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HW 3

1. A plane partition is an infinite 2D-array
$\pi = (\pi_{i,j})_{i=1,2,\dots}^{j=1,2,\dots}$ of nonnegative integers $\pi_{i,j} \in \mathbb{N}$
such that only finitely many entries are nonzero and the entries are weakly decreasing along rows and down columns in the sense that $\pi_{i,j} \geq \pi_{i',j'}$ if $i \leq i'$ and $j \leq j'$.

The size $|\pi|$ of π is the sum of the entries:
$|\pi| := \sum_{i,j \geq 1} \pi_{i,j}$

Prove that

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$$\sum_{\pi \text{ a plane partition}} q^{|\pi|} = \prod_{i \geq 1} \frac{1}{(1-q^i)^i}$$

Proof:

I don't understand this first line

$$\sum_{\pi \text{ a plane partition}} q^{|\pi|} = \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \prod_{k=1}^{\infty} \overline{[i+j+k-1]}$$

The algebraic manipulations after this are okay, but use words to explain where this formula is coming from. -2 pts

$$= \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{[i+j][i+j+1][i+j+2]\cdots}{[i+j-1][i+j][i+j+1][i+j+2]\cdots}$$

$$= \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{1}{[i+j-1]}$$

$$= \prod_{i=1}^{\infty} \frac{1}{[i][i+1][i+2][i+3]\cdots}$$

$$= \prod_{i \geq 1} \frac{1}{[i]^i} = \prod_{i \geq 1} \frac{1}{(1-q^i)^i}$$

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2 (a) Antichains have the greatest

9/10 number of the linear extension, because the elements can appear in any order without violating the requisite relationships. Chains have the least since the elements in a chain can only be listed in one order for a linear Extension.

Good

(b) let $g : L(P) \to L(P^*)$ defined by $g(e) = g(e_1, e_2, \dots, e_n) = (e_n, \dots, e_2, e_1)$.

for any linear extension $e \in L(P)$ (i.e. writing the linear extension backwards).

This uniquley definds a new list e^* that has all relationships from e reversed, so $e_i < e_j \Rightarrow e_j^* < e_i^*$. Thus e^* is a linear extension of P^*. (since $e_i \le e_j \Rightarrow i \le_P j$ and $e_i \le e_j \Rightarrow e_j^* \le e_i^* \Rightarrow j \le_{P^*} i$).

Since g is reversible $\Rightarrow$ we have the number of $L(P)$ = the number of $L(P^*)$

Good

(c) Since P and Q are in disjoint union, we can combine arbitrary linear extensions of P and Q to form a linear extension of $P \cup Q$.

then we have to determine the number of ways that we can combine the linear extensions that we chose into a simple list. Assume that we want to insert $p \in L(P)$ into $q \in L(Q)$. So each element of P will be placed before elements of q or after the last element of the list. this process will be repeated for all elements of P. Thus $\Rightarrow \binom{n+m+1-1}{n} = \binom{n+m}{n}$

I don't quite understand your explanation for why (n+m choose n) is the right formula. -1pt

ways $\Rightarrow$ (the number of linear extensions of P) $\cdot$ (the number of linear extensions of Q) $\cdot \binom{n+m}{n}$ linear extensions of $P \cup Q$.

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#3# Choosing k numbers from $[n]$ that appear in the 1st part of the SYT determines the entire SYT, and Because there is only one way to arrange the numbers along the raw, and the same for the columns. And since we have to arrange in the upper left corner

I think you mean we have to have a 1 in the upper-left corner. But yes.

$\exists$ $(k-1)$ choices from $(n-1)$ values

$$\Longrightarrow \quad f^{\lambda} = \binom{n-1}{k-1} \quad \text{for} \quad \lambda = (k, \underbrace{1, 1, \dots, 1}_{n-k})$$

for $1 \leq k \leq n$.

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~~4~~ we know that $M \xrightarrow{RSK} (P, Q)$

$\Rightarrow M \xrightarrow{RSK} (Q, P)$. Assume M is the permutation matrix

$\Longrightarrow M^T = M$. Also, because M is premutation

Should explain that for permutatiion matrices, transposition = inversion of permutation. -1pt

matrix $\Longrightarrow RSK(M) = RS(M)$.

And, because $M \xrightarrow{RSK} (P, Q) = M \xrightarrow{RSK} (Q, P)$

$\Longrightarrow P = Q$.

Therefore, $M \xrightarrow{RS} (P, P)$

which means $\exists$ a bijection btwn involutions in S_n and SYT of $sh(\lambda \vdash n)$

$\Longrightarrow$ the number $\{\sigma \in S_n \mid \sigma = \sigma^{-1}\}$

$= \sum_{\lambda \vdash n} f^{\lambda}$.

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For permutation $\sigma \in S_n$, we could maximize $\min(lis(\sigma), lds(\sigma))$ as follows:

If $n = 2k+1$, then we consider the permutations that is both increasing sequence $1, 2, \ldots, k+1$ and the decreasing seq $2k+1, 2k, \ldots, k+1$)

Then $\min(lis(\sigma), lds(\sigma)) = k$.

For $n = 2k \Rightarrow$ the permutations will be the sequences $1, 2, \ldots, k$ and $2k, 2k-1, \ldots, k+1$, giving $\min(lis(\sigma), lds(\sigma)) = k$.

This is a good heuristic, but why is this the maximum achievable.
Answer: because an increasing and a decreasing subsequence can intersect in at most one element. You should explain that. -1pt