Howard Math 274, HW# 1,

Spring 2022; Instructor: Sam Hopkins; Due: Friday, February 11th

1. A k-ary necklace of length n is a rotation equivalence class of colorings of the vertices of an n-gon with k colors. Use unweighted Pólya counting to show that the number of k-ary necklaces of length n is

$$\frac{1}{n} \sum_{d|n} \varphi(d) k^{\frac{n}{d}}.$$

This formula uses some notation from number theory: $d \mid n$ means "d divides n"; and $\varphi(d)$ is Euler's totient function, the number of $1 \le j \le d$ with gcd(d, j) = 1.

- 2. Continuing the previous problem, now using weighted Pólya counting: how many ways, up to rotation, can the vertices of a hexagon be colored with 2 red, 2 green, and 2 blue vertices?
- 3. There are 24 orientation-preserving symmetries of a cube—they are all spatial rotations. Use unweighted Pólya counting to give a formula for the number of ways, up to orientation-preserving symmetries, to color the faces of a cube with k colors.
 - **Hint 1**: Your formula should be a polynomial in k.
 - Hint 2: This group of symmetries is *abstractly* isomorphic to the symmetric group S_4 (but of course there are six, not four, faces of a cube); for more information on this group see for instance the Wikipedia page https://en.wikipedia.org/wiki/Octahedral_symmetry.
- 4. Continuing the previous problem, now using weighted Pólya counting: how many ways, up to orientation-preserving symmetries, can the faces of a cube be colored with 2 red, 2 green, and 2 blue faces?
- 5. Let $\mathcal{M}_{n\times m}(k)$ be the set of $n\times m$ matrices with entries from the set $\{1,2,\ldots,k\}$. For example,

$$\begin{pmatrix} 2 & 3 & 4 & 2 \\ 3 & 1 & 2 & 3 \\ 4 & 3 & 5 & 2 \end{pmatrix} \in \mathcal{M}_{3\times 4}(5).$$

The symmetric group S_n acts on $\mathcal{M}_{n\times m}(k)$ by permuting rows: e.g., for $\sigma=(1,2)(3)\in S_3$,

$$\sigma \cdot \begin{pmatrix} 2 & 3 & 4 & 2 \\ 3 & 1 & 2 & 3 \\ 4 & 3 & 5 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 & 3 \\ 2 & 3 & 4 & 2 \\ 4 & 3 & 5 & 2 \end{pmatrix}.$$

Let $\widetilde{\mathcal{M}}_{n\times m}(k)$ denote the set of S_n -equivalence classes of $\mathcal{M}_{n\times m}(k)$. Give a formula (in terms of n, m, and k) for $\#\widetilde{\mathcal{M}}_{n\times m}(k)$.

Hint: To simplify your formula you may use the fact, which we proved last semester, that the (unsigned) Stirling numbers of the 1st kind $c(n,j) := \#\{\sigma \in S_n : \sigma \text{ has } k \text{ cycles}\}$ have generating function $\sum_{j=1}^n c(n,j)t^j = t(t+1)\cdots(t+n-1)$.

Hard bonus problem, just to think about: What if I'm allowed to independently permute both rows and columns of the matrix?