

2/6

Trigonometric substitution § 7.3

It is often possible to compute integrals involving $(a^2 - x^2)$ by writing $x = \sin(u)$ so that $(a^2 - x^2) = (a^2 - \sin^2 u) = a^2(1 - \sin^2 u) = a^2 \cos^2 u$.

E.g. Let's compute $\int \frac{1}{\sqrt{1-x^2}} dx$ this way.

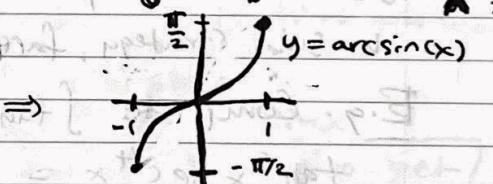
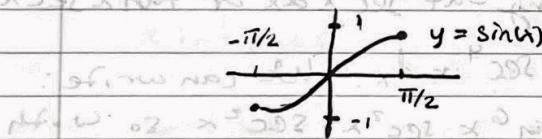
Write $x = \sin(u) \Rightarrow dx = \cos(u) du$ so that

$$\begin{aligned}\int \frac{1}{\sqrt{1-x^2}} dx &= \int \frac{1}{\sqrt{1-\sin^2 u}} \cos(u) du = \int \frac{1}{\sqrt{\cos^2 u}} \cos(u) du \\ &= \int \frac{1}{\cos(u)} \cos(u) du = \int du = u + C\end{aligned}$$

This is the answer in terms of u , but we want the x answer.

Since $x = \sin(u) \Rightarrow u = \arcsin(x)$ (also written $\sin^{-1}(x)$),

Recall: $y = \arcsin(x) \Leftrightarrow \sin(y) = x$ for $-\pi/2 \leq y \leq \pi/2$
inverse function



e.g. since $\sin(\pi/2) = 1$ have $\arcsin(1) = \pi/2$, etc..

$$\text{Thus, } \int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + C$$

Notice: for this problem we used a u -substitution but it was a "reverse" u -substitution where we wrote $x = f(u)$ instead of $u = f(x)$. This is okay as long as you correctly compute the differential $dx = f'(u) du$.

Trig substitutions can be very useful when dealing with circles and related shapes..

E.g. Let's compute the area of a circle of radius r using an integral.

The equation of a circle is $x^2 + y^2 = r^2$.

If we solve for y we get $y = \sqrt{r^2 - x^2}$,
and the area under this curve = $\frac{1}{2}$ area of circle:



So area of circle of radius r = $2 \cdot \int_{-r}^r \sqrt{r^2 - x^2} dx$. Let's solve this integral by trig. sub.

Since we see $r^2 - x^2$ we set $x = r \cdot \sin(\theta) \Rightarrow dx = r \cos(\theta) d\theta$.

$$\begin{aligned}\Rightarrow \int \sqrt{r^2 - x^2} dx &= \int \sqrt{r^2 - r^2 \sin^2(\theta)} r \cos(\theta) d\theta \\ &= \int r \sqrt{1 - \sin^2(\theta)} r \cos(\theta) d\theta = r^2 \int \cos \theta \cos \theta d\theta = r^2 \int \cos^2 \theta d\theta.\end{aligned}$$

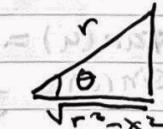
How to solve $\int \cos^2 \theta d\theta$? We can do int. by parts:

$$\begin{aligned}\int \frac{\cos \theta \cos \theta d\theta}{u} &= \frac{\cos \theta \sin \theta}{u} - \int \frac{\sin \theta \sin \theta d\theta}{v} = \cos \theta \sin \theta + \int \sin^2 \theta d\theta \\ &= \cos \theta \sin \theta + \int (1 - \cos^2 \theta) d\theta = \cos \theta \sin \theta + \sin \theta - \int \cos^2 \theta d\theta\end{aligned}$$

$$\Rightarrow 2 \int \cos^2 \theta d\theta = \cos \theta \sin \theta + \theta \Rightarrow \int \cos^2 \theta d\theta = \frac{1}{2} (\cos \theta \sin \theta + \theta)$$

$$\text{So } \Rightarrow \int \sqrt{r^2 - x^2} dx = r^2 / 2 (\cos \theta \sin \theta + \theta) \text{ when } x = r \sin \theta$$

Picture of relationship between r & θ :



$$\sin \theta = \frac{x}{r}$$

$$\cos \theta = \frac{\sqrt{r^2 - x^2}}{r}$$

$$\theta = \arcsin\left(\frac{x}{r}\right).$$

$$\text{Thus } \Rightarrow \int \sqrt{r^2 - x^2} dx = r^2 / 2 \left(\frac{\sqrt{r^2 - x^2}}{r} \frac{x}{r} + \arcsin\left(\frac{x}{r}\right) \right)$$

$$= \frac{x}{2} \sqrt{r^2 - x^2} + r^2 / 2 \arcsin\left(\frac{x}{r}\right).$$

$$\Rightarrow \frac{1}{2} \text{ area of circle} = \int_{-r}^r \sqrt{r^2 - x^2} dx = \left[\frac{x}{2} \sqrt{r^2 - x^2} + r^2 / 2 \arcsin\left(\frac{x}{r}\right) \right]_{-r}^r$$

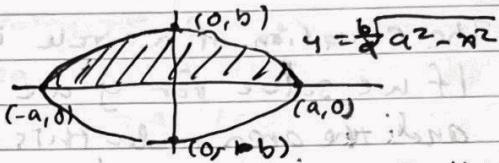
$$= (0 + \frac{r^2}{2} \arcsin(1)) - (0 + \frac{r^2}{2} \arcsin(-1)) = \frac{r^2}{2} \left(\frac{\pi}{2} - \frac{-\pi}{2} \right) = \frac{\pi r^2}{2}$$

2/8

E.g. We can find the area of an ellipse very similarly...

Ellipse equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



$\Rightarrow y = \frac{b}{a} \sqrt{a^2 - x^2}$ is upper curve of ellipse

$$\begin{aligned} \Rightarrow \text{area of ellipse} &= \int_{-a}^a \frac{b}{a} \sqrt{a^2 - x^2} dx \quad \text{take } x = a \sin \theta \\ &= \frac{b}{a} \left(\int_{-\pi/2}^{\pi/2} \sqrt{a^2 - a^2 \cos^2 \theta} d\theta \right) = \frac{b}{a} \left(\frac{\pi a^2}{2} \right) = \boxed{\frac{ab\pi}{2}} \end{aligned}$$

Sometimes we see expressions of form $(a^2 + x^2)$, in that case we take $x = a \tan(u)$ because of identity $1 + \tan^2 \theta = \sec^2 \theta$

E.g. Let's compute $\int \frac{1}{(1+x^2)^2} dx$ with a trig. sub.

We let $x = \tan(u) \Rightarrow dx = \sec^2(u) du$

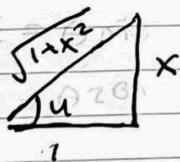
(recall: $d/dx (\tan(u)) = \sec^2(u)$)

$$\text{Thus } \int \frac{1}{(1+x^2)^2} dx = \int \frac{1}{(1+\tan(u))^2} \sec^2(u) du = \int \frac{1}{(\sec^2(u))^2} \sec^2(u) du$$

$$= \int \frac{1}{\sec^2(u)} du = \int \cos^2(u) du = \frac{\sin(u) \cos(u) + u}{2} + C$$

we just saw this

draw picture
of relationship
between x and u :



$$\tan(u) = x$$

$$\sin(u) = \frac{x}{\sqrt{1+x^2}}$$

$$\cos(u) = \frac{1}{\sqrt{1+x^2}} \quad (\text{or } \tan^{-1}(x))$$

$$\Rightarrow \int \frac{1}{(1+x^2)^2} dx = \frac{1}{2} \left(\frac{x}{\sqrt{1+x^2}} \times \frac{1}{\sqrt{1+x^2}} + \arctan(x) \right) + C$$

$$= \frac{1}{2} \left(\frac{x}{1+x^2} + \tan^{-1}(x) \right) + C \quad \checkmark$$

Exercise: What if we did $\int \frac{1}{(4+x^2)^2} dx$ instead?

Or even simpler: $\int \frac{1}{4+x^2} dx$.

§ 7.4

Integration of rational functions by partial fractions

Recall that a rational function is $f(x) = \frac{P(x)}{Q(x)}$ where $P(x), Q(x)$ polynomials.

We will now describe procedure for computing $\int \frac{P(x)}{Q(x)} dx$.

① Recall that the degree of a polynomial $P(x)$ is highest power of x in $P(x)$: e.g. $\deg(P(x)) = 3$ for $P(x) = x^3 + 5x + 4$.

If $\deg(P(x)) \geq \deg(Q(x))$ then we can use long division

to write $\frac{P(x)}{Q(x)} = \frac{S(x)}{Q(x)} + R(x)$ where $\deg(S(x)) < \deg(Q(x))$.

$$\text{E.g. } \frac{2x^3 + 1}{x^2 - 1} = 2x + \frac{2x + 1}{x^2 - 1}$$

Since it is easy to integrate polynomials, from now on assume $\deg(P(x)) < \deg(Q(x))$.

① First suppose the denominator $Q(x)$ factors into distinct linear terms.

$$\text{E.g. w/ } \frac{P(x)}{Q(x)} = \frac{2x+1}{x^2-1} = \frac{2x+1}{(x+1)(x-1)} \leftarrow \begin{matrix} \text{distinct} \\ \text{linear factors.} \end{matrix}$$

$$\text{Then we write: } \frac{P(x)}{(x-a)(x-b)\dots(x-z)} = \frac{A}{x-a} + \frac{B}{x-b} + \dots + \frac{Z}{x-z}.$$

$$\text{E.g. } \frac{2x+1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \text{ for some } A, B \in \mathbb{R} \quad \text{we need to solve for:}$$

$$\text{multiply by } Q(x) \Rightarrow 2x+1 = A(x-1) + B(x+1)$$

$$2x+1 = (A+B)x + (-A+B)1$$

$$\text{equate coeffs } \begin{cases} A+B=2 \\ -A+B=1 \end{cases} \quad \begin{matrix} A=1 \\ B=1+A \end{matrix}$$

$$A+A+1=2 \Rightarrow A=\frac{1}{2} \Rightarrow B=1+\frac{1}{2}=\frac{3}{2}$$

$$\text{So } \frac{2x+1}{(x+1)(x-1)} = \frac{1/2}{x+1} + \frac{3/2}{x-1} \leftarrow \begin{matrix} \text{we can integrate these!} \\ \text{using logarithms!} \end{matrix}$$

$$\text{Thus, } \int \frac{2x+1}{(x+1)(x-1)} dx = \int \frac{1/2}{x+1} dx + \int \frac{3/2}{x-1} dx$$

$$= \frac{1}{2} \ln(x+1) + \frac{3}{2} \ln(x-1) + C$$

NOTE: In general $\int \frac{1}{x+a} = \ln(x+a)$ (easy u-sub).

2/10

② If $Q(x)$ has repeated linear factors, partial fractions is slightly more complicated... let's see an example:

E.g. For $\frac{P(x)}{Q(x)} = \frac{2x+1}{(x-1)^2}$ we write:

$$\text{mult. by } Q(x) \quad \frac{2x+1}{(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} \quad \text{in general we have powers } (x-a)^r$$

Then we solve for $A, B \in \mathbb{R}$ as before:

$$2x+1 = A(x-1) + B \quad \begin{array}{l} \text{equate coeffs} \\ \Rightarrow A = 2 \quad -A+B=1 \\ B = 1+A \\ B = 3, \end{array}$$

Thus
$$\int \frac{2x+1}{(x-1)^2} dx = \int \frac{2}{(x-1)} dx + \int \frac{3}{(x-1)^2} dx$$

$= 2 \ln(x-1) - 3(x-1)^{-1} + C$

recall to integrate

So in general we will get terms

like $\ln(x+a)$ and $(x+a)^{-r}$.

③ If $Q(x)$ has irreducible quadratic factors, then partial fractions won't work; instead need trig. sub.

E.g. For $\int \frac{1}{x^2+4} dx$ cannot write $(x^2+4) = (x+a)(x+b)$ for real #'s a, b since

Instead, use $x = 2\tan\theta$ would need $\sqrt{a^2+b^2}$ of reals.

$$\Rightarrow dx = 2\sec^2\theta d\theta$$

$$\Rightarrow \int \frac{1}{x^2+4} dx = \int \frac{1}{4\tan^2\theta+4} 2\sec^2\theta d\theta = \frac{1}{2} \int \frac{1}{\tan^2\theta+1} \sec^2\theta d\theta$$

$$= \frac{1}{2} \int \frac{1}{\sec^2\theta} \sec^2\theta d\theta = \frac{1}{2} \int d\theta = \frac{1}{2}\theta + C$$

$$= \frac{1}{2} \arctan\left(\frac{x}{2}\right) + C \quad \text{since } \tan\theta = \frac{x}{2}.$$

Summary of strategies for Integration § 7.5

We have now learned many integration techniques. When presented w/ an integral, it can be tricky to decide what to do!

Here are some general guidelines:

- ① Know and recognize basic integrals such as

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \int \ln(x) dx = x \ln(x) - x, \int e^x dx = e^x, \int \sin(x) dx = -\cos(x)$$

$$\int \cos(x) dx = \sin(x), \int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x), \int \frac{1}{1+x^2} dx = \arctan(x), \dots$$

- ② If you see a function $f(x)$ and its derivative $f'(x)$ in the integrand, try u-substitution.

- ③ If the integrand is a product of two terms (especially, a polynomial times exponential or trig function...) try integration by parts

- ④ For things like $\int \sin^n x \cos^m x dx$ use the trick we learned of exploiting $\boxed{\sin^2 x + \cos^2 x = 1}$

- ⑤ If you see $a^2 - x^2$ appear, try trig. sub. $x = a \sin(\theta)$. If you see $a^2 + x^2$, try trig. sub. $x = a \tan(\theta)$.

- ⑥ For a rational function $\frac{P(x)}{Q(x)}$, try the technique of partial fraction decomposition.

Sometimes you may need to apply multiple of these steps, and sometimes multiple times.

Even integrals that look similar can require different strategies!

$$\int \frac{x}{x^2+1} dx$$

u-sub w/
 $u = x^2 + 1$

$$\int \frac{1}{x^2+1} dy$$

trig sub
 $y = \tan(\theta)$

$$\int \frac{1}{x^2-1} dx$$

partial
fractions!

2/13

Approximate Integration §7.7

Sometimes a definite integral is difficult or impossible to evaluate exactly, and we'd like to get an approximation.

Recall how the definite integral is defined:

- we break $[a, b]$ into n sub intervals $[x_i, x_{i+1}]$ of width $\Delta x = \frac{b-a}{n}$ (so $x_i = a + i\Delta x$ for $i=0, 1, \dots, n$)
- for each sub interval $[x_{i-1}, x_i]$ we select a point $x_i^* \in [x_{i-1}, x_i]$ (so we get n points x_1^*, \dots, x_n^*)
- we let $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$.

We can thus get an approximation for $\int_a^b f(x) dx$ by fixing a finite value of n and choosing particular x_i^* .

In Calc 1 we saw the left- and right-endpoint approximations.

$$\int_a^b f(x) dx \approx L_n = \sum_{i=1}^n f(x_{i-1}) \Delta x \text{ and } \int_a^b f(x) dx \approx R_n = \sum_{i=1}^n f(x_i) \Delta x.$$

A better approximation is to let $x_i^* = \bar{x}_i = \frac{x_{i-1} + x_i}{2}$ be the midpoint of the sub-intervals, giving the midpoint approx.:

$$\int_a^b f(x) dx \approx M_n = \sum_{i=1}^n f(\bar{x}_i) \Delta x.$$

Fig: Let's approx. $\int_{-2}^4 x^3 - 2x + 4 dx$ using midpoint approx.

With $n=3$ sub intervals: $\Delta x = \frac{4-(-2)}{3} = \frac{6}{3} = 2$

$y = f(x)$
 $= x^3 - 2x + 4$ The intervals are therefore,

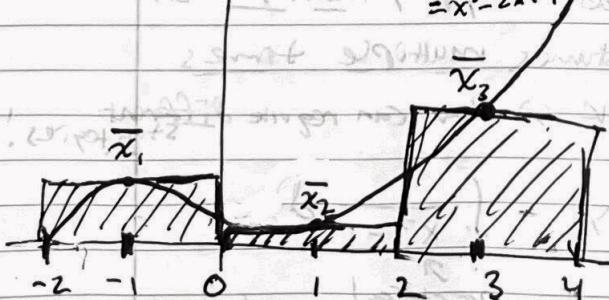
$$[-2, 0], [0, 2], [2, 4]$$

with midpoints $\bar{x}_1 = -1, \bar{x}_2 = 1, \bar{x}_3 = 3$

$$f(-1) = (-1)^3 - 2(-1) + 4 = 5$$

$$f(1) = (1)^3 - 2(1) + 4 = 3$$

$$f(3) = (3)^3 - 2(3) + 4 = 25$$



$$\text{So } M_3 = 5 \cdot 2 + 3 \cdot 2 + 25 \cdot 2$$

$$= 33 \cdot 2 = \boxed{66}$$

Another good approx. of $\int_a^b f(x) dx$ is the trapezoid approx.:

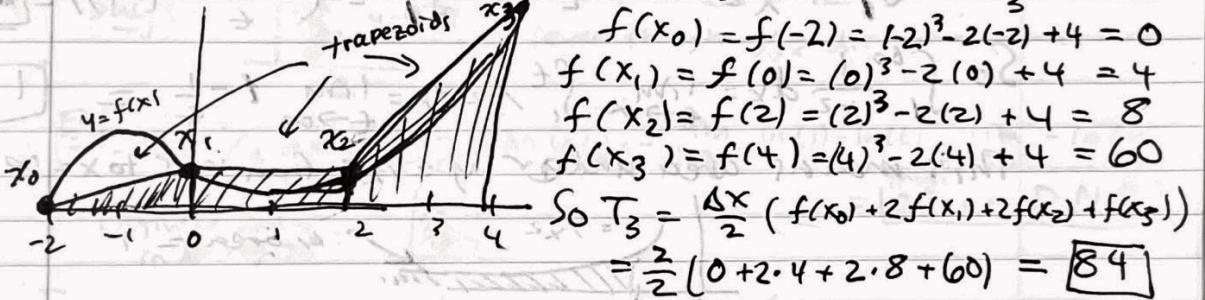
$$\int_a^b f(x) dx \approx T_n = \frac{\Delta x}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n))$$

↑
2's everywhere except x_0 and x_n

It is called "trapezoid" approx. because unlike other approx's using rectangles, it breaks area under curve into trapezoids.

E.g. Let's approx. $\int_{-2}^4 x^3 - 2x + 4 dx$ using trapezoid approx.

with $n=3$ subintervals: again $\Delta x = \frac{4 - (-2)}{3} = 2$



The error of an approx. is how much we need to add to get $\int_a^b f(x) dx$.

$$\text{error} = \int_a^b f(x) dx - \text{approx.}$$

E.g. We can compute the true value of $\int_{-2}^4 x^3 - 2x + 4 dx$ is

$$\begin{aligned} \int_{-2}^4 x^3 - 2x + 4 dx &= \left[\frac{x^4}{4} - x^2 + 4x \right]_{-2}^4 = \left(\frac{4^4}{4} - 4^2 + 4(4) \right) - \left(\frac{(-2)^4}{4} - (-2)^2 + 4(-2) \right) \\ &= (64 - 16 + 16) - (4 - 4 - 8) = 72 \end{aligned}$$

Thus error of $M_3 = 72 - 66 = 6$, error of $T_3 = 72 - 84 = -12$ //

In general: error of M_n and of T_n have opposite sign,

(error of M_n) is about $1/2$ (error of T_n),

and $|\text{error of } M_n|$ and $|\text{error of } T_n| \sim \frac{1}{n^2}$,

meaning if we double n, error gets cut in four.

See book for Simpson's rule which is slightly better error than M_n/T_n but significantly more complicated //

Improper integrals § 7.8

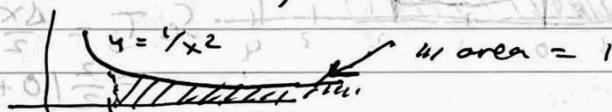
Sometimes we want to find the area under a curve as the curve goes off to infinity. This is called an improper integral:

$$\int_a^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx.$$

$$\text{E.g.: } \int_1^t \frac{1}{x^2} dx = \left[-x^{-1} \right]_1^t = \left(-\frac{1}{t} - (-1) \right) = \boxed{1 - \frac{1}{t}}$$

$$\text{So } \int_1^{\infty} \frac{1}{x^2} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2} dx = \lim_{t \rightarrow \infty} 1 - \frac{1}{t} = \boxed{1}.$$

This means area under $y = 1/x^2$ from $x=1$ to $x=\infty$ is 1:



$$\text{E.g.: On the other hand, } \int_1^{\infty} \frac{1}{x} dx = \left[\ln(x) \right]_1^t = \ln(t) - \ln(1) = \boxed{\ln(t)}$$

$$\text{So } \int_1^{\infty} \frac{1}{x} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x} dx = \lim_{t \rightarrow \infty} \ln(t) = \boxed{\infty \text{ or D.N.E.}}$$

We see that $\int_a^{\infty} f(x) dx$ need not exist as a limit!

Similarly, we define $\int_{-\infty}^a f(x) dx = \lim_{t \rightarrow -\infty} \int_t^a f(x) dx$ and

2-sided improper integral $\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^{\infty} f(x) dx$.

$$\text{E.g.: To compute } \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx \text{ write } \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^0 \frac{1}{1+x^2} dx + \int_0^{\infty} \frac{1}{1+x^2} dx.$$

$$\text{Recall: } \int \frac{1}{1+x^2} dx = \arctan(x)$$

$$\text{So } \int_0^{\infty} \frac{1}{1+x^2} dx = \lim_{t \rightarrow \infty} \left[\arctan(x) \right]_0^t = \lim_{t \rightarrow \infty} \arctan(t) - \arctan(0) = \pi/2$$

$$\text{And similarly } \int_{-\infty}^0 \frac{1}{1+x^2} dx = \pi/2, \text{ so } \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi/2 + \pi/2 = \boxed{\pi}$$

Another kind of improper integral is when the integrand is discontinuous.

Suppose $f(x)$ is continuous on $(a, b]$ but discontinuous at $x=a$.

Then we define $\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$.

E.g. $\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{t \rightarrow 0^+} \left[2\sqrt{x} \right]_t^1 = \lim_{t \rightarrow 0^+} 2 - 2\sqrt{t} = 2$

Says:  this area = 2
(even though $1/\sqrt{x}$ discontinuous at $x=0$)

E.g. $\int_0^1 \frac{1}{x} dx = \lim_{t \rightarrow 0^+} [\ln(x)]_t^1 = \lim_{t \rightarrow 0^+} \ln(1) - \ln(t) = \lim_{t \rightarrow 0^+} -\ln(t) = \infty$ or D.N.E.

Infinite area on ~~region~~: 

Similarly, we define $\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$ for an $f(x)$ that is discontinuous at $x=b$, and if $f(x)$ is continuous on $[a, b]$ except at c then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx \quad \text{if these are convergent.}$$

E.g. For $\int_{-1}^1 \frac{1}{\sqrt{|x|}} dx$, we notice discontinuity at $x=0$;

 and write $\int_{-1}^1 \frac{1}{\sqrt{|x|}} dx = \int_{-1}^0 \frac{1}{\sqrt{|x|}} dx + \int_0^1 \frac{1}{\sqrt{|x|}} dx = 2+2 = 4$
by symmetry both areas are same

E.g. For $\int_{-1}^1 \frac{1}{x^2} dx$, notice discontinuity at $x=0$.

and write $\int_{-1}^1 \frac{1}{x^2} dx = \int_{-1}^0 \frac{1}{x^2} dx + \int_0^1 \frac{1}{x^2} dx = \lim_{t \rightarrow 0^+} [-x^{-1}]_0^t + \lim_{t \rightarrow 0^+} [-x^{-1}]_0^t = \infty + \infty \text{ so } \boxed{\text{D.N.E.}}$

WARNING: If you did $\int_{-1}^1 \frac{1}{x^2} dx = [-x^{-1}]_{-1}^1 = -1 - (-1) = 0$

That would give wrong answer because

You did not notice the discontinuity!