Total: 47 points + 5 bonus pts for presenting = 52/50

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1 | Unweighted Pólya counting tells us the number of k-ary necklaces of length n is $1/\#G \sum_{g\in G} (\#Y)^{c(g)}$. It is apparent from the definition of the necklaces that $(\#G) = n$ and $(\#Y) = k$. Consider necklaces made of a repeated subnecklace (e.g. a square colored $\begin{smallmatrix} R-B \\ B-R \end{smallmatrix}$ is made of the subnecklace R-B repeated twice). We must adjust for the number of times such a coloring can be rotated. If the subnecklace is repeated d times, then the necklace can be rotated n/d times (i.e. is fixed by a rotation of order d, which there are $\varphi(d)$ of). Since the subnecklace must be repeated evenly, we sum over factors of n, obtaining the formula $\frac{1}{n}\sum_{d\mid n}\varphi(d)k^{n/d}$.

why $\varphi(d)$?

This is basically the right type of reasoning, but you're not using (unweighted) Polya counting here. To use Polya counting, think about the number of cycles of elements of the group <(1,2,…,n)> generated by an n-cycle. How many cycles various elements in this group have depends on things like gcd's. E.g., if we square (1,2,3,4,5,6) we get (1,3,5)(2,4,6) which has 2 3-cycles because gcd(6,2) = 2. Similarly, the 4th power of (1,2,…,6) consists of 2 3-cycles as well, since gcd(6,4)=2 as well. But if we look at instead the 3rd power of (1,2,…,6) we get (1,4)(2,5)(3,6) which consists of 3 2-cycles, since gcd(6,3) = 3. [-2pts]

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2 | Weighted Pólya counting gives us $\frac{1}{6}(t_1^6 + t_6 + t_3^2 + t_2^3 + t_3^2 + t_6) = \frac{1}{6}(t_1^6 + t_2^3 + 2t_3^2 + 2t_6) \Rightarrow \frac{1}{6}\left[(y_1+y_2+y_3)^6 + (y_1^2+y_2^2+y_3^2)^3 + 2(y_1^3+y_2^3+y_3^3)^2 + 2(y_1^6+y_2^6+y_3^6)\right]$. Since we want the coeff. of $y_1^2y_2^2y_3^2$, we can ignore $2(y_1^3+y_2^3+y_3^3)^2 + 2(y_1^6+y_2^6+y_3^6)$. There are $\binom{6}{2}\binom{4}{2}$ ways to get $y_1^2y_2^2y_3^2$ from $(y_1+y_2+y_3)^6$, and a $(3\cdot 2)$ ways to get $y_1^2y_2^2y_3^2$ from $(y_1^2+y_2^2+y_3^2)^3$, for a total of $\frac{1}{6}(90+6) = \frac{1}{6}(96) = \boxed{16 \text{ colorings}}$

Good, although you could explain exactly how you got the cycle index polynomial in this case.
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$= 1/6(\overset{729}{k^6} + \overset{27}{k^3} + \overset{18}{2k^2} + \overset{6}{2k}) \overset{3}{=} 780/6 = 1$

$\overset{3}{=} 255/5 = 51$

$= 1/5(k^5 + 4k) \overset{2}{=} 40/5 = 8$ $\begin{smallmatrix} 0 & 1 & 2 & 2 \\ 3 & 3 & 4 & 5 \end{smallmatrix}$

$= \frac{1}{4}\sum_{\{1,2,4\}}\varphi(d)k^{n/d}$

$= \frac{1}{4}(\overset{81}{k^4} + \overset{9}{k^2} + \overset{6}{2k})$

$= 1/4(96) = 24$

Pólya U | # orbits$(G \curvearrowright Y^X) = 1/\#G \sum_{g\in G}(\#Y)^{c(g)}$

W | $P(y_1,\dots,y_k) = Z_G\left(\sum_{i\in Y} y_i, \sum_{i\in Y} y_i^2, \dots, \sum_{i\in Y} y_i^n\right)$

3 The rotations of a cube can be divided into the following types:

- Rotation about an axis between opposite corners (8 rotations, 2 orbits, k^2 fixed)
- Rotation about an axis between opposite edges (6 rotations, 3 orbits, k^3 fixed)
- Rotation about an axis between the centers of opposite faces
  - 90° rotation (6 rotations, 3 orbits, k^3 fixed)
  - 180° rotation (3 rotations, 4 orbits, k^4 fixed)
- Identity (1 rotation, 6 orbits, k^6 fixed)

I was confused at first what you meant by "k^6 fixed" but now I see that you mean that there are k^6 colorings fixed by this element, which is correct. Got it.

Thus from unweighted Pólya counting we get $\frac{1}{24}(k^6+3k^4+12k^3+8k^2)$ colorings.

Good. 10/10

4 Weighted Pólya counting gives us $\frac{1}{24}(t_1^6+3t_1^2t_2^2+6t_1^2t_4+6t_2^3+8t_3^2)$

$\Rightarrow \frac{1}{24}\left[(y_1+y_2+y_3)^6+3(y_1+y_2+y_3)^2(y_1^2+y_2^2+y_3^2)^2+6(y_1+y_2+y_3)^2(y_1^4+y_2^4+y_3^4)\right.$
$\left.+6(y_1^2+y_2^2+y_3^2)^3+8(y_1^3+y_2^3+y_3^3)^2\right].$

We want the coeff. of $y_1^2y_2^2y_3^2$, so we can ignore $6(y_1+y_2+y_3)^2(y_1^4+y_2^4+y_3^4)$ and $8(y_1^3+y_2^3+y_3^3)^2$. There are $\binom{6}{2}\binom{4}{2}$ ways to get $y_1^2y_2^2y_3^2$ from $(y_1+y_2+y_3)^6$, $3\cdot(3\cdot 2)$ ways to get $y_1^2y_2^2y_3^2$ from $(y_1+y_2+y_3)^2(y_1^2+y_2^2+y_3^2)^2$, and $6\cdot(3\cdot 2)$ ways to get $y_1^2y_2^2y_3^2$ from $(y_1^2+y_2^2+y_3^2)^3$, for a total of $\frac{1}{24}(90+18+36)=\frac{1}{24}(144)=\boxed{6 \text{ colorings}}$

Very good, though again could explain how you got the cycle index polynomial. 10/10

5 We can consider $M_{n\times m}(k)$ to be a collection of n 'gemstones' (acted on by S_n), each with m ordered colorable facets. Nice way of putting it.
Unweighted Pólya counting tells us $\#\tilde{M}_{n\times m}(k)=\frac{1}{|S_n|}\sum_{\mu\in S_n}(\#Y)^{c(g)}$. $|S_n|=n!$. $\#Y$, the number of colorings for each gemstone, is k^m. The unsigned Stirling numbers of the first kind $c(n,j)$ count permutations of S_n with j cycles, so we can sum over $1\le j\le n$ instead of over $\mu\in S_n$. Thus we obtain the formula: $\frac{1}{n!}\sum_{j=1}^{n}c(n,j)(k^m)^j$.

Good, though you should write down what happens when we simplify using the generating function for the Stirling numbers from the hint (we just get a binomial coefficient, like you saw in your presentation). [-1pt]

$\frac{1}{n!}\sum_{\mu\in S_n}(k^m)^{c(g)}=\frac{1}{n!}\sum_{j=1}^{n}c(n,j)(k^m)^j$

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