10/17 <u>Implicit differentiation</u> §3.5

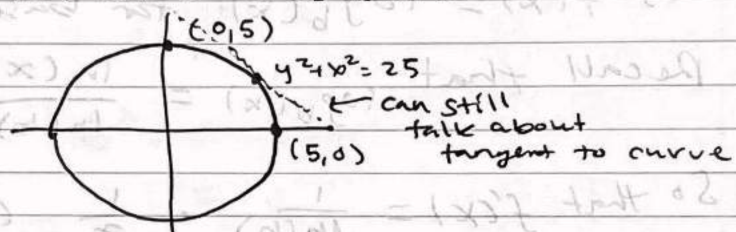
We've been studying curves of form $y = f(x)$.

But can also consider equations like

$$(*) \quad y^2 + x^2 = 25$$

where y is defined "implicitly" in terms of x.

The equation $(*)$ defines a circle of radius 5:



Even though this is not exactly the graph of a function (it doesn't pass the horizontal line test), we can still make sense of the derivative $y' = dy/dx$ at any point (x,y) on this curve: we still can take the <u>slope</u> of the <u>tangent</u> to the curve at (x,y).

—

How can we find $\frac{dy}{dx}$ when y is defined implicitly in terms of x? It turns out we can use the <u>chain rule</u> to do this <u>without</u> having to <u>solve</u> for y in terms of x!

<u>E.g.</u> what is the slope of tangent to circle

$$x^2 + y^2 = 25 \text{ at the point } (x,y) = (3,4)?$$

Let's use <u>implicit</u> <u>differentiation</u>: this means we take the equation

$$x^2 + y^2 = 25$$

and apply d/dx to both sides of it:

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

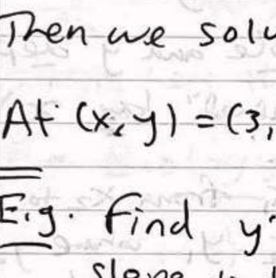
$$2x + \underbrace{2y \cdot \frac{dy}{dx}}_{\text{this part we got from the chain rule!}} = 0$$

slope of tangent $= \frac{-3}{4}$

Then we solve for $\frac{dy}{dx}$: $\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$.

At $(x,y) = (3,4)$ this gives $\frac{dy}{dx} = \frac{-3}{4}$. ✓

E.g. Find y' if $x^3+y^3 = 6xy$. What is slope to tangent of this curve at $(x,y) = (3,3)$?



$x^3+y^3 = 6xy$ (called 'folium of Descartes')

A: We implicitly differentiate $x^3+y^3 = 6xy$:

$$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(6xy)$$

$$3x^2 + \frac{d}{dx}(y^3) = \underbrace{6x \cdot \frac{d}{dx}(y) + y \cdot 6}_{\text{product rule}}$$

$$3x^2 + \underbrace{3y^2\frac{dy}{dx}}_{\text{chain rule}} = 6x\frac{dy}{dx} + 6y$$

Solve for $\frac{dy}{dx}$: $\frac{dy}{dx}(3y^2-6x) = 6y - 3x^2$

$$\text{So } y' = \frac{dy}{dx} = \frac{6y-3x^2}{3y^2-6x} = \frac{2y-x^2}{y^2-2x}$$

At $(x,y) = (3,3)$ this gives

$$\frac{dy}{dx} = \frac{6-9}{9-6} = \frac{-3}{3} = -1$$ (looks correct on graph).

Note: No way we could solve $x^3+y^3 = 6xy$ for y (unlike circle example) so we have to differentiate implicitly.

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§ 3.7 + 3.8

10/19

Rates of change & exponential growth in the sciences

Lets take a minute to review the importance of the derivative to the sciences more broadly.

Suppose $y = f(x)$ models something in the sciences; recall x is independent variable and y dependent variable (we think of y as being "determined" by x).

The change in x $\Delta x = x_2 - x_1$ from x_2 to x_1 causes a change in y $\Delta y = y_2 - y_1$ where $y_2 = f(x_2)$ and $y_1 = f(x_1)$.

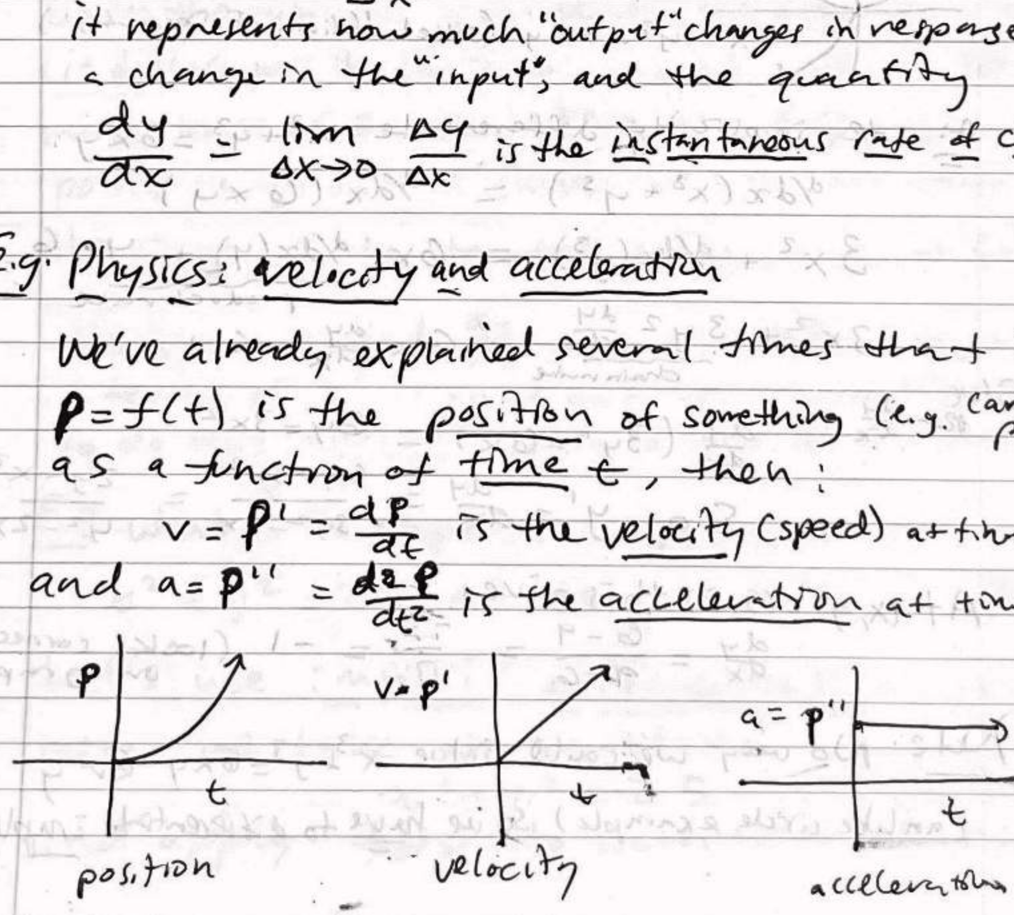
The quantity $\frac{\Delta y}{\Delta x}$ is the (average) rate of change; it represents how much "output" changes in response to a change in the "input", and the quantity $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ is the instantaneous rate of change.

E.g. Physics: velocity and acceleration

We've already explained several times that if $p = f(t)$ is the position of something (e.g. car or particle) as a function of time t, then:

$v = p' = \frac{dp}{dt}$ is the velocity (speed) at time t

and $a = p'' = \frac{d^2p}{dt^2}$ is the acceleration at time t



E.g. Economics: marginal cost (or revenue, etc.)

If $y = f(x)$ represents the total cost for a firm to produce x units of a product, the derivative $\frac{dy}{dx}$ = marginal cost, the cost of producing one new unit.

(Notice here that the dependent variable is not time!)

E.g. Biology: population growth

If $n = f(t)$ is the size (# of organisms) in a population at time t, then derivative $\frac{dn}{dt}$ = (instantaneous) growth rate, telling us rate at which pop. is growing or shrinking.

Exponential growth

Building on that biology example, a common situation in the sciences is that the rate of change of $y = f(x)$ is proportional to the value of y, i.e.:

$$(*) \quad \boxed{\frac{dy}{dx} = k \cdot y}$$

If $k > 0$, this (differential) equation represents exponential growth

and if $k < 0$, this equation represents exponential decay

Which kinds of functions $y = f(x)$ solve the equation $(*)$?

Well, $y = e^{kx}$ has $\frac{dy}{dx} = e^{kx} \cdot \frac{d}{dx}(kx)$ chain rule

$= ke^{kx} = ky$

and more generally, $y = C \cdot e^{kt}$

for any constant C will have $\frac{dy}{dx} = k \cdot y$.

Theorem: The only solutions to (*) are $y = C \cdot e^{kt}$.

(You would learn the proof of this theorem in a basic class on differential equations...)

Note: The constant $C = y(0)$ since

$$y(0) = C \cdot e^{k0} = C \cdot 1 = C.$$

This C usually represents the "initial ~~population~~ amount"
e.g. "initial population" or "principal."

E.g. The population function $n = f(t)$ of a bacteria colony might satisfy $\frac{dn}{dt} = kn$ for $k > 0$ since amount of population growth is proportional to pop. size.

E.g. The amount of money $y = f(t)$ in some investment that gives constant rate of return satisfies $\frac{dy}{dt} = ky$ for $k > 0$ (remember how we defined e in terms of interests...)

E.g. The mass $m = f(t)$ of a radioactive substance experiences exponential decay over time,

i.e. $\frac{dm}{dt} = k \cdot m$ for some $k < 0$.

10/24 Related rates §3.9

Suppose that we have two functions $f(t)$ and $g(t)$ (where the dependent variable t represents time, say).
It may be easier to measure how one of them, say $g(t)$, is changing over time, but we may really care about how the other one, $f(t)$, is changing.
If the two functions f and g are related in some way (say, by geometry...) then their rates of change are also related (by using the chain rule!).
This is the general idea of related rates, but it is easiest to see in examples:

E.g. Suppose that a spherical balloon is filling with air.

Let $V(t)$ = volume of balloon at time t (in seconds) and $r(t)$ = radius of balloon at time t.

It is probably easier to measure the volume, but perhaps we want to know how the radius is changing over time.

Given → Suppose that $\boxed{\frac{dV}{dt} = 100 \text{ cm}^3/\text{s}}$, i.e. volume increasing at constant rate of $100 \text{ cm}^3/\text{s}$.

What is the rate at which radius is increasing when the radius is $r = 25$ cm?

Want to find → i.e., $\boxed{\text{what is } \frac{dr}{dt} \text{ when } r = 25 \text{ cm?}}$

To find this out, we need to know how volume is related to radius.

So recall that the volume of a sphere is given by:

$$V = 4/3 \cdot \pi \cdot r^3$$

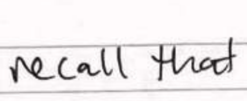
Then, to figure out how $\frac{dV}{dt}$ and $\frac{dr}{dt}$ are related, differentiate:

$$d/dt(V) = d/dt\left(4/3\,\pi r^3\right) \quad \text{remember chain rule}$$

$$\frac{dV}{dt} = \frac{4}{3}\pi\, 3r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{dV}{dt} \cdot \frac{1}{4\pi r^2}$$

So w/ $\frac{dV}{dt} = 100\ \text{cm}^3/\text{s}$ and $r = 25$ cm get $\frac{dr}{dt} = 100 \cdot \frac{1}{4\pi(25)^2}$ cm/s $= \frac{1}{25\pi} \approx 0.0127$.

Eg. 

A 10 ft ladder rests against a wall, and the ladder is sliding away from the wall at rate of 4 ft/s. How fast is it sliding down the wall, when its bottom is 6 ft from wall?

Let $x(t)$ = distance of bottom of ladder from wall.
$y(t)$ = height of top of ladder on wall.

Given: $\frac{dx}{dt} = 4$ ft/s Find: $\frac{dy}{dt}$ when $x = 6$ ft.

How are x and y related? By Pythagorean Thm:

$$x^2 + y^2 = (10\text{ ft})^2 = 100\text{ ft}^2$$

So

$$d/dt(x^2+y^2) = d/dt(100) = 0$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}$$

When $x = 6$ ft, have $y = \sqrt{100 - x^2} = \sqrt{64} = 8$ ft,

So then $\frac{dy}{dt} = \frac{-6}{8}(4\text{ ft/s}) = -3$ ft/s ✓

10/25

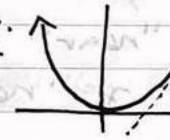
Linear approximation §3.10

Let $f(x)$ be a function differentiable at $x = a$.
The tangent line to the curve $y = f(x)$ at $(x,y) = (a, f(a))$ is the best linear approximation to $f(x)$ near a.
It's equation is given by

$$\boxed{L(x) = f(a) + (x-a)\cdot f'(a)}.$$

We write "$f(x) \approx f(a) + (x-a)\cdot f'(a)$" to mean $f(x)$ approximately equals the value of this line.

Eg.



$y = f(x) = x^2$

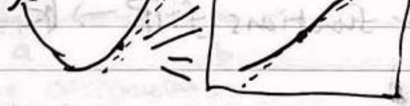
equation of tangent to $f(x) = x^2$ at point $(1,1)$ is

$L(x) = f(a) + (x-a)\cdot f'(a)$

$\Rightarrow \quad = 1 + (x-1)\cdot 2 = 2x - 1.$

Line $y = 2x-1$ is "close" to $y = x^2$ at x values near $x=1$

If we "zoom in" near the point $x=a$, $y=f(a)$:



the curve looks very close to the tangent line
This is why the approx. is useful.

In many applied situations we may be able to compute $f(a)$ and $f'(a)$, but $f(x)$ may be complicated so $L(x) = f(a) + (x-a)\cdot f'(a) \approx f(x)$ is easier to work with.

Sometimes linear approximation is phrased using the language of "differentials".

$$\boxed{dy = f'(x)\cdot dx} \quad \left(\text{think } \frac{dy}{dx} = f'(x) \text{ and "multiply" by } dx\right)$$

This relates to the "approximation":

$$\boxed{\underset{(f(x)-f(a))}{\Delta y} \approx f'(x)\cdot \underset{(x-a)}{\Delta x}} \quad \leftarrow \text{recall how } \Delta x, \Delta y \text{ relate to } dx \text{ and } dy.$$

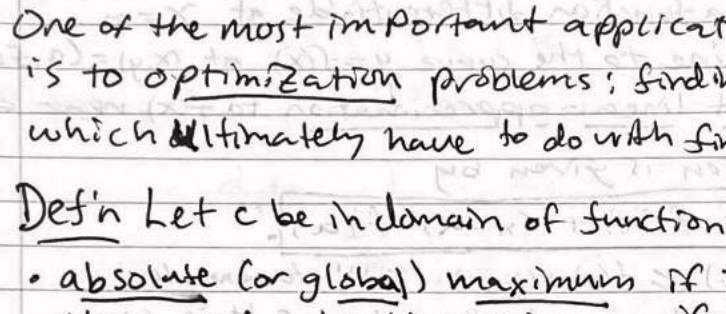
10/26 Maximum and minimum values §4.1

One of the most important applications of calculus is to optimization problems: finding "best" or "cheapest" option, which ultimately have to do with finding maxima & minima.

Def'n Let c be in domain of function f. Say $f(c)$ is

- absolute (or global) maximum if $f(c) \geq f(x)$ $\forall x$ in domain,
- absolute (or global) minimum if $f(c) \leq f(x)$ $\forall x$ in domain,
- local maximum if $f(c) \geq f(x)$ for x "near" c,
- local minimum if $f(c) \leq f(x)$ for x "near" c.

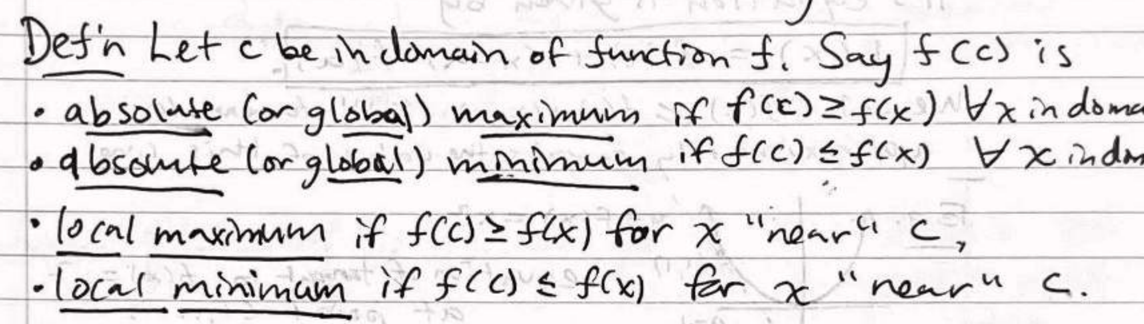
E.g.



(Note that global min./max. are also necessarily local min./max.)

Behavior of min./max. for functions $f: \mathbb{R} \to \mathbb{R}$ can be very complicated:

E.g.



And of course we've also seen in examples like this that local min./max. do not have to be global min/max.

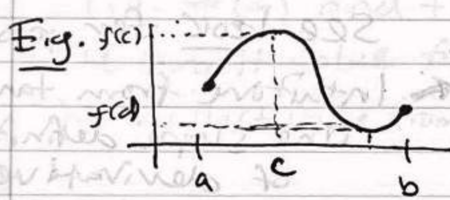
Things are much better when we restrict domain of f to be a closed interval $[a, b]$:

global min./max.
are also called "extreme values"

↑

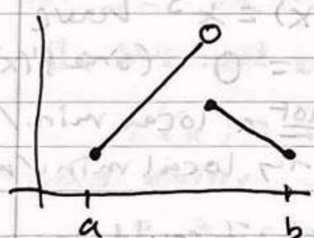
Theorem (Extreme Value Theorem) Let f be a continuous function defined on a closed interval $[a,b]$. Then f attains a global max. value $f(c)$ and global min. value $f(d)$ at some points $c, d \in [a,b]$.

E.g.

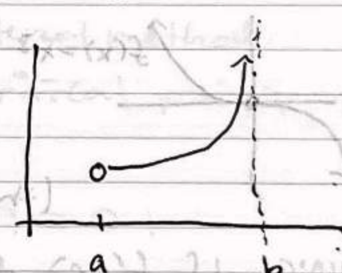


NOTE: can attain max. or min. multiple times, e.g. with a constant function.

WARNING: Both the fact that f is continuous + fact that it's domain is a closed interval are crucial for the extreme value theorem:



not continuous + no global max.



defined on open interval (a,b) and no max. or min.

But as long as we stick to continuous functions on closed intervals, extreme value thm. says we will achieve extreme values (Its proof is difficult... skipped!)

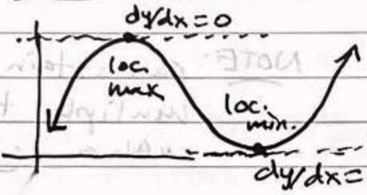
But... how do we find the extreme values that the extreme value thm says exist? We use calculus, specifically: the derivative!

10/28

We mentioned before that at (local) min./max.,
the derivative must be zero:

Thm (Fermat) If f has local min./max. at c,
and if $f'(c)$ exists, then $f'(c) = 0$.

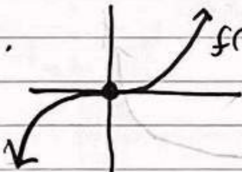
E.g.



See book for proof!
Intuitive from tangent line slope definition of derivative...

WARNING: The converse of this thm is not true,
i.e., if $f'(c)=0$ it does not mean c is a max./min.

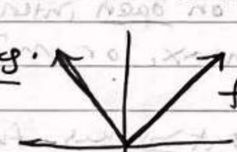
E.g.



for $f(x)=x^3$ have
$f'(0) = 0$ (since $f'(x)=3x^2$)
but 0 is not a local min./max.
(there aren't any local min./max.'s)

WARNING: If $f'(c)$ does not exist, it could be a min./max.!

E.g.



for $f(x)=|x|$ (absolute value),
we explained before why
$f'(0)$ does not exist,
but 0 is a global minimum.

Def'n A critical point (or critical number) of a function f is
a value $x=c$ where either:
- $f'(c) = 0$
- or $f'(c)$ does not exist.

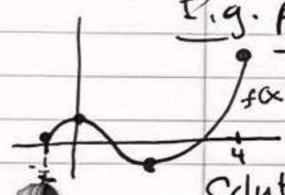
We can use critical points to find extreme values:

The Closed Interval Method

To find the absolute maximum and minimum of a continuous function f defined on a closed interval $[a,b]$:

1. Find the values of f at the critical points of f in (a,b).
2. Find the values of f at the endpoints of the interval. (i.e. $f(a)$ and $f(b)$).
3. The largest value from steps 1+2 is the absolute maximum. The smallest value from steps 1+2 is the absolute minimum.

E.g. Problem: Find the absolute maximum and minimum of



$$f(x) = x^3 - 3x^2 + 1 \text{ on interval } -\tfrac{1}{2} \le x \le 4.$$

Solution: we use closed interval method.

1. We need to find the critical points. So we compute

$$f'(x) = 3x^2 - 6x$$

and solve for $f'(x) = 0$:

$$3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 2.$$

The critical points are $x=0$ and $x=2$. Their values are:

$\boxed{f(0) = 0^3 - 3\cdot 0^2 + 1 = 1}$ and $\boxed{f(2) = 2^3 - 3\cdot 2^2 + 1 = -3}$

2. We compute the values of f on the end points:

$\boxed{f(-\tfrac{1}{2}) = (-\tfrac{1}{2})^3 - 3\cdot(-\tfrac{1}{2})^2 + 1 = \tfrac{1}{8}}$

and $\boxed{f(4) = 4^3 - 3\cdot 4^2 + 1 = 17}$

3. The absolute max. is the biggest circled # above, i.e. $\boxed{max = 17}$ (at $x=4$)

The absolute min. is the smallest circled # above, i.e. $\boxed{min = -3}$ (at $x=2$).

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