } \square} \vec{x}^{c_0(T)}$ and an SSYT

of sh. $\boxed{\square \square \square}$ is $\boxed{\substack{i_1 \\ i_2 \\ \vdots \\ i_n}}$ w/ $i_1 \leq \dots \leq i_n$

$$\text{So } S_{(n)} = \sum_{i_1 \leq \dots \leq i_n} x_{i_1} \dots x_{i_n} = h_n.$$

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But for other shapes than single row/column, not clear that S_λ is symmetric. We prove this now.

DEF'N Let T be a SSYT. The i^{th} Bender-Knuth involution (for $i=1, 2, \dots$) applied to T , denoted $b_i(T)$, is the following operation:

- first, "freeze" all entries i ^(directly) above an $i+1$, and all $i+1$'s ^(directly) below an i ,
- then, in each row, if there are a unfrozen i 's and b unfrozen $i+1$'s in this row, modify these entries so that there are b unfrozen i 's and a unfrozen $i+1$'s (in unique way that preserves SSYT-ness).

e.g. let's apply b_4 to

$T =$	1 1 1 1 2 2 2 3 3 3 4
	2 2 3 3 3 3 3 4 4 5
	3 4 4 4 5 5 5 5
	5 5 5 6
	6 6

$\square = \text{frozen}$

1 unfrozen 4
2 unfrozen 5's in 3rd row

$b_4(T) =$	1 1 1 1 2 2 2 3 3 5
	2 2 3 3 3 3 4 4 4 5
	3 4 4 4 4 5 5 5
	4 5 5 6
	6 6

2 unfrozen 4's
1 unfrozen 5 in 3rd row,
etc...

(of same shape as T !)

Prop. $b_i(T)$ is an SSYT with $\text{co}(b_i(T)) = (i, i+1) \cdot \text{co}(T)$, i.e., # i 's in T = # $i+1$'s in $b_i(T)$ and vice-versa.

Also, $b_i(b_i(T)) = T$.

Pf. All statements are relatively straight forward. To see $\text{co}(b_i(T)) = (i, i+1) \cdot \text{co}(T)$, note that frozen i 's + $i+1$'s come in pairs that cancel, while unfrozens get switched. \square

Cor For any λ , s_λ is a symmetric function.

Pf: Bender-Knuth involutions show that $(i, i+1) \cdot s_\lambda = s_\lambda$

$$\left(\text{since } \sum_{T: \text{SSYT} \text{ sh}(T)=\lambda} \vec{x}^{\text{co}(T)} = \sum_{T: \text{sh}(T)=\lambda} \vec{x}^{\text{co}(\text{bi}(T))} = \sum_{T: \text{sh}(T)=\lambda} \vec{x}^{(i, i+1), \text{co}(T)} = (i, i+1) \cdot \sum_{T: \text{sh}(T)=\lambda} \vec{x}^{\text{co}(T)} \right)$$

But then note that any permutation $\sigma \in S_n$ is a composition of adjacent transpositions $\sigma = (i_1, i_1+1) \cdot (i_2, i_2+1) \cdots (i_l, i_l+1)$

(Think about sorting ~~any~~ numbers in a line: 7 1 3 2 5 6 4, can always do it by swapping adjacent positions.)

So $\sigma \cdot s_\lambda = s_\lambda$ for any $\sigma \in S_n$, so s_λ is symmetric! \square

Thm $\{s_\lambda : \lambda \vdash n\}$ is a basis of $\text{Sym}(n)$.

Pf: Just proved that s_λ for $\lambda \vdash n$ is symmetric, and that it has degree n is clear. Since there are correct # of s_λ for a basis, what we need to show is that they span all of $\text{Sym}(n)$. We do this, like with the other bases, by a triangularity argument. So write

$$s_\lambda = \sum_{\mu} K_{\lambda, \mu} m_\mu.$$

Note that $K_{\lambda, \mu} := \# \text{SSYT w/ sh}=\lambda \text{ and } \text{co}=\mu$.

We claim that $K_{\lambda, \mu} \neq 0 \Rightarrow \mu \leq \lambda$ in lex. order.

Indeed, there is 1 tableau counted by $K_{\lambda, \mu} = 1$: we have all i 's in the i^{th} row

e.g.,

1	1	1	1
2	2	2	
3	3	3	
4			

 Now suppose $\mu \neq \lambda$ and $K_{\lambda, \mu} \neq 0$. Let j be smallest # s.t. $\mu_j \neq \lambda_j$. Then $\lambda_i = \mu_i$ if $i < j$

all i 's in row i for $i < j$. So a tableau counted by $K_{\lambda, \mu}$ has λ_j 's in row j and μ_j 's in row j . $\Rightarrow \mu \leq \lambda$. \square

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Expanding Schur functions in the other bases

From what we just explained, we have

$$s_\lambda = \sum_{\mu} k_{\lambda, \mu} m_\mu$$

where $k_{\lambda \mu} := \#\{ \text{SSYT } T : s_h(T) = \lambda, c_o(T) = \mu \}$
↳ called the Kostka numbers

But we also have the e_μ , h_μ , and p_μ bases, so
can ask what s_λ looks like in these bases.

The expansion of Schurs into power sums:

$$s_\lambda = \sum_{\mu} z_\mu^\lambda \chi^\lambda(\mu) p_\mu \quad z_\mu = 1^{m_1} m_1! 2^{m_2} m_2! \dots$$

is perhaps the most important one, because

$\chi^\lambda(\mu)$ = "character of irreducible representation of S_n
indexed by $\lambda \vdash n$ at permutation of cycle type μ ."

There is a combinatorial formula for $\chi^\lambda(\mu)$ called
the "Murnaghan-Nakayama rule"; See Ch. 7 Stanley EC2
But it's beyond scope of this class...

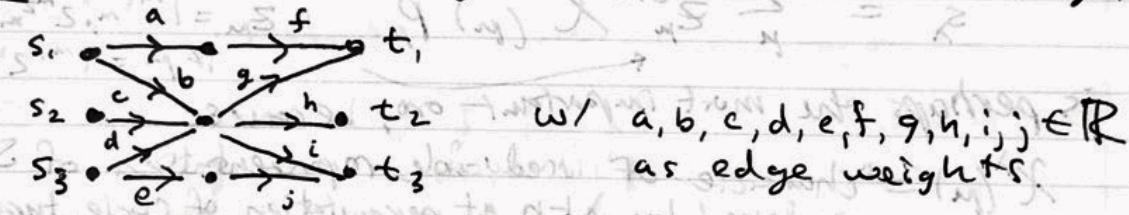
Instead we'll focus on the expansion of s_λ into e_μ/h_μ .

The formula for writing s_λ in the e_μ 's/ h_μ 's is called
the Jacobi-Trudi formula and it expresses
 s_λ as a determinant.

To prove the J-T formula, we will apply another
result: the Lindström-Gessel-Viennot lemma,
which is itself a very powerful enumerative
tool worth knowing about.

DEF'N A directed graph (or digraph) $G = (V, E)$ has vertices set V and directed edge set E , where a directed edge $e = (u, v)$ is an ordered pair of vertices we draw as an arrow $\xrightarrow{e} u \rightarrow v$. We say G is acyclic if it has no directed cycles:  An network is an acyclic digraph w/ distinguished source vertices s_1, s_2, \dots, s_n and target vertices t_1, \dots, t_m , and a weight function $wt: E \rightarrow \mathbb{R}$ on edges.

e.g. Here is an acyclic network w/ 3 sources + targets:



DEF'N A path P in a digraph is a sequence of edges $s \xrightarrow{e_1} e_2 \xrightarrow{e_2} \dots \xrightarrow{e_n} t$ connecting s to t .

We define the weight of P to be $wt(e_1) \cdot \dots \cdot wt(e_n)$.

The path matrix of network G is $n \times n$ matrix M with $M_{i,j} := \sum_{\text{paths } P: s_i \text{ to } t_j} wt(P)$.

To a tuple (P_1, \dots, P_n) of paths we associate weight $wt(P_1) \cdot \dots \cdot wt(P_n)$. We say the tuple is nonintersecting if all the vertices in P_i and P_j are disjoint, for every $i \neq j$.

Thm (Lindström-Gessel-Viennot Lemma)

Let M be the path matrix of acyclic network G .

$$\text{Then } \det(M) = \sum_{\substack{\text{non-intersecting tuple} \\ T = (P_1, \dots, P_n)}} \text{sgn}(\sigma) \cdot w^+(T).$$

$$P_i : s_i \rightarrow t_{\sigma(i)} \quad \# \text{ inversions}(\sigma)$$

(Recall for a permutation $\sigma \in S_n$, $\text{sgn}(\sigma) = (-1)^{\# \text{ inversions}(\sigma)}$)

E.g. w/ G the network from previous example

path matrix
is

$$M = \begin{bmatrix} s_1 & \xrightarrow{+} & t_1 & t_2 & t_3 \\ af + bg & bh & bi \\ cg & ch & ci \\ dg & dh & di + ej \end{bmatrix}$$

$$\text{and } \det(M) = ((af + bg)(ch)(di + ej) + (bh)(ci)(dg) + (bi)(cg)(dh)) \\ - ((bi)(ch)(dg) + (bh)(cg)(di + ej) + (af + bg)(ci)(dh))$$

$$= (af)(ch)(ej) \quad \text{wt of unique tuple of no non-int. lattice paths}$$

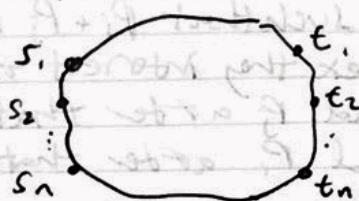
NOTE: $\text{sgn} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = +1$

In this example, we see a very important special case:

Cor (Planar LGV Lemma)

Suppose network G is planar (i.e., edges only cross at vertices)

drawn in a disc w/ sources s_1, \dots, s_n and targets t_n, \dots, t_1 on boundary (in counter-clockwise order), like



Then,

$$\det(M) = \sum_{\substack{\text{non-intersecting} \\ T = (P_1, \dots, P_n)}} w^+(T).$$

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Pf of LGV Lemma: (from Ch. 2 of Saglam)

We will use a technique from last semester:

(Sign-reversing involution)

First, we use the "Leibniz formula" for determinant:

$$\det(M) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n M_{i, \sigma(i)}$$

$$= \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{\substack{i=1 \\ \text{path}}} \sum_{\substack{\text{end} \\ \text{path}}} \text{wt}(P)$$

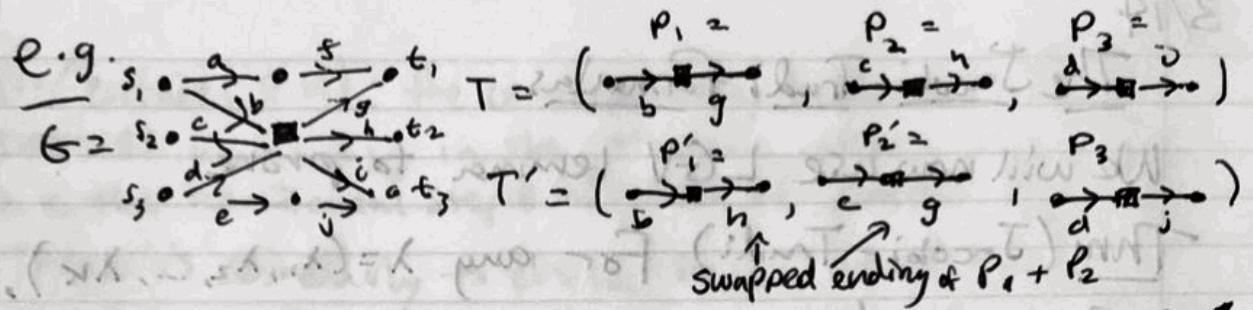
$$= \sum_{\sigma \in S_n} \text{sgn}(\sigma) \sum_{\substack{\text{tuple of paths} \\ T}} \text{wt}(T)$$

$$= \sum_{\substack{\text{tuple of paths} \\ T = (P_1, \dots, P_n), P_i: s_i \rightarrow t_{\sigma(i)}}} \text{sgn}(\sigma) \cdot \text{wt}(T)$$

So $\det(M)$ is naturally the generating function of all tuples of paths connecting sources to sinks.

To show that this sum can be taken over only the non-intersecting tuples, we will cancel all the intersecting tuples, by defining an appropriate sign-reversing involution:

- Given an intersecting tuple $T = (P_1, \dots, P_n)$, let (i, j) be lex. smallest pair such that $P_i \cap P_j$ intersect, and let v be the last vertex they intersect at. Define P'_i to be P_i up to v , and P''_i after that, and P'_j to be P_j up to v , and P''_j after that. Set $T' := (P_1, P_2, \dots, P'_i, \dots, P'_j, \dots, P_n)$.



Then $T \mapsto T'$ is an involution, and if $\sigma = \sigma^*$ are the permutations corresponding to T, T' we have $\text{sgn}(\sigma) = -\text{sgn}(\sigma^*)$ because $\sigma^* = \sigma^{(i,j)}$

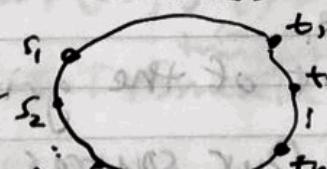
[Exercise: check this fact about permutation signs.]

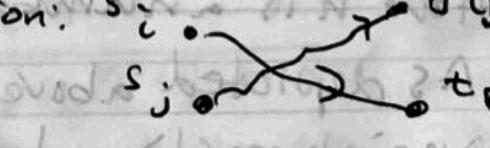
Also, T and T' use same edges, so $\text{wt}(T) = \text{wt}(T')$.

Thus $T \mapsto T'$ is a sign-reversing involution on all intersecting tuples, so the intersecting tuples cancel in the sum and we get

$$\det(M) = \sum_{\text{tuples}} \text{sgn}(\sigma) \cdot \text{wt}(T) = \sum_{\substack{\text{non-intersecting} \\ T = (P_1, \dots, P_n): P_i: S_i \rightarrow T_{\sigma(i)}}} \text{sgn}(\sigma) \cdot \text{wt}(T).$$

Pf of planar LGV corollary:

If G looks like  and is planar,

then for any $T = (P_1, \dots, P_n)$ whose σ is not $(1 \ 2 \ \dots \ n)$ we will have an intersection: $s_i \xrightarrow{\sigma(i)} t_{\sigma(i)}$
 there will be some $i < j$ with $\sigma(i) > \sigma(j)$. 

So for planar networks like this, we only need to sum over T 's with $\sigma = \text{identity}$, which have $\text{sgn}(\sigma) = +1$.

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The Jacobi-Trudi formulas

We will now use LGV lemma to prove.

Thm (Jacobi-Trudi): For any $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$,

(a) $S_\lambda = \det(h_{\lambda_i - i + j})_{1 \leq i, j \leq k}$

(b) $S_\lambda = \det(\ell_{\lambda_i - i + j})_{1 \leq i, j \leq k}$

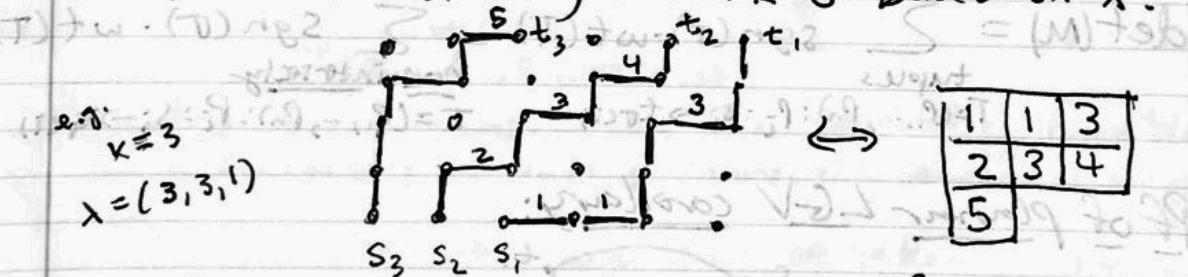
e.g.

$$S_{(2,1)} = \det \begin{bmatrix} h_2 & h_3 \\ h_0 & h_1 \end{bmatrix} = h_2 h_1 - h_3 \cdot 1 = M_{(2,1)} + 2M_{(1,1,1)}$$

NOTE $h_r = e_r = 0$ for $r < 0$ in this formula.

Pf of Jacobi-Trudi: We prove (a); (b) is similar.

Construct the following network G based on λ :



The network is a part of the grid \mathbb{Z}^2 , w/ all edges

directed right and up. Our sources are $s_i = (k-i, 1)$ and targets are $t_i = (k-i+\lambda_i, n)$ for $i=1, 2, \dots, k$.
(Here n is a number will will send $\rightarrow \infty$.)

As depicted above, tuples (P_1, \dots, P_k) of non-intersecting lattice paths w/ $P_i: s_i \rightarrow t_i$ correspond bijectively to SSYT of shape $= \lambda$.

(w/ entries $\leq n$): the correspondingly tableau T has entries $i_1, \dots, i_{\lambda_j}$ in the j^{th} row, where $i_1, \dots, i_{\lambda_j}$ are the horizontal step heights of path P_j .

There is something to check here: that the non-intersectingness of the paths is equivalent to the SSYT condition. That's an exercise for you...

This bijection tells us what the edge weights of our network should be: vertical steps have $\text{wt} = 1$, and a horizontal step at height i has $\text{wt} = x_i$.

Thus the LGV Lemma applied to this network says

$$S_\lambda(x_1, \dots, x_n) = \det(M), \text{ where } M_{i,j} = \sum_{\substack{\text{paths} \\ P: s_i \rightarrow t_j}} \text{wt}(P)$$

But it's not hard to see that $t_j = (k-j+\lambda_j, n)$

$$\sum_{\substack{\text{paths} P: s_i \rightarrow t_j}} \text{wt}(P) = \sum_{\substack{\text{paths} P: s_i \rightarrow t_j \\ s_i = (k-i, 1)}} \text{wt}(P) = h_{\lambda_j - j + i}(x_1, \dots, x_n)$$

choose any size
 $\lambda_j - j + i$ multiset of
heights of steps.

$$\text{So } S_\lambda(x_1, \dots, x_n) = \det(h_{\lambda_j - j + i}) = \det(h_{\lambda_i - i + j}),$$

and we get the J-T formula in limit $n \rightarrow \infty$ \square

Rmk: Can use J-T formula (+ some determinant manipulation)

$$\hookrightarrow \text{show } S_\lambda(x_1, \dots, x_n) = \frac{\det(x_j^{\lambda_j + n - i})}{\det(x_j^{n - i})} = \prod_{1 \leq i < j \leq n} (x_j - x_i)$$

which is actually the original definition of the Schur polynomials from late 19th/early 20th century.
"Bialternant definition"