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Trigonometric substitution § 7.3

It is often possible to compute integrals involving  $(a^2 - x^2)$  by writing  $x = a \sin(u)$  so that  $(a^2 - x^2) = (a^2 - a^2 \sin^2 u) = a^2(1 - \sin^2 u) = a^2 \cos^2 u$ .

E.g. Let's compute  $\int \frac{1}{\sqrt{1-x^2}} dx$  this way.

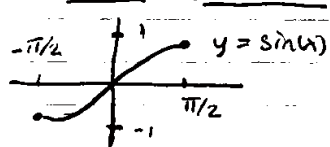
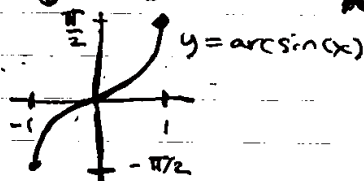
Write  $x = \sin(u) \Rightarrow dx = \cos(u) du$  so that

$$\begin{aligned} \int \frac{1}{\sqrt{1-x^2}} dx &= \int \frac{1}{\sqrt{1-\sin^2(u)}} \cos(u) du = \int \frac{1}{\sqrt{\cos^2(u)}} \cos(u) du \\ &= \int \frac{1}{\cos(u)} \cos(u) du = \int du = u + C \end{aligned}$$

This is the answer in terms of  $u$ , but we want the  $x$  answer.

Since  $x = \sin(u) \Rightarrow u = \arcsin(x)$  (also written  $\sin^{-1}(x)$ ).

Recall:  $y = \arcsin(x) \Leftrightarrow \sin(y) = x$  for  $-\pi/2 \leq y \leq \pi/2$  (inverse function)

 $\Rightarrow$ 

e.g. since  $\sin(\pi/2) = 1$  have  $\arcsin(1) = \pi/2$ , etc...

Thus,  $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + C$

**Notice:** for this problem we used a  $u$ -substitution but it was a "reverse"  $u$ -substitution where we wrote  $x = f(u)$  instead of  $u = f(x)$ . This is okay as long as you correctly compute the differential  $dx = f'(u) du$ .

Trig substitutions can be very useful when dealing with circles and related shapes...

Eg. Let's compute the area of a circle of radius  $r$  using an integral.

The equation of a circle is  $x^2 + y^2 = r^2$ .

If we solve for  $y$  we get  $y = \sqrt{r^2 - x^2}$ ,  
and the area under this curve =  $1/2$  area of circle:



So area of circle of radius  $r = 2 \cdot \int_{-r}^r \sqrt{r^2 - x^2} dx$ . Let's solve this integral by trig. sub.

Since we see  $r^2 - x^2$  we set  $x = r \cdot \sin(\theta) \Rightarrow dx = r \cos(\theta) d\theta$ .

$$\begin{aligned} \Rightarrow \int \sqrt{r^2 - x^2} dx &= \int \sqrt{r^2 - r^2 \sin^2(\theta)} r \cos(\theta) d\theta \\ &= \int r \sqrt{1 - \sin^2(\theta)} r \cos(\theta) d\theta = r^2 \int \cos \theta \cos \theta d\theta = r^2 \int \cos^2 \theta d\theta \end{aligned}$$

How to solve  $\int \cos^2 \theta d\theta$ ? We can do int. by parts:

$$\begin{aligned} \int \underbrace{\cos \theta}_u \underbrace{\cos \theta d\theta}_{dv} &= \frac{\cos \theta \sin \theta}{u} - \int \frac{\sin \theta}{v} \frac{\sin \theta d\theta}{du} = \cos \theta \sin \theta + \int \sin^2 \theta d\theta \\ &= \cos \theta \sin \theta + \int (1 - \cos^2 \theta) d\theta = \cos \theta \sin \theta + \int d\theta - \int \cos^2 \theta d\theta \end{aligned}$$

$$\Rightarrow 2 \int \cos^2 \theta d\theta = \cos \theta \sin \theta + \theta \Rightarrow \int \cos^2 \theta d\theta = \frac{1}{2} (\cos \theta \sin \theta + \theta)$$

$$\text{So } \Rightarrow \int \sqrt{r^2 - x^2} dx = r^2/2 (\cos \theta \sin \theta + \theta) \text{ when } x = r \sin \theta$$

Picture of relationship between  $r$  &  $\theta$ :



$$\sin \theta = \frac{x}{r}$$

$$\cos \theta = \frac{\sqrt{r^2 - x^2}}{r}$$

$$\theta = \arcsin\left(\frac{x}{r}\right)$$

$$\begin{aligned} \text{Thus } \Rightarrow \int \sqrt{r^2 - x^2} dx &= r^2/2 \left( \frac{\sqrt{r^2 - x^2}}{r} \frac{x}{r} + \arcsin\left(\frac{x}{r}\right) \right) \\ &= \frac{x}{2} \sqrt{r^2 - x^2} + r^2/2 \arcsin\left(\frac{x}{r}\right). \end{aligned}$$

$$\Rightarrow 1/2 \text{ area of circle} = \int_{-r}^r \sqrt{r^2 - x^2} dx = \left[ \frac{x}{2} \sqrt{r^2 - x^2} + r^2/2 \arcsin\left(\frac{x}{r}\right) \right]_{-r}^r$$

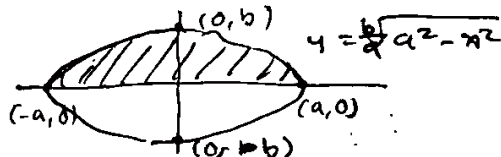
$$= \left( 0 + \frac{r^2}{2} \arcsin(1) \right) - \left( 0 + \frac{r^2}{2} \arcsin(-1) \right) = \frac{r^2}{2} \left( \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right) = \frac{\pi r^2}{2}$$

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E.g. We can find the area of an ellipse very similarly.

Ellipse equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



$\Rightarrow y = \frac{b}{a} \sqrt{a^2 - x^2}$  is upper curve of ellipse

$$\begin{aligned} \Rightarrow \frac{1}{2} \text{ area of ellipse} &= \int_{-a}^a \frac{b}{a} \sqrt{a^2 - x^2} dx \quad \text{take } x = a \sin \theta \\ &= \frac{b}{a} \left( \int_{-\pi/2}^{\pi/2} \sqrt{a^2 - x^2} dx \right) = \frac{b}{a} \left( \frac{\pi a^2}{2} \right) = \boxed{\frac{ab\pi}{2}} \end{aligned}$$

take  $x = a \sin \theta$   
 $dx = a \cos \theta d\theta$   
as before and do same steps

Sometimes we see expressions of form  $(a^2 + x^2)$ , in that case we take  $x = a \tan(u)$  because of identity  $\boxed{1 + \tan^2 \theta = \sec^2 \theta}$

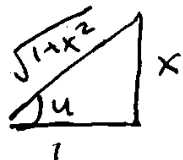
E.g. Let's compute  $\int \frac{1}{(1+x^2)^2} dx$  with a trig. sub.

We let  $x = \tan(u) \Rightarrow dx = \sec^2(u) du$   
(recall:  $d/dx (\tan(u)) = \sec^2(u)$ )

$$\begin{aligned} \text{Thus } \int \frac{1}{(1+x^2)^2} dx &= \int \frac{1}{(1+\tan^2(u))^2} \sec^2(u) du = \int \frac{1}{(\sec^2(u))^2} \sec^2(u) du \\ &= \int \frac{1}{\sec^2(u)} du = \int \cos^2(u) du = \frac{\sin(u) \cos(u)}{2} + u + C \end{aligned}$$

we just saw this

draw picture of relationship between  $x$  and  $u$ :



$$\begin{aligned} \tan(u) &= x \\ \sin(u) &= \frac{x}{\sqrt{1+x^2}} \end{aligned}$$

$$\cos(u) = \frac{1}{\sqrt{1+x^2}}$$

$$u = \arctan(x) \quad (\text{or } \tan^{-1}(x))$$

$$\begin{aligned} \Rightarrow \int \frac{1}{(1+x^2)^2} dx &= \frac{1}{2} \left( \frac{x}{\sqrt{1+x^2}} \times \frac{1}{\sqrt{1+x^2}} + \arctan(x) \right) + C \\ &= \frac{1}{2} \left( \frac{x}{1+x^2} + \tan^{-1}(x) \right) + C \quad \checkmark \end{aligned}$$

Exercise: What if we did  $\int \frac{1}{(4+x^2)^2} dx$  instead?

or even simpler:  $\int \frac{1}{4+x^2} dx$

Integration of rational functions by partial fractions

Recall that a rational function is  $f(x) = \frac{P(x)}{Q(x)}$  where  $P(x), Q(x)$  polynomials.

We will now describe procedure for computing  $\int \frac{P(x)}{Q(x)} dx$ .

① Recall that the degree of a polynomial  $P(x)$  is highest power of  $x$  in  $P(x)$ : e.g.  $\deg(P(x)) = 3$  for  $P(x) = x^3 + 5x + 4$ .

If  $\deg(P(x)) \geq \deg(Q(x))$  then we can use long division to write  $\frac{P(x)}{Q(x)} = \frac{S(x)}{Q(x)} + R(x)$  where  $\deg(S(x)) < \deg(Q(x))$ .

E.g.  $\frac{2x^3+1}{x^2-1} = 2x + \frac{2x+1}{x^2-1}$

Since it is easy to integrate polynomials, from now on assume  $\deg(P(x)) < \deg(Q(x))$ .

① First suppose the denominator  $Q(x)$  factors into distinct linear terms.

E.g. w/  $\frac{P(x)}{Q(x)} = \frac{2x+1}{x^2-1} = \frac{2x+1}{(x+1)(x-1)}$  ← distinct linear factors.

Then we write:  $\frac{P(x)}{(x-a)(x-b)\dots(x-z)} = \frac{A}{x-a} + \frac{B}{x-b} + \dots + \frac{Z}{x-z}$ .

E.g.  $\frac{2x+1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$  for some  $A, B \in \mathbb{R}$   
we need to solve for:

multiply by  $Q(x) \Rightarrow 2x+1 = A(x-1) + B(x+1)$   
 $2x+1 = (A+B)x + (-A+B)$

equating coeffs  $\Rightarrow A+B=2$  and  $B-A=1$   
 $A+A+1=2 \quad B=1+A$

$2A=1 \Rightarrow A=1/2 \Rightarrow B=1+1/2=3/2$

So  $\frac{2x+1}{(x+1)(x-1)} = \frac{1/2}{x+1} + \frac{3/2}{x-1}$  ← we can integrate these! using logarithms!

Thus,  $\int \frac{2x+1}{(x+1)(x-1)} dx = \int \frac{1/2}{x+1} dx + \int \frac{3/2}{x-1} dx$   
 $= 1/2 \ln(x+1) + 3/2 \ln(x-1) + C$

NOTE: In general  $\int \frac{1}{x+a} = \ln(x+a)$  (easy u-sub).

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② If  $Q(x)$  has repeated linear factors, partial fractions is slightly more complicated... let's see an example:

E.g. For  $\frac{P(x)}{Q(x)} = \frac{2x+1}{(x-1)^2}$  <sup>repeated factor</sup> we write:

mult. by  $Q(x)$   $\left( \frac{2x+1}{(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} \right)$   $\leftarrow$  in general we have powers  $(x-a)^r$  up to the multiplicity in  $Q(x)$

Then we solve for  $A$  &  $B \in \mathbb{R}$  as before:

$$2x+1 = A(x-1) + B$$

$$2x+1 = Ax + (-A+B)$$

$\Rightarrow$   $A = 2$  &  $-A+B=1$   
 $B = 1+A$   
 $B = 3$

Thus  $\int \frac{2x+1}{(x-1)^2} dx = \int \frac{2}{(x-1)} dx + \int \frac{3}{(x-1)^2} dx$

$$= 2 \ln(x-1) - 3(x-1)^{-1} + C$$

$\nwarrow$  recall to integrate  $\frac{1}{(x-a)^r}$  for  $r \geq 2$  use simple  $u$ -sub.  $u = x-a$ .

So in general we will get terms like  $\ln(x+a)$  and  $(x+a)^{-r}$ .

③ If  $Q(x)$  has irreducible quadratic factors, then partial fractions won't work: instead need trig. sub.

E.g. For  $\int \frac{1}{x^2+4} dx$  cannot write  $(x^2+4) = (x+a)(x+b)$  for real #'s  $a, b$  since would need  $\sqrt{\text{of neg.}}$

Instead, use  $x = 2 \tan \theta$   
 $\Rightarrow dx = 2 \sec^2 \theta d\theta$

$$\Rightarrow \int \frac{1}{x^2+4} dx = \int \frac{1}{4 \tan^2 \theta + 4} 2 \sec^2 \theta d\theta = \frac{1}{2} \int \frac{1}{\tan^2 \theta + 1} \sec^2 \theta d\theta$$

$$= \frac{1}{2} \int \frac{1}{\sec^2 \theta} \sec^2 \theta d\theta = \frac{1}{2} \int d\theta = \frac{1}{2} \theta + C$$

$$= \frac{1}{2} \arctan\left(\frac{x}{2}\right) + C \quad \leftarrow \text{since } \tan \theta = \frac{x}{2}$$

## Summary of strategies for integration § 7.5

We have now learned many integration techniques. When presented w/ an integral, it can be tricky to decide what to do!

Here are some general guidelines:

- ① know and recognize basic integrals such as

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \int \frac{1}{x} = \ln(x), \int e^x dx = e^x, \int \sin(x) dx = -\cos(x) \\ \int \cos(x) dx = \sin(x), \int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x), \int \frac{1}{1+x^2} dx = \arctan(x), \dots$$

- ② If you see a function  $f(x)$  and its derivative  $f'(x)$  in the integrand, try u-substitution.

- ③ If the integrand is a product of two terms (especially, a polynomial times exponential or trig function...) try integration by parts

- ④ For things like  $\int \sin^n x \cos^m x dx$  use the trick we learned of exploiting  $\boxed{\sin^2 x + \cos^2 x = 1}$

- ⑤ If you see  $a^2 - x^2$  appear, try trig. sub.  $x = a \sin(\theta)$ .  
If you see  $a^2 + x^2$ , try trig. sub.  $x = a \tan(\theta)$ .

- ⑥ For a rational function  $\frac{p(x)}{q(x)}$ , try the technique of partial fraction decomposition.

Sometimes you may need to apply multiple of this steps, and sometimes multiple times.

Even integrals that look similar can require different strategies!:

$$\begin{array}{ccc} \int \frac{x}{x^2+1} dx & \int \frac{1}{x^2+1} dx & \int \frac{1}{x^2-1} dx \\ \uparrow & \uparrow & \uparrow \\ \text{u-sub w/} & \text{trig sub} & \text{partial} \\ u = x^2+1 & x = \tan(\theta) & \text{fractions!} \end{array}$$

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## Approximate Integration §7.7

Sometimes a definite integral is difficult or impossible to evaluate exactly, and we'd like to get an approximation.

Recall how the definite integral is defined:

- we break  $[a, b]$  into  $n$  sub intervals  $[x_i, x_{i+1}]$  of width  $\Delta x = \frac{b-a}{n}$  (so  $x_i = a + i\Delta x$  for  $i=0, 1, \dots, n$ )
- for each sub interval  $[x_{i-1}, x_i]$  we select a point  $x_i^* \in [x_{i-1}, x_i]$  (so we get  $n$  points  $x_1^*, \dots, x_n^*$ )
- we let  $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$ .

We can thus get an approximation for  $\int_a^b f(x) dx$  by fixing a finite value of  $n$  and choosing particular  $x_i^*$ .

In Calc I we saw the left- and right-endpoint approximations:

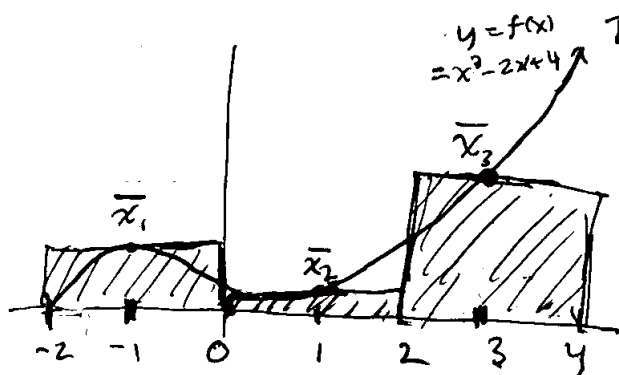
$$\int_a^b f(x) dx \approx L_n = \sum_{i=1}^n f(x_{i-1}) \Delta x \quad \text{and} \quad \int_a^b f(x) dx \approx R_n = \sum_{i=1}^n f(x_i) \Delta x$$

A better approximation is to let  $x_i^* = \bar{x}_i = \frac{x_{i-1} + x_i}{2}$  be the midpoint of the sub-intervals, giving the midpoint approx.:

$$\int_a^b f(x) dx \approx M_n = \sum_{i=1}^n f(\bar{x}_i) \Delta x$$

Ex: Let's approx.  $\int_{-2}^4 x^3 - 2x + 4 dx$  using midpoint approx.

with  $n=3$  sub intervals:  $\Delta x = \frac{4 - (-2)}{3} = \frac{6}{3} = 2$



The intervals are therefore:

$$[-2, 0], [0, 2], [2, 4]$$

with midpoints  $\bar{x}_1 = -1, \bar{x}_2 = 1, \bar{x}_3 = 3$

$$f(-1) = (-1)^3 - 2(-1) + 4 = 5$$

$$f(1) = (1)^3 - 2(1) + 4 = 3$$

$$f(3) = (3)^3 - 2(3) + 4 = 25$$

$$\text{So } M_3 = 5 \cdot 2 + 3 \cdot 2 + 25 \cdot 2$$

$$= 33 \cdot 2 = \boxed{66}$$

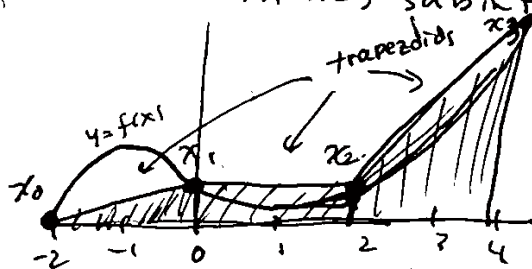
Another good approx. of  $\int_a^b f(x) dx$  is the trapezoid approx.:

$$\int_a^b f(x) dx \approx T_n = \frac{\Delta x}{2} (f(x_0) + \underbrace{2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1})}_{2's \text{ everywhere except } x_0 \text{ and } x_n} + f(x_n))$$

It is called "trapezoid" approx. because unlike other approx's using rectangles, it breaks area under curve into trapezoids:

Ex 9: Let's approx.  $\int_{-2}^4 x^3 - 2x + 4 dx$  using trapezoid approx.

with  $n=3$  subintervals: again  $\Delta x = \frac{4 - (-2)}{3} = 2$



$$\begin{aligned} f(x_0) &= f(-2) = (-2)^3 - 2(-2) + 4 = 0 \\ f(x_1) &= f(0) = (0)^3 - 2(0) + 4 = 4 \\ f(x_2) &= f(2) = (2)^3 - 2(2) + 4 = 8 \\ f(x_3) &= f(4) = (4)^3 - 2(4) + 4 = 60 \end{aligned}$$

$$\begin{aligned} \text{So } T_3 &= \frac{\Delta x}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)) \\ &= \frac{2}{2} (0 + 2 \cdot 4 + 2 \cdot 8 + 60) = \boxed{84} \end{aligned}$$

The error of an approx. is how much we need to add to get  $\int_a^b f(x) dx$ :

$$\text{Error} = \int_a^b f(x) dx - \text{approx.}$$

Ex 9: We can compute the true value of  $\int_{-2}^4 x^3 - 2x + 4 dx$  is

$$\begin{aligned} \int_{-2}^4 x^3 - 2x + 4 dx &= \left[ \frac{x^4}{4} - x^2 + 4x \right]_{-2}^4 = \left( \frac{4^4}{4} - 4^2 + 4(4) \right) - \left( \frac{(-2)^4}{4} - (-2)^2 + 4(-2) \right) \\ &= \left( \frac{64}{4} - 16 + 16 \right) - \left( \frac{4}{4} - 4 - 8 \right) = \boxed{72} \end{aligned}$$

$$\text{Thus error of } M_3 = 72 - 66 = \boxed{6}, \text{ error of } T_3 = 72 - 84 = \boxed{-12}$$

In general: error of  $M_n$  and of  $T_n$  have opposite sign,

(error of  $M_n$ ) is about  $1/2$  (error of  $T_n$ ),

and  $|\text{error of } M_n|$  and  $|\text{error of } T_n| \sim \frac{1}{n^2}$ ,

meaning if we double  $n$ , error gets cut in four.

See book for Simpson's rule which is slightly better error than  $M_n/T_n$  but significantly more complicated...



## Improper integrals § 7.8

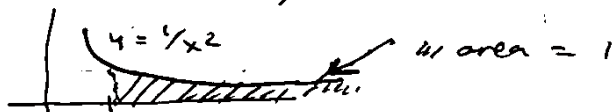
Sometimes we want to find the area under a curve as the curve goes off to infinity. This is called an improper integral:

$$\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

E.g.:  $\int_1^t \frac{1}{x^2} dx = \left[ -x^{-1} \right]_1^t = \left( -\frac{1}{t} - (-1) \right) = \boxed{1 - \frac{1}{t}}$

So  $\int_1^\infty \frac{1}{x^2} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2} dx = \lim_{t \rightarrow \infty} 1 - \frac{1}{t} = \boxed{1}$

This means area under  $y = 1/x^2$  from  $x=1$  to  $x=\infty$  is 1.



E.g.: On the other hand,  $\int_1^t \frac{1}{x} dx = \left[ \ln(x) \right]_1^t = \ln(t) - \ln(1) = \boxed{\ln(t)}$

So  $\int_1^\infty \frac{1}{x} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x} dx = \lim_{t \rightarrow \infty} \ln(t) = \boxed{\text{"}\infty\text{" or D.N.E.}}$

We see that  $\int_a^\infty f(x) dx$  need not exist as a limit!

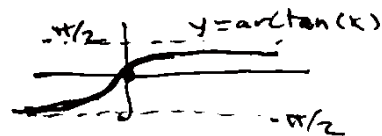
Similarly, we define  $\int_{-\infty}^a f(x) dx = \lim_{t \rightarrow -\infty} \int_t^a f(x) dx$  and

2-sided improper integral  $\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^\infty f(x) dx$ .

E.g.: To compute  $\int_{-\infty}^\infty \frac{1}{1+x^2} dx$  write  $\int_{-\infty}^\infty \frac{1}{1+x^2} dx = \int_{-\infty}^0 \frac{1}{1+x^2} dx + \int_0^\infty \frac{1}{1+x^2} dx$ .

Recall:  $\int \frac{1}{1+x^2} dx = \arctan(x)$

So  $\int_0^\infty \frac{1}{1+x^2} dx = \lim_{t \rightarrow \infty} \left[ \arctan(x) \right]_0^t = \lim_{t \rightarrow \infty} \arctan(t) - \arctan(0) = \pi/2$



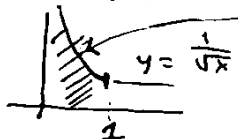
And similarly  $\int_{-\infty}^0 \frac{1}{1+x^2} dx = \pi/2$ , so  $\int_{-\infty}^\infty \frac{1}{1+x^2} dx = \pi/2 + \pi/2 = \boxed{\pi}$

Another kind of improper integral is when the integrand is discontinuous.


Suppose  $f(x)$  is continuous on  $(a, b]$  but discontinuous at  $x=a$ .

Then we define  $\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$ .

E.g.:  $\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{t \rightarrow 0^+} \left[ 2\sqrt{x} \right]_t^1 = \lim_{t \rightarrow 0^+} 2 - 2\sqrt{t} = \boxed{2}$  ←


Says:  this area = 2  
(even though  $1/\sqrt{x}$  is discontinuous at  $x=0$ )

E.g.:  $\int_0^1 \frac{1}{x} dx = \lim_{t \rightarrow 0^+} [\ln(x)]_t^1 = \lim_{t \rightarrow 0^+} \ln(1) - \ln(t) = \lim_{t \rightarrow 0^+} -\ln(t) = \boxed{\infty} \text{ or D.N.E.}$

Infinite area in region: 

Similarly we define  $\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$  for an  $f(x)$  that is discontinuous at  $x=b$ , and if  $f(x)$  is continuous on  $[a, b]$  except at  $c$  then

$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$  if these are convergent.

E.g.: For  $\int_{-1}^1 \frac{1}{\sqrt{|x|}} dx$ , we notice discontinuity at  $x=0$ ;  
 and write  $\int_{-1}^1 \frac{1}{\sqrt{|x|}} dx = \int_{-1}^0 \frac{1}{\sqrt{|x|}} dx + \int_0^1 \frac{1}{\sqrt{x}} dx = 2 + 2 = \boxed{4}$   
by symmetry both are same

E.g. For  $\int_{-1}^1 \frac{1}{x^2} dx$ , notice discontinuity at  $x=0$ .  
and write  $\int_{-1}^1 \frac{1}{x^2} dx = \int_{-1}^0 \frac{1}{x^2} dx + \int_0^1 \frac{1}{x^2} dx = \lim_{t \rightarrow 0^-} [-x^{-1}]_{-1}^0 + \lim_{t \rightarrow 0^+} [-x^{-1}]_0^1 = \infty + \infty = \boxed{\text{D.N.E.}}$

WARNING: If you did  $\int_{-1}^1 \frac{1}{x^2} dx = [-x^{-1}]_{-1}^1 = -1 - (-1) = 0$

That would give wrong answer because

you did not notice the discontinuity!