Total: 57/60

Homework 1 — Combinatorics 1

To be more precise you should say "and then partitioning the remaining n into at most k parts can be done in p_1(n) + … p_k(n) ways" to be more precise, but yes this is the right way to think of it. 10/10

#1 Let $p_k(n)$ denote the number of partitions of $n$ into $k$ parts. Prove bijectively that $p_0(n) + p_1(n) + p_2(n) + \dots + p_k(n) = p_k(n+k)$.

Let $p_k(n)$ denote the number of partitions of $n$ into $k$ parts. $p_k(n+k)$ can be expressed by putting one value into each of the $k$ parts, i.e. $p_k(k)$, and then partitioning can be done in $p_0(n) + p_1(n) + p_2(n) + \dots + p_k(n)$ ways.

$\therefore p_0(n) + p_1(n) + p_2(n) + \dots + p_k(n) = p_k(n+k)$ ■

#2 Fix natural numbers $k, n$. Let $[n]$ denote the set $[n] := \{1, 2, \dots, n\}$. Give a simple formula for the number of ordered $k$-tuples $(T_1, \dots, T_k)$ of subsets of $[n]$ satisfying

- $T_i \cap T_j = \emptyset$ for all $i \neq j$ (i.e. they are disjoint);
- $\bigcup_{i=1}^{k} T_i = [n]$ (i.e. their union is the whole set $[n]$).

Yes, correct, but important to note that we are considering *distinguishable* balls and boxes here. If it were indistinguishable balls and boxes, we would get partitions as in the last problem.

10/10

If we look at this as a "balls into buckets" problem, then there are $n$ balls that we need to put into $k$ buckets. Since there is no limit on how many balls we can put into a bucket, we have $k$ buckets that we can pick from $n$ times, which gives us the equation $k^n$. Notice that, with this equation, there are no quantum balls — no single ball shows up in two buckets — so $T_i \cap T_j = \emptyset$ for all $i \neq j$, and that all of the balls are in a bucket so $\bigcup_{i=1}^{k} T_i = [n]$.

#3 Show that $\sum_{n_1,\dots,n_k \geq 0} \min(n_1, \dots, n_k) x_1^{n_1} x_2^{n_2} \dots x_k^{n_k} = \frac{x_1 x_2 \dots x_k}{(1-x_1)(1-x_2)\dots(1-x_k)(1-x_1x_2\dots x_k)}$

The R.H.S. $= \frac{x_1 x_2 \dots x_k}{(1-x_1)(1-x_2)\dots(1-x_k)(1-x_1x_2\dots x_k)} =$

$x_1 x_2 \dots x_k \cdot \frac{1}{1-x_1} \cdot \frac{1}{1-x_2} \cdot \dots \cdot \frac{1}{1-x_k} \cdot \frac{1}{1-x_1x_2\dots x_k} =$

(1) $x_1 x_2 \dots x_k (1 + x_1 + x_1^2 + \dots)(1 + x_2 + x_2^2 + \dots) \cdot \dots \cdot (1 + x_1 x_2 \dots x_k + x_1^2 x_2^2 \dots x_k^2 + \dots)$.

This is equal to $\sum_{n_1, n_2, \dots, n_k \geq 0} a(n_1, n_2, \dots, n_k) x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$, where $a(n_1, n_2, \dots, n_k)$ is a coefficient representing the number of ways to obtain a particular combination of powers.

Could be expressed a little more formally (instead of just doing an example) but this is completely the right idea.

10/10

Notice that there are only the smallest power of $x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$ ways to make $x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$ because in the multiplication in (1), the exponential value chosen in the last quantity, i.e. $(1 + x_1 x_2 \dots x_k + x_1^2 x_2^2 \dots x_k^2 + \dots)$, is responsible for fixing the rest of the choices. For example, there are only 3 different ways of making $x_1^4 x_2^3$ because you can only choose $1$, $x_1 x_2$ or $x_1^2 x_2^2$ in the third term to get $x_1^4 x_2^3$ so $a(n_1, n_2, \dots, n_k) = \min(n_1, n_2, \dots, n_k) \Rightarrow$

$$\frac{x_1 x_2 \dots x_k}{(1-x_1)(1-x_2)\dots(1-x_k)(1-x_1 x_2 \dots x_k)} = \sum_{n_1, \dots, n_k \geq 0} \min(n_1, \dots, n_k) x_1^{n_1} x_2^{n_2} \dots x_k^{n_k} \blacksquare$$

#4 Let $\bar{c}(n, m)$ denote the number of compositions of $n$ into parts of size at most $m$. Show that $\sum_{n \geq 0} \bar{c}(n, m) x^n = \frac{1-x}{1-2x+x^{m+1}}$.

Let $\bar{c}(n, m)$ denote the number of compositions of $n$ into parts of size at most $m$. Since $\sum_{n \geq 0} c_k(n) x^n = (x + x^2 + x^3 + \dots)^k = \left(\frac{1}{1-x} - 1\right)^k$, so $\sum_{n \geq 0} \bar{c}_k(n, m) x^n = (x + x^2 + x^3 + \dots + x^m)^k = \left(\frac{1}{1-x} - 1 - \frac{x^{m+1}}{1-x}\right)^k = \left(\frac{1}{1-x} - \frac{1-x}{1-x} - \frac{x^{m+1}}{1-x}\right)^k = \left(\frac{x - x^{m+1}}{1-x}\right)^k$.

(Strictly speaking you didn't explain what c_k(n,m) is, but I understand.)

$\bar{c}(n, m) = \sum_{k \geq 0} \bar{c}_k(n)$, so $\sum_{n \geq 0} \bar{c}(n, m) x^n = \sum_{n \geq 0} \left(\sum_{k=0}^{m} \bar{c}_k(n, m)\right) x^n = \sum_{k \geq 0} \left(\sum_{n \geq 0} \bar{c}_k(n, m) x^n\right) = \sum_{k \geq 0} \left(\frac{x - x^{m+1}}{1-x}\right)^k = \frac{1}{1 - \left(\frac{x - x^{m+1}}{1-x}\right)} = \frac{1}{1 - \left(\frac{x - x^{m+1}}{1-x}\right)} \cdot \frac{1-x}{1-x} = \frac{1-x}{1-2x+x^{m+1}}$.

Nice proof. 10/10

$\therefore \sum_{n \geq 0} \bar{c}(n, m) x^n = \frac{1-x}{1-2x+x^{m+1}} \blacksquare$

#5 Prove that, for any $n \geq 0$, $4^n = \sum_{n \geq 0} \binom{2k}{k} \binom{2(n-k)}{n-k}$.

By definition, the generating function of $4^n$ is $\frac{1}{1-4x}$.

Yes.

Scan got a little cut off here…

As was shown in class, $\frac{1}{\sqrt{1-4x}} = (1-4x)^{-\frac{1}{2}} = \sum_{k\geq 0}\binom{-\frac{1}{2}}{k}(-4)^k x^k =$ Yes.

$\sum_{k\geq 0}\left(\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\dots(\frac{-(2k-1)}{2})}{k!}(-4)^k\right)x^k = \sum_{k\geq 0}\left(\frac{2^k(1)(3)(5)\dots(2k-1)}{k!}\right)x^k$

$= \sum_{k\geq 0}\left(\frac{(1)(3)(5)\dots(2k-1)}{k!}\cdot\frac{(2)(4)(6)\dots(2k)}{k!}\right)x^k = \sum_{k\geq 0}\frac{(2k)!}{k!\,k!}x^k = \sum_{k\geq 0}\binom{2k}{k}x^k.$

No.. this last sentence is not the way to think about it: the point is that multiplication of power series is convolution of the coefficients, and the expression with products of central binomials is exactly a convolution.

Note that the generating function of $\binom{2(n-k)}{n-k}$ is the same as that of $\binom{2k}{k}$, so on the R.H.S. we have $\frac{1}{\sqrt{1-4x}}\cdot\frac{1}{\sqrt{1-4x}} = \frac{1}{1-4x}$ which is the generating function of $4^n$.

$\therefore \forall n \geq 0,\ 4^n = \sum_{n\geq 0}\binom{2k}{k}\binom{2(n-k)}{n-k}$ ■

8/10

#6 Let $n \geq 1$, and let $ODD(n)$ denote the subset of permutations in the symmetric group $G_n$ with no cycles of even size. Prove that $\sum_{\sigma\in ODD(n)} 2^{\#cycles(\sigma)} = 2\cdot n!$.

$\sum_{\sigma\in ODD(n)} 2^{\#cycles(\sigma)}$ is equal to Touchard's theorem where $t_1 = t_3 = t_5 = \dots = 2$ and

$t_2 = t_4 = t_6 = \dots = 0 \implies \sum_{\sigma\in ODD(n)} 2^{\#cycles(\sigma)} = 2\left(\frac{x}{1}\right) + 2\left(\frac{x^3}{3}\right) + 2\left(\frac{x^5}{5}\right) + \dots = e^{\ln(1+x)-\ln(1-x)} =$

$e^{\ln\left(\frac{1+x}{1-x}\right)} = \frac{1+x}{1-x} = \frac{1}{1-x} + \frac{x}{1-x} = (1+x+x^2+\dots)+(x+x^2+x^3+\dots) = 1 + 2(x+x^2+x^3+\dots)$

$= 1 + \frac{2}{1-x} = 1 + 2(1-x)^{-1} = 1 + 2\sum_{n\geq 0}\frac{(-1)(-2)(-3)\dots(-n)}{n!}(-x)^n = 1 + 2\sum_{n\geq 0} n!\left(\frac{x^n}{n!}\right) = 1 + \sum_{n\geq 0} 2n!\left(\frac{x^n}{n!}\right)$

$\therefore \sum_{\sigma\in ODD(n)} 2^{\#cycles(\sigma)} = 2n!$ ■

Right basic idea but strictly speaking what you wrote doesn't make sense because you start your sequence of equalities with
sum_{sigma in ODD(n)} 2^{#cycles(sigma)},
which is just a number, but then you end with a function of x. You should start the equalities with
sum_{n >= 0} (x^n/n!) * sum_{sigma in ODD(n)} 2^{#cycles(sigma)},
then the argument would be 100% correct.

9/10

Several blank pages here… probably because of the scanning program you used?