§ 4.2, [redacted]

10/31: **The Mean Value Theorem and its consequences**

The IVT and EVT are important results about continuous f.
The Mean Value Theorem is a 3rd important result for differentiable f.

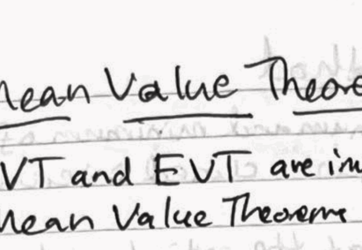
Theorem (Mean Value Theorem) Let f be defined on closed interval $[a,b]$ such that:
- f is continuous on $[a,b]$
- f is differentiable on (a,b).

Then there exists some c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Notice that $\frac{f(b)-f(a)}{b-a}$ is the slope of the line from $(a,f(a))$ to $(b,f(b))$

E.g.

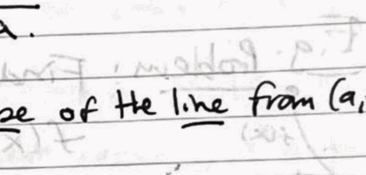


← the mean value theorem says there is some point c where the slope of the tangent is the same as the slope of line connecting the endpoints

Since $\frac{f(b)-f(a)}{b-a}$ is also the "average" (or "mean") rate of change of f, Thm can also be thought of as saying somewhere instantaneous rate of change = average rate of change.

Pf idea: The case where $f(a)=f(b)$ is called Rolle's thm

It says that if f looks like:



then it has a local max. or min. in (a,b), which follows from EVT.

The more general case when $f(a) \neq f(b)$ can be reduced to Rolle's theorem by "tilting your head".
See the book for the full proof. . .

The Mean Value Theorem has many consequences, including:

Thm If $f'(x)=0$ for all x in (a,b), then f is constant on all of (a,b).

Pf: Choose any points $x_1 < x_2$ in (a,b). Then by the MVT there is some c with $x_1 < c < x_2$ such that $f(x_2)-f(x_1) = f'(c)(x_2 - x_1)$. But by assumption $f'(c)=0$, so $f(x_2)=f(x_1)$. ☒

Cor If $f'(x)=g'(x)$ for all x in (a,b), then $f(x)=g(x)+c$ for some constant $c \in \mathbb{R}$, for all $x \in (a,b)$.

Pf: Apply previous theorem to $f-g$. ☒

What the derivative says about shape of graph §4.3

We can now prove:

Thm. If $f'(c)>0$ on some interval, then f is increasing (on that interval).

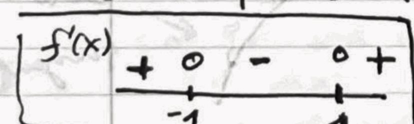
- If $f'(c)<0$ on some interval, then f is decreasing.

Pf: Very similar to proof of previous theorem, but now $f'(c)>0$ means $f(x_2)>f(x_1)$ (increasing). ☒

E.g. This can help us draw graph of f.
Consider $f(x)=x^3-3x$, so $f'(x)=3(x^2-1)=3(x+1)(x-1)$

We know critical points are $x=-1$ and $x=1$.

Choose points "inbetween": e.g. $x=0 \Rightarrow f'(0)=3(-1)=-3<0$
$x=-2 \Rightarrow f'(-2)=3(4-1)>0$
$x=2 \Rightarrow f'(2)=3(4-1)>0$



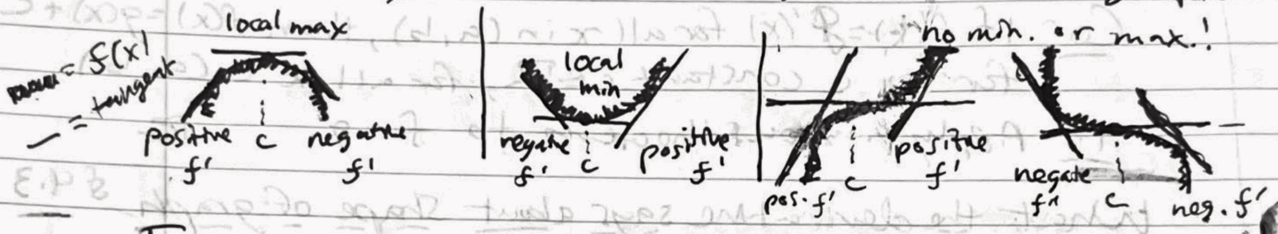
$\Rightarrow$ So from $-\infty$ to -1, f is increasing, from -1 to 1 f is decreasing, from 1 to ∞ f is increasing.

11/2 §4.3

The sign of $f'(x)$ dictating increasing vs. decreasing also means we can use the derivative to identify local min. & max.:

Thm (First Derivative Test). Let c be a critical point of f. continuous fn.
1) If f' changes from negative to positive at c, c is a local min.
2) If f' changes from positive to negative at c, c is a local max.
3) If f' has same sign to the left and right of c (i.e. both positive or both negative) then c is not a local min. or max.

Can easily remember this if you think of the graph:



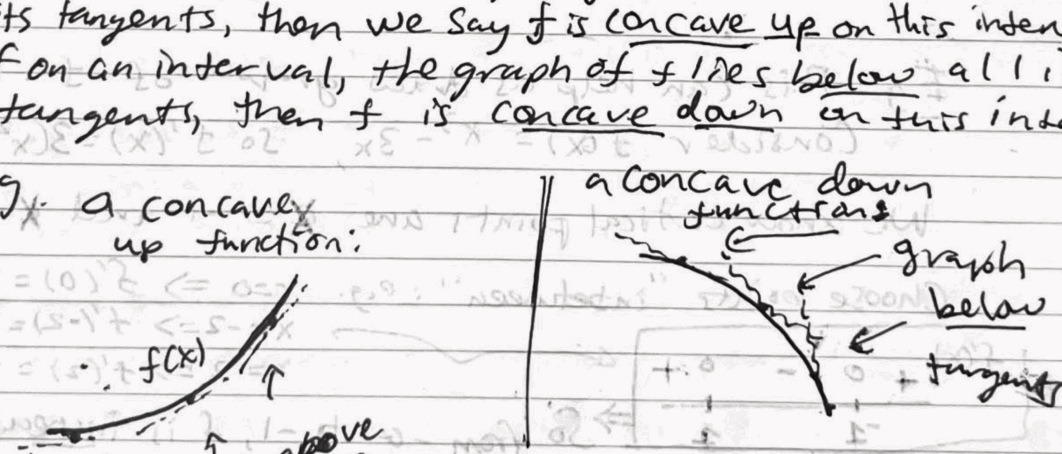
E.g. W/ $f(x) = x^3 - 3x$ as before, we found sign chart of f' to be $\begin{array}{ccccc} + & 0 & - & 0 & + \\ & -1 & & 1 & \end{array}$

So -1 is a local max, and 1 a local min

The second derivative f'' also has important info about shape of graph of f.

Def'n If on some interval, the graph of f lies above all its tangents, then we say f is concave up on this interval. If on an interval, the graph of f lies below all its tangents, then f is concave down on this interval.

E.g.

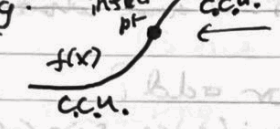


Thm If $f''(x)>0$ on an interval, then f is concave up there.
If $f''(x)<0$ on an interval, then f is concave down there.

Pf: See book. Similar to increasing/decreasing for f'. $\blacksquare$

Def'n A point c where f switches from concave up to concave down, or vice-versa, is called an inflection point.

E.g.



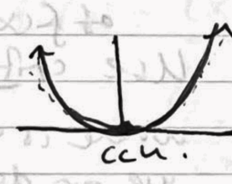
this is an inflection point, it can tell you when process is changing from "exponential growth" to "leveling off"

Note: Can find inflection points by setting $f''(x)=0$ (like with finding critical points by $f'(x)=0$)

The second derivative can also help identify min.'s/max.'s:

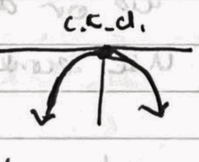
Theorem (Second Derivative Test) Let c be a critical point of f.
- If f is concave up at c, then c is a local min.
- If f is concave down at c, then c is a local max.

E.g. $f(x)=x^2 \Rightarrow$



$c=0$ is a c.p.
and $f''(0)=2>0$
so c.c.u. $\Rightarrow$ local min

$f(x)=-x^2$

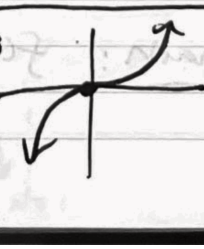


$c=0$ is a c.p.
and $f''(0)=-2<0$
so c.c.d. $\Rightarrow$ local max.

These examples show why 2nd deriv. test is true too!

WARNING: If $f''(c)=0$ (so f is neither c.c.u or c.c.d at c) then 2nd deriv. test is inconclusive, so could be min or max or neither:

E.g. $f(x)=x^3$



at $c=0$ c.p. have $f''(c)=0$, and 0 is neither local min. nor local max!!

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Summary of curve sketching §4.5

Now that we have the tools of the 1st and 2nd derivatives, we can give very reasonable sketches of graphs of most f.

Let us summarize the main things to depict in a sketch of $f(x)$:

A Domain - where is $f(x)$ defined?

B Intercepts - where does graph cross x- and y-axes? i.e., where is $f(x)=0$ and what is $f(0)$?

C Symmetry - Is $f(x)$ even or odd?
and Periodicity Is it periodic (like sin/cos)?

D Asymptotes - Does $f(x)$ have vertical or horizontal asymptotes? where? (Remember: limits at or $= \infty$)

E Increasing/Decreasing - Where is $f(x)$ increasing or decreasing? To figure this out, we look at $f'(x)$, where it is >0 and <0.

F (Local) Minima/Maxima - where are the minima/maxima of $f(x)$? What are the values there? Use critical points ($f'(x)=0$) to find.

G - Concavity and points of Inflection - where is the graph of $f(x)$ concave up or down? where are the inflection points? Use second derivative $f''(x)$ for these.

E.g. Let's use these guidelines to sketch graph of

$$f(x) = \frac{2x^2}{x^2-1}$$

A Domain: $f(x)$ not defined when $x^2-1=0$, i.e. at $x=\pm 1$.

B Intercepts: $f(0) = 0$, and this is only point on x or y-axes.

C Symmetry: This is an even function since $f(-x) = f(x)$ (= symmetric over y-axis)

D Asymptotes: $\lim_{x\to\infty} \frac{2x^2}{(x^2-1)} = \lim_{x\to-\infty} \frac{2x^2}{(x^2-1)} = 2$

so horizontal asymptote at $y=2$.

Also, vertical asymptotes at $x=1$ and $x=-1$ (since denominator goes to 0 there).

E Increasing/Decreasing: $f'(x) = \frac{(x^2-1)\cdot 4x - 2x^2 \cdot 2x}{(x^2-1)^2}$ by quot. rule

$= -\frac{4x}{(x^2-1)^2}$

This is <0 for $x>0$ and >0 for $x<0$ so...

f decreasing when $x>0$ and f increasing when $x<0$

F Min/max. critical points are only 0 (where $f'(x)=0$) and have $f(0)=0$. It is a local max by 1st deriv. test (we go from increasing to decreasing at $x=0$)

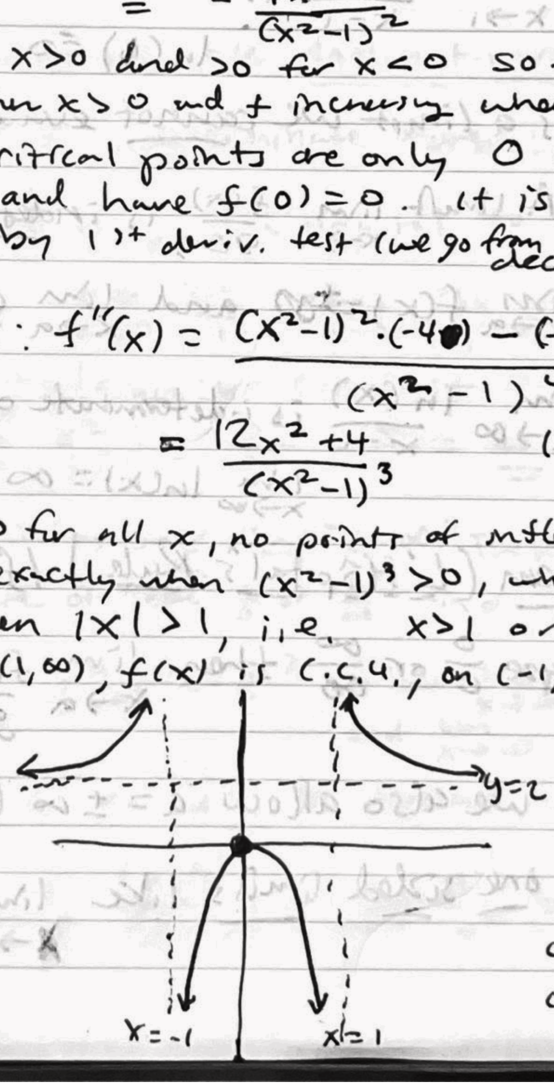
G - Concavity/points of inflection: $f''(x) = \frac{(x^2-1)^2\cdot(-4) - (-4x)(2(x^2-1)\cdot 2x)}{(x^2-1)^4}$

$= \frac{12x^2+4}{(x^2-1)^3}$ (by quot. rule)

Since $12x^2+4>0$ for all x, no points of inflection, and $f''(x)>0$ exactly when $(x^2-1)^3>0$, which is when $|x|>1$, i.e. $x>1$ or $x<-1$

So on $(-\infty,-1)\cup(1,\infty)$, $f(x)$ is c.c.u., on $(-1,1)$ c.c.d.

Altogether this gives us the sketch: $\Rightarrow$



intercept + local max at (0,0) ✓

increasing for $x<0$ ✓

decreasing for $x>0$ ✓

c.c.u. on $(-\infty,-1)\cup(1,\infty)$ ✓

c.c.d on $(-1,1)$ ✓

11/7 L'Hôpital's Rule §4.4

Recall that the derivative was defined as a limit. Surprisingly, the derivative can also help us compute certain limits. The kinds of limits the derivative helps with are those in "indeterminate form," which basically means $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Def'n A limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ is said to be of indeterminate form of type $\frac{0}{0}$ if $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$.

E.g. $\lim_{x\to 1} \frac{\ln(x)}{x-1}$ is indeterminate of type $\frac{0}{0}$ since $\ln(1) = 0$ and $1-1=0$.

This is a limit we cannot evaluate just by "plugging in."

Def'n A limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ is indeterminate of type $\frac{\infty}{\infty}$ if $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$

E.g. $\lim_{x\to\infty} \frac{\ln(x)}{x-1}$ is indeterminate of type $\frac{\infty}{\infty}$ since $\lim_{x\to\infty} \ln(x) = \infty$ and $\lim x-1 = \infty$

Theorem (L'Hôpital's Rule) If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is indeterminate of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Note: Here we also allow $a = \pm\infty$ (limits at infinity) or one sided limits like $\lim_{x\to a^+} \frac{f(x)}{g(x)}$, etc.

E.g. Since $\lim_{x \to 1} \frac{\ln(x)}{x-1}$ is indeterminate of type $\frac{0}{0}$,
we can apply L'Hôpital's rule to compute:

$$\lim_{x \to 1} \frac{\ln(x)}{x-1} = \lim_{x \to 1} \frac{d/dx(\ln(x))}{d/dx(x-1)} = \lim_{x \to 1} \frac{1/x}{1} = \lim_{x \to 1} \frac{1}{x} = \underline{\underline{1}}.$$

E.g. Since $\lim_{x \to \infty} \frac{\ln(x)}{x-1}$ is indeterminate of type $\frac{\infty}{\infty}$,
we can apply L'Hôpital:

$$\lim_{x \to \infty} \frac{\ln(x)}{x-1} = \lim_{x \to \infty} \frac{1/x}{1} = \lim_{x \to \infty} \frac{1}{x} = \underline{\underline{0}}.$$

WARNING: L'Hôpital's rule does not work if the limit is not of indeterminate form:

E.g. If we tried to apply L'Hôpital to $\lim_{x \to 0} \frac{x^2}{x+1}$
we would write "$\lim_{x \to 0} \frac{x^2}{x+1} = \lim_{x \to 0} \frac{2x}{1} = 0$"
but this is wrong since we can just plug in 0 to see that $\lim_{x \to 0} \frac{x^2}{x+1} = \frac{0^2}{0+1} = \frac{0}{1} = \underline{\underline{1}}$.

E.g. Sometimes limits look like "$0 \cdot \infty$." These are really indeterminate of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ "in disguise"

If we look at $\lim_{x \to \infty} x \cdot e^{-x}$ we have $\lim_{x \to \infty} x = \infty$
and $\lim_{x \to \infty} e^{-x} = 0$

We can re-write e^{-x} as $\frac{1}{e^x}$ to then use L'Hôpital:

$$\lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{d/dx(x)}{d/dx(e^x)} = \lim_{x \to \infty} \frac{1}{e^x} = \underline{\underline{0}}.$$

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