



Upho lattices and their cores

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Finite & infinite graded posets

A finite poset P is **n -graded** if $P = \sqcup_{i=0}^n P_i$ where all maximal chains are of form $x_0 < x_1 < \dots < x_n$ with $x_i \in P_i$ for all i . Its **rank generating** and **(reciprocal) characteristic polynomials** are

$$F(P; x) = \sum_{i=0}^n \#P_i x^i = \sum_{p \in P} x^{\rho(p)}$$

$$\chi(P; x) = \sum_{p \in P} \mu(\hat{0}, p) x^{\rho(p)}$$

For $B_n =$ **Boolean lattice** of subsets of $[n]$:

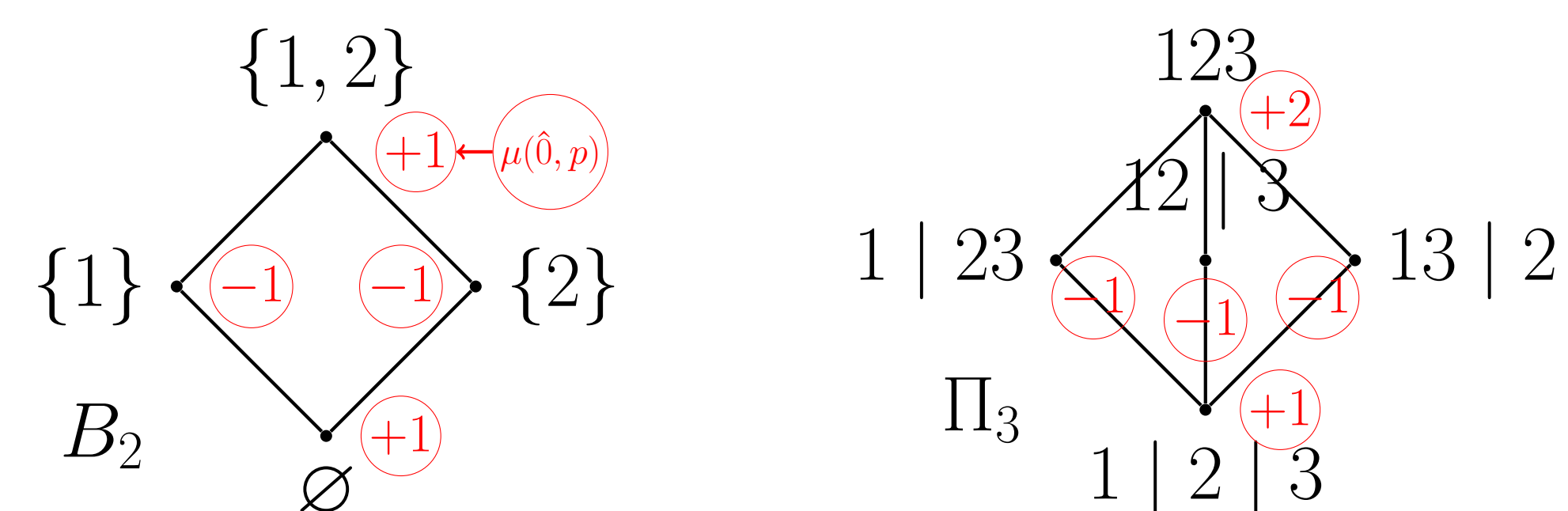
$$F(B_n; x) = \sum_{k=0}^n \binom{n}{k} x^k = (1+x)^n$$

$$\chi(B_n; x) = \sum_{k=0}^n (-1)^k \binom{n}{k} x^k = (1-x)^n$$

For $\Pi_n =$ **partition lattice** of set partitions of $[n]$:

$$F(\Pi_n; x) = \sum_{k=0}^n S(n, n-k) x^k$$

$$\chi(\Pi_n; x) = \sum_{k=0}^n s(n, n-k) x^k = \prod_{i=1}^{n-1} (1-ix)$$



An infinite poset \mathcal{P} is **finite type \mathbb{N} -graded** if $\mathcal{P} = \sqcup_{i=0}^{\infty} P_i$ where all maximal chains are of form $x_0 < x_1 < \dots$ with $x_i \in P_i$ for all i and where $\#P_i < \infty$ for all i . We again define

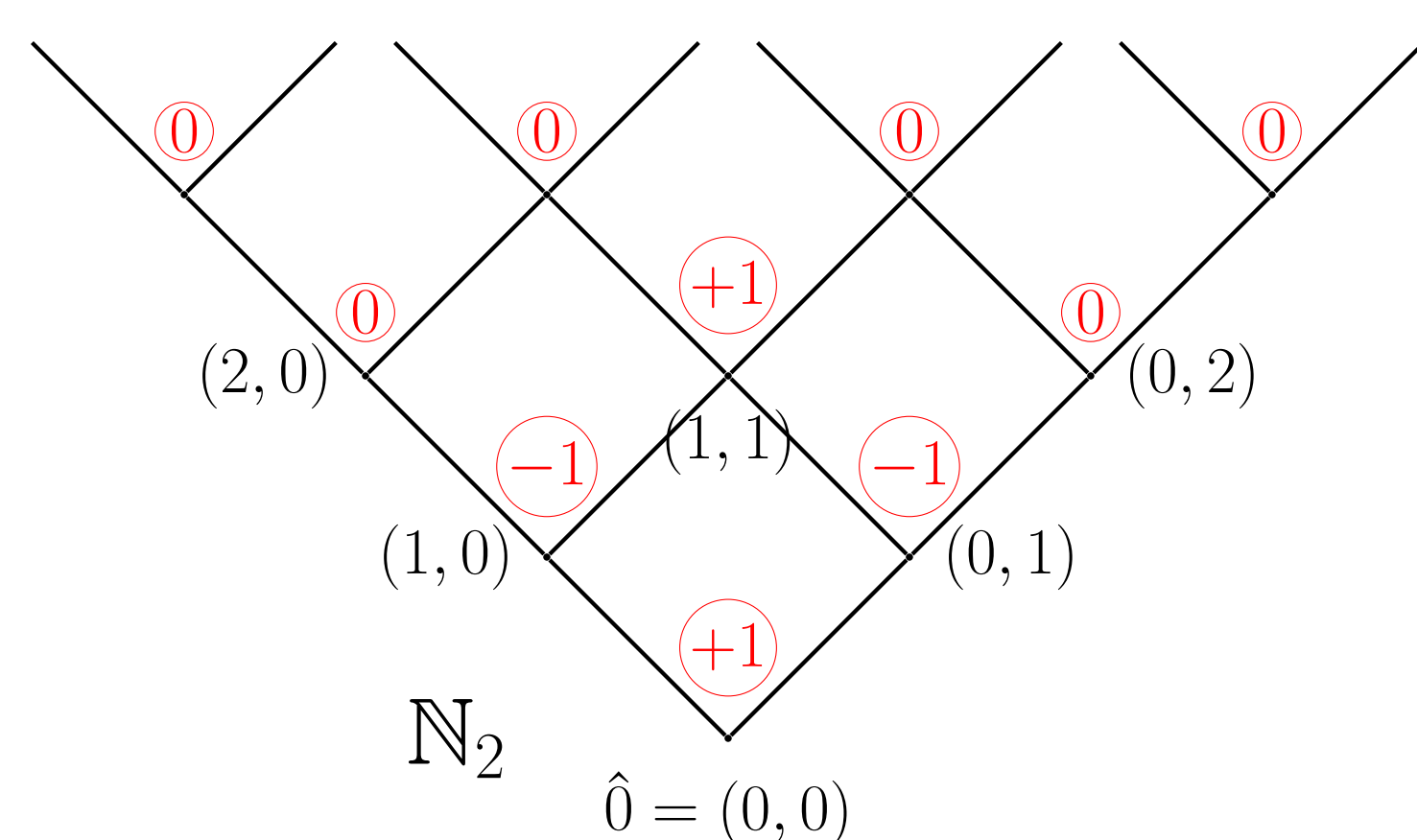
$$F(\mathcal{P}; x) = \sum_{p \in \mathcal{P}} x^{\rho(p)}$$

$$\chi(\mathcal{P}; x) = \sum_{p \in \mathcal{P}} \mu(\hat{0}, p) x^{\rho(p)}$$

For $\mathcal{P} = \mathbb{N}^n$:

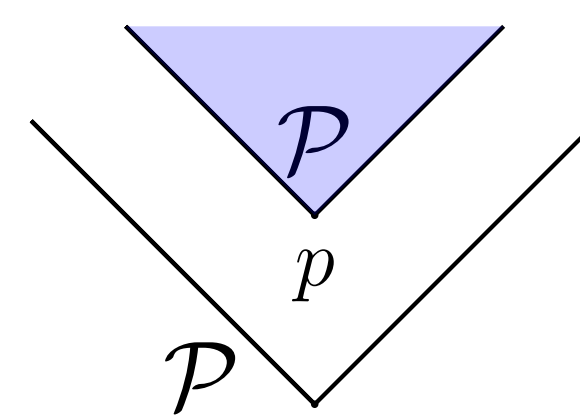
$$F(\mathbb{N}^n; x) = \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} x^k = \frac{1}{(1-x)^n}$$

$$\chi(\mathbb{N}^n; x) = (1-x)^n$$

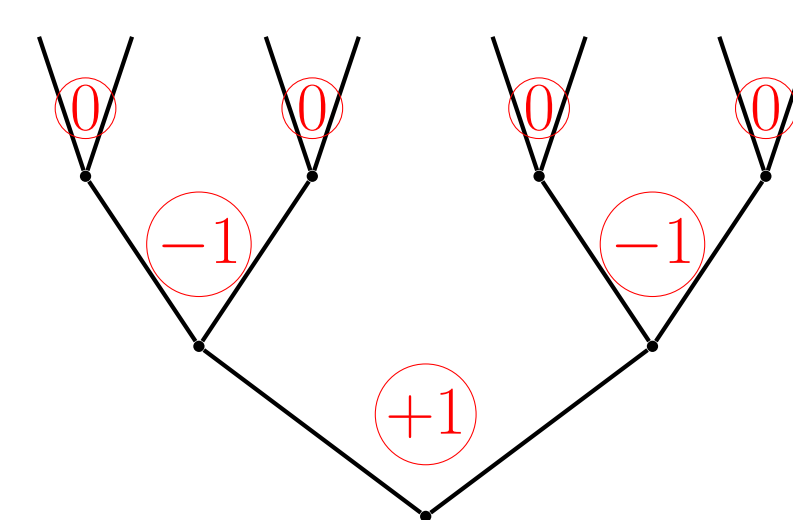


Upho posets

A poset \mathcal{P} is **upper homogeneous**, or “**upho**,” if for every $p \in \mathcal{P}$ the **principal order filter** $V_p = \{q : q \geq p\}$ is isomorphic to whole poset \mathcal{P} . Looking up from each $p \in \mathcal{P}$, we see a copy of \mathcal{P} :



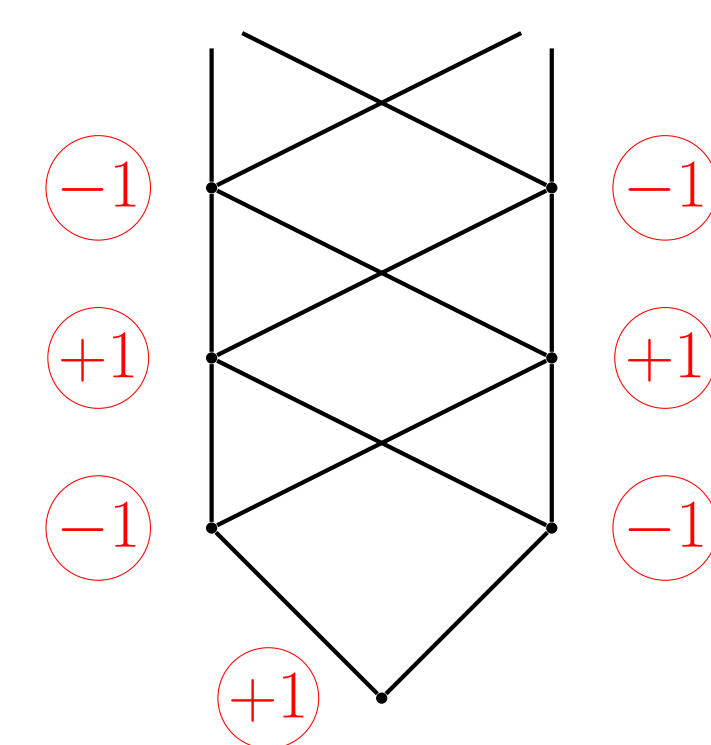
We consider **finite type \mathbb{N} -graded** upho posets. Since \mathbb{N} is upho, and upho-ness is preserved by direct product, \mathbb{N}^n is upho for any $n \geq 1$. Other examples: \mathcal{P} = the infinite binary tree poset:



$$F(\mathcal{P}; x) = \sum_{n \geq 0} 2^n x^n = \frac{1}{1-2x}$$

$$\chi(\mathcal{P}; x) = 1-2x$$

\mathcal{P} = the upho poset with $\#P_i = 2$ for all $i \geq 1$:



$$F(\mathcal{P}; x) = 1 + \sum_{n \geq 1} 2 x^n = \frac{1+x}{1-x}$$

$$\chi(\mathcal{P}; x) = 1 + \sum_{n \geq 1} (-1)^n 2 x^n = \frac{1-x}{1+x}$$

These examples with two **atoms** have obvious generalizations to any number $r \geq 1$ of atoms.

From the above examples, it is not hard to guess:

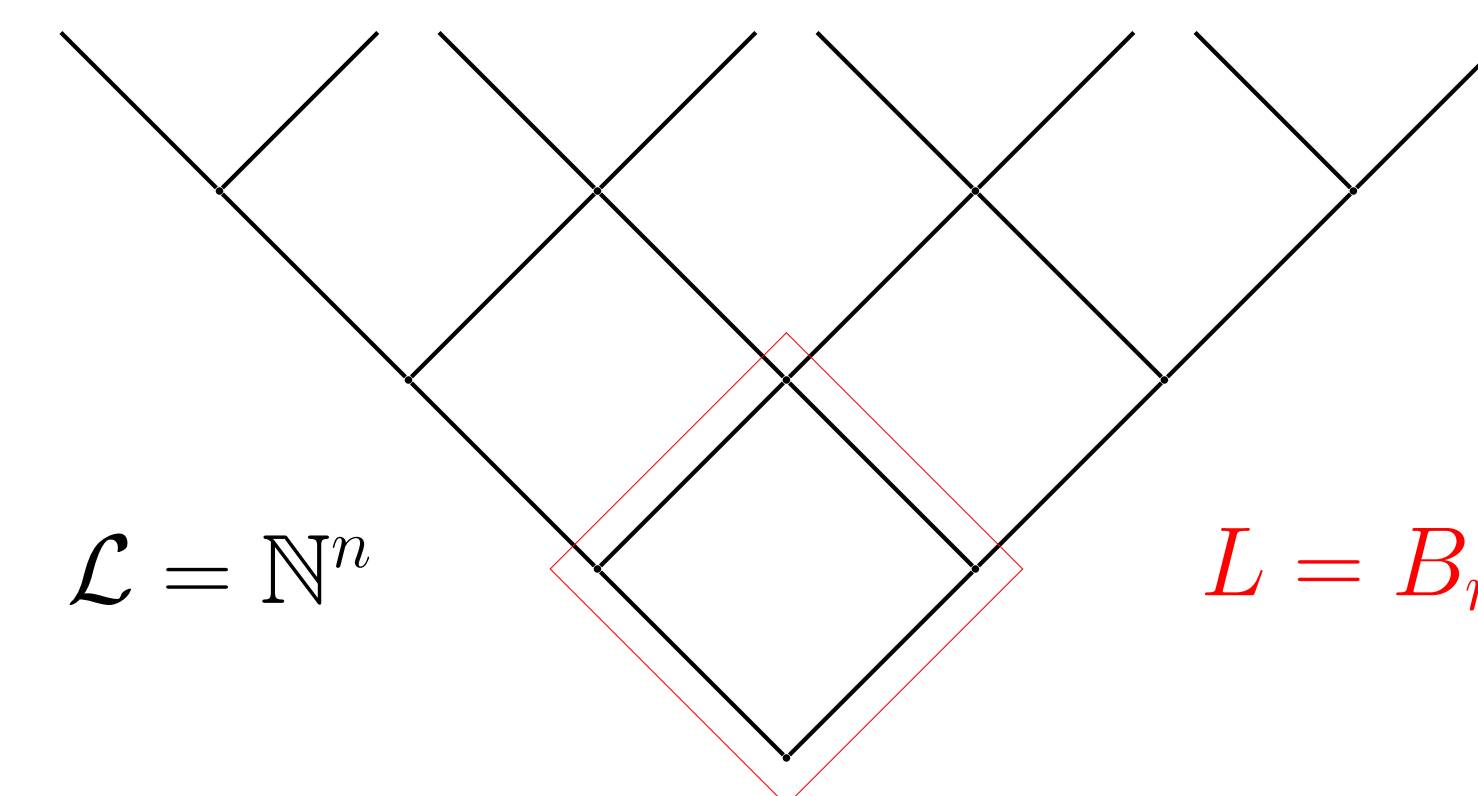
Theroem (H. 2020)

For \mathcal{P} an upho poset, $F(\mathcal{P}; x) = \chi(\mathcal{P}; x)^{-1}$.

This is easy to prove by **Möbius inversion**.

Upho lattices and cores

For an upho **lattice** \mathcal{L} we let $L = [\hat{0}, s_1 \vee \dots \vee s_r]$ be the interval from its minimum to the join of its atoms s_1, \dots, s_r , which we call its **core**.



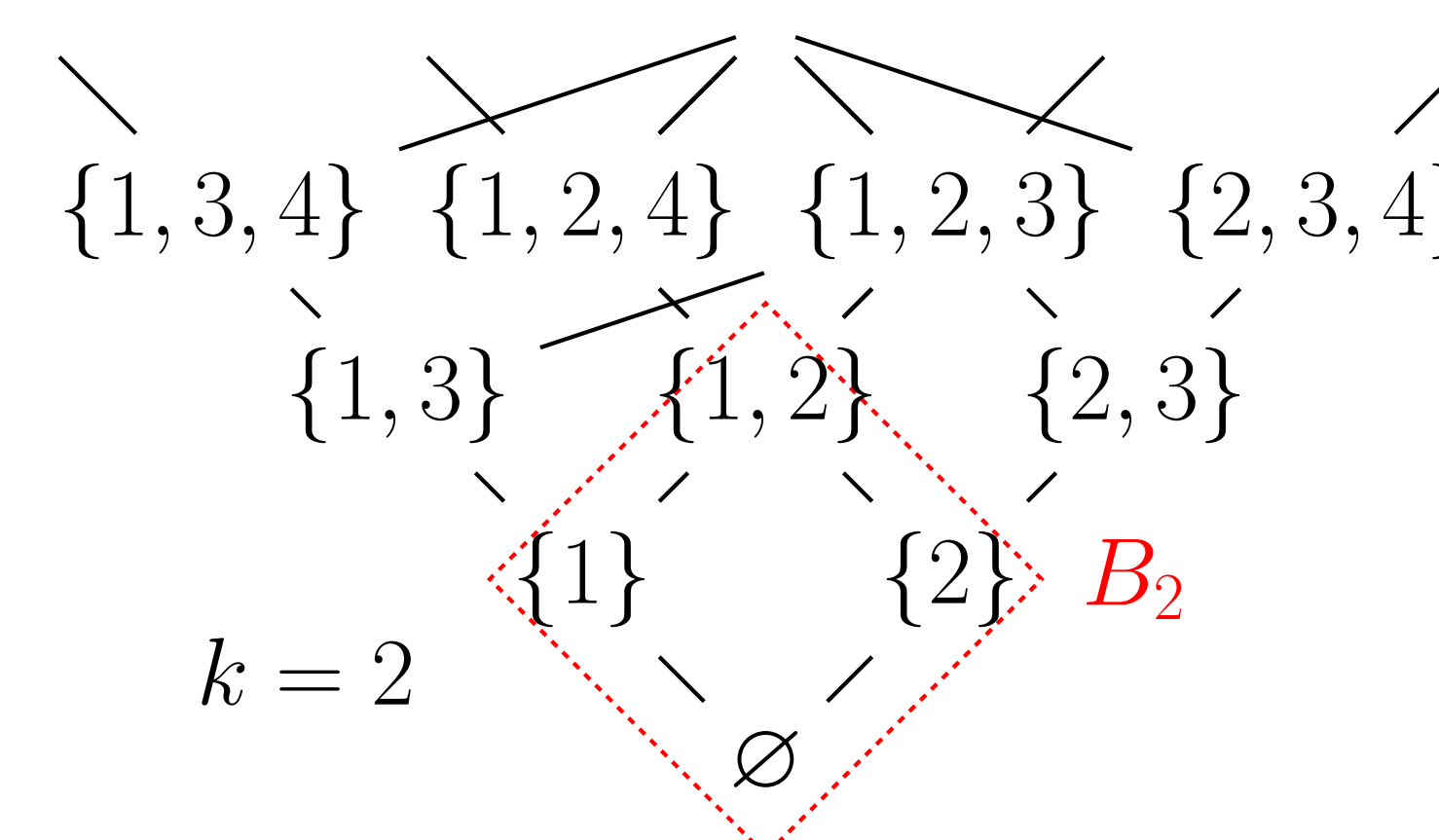
Corollary (from cross-cut thm.)

\mathcal{L} upho lattice, core $L \Rightarrow F(\mathcal{L}; x) = \chi(L; x)^{-1}$.

Note: the core does not determine the upho lattice, i.e., a given L can be a core of more than one \mathcal{L} .

For example, fix $k \geq 1$ and let

$\mathcal{L} = \{\text{finite } A \subseteq \{1, 2, \dots\} : \max(A) < \#A + k\}$, ordered by inclusion.



This \mathcal{L} is an upho lattice with core $L = B_k$, but it is **not** isomorphic to \mathbb{N}^k (for $k \geq 2$).

Nevertheless, we are still interested in knowing:

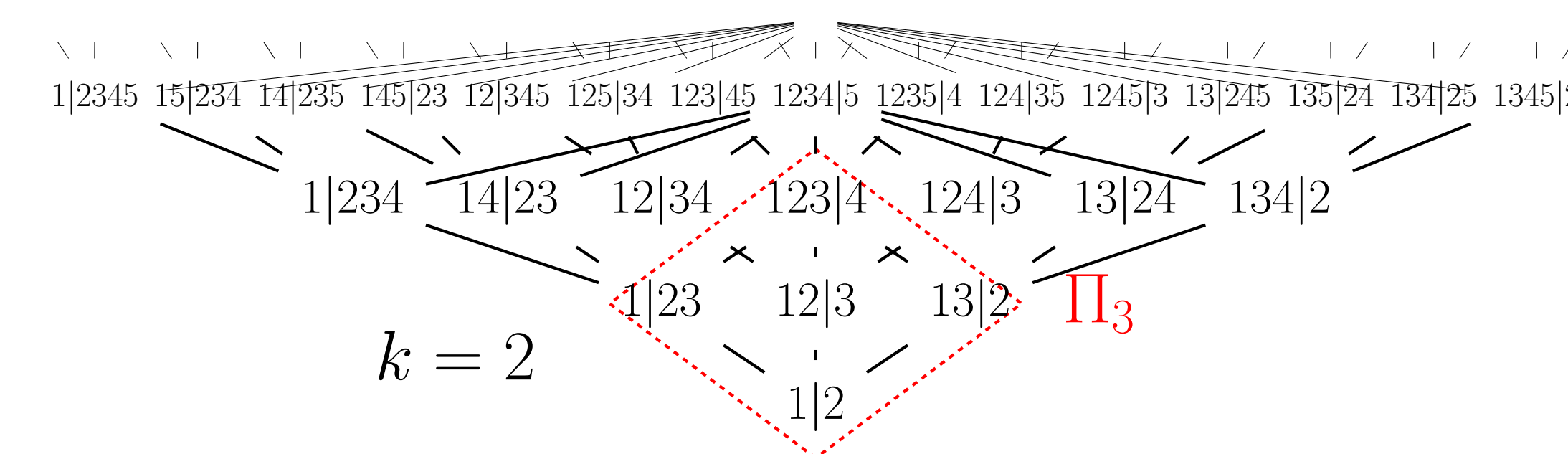
Main question

Which finite lattices L are cores of upho lattices?

For example, we know the Boolean lattice B_n is a core, for any $n \geq 1$. We cannot fully answer this question, but we can provide positive and negative examples, showing it is subtle.

Combinatorial examples of cores

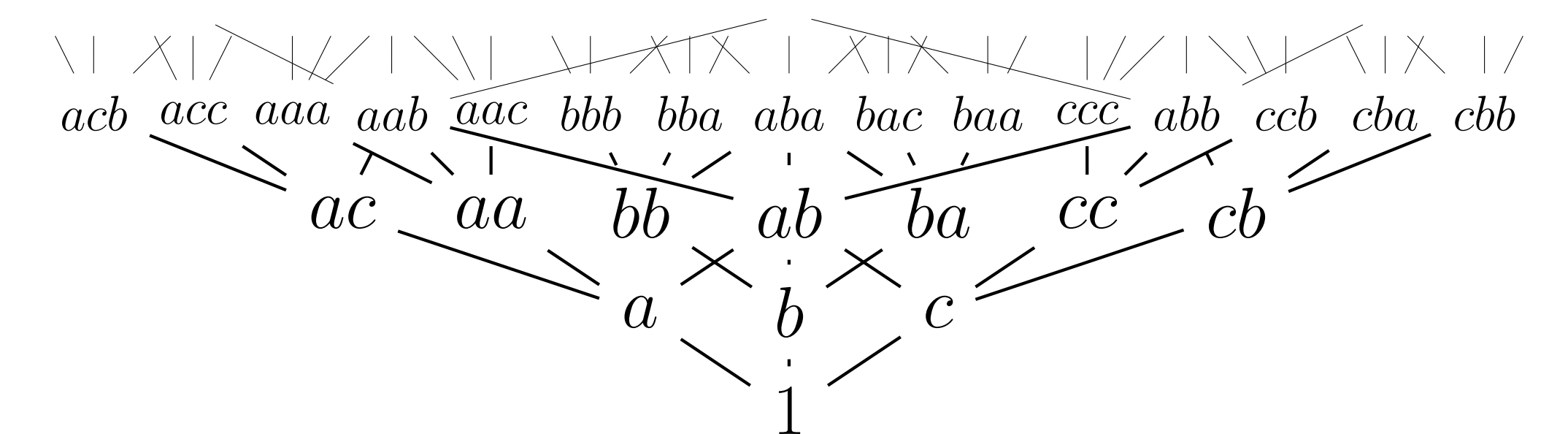
Fix $k \geq 1$ and let \mathcal{L} be the set partitions of $[n]$ (for any $n \geq k$) into k blocks, ordered by refinement:



This \mathcal{L} is an upho lattice with core $L = \Pi_{k+1}$. Similar construction exists for any **uniform sequence** of **supersolvable geometric lattices**.

Algebraic examples of cores

Consider the monoid $M = \langle a, b, c \mid ab = bc = ca \rangle$, ordered by left divisibility:



This is an upho lattice. Same is true for any (homogeneous) **Garside monoid**. Hence, **weak order** and **noncrossing partition lattice** of any **finite Coxeter group** are cores.

Non-examples of cores

Lemma

If L is a core of an upho lattice, then the power series $\chi(L; x)^{-1}$ has all positive coefficients.

If L is the **face lattice** of **octahedron**, then $[x^{13}] \chi(L; x)^{-1} = -123704$, so L is not a core. More generally, face lattices of n -dimensional **cross polytopes** and **hypercubes** aren't cores ($n \geq 3$).

If L is the **lattice of flats** of **uniform matroid** $U(3, 4)$, then $[x^7] \chi(L; x)^{-1} = -80$, so L is not a core. More generally, lattice of flats of $U(k, n)$ is not a core for $2 < k < n$.