Total: 49/50 + 5 bonus points for presenting = 54/50

Caleb Dahme

1 Note that a 180° rotation of a square RPP (sh $\lambda = n \times n$) produces the upper left corner of a plane partition, so consider the formula $\sum_{\pi \in RPP(\lambda)} q^{|\pi|} = \prod_{u \in \lambda} \frac{1}{1-q^{h(u)}}$ with regards to such square RPP's.
Notice that for $n \times n$ RPP's, there are i boxes with hook length i for $i \leq n$. However, once we flip the RPP to form the upper left corner of the plane partition, this is true for all i, since the hook length would be equivalent to the number of boxes directly up & directly to the left (including itself). Thus there are i copies of $\frac{1}{1-q^i}$ for all i in the product, so we obtain $\sum_{\pi \text{ a plane partition}} q^{|\pi|} = \prod_{i \geq 1} \frac{1}{(1-q^i)^i}$.

Ok. 10/10

4 From toggle-based RSK, we know $M \xmapsto{RSK} (P,Q) \Rightarrow M^t \xmapsto{RSK} (Q,P)$.

10/10

Suppose M is the permutation matrix of an involution in S_n. Then $M^t = M$ ($\because$ each 1 in M is either at $M_{i,i}$ or at $M_{i,j}$ with a matching 1 at $M_{j,i}$). Also, since M is a permutation matrix, $RSK(M)$ is the same as $RS(M)$. But since $M \xmapsto{RSK} (P,Q) = M \xmapsto{RSK} (Q,P)$, we must have $P = Q$, so we're really looking at $M \xmapsto{RS} (P,P)$, i.e. a bijection between involutions in S_n and SYT of $sh(\lambda \vdash n)$.
Therefore $\#\{\sigma \in S_n \mid \sigma = \sigma^{-1}\} = \sum_{\lambda \vdash n} f^\lambda$.

Yes. (Put more simply: inversion of permutations corresponds to transposition of permutation matrix.)

RSK on perm. matrix becomes RS

$M \xmapsto{RSK} (P,Q)$, then $M^t \xmapsto{RSK} (Q,P)$

$M = M^t$ iff $P = Q$, i.e. M is an involution

2a Antichains have the greatest number of linear extensions, since the elements can appear in any order without violating the requisite relationships. Chains have the least, since the elements in a chain can only be listed in one order for a linear extension. Good.

10/10

b Consider $f : \mathcal{L}(P) \to \mathcal{L}(P^*)$ defined by $f(E) = f(E_1, E_2, \dots, E_n) = (E_n, \dots, E_2, E_1)$ for any linear extension $E \in \mathcal{L}(P)$ (i.e. writing the linear extension backwards). This uniquely defines a new list E' that has all the relationships from E reversed, so $E_i \le E_j \Rightarrow E'_j \le E'_i$. Thus E' is a linear extension of P^* (since $E_i \le E_j \Rightarrow i \le_P j$ and $E_i \le E_j \Rightarrow E'_j \le E'_i \Rightarrow j \le_{P^*} i$). Since f is reversible, we must have $\#\mathcal{L}(P) = \#\mathcal{L}(P^*)$. Good.

c Since P and Q are in disjoint union, we can combine arbitrary linear extensions of P & Q to form a linear extension of $P \cup Q$. Then we need only determine how many ways we can combine the chosen linear extensions into a single list. Suppose we wanted to insert $p \in \mathcal{L}(P)$ into $q \in \mathcal{L}(Q)$. Each element of p could be placed before any element of q, or after the last element of the list; this process is then repeated for each element of p, and is analogous to putting n balls into $m+1$ boxes, or the 'stars & bars' problem with n stars & $m+1$ bars. Thus we obtain $\binom{n+m+1-1}{n} = \binom{n+m}{n}$ ways to weave the lists together while maintaining the order within each list. Thus, there are $\#\mathcal{L}(P) \cdot \#\mathcal{L}(Q) \cdot \binom{n+m}{n}$ linear extensions of $P \cup Q$.

Good.

3 Note that, since a SYT has a $<$ relationship both along rows & columns, choosing which k numbers from $[n]$ appear in the first part of the SYT determines the entire SYT, since there's only one way to arrange the numbers along the row, and the same is true for the remaining values used to populate the column. However, since 1 must go in the upper-left corner, there are really only $k-1$ choices from $n-1$ values. Therefore $f^\lambda = \binom{n-1}{k-1}$ for hook-shaped partitions $\lambda = (k, \underbrace{1, \dots, 1}_{n-k})$ for $1 \le k \le n$.

10/10

Very nice. Can also use Hook Length Formula, but this argument is simpler.

9/10

5 For $\sigma \in S_n$, we can maximize $\min(\text{lis}(\sigma), \text{lds}(\sigma))$ with the following construction: If $n = 2k+1$, consider permutations that contain both the increasing sequence $1, 2, \ldots, k+1$ and the decreasing sequence $2k+1, 2k, \ldots, k+1$. (e.g. $\sigma = (1, 2k+1, 2, 2k, 3, \ldots, k+1)$ or $\sigma' = (1, 2, \ldots, k, 2k+1, 2k, \ldots, k+1)$.) Then $\min(\text{lis}(\sigma), \text{lds}(\sigma)) = k$, and any permutation not of this form will shorten either $\text{lis}(\sigma)$ or $\text{lds}(\sigma)$ (since extending either $\text{lis}(\sigma)$ or $\text{lds}(\sigma)$ decreases $\min(\text{lis}(\sigma), \text{lds}(\sigma))$).
For $n = 2k$, the relevant permutations will contain the sequences $1, 2, \ldots, k$ and $2k, 2k-1, \ldots, k+1$, giving $\min(\text{lis}(\sigma), \text{lds}(\sigma)) = k$.
(Even though one of $k, k+1$ is followed by the other, we care about the min. length of the two sequences, so we ignore whichever of the two is extended.)

This is a good heuristic, but why is this the maximum? Answer: because an increasing and decreasing subsequence can intersect in at most one element. You should include that explanation.