Order polynomial product formulas and poset dynamics

University of Minnesota Combinatorics Seminar

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UMN

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Section 1

Introduction

Two approaches in math

studying general objects:
all algebraic varieties, ...

studying special objects:
a particular PDE, ...

$\uparrow$

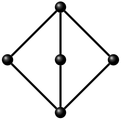
this talk

Posets

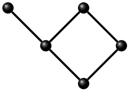
The objects I'm interested in are *(finite) posets* (partially ordered sets).

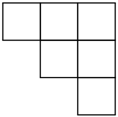
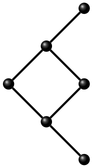
Posets are a unifying theme in modern enumerative & algebraic combinatorics (see, e.g., Stanley's *Enumerative Combinatorics*).

Posets are represented via their *Hasse diagrams*:



Young diagram shapes and *shifted shapes* are natural examples of posets:

 $\Leftrightarrow$ 

 $\Leftrightarrow$ 

The main heuristic

Over the past couple years I've had success developing and applying the following heuristic for finding special posets:

posets with good dynamical properties
$=$ **posets with order polynomial product formulas**

Here the *order polynomial* is a certain enumerative invariant of a poset.

Meanwhile, *good dynamical behavior* means good behavior of *promotion of linear extensions* and *rowmotion of order ideals/P-partitions*.

The rest of the talk will explain this heuristic, and the examples it produces.

Section 2

Order polynomial product formulas

Plane partitions

A *$a \times b$ plane partition* is an $a \times b$ array of nonnegative integers that are weakly decreasing in rows and columns.

Let $\mathcal{PP}^m(a \times b) := \{a \times b \text{ plane partitions with entries} \leq m\}$:

$$\begin{array}{|c|c|c|c|} \hline 5 & 2 & 1 & 0 \\ \hline 5 & 1 & 0 & 0 \\ \hline \end{array} \in \mathcal{PP}^5(2 \times 4)$$

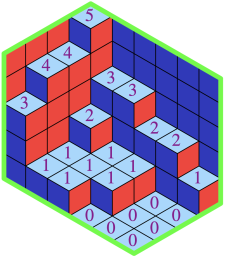
Theorem (MacMahon's formula (c.1915) for plane partitions in a box)

$$\sum_{\pi \in \mathcal{PP}^m(a \times b)} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{(1 - q^{i+j+m-1})}{(1 - q^{i+j-1})},$$

where $|\pi| = \sum \pi_{i,j}$ is the size of the plane partition π.

Other guises of plane partitions

Plane partitions have a beautiful $3D$ representation:



$\Leftrightarrow$

5	3	3	2	2	1
4	2	1	1	1	0
4	1	1	1	0	0
3	1	1	0	0	0

In this way they correspond to *lozenge tilings* of regions of the triangular lattice, and are a special case of the *dimer model* in statistical mechanics.

Plane partitions are also intimately related to the *representation theory of classical groups*, because $\mathcal{PP}^m(a \times b)$ indexes a basis of the irreducible representation V^λ of $\mathfrak{sl}(a+b)$ with highest weight $\lambda = m^a$.

P-partitions and order polynomials

For P a poset, a *P-partition* is a weakly order-reversing map $P \to \mathbb{N}$.

Let $\mathcal{PP}^m(P) := \{P\text{-partitions with entries} \leq m\}$, and define the *order polynomial* $\Omega_P(m)$ of P by

$$\Omega_P(m) := \#\mathcal{PP}^m(P) \text{ for all } m \in \mathbb{N}.$$

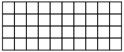
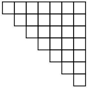
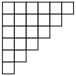
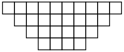
Basic facts:

- $\Omega_P(m)$ is a polynomial in m of degree $\#P$.
- Its leading coefficient is $e(P)/\#P!$, where $e(P)$ is the number of *linear extensions* of P (total orderings extending the partial order).

Many P have product formulas for $e(P)$: e.g., *Hook Length Formulas*.

Our question: which P have product formulas for $\Omega_P(m)$?

Shapes with order polynomial product formulas

Rectangle 	$\prod_{i=1}^{a}\prod_{j=1}^{b}\frac{m+i+j-1}{i+j-1}$	$\mathfrak{sl}(n)$	MacMahon c. 1915
Shifted staircase 	$\prod_{1\leq i\leq j\leq n}\frac{m+i+j-1}{i+j-1}$	$\mathfrak{so}(2n+1)$	Conj. MacMahon 1896, Andrews/Macdonald c. 1977 *"symmetric plane partitions"*
Staircase 	$\prod_{1\leq i\leq j\leq n}\frac{i+j+2m}{i+j}$	$\mathfrak{sp}(2n)$	Proctor 1988 *"symmetric, self-complementary plane partitions"*
Shifted Trapezoid 	$\prod_{i=1}^{k}\prod_{j=1}^{2n-k+1}\frac{m+i+j-1}{i+j-1}$	$\mathfrak{sp}(2n)$	Proctor 1983 *"transpose-complementary plane partitions"*

Shifted double staircase

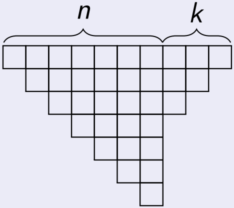
Recently with Tri Lai we found the first new family of posets with an order polynomial product formula since the 80s:

Theorem (Hopkins–Lai 2020)

We have

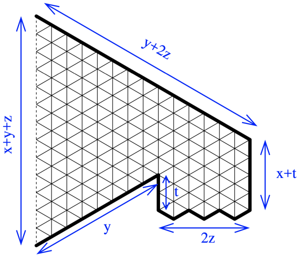
$$\Omega_P(m) = \prod_{1 \le i \le j \le n} \frac{m+i+j-1}{i+j-1} \prod_{1 \le i \le j \le k} \frac{m+i+j}{i+j},$$

for P a shifted double staircase shaped poset:

$P =$ 

Lozenge tilings of flashlight region

We actually prove a more general tiling theorem:

$F_{x,y,z,t} =$ 

Theorem (Hopkins–Lai 2020, cf. Ciucu 2019)

The number of lozenge tilings of $F_{x,y,z,t}$ is

$$\prod_{1\leq i\leq j\leq y+z} \frac{x+i+j-1}{i+j-1} \prod_{1\leq i\leq j\leq z} \frac{x+i+j}{i+j} \prod_{i=1}^{t}\prod_{j=1}^{z} \frac{(x+z+2i+j)}{(x+2i+j-1)}.$$

We prove this via *Kuo condensation*, a powerful dimer recurrence technique.

More about SDS order polynomial formula

Two aspects of the shifted double staircase order polynomial product formula are even more interesting than the result itself:

- Okada, 2020, in preparation, proved a remarkable algebraic extension of this product formula involving Lie group characters, suggesting it has some deeper representation theoretic meaning.
- It was discovered via the aforementioned heuristic relating product formulas and poset dynamics, as I'll explain in the next section.

Section 3

Poset dynamics: promotion and periodicity

Promotion of SYTs

Standard Young Tableaux (SYTs) of a shape λ with n boxes are bijective fillings of the boxes with $1, \ldots, n$, increasing in rows and columns.

Promotion is the following invertible operation on these SYTs:

- Delete the entry 1.
- Slide boxes into the resulting hole.
- Decrement all entries.
- Fill the hole with n.

Example

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} \xrightarrow{\text{del. } 1} \begin{array}{|c|c|c|}\hline \bullet & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} \xrightarrow{\text{slide}} \begin{array}{|c|c|c|}\hline \bullet & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} \xrightarrow{\text{decr.}} \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & \bullet \\ \hline \end{array} \xrightarrow{\text{fill } 6} \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array} = \mathrm{Pro}(T)$$

Together with *evacuation*, first defined by Schützenberger to study *RSK algorithm*. Straightforward extension to linear extensions of any poset.

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Shapes with good promotion behavior

Promotion behaves chaotically for most shapes, but:

Theorem

- *(Schützenberger 1977) For P a rectangle, $\mathrm{Pro}^{\#P}$ is the identity.*
- *(Edelman–Greene 1987) For P a staircase, $\mathrm{Pro}^{\#P}$ is transposition.*
- *(Haiman 1992) For P a shifted trapezoid or shifted double staircase, $\mathrm{Pro}^{\#P}$ is the identity.*
- *(Haiman–Kim 1992) These are the* **only** *four families of shapes with good promotion behavior.*

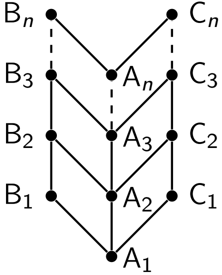
This theorem led to "good dynamics $=$ $\Omega_P(m)$ product formula" heuristic.

Stanley's Question (2009): Any other posets P with good Pro behavior?

The $V(n)$ poset

Let's explore Stanley's question using "the other direction" of the heuristic.

Let $V(n)$ be the following poset:



Theorem (Kreweras–Niederhausen, 1981)

$$\Omega_{V(n)}(m) = \frac{\prod_{i=1}^{n}(m+1+i)\prod_{i=1}^{2n}(2m+i+1)}{(n+1)!(2n+1)!}.$$

Kreweras words and walks

Linear extensions of $V(n)$ correspond to *words* with n A's, n B's, and n C's such that every prefix has as many A's as B's and as many A's as C's:

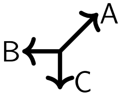
AABBCACCB

This variant of *"ballot sequences"* was enumerated by Kreweras:

Theorem (Kreweras, 1965)

$$e(V(n)) = \frac{4^n}{(n+1)(2n+1)}\binom{3n}{n}$$

In turn, these words correspond to *walks* in $\mathbb{N}^2$ from the origin to itself with steps of the form $(1,1)$, $(-1,0)$, and $(0,-1)$:



These *Kreweras walks* are a fundamental example of *"walks with small steps in the quarter plane"* (see Bousquet-Mélou–Mishna).

Promotion of the $V(n)$ poset

Example

$$
\begin{aligned}
w &= \mathsf{AABBCACCB}\\
\mathrm{Pro}(w) &= \mathsf{ABACACCBB}\\
\mathrm{Pro}^2(w) &= \mathsf{AACACCBBB}\\
\mathrm{Pro}^3(w) &= \mathsf{ACACABBBC}\\
\mathrm{Pro}^4(w) &= \mathsf{AACABBBCC}\\
\mathrm{Pro}^5(w) &= \mathsf{ACABBACCB}\\
\mathrm{Pro}^6(w) &= \mathsf{AABBACCBC}\\
\mathrm{Pro}^7(w) &= \mathsf{ABAACCBCB}\\
\mathrm{Pro}^8(w) &= \mathsf{AAACCBCBB}\\
\mathrm{Pro}^9(w) &= \mathsf{AACCBABBC}
\end{aligned}
$$

Recently with Martin Rubey, we addressed Stanley's question by showing that $V(n)$ has good behavior of promotion:

Theorem (Hopkins–Rubey, 2020)

For $P = V(n)$, $\mathrm{Pro}^{\#P}$ is reflection across the vertical axis of symmetry.

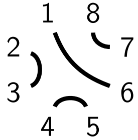
Promotion and rotation of webs

Webs are certain planar graphs that Kuperberg introduced to study the invariant theory of Lie algebras and quantum groups.

Previously work of White, Petersen–Pylyavskyy–Rhoades, and Tymoczko represented promotion of two- & three-rowed SYTs as rotation of webs:

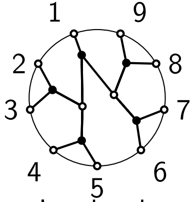
1	2	4	7
3	5	6	8

$\Leftrightarrow$

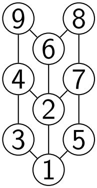


1	2	6
3	4	8
5	7	9

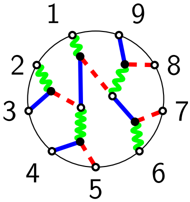
$\Leftrightarrow$



We did similarly for $V(n)$ linear extensions using edge-colored webs:



$\Leftrightarrow$



Section 4

Poset dynamics: rowmotion and orbit structure

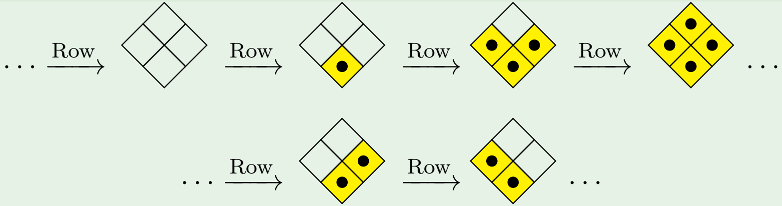
Rowmotion of order ideals

There's another poset operation which enters into the dynamics heuristic.

We use $\mathcal{J}(P)$ to denote the *order ideals* (downwards-closed subsets) of P.

Rowmotion sends $I \in \mathcal{J}(P)$ to the order ideal generated by the minimal elements of the complement $P \setminus I$.

Example



Toggling

For $p \in P$, *toggling at p* is the following operation on $\mathcal{J}(P)$:

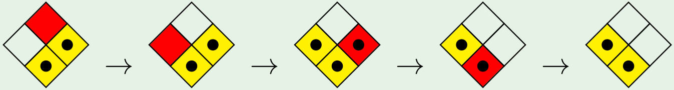
$$\tau_p(I) = \text{add } p \text{ to } I \text{ or remove } p \text{ from } I\text{, if it's still an order ideal.}$$

Cameron–Fon-der-Flaass, 1995 showed that

$$\mathrm{Row} = \tau_{p_1} \cdot \tau_{p_2} \cdots \tau_{p_n}$$

where $p_1, \ldots, p_n$ is any linear extension of P.

Example

$I =$  $= \mathrm{Row}(I)$

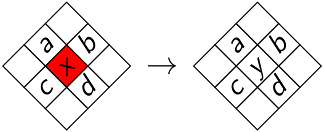
Piecewise-linear rowmotion

There's a natural identification $\mathcal{J}(P) \simeq \mathcal{PP}^1(P)$ (the indicator function).

In 2013, Einstein–Propp introduced a *piecewise-linear* extension of rowmotion $\mathrm{Row}\colon \mathcal{PP}^m(P) \to \mathcal{PP}^m(P)$ for any m:

$$\mathrm{Row} := \tau^{\mathrm{PL}}_{p_1} \cdot \tau^{\mathrm{PL}}_{p_2} \cdots \tau^{\mathrm{PL}}_{p_n}$$

where the *piecewise-linear toggle* τ^{PL}_p is



$\min(a,b) - -x - - - y - - \max(c,d)$

with $y = \max(a,b) + \min(c,d) - x$.

Can be seen as a PL map on the *order polytope* of P.

Cyclic sieving

Grinberg–Roby, 2015 established periodicity of piecewise-linear rowmotion for many P we've seen: *rectangles*, *shifted staircases*, and *staircases*.

But can ask for even more refined information, such *orbit structure*.

A very compact way to record orbit structure of a cyclic action is via the *cyclic sieving phenomenon (CSP)*:

Definition

For $C = \langle c \rangle$ a $\mathbb{Z}/n\mathbb{Z}$-action on a finite set X, and $f(q) \in \mathbb{N}[q]$ a polynomial, we say (X, C, f) *exhibits CSP* if for all k,

$$\#X^{c^k} = f(\zeta^k)$$

with $\zeta := e^{2\pi i/n}$ a primitive nth root of unity.

Cyclic sieving example: subset rotation and q-binomials

Theorem (Reiner–Stanton–White, 2004)

With $X = \{\textit{size } k \textit{ subsets of } \{1,\ldots,n\}\}$*, and* $C = \mathbb{Z}/n\mathbb{Z}$ *acting on* X *by rotating values,* (X, C, f) *exhibits CSP, where* $f(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$ *is the q-binomial coefficient.*

Example ($n = 4$, $k = 2$)

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4 \Rightarrow \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{q:=1} = 6,\ \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{q:=\pm i} = 0,\ \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{q:=-1} = 2$$

$c^0(S) = S$	$c^1(S) = S$	$c^2(S) = S$	$c^3(S) = S$
$\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$	(none)	$\{1,3\}$, $\{2,4\}$	(none)

Rhoades's CSP for rectangle rowmotion

Theorem (Rhoades, 2010)

$(\mathcal{PP}^m(a \times b), \langle \mathrm{Row} \rangle, f)$ *exhibits CSP, where*

$$f = \sum_{\pi \in \mathcal{PP}^m(a \times b)} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{(1 - q^{i+j+m-1})}{(1 - q^{i+j-1})},$$

is MacMahon's size generating function of plane partitions in a box.

Case $m = 1$ recovers subset rotation CSP.

Implies that **every** symmetry class has a product formula.

Rhoades used Lusztig's *dual canonical basis* of $\mathfrak{sl}_n$ representations to prove this CSP. Recently Shen–Weng gave a new proof using *cluster algebras*.

Conjectural extension of Rhoades's CSP

Conjecture (Hopkins, 2020)

For the P with good behavior of PL rowmotion, $(\mathcal{PP}^m(P), \langle \mathrm{Row} \rangle, \Omega_P(m))$ exhibits CSP, where

$$\Omega_P(m;q) := \prod_{\alpha \text{ root of } \Omega_P(m)} \frac{(1-q^{\kappa(m-\alpha)})}{(1-q^{-\kappa\alpha})}, \qquad (\kappa{:=}\min\{k{>}0\colon\, k\alpha{\in}\mathbb{Z}\forall\alpha\})$$

is the natural q-analog of the product formula for $\Omega_P(m)$.

Directly connects dynamics to order polynomial product formula.

Not clear why $\Omega_P(m;q) \in \mathbb{N}[q]$! (Cf. Stanton's "Fake Gaussian sequences")

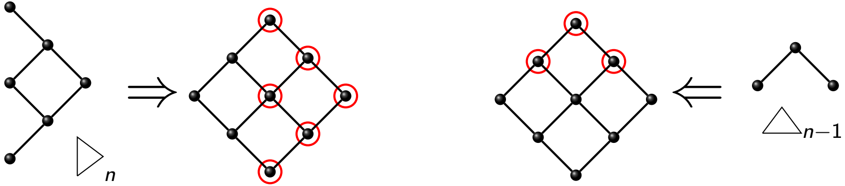
I have not been able to prove this conjecture, but have proved some "morally similar" results...

Embedding staircases into the square

Arguments of Grinberg–Roby, 2015 give the following embeddings:

Lemma

- *There is a* Row*-equivariant bijection between* $\mathcal{PP}^m(\triangleright_n)$ *and the subset of* $\pi \in \mathcal{PP}^m(n \times n)$ *for which* $\pi = \mathrm{Tr}(\pi)$ *(the transpose of* π*).*
- *There is a* Row*-equivariant bijection between* $\mathcal{PP}^M(\triangle_{n-1})$ *and the subset of* $\pi \in \mathcal{PP}^{2M}(n \times n)$ *for which* $\mathrm{Row}^n(\pi) = \mathrm{Tr}(\pi)$*.*



Fixed point counts for $\langle \mathrm{Row}, \mathrm{Tr} \rangle$

Theorem (Hopkins, 2019)

For all k, we have

$$\#\{\pi \in \mathcal{PP}^m(n \times n) \colon \mathrm{Row}^k(\pi) = \mathrm{Tr}(\pi)\} = f(\zeta^k),$$

where $\zeta := e^{\pi i/n}$ is a primitive $(2n)$th root of unity and

$$f(q) := \sum_{\substack{\pi \in \mathcal{PP}^m(n\times n), \\ \mathrm{Tr}(\pi)=\pi}} q^{|\pi|} = \prod_{1\leq i<j\leq n} \frac{(1-q^{2(i+j+m-1)})}{(1-q^{2(i+j-1)})} \prod_{i=1}^{n} \frac{(1-q^{2i+m-1})}{(1-q^{2i-1})}.$$

To prove this I studied how certain involutive automorphisms of the quantized enveloping algebra behave on the dual canonical basis.

Section 5

Conclusion

Recap of heuristic

The heuristic

posets with good dynamical properties
$=$ **posets with order polynomial product formulas**

has been successfully applied "in both directions," and led to the first new examples of these special posets in many years.

Many of the conjectures this heuristic produces remain open.

Moreover, the heuristic has also pointed the way to *interesting algebra* underlying the remarkable combinatorial phenomena.

Things I didn't have time to discuss...

- Many of these posets have a direct connection to the representation theory of Lie algebras: e.g., the *minuscule posets* and *root posets*.
- We can also look for CSPs for promotion of linear extensions. I have conjectured that $(\{\mathrm{lin.\ ext.'s\ of\ } P\}, \langle \mathrm{Pro} \rangle, e(P;q))$ exhibits CSP for the relevant posets P, where

$$e(P;q) := (1-q^{\kappa})(1-q^{2\kappa})\cdots(1-q^{\#P\cdot\kappa}) \lim_{m\to\infty} \Omega_P(m).$$

- Einstein–Propp in fact introduced a further *birational* lift of rowmotion, which has since received significant attention.
- It's natural to look for invariant functions of these actions, but they are hard to find in practice. There's been more success studying the "dual" notion of *homomesies*: functions with constant orbit averages.

Thank you!

- These slides are on my website.
- See my survey `arXiv:2006.01568` for references.