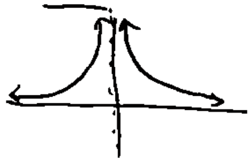
9/19

More ways limits can fail to exist: §2.2

So far we only saw one example of a limit not existing, and it was when the two one-sided limits disagreed.

But limits can fail to exist for many reasons.

E.g. Consider $f(x) = \frac{1}{x^2}$. for x near zero,



$f(x)$ will be a big positive #, and gets bigger + bigger as x gets closer+closer to 0.

So $\lim_{x \to 0} \frac{1}{x^2}$ does not exist.

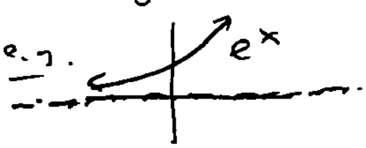
In this case we write $\lim_{x \to 0} \frac{1}{x^2} = \infty$ to mean that as x gets closer to zero (on either side), $f(x)$ can be made arbitrarily big.

Note: $\lim_{x \to a} f(x) = \infty$ (or $\lim_{x \to a} f(x) = -\infty$) counts as the limit not existing.

Compare: If $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$

then $f(x)$ has a "horizontal asymptote at $y = L$."

e.g.

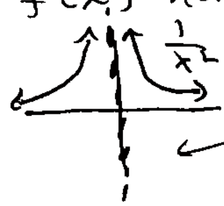


hor.z. asymptote at $y = 0$

If $\lim_{x \to a} f(x) = \infty$ or $\lim_{x \to a} f(x) = -\infty$

then $f(x)$ has a "vertical asymptote at $x = a$"

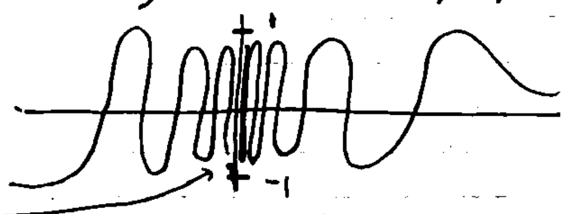
e.g.



vertical asymptote at $x = 0$.

Limits can fail to exist for even more "complicated" reasons:

<u>E.g.</u> Let $f(x) = \sin\left(\frac{1}{x}\right)$. We saw a while ago that its graph looks like:



as we bring x closer and closer to zero, $\frac{1}{x}$ passes through many values, so $\sin\left(\frac{1}{x}\right)$ passes through many periods. In each period, it attains a max. value of 1 and a min. value of -1.

Thus, <u>near zero</u>, there are <u>many x</u> for which $\sin\left(\frac{1}{x}\right) = 1$ and many for which $\sin\left(\frac{1}{x}\right) = -1$.

Since it oscillates rapidly between these values, there is no single value that $f(x)$ approaches as x gets closer to zero. Thus,

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) \text{ does not exist.}$$

In fact, neither of $\lim\limits_{x \to 0^-} \sin\left(\frac{1}{x}\right)$ or $\lim\limits_{x \to 0^+} \sin\left(\frac{1}{x}\right)$ exist either. (So this limit fails to exist not because of a disagreement between one-sided limits or because function is unbounded, but because it "oscillates too much"...)

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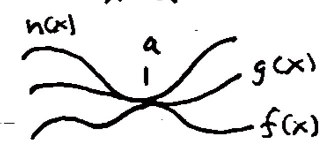
The Squeeze Theorem §2.3

Sometimes we can calculate a limit for a function $f(x)$ by comparing it to other functions in size.

Thm If $f(x) \le g(x)$ for x near a (except possibly at a) and the limits of f + g at a both exist, then
$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

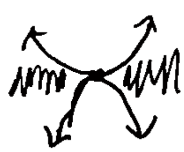
Thm (Squeeze Theorem) If $f(x) \le g(x) \le h(x)$ for x near a (except possibly at a) and $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$ then $\lim_{x\to a} g(x) = L$.

"picture":



Eg. Let's use the squeeze theorem to compute
$$\lim_{x\to 0} x^2 \sin\left(\tfrac{1}{x}\right).$$
Note we cannot use product law for limits here since $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist. But...

Since $\sin(\frac{1}{x})$ is always between -1 and 1, have
$$-x^2 \le x^2 \sin\left(\tfrac{1}{x}\right) \le x^2 \quad \forall x$$



so that we can apply squeeze thm with $\lim_{x\to 0} -x^2 = \lim_{x\to} x^2 = 0$

and thus $\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$.

(Even though $x^2 \sin(\frac{1}{x})$ "oscillates" a lot as $x \to 0$, amplitude of waves gets smaller and smaller ...)

9/21 <u>More about one-sided limits + limits at ∞</u> §2.6

Basically all of the laws/theorems for limits also hold for one-sided limits and limits at infinity.

<u>E.g.</u> we have $\lim_{x\to a^-} f(x)+g(x) = \lim_{x\to a^-} f(x) + \lim_{x\to a^-} g(x)$

$\lim_{x\to\infty} f(x)\cdot g(x) = \lim_{x\to\infty} f(x) \cdot \lim_{x\to\infty} g(x)$, etc.

(when these limits exist) and even

<u>Thm</u> If $f(x) \leq g(x)$ then $\lim_{x\to\infty} f(x) \leq \lim_{x\to\infty} g(x)$

and versions of the squeeze thm, etc.

Perhaps the one additional law for limits at ∞ is:

<u>Prop.</u> For any integer $r>0$, have

$$\lim_{x\to\infty} \frac{1}{x^r} = \lim_{x\to-\infty} \frac{1}{x^r} = 0.$$

(For $r>0$ have $\lim_{x\to\infty} x^r = \infty$ and $\lim_{x\to-\infty} x^r = \begin{cases} +\infty & r \text{ even} \\ -\infty & r \text{ odd} \end{cases}$)

Let's see one example how to use this:

<u>E.g.</u> $$\lim_{x\to\infty} \frac{3x^2-2x+4}{5x^2+x-7} \overset{\text{divide top + bottom by } x^2}{=} \lim_{x\to\infty} \frac{3-\frac{2}{x}+\frac{4}{x^2}}{5+\frac{1}{x}-\frac{7}{x^2}}$$

quotient law

$$= \frac{\lim_{x\to\infty} 3-\frac{2}{x}+\frac{4}{x^2}}{\lim_{x\to\infty} 5+\frac{1}{x}-\frac{7}{x^2}} = \cdots = \frac{3-2\cdot 0+4\cdot 0}{5+1\cdot 0-7\cdot 0}$$

<u>upshot</u>: only "leading term" matters at ∞. $= 3/5$.

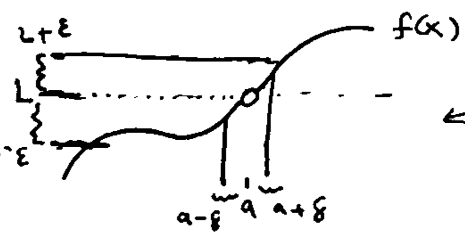
Precise definition of limit. §2.4

The way we defined a limit so far has been a little vague because of imprecise terms like "near" and "close to". The precise definition of a limit is:

Def'n Let $f(x)$ be a function defined on an open interval containing $a \in \mathbb{R}$, except possibly at a itself.
We say $\lim_{x\to a} f(x) = L$ for a number $L \in \mathbb{R}$ if:
- for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all x with $0 < |x-a| < \delta$, we have that $|f(x) - L| < \varepsilon$.

Think: However close ($\varepsilon > 0$) we desire the output ($f(x)$) to be to the ~~target~~ limit value (L), we can get that close by requiring the input (x) to be close ($\delta > 0$) to limit point (a).

Graphically:

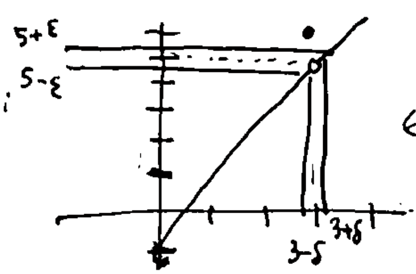


← output $f(x)$ between $L-\varepsilon$ and $L+\varepsilon$ when input x between $a-\delta$ and $a+\delta$.

Let's see an example of showing that

$$\lim_{x\to 3} f(x) = 5 \quad \text{when } f(x) = \begin{cases} 2x-1 & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$

Graphically:



← looks like we can find a narrow band of inputs to fall into a narrow band of outputs.

Think: my "enemy" gives me $\varepsilon > 0$. I need to find a $\delta > 0$ so that $|f(x) - 5| < \varepsilon$,

i.e. $5 - \varepsilon < f(x) < 5 + \varepsilon$,

for all x with $0 < |x - 3| < \delta$,

i.e. $3 - \delta < x < 3 + \delta$ and $x \neq 3$.

A good choice for this $f(x)$ is $\delta = \frac{\varepsilon}{2}$.

Indeed, if $3 - \delta < x < 3 + \delta$ (and $x \neq 3$)

that means $3 - \frac{\varepsilon}{2} < x < 3 + \frac{\varepsilon}{2}$

So that $6 - \varepsilon < 2x < 6 + \varepsilon$

i.e., $5 - \varepsilon < 2x - 1 < 5 + \varepsilon$

which is $5 - \varepsilon < f(x) < 5 + \varepsilon$,

what we wanted to show!

We see how this definition captures the idea of "the function gets close to a particular value at inputs near where we want to compute the limit" precisely. But in practice finding the "right" δ in terms of ε can be quite tricky. Let us give one example of what formally proving the limit laws looks like using the ε-δ definition of limit.

Thm If $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist,

then $\lim\limits_{x\to a} (f(x) + g(x)) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$,

<u>Pf</u>: Let $L_1 = \lim_{x\to a} f(x)$ and $L_2 = \lim_{x\to a} g(x)$.
We want to show that $\lim_{x\to a}(f(x)+g(x)) = L_1 + L_2$.
So let $\varepsilon > 0$ be given.
By supposition that $\lim_{x\to a} f(x) = L_1$, there is a $\delta_1 > 0$ s.t. $|f(x) - L_1| < \varepsilon/2$ for all

$$0 < |x-a| < \delta_1.$$

Similarly, there is a $\delta_2 > 0$ s.t. $|g(x) - L_2| < \frac{\varepsilon}{2}$ for all

$$0 < |x-a| < \delta_2.$$

Set $\delta = \min(\delta_1, \delta_2)$. Then for all $0 < |x-a| < \delta \le \delta_1, \delta_2$ have

$$|f(x) - L_1| < \frac{\varepsilon}{2} \text{ and } |g(x) - L_2| < \frac{\varepsilon}{2}$$

So that

$$\begin{aligned} &|(f(x)+g(x)) - (L_1+L_2)| \\ &= |(f(x)-L_1) + (g(x)-L_2)| \\ &\le |f(x)-L_1| + |g(x)-L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

which is exactly what we wanted to show. ⊠

See that ε-δ style arguments can be rather tricky, and also sometimes tedious.

<u>Exercise</u>: Prove another limit law using ε-δ definition of limit.

From now on, we will compute limits (and derivatives, etc.) using the laws, and not ε-δ definition.

9/23 Continuity §2.5

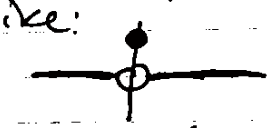
Recall that we say $f(x)$ is continuous at a if $f(a) = \lim_{x\to a} f(x)$. This requires 3 things:

- $f(x)$ is defined at $x=a$, i.e. $a \in$ domain of f,
- $\lim_{x\to a} f(x)$ exists,
- and $f(a)$ and $\lim_{x\to a} f(x)$ are the same number.

If $f(x)$ is not continuous at a we say it is discontinuous there.

Most of the examples of discontinuity we've seen were piecewise functions like:

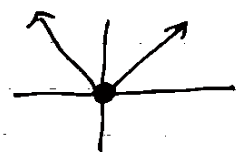
$$f(x) = \begin{cases} 1 & \text{if } x=0 \\ 0 & \text{otherwise} \end{cases}$$



where the function "jumps" suddenly at a point.

But note that not all piecewise functions are discontinuous, e.g. the absolute value function

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



is continuous even at $x=0$.

The reason examples of discontinuity we've seen look "contrived" is because

Thm The following kinds of functions are continuous at all points in their domain:

- polynomials
- rational functions
- root functions like $\sqrt{x}$
- trig functions like $\sin(x)$ and $\cos(x)$
- exponentials like e^x
- logarithms like $\ln(x)$.

Furthermore...

Thm If f and g are continuous at a, then so are
• $f+g$ • $f-g$ • $f\cdot g$ • $\frac{f}{g}$ if $g(a)\neq 0$ • $c\cdot f$ for any constant $c\in\mathbb{R}$.

And we can even say the following about composition:

Thm If $\lim_{x\to a} g(x) = b$ and f is continuous at b,
then $\lim_{x\to a} f(g(x)) = f(b)$ $\left(= f\left(\lim_{x\to a} g(x)\right)\right)$

"Can push the limit thru continuous functions"

Cor If g is continuous at a and f is continuous at g(a)
then composite $f\circ g$ is continuous at a.

Upshot: All the ways of combining all the "normal" functions we've considered give functions continuous at all pts in their domain.
So... to compute a limit of a function like this, try plugging in!

E.g. $\lim_{x\to 0} \sin\left(\frac{\pi}{2}\cdot e^x\right) = \sin\left(\frac{\pi}{2}\cdot e^0\right) = \sin\left(\frac{\pi}{2}\right) = 1.$

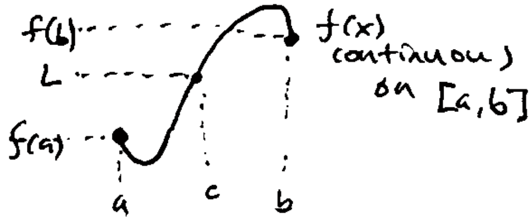
Warning: Remember $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ D.N.E.
But 0 is not in the domain of $\sin\left(\frac{1}{x}\right)$.

One more important property of continuous functions:

Thm Let f(x) be continuous on some closed interval [a, b].
Then for every L w/ $f(a)\leq L\leq f(b)$, there is a $c\in[a,b]$ w/ $f(c)=L$.

Called the "intermediate value theorem"

It says f takes on all values "intermediate" between f(a) and f(b).



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9/26 **The derivative as a function** §2.8

Recall that we defined the derivative of $f(x)$ at a in 2 ways:

- the slope of the tangent to the curve $y=f(x)$ at $(a, f(a))$
- the limit $\displaystyle\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

We were thinking of the point a as fixed. But now let us consider the point we're taking the derivative at to vary. Thus, we define the derivative at x:

$$f'(x) := \lim_{K\to x} \frac{f(K)-f(x)}{K-x}$$

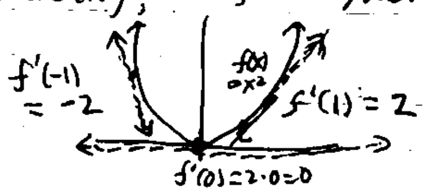
We think of $f'(x)$ as a new function defined from $f(x)$.

E.g. Let's compute $f'(x)$ for $f(x)=x^2$.

$$f'(x) = \lim_{K\to x} \frac{f(K)-f(x)}{K-x} = \lim_{K\to x} \frac{K^2-x^2}{K-x}$$

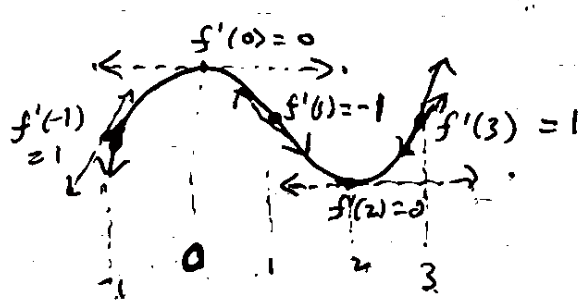
"difference of squares" $\rightsquigarrow$ $$= \lim_{K\to x} \frac{\cancel{(K-x)}(K+x)}{\cancel{(K-x)}} = \lim_{K\to x} K+x = \underline{\underline{2x}}$$

Graphically, this answer makes sense in terms of tangent lines:



← tangent slope is negative for $x<0$, positive for $x>0$

E.g. We can estimate $f'(x)$ from a graph of $f(x)$ using tangents:



← We see that

- $f'(x)>0 \iff f$ is increasing at x
- $f'(x)<0 \iff f$ is decreasing at x
- $f'(x)=0 \iff f$ has a local min./max. at x

Def'n We say $f(x)$ is differentiable at x if $f'(x)$ exists.
Since it's a limit, it doesn't have to exist!
In fact, we have the following important theorem

Theorem If $f(x)$ is differentiable at x, then it is continuous at x.

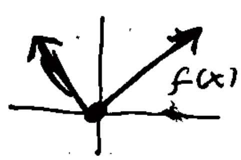
E.g. Let $f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 1 \end{cases}$

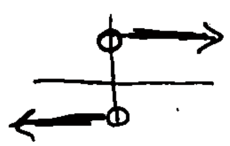
Then, since $f(x)$ is not continuous at $x=0$, $f'(0)$ does not exist ("is not defined" or).

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h - 0} = \lim_{h \to 0} \frac{0-1}{h} = \lim_{h \to 0} \frac{-1}{h} \text{ D.N.E.}$$

But... there are other ways $f(x)$ can fail to be differentiable

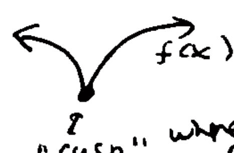
E.g. Let $f(x) = |x|$. We mentioned before that $f(x)$ is continuous at $x=0$. But it is not differentiable at $x=0$.





Indeed, for $x>0$, have $f'(x) = 1$ since slope is clearly $= 1$.
For $x < 0$, have $f'(x) = -1$ since slope is $= -1$.
But at $x = 0$, slopes on left- and right-hand sides disagree, so cannot define derivative as a single number.

In general, a major way differentiability can fail is if there is a "crease" (or "cusp") $\Rightarrow$



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