# Math 210 (Modern Algebra I), HW# 3,

Fall 2024; Instructor: Sam Hopkins; Due: Wednesday, September 25th

1. For $p$ a prime number, a group $G$ is called a *$p$-group* if every element has order a power of $p$. Prove that a finite abelian $p$-group is generated by its elements of maximal order.

2. Let $G$ be a group. Recall that an automorphism $\varphi \in \mathrm{Aut}(G)$ is called *inner* if it is conjugation by some fixed $h \in G$, i.e., is of the form $\varphi\colon g \mapsto hgh^{-1}$. Also recall that the *center* of $G$ is $Z(G) = \{g \in G\colon gx = xg \text{ for all } x \in G\}$.

   (a) Prove that $\mathrm{Inn}(G)$, the set of inner automorphisms of $G$, is a subgroup of $\mathrm{Aut}(G)$. (In fact it is a normal subgroup, but you do not need to prove that.)

   (b) Prove that $Z(G)$ is a normal subgroup of $G$.

   (c) Prove that $G/Z(G)$ is isomorphic to $\mathrm{Inn}(G)$.

3. An action of a group $G$ on a set $S$ is called *transitive* if for every $x, y \in S$ there is a $g \in G$ such that $g \cdot x = y$. An action of a group $G$ on a set $S$ is called *free* if $g \cdot x = x$ for some $x \in S$ and $g \in G$ implies $g = e$. In what follows, let $S = \{1, 2, \ldots, n\}$ and let $G$ be a finite group.

   (a) Suppose $G$ acts transitively on $S$. Prove that $n$ divides the order of $G$.

   (b) Suppose $G$ acts freely and transitively on $S$. Prove that the order of $G$ is exactly $n$.

   (c) Give an example, for each $n \geq 1$, of such a $G$ acting freely and transitively on $S$.

4. The *cycle type* of a permutation $\sigma \in S_n$ in the symmetric group on $n$ letters is the list $m_1(\sigma), m_2(\sigma), \ldots, m_n(\sigma)$ where $m_i(\sigma)$ is the number of $i$-cycles in $\sigma$'s cycle decomposition.

   (a) Prove that two permutations in $S_n$ are in the same conjugacy class if and only if they have the same cycle type.

   (b) Prove that the cardinality of the conjugacy class of $\sigma \in S_n$ is $\dfrac{n!}{1^{m_1}\, m_1!\, 2^{m_2}\, m_2! \,\cdots\, n^{m_n}\, m_n!}$ where $m_i = m_i(\sigma)$ are the numbers in the cycle type of $\sigma$.

5. Let $G$ be a finite group of order $pq$ for distinct primes $p < q$. Prove that $G$ is not simple, i.e., that it has a normal subgroup $N \trianglelefteq G$ other than $\{e\}$ and $G$.
   **Hint**: use the Sylow theorems; specifically, show that any Sylow $q$-subgroup is normal in $G$.