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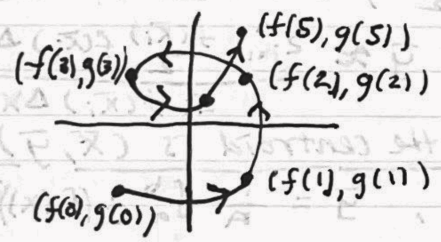
Parametric Equations § 10.1

The 1^{st} half of the semester for Calc II focused on integration.
In 2^{nd} half we explore other topics, starting with Chapter 10 on parametric equations & polar coordinates.

Up until now we have considered curves of the form $y = f(x)$ (or more rarely, $f(x, y) = 0$).
A parametrized curve is defined by two equations:

$$x = f(t) \quad \text{and} \quad y = g(t)$$

where t is an auxilliary variable. Often we think of t as time, so the curve describes motion of a particle where at time t particle is at position $(f(t), g(t))$:

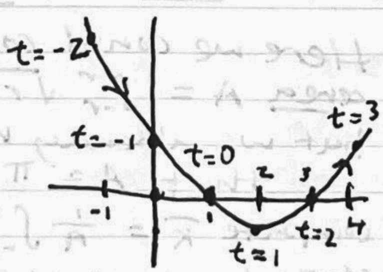


$\Leftarrow$ In this picture the arrows $\rightarrow$ show movement of particle over time

E.g. Consider parametrized curve $\boxed{x = t+1, \ y = t^2 - 2t}$.
We can make a chart with various values of t:

t	x	y
-2	-1	8
-1	0	3
0	1	0
1	2	-1
2	3	0
3	4	3

$\Rightarrow$



$\Leftarrow$ plot of points $(f(t), g(t))$ for $t = -1, 0, 1, \ldots, 4$ looks like a parabola

In this case, we can eliminate the variable t:

$$x = t + 1 \Rightarrow t = x - 1$$

$$y = t^2 - 2t \Rightarrow y = (x-1)^2 - 2(x-1) = x^2 - 4x + 3$$

So this parametrized curve is just $\boxed{y = x^2 - 4x + 3}$

initial time → initial point is $(f(0), g(0))$

terminal time → terminal point is $(f(2\pi), g(2\pi))$

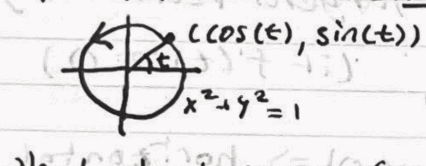
E.g. Consider the parametric curve:

$$x = \cos(t), \quad y = \sin(t) \quad \text{for} \quad 0 \le t \le 2\pi$$

How can we visualize this curve?

Notice that $x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$,

so this parametrizes a circle $x^2 + y^2 = 1$.



⇐ here t = angle (in radians) of point $(\cos(t), \sin(t))$ on unit circle

E.g. What about $x = \cos(2t)$, $y = \sin(2t)$, $0 \le t \le 2\pi$?

Notice we still have $x^2 + y^2 = \cos^2(2t) + \sin^2(2t) = 1$,

so the parametrized curve still traces a circle:

⇐ But now the parametrized curve traces the circle twice: once for $0 \le t \le \pi$ and once for $\pi \le t \le 2\pi$

Can think of this particle as moving "faster" than the last one.

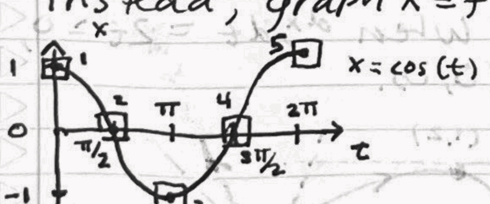
We see same curve can be parametrized in different ways!

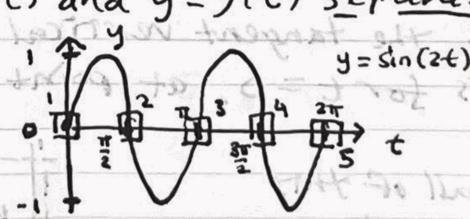
E.g. Consider the curve $x = \cos(t)$, $y = \sin(2t)$.

It's possible to eliminate t to get $y^2 = 4x^2 - 4x^4$,

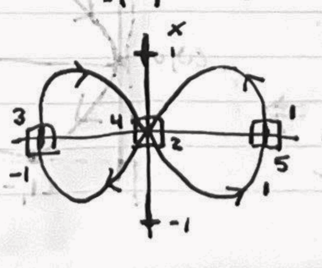
but that equation is hard to visualize.

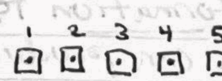
Instead, graph $x = f(t)$ and $y = g(t)$ separately:





Then combine into one picture showing $(f(t), g(t))$: ⇒





⇐ are "snapshots" of the particle as it traces the curve

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Calculus with parametrized curves §10.2

Much of what we have done with curves of form $y=f(x)$ in calculus can also be done for parametrized curves:

Tangent vectors: Let $(x,y)=(f(t),g(t))$ be a curve. Then, at time t, the slope of tangent vector is given by:

$$\frac{dy}{dx} \underset{\text{chain rule}}{=} \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)} \quad (\text{if } f'(t) \neq 0)$$

If $dy/dt = 0$ (and $dx/dt \neq 0$) $\Rightarrow$ horizontal tangent

If $dx/dt = 0$ (and $dy/dt \neq 0$) $\Rightarrow$ vertical tangent

E.g. Consider curve $x=t^2$, $y=t^3-3t$.

First, notice that when $t=\pm\sqrt{3}$ we have

$$x=t^2=3 \quad \text{and} \quad y=t^3-3t=t(t^2-3)=0,$$

so curve passes thru $(3,0)$ at two times $t=\sqrt{3}$ and $t=-\sqrt{3}$.

We then compute that:

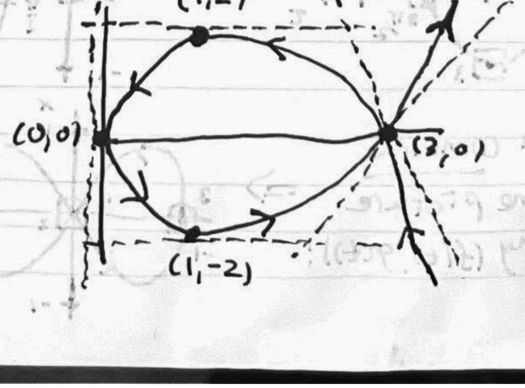
$$dy/dx = \frac{dy/dt}{dx/dt} = \frac{3t^2-3}{2t} \begin{cases} = -6/2\sqrt{3} = -\sqrt{3} \text{ at } t=-\sqrt{3} \\ = 6/2\sqrt{3} = \sqrt{3} \text{ at } t=\sqrt{3} \end{cases}$$

So two tangent lines, of slopes $\pm\sqrt{3}$, for curve at $(3,0)$.

When is the tangent horizontal? When $dy/dt = 3t^2-3=0$ which is for $t=\pm 1$, at points $(1,2)$ and $(1,-2)$

When is the tangent vertical? When $dx/dt = 2t = 0$, which is for $t=0$, at point $(0,0)$.

Putting all of this information together, we can produce a pretty good sketch of the curve $\Rightarrow$



Arc lengths: We saw several times how to find lengths of curves by breaking into line segments:

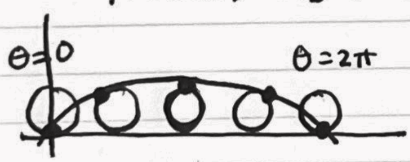
$\Leftarrow$ recall length of each small segment
$= \sqrt{(\Delta x)^2 + (\Delta y)^2}$

For a parametrized curve $(x, y) = (f(t), g(t))$ with $\alpha \le t \le \beta$

we get that length of curve $= \int_\alpha^\beta \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \boxed{\int_\alpha^\beta \sqrt{f'(t)^2 + g'(t)^2}\, dt.}$

Exercise: Using parametrization $x = \cos(t), y = \sin(t)$, $0 \le t \le 2\pi$, show circumfrence of unit circle $= 2\pi$ using this formula.

E.g. The cycloid is the path a point on unit circle traces as the circle rolls:



$\Leftarrow$ think of this as an animation of a rolling circle, with point • marked where angle $\theta =$ "time"

The cycloid is parametrized by:

$$x = \theta - \sin\theta, \quad y = 1 - \cos\theta \quad \text{for } 0 \le \theta \le 2\pi$$

Q: What is the arclength of the cycloid?

A: We compute $\frac{dx}{d\theta} = 1 - \cos\theta$, $\frac{dy}{dx} = \sin\theta$ so that

$$\sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2} = \sqrt{2(1-\cos\theta)}$$

using trig identity $\frac{1}{2}(1 - \cos 2x) = \sin^2 x$

$$= \sqrt{4\sin^2(\theta/2)}$$
$$= 2\sin(\theta/2)$$

$\Rightarrow$ length of cycloid $= \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

$$= \int_0^{2\pi} 2\sin\left(\tfrac{\theta}{2}\right) d\theta = \left[-4\cos\left(\tfrac{\theta}{2}\right)\right]_0^{2\pi}$$

$$= ((-4 \cdot -1) - (-4 \cdot 1)) = \underline{8}.$$