Symmetry of Narayana Numbers and Rowvacuation of Root Posets

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Wednesday, October 28th, 2020

BIRS 2020 Online Workshop on Dynamical Algebraic Combinatorics

see slides + write-up on my webpage

(this talk is being recorded)

ALTERNATE TALK:
"Order polynomial product formulas + poset dynamics"

W Weyl group, Φ root system.

The W-Catalan number

$$\mathrm{Cat}(W) = \prod_{i=1}^{r} \frac{h+d_i}{d_i} \quad \text{is} \ldots$$

Reading

1) # vertices of W-associahedron
= # facets of cluster complex

Simion, Fomin, Zelevinsky, Reading...

2) # "noncrossing partitions"
= # $[e, c]_{\mathrm{abs}}$

Reiner, Athanasiadis, Brady-Watt, Bessis, ...

Shi

Cellini-Papi

3) # "nonnesting partitions"
= # antichains of Φ^+

Postnikov

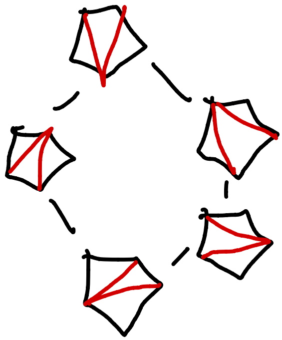
4) # W-orbits of $Q/(h+1)Q$

Haiman

5) # dominant regions of Shi arrangement

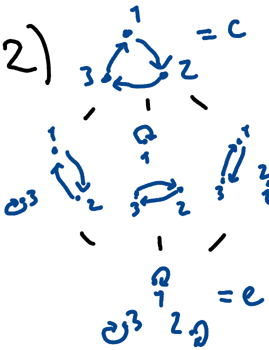
Shi

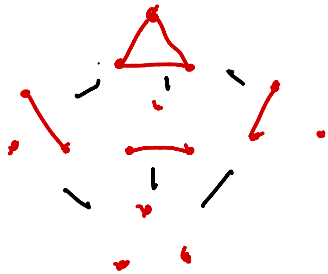
In Type A, we get familiar objects...

1) 

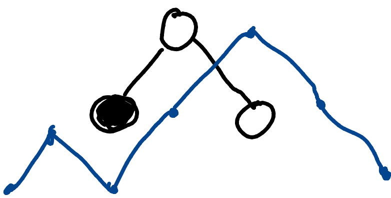
vertices of associahedron

= triangulations of n-gons

2) 

= 

lattice of noncrossing partitions

3) 

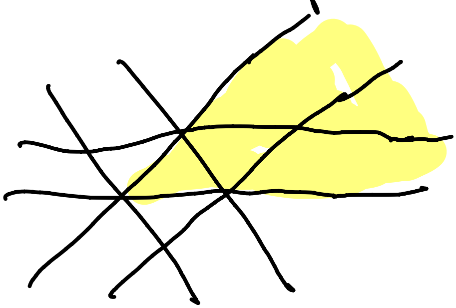
root poset antichains

= Dyck paths

4) $\mathbb{Q}/(h+1)\mathbb{Q}$ = parking fn's =

$$\left\{\begin{matrix} 210 & 201 & 120 \\ 102 & 021 & 012 \\ 200 & 020 & 002 \\ 110 & 101 & 011 \\ 100 & 010 & 001 \\ & 000 & \end{matrix}\right\}$$

W-orbits = decreasing parking fn's →

5)  ← Shi arr.

 = dominant cone

The W-Narayana number

$Nar(W,k)$ for $k=0,1,\dots,r$ is...

1) h_k(W-associahedron)

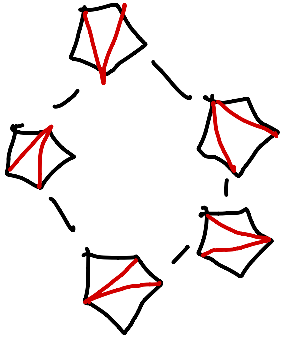
2) # noncrossing partitions of rank k

3) # antichains $A \in \mathcal{A}(\Phi^+)$ of cardinality k

4) # W-orbits of $Q/(h+1)Q$ w/ stabilizer of rank k

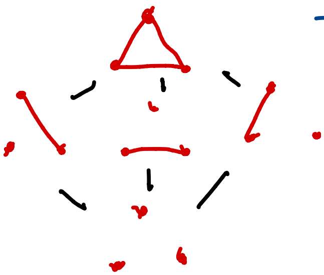
5) h_k(complex dual to dominant regions of Shi arrangement)

In Type A, ...

1) 

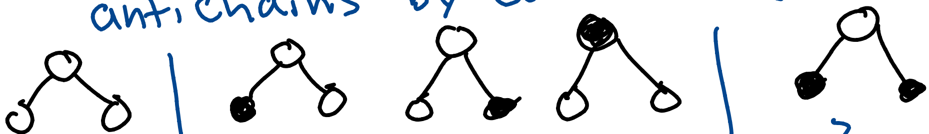
$f = (1, 5, 5, 1)$

$h = (1, 3, 1)$

2) 

rank	#
2	1
1	3
0	1

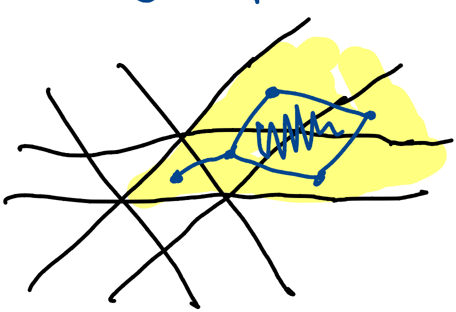
3) antichains by cardinality:



0	1	2

4) decreasing pf's by stabilizer rank:

210	200 110 100	000
0	1	2

5) 

$f = (1, 5, 5, 1)$

$h = (1, 3, 1)$

Curious symmetry of Narayana #'s :

$$\boxed{\mathrm{Nar}(W, k) = \mathrm{Nar}(W, r-k)}$$

1) For W-associahedron, follows from Dehn-Sommerville relations

2) For noncrossing partitions, follows from Self-duality of the lattice

3) For nonnesting partitions, ...
???

Panyushev's Problem:

Explain ??? bijectively!

Conjecture[#1] (Panyushev, "ad-nilpotent ideals...")
There's a "natural" involution
$$\mathfrak{P} : \mathcal{A}(\Phi^+) \to \mathcal{A}(\Phi^+) \text{ for which}$$
$$\# A + \# \mathfrak{P}(A) = r \quad \forall\, A \in \mathcal{A}(\Phi^+).$$

Panyushev could not define $\mathfrak{P}$ in general, but did give a definition for Type A, which turns out to be equivalent to the Lalanne-Kreweras involution on Dyck paths (subject of Mike's talk).

Types B/C follow easily from A via "folding" (a.k.a. symmetry).

But in a follow-up paper...

Conjecture #2 (Panyushev, "Orbits of antichains...")

Rowmotion $\mathrm{Row}: \mathcal{A}(\Phi^+) \to \mathcal{A}(\Phi^+)$ satisfies:

- $\mathrm{Row}^{2h}(A) = A$, and
- $\sum_{i=0}^{2h-1} \#\mathrm{Row}^i(A) = hr \quad \forall A \in \mathcal{A}(\Phi^+)$.

Conj. #1

Can partition $\mathcal{A}(\Phi^+)$ into subsets of size dividing 2 s.t. avg. of # in any part $= \frac{r}{2}$

Conj. #2

Can partition $\mathcal{A}(\Phi^+)$ into subsets of size dividing $2h$ s.t. avg. of # in any part $= \frac{r}{2}$

Conj. #2 was proved by Armstrong-Stump-Thomas.

It initiated a lot of recent DAC (homomesy, etc).

Rowmotion + Rowvacuation:

Recall (Cameron-Fon-der-Flaass, Striker-Williams)

Row: $\underline{J}(P) \to \underline{J}(P) = t_0 t_1 t_2 \cdots t_{r(P)}$,

where P is a graded poset of rank $r(P)$

and $t_i :=$ order ideal toggle at all elem.'s of rank $= i$

Rowvacuation Rvac: $\underline{J}(P) \to \underline{J}(P)$ is

$$\text{Rvac} := t_{r(P)} (t_{r(P)-1} t_{r(P)}) \cdots (t_1 \cdots t_{r(P)}) (t_0 \cdots t_{r(P)})$$

Note: Just like promotion + evacuation,

$t_i^2 = \text{identity}$

$+ \; t_i t_j = t_j t_i$ if $|i-j| > 1$

$\Rightarrow$

- $\text{Rvac}^2 = \text{identity}$
- $\text{Rvac} \cdot \text{Row} = \text{Row}^{-1} \cdot \text{Rvac}$

Antichain versions of Row, Rvac:

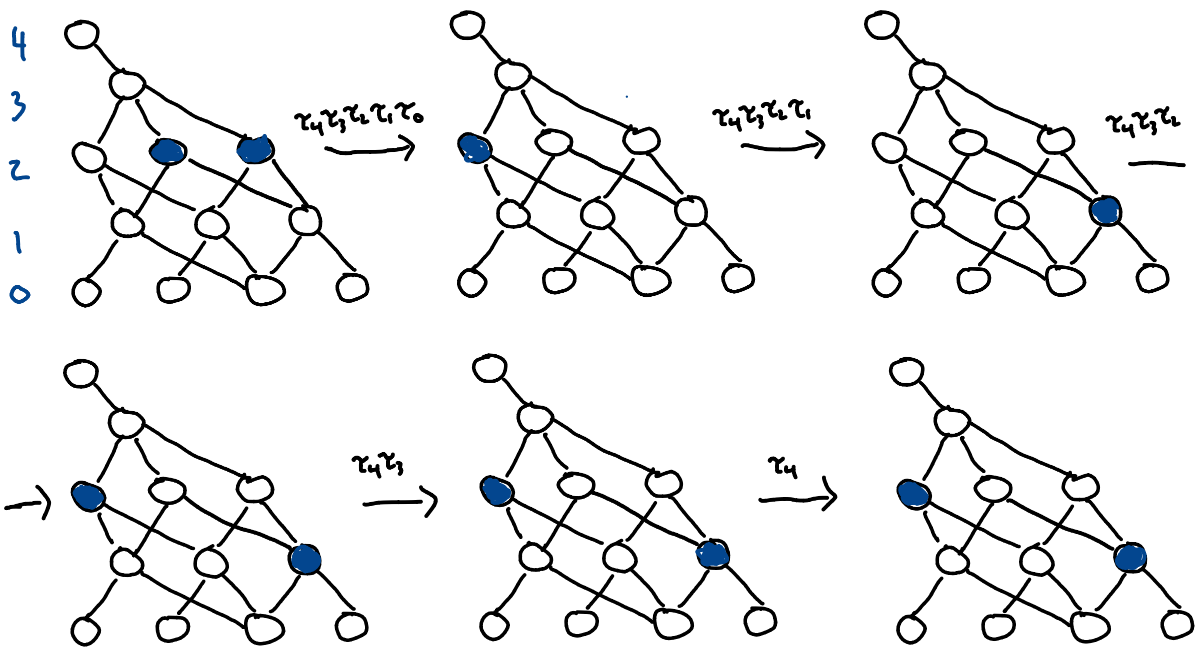
Prop. (Joseph) Row, Rvac: $\underline{\mathcal{A}}(P) \to \underline{\mathcal{A}}(P)$

satisfy $\mathrm{Row} = \tau_{r(P)} \cdots \tau_1 \tau_0$

$$\mathrm{Rvac} = \tau_{r(P)} \left(\tau_{r(P)} \tau_{r(P)-1}\right) \cdots \left(\tau_{r(P)} \cdots \tau_1\right) \left(\tau_{r(P)} \cdots \tau_0\right)$$

where $\tau_i :=$ antichain toggle at all elem.'s of rank $= i$

E.g. $P = \Phi^+(D_4)$



Conjecture For Φ of classical type A, B, C or D, $\#A + \#\mathrm{Rvac}(A) = r$ $\forall A \in \mathcal{A}(\Phi^+)$, so Rowvacuation is Panyushev's involution $\mathcal{P}$.

For Types A, B/C definitely true b/c ...

Theorem (H.-Joseph) $\mathrm{Rvac}: \mathcal{A}(\Phi^+(A_n)) \to \mathcal{A}(\Phi^+(A_n))$ is the Lalanne-Kreweras involution.

For D_n, verified for $n \leq 8$. ✓

Unfortunately fails for exceptional types F_4, E_n ... but even then works for 80%+ of antichains...

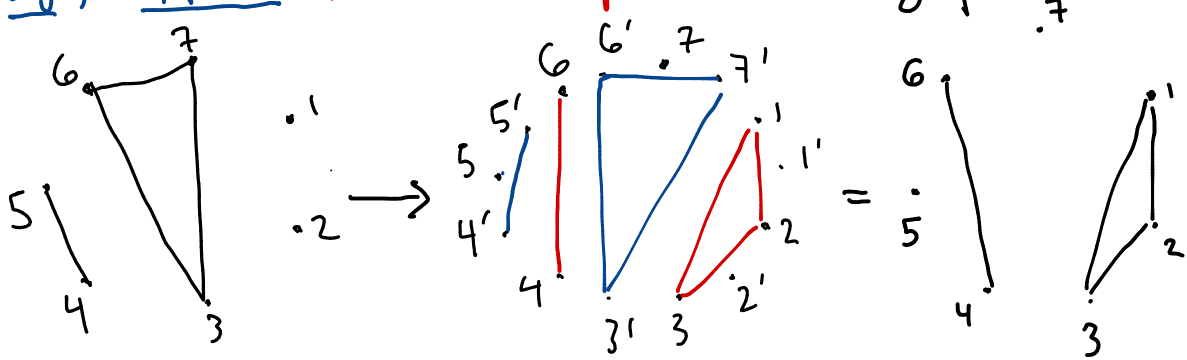
What's going on here?! Can it be fixed?!

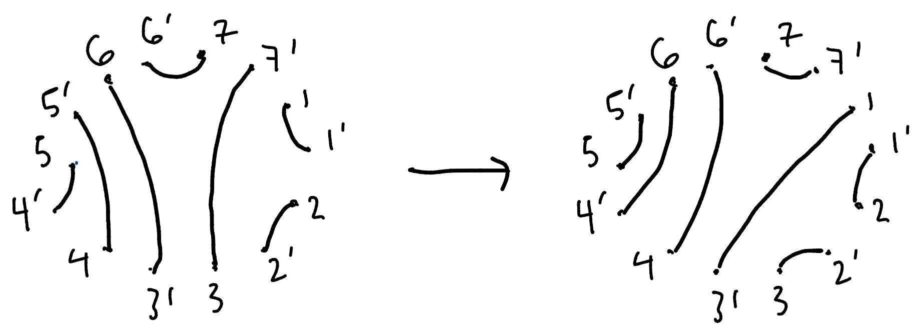
To resolve conjecture, helpful to review Armstrong-Stump-Thomas (AST) results...

Let $NC(W) = NC(W,c) := [e,c]_{abs}$, and recall Kreweras complement $Krew: NC(W) \to NC(W)$ is $Krew(w) := cw^{-1}$.

It's a self-duality of lattice $NC(W)$.

E.g., Type A: Kreweras comp. of noncrossing partitions





= rotation of noncrossing matchings

Answering a conjecture of Bessis-Reiner...

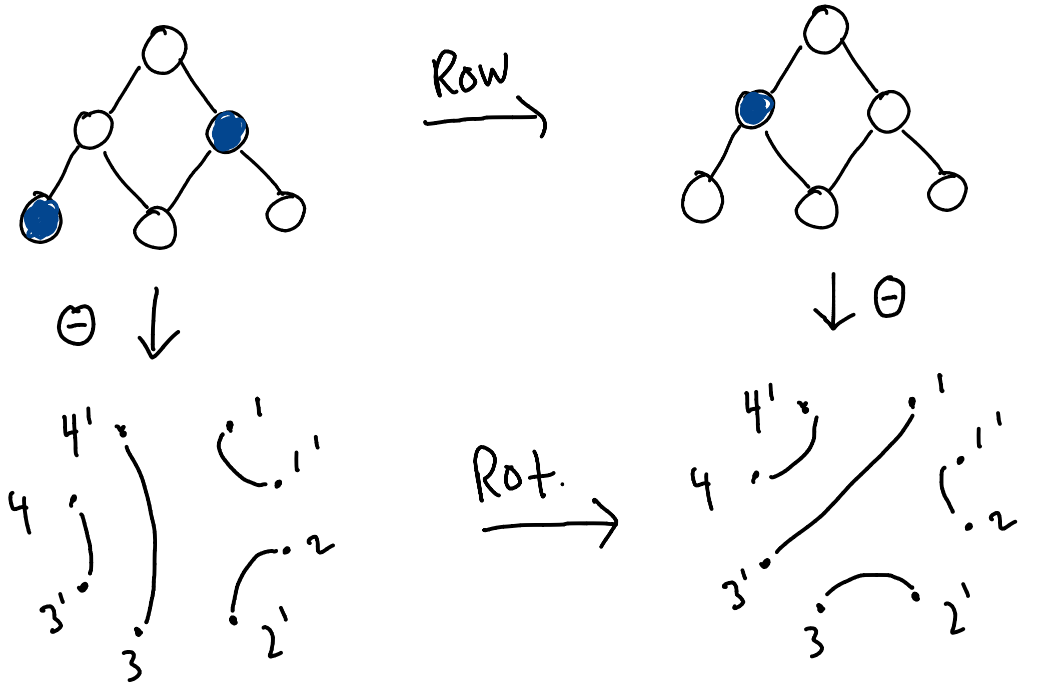
Theorem (AST) There's a unique bijection

$$\Theta : \mathcal{A}(\Phi^+) \to NC(W) \quad \text{s.t.}$$

- (base case)
- (equivariance) $\Theta \cdot \mathrm{Row} = \mathrm{Krew} \cdot \Theta$ ← Key property
- (parabolic induction)

Good	Bad
• uniformly stated	• not uniformly proved
• computationally efficient	• parabolic induction is hiding a lot...
• involves interesting dynamics	• Classical type proofs are via special models and are complicated!

Eg. Type A:



How to work rowvacuation into this?

Define Flip: $NC(W) \to NC(W)$ by

$$\mathrm{Flip}(w) := g w^{-1} g^{-1}$$

where $g \in W$ is correct involution for c:

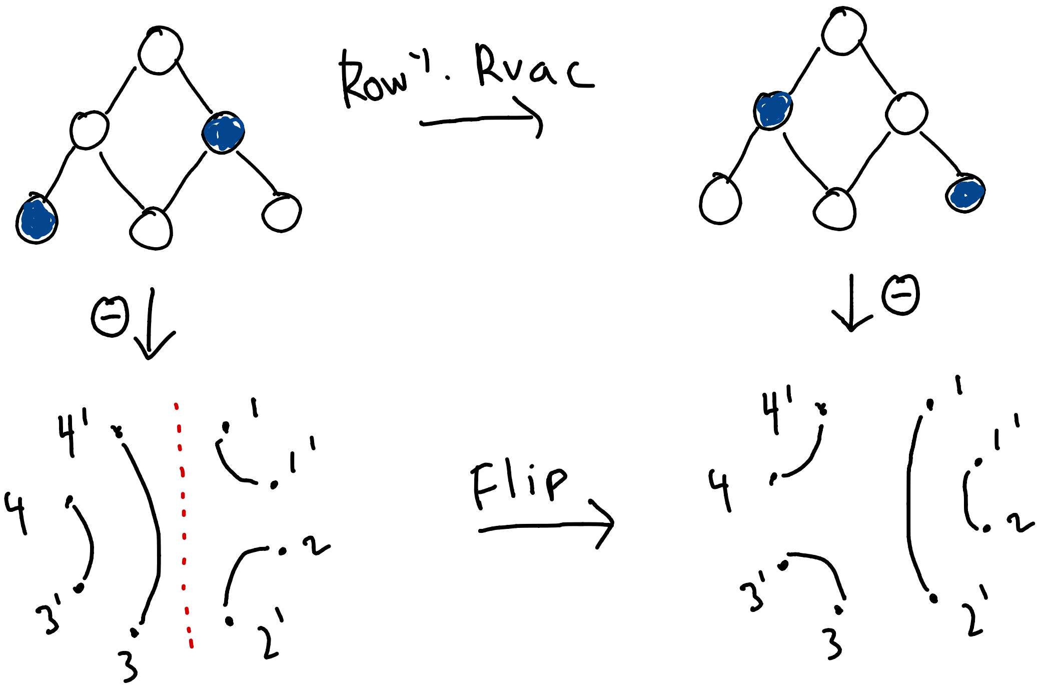
- c = standard Cox. elem. in Type A $\Rightarrow g = w_0$
- (N. Williams) $c = c_L c_R$ bipartite $\Rightarrow g = c_L$

Theorem For AST bijection $\Theta : A(\Phi^+) \to NC(W)$,

$$\Theta \cdot \mathrm{Row}^{-1} \cdot \mathrm{Rvac} = \mathrm{Flip} \cdot \Theta.$$

Follows just from general properties of Θ!

Eg. Type A: Flip = flip accross diameter



Idea: with a precise understanding of AST bij. Θ in Type D, could use this to resolve rowvacuation conjecture.

Thank you!

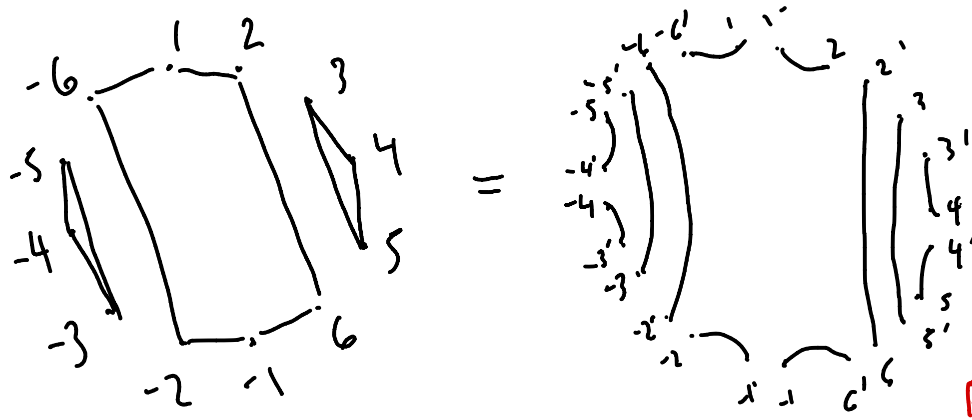
and

a Special Thanks

to the Organizers!

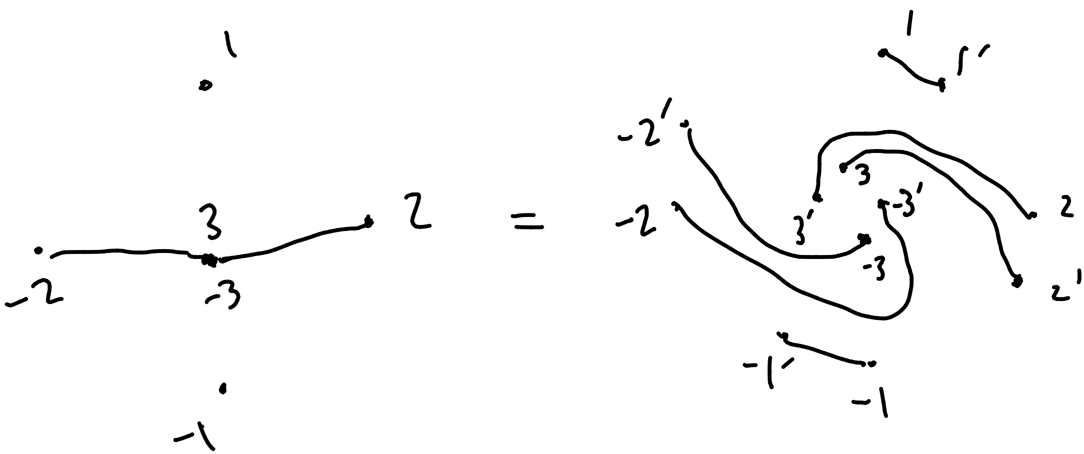
Secret bonus slide: NC(W) for W = B/D

Type B = 180° sym. Type A ✓



Reiner '97

Type D = complicated!



Athanasiadis-Reiner '04