Involutions on Dyck paths, and piecewise linear & birational lifts

ACPMS special 1-day seminar on Birational Combinatorics

Sam Hopkins (Howard University)
based on joint work with Michael Joseph (Dalton State College)

June 8th, 2022

Section 1

Catalan numbers, Dyck paths, Naryana numbers, and the Lalanne–Kreweras involution

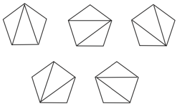
Catalan numbers

The **Catalan numbers** C_n are a famous sequence of numbers

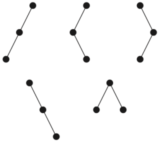
$$1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430, \ldots,$$

which count numerous combinatorial collections including:

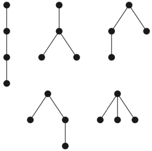
triangulations of an $n+2$-gon



binary trees with n nodes



plane trees with $n+1$ nodes



bracketings of $n+1$ terms

a(b(cd)) a((bc)d)
(ab)(cd) (a(bc))d
((ab)c)d

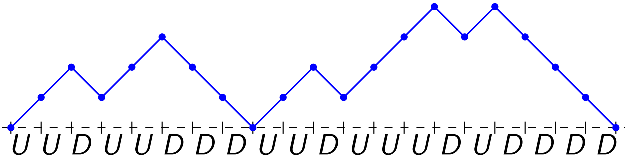
There is a well-known product formula for the Catalan numbers:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Dyck paths

The interpretation of C_n I want to focus on is in terms of Dyck paths.

A **Dyck path** of length $2n$ is a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to $(2n,0)$ consisting of n *up steps* $U=(1,1)$ and n *down steps* $D=(1,-1)$ that never goes below the x-axis:



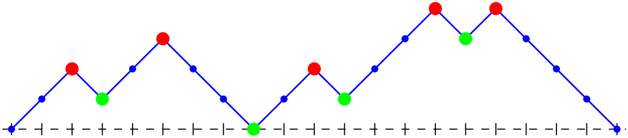
$n=10$

The number of Dyck paths of length $2n$ is C_n:

Peaks and valleys in Dyck paths

Dyck paths look like mountain ranges. So we use some topographic terminology when working with Dyck paths.

A **peak** in a Dyck path is an up step that is immediately followed by a down step; a **valley** is a down step immediately followed by an up step.



Here the peaks are marked by red circles and the valleys by green circles.

It's easy to see that a Dyck path which has k valleys has $k + 1$ peaks.

Narayana numbers

The **Narayana number** $N(n,k)$ is the number of Dyck paths of length $2n$ with exactly k valleys.

$n \setminus k$	0	1	2	3
1	1			
2	1	1		
3	1	3	1	
4	1	6	6	1

$\leftarrow$ array of $N(n,k)$

Evidently, the Narayana numbers $N(n,k)$ refine the Catalan number C_n:

$$C_n = \sum_{k=0}^{n-1} N(n,k).$$

They are named after *T.V. Narayana*, who in 1959 showed that

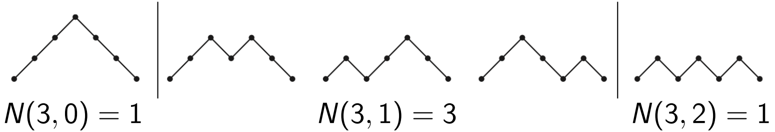
$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}.$$

Symmetry of Narayana numbers

From Narayana's formula, it follows immediately that

$$N(n,k) = N(n, n-1-k)$$

for all k. That is, the sequence of Narayana numbers is *symmetric*.

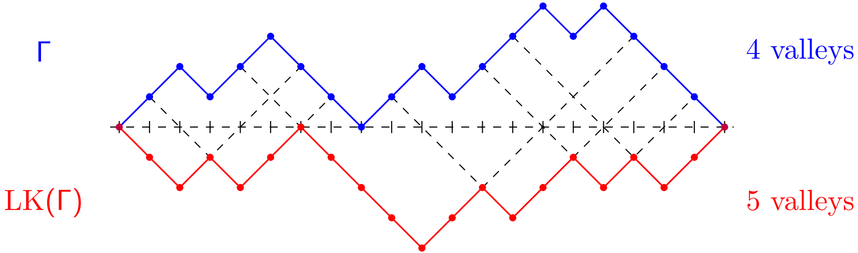


However, it is not combinatorially obvious why the number of Dyck paths with k valleys should be the same as the number with $n-1-k$ valleys.

The Lalanne–Kreweras involution

The **Lalanne–Kreweras involution** is a map on Dyck paths which combinatorially demonstrates the symmetry of the Narayana numbers:

$$\#\mathrm{valleys}(\Gamma) + \#\mathrm{valleys}(\mathrm{LK}(\Gamma)) = n - 1.$$



As depicted above, to compute the LK involution of a Dyck path Γ, we draw dashed lines emanating from the middle of every double up step and every double down step of Γ, at -45° and 45° respectively; these dashed lines intersect at the valleys of (an upside copy of) the Dyck path $\mathrm{LK}(\Gamma)$.

Section 2

Poset description of LK

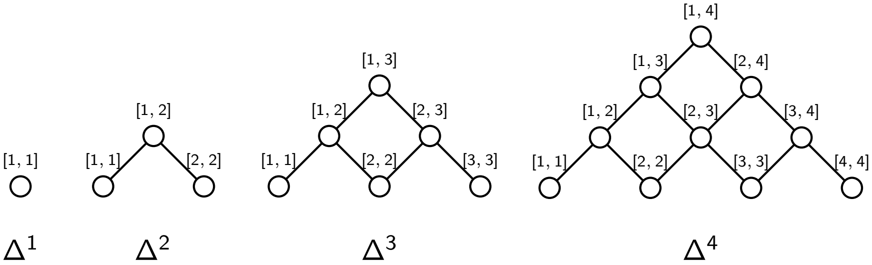
The poset Δ^{n-1}

We can reinterpret the LK involution using a partially ordered set Δ^{n-1}.

Δ^{n-1} is the poset whose elements are **intervals** $[i,j] := \{i, i+1, \ldots, j\}$ with $1 \leq i \leq j \leq n-1$, and with the partial order given by **inclusion**:

$$[i,j] \leq [i',j'] \iff [i,j] \subseteq [i',j'] \iff i \leq i' \leq j' \leq j$$

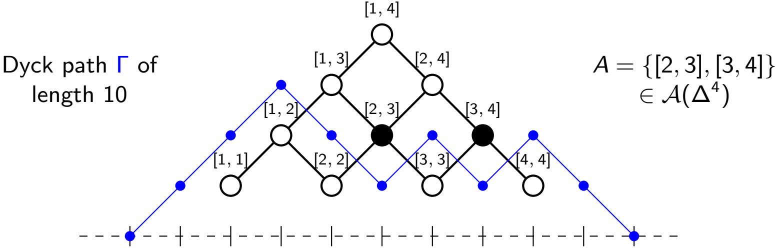
Δ^{n-1} has a "triangular" Hasse diagram:



Dyck paths are antichains in Δ^{n-1}

Recall that an **antichain** $A \subseteq P$ of a poset P is a subset of pairwise incomparable elements. We use $\mathcal{A}(P)$ to denote the set of antichains of P.

The Dyck paths of length $2n$ are in bijection with the antichains of Δ^{n-1}:



The number of valleys of Dyck path Γ is the cardinality of antichain A.

Thus, via this bijection, we can view the LK involution as an involution on antichains $\mathrm{LK}\colon \mathcal{A}(\Delta^{n-1}) \to \mathcal{A}(\Delta^{n-1})$ which satisfies

$$\#A + \#\mathrm{LK}(A) = n - 1.$$

The LK involution on antichains

D. Panyushev gave a simple description of the LK involution on $\mathcal{A}(\Delta^{n-1})$:

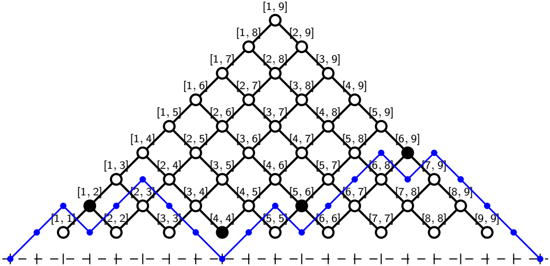
Theorem (Panyushev, 2004)

Let $A = \{[i_1, j_1], [i_2, j_2], \ldots, [i_k, j_k]\} \in \mathcal{A}(\Delta^{n-1})$ *with* $i_1 < i_2 < \cdots < i_k$. *Then* $\mathrm{LK}(A) = \{[i'_1, j'_1], [i'_2, j'_2], \ldots, [i'_{n-1-k}, j'_{n-1-k}]\} \in \mathcal{A}(\Delta^{n-1})$, *where*

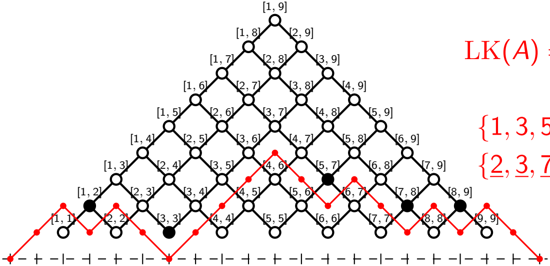
- $\{i'_1 < i'_2 < \cdots < i'_{n-1-k}\} = \{1, 2, \ldots, n-1\} \setminus \{j_1, j_2, \ldots, j_k\}$;
- $\{j'_1 < j'_2 < \cdots < j'_{n-1-k}\} = \{1, 2, \ldots, n-1\} \setminus \{i_1, i_2, \ldots, i_k\}$.

From Panyushev's description, it is immediate that this operation is an involution (i.e., $\mathrm{LK}^2(A) = A$), and that $\#A + \#\mathrm{LK}(A) = n - 1$.

The LK involution on antichains: example



$A = \{[1, \underline{2}], [4, \underline{4}], [5, \underline{6}], [6, \underline{9}]\}$



$\mathrm{LK}(A) = \{[1, \underline{2}], [3, \underline{3}], [5, \underline{7}], [7, \underline{8}], [8, \underline{9}]\}$

$\{1, 3, 5, 7, 8\} = \{1, \ldots, 9\} \setminus \{\underline{2}, \underline{4}, \underline{6}, \underline{9}\}$

$\{\underline{2}, \underline{3}, \underline{7}, \underline{8}, \underline{9}\} = \{1, \ldots, 9\} \setminus \{1, 4, 5, 6\}$

Section 3

Toggling

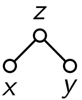
Toggling for antichains

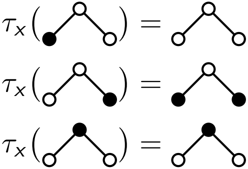
Our first new result gives another expression for the LK involution in terms of certain "local" involutions called **toggles**.

Let P be a poset and $A \in \mathcal{A}(P)$ an antichain. Let $p \in P$ be any element. The **toggle of** p **in** A is the antichain $\tau_p(A) \in \mathcal{A}(P)$, where

$$\tau_p(A) := \begin{cases} A \setminus \{p\} & \text{if } p \in A; \\ A \cup \{p\} & \text{if } p \notin A \text{ and } A \cup \{p\} \text{ remains an antichain;} \\ A & \text{otherwise.} \end{cases}$$

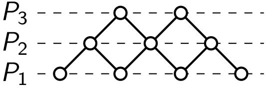
In other words, we "toggle" the status of p in A, if possible:

$P =$ 



Toggling in ranked posets

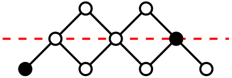
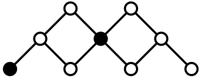
A poset P is **ranked** if we can write $P = P_1 \sqcup P_2 \sqcup \cdots \sqcup P_r$ so that all the edges of the Hasse diagram of P are from P_i (below) to P_{i+1} (above):



Since τ_p and τ_q commute if p and q are incomparable, and all the elements within a rank are incomparable, we can define

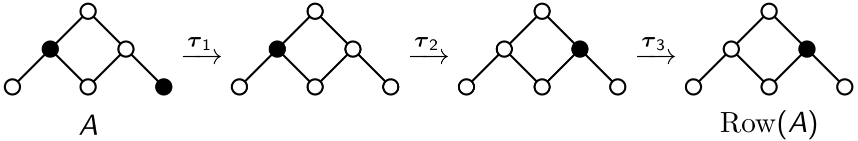
$$\boldsymbol{\tau}_i := \prod_{p \in P_i} \tau_p$$

to be the composition of all toggles at rank i, for $i = 1, \ldots, r$:

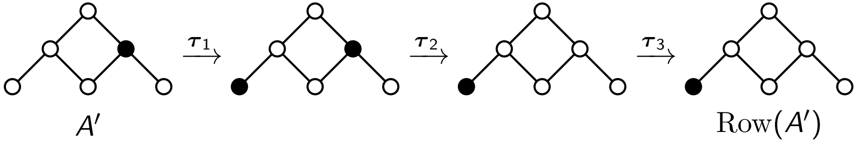
$\boldsymbol{\tau}_2$ () = 

Rowmotion

Rowmotion $\mathrm{Row} := \boldsymbol{\tau}_r \cdots \boldsymbol{\tau}_2 \boldsymbol{\tau}_1 \colon \mathcal{A}(P) \to \mathcal{A}(P)$ is the composition of all rank toggles from bottom to top:



Rowmotion has been studied by many authors (Cameron–Fon-Der-Flaass, Striker–Williams, Propp–Roby, etc...) in emerging subfield of **dynamical algebraic combinatorics**. Rowmotion is invertible, but not an involution:



(Actually, $\mathrm{Row} \colon \mathcal{A}(\Delta^{n-1}) \to \mathcal{A}(\Delta^{n-1})$ has order $2n$.)

The LK involution as a composition of toggles

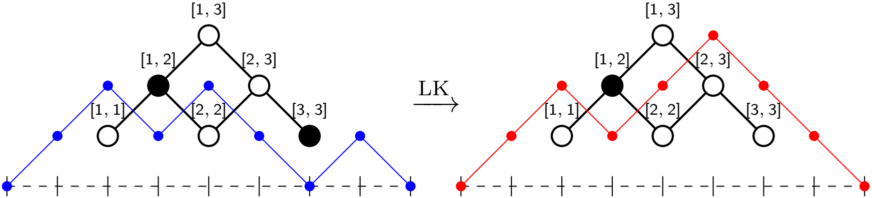
We showed the Lalanne–Kreweras involution can also be written as a composition of rank toggles:

Theorem (H.–Joseph, 2022)

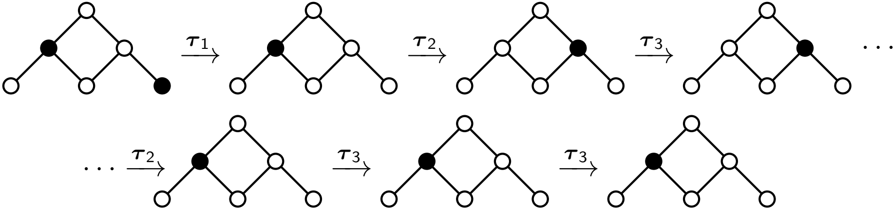
The LK involution $\mathrm{LK}\colon \mathcal{A}(\Delta^{n-1}) \to \mathcal{A}(\Delta^{n-1})$ *can be written as the following composition of toggles:*

$$\mathrm{LK} = (\boldsymbol{\tau}_{n-1})(\boldsymbol{\tau}_{n-1}\boldsymbol{\tau}_{n-2})\cdots(\boldsymbol{\tau}_{n-1}\cdots\boldsymbol{\tau}_3\boldsymbol{\tau}_2)(\boldsymbol{\tau}_{n-1}\cdots\boldsymbol{\tau}_2\boldsymbol{\tau}_1)$$

The LK involution as a composition of toggles: example



$A = \{[1,2],[3,3]\}$ $\mathrm{LK}(A) = \{[1,2]\}$



Rowvacuation

For any ranked poset P, can define **rowvacuation** $\mathrm{Rvac}\colon \mathcal{A}(P) \to \mathcal{A}(P)$ by same formula: $\mathrm{Rvac} := (\boldsymbol{\tau}_r)(\boldsymbol{\tau}_r\boldsymbol{\tau}_{r-1})\cdots(\boldsymbol{\tau}_r\cdots\boldsymbol{\tau}_3\boldsymbol{\tau}_2)(\boldsymbol{\tau}_r\cdots\boldsymbol{\tau}_2\boldsymbol{\tau}_1)$.

General algebraic properties of the toggles imply:

Proposition

$\langle \mathrm{Row}, \mathrm{Rvac} \rangle$ *gives a dihedral group action on* $\mathcal{A}(P)$*, i.e.,*

- $\mathrm{Rvac} \cdot \mathrm{Row} = \mathrm{Row}^{-1} \cdot \mathrm{Rvac}$;
- Rvac *is an involution.*

These names come from Schützenberger's **promotion** and **evacuation** operators acting on the linear extensions of a poset, which can be defined similarly and satisfy analogous properties.

Section 4

Piecewise linear and birational lifts

Lifting combinatorial constructions: overview

Why did we want to write the LK involution as a composition of toggles? In order to **extend** it to the **piecewise linear** and **birational** realms...

A recent trend has been to take some combinatorial construction and realize it as an expression involving $+$ and $-$ and $\min$ and $\max$, and then "de-tropicalize" that PL expression to get a birational transformation.

For example, in 2013, Einstein and Propp introduced **piecewise-linear** and **birational** lifts of **rowmotion**. Remarkably, many theorems lift:

Theorem (Grinberg–Roby, 2015)

The piecewise-linear and birational lifts of $\mathrm{Row}\colon \mathcal{A}(\Delta^{n-1}) \to \mathcal{A}(\Delta^{n-1})$ *still have order* $2n$.

This is surprising, because for other posets these lifts of rowmotion will not even have finite order!

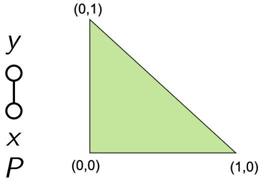
The chain polytope of a poset

In 1986, Richard Stanley associated to any poset P two polytopes in $\mathbb{R}^P$, the **order polytope** $\mathcal{O}(P)$ and the **chain polytope** $\mathcal{C}(P)$.

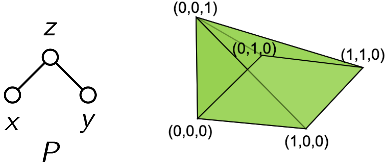
The **chain polytope** $\mathcal{C}(P)$ has facets

$$\begin{aligned} 0 \le x_p, &\qquad \forall p \in P \\ \sum_{p \in C} x_p \le 1, &\qquad \forall C = \{x_1 < x_2 < \cdots < x_k\} \subseteq P \text{ a maximal chain.} \end{aligned}$$

Stanley proved that the **vertices** of $\mathcal{C}(P)$ are precisely the **indicator functions of antichains** $A \in \mathcal{A}(P)$:



$\mathcal{C}(P)$



$\mathcal{C}(P)$

Piecewise linear toggling

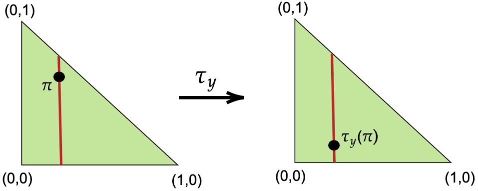
To define the PL extension of rowmotion, Einstein and Propp (c.f. Joseph) introduced a **piecewise linear extension** of the **toggles** τ_p.

For $p \in P$, the **PL toggle** $\tau_p^{\mathrm{PL}} \colon \mathcal{C}(P) \to \mathcal{C}(P)$ is defined by

$$\tau_p^{\mathrm{PL}}(\pi)(q) := \begin{cases} \pi(q) & \text{if } q \neq p; \\ 1 - \max\left\{ \sum_{r \in C} \pi(r) \colon \begin{array}{l} C \subseteq P \text{ a maximal} \\ \text{chain with } p \in C \end{array} \right\} & \text{if } p = q. \end{cases}$$

Restricted to the vertices of the chain polytope $\mathcal{C}(P)$, it is the same as τ_p.

Geometrically, τ_p **reflects** π within line segment in $\mathcal{C}(P)$ in direction x_p:



The PL LK involution

As before, for a ranked poset P we use $\boldsymbol{\tau}_i^{\mathrm{PL}} := \prod_{p \in P_i} \tau_p^{\mathrm{PL}}$ to denote the composition of all toggles at rank i.

We define the **PL LK involution** $\mathrm{LK}^{\mathrm{PL}} \colon \mathcal{C}(\Delta^{n-1}) \to \mathcal{C}(\Delta^{n-1})$ to be

$$\mathrm{LK}^{\mathrm{PL}} := (\boldsymbol{\tau}_{n-1}^{\mathrm{PL}})(\boldsymbol{\tau}_{n-1}^{\mathrm{PL}}\boldsymbol{\tau}_{n-2}^{\mathrm{PL}}) \cdots (\boldsymbol{\tau}_{n-1}^{\mathrm{PL}} \cdots \boldsymbol{\tau}_{3}^{\mathrm{PL}}\boldsymbol{\tau}_{2}^{\mathrm{PL}})(\boldsymbol{\tau}_{n-1}^{\mathrm{PL}} \cdots \boldsymbol{\tau}_{2}^{\mathrm{PL}}\boldsymbol{\tau}_{1}^{\mathrm{PL}})$$

By prior theorem, it's same as LK when restricted to the vertices of $\mathcal{C}(P)$.

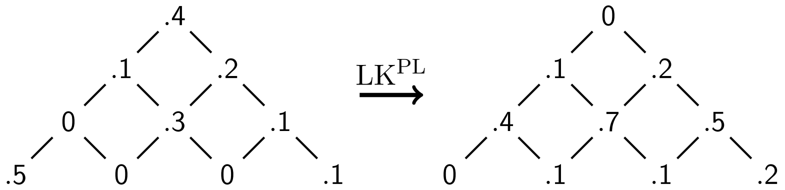
Theorem (H.–Joseph, 2022)

(1) $\mathrm{LK}^{\mathrm{PL}}$ *is an involution.*

(2) *For any* $\pi \in \mathcal{C}(\Delta^{n-1})$, $\sum_{p \in P} \pi(p) + \sum_{p \in P} \mathrm{LK}^{\mathrm{PL}}(\pi)(p) = n - 1$.

Observe that (2) is an extension of the fact that LK combinatorially exhibits the symmetry of the Narayana numbers.

The PL LK involution: example



$.1 = 1 - \max(0 + 0 + .1 + .4, 0 + .3 + .1 + .4, 0 + .3 + .2 + .4)$

We can check that

$$(.5+0+0+.1+0+.3+.1+.1+.2+.4)+(0+.1+.1+.2+.4+.7+.5+.1+.2+0) =$$

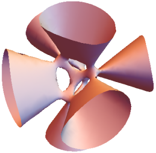
$$1.7 + 2.3 = 4$$

Tropical geometry

Algebraic geometry studies **polynomial** expressions like

$$x^3y + y^3z + z^3x$$

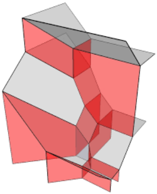
that give "curvy" hypersurfaces



Tropical geometry studies **piecewise linear** expressions like

$$\max(3x + y, 3y + z, 3z + x)$$

that give "flat" polytopal complexes



$$(\times, +) \to (+, \max) = \text{"tropicalization"}$$
$$(+, \max) \to (\times, +) = \text{"de-tropicalization"}$$

Birational toggling

Einstein–Propp (c.f. Joseph–Roby) also introduced a **birational extension** of the **toggles** τ_p (and rowmotion), via de-tropicalization.

For $p \in P$, the **birational toggle** $\tau_p^{\mathrm{B}} \colon \mathbb{C}^P \dashrightarrow \mathbb{C}^P$ is

$$\tau_p^{\mathrm{B}}(\pi)(q) := \begin{cases} \pi(q) & \text{if } q \neq p; \\ \kappa \cdot \Big(\displaystyle\prod_{\substack{C \subseteq P \\ \text{max. chain,} \\ p \in C}} \sum_{r \in C} \pi(r) \Big)^{-1} & \text{if } p = q, \end{cases}$$

where $\kappa \in \mathbb{C}$ is some fixed constant.

The birational toggle τ_p^{B} tropicalizes to the PL toggle τ_p^{PL}.

The birational LK involution

As before, if P is ranked we set $\boldsymbol{\tau}_i^{\mathrm{B}} := \prod_{p \in P_i} \tau_p^{\mathrm{B}}$.

We define the **birational LK involution** $\mathrm{LK}^{\mathrm{B}} \colon \mathbb{C}^{\Delta^{n-1}} \dashrightarrow \mathbb{C}^{\Delta^{n-1}}$ by

$$\mathrm{LK}^{\mathrm{B}} := (\boldsymbol{\tau}_{n-1}^{\mathrm{B}})(\boldsymbol{\tau}_{n-1}^{\mathrm{B}}\boldsymbol{\tau}_{n-2}^{\mathrm{B}}) \cdots (\boldsymbol{\tau}_{n-1}^{\mathrm{B}} \cdots \boldsymbol{\tau}_3^{\mathrm{B}}\boldsymbol{\tau}_2^{\mathrm{B}})(\boldsymbol{\tau}_{n-1}^{\mathrm{B}} \cdots \boldsymbol{\tau}_2^{\mathrm{B}}\boldsymbol{\tau}_1^{\mathrm{B}})$$

It tropicalizes to $\mathrm{LK}^{\mathrm{PL}}$.

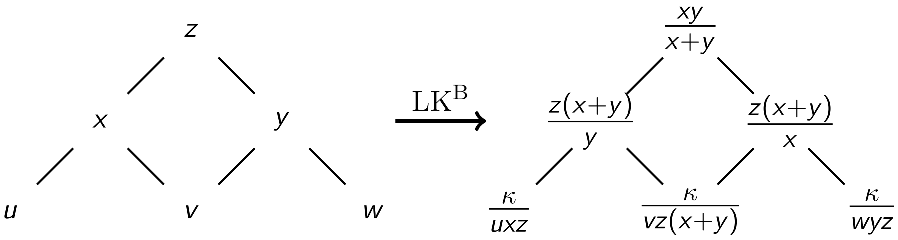
Theorem (H.–Joseph, 2022)

(1) LK^{B} *is an involution.*

(2) *For any* $\pi \in \mathbb{C}^{\Delta^{n-1}}$, $\prod_{p \in P} \pi(p) \cdot \prod_{p \in P} \mathrm{LK}^{\mathrm{B}}(\pi)(p) = \kappa^{n-1}$.

Note that (2) is the birational analog of the fact that LK combinatorially exhibits the symmetry of the Narayana numbers.

The birational LK involution: example



We can check that this operation really is an involution; e.g.,

$$\frac{z'(x'+y')}{y'} = \frac{\frac{xy}{x+y} \cdot \left(\frac{z(x+y)}{y} + \frac{z(x+y)}{x}\right)}{\frac{z(x+y)}{x}} = \frac{zx+zy}{\frac{z(x+y)}{x}} = \frac{z(x+y)}{\frac{z(x+y)}{x}} = x.$$

And if we multiply together all the above values, we get κ^3.

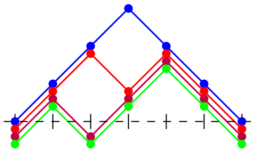
Section 5

Conclusion: so what?

What do the lifts do for us?

(1) **They are more general**
All birational identities tropicalize. But PL identities do *not* always de-tropicalize. So a result proved at the birational level is a strictly stronger result. (Ask Tom and Darij about noncommutative stuff...)

(2) **They lead to further statistical symmetry results**
For any $m \geq 1$, the points in $\frac{1}{m}\mathbb{Z}^{\Delta^{n-1}} \cap \mathcal{C}(\Delta^{n-1})$ correspond to m-tuples of nested Dyck paths:



$n = 3$

$m = 4$

3 valleys

The PL LK involution implies that the generating function over these m-tuples for the (total) number of valleys statistic is still symmetric.

What do the lifts do for us?

(3) **They give new ways of looking at combinatorial constructions**
Writing LK as a composition of toggles leads us to consider this same composition of toggles (i.e., rowvacuation) for other posets.
Δ^{n-1} is the **root poset** of Type A_{n-1}. For any root system Φ, can define Φ-**Narayana numbers** $N(\Phi, k)$ using antichains in the root poset Φ^+, and they are again *symmetric*: $N(\Phi, k) = N(\Phi, r-k)$.

Theorem (Defant-H., 2021)

For a root system Φ of classical type A, B, C, or D, rowvacuation is an involution on $\mathcal{A}(\Phi^+)$ which combinatorially exhibits the symmetry of the Φ-Narayana numbers.

Unfortunately, this *fails* for exceptional root systems!

What do the lifts do for us?

(4) **They give potentially interesting algebro-geometric things**
This is more speculative, but... birational lifts of combinatorial constructions give interesting birational endomorphisms $\mathbb{C}^N \dashrightarrow \mathbb{C}^N$ (of finite order). Could be worth looking at the variety of **fixed points**. See also: our conjectural polynomial **invariants** of birational LK!

(5) **They suggest connections to algebra**
Birational rowmotion has been related to the *Zamolodchikov Periodicity Conjecture*, *Geometric Crystals* and *Geometric RSK*, etc. So far I don't know of any fancy algebraic connections like this for rowvacuation, but there could definitely be some...

Thank you!

these slides are on the conference website
and the paper on the arXiv: `arXiv:2012.15795`

Exercises 231

6.24. [?] Explain the significance of the following sequence:

un, dos, tres, quatre, cinc, sis, set, vuit, nou, deu, . . .

R. Stanley, *Enumerative Combinatorics*, Vol. 2