

# Rank and characteristic generating functions of upper homogeneous posets

Sam Hopkins  
[Howard University]

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based on  
arXiv: 2202.12103

(these slides are on my website  
[samuelhopkins.com](http://samuelhopkins.com))

A finite poset  $P$  is graded of rank  $n$  if we can write  $P = P_0 \sqcup P_1 \sqcup \dots \sqcup P_n$  such that every maximal chain looks like  $p_0 < p_1 < \dots < p_n$ ,  $p_i \in P_i$ .

Rank function  $\text{rk}: P \rightarrow \mathbb{N}$  is  $\text{rk}(p) = i \Leftrightarrow p \in P_i$ .

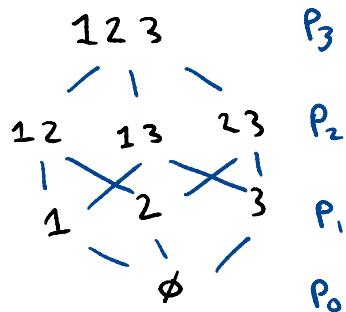
Rank generating polynomial is

$$F_P(x) = \sum_{i \geq 0} \#P_i \cdot x^i = \sum_{p \in P} x^{\text{rk}(p)}$$

e.g.

$P$  = Boolean lattice  $B_n$  of subsets of  $[n] = \{1, 2, \dots, n\}$

$$F_{B_n}(x) = \sum_{k=0}^n \binom{n}{k} x^k = (1+x)^n$$



Suppose further that  $P$  has minimum  $\hat{0}$ .

Then the characteristic polynomial is

$$\chi_p(x) = \sum_{p \in P} \mu(\hat{0}, p) x^{\text{rk}(p)} \quad \text{M\"obius fn.!}$$

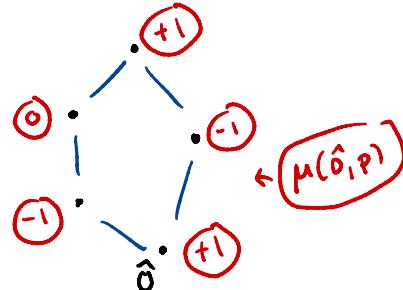
Rmk: Often reciprocal polynomial  $\sum_p \mu(\hat{0}, p) x^{n-\text{rk}(p)}$  is studied instead, since it better matches e.g. char. poly. of a matroid.

## Review of Möbius functions:

The Möbius function  $\mu(p, q)$  for  $p \leq q \in P$  is the inverse of the zeta function  $\zeta(p, q) = 1$  in  $\mathcal{I}(P)$ .

It can be computed recursively by:

↑  
incidence  
algebra!



$$\cdot \mu(p, p) = 1$$

$$\cdot \mu(p, q) = - \sum_{p \leq r < q} \mu(p, r) \text{ for } p < q.$$

Recall Möbius inversion:

$$f, g : P \rightarrow R \quad \begin{matrix} \text{(abelian} \\ \text{gp.)} \end{matrix} \quad f(p) = \sum_{q \geq p} g(q) \Leftrightarrow g(p) = \sum_{q \geq p} \mu(p, q) f(q)$$

Also "Philip Hall's Thm.":

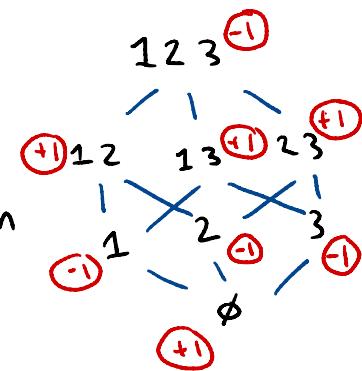
$$\mu(p, q) = c_0 - c_1 + c_2 - c_3 + \dots$$

↙ Euler characteristic!

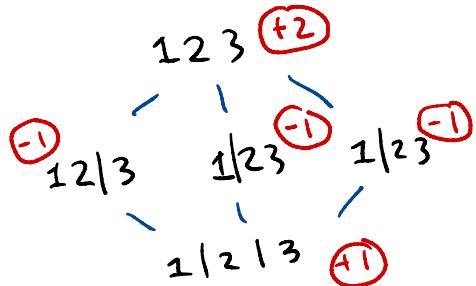
where  $c_i = \# \text{ chains } p = p_0 < p_1 < \dots < p_i = q$ .

e.g. In  $B_n$ ,  $\mu(s, T) = (-1)^{\# T \setminus s}$

so  $\chi_{B_n}(x) = \sum_{k=0}^n (-1)^k \binom{n}{k} x^k = (1-x)^n$



e.g.  $P = \text{lattice } \mathcal{T}\mathcal{I}_n$  of  
set partitions of  $[n]$   
ordered by refinement



$$F_{\mathcal{T}\mathcal{I}_n}(x) = \sum_{k=0}^{n-1} S(n, n-k) x^k$$

↑ Stirling # of 2<sup>nd</sup> kind

$$\chi_{\mathcal{T}\mathcal{I}_n}(x) = (1-x)(1-2x)\cdots(1-(n-1)x) = \sum_{k=0}^{n-1} \Delta(n, n-k) x^k$$

↑ Stirling # of 1<sup>st</sup> kind

Now we try to do same for certain infinite posets.

(Infinite) poset  $P$  is  $\mathbb{N}$ -graded if  $P = P_0 \sqcup P_1 \sqcup P_2 \sqcup \dots$   
s.t. every max'l chain is  $p_0 < p_1 < p_2 < \dots$ ,  $p_i \in P_i$ .

As before, rank fn.  $\text{rk}: P \rightarrow \mathbb{N}$  is  $\text{rk}(p) = i \Leftrightarrow p \in P_i$ .

Say  $P$  is finite type if  $\# P_i < \infty \forall i$ .

Then we can form the rank generating function

$$F_P(x) = \sum_{i \geq 0} \# P_i x^i = \sum_{p \in P} x^{\text{rk}(p)}$$

Suppose further  $P$  has minimum  $\hat{0}$ .

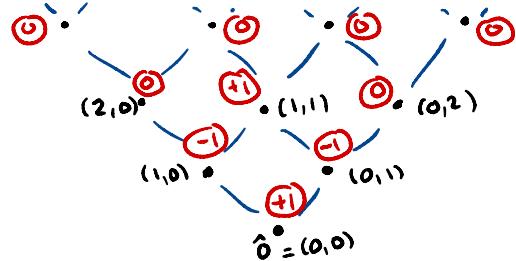
Then define the characteristic generating fn.

$$\chi_p(x) = \sum_{p \in P} \mu(\hat{0}, p) x^{\text{rk}(p)}$$

e.g.  $P = \mathbb{N}^2$

$$F_P(x) = \sum_{k \geq 0} (k+1) x^k$$

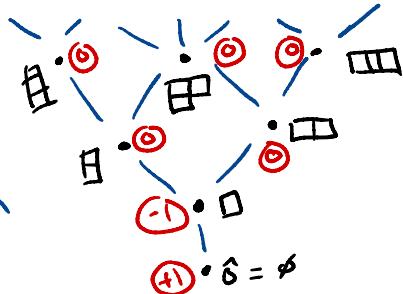
$$X_P(x) = 1 - 2x + x^2$$



e.g.  $P = \mathcal{Y}$ , Young's lattice of partitions

$$F_P(x) = \sum_{n \geq 0} P(n) x^n$$

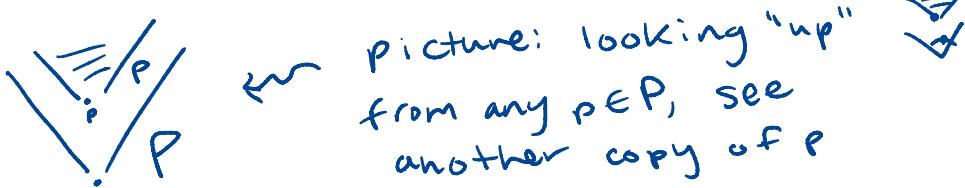
$$X_P(x) = 1 - x$$



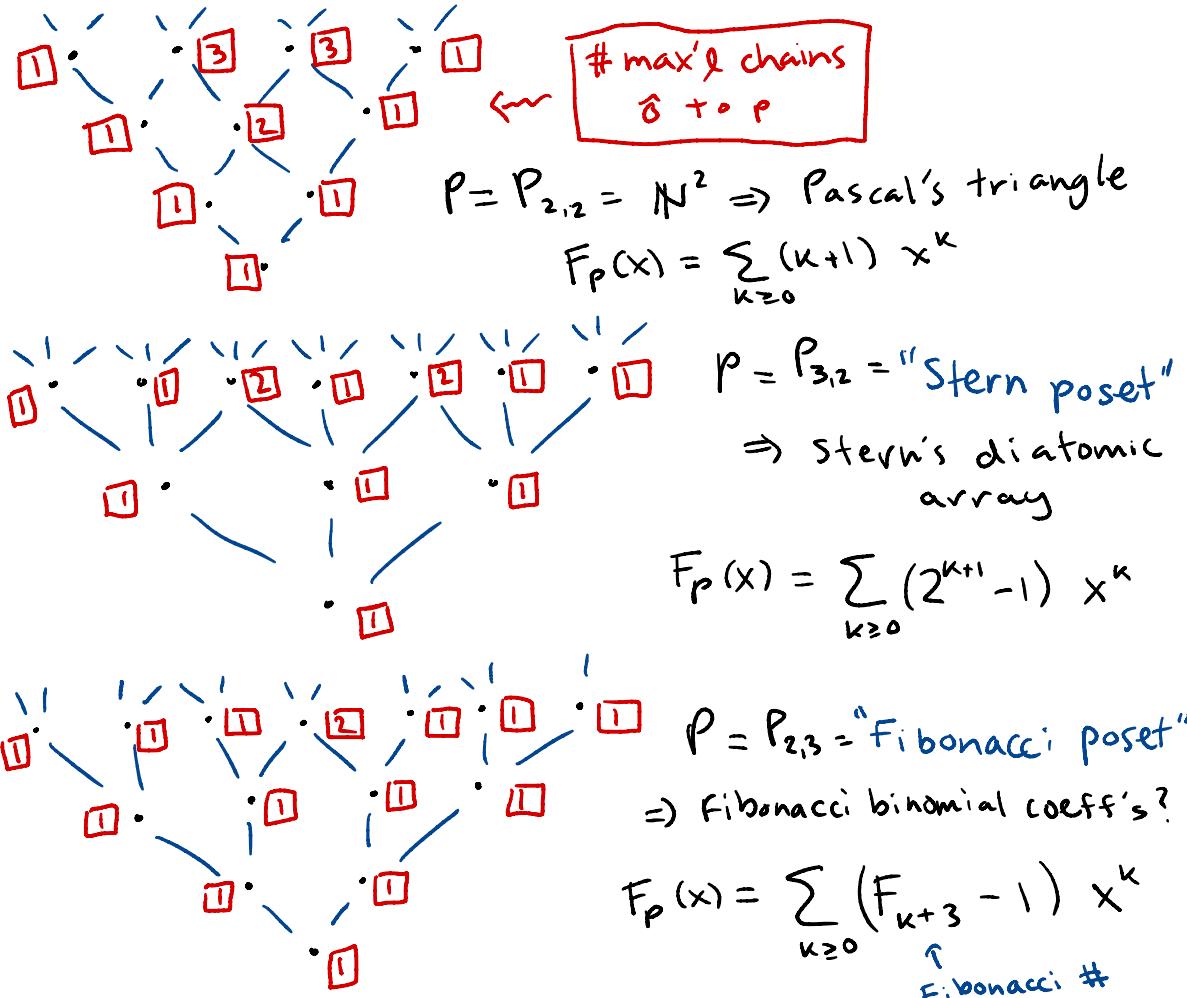
Stanley recently introduced following class of posets:

Say  $P$  is **upper homogeneous** or "**upho**" if

$V_p \cong P \forall p \in P$ , where  $V_p = \{q : q \geq p\}$  is the **principal order filter** generated by  $p$ .



Stanley was interested in certain analogs of **Pascal's triangle** coming from some **planar upho posets**:  
(= planar Hasse diagram)



An MIT PRIMES gp. looked at rank g.f.'s of upho posets:

Thm (Gao - Guo - Seetharaman - Seidel, 2022)

For a planar upho poset  $P$ , have  $F_P(x) = \frac{1}{Q(x)}$

where  $Q(x) = 1 - bx + c_2 x^2 + c_3 x^3 + \dots + c_n x^n$

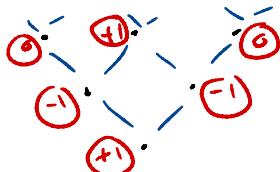
is a polynomial w/  $b, c_2, \dots, c_n \in \mathbb{N}$ ,  $c_2 + \dots + c_n - b \leq 0$ .

Thm (GGSS, 2022)

There are uncountably many  $F_p(x)$ 's among all upho posets  $P$ .  $\curvearrowleft$  read: very bad!

I recently noticed that for upho  $P$ , there is a very simple relationship between  $F_p(x)$  and  $X_p(x)$ :

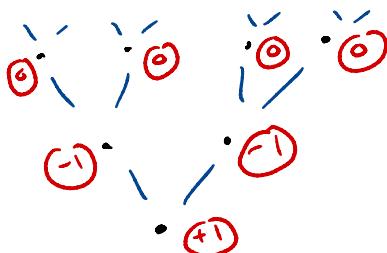
e.g.



$$P = \mathbb{N}^2$$

$$F_p(x) = \sum_{k \geq 0} (k+1) x^k = \frac{1}{(1-x)^2}$$

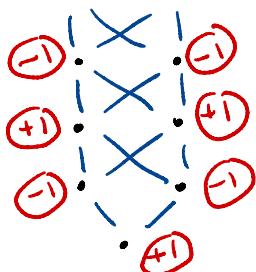
$$X_p(x) = 1 - 2x + x^2 = (1-x)^2$$



$$P = \text{"binary tree"}$$

$$F_p(x) = \sum_{k \geq 0} 2^k x^k = \frac{1}{1-2x}$$

$$X_p(x) = 1 - 2x$$



$$P = \text{"bowtie poset"}$$

$$F_p(x) = 1 + \sum_{k \geq 1} 2 x^k = \frac{1+x}{1-x}$$

$$X_p(x) = 1 + \sum_{k \geq 1} (-1)^k 2 x^k = \frac{1-x}{1+x}$$

Thm (H., 2022)

For  $P$  upho, we have  $F_p(x) = X_p(x)^{-1}$ .

Pf: Set  $f(p) = x^{\text{rk}(p)}$  and  $g(p) = \sum_{q \geq p} f(q) = \sum_{q \geq p} x^{\text{rk}(q)}$  if  $p \in P$ .

By Möbius inversion,

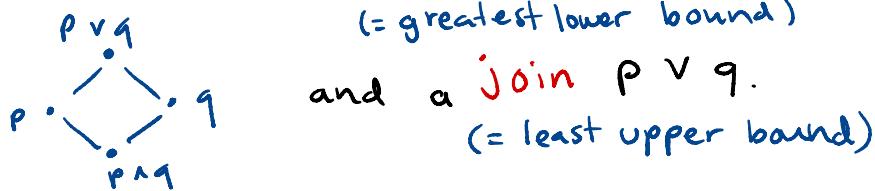
$$1 = x^0 = f(\hat{0}) = \sum_{q \geq \hat{0}} \mu(\hat{0}, q) g(q),$$

but since  $\vee_q \simeq P \setminus q$ ,  $g(q) = x^{\text{rk}(q)} \cdot F_p(x)$ , so

$$1 = \sum_{q \in P} \mu(\hat{0}, q) x^{\text{rk}(q)} F_p(x) = F_p(x) \cdot \chi_p(x)$$



Recall that a **lattice** is a poset  $P$  where every pair  $p, q \in P$  has a **meet**  $p \wedge q$ ,



Lattices have good Möbius fn.'s, e.g.:

Cor (to Rota's crosscut thm)

If  $P$  is a finite lattice, w/ minimum  $\hat{0}$ ,

maximum  $\hat{1}$ , and **atoms**  $a_1, \dots, a_m$

(= covers of  $\hat{0}$ :  $a_1 \swarrow \searrow \dots \swarrow \searrow a_m$ )

then  $\mu(\hat{0}, \hat{1}) = 0$  if  $\hat{1} \neq a_1 \wedge \dots \wedge a_m$ .

↑  
join of all atoms

Hence... If  $P$  is an upho lattice, then

$$F_P(x) = X_P(x)^{-1} = X_{P'}(x)^{-1}, \text{ where}$$

$P'$  = Subposet of  $P$  below joins of atoms

is a finite graded lattice ...

so  $F_P(x)^{-1} = X_{P'}(x)$  is a polynomial.

(In fact, can replace "lattice" by "meet semilattice" above. And all planar upho  $P$  are meet semilattices.)

Note Upgo lattice  $P$  is not determined by this finite graded lattice  $P'$ :

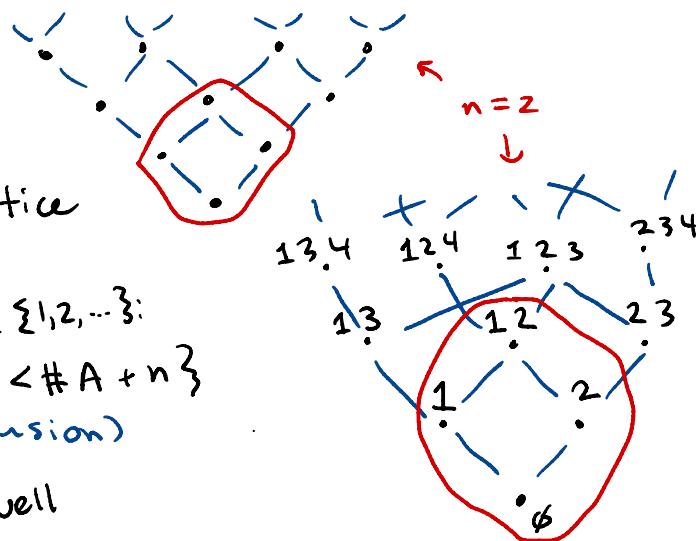
e.g.  $P = \mathbb{N}^n$

$$\Rightarrow P' = B_n$$

Boolean lattice

but...  $P = \left\{ \text{finite } A \subseteq \{1, 2, \dots\} : \max(A) < \#A + n \right\}$   
(order = inclusion)

$$\Rightarrow P' = B_n \text{ as well}$$



Nevertheless, still interesting to ask...

maybe at  
an REU...

Q: When can we extend a finite graded lattice  $P'$  to an upho lattice  $P$ ?

e.g. Just saw 2 ways for  $P' = B_n \dots$

Note Since  $F_p(x) = X_{P'}(x)^{-1}$ , we need

$X_{P'}(x)^{-1}$  to have nonnegative coeff.'s,

so a "random" finite lattice  $P'$  does not extend.

e.g. fix  $n \geq 1$  and a prime  $p$  and set

$P =$  subgroups of  $\mathbb{Z}^n$  of index  $p^k$  ( $k \geq 0$ )  
ordered by reverse inclusion.

Then  $P$  is an upho lattice

and  $P' = B_n(p)$

= lattice of subspaces

of  $(\mathbb{Z}/p\mathbb{Z})^n$ , with

$$X_{P'}(x) = (1-x)(1-px)\cdots(1-p^{n-1}x) = F_p(x)^{-1}.$$



Rmk: More generally,  $R$  is a local P.I.D. with finite residue field, then submodules of  $R^d$  of finite colength (ordered by rev. inclusion) give an upho lattice.

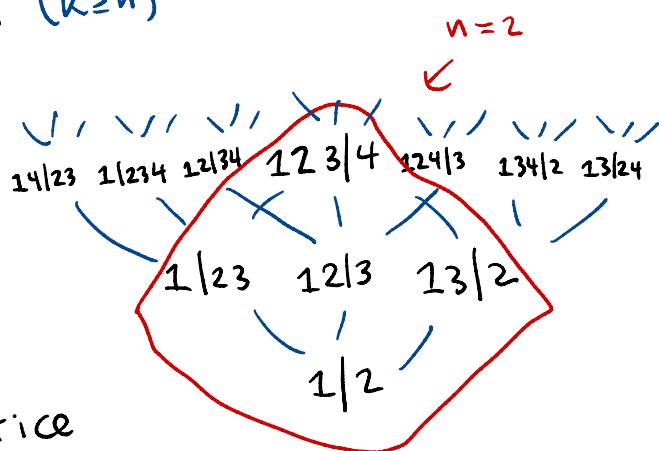
Conjecture (Stanley)

All\* **modular** upho lattices are of this form.

$$\begin{aligned} & \stackrel{\leftarrow}{\text{rk}} (\text{rk}(p) + \text{rk}(q)) \\ & = \text{rk}(p \cap q) + \text{rk}(p \vee q) \quad \forall p, q \in P \end{aligned}$$

e.g. Fix  $n \geq 1$  and set  
 $P = \text{partitions of } [k] \quad (k \geq n)$   
 into  $n$  blocks

where  $\pi_1 \leq \pi_2$  if  
 $\forall B_1 \in \pi_1, \exists B_2 \in \pi_2$   
 s.t.  $B_1 \subseteq B_2$ .



Then  $P$  is an upho lattice

and  $P' = \prod_{n+1}$ , so that  
 (set partition lattice)

$$F_p(x) = \sum_{k \geq n} S(k, n) x^{k-n} = \frac{1}{(1-x)(1-2x)\cdots(1-nx)} = x_{\prod_{n+1}}(x)^{-1}.$$

Finite poset  $P$ , graded of rank  $n$ , w/ min.  $\hat{0}$  & max.  $\hat{1}$   
 is called **uniform** if for all  $i = 0, 1, \dots, n$   
 there exists a single poset  $Q_i$  such that  
 $V_p = [\rho, \hat{1}] \cong Q_i \quad \forall p \in P \text{ with } \text{rk}(p) = n-i.$

e.g.  $P = B_n, B_n(q), \Pi_{n+1}$  are all uniform  
 (with  $Q_i = B_i, B_i(q), \Pi_{i+1}$  respectively).

For such uniform  $P$ , define the **Whitney #'**s  
 of the 1<sup>st</sup> and 2<sup>nd</sup> kind  $v(i, j)$  and  $V(i, j)$   
 by  $X_{Q_i}(x) = \sum_{j=0}^i v(i, i-j) x^j$   
 and  $F_{Q_i}(x) = \sum_{j=0}^i V(i, i-j) x^j$

Thm (Dowling, 1971)  $\leftarrow$  See Stanley EC1  
 exercise 3.130

For  $P$  uniform, the matrices

$(v(i, j))_{\substack{i=0, \dots, n \\ j=0, \dots, n}}$  and  $(V(i, j))_{\substack{i=0, \dots, n \\ j=0, \dots, n}}$   
 are inverses of one another.

e.g. For  $P = T_{n+1}$ , says the Stirling # matrices  $(S(i,j))$  and  $(S(i,j))$  are inverse:

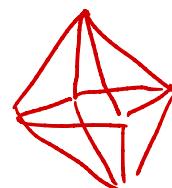
$$n=4 \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -6 & 11 & -6 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{bmatrix} \quad \text{well-known!}$$

Uniform posets are "as close" to being upho as finite posets can be, and the Whitney # thm. "feels similar" to the thm. that  $F_p(x)$  and  $X_p(x)$  are inverse for upho  $P$ .

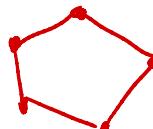
So trying to extend uniform lattices  $P'$  to upho lattices  $P$  might make sense...

e.g. Can you extend ...

$P' =$  face lattice of  
cross polytope ?



$P' =$  bond lattice of  
odd cycle graph ?



Thank You!

And can't wait to see  
all of you **in person**  
at **OPAC 2022** in  
**just a couple weeks ...**

