Total: 47/50 + 5 bonus points from presentation = 52/50

1 By the Matrix-Tree Theorem the number of spanning trees of $K_{n,m} = \det(L(K_{n,m})^{i,i})$. CalebDeBose

$L(K_{n,m})$ is of the form $\begin{bmatrix} m & & 0 & -1 & \cdots & -1 \\ & \ddots & & \vdots & & \vdots \\ 0 & & m & -1 & \cdots & -1 \\ -1 & \cdots & -1 & n & & 0 \\ \vdots & & \vdots & & \ddots & \\ -1 & \cdots & -1 & 0 & & n \end{bmatrix}$, which is of block form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where

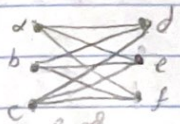
$A = \begin{pmatrix} m & & 0 \\ & \ddots & \\ 0 & & m \end{pmatrix}$ of size $n \times n$, $B = -J_{n,m}$, $C = -J_{n,m}$, and $D = \begin{pmatrix} n & & 0 \\ & \ddots & \\ 0 & & n \end{pmatrix}$ of size $m \times m$.

W/og, consider $L(K_{n,m})^{n+1,n+1}$. Then A doesn't change, $B = -J_{n,(m-1)}$, $C = -J_{(m-1),n}$, and D has size $(m-1) \times (m-1)$. Good.

$\det(L(K_{n,m})^{n+1,n+1}) = \det(A - BD^{-1}C) \cdot \det(D)$. Note that $D^{-1} = \frac{1}{n} I_{(m-1),(m-1)}$, so $BD^{-1}C = \frac{m-1}{n} J_{n \times n}$. Thus $A - BD^{-1}C = \begin{pmatrix} m-\left(\frac{m-1}{n}\right) & & \left(\frac{1-m}{n}\right) \\ & \ddots & \\ \left(\frac{1-m}{n}\right) & & m-\left(\frac{m-1}{n}\right) \end{pmatrix}$, so we have $\det(A-BD^{-1}C) = m^{n-1}$. $\det(D) = n^{m-1}$, so $\det(L(K_{n,m})^{n+1,n+1}) = m^{n-1} n^{m-1}$. ∎

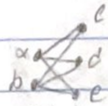
Okay (could explain det(A-BD^-1C) little more) 10/10

$$\det(A-BD^{-1}C) = -\frac{nm^n}{n} + m^n + \frac{nm^{n-1}}{n} = -m^n + m^n + m^{n-1} = m^{n-1}.$$



$3^2 \cdot 3^2 = 81$

$$\begin{bmatrix} 3 & 0 & 0 & -1 & -1 & -1 \\ 0 & 3 & 0 & -1 & -1 & -1 \\ 0 & 0 & 3 & -1 & -1 & -1 \\ -1 & -1 & -1 & 3 & 0 & 0 \\ -1 & -1 & -1 & 0 & 3 & 0 \\ -1 & -1 & -1 & 0 & 0 & 3 \end{bmatrix}$$



$2^{3-1} \cdot 3^{2-1}$

$2^2 3 = 12$

$$\begin{bmatrix} 3 & 0 & -1 & -1 & -1 \\ 0 & 3 & -1 & -1 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{bmatrix}$$

2 Consider bipartite graph $K_{n,m}$. Since none of the vertices in either group n or m are connected to vertices in the same group, all closed walks must be of even length ℓ. Since the final step is forced, we can ignore it. Since the starting point is indeterminate, let us come back to the first step later. This leaves $\ell - 2$ steps to consider. Since exactly half of each of those is made to each side, there are $m^{\ell/2-1}$ and $n^{\ell/2-1}$ steps in each closed path. If we start on the n side, this gives us m possible first choices, and $m(mn)^{\ell/2-1}$ closed paths. Similarly, starting on the m side gives $n(mn)^{\ell/2-1}$ closed paths, for a total of $(m+n)(mn)^{\ell/2-1}$ total closed paths on $K_{m,n}$.

True.

Not quite: should be (mn)^{ell/2} for both starting on left and right = 2(mn)^{ell/2} total

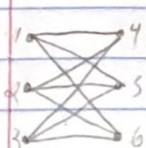
The adjacency matrix $A_{K_{n,m}}$ is of block form $A = O_{n,n}$; $B = J_{n,m}$; $C = J_{m,n}$; $D = O_{m,m}$. The characteristic polynomial of $A_{K_{n,m}} = \det(tI - A_{K_{n,m}})$. $tI - A_{K_{n,m}}$ is of block form with $A = tI_{n,n}$; $B = -J_{n,m}$; $C = -J_{m,n}$; $D = tI_{m,m}$, so $\det(tI - A_{K_{m,n}})$ $= \det(A - BD^{-1}C) \cdot \det(D)$. $D^{-1} = \frac{1}{t} I_{m,m}$, so $BD^{-1}C = \frac{m-1}{t} J_{n,n}$, so $A - BD^{-1}C = tI_{n,n} - \frac{m-1}{t} J_{n,n}$, so $\det(A - BD^{-1}C) = t^n - mnt^{n-2}$. $\det(D) = t^m$, so $\det(tI - A_{K_{m,n}}) = t^{m+n} - mnt^{m+n-2}$ $= t^{m+n-2}(t^2 - mn)$. The roots of this are 0 and $\pm\sqrt{mn}$, which comprise the eigenvalues of $A_{K_{m,n}}$.

Ah, okay… you just directly computed eigenvalues of matrix. But problem wanted you to use the walk computation + transfer matrix method to get eigenvalues instead.

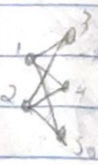
9/10

$$\left(m^{\ell/2} n^{\ell/2-1}\right) + \left(m^{\ell/2-1} n^{\ell/2}\right) = (m+n)(mn)^{\ell/2-1}$$

Hint: trace(A) & Characteristic Polynomial

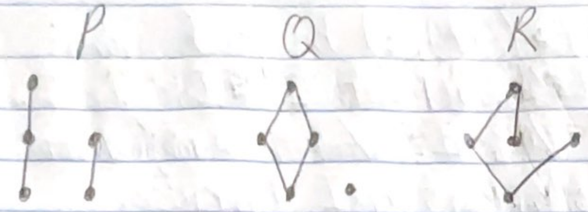


$$\begin{pmatrix} 0&0&0&1&1&1 \\ 0&0&0&1&1&1 \\ 0&0&0&1&1&1 \\ 1&1&1&0&0&0 \\ 1&1&1&0&0&0 \\ 1&1&1&0&0&0 \end{pmatrix}$$

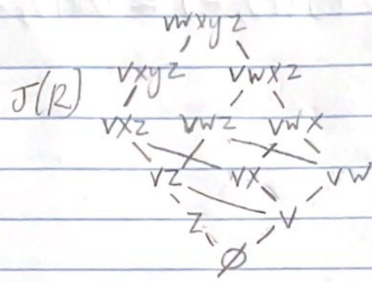


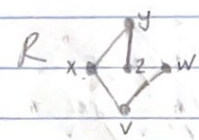
$$\begin{pmatrix} 0&0&1&1&1 \\ 0&0&1&1&1 \\ 1&1&0&0&0 \\ 1&1&0&0&0 \\ 1&1&0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&1&1&1 \\ 0&0&1&1&1 \\ 1&1&0&0&0 \\ 1&1&0&0&0 \\ 1&1&0&0&0 \end{pmatrix} = \begin{pmatrix} 3&3&0&0&0 \\ 3&3&0&0&0 \\ 0&0&2&2&2 \\ 0&0&2&2&2 \\ 0&0&2&2&2 \end{pmatrix}$$

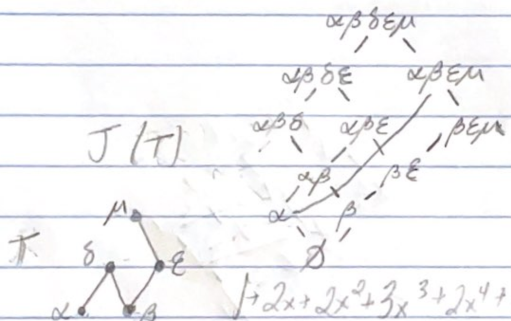
3



Yes (but could write a little more) 10/10



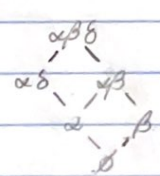


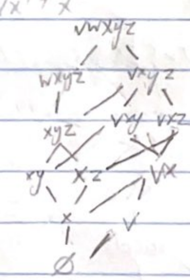


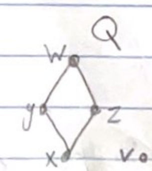
$1+2x+2x^2+3x^3+2x^4+x^5$

$1+2x+2x^2+2x^3+2x^4+x^5$

this one doesn't work (as you show)



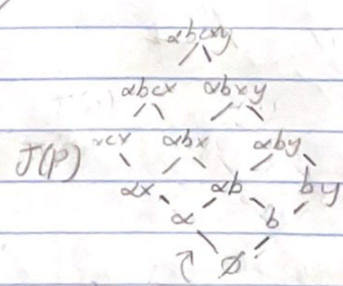


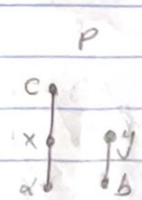


$(1+x)(1+x+2x^2+x^3+x^4)$

Doesn't exist $(1+x^2)(1+2x+2x^2+x^3)$

yes





(I is closed under going down)

$$\sum_{I\in J(P)} x^{\#I} = (1+x)(1+x^2)(1+x+x^2) = (1+x+x^2+x^3)(1+x+x^2) = (1+2x+3x^2+3x^3+2x^4+x^5)$$

$$= (1+x+x^2+x^3+x+x^2+x^3+x^4+x^2+x^3+x^4+x^5)$$

4 Define $f:(J(P),\subseteq)\to(A(P),\preceq)$ by the following:

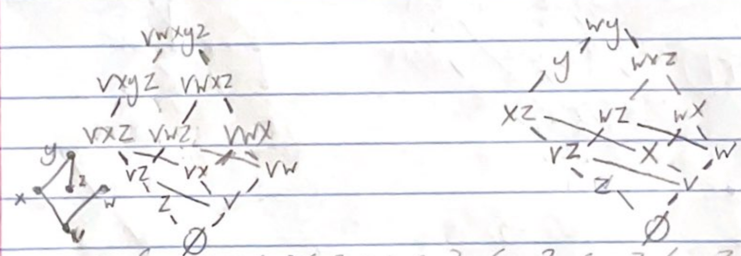
$$f(I)=\begin{cases}\{\hat{1}\} & \text{if } \hat{1}\in I \text{ and } \hat{1}\notin I \text{ for } I' \lessdot I.\\ I & \text{if } I=\emptyset \text{ or } I \text{ is a collection of minimal elements}\\ I\setminus I_m & \text{otherwise}\end{cases}$$

where $I_m := \{z\in I \mid z = A\wedge b_i$, where $A := \{x\in I \mid x \text{ is not a minimal element of } P\}$ and $b\in B := \{y\in I \mid y \text{ is a minimal element of } P\}\}$

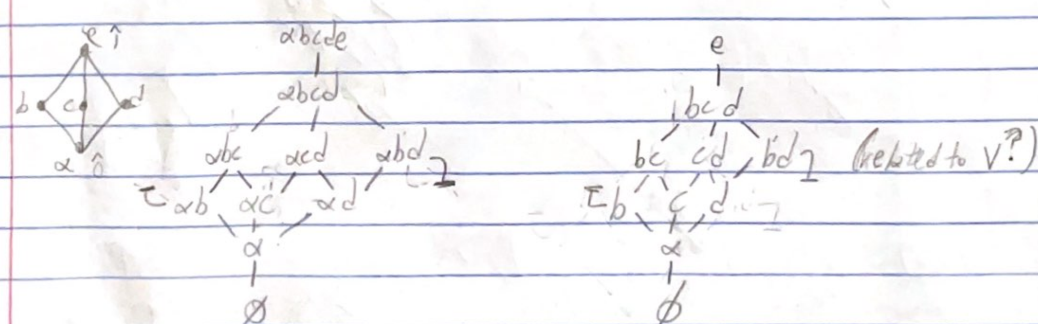
Thus $(J(P),\leq)\cong(A(P),\preceq)$.

Right basic idea but your map is not quite the correct one:
easier to say take I to A = maximal elements of I,
with inverse take A to I = everything less than something in A

8/10



$A(P) = \{\emptyset, \{v\}, \{w\}, \{x\}, \{y\}, \{z\}, \{vz\}, \{xz\}, \{wx\}, \{wz\}, \{wy\}, \{xwz\}\}$



$A(P) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{bc\}, \{bd\}, \{cd\}, \{bcd\}\}$

5 $f_L(m,t) := \#\{(t_1, \dots, t_m) \in L^m : t_1 \wedge t_2 \wedge \dots \wedge t_m = t\}$ for $t \in L$.

$g_L(m,t) := \#\{(t_1, \dots, t_m) \in L^m : t_1 \wedge t_2 \wedge \dots \wedge t_m \geq t\}$ for $t \in L$.

$g_L(m,y) = \sum_{x \in P : x \leq y} f_L(m,x)$ $\forall y \in P$, so $f_L(m,y) = \sum_{x \in P : x \leq y} \mu(x,y) g(m,x)$

$g(m,t) = \#\{\text{size } m \text{ subsets of } L^m \text{ above } t\} = (\#\{s \in L : s \geq t\})^m$, so

$f_L(m) = \sum_{t \in L} \mu(\hat{0}, t) \cdot (\#\{s \in L : s \geq t\})^m$.

Good. 10/10