Howard Math 157: Calculus II Spring 2024

Instructor: Sam Hopkins (sam.hopkins@howard.edu)
(call me "Sam")

1/8 Logistics:

Classes: MWRF 10:10-11 am ASB-B #100

Office HRs: R 9-10 am Annex III - #220
or by appointment - email me!

website: samuelfhopkins.com/classes/157.html

Text: Calculus, Early Transcendentals by Stewart, 9e

Grading: 35% (in-person) quizzes
45% three (in-person) midterms
20% (in-person) final exam

There will be 11 in-person quizzes taken on Thursdays (about 20 mins, we will go over answers in class).
Your lowest 2 scores will be dropped (so 9/11 count).
The 3 midterms will happen in-class, also on Thursdays.
The final will take place during finals week.
This is an in-person class; all assessments must be taken in-person!

Beyond that, I will assign additional practice problems from the book.

and I expect you to SHOW UP TO CLASS + PARTICIPATE! :)

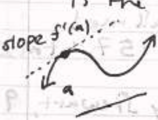
which means ASK QUESTIONS!

Overview of the course:

In Calculus I we learned two important and related operations on functions $f(x): \mathbb{R} \to \mathbb{R}$:

- differentiation and • integration.

The derivative $f'(a)$ of $f(x)$ at a point $x = a$ is the slope of the tangent to $y = f(x)$ at $(a, f(a))$.



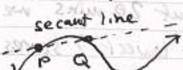
It is also the "instantaneous rate of change" of the function $f(x)$ at $x = a$.

The integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from $x = a$ to $x = b$:

area (▨) $= \int_a^b f(x)\,dx$

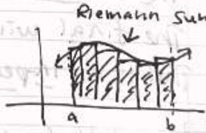
Both the derivative and integral are formally defined as limits:

- the derivative is the limit of slopes of secant lines approximating the tangent:



$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

- the integral is the limit of Riemann sums (= rectangles) approximating area under curve:



$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

The Fundamental Theorem of Calculus says that differentiation and integration are inverse operations:

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F'(x) = f(x)$.

In Calculus II we will continue to study derivatives & integrals.

Some of the things we will learn are:

- Applications of integration:

In Calc I we learned many applications of derivatives (minimums & maximums, concavity, etc.)
In Calc II we will learn more things we can compute using integrals (beyond area under curve) like
  - volumes (3D version of area)
  - lengths (1D version of area)

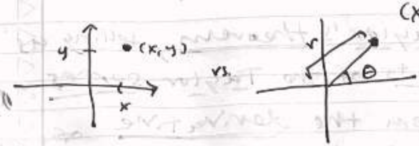
Also, FTC says that integral represents net change, so we will study some physical applications of integrals like to work (in the sense of force).

- Techniques for integration:

Using rules for differentiation like product and chain rules, we know how to take the derivative of "any" function,
e.g. $d/dx\,(x \sin(e^{x^2} + 5x - 6))$

But... integrating a "random" function like this can be really hard or not even possible. We will learn more techniques for computing integrals, when possible.
[Recall that we already learned one technique: u-substitution.]

- Polar coordinates: We are used to working with (x, y) aka. "Cartesian coordinates"



Polar coordinates (r, θ) are a different system where we can also do calculus.

• Taylor series:

How do we evaluate a function $f(x)$ at a particular value, e.g. compute $f(1.5)$?

If $f(x)$ is a polynomial like $f(x) = 6x^2 - 2x + 3$ we can use arithmetic: $f(1.5) = 6(1.5)^2 - 2(1.5) + 3 = \dots$

If it is a rational function like $f(x) = \frac{x+1}{x^2-2}$ we can use division similarly: $f(1.5) = \frac{1.5+1}{(1.5)^2-2} = \dots$

But what about something like $f(x) = \sin(x)$ or $f(x) = e^x$? How to compute $e^{1.5}$? What does your calculator even do?

Even though e^x is not a polynomial, it has a representation as a kind of "infinite" polynomial:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots$$

This is called a Taylor series, and it lets us compute things like $e^{1.5}$ (at least approximately).

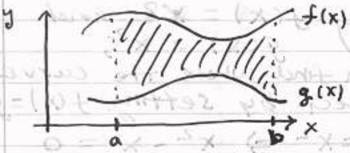
We will learn how to deal with these kind of infinite sums called series (specifically, power series) and related mathematical constructions called sequences.

We will also learn Taylor's theorem, telling us that the coefficients of the Taylor series can be computed from the derivative of the function (which is where calculus comes in!).

1/11

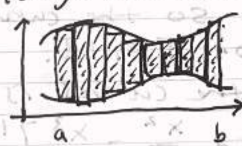
Area between curves (§6.1 of textbook)

The integral computes the area under a curve.
What if we have two curves, $y = f(x)$ and $y = g(x)$,
and we want to know the area between the curves?



Specifically, suppose that $f(x) \geq g(x)$ for all x in some closed interval from $x = a$ to $x = b$.

Then, as with the integral, we can define the area between the curves on $[a,b]$ by approximating it with a large number of thin rectangles:



Let $\Delta x = \frac{b-a}{n}$ (for some $n \geq 1$)
and let $x_i = a + i \cdot \Delta x$ for $i = 0, 1, \ldots, n$
So that $[a,b]$ is divided into n sub-intervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$

For each sub-interval, choose a $x_i^* \in [x_{i-1}, x_i]$,
and consider the thin rectangles of width Δx and
height $= f(x_i^*) - g(x_i^*)$ ← difference in hts of two curves at $x = x_i^*$

Then area between curves from $x=a$ to $x=b$ $\approx \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*))\,\Delta x$

and is exactly → $= \lim_{n\to\infty} \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*))\,\Delta x$

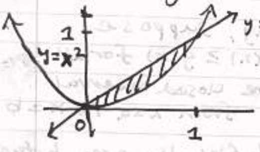
$$\boxed{= \int_a^b f(x) - g(x)\,dx}$$

So... area between two curves can be computed as integral of difference function

Note: If we let $g(x) = 0$ be the function corresponding to the x-axis $y = 0$, then we recover the area under the curve as $\int_a^b f(x)\,dx$ from

E.g.: Let's compute the area bounded by the curves $y = x$ and $y = x^2$.

Since the problem does not tell us the <u>bounds</u> of integration, let us sketch the curves:



Letting $f(x) = x$, and $g(x) = x^2$, we can find where the curves intersect by setting $f(x) = g(x)$

$$\Rightarrow x = x^2 \Rightarrow x^2 - x = 0$$
$$\Rightarrow x(x-1) = 0$$
$$\Rightarrow x = 0 \text{ or } x = 1$$

Also, choosing $x = \frac{1}{2}$, we see that between $x=0$ and $x=1$, $f(x) = \frac{1}{2} \geq g(x) = \frac{1}{4}$, so the curve $y = f(x)$ is <u>above</u> $y = g(x)$ on $[0,1]$.

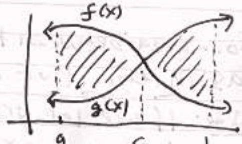
Thus, the area bounded by the curves is

$$\int_a^b f(x) - g(x)\,dx = \int_0^1 x - x^2\,dx = \frac{x^2}{2} - \frac{x^3}{3}\Big]_0^1$$
$$= \left(\frac{1^2}{2} - \frac{1^3}{3}\right) - \left(\frac{0^2}{2} - \frac{0^3}{3}\right) = \frac{1}{2} - \frac{1}{3} = \boxed{\frac{1}{6}}.$$

If on the interval $[a,b]$, sometimes $f(x) > g(x)$ and sometimes $g(x) > f(x)$, then to correctly find area between them, we need to take <u>absolute value</u> of difference:

$$\text{area between curves} = \int_a^b |f(x) - g(x)|\,dx.$$

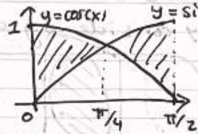
In practice, we break up this integral into the parts where $f(x) \geq g(x)$ and where $g(x) \geq f(x)$



$$\Rightarrow \int_a^c f(x) - g(x)\,dx + \int_c^b g(x) - f(x)\,dx$$

E.g.: Compute the area between $y = f(x) = \cos(x)$ and $y = g(x) = \sin(x)$ for $x = 0$ to $x = \pi/2$.

Again, good idea to sketch curves to see what's going on:



$\cos(0) = 1 > 0 = \sin(0)$, but $\sin(\pi/2) = 1 > 0 = \cos(\pi/2)$, so which curve is on top changes from $x=0$ to $x=\pi/2$

in fact, have $\cos(\pi/4) = \sin(\pi/4)$ (by symmetry, or isosceles right triangle...)

Thus...

area between $y=\cos(x)$ and $y=\sin(x)$ from $x=0$ to $x=\pi/2$ $= \int_0^{\pi/4} \cos(x) - \sin(x)\,dx + \int_{\pi/4}^{\pi/2} \sin(x) - \cos(x)\,dx$

$= \sin(x) + \cos(x)\Big]_0^{\pi/4} + -\cos(x) - \sin(x)\Big]_{\pi/4}^{\pi/2}$

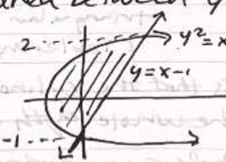
$= \left(\sin(\pi/4) + \cos(\pi/4) - \sin(0) - \cos(0)\right) + \left(-\cos(\pi/2) - \sin(\pi/2) + \cos(\pi/4) + \sin(\pi/4)\right)$

$= \left(1/\sqrt{2} + 1/\sqrt{2} - 0 - 1\right) + \left(-0 - 1 + 1/\sqrt{2} + 1/\sqrt{2}\right) = \boxed{2\sqrt{2} - 2}$ //

E.g.: Sometimes it is easier to integrate w.r.t. y variable

Let's find area between $y = x - 1$ and $y^2 = x + 1$:

We sketch the curves: they intersect at $y = -1$ and $y = 2$



$x = y^2 - 1 = g(y)$

and $x = y + 1 = f(y)$

set equal $y^2 - 1 = y + 1$

$\Rightarrow y^2 - y - 2 = 0$

$\Rightarrow (y-2)(y+1) = 0$

$\Rightarrow y = 2$ or $y = -1$

Then, since $y = x - 1$ is to right of $y^2 = x + 1$ for $y = -1$ to $y = 2$:

area between curves $= \int_{-1}^{2} f(y) - g(y)\,dy = \int_{-1}^{2} (y+1) - (y^2 - 1)\,dy$

$= \int_{-1}^{2} -y^2 + y + 2\,dy = \frac{-y^3}{3} + \frac{y^2}{2} + 2y\Big]_{-1}^{2} = \left(\frac{-8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right)$

$= \boxed{4.5}$ //

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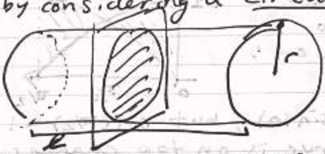
Volumes (§6.2)

Volumes are the 3-dimensional version of areas.
Let's start by considering a circular cylinder:



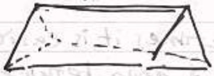
The cross-section (= intersection w/ y,z-plane) of this cylinder at any x-coordinate is a circle (of radius r).
We thus define the volume of the cylinder

to be = area of cross-section × length of cylinder

$= \pi r^2 \cdot \ell.$

We can also consider cylinders whose cross-sections are other shapes, e.g., rectangles or triangles:

rectangular prism
(or rectangular cylinder)



triangular cylinder
('Toblerone' bar)

The important thing is that the cylinder has a certain length and across the whole length cross-sections are same.

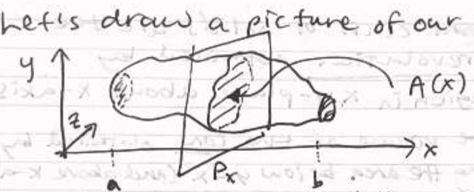
Thus, for any cylinder we define

Volume of cylinder = area of cross-section × length.

E.g.: Volume of rectangular prism = width × height × length.

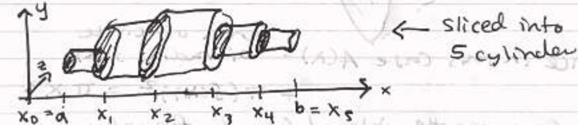
Q: What if the cross-section of our solid is not constant?

Let's draw a picture of our solid:



Suppose the solid extends between $x=a$ and $x=b$, and let $A(x)$ for $a \le x \le b$ be the area of the cross-section obtained by intersecting with plane P_x perpendicular to x-axis at that point.

We can approximate the volume by dividing the solid into several short cylinders:



As w/ integral, we break up the interval $[a,b]$ into n sub-intervals $[x_{i-1}, x_i]$, $i = 1, \dots, n$, $x_i = x_{i-1} + \Delta x$

Then the volume of the solid is $\approx \sum_{i=1}^{n}$ area of cross-section of i^{th} short cylinder $\times \Delta x$

$$= \sum_{i=1}^{n} A(x_i^*) \Delta x$$

and is exactly $= \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*) \Delta x$

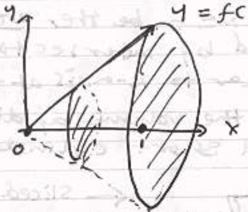
$$\boxed{= \int_a^b A(x)\, dx}$$

This lets us compute volume as an integral!

An important class of solids are the solids of revolution obtained by rotating a region in x,y-plane about x-axis;

E.g. Find the volume of the cone obtained by rotating the area below $y=x$ (and above x-axis) from $x=0$ to $x=1$ about the x-axis.

Sketch:



at any x with $0 \le x \le 1$ ⇐ cross-section of cone is a circle of radius $f(x)=x$

Since in this case $A(x) =$ area of circle of radius $f(x)$

$$= \pi (f(x))^2 = \pi x^2$$

we can use the integral formula for volume to get

$$\text{Volume of cone} = \int_0^1 \pi x^2 \, dx = \frac{\pi}{3} x^3 \Big]_0^1 = \boxed{\frac{\pi}{3}}$$

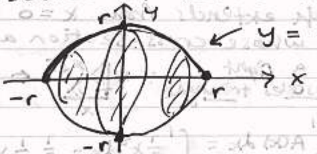
We see that in general the volume of a solid of revolution obtained by rotating the area below the curve $y=f(x)$ from $x=a$ to $x=b$ about the x-axis is

$$\boxed{= \int_a^b \pi \, (f(x))^2 \, dx}$$

since every cross-section is a circle of radius $= f(x)$

E.g.: Find the volume of a sphere of radius r using an integral.

To do this, we have to realize the sphere as a solid of revolution:

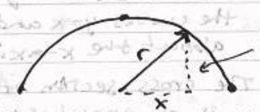


$\leftarrow y = \sqrt{r^2 - x^2}$

We see that a sphere is obtained by rotating a semicircle of radius r about x-axis, and

semicircle of radius r = area below curve $y = \sqrt{r^2 - x^2}$ from $x = -r$ to $x = r$

(Think:



$y = \sqrt{r^2 - x^2}$ since $x^2 + y^2 = r^2$ by Pythagorean Thm.)

Thus, according to the formula for volume of a solid of revolution, we have:

volume of sphere of radius r $= \int_{-r}^{r} \pi \left(\sqrt{r^2 - x^2}\right)^2 dx$

$= \pi \int_{-r}^{r} (r^2 - x^2)\, dx$

$= \pi \left(r^2 x - \frac{x^3}{3} \Big]_{-r}^{r} \right)$

$= \pi \left(\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 - \frac{-r^3}{3}\right) \right)$

$= \pi \left(2r^3 - \frac{2}{3} r^3\right) = \boxed{\frac{4}{3}\pi r^3}$. //

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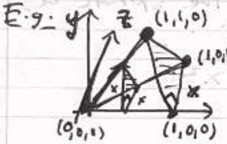
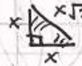
<u>More</u> <u>about</u> <u>volumes</u> §6.2

Solids of revolutions have cross-sections that are <u>circles</u> (or annulus, see below...)
but the formula $\int_a^b A(x)\,dx$ for volume works for <u>other</u> <u>shapes</u> too...

E.g.:



Let's consider the <u>triangular</u> <u>cone</u> which extends from $x=0$ to $x=1$ and whose cross-section at x is a <u>right</u> <u>isosceles</u> <u>triangle</u>:

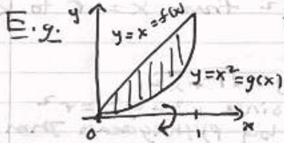


⇐ area $= A(x) =$ $\frac{1}{2}$ base × height $= \frac{1}{2}x^2$

Then the volume of this triangular cone $= \int_0^1 A(x)\,dx = \int_0^1 \frac{1}{2}x^2\,dx = \frac{1}{2}\,\frac{1}{3}x^3\Big]_0^1 = \boxed{\frac{1}{6}}$.

Returning to <u>solids</u> of <u>revolution</u> ... we can also rotate the region <u>between</u> two curves about an axis.

E.g.



Let's rotate the region between the curves $y=x$ and $y=x^2$ from $x=0$ to $x=1$ about the x-axis to make a solid.

The cross-section of this solid is an <u>annulus</u>: the region between two circles

"annulus" a.k.a. "washer" shape →

← area of annulus is $\pi(r_2^2 - r_1^2)$

In the case of region between two curves $y=f(x)$ and $y=g(x)$, the area of this cross-section is $A(x) = \pi(f(x)^2 - g(x)^2)$.

So the volume of the solid is $= \int_a^b \pi(f(x)^2 - g(x)^2)$.

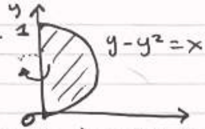
In above example with $f(x) = x$ and $g(x) = x^2$,

we get volume $= \int_0^1 \pi(x^2 - (x^2)^2)\,dx = \int_0^1 \pi(x^2 - x^4)\,dx$

$= \pi\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \boxed{\pi\,\frac{2}{15}}$

Sometimes we want to rotate across $\underline{y\text{-axis}}$ instead of x-axis.

E.g.

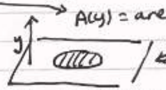


How can we compute the volume of the solid obtained by rotating the region between y-axis and curve $y-y^2=x$ about the $\underline{y\text{-axis}}$?

We just do same thing we've been doing, but with respect to y!

$$\text{volume of solid} = \int_a^b A(y)\,dy$$

A(y) = area of y cross-section



$$= \int_0^1 \pi(y-y^2)^2\,dy$$

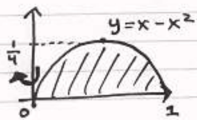
since y-cross-section is circle of radius $f(y)=y-y^2$

$$= \int_0^1 \pi(y^2-2y^3+y^4)\,dy$$

$$= \pi\left[\tfrac{1}{3}y^3 - \tfrac{2}{4}y^4 + \tfrac{1}{5}y^5\right]_0^1 = \pi\left(\tfrac{1}{3}-\tfrac{1}{2}+\tfrac{1}{5}\right) = \boxed{\frac{\pi}{30}}.$$

What about the following solid of revolution problem?

E.g.



Compute the volume of solid obtained by rotating region below $y=x-x^2$ (and above x-axis) about the $\underline{y\text{-axis}}$.

To do this following the method above, we would have to realize this region as the region between two curves

- $x=f(y)$ and $x=g(y)$ and <u>integrate</u> <u>w.r.t.</u> <u>y</u>.

(To find $f(y)$ and $g(y)$ we need to "invert" $y=x-x^2$ using the quadratic formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\Rightarrow f(y)=\frac{1+\sqrt{1-4y}}{2}$ and $g(y)=\frac{1-\sqrt{1-4y}}{2}$.)

But... there is a <u>better approach</u> using integration <u>w.r.t. X</u>

...

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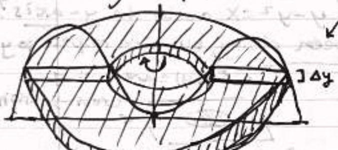
The method of cylindrical shells §6.3

To compute the volume of a solid of revolution obtained by rotating the region below $y=f(x)$ about the y-axis, using the previous method, we break into "thin washers":



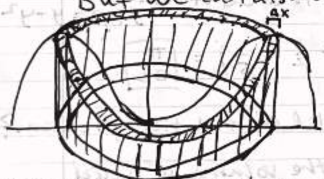
thickness of "washer" $= \Delta y$

volume of washer $= \Delta y \times$ area of annulus

$= \Delta y \cdot \pi (r_2^2 - r_1^2)$

$\Rightarrow \int \text{~~~} \, dy$ (integrate w.r.t. y)

But we can also break this solid into hollow cylindrical shells:



⇝ these shells look like:

gap in radii $= \Delta x$

height $= f(x_i^*)$

(think: very thin tubes of toilet paper)

(please see the textbook for better 3D pictures!)

By breaking the solid into many cylindrical shells in this way we obtain the following formula:

$$\text{Volume of solid} \approx \sum_{i=1}^{n} \text{volume of } i^{\text{th}} \text{ shell}$$

$$= \sum_{i=1}^{n} \text{height of } i^{\text{th}} \text{ shell} \cdot \text{area of annulus base}$$

$$= \sum_{i=1}^{n} f(x_i^*) \cdot \pi\left((x_i^* + \Delta x)^2 - (x_i^*)^2 \right)$$

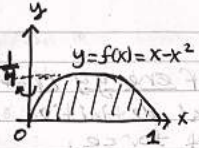
$$= \sum_{i=1}^{n} f(x_i^*) \cdot 2\pi x_i^* \Delta x + \sum_{i=1}^{n} f(x_i^*) \pi (\Delta x)^2$$

and in the limit $n \to \infty$ we get

$$\text{Volume} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) 2\pi x_i^* \Delta x + \underbrace{\sum_{i=1}^{n} f(x_i^*) \pi (\Delta x)^2}_{\to 0 \text{ as } n \to \infty}$$

$$\boxed{= \int_a^b 2\pi f(x) \cdot x \, dx}$$

E.g.

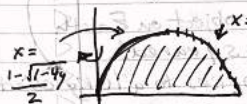


Returning to the example of solid obtained by rotating region below $y=x-x^2$ about <u>y-axis</u>,

its Volume $= \int_a^b 2\pi f(x)\, x\, dx = \int_0^1 2\pi (x-x^2)x\, dx$

$= 2\pi \int_0^1 x^2 - x^3\, dx = 2\pi \left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \boxed{\frac{\pi}{6}}$

Using the "washer" method instead, we would have to compute:



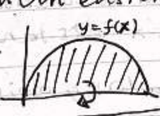
Volume $= \int_0^{1/4} \pi\left(\left(\frac{1+\sqrt{1-4y}}{2}\right)^2 - \left(\frac{1-\sqrt{1-4y}}{2}\right)^2\right) dy$

$= \cdots = \pi/6$

which involves much harder algebra!

<u>Upshot</u>: Both the "<u>disks</u>/<u>washers</u>" method and the "<u>cylindrical</u> <u>shells</u>" method will work to compute the volume of a solid of revolution, but often one will lead to a much easier integral.

E.g.



for region below curve $y=f(x)$ rotated about <u>x-axis</u>, use "<u>disk</u>/<u>washer</u>" method to get

$$\boxed{\underline{\text{volume}} = \int_a^b \pi\,(f(x))^2\, dx}$$

E.g.

for region below curve $y=f(x)$ rotated about <u>y-axis</u> use "<u>cylindrical</u> <u>shells</u>" method to get

$$\boxed{\underline{\text{volume}} = \int_a^b 2\pi\, f(x)\cdot x\, dx}$$

For <u>other</u> kinds of regions ... <u>guess</u> or <u>try</u> <u>both</u> methods! //

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Work §6.4

Intuitively, work is the amount of energy spent accomplishing some task. The formal definition in physics depends on the notion of force.

You can think of force as the push/pull on an object. Its formal definition comes from Newton's 2^{nd} Law:

$$F = ma$$

$$\text{Force} = \text{Mass} \times \text{Acceleration}$$

mass

E.g. The acceleration due to gravity of an object on Earth is $9.8\ m/s^2$ (meters per second squared).

So the amount of force that gravity applies to a 10 kg object is $10\,kg \times 9.8\,m/s^2 = 98\ kg\,m/s^2 = 98$ Newtons $= 98\ N$

this is called weight   SI unit of force

Work is force applied over a distance. Specifically, if an object moves a distance d while experiencing a constant force F (i.e., constant acceleration & mass), we define work done $= Fd$ = Force × distance.

E.g. What is the work done lifting a 10 kg object 100 m in the air? We use the formula:

work = force × distance = $98\ N \times 100\ m$

$= 9800\ Nm$

$= 9800$ Joules

$= 9800\ J$

to lift an object we must counteract gravity

SI unit of energy

But what if the object experiences a non-constant force? How do we find work done? We need calculus!

Suppose our object moves from $x=a$ to $x=b$ and at each point x in between experiences force $f(x)$.
As usual, we can approximate the work done breaking the interval $[a,b]$ into sub-intervals $[x_0,x_1], [x_1,x_2], \dots, [x_{n-1}, x_n]$ (of width $\Delta x = \frac{b-a}{n}$) and selecting sample points x_i^* in $[x_{i-1}, x_i]$.
The work done moving across the i^{th} sub-interval is

$$W_i \approx \underbrace{f(x_i^*) \cdot \Delta x}_{\text{force} \times \text{distance}}$$

So the total work is then approximately:

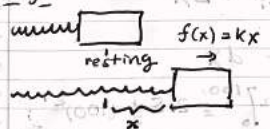
$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} f(x_i^*) \Delta x.$$

We get an exact value for work as an integral:

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \Delta x = \boxed{\int_a^b f(x)\,dx}$$

work = integral of force over distance

E.g.



Hooke's Law says that the force needed to maintain a spring stretched a distance x from its resting state is given by

$$f(x) = k \cdot x$$

where k is the "spring constant".

Q: Suppose a spring has a spring constant of $k = 10 \frac{N}{m}$. How much work is done stretching this spring 0.5 m?

A: At a stretch distance of x (meters), we need to apply force

$$f(x) = kx = 10x \text{ N} \quad \text{by Hooke's Law.}$$

So Work = integral of force over distance $= \int_0^{0.5} f(x)\,dx = \int_0^{0.5} 10x\,dx = 10 \frac{1}{2} x^2 \Big]_0^{0.5}$

$$= 10 \cdot \frac{1}{2} \cdot 0.25 = \boxed{1.25 \text{ J}}$$

E.g.  A 100 meter cable hangs off a building. Its weight is 250 Newtons. How much work is done lifting the rope to the top of the building?

Let's show two (related) approaches to this problem:

① Break the cable into n intervals of length $\Delta x = \frac{100}{n}$ m.
Let x_i^* be a point in the i^{th} interval.
All the points in the i^{th} interval must be raised $\approx x_i^*$ meters to bring them to the top.

Since the $\underline{\text{weight density}}$ of the cable is $\frac{250\text{ N}}{100\text{ m}} = 2.5 \frac{\text{N}}{\text{m}}$

the $\underline{\text{weight}}$ of the i^{th} segment is $2.5 \frac{\text{N}}{\text{m}} \cdot \Delta x$ m

So total work $W \approx \sum_{i=1}^{n} 2.5 \cdot x_i^* \cdot \Delta x$

and taking limit $n \to \infty$ gives $W = \int_0^{100} 2.5\, x \, dx$

$$= 2.5 \cdot \tfrac{1}{2} x^2 \Big]_0^{100} = 2.5 \tfrac{1}{2} (100)^2$$
$$= \boxed{12500 \text{ J}}$$

② After we have pulled up x meters of the cable, there is $(100 - x)$ meters left, and this weighs
$$f(x) = \underbrace{2.5}_{\text{weight density}} \cdot (100 - x) \text{ N}.$$

Integrating this force over the distance gives:
$$\int_0^{100} 2.5\,(100 - x)\,dx = -\tfrac{1}{2}\, 2.5\, (100 - x)^2 \Big]_0^{100}$$
$$= \boxed{12500 \text{ J}}$$

simple u-sub. to anti-differentiate

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Average value of a function §6.5

To compute the average of a finite list $y_1, y_2, \dots, y_n \in \mathbb{R}$ of real numbers, we add them up and then divide by the number of items in the list:

$$\text{average value} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

E.g. To compute the average height of a person in this room, we sum the heights of all people and then divide by # of people.

But what about computing: The average temperature during a day. A day has ∞-many times, so we cannot just add all the temperatures and divide.
Instead, we approximate by choosing n times to measure temperature at, then let $n \to \infty$.

Def'n If $f(x)$ is a continuous function on $[a,b]$, pick some n and let $x_0 = a$, $x_i = x_{i-1} + \Delta x$ for $i = 1, \dots, n$, where $\Delta x = \frac{b-a}{n}$ as usual. To approximate the average of $f(x)$ on $[a,b]$, we sample f at the points $x_1, x_2, \dots, x_n$ and average these samples:

$$\text{avg. value of } f(x) \text{ on } [a,b] \approx \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$$

And to define average exactly, we let $n \to \infty$:

$$\text{avg. value of } f(x) \text{ on } [a,b] = \lim_{n\to\infty} \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$$

$$= \lim_{n\to\infty} \frac{\sum_{i=1}^{n} f(x_i)}{n} = \frac{1}{b-a} \sum_{i=1}^{n} f(x_i)\,\Delta x \quad \text{since } \Delta x = \frac{b-a}{n}$$

$$\boxed{= \frac{1}{b-a} \int_a^b f(x)\,dx}$$

"average of function on interval = $\dfrac{\text{integral of function on interval}}{\text{length of interval}}$"

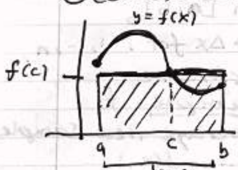
E.g.: Let's compute the average of $f(x) = 1 + x^2$ on $[-1,2]$.

$$\text{avg.} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2-(-1)}\int_{-1}^{2} 1 + x^2\,dx$$
$$= \frac{1}{3}\left[x + \frac{1}{3}x^3\right]_{-1}^{2} = \frac{1}{3}\left(\left(2+\frac{8}{3}\right) - \left(-1-\frac{1}{3}\right)\right) = \frac{1}{3}\cdot\frac{18}{3} = \boxed{2}.$$

<u>Thm</u> (Mean Value Theorem for Integrals)
If $f(x)$ is a continuous function defined on $[a,b]$, then there exists a point c with $a \le c \le b$ s.t.
$$f(c) = f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

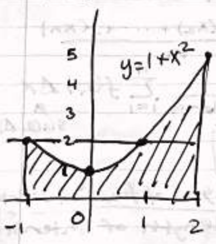
MVT for integrals says there is some time during the day when the temperature is exactly the <u>average temperature</u> for that day.

Geometrically:



MVT for integrals says that there is a c in $[a,b]$ s.t. area under curve $y = f(x)$ from $x=a$ to $x=b$ is same as area of <u>rectangle</u> of <u>height</u> $f(c)$ and <u>width</u> $b-a$.

E.g.:



Since the average of $f(x) = 1 + x^2$ on $[-1,2]$ is $f_{avg} = 2$, MVT for integrals says there is some c in $[-1,2]$ s.t. $f(c) = 2$. Actually, there are <u>two</u> such c's: $c = -1$ and $c = 1$ (Since $1 + (-1)^2 = 1 + 1^2 = 2$.)
Could solve for c by setting
$$2 = f(c) = 1 + c^2 \Rightarrow c = \pm 1.$$

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