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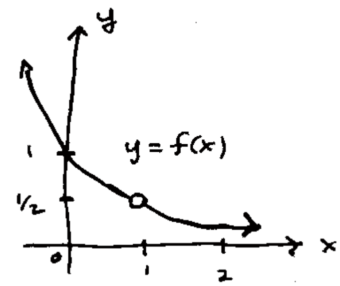
Intro to limits and derivatives §2.1 + 2.2

So far we have reviewed functions, and hopefully you had seen most of that material before in algebra/pre-calculus. Today, we will introduce calculus in earnest.

The first important notion in calculus is a limit.

Consider the function

$$f(x) = \frac{x-1}{x^2-1}$$



If we graph it near $x=1$, it looks something like this ⇒

Notice the "O" at $x=1$: this shows that $x=1$ is not in the domain of f (because we would divide by zero at $x=1$).

However, it looks like there is a value $f(x)$ "should" take at $x=1$: the value $1/2$.

At x values near 1, $f(x)$ gets close to $1/2$, and it gets closer to $1/2$ the nearer to $x=1$ we get.

We express this by $\displaystyle\lim_{x\to 1} \frac{x-1}{x^2-1} = \frac{1}{2}$

or in words, "the limit of $f(x)$ as x goes to 1 is $1/2$."

Def'n (Intuitive definition of a limit)

The limit of $f(x)$ at $x=a$ is L, written

$$\lim_{x\to a} f(x) = L$$

if we can force $f(x)$ to be as close to L as we want by requiring the input to be sufficiently close, but not equal, to a.

Notice how the definition of the limit does not require $f(x)$ to be defined at $x=a$, or for $f(a)$ to equal the limit $\lim_{x \to a} f(x)$ if it is defined. But... if this is the case we say $f(x)$ is continuous at a.

Def'n $f(x)$ is continuous at a point $x=a$ in its domain if $f(a) = \lim_{x \to a} f(x)$.

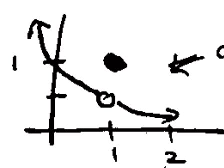
Most of the functions we've looked at so far, like x^n, $\sqrt[n]{x}$, $\sin(x)$, $\cos(x)$, e^x, $\ln(x)$, etc. are continuous at all points in their domain.

Very roughly, this means we can "draw the graph without lifting our pencil."

For an example of a function that is not continuous (i.e., discontinuous) at a point in its domain:

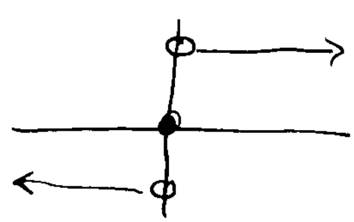
E.g. Let $f(x) = \begin{cases} \frac{x-1}{x^2-1} & \text{if } x \neq 1, -1 \\ 1 & \text{if } x = 1 \end{cases}$

The graph of $f(x)$ near $x=1$ is



and since $\lim_{x \to 1} f(x) = \frac{1}{2} \neq 1 = f(1)$, it's discontinuous at $x=1$.

E.g. Let $f(x) = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$



Then $\lim_{x \to 0} f(x)$ does not exist, because for values of x slightly more than 0, $f(x) = 1$, while for values of x slightly less than 0, $f(x) = -1$. Does not get close to a single value near $x = 0$!

This last example is related to one-sided limits:

Def'n We write $\lim_{x \to a^-} f(x) = L$ and say the left-hand limit of $f(x)$ at $x=a$ is L (or "limit as x approaches a from the left") if we can make $f(x)$ as close to L as we want by requiring x to be sufficiently close to and less than a.

We write $\lim_{x \to a^+} = L$ and say the right-hand limit is L for analogous thing but with values greater than a.

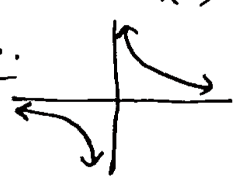
E.g. With $f(x)$ as in previous example, we have
$$\lim_{x \to 0^-} f(x) = -1 \text{ and } \lim_{x \to 0^+} f(x) = 1.$$

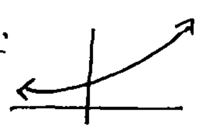
Note $\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$.

Related to one-sided limits are limits at infinity.

Def'n We write $\lim_{x \to \infty} f(x) = L$ if we can make $f(x)$ arbitrarily close to L by requiring x to be big enough.

We write $\lim_{x \to -\infty} f(x) = L$ if same but for x small enough.

E.g.  For $f(x) = 1/x$, we have
$$\lim_{x \to \infty} f(x) = 0 = \lim_{x \to -\infty} f(x)$$

E.g.  For $f(x) = e^x$, we have
$$\lim_{x \to -\infty} f(x) = 0 \quad (\text{but not as } x \to \infty).$$

E.g. When we defined $e = \lim_{n \to \infty} (1 + 1/n)^n$, we were using limit at infinity of $f(n) = (1 + 1/n)^n$.

We can check
$$f(1) = (1+1)^1 = 2$$
$$f(2) = (1 + 1/2)^2 = 2.25$$
$$\vdots$$
$$f(100) = 2.7048\ldots$$
and it gets closer to $e = 2.71\ldots$ as $n \to \infty$. //

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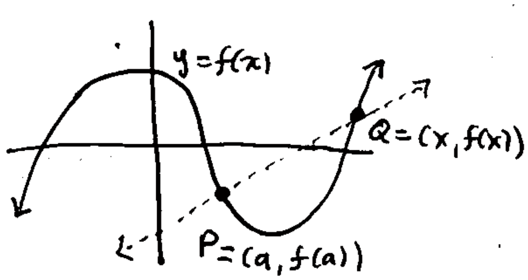
Derivative as a limit §2.1, 2.7

If most "normal" functions we work with are continuous at all points in their domain, you might wonder why we define limits at all, especially for points not in domain.

Reason is we want to define the derivative as a limit, and this naturally involves a limit that is "0/0" (so not computable just by "plugging in values").

Recall our discussion from 1st day of class:



We have a point P on a curve, i.e. graph of function $f(x)$. Assume $P=(a,f(a))$ is fixed. For another point Q on the curve, with $Q=(x,f(x))$, what is the slope of the secant line from P to Q?

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f(x)-f(a)}{x-a}$$

Recall that the tangent line of the curve at P is the limit of the secant line as we send Q to P. So what is the slope of the tangent line at P?

$$\text{slope of tangent} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

This is the derivative of $f(x)$ at $x=a$!

Def'n The derivative of $f(x)$ at a point $x=a$ in its domain is $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$.

E.g. Let's compute the derivative of $f(x)=x^2$ at point $x=1$. We need to compute

$$\lim_{x \to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x \to 1} \frac{x^2-1}{x-1}$$

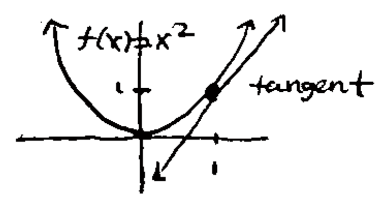
To do this, we use the algebraic trick:

$$(x^2-1) = (x+1)(x-1)$$

So $\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}} = \lim_{x \to 1} (x+1) = \underline{2}$.

We will justify all these steps later when we talk about rules for computing limits (but it should match $\lim_{x \to 1} \frac{x-1}{x^2-1} = 1/2$ from before...)

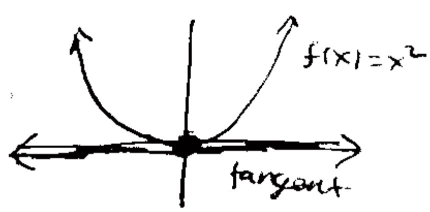
And it looks reasonable that the slope of the tangent at $x=1$ is 2:



E.g. If instead we compute the derivative of $f(x)=x^2$ at point $x=0$, we get

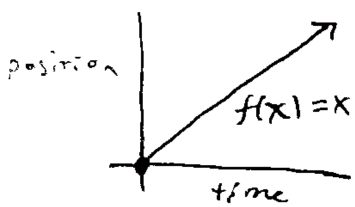
$$\lim_{x \to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0} \frac{x^2-0}{x-0} = \lim_{x \to 0} \frac{x^2}{x} = \lim_{x \to 0} x = \underline{0}.$$

and again it looks like the slope of tangent at $x=0$ is zero (horizontal):

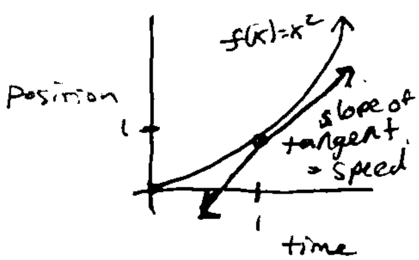


But why do we care about derivatives?
They tell us "instantaneous rate of change."

E.g. Suppose a car's position in meters (away from some initial point) after x seconds is given by function $f(x)$.
How can we find the speed of the car at time $x = a$?



If $f(x) = x$, so that the car is moving at a constant rate of 1 m/s, then clearly at any time its speed is this value of 1 m/s.



But what if instead $f(x) = x^2$ (which represents an accelerating car).
To find the speed at time $x = 1$, we could measure its position at time $x=1$ and $x=b$ for b a little more than 1.
We then compute:

$$\underset{\text{at } x=1}{\text{speed}} \approx \frac{f(b)-f(1)}{b-1} \leftarrow \text{rate of growth} : \frac{\text{rise}}{\text{run}}$$

To be super accurate, we want b to be very close to 1, so the best definition of speed at time 1 is:

$$\lim_{b \to 1} \frac{f(b)-f(1)}{b-1}, \text{ i.e., the derivative of } f(x) \text{ at } x=1!$$

We saw before that the derivative of $f(x) = x^2$ at $x = 1$ is 2, so the accelerating car is moving faster than the constant speed car at time $x=1$. However, at time $x=0$, the derivative is 0, because car is just starting to move!

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Rules for limits §2.3

The following rules for limits allow us to compute many limits in practice:

Thm (Limit Laws) Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist.

Then: 1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

3. $\lim_{x\to a}[c\cdot f(x)] = c\cdot\lim_{x\to a} f(x)$ for any constant $c\in\mathbb{R}$

4. $\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$

5. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ as long as $\lim_{x\to a} g(x)\neq 0$.

"Limit of sum is sum of limits," et cetera ...

Together with

Thm (Base Case Limits)

$\lim_{x\to a} c = c$ for any constant $c\in\mathbb{R}$,

and $\lim_{x\to a} x = a$

these laws tell us that:

Thm . If $P(x)$ is a polynomial, then $\lim_{x\to a} P(x) = P(a)$

. If $\frac{P(x)}{Q(x)}$ is a rational function (ratio of polynomials) and a is in its domain, then $\lim_{x\to a}\frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$.

"Can evaluate limits of polynomials/rational fn's by plugging in."

Let's see how we can use these laws to show

E.g. $\lim_{x\to 1} \frac{x-1}{x^2-1} = 1/2$

Pf: $\lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{x-1}{(x+1)(x-1)}$ "difference of squares"

$= \lim_{x\to 1} \frac{1}{x+1} \cdot \lim_{x\to 1} \frac{x-1}{x-1}$ "product of limits"

$= 1/2 \cdot 1$ ∎

How do we know $\lim_{x\to 1} \frac{x-1}{x-1} = 1$? Notice that $\frac{x-1}{x-1} = 1$ for any $x \neq 1$. We need one more rule:

Thm If $f(x) = g(x)$ for all $x \neq a$, then
$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x).$$

This makes sense because remember that:
"the limit of $f(x)$ at $x=a$ only cares about $f(x)$ near $x=a$, not what happens exactly at $x=a$."
This rule lets us "cancel factors" in a limit!

Also have:
Thm (Limits of powers/roots) For any positive integer n,
$$\lim_{x\to a} [f(x)^n] = \left(\lim_{x\to a} f(x)\right)^n \text{ and } \lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$$
(whenever the right-hand side is defined).

These tell us: if $f(x)$ is any "algebraic function" (built out of powers and roots, together with addition/subtraction/multiplication/division) and a is in domain of $f(x)$, then $\lim_{x\to a} f(x) = f(a)$. ∎

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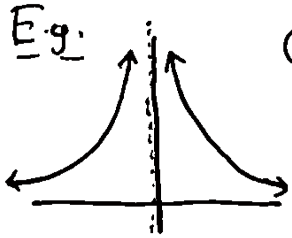
More ways limits can fail to exist §2.2

So far we've only seen one example of a limit not existing, and it was when the 2 one-sided limits disagreed.

But limits can fail to exist for many reasons.

E.g.



Consider $f(x) = \frac{1}{x^2}$. For x near zero, $f(x)$ will be a big positive number, and it gets bigger & bigger as x gets closer & closer to 0.

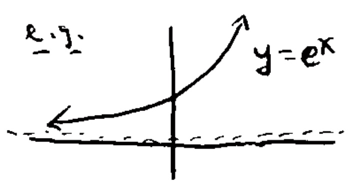
So $\lim_{x \to 0} \frac{1}{x^2}$ does not exist.

In this case we write $\lim_{x \to 0} \frac{1}{x^2} = \infty$ to mean that as x gets closer to 0 (on either side), $f(x)$ becomes arbitrarily large.

Note: $\lim_{x \to a} f(x) = \infty$ (or $\lim_{x \to a} f(x) = -\infty$) counts as the limit not existing (since it is not a finite number)

Compare: If $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, then $f(x)$ has a "horizontal asymptote at $y = L$"

e.g.



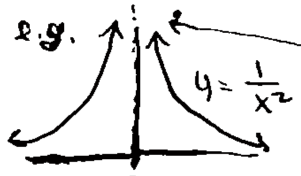
horiz. asymptote at $y = 0$.

one-sided limits

If $\lim_{x \to a} f(x) = \pm\infty$ (or $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$)

then $f(x)$ has a "vertical asymptote at $x = a$"

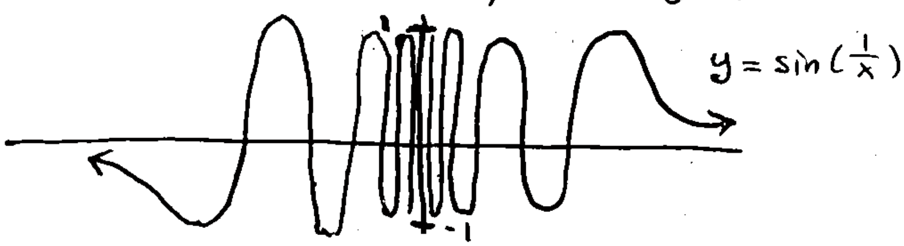
e.g.



vertical asymptote at $x = 0$

Limits can fail to exist for even more "complicated" reasons:

E.g. Let $f(x) = \sin(\frac{1}{x})$, whose graph looks like:



As x gets closer and closer to zero, $\frac{1}{x}$ passes through many multiples of 2π, so $\sin(\frac{1}{x})$ passes thru many periods. In each period, it attains a max. value of $+1$, and also a min. value of -1.

Thus, near zero, there are ∞-many x for which $\sin(\frac{1}{x}) = 1$, and ∞-many for which $\sin(\frac{1}{x}) = -1$. Since it oscillates rapidly between these values, there is no single value that $f(x)$ approaches as x gets close to zero. Therefore, the limit

$\lim_{x \to 0} \sin(1/x)$ does not exist.

In fact, neither of $\lim_{x \to 0^-} \sin(\frac{1}{x})$ or $\lim_{x \to 0^+} \sin(\frac{1}{x})$ exist, so this limit fails to exist not because of a disagreement between one-sided limits, or because the function goes off to $\pm\infty$, but for a more complicated reason...

The Squeeze Theorem §2.3

Sometimes we can calculate a limit for a function $f(x)$ by comparing it (in size) to other functions.

Thm If $f(x) \le g(x)$ for x near a (except possibly at a) and the limits of f & g at a both exist, then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$$

Thm (Squeeze Theorem) If $f(x) \le g(x) \le h(x)$ for x near a (except possibly at a), and $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$ then also $\lim_{x\to a} g(x) = L$.

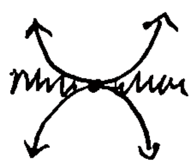
Picture: "squeeze"

Eg: Let's use the squeeze theorem to compute

$$\lim_{x\to 0} x^2 \sin(1/x).$$

Note we cannot use product law for limits here since $\lim_{x\to 0} \sin(1/x)$ does not exist. But...

Since $\sin(1/x)$ is always between -1 and 1, have



$$-x^2 \le x^2 \sin(1/x) \le x^2 \quad \text{for all } x$$

so that we can apply squeeze thm with $\lim_{x\to 0} -x^2 = 0 = \lim_{x\to 0} x^2$

to conclude that $\lim_{x\to 0} x^2 \sin(1/x) = 0$ as well.

(Even though $x^2 \sin(1/x)$ "oscillates" a lot as $x \to 0$, the amplitudes of the waves get smaller and smaller...!)