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Specializations of symmetric functions

So far we haven't done much counting w/ symmetric fn's.
One way to get interesting sequences of #'s from sym fn's is by specializing them, i.e., plugging in values.

Prop: (a) $e_k(1, 1, \dots, 1, 0, 0, \dots) = \binom{n}{k}$

(b) $h_k(1, 1, \dots, 1) = \binom{\binom{n}{k}}{k} = \binom{n+k-1}{k}$

Pf: Recall $e_k(x_1, \dots, x_n) = \sum_{\substack{i \in S \\ S \subseteq [n], |S|=k}} \prod_{i \in S} x_i$ product of k distinct variables, so clearly

Setting $x_i = 1 \forall 1 \leq i \leq n$, gives $e_k(1, 1, \dots, 1) = \binom{n}{k}$.

Similarly, $h_k(1, 1, \dots, 1) = \binom{\binom{n}{k}}{k} = \# k\text{-multisets of } [n] = \{1, 2, \dots, n\}$

Recall that from "Stars and bars" we showed that

$$\binom{\binom{n}{k}}{k} = \binom{n+k-1}{k} \text{ e.g. } \{1, 1, 3, 4, 4\} \subseteq [5] \text{ multi-subset}$$

$$\Leftrightarrow * * | | * | * * |$$

1's 2's 3's 4's 5's

In fact, we can similarly get the q -binomials;

DEF'N The q -binomial coefficient $\begin{bmatrix} a+b \\ b \end{bmatrix}_q$ is the g.f.

$$\begin{bmatrix} a+b \\ b \end{bmatrix}_q := \sum_{\lambda \subseteq a \times b} q^{|\lambda|}$$
 of partitions in an $a \times b$ rectangle.

e.g. $\begin{bmatrix} 2+2 \\ 2 \end{bmatrix}_q = q^4 + q^3 + 2q^2 + q + 1$ since $\emptyset, \square, \square, \square, \square, \square \subseteq \square$

We showed last semester that

$$\begin{bmatrix} a+b \\ b \end{bmatrix}_q = \frac{[a+b]_q!}{[a]_q! [b]_q!} \text{ where }$$

"q-number"

"q-factorial"

$$\begin{aligned} \text{Def: } [n]_q &= 1 + q + \dots + q^{n-1} = \frac{(1-q^n)}{(1-q)} \text{ and } [n]_q! = [n]_q \cdot [n-1]_q \cdots [1]_q \\ \text{e.g. } \left[\begin{matrix} 2+2 \\ 2 \end{matrix} \right]_q &= \frac{[4][3][2][1]}{[2][1][2][1]} = \frac{(1+q+q^2) \cdot \frac{(1-q^4)}{(1-q^2)}}{(1+q+q^2)(1+q^2)} = (1+q+q^2)(1+q^2). \end{aligned}$$

Thm (a) $e_k(1, q, \dots, q^{n-1}) = q^{\binom{k}{2}} \cdot \left[\begin{matrix} n \\ k \end{matrix} \right]_q$

(b) $h_k(1, q, \dots, q^{n-1}) = \left[\begin{matrix} n+k-1 \\ k \end{matrix} \right]_q$

Pf: We do (b) first. Observe that

$$h_k(1, q, \dots, q^{n-1}) = \sum_{\substack{1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n}} q^{\sum_{j=0}^{k-1} (i_j - 1)}$$

To any such k -multiset $1 \leq i_1 \leq \dots \leq i_k \leq n$, let's associate a partition λ inside the $(n-1) \times k$ rectangle, as follows:

Q.E.D. $n=3$ $k=5$ $\lambda = \begin{array}{|c|c|c|c|c|c|} \hline & 2 & 2 & & & \\ \hline 3 & | & | & 3 & | & | \\ \hline 3 & | & | & 3 & | & | \\ \hline \end{array} \leftrightarrow \{1, 1, 2, 3, 3\} \subseteq [3]$

i.e., the values of the multiset tell us heights of horizontal steps on SE border of partition λ (where ht 1 = top and ht n = bottom).

Under this correspondence, $|\lambda| = \sum_{j=0}^{k-1} (i_j - 1)$ (number of boxes).

$$\text{So indeed } h_k(1, q, \dots, q^{n-1}) = \left[\begin{matrix} n+k-1 \\ k \end{matrix} \right]_q.$$

For (a): Similar. Can use trick of changing k -subset

$i_1 < i_2 < \dots < i_k$ to k -multisubset $i_1 < i_2 - 1 < \dots < i_k - (k-1)$. The difference $1 + 2 + \dots + (k-1) = \binom{k}{2}$ explains factor of $q^{\binom{k}{2}}$.

Cor ("Principal specialization" of e_k and h_k)

$$(a) e_k(1, q, q^2, \dots) = q^{\binom{k}{2}} \cdot \frac{1}{(1-q)(1-q^2) \cdots (1-q^k)}$$

$$(b) h_k(1, q, q^2, \dots) = \frac{1}{(1-q)(1-q^2) \cdots (1-q^k)}.$$

Pf: Note $\lim_{n \rightarrow \infty} \left[\begin{matrix} n \\ k \end{matrix} \right]_q = \lim_{n \rightarrow \infty} \frac{[n]_q!}{[n-k]_q! [k]_q!} = \lim_{n \rightarrow \infty} \frac{(1-q^n)(1-q^{n-1}) \cdots (1-q^{n-k+1})}{(1-q^k)(1-q^{k-1}) \cdots (1-q)} = \frac{1}{(1-q)(1-q^2) \cdots (1-q^k)}.$

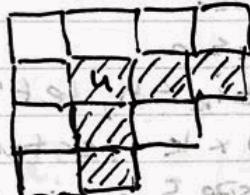
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Principal specialization of Schur functions

We could ask about specializations of other sym-fn's, like P_λ 's or m_λ 's. ^(HW #21) But now we discuss S_λ 's:

DEF'N Let λ be a partition, viewed as a Young diagram, and let $u \in \lambda$ be a box of the Young diagram. The hook of u is all boxes below or to the right of u , together with u itself.

e.g.



boxes in

$$\boxed{u} = \text{hook} + \text{length}(u) = 5$$

The hook length $h(u) := \# \text{ boxes in hook of } u$.

Thm (Principal specialization of Schurfunction)

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition. Then,

$$S_\lambda(1, q, q^2, \dots) = q^{b(\lambda)} \cdot \prod_{u \in \lambda} \frac{1}{1 - q^{h(u)}}$$

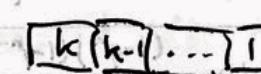
$$\text{where } b(\lambda) = 0 \cdot \lambda_1 + 1 \cdot \lambda_2 + 2 \cdot \lambda_3 + \dots = \sum_{i=1}^k (i-1) \cdot \lambda_i.$$

e.g.

$$P_k(1, q, q^2, \dots) = S_{(1^k)}(1, q, \dots) = q^{\binom{k}{2}} \cdot \frac{1}{(1-q)(1-q^{k-1}) \cdots (1-q)}$$

Since hook lengths are  for single column.

$$\text{Similarly, } h_k(1, q, \dots) = S_{(k)}(1, q, \dots) = \frac{1}{(1-q^k) \cdots (1-q)}$$

Since  are hook lengths for single row.

We saw these cases last class.

e.g. Let $\lambda = (2, 1)$. Then

$$S_\lambda(1, q, q^2, \dots) = \sum_{\substack{\text{sum of entries of } T - |\lambda| \\ \text{ssYT } T, \text{ sh}(T) = \lambda}} q^{\text{sum of entries of } T - |\lambda|}$$

$$= q + 2q^2 + 3q^3 + 5q^4 + \dots$$

1	1	1, 2
2	3	4, 3, 2
		5, 4, 3, 2

$$= q \cdot \frac{1}{(1-q)^2(1-q^3)} = q^{b(\lambda)} \cdot \prod_{u \in \lambda} \frac{1}{1-q^{h(u)}} \text{ since hook length.}$$

In fact, even have a "finite version":

Thm (Stanley's hook-content formula)

$$S_\lambda(1, q, q^2, \dots, q^{n-1}) = q^{b(\lambda)} \cdot \prod_{u \in \lambda} \frac{1-q^{c(u)+n}}{1-q^{h(u)}}$$

where $c(u) := j-i$ for box $u = (i, j)$.

As before can get principal specialization via limit $n \rightarrow \infty$.

Pf sketch: Starts w/ the "bialternant formula":

$$S_\lambda(x_1, x_2, \dots, x_n) = \frac{\det(x_j^{\lambda_i+n-i})}{\prod_{1 \leq i < j \leq n} (x_j - x_i)}, \text{ makes}$$

Substitution $x_i \rightarrow q^{i-1}$ does some algebraic manipulations of the determinant. (See Stanley EC2).

3/21 We'd prefer a combinatorial proof, which we will give for the principal specialization. Starting point:

DEF'N A reverse plane partition of shape λ is a

filling of the boxes of λ with nonnegative integers that is weakly increasing in both rows and columns.

e.g.

0	0	2
0	1	3
1	1	

is an r.p.p. of sh = (3, 3, 2).

Let $RPP(\lambda) :=$ set of r.p.p.'s of shape λ . There is a simple bijection $\phi: SSYT(\lambda) \rightarrow RPP(\lambda)$ that subtracts i from all boxes in i^{th} row:

e.g. $\phi \left(\begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & \\ 3 & 5 & \end{array} \right) = \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & \\ 0 & 2 & \end{array}$

Notice that via this bijection, sum of entries in π $= b(\lambda) + |\lambda| + \text{sum of entries in } \phi(\pi)$. So principal specialization of S_λ is equivalent to ...

$$\text{Thm } \sum_{\pi \in RPP(\lambda)} q^{|\pi|} = \prod_{u \in \lambda} \frac{1}{1 - q^{h(u)}}$$

where $|\pi| :=$ sum of entries of r.p.p. π .

We will explain a bijection proof of this thm.

To prove thm, it is enough to construct a bijection $\phi: RPP(\lambda) \rightarrow \{\text{arbitrary N-fillings A of boxes of } \lambda\}$

s.t. sum of entries $= \sum_{u \in \lambda} A(u) \cdot h(u)$ for $A = \phi(\pi)$
 $\therefore \text{wt}(A)$

Why? Because then:

$$\begin{aligned} \sum_{\pi \in RPP(\lambda)} q^{|\pi|} &= \sum_{A \text{ an N-filling of } \lambda} q^{\text{wt}(A)} = \sum_{A} \prod_{u \in \lambda} q^{A(u) \cdot h(u)} \\ &= \prod_{u \in \lambda} \left(1 + q^{h(u)} + q^{2 \cdot h(u)} + \dots \right) = \prod_{u \in \lambda} \frac{1}{1 - q^{h(u)}} \quad \checkmark \end{aligned}$$

choose each value $A(u)$ independently

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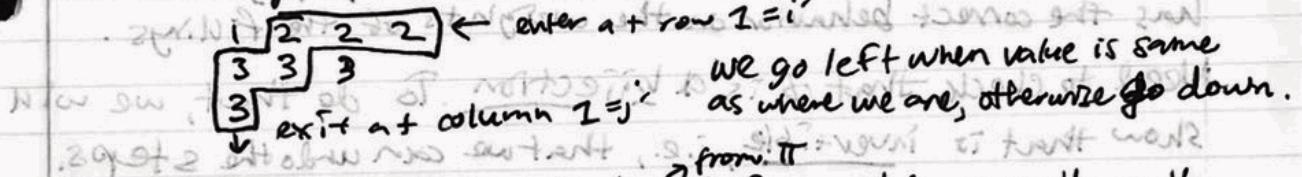
The bijection ϕ is called the Hillman-Graafl algorithm. It is defined via a series of steps. We start off by writing our $\text{PPRP}(\lambda)$ next to the all 0's filling:

$$\text{PPR}(\lambda) \rightarrow \pi = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & \\ \hline 3 & & & \\ \hline \end{array} \leftarrow \pi_0 \quad \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \\ \hline 0 & & & \\ \hline \end{array} = A_0 \quad \pi = \pi_0 \leftarrow$$

Then we find a path of boxes in π ($=\pi_0$) as follows:

- Start at northeastern most box (i, j) for which $\pi(i, j) \neq 0$,
- if we're at box (i, j) , then
 - move to $(i, j-1)$ if $\pi(i, j) = \pi(i, j-1)$
 - move to $(i+1, j)$ otherwise
- repeat the previous until we exit λ (by leaving south out of a column).

For example, w/ the above example we get this path



Then we define π_1 by subtracting 1 from all boxes on the path, and we define A_1 by adding 1 to A_0 in position (i', j') , where $i' = \text{row we entered at}$, and $j' = \text{column we exited at}$.

Thus,

$$\pi_1 = \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & \\ 2 & & & \end{array} \quad A_1 = \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \\ 0 & & & \end{array} \leftarrow \text{added 1 to } (1,1)$$

Notice that the # of boxes in the path must be the same as the number of boxes in the hook of (i', j') :

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \vdots & \ddots & \vdots \\ \bullet & \bullet & \bullet \end{array} = \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{array} \quad \text{because the path is a "ribbon"}$$

So we have that $|\pi_0| - |\pi_1| = \text{wt}(A_1) - \text{wt}(A_0)$.

Then we repeat: find a path in π , using the same rules,

and define π_2 and A_2 from π_1 and A_1 in same way:

$$\pi_1 = \boxed{\begin{matrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 \\ 2 \end{matrix}} \leftarrow A_1 = \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 \end{matrix} \Rightarrow \pi_2 = \boxed{\begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 \\ 1 \end{matrix}} \leftarrow A_2 = \begin{matrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 \end{matrix}$$

$$\Rightarrow \pi_3 = \boxed{\begin{matrix} 0 & 0 & 0 \\ 1 & 2 & 2 \\ 1 \end{matrix}} \leftarrow A_3 = \begin{matrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 \end{matrix} \Rightarrow \pi_4 = \boxed{\begin{matrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 \end{matrix}} \leftarrow A_4 = \begin{matrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 \end{matrix}$$

$$\Rightarrow \pi_5 = \boxed{\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 \end{matrix}} \leftarrow A_5 = \begin{matrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 \end{matrix}$$

We stop when we reach π_k being all 0's. Then we set $\phi(\pi) := A_k$.

Note that $|\pi| = (\pi_0 - |\pi_1|) + (\pi_1 - |\pi_2|) + \dots + (\pi_{k-1} - |\pi_k|)$

$$= (\text{wt}(A_k) - \text{wt}(A_{k-1})) + \dots + (\text{wt}(A_1) - \text{wt}(A_0)) = \text{wt}(A_k) = \text{wt}(\pi)$$

So indeed we defined map $\phi: \text{RPP}(\lambda) \rightarrow \{\text{NN-filings } A\}$ which has the correct behavior on the weights of the fillings.

Need to check that ϕ is a bijection. To do that, we will show that is invertible, i.e., that we can undo the steps.

To explain inverse procedure:

1) Note that if we increment (i'_1, j'_1) before (i'_2, j'_2) , then either $i'_1 \leq i'_2$, or $i'_1 = i'_2$ and $j'_1 \geq j'_2$.

This tells us the reverse order to decrement values of A in the ~~reverse~~ inverse procedure

2) Show that we can build reverse of any path by entering at bottom of column j' , and moving right when entry to the right is same, otherwise move up. (stopping when we reach right of row i').

For the details of this proof of bijectivity, see Sagan.

Main takeaway: we can "locally" reverse the steps.