Howard Math 274, HW# 3,

Spring 2022; Instructor: Sam Hopkins; Due: Friday, April 22nd

1. A *plane partition* is an infinite $2D$-array $\pi = (\pi_{i,j})_{i=1,2,\ldots}^{j=1,2,\ldots}$ of nonnegative integers $\pi_{i,j} \in \mathbb{N}$ such that only finitely many entries are nonzero and the entries are weakly *decreasing* along rows and down columns in the sense that $\pi_{i,j} \geq \pi_{i',j'}$ if $i \leq i'$ and $j \leq j'$. The *size* $|\pi|$ of π is the sum of the entries: $|\pi| := \sum_{i,j\geq 1} \pi_{i,j}$. Prove that

$$\sum_{\pi \text{ a plane partition}} q^{|\pi|} = \prod_{i\geq 1} \frac{1}{(1-q^i)^i} \tag{1}$$

Hint: recall we proved the following product formula for *reverse* plane partitions of shape λ:

$$\sum_{\pi \in \mathrm{RPP}(\lambda)} q^{|\pi|} = \prod_{u\in\lambda} \frac{1}{1-q^{h(u)}} \tag{2}$$

where $h(u)$ is the hook length of the box u. Observe that a 180° rotation of a reverse plane partition of shape $\lambda = n \times n = (\overbrace{n, n, \ldots, n}^{n})$ is the same as a plane partition whose nonzero entries fit in the upper-left $n \times n$ square. Then deduce (1) from (2) by taking the limit $n \to \infty$.

2. Recall that a linear extension of a (finite) poset P is a list $p_1, \ldots, p_n$ of all its elements (each appearing once) where $p_i \leq p_j$ implies $i \leq j$. $\mathcal{L}(P)$ denotes the set of linear extensions of P.

   (a) Among posets P with n elements, which has the greatest number $\#\mathcal{L}(P)$ of linear extensions? Which has the least?

   (b) The *dual* P^* of a poset P is the poset with the same elements but the reverse order: $p \leq_P q \Leftrightarrow q \leq_{P^*} p$. Prove that $\#\mathcal{L}(P) = \#\mathcal{L}(P^*)$.

   (c) The *(disjoint) union* $P \cup Q$ of two posets P and Q is the poset whose elements are the elements in the union of the two sets, where the order within P and within Q is the same, but all $p \in P$ are incomparable to all $q \in Q$. Give a formula for $\#\mathcal{L}(P \cup Q)$ in terms of $\#\mathcal{L}(P)$, $\#\mathcal{L}(Q)$, and $n = \#P$ and $m = \#Q$.

3. Recall that f^λ denotes the number of Standard Young Tableaux of shape λ. Give a simple formula for f^λ in the case of a *hook* shaped partition $\lambda = (k, \overbrace{1, 1, \ldots, 1}^{n-k})$ for $1 \leq k \leq n$.

4. We used the Robinson-Schensted algorithm to prove that $\sum_{\lambda\vdash n} (f^\lambda)^2 = n!$, the number of permutations in the symmetric group S_n. Prove that $\sum_{\lambda\vdash n} f^\lambda = \#\{\sigma \in S_n \colon \sigma = \sigma^{-1}\}$, the number of *involutions* in S_n. **Hint**: use a symmetry property of RS(K) we discussed.

5. For $\sigma \in S_n$, let $\mathrm{lis}(\sigma)$ (resp., $\mathrm{lds}(\sigma)$) denote the length of the longest increasing (resp., decreasing) subsequence in σ. The Erdős-Szekeres theorem says $\max(\mathrm{lis}(\sigma), \mathrm{lds}(\sigma)) \geq \sqrt{n}$ for permutations $\sigma \in S_n$. Describe a permutation maximizing $\min(\mathrm{lis}(\sigma), \mathrm{lds}(\sigma))$ among $\sigma \in S_n$.