Howard Math 156: Calculus I   Fall 2022
Instructor: Sam Hopkins (sam.hopkins@howard.edu)
(Call me "Sam")

8/22 Logistics

Classes: MTWF, 2:10-3pm, ASB-B ~~[illegible]~~ #105
Office hrs: Tue 1-2pm, Annex III - #220
Or by appointment (email me!)

Website: samuelfhopkins.com/classes/156.html

Text: Calculus, Early Transcendentals by Stewart, 9e

Grading: 40% (in-person) quizzes
40% two (in-person) midterms
20% final exam

There will be 12 in-person quizzes taken on Tuesdays.
(About 20 mins, we will then go over them ~~for~~ rest of class).
Your lowest 2 scores will be dropped (so 10/12 count).

The 2 midterms will happen in-class, also on Tuesdays.

The final will be during finals week

Beyond that, I may assign additional HW (not graded)
and I expect you to SHOW UP TO CLASS
+ PARTICIPATE! :)

that means... Interrupt me by
ASKING QUESTIONS!

(and please say your names when you ask a question
so I learn to put names to faces)

What is calculus about?

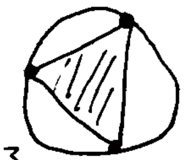
Calculus is different from the math you've seen. It deals with change, with infinities (and infinitesimals) and with limiting processes.

It's good to have a preview of all this new stuff. Let's go over the book's introduction to calculus...

Area of a circle:

We all know that the area of a circle of radius R is πR^2, where $\pi = 3.14159\ldots$ is a special number.

But how would you figure this out if you didn't know?



n=3

← You could try to approximate the area by using a simpler shape, like a regular triangle whose area you already know how to compute.

But this clearly leaves some area out... so you might consider instead regular 4-gon, 5-gon, ...

n=4 n=5 n=6

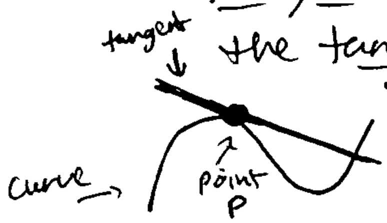
... each inscribed regular n-gon gives a better and better approximation to the area of the circle, and the true area can be calculated by taking a limit as n goes to ∞! infinity.

We won't study this exact problem, but we will consider the area under a curve:

Can also be obtained by a limit of simpler shapes: →

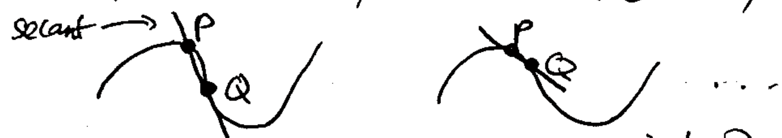
Put thin rectangles under the curve!

Tangent to a curve: How would you find the tangent line to a curve at a point?



The tangent is the line that "just touches" the curve at that point...

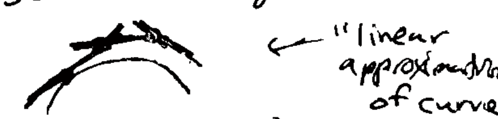
Calling the point P, can draw secant line through P and Q, another nearby point on curve:



As we move the ~~[illegible]~~ point Q closer and closer to P, we get a better and better approx. of the tangent. In the limit, the secant line becomes the tangent line.

Why care about tangents to curves? They tell us about velocity and acceleration in physics (and rates of change in sciences in general).

Also, allow us to approximate whole curve: ("Newton's method"... used by NASA!)



Big idea of calculus:

Even though the area problem and the tangent line problem seem pretty different, they are actually... the same problem or more precisely... the opposite problems!

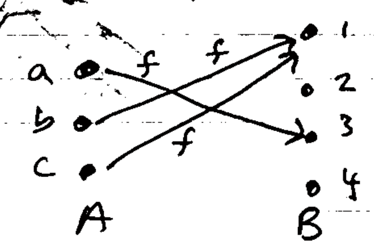
This semester, we will learn why (+how)! //

8/24 Functions (§1.1 of text book)

functions are the basic thing we will study in calculus.
They are fundamental in all sciences as models
e.g. If we produce x units of some product
our revenue may be given by function
$R(x) = p \cdot x$ where p = price of product
(very simple linear model, doesn't take into account costs)
We will see derivative $R'(x)$ (slope of tangent at point x)
is what economists would call "marginal revenue"

But what is a function?
formally, a function f between two sets A and B
is a relation between the elements of A and B
such that every element of A is related to a unique element of B

e.g. $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$



$f(a) = 3$
$f(b) = 1$
$f(c) = 1$

The set A is called the domain of f and set B
is called the codomain. The range of
f is the set of all $f(x)$ for $x \in A$.
E.g. Range for f above is $\{1, 3\}$

The function is called <u>one</u>-<u>to</u>-<u>one</u> if every element in range is related to a <u>unique</u> $x \in A$.

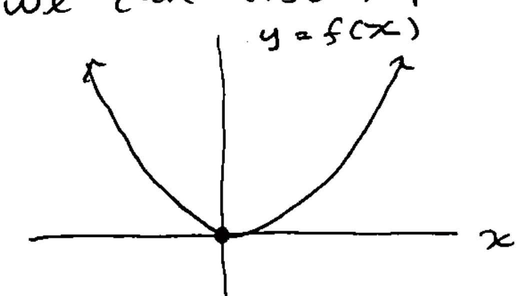
<u>E.g.</u> example f above <u>not</u> one-to-one since $f(b) = 2$ and $f(c) = 2$.

That is the formal mathematical definition of a function, but we will normally work just with functions f whose domain and range are subsets of the real numbers $\mathbb{R}$. Then we have several other ways to represent such an f than an "<u>arrow</u> diagram" or <u>chart</u> (and we have to because there are uncountably infinite # of real numbers).

You are probably used to functions defined by an <u>algebraic formula</u> like

$$f(x) = x^2$$

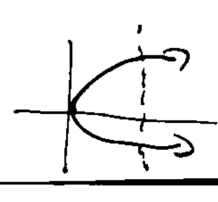
which we can also represent by a <u>graph</u>



How do we know if a graph represents a function?

"<u>Vertical line test</u>": graph represents a function $\Leftrightarrow$ each vertical line intersects ≤ 1 point

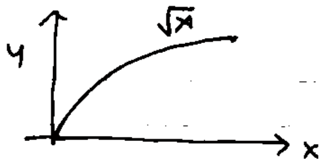
<u>e.g.</u>



graph $x = y^2$ <u>NOT</u> a function because <u>vert.</u> <u>line</u> $x = 4$ intersects two points!

The domain of $f(x) = x^2$ is all of the real numbers also denoted $(-\infty, \infty)$. The range is the nonnegative reals, also denoted $[0, \infty)$.

What about $f(x) = \sqrt{x}$?



we mean positive square root when we write this

The domain is $[0, \infty)$ and range is also $[0, \infty)$.

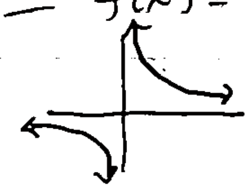
In general to find the domain of a function, you think about what values you're allowed to plug into it. With a square root, need nonneg. #s.

E.g. domain of $\sqrt{x-1}$ is $\{x \in \mathbb{R} \mid x \geq 1\} = [1, \infty)$

If you have a denominator, it cannot be zero.

E.g. $f(x) = 1/x$

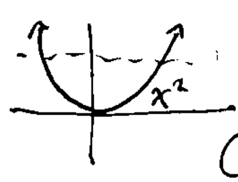
(hyperbola)



has domain (and range) $(-\infty, 0) \cup (0, \infty) = \{x \in \mathbb{R} \mid x \neq 0\}$

We can also test one-to-one-ness graphically using the "horizontal line test": f is one-to-one $\Longleftrightarrow$ graph of function f has every horizontal line intersect ≤ 1 point

e.g.



$f(x) = x^2$ is NOT one-to-one.

Q: what about $f(x) = x^3$?

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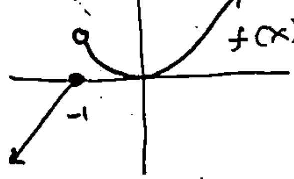
Not every function is determined by a single formula.
We can define a piecewise function like

$$f(x) = \begin{cases} x+1 & \text{if } x \le -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

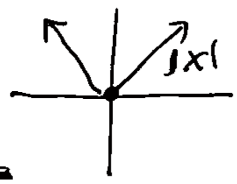
The graph of $y = f(x)$ has two parts:



(can use ○ to denote a 'discontinuity') empty circle

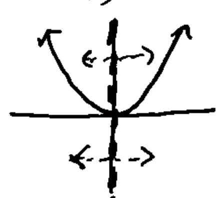
Another important piecewise function is absolute value

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

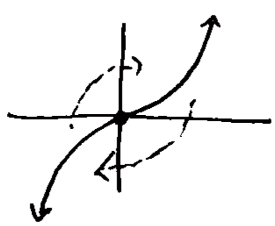


← graph of $|x|$ has two parts, but they 'touch' each other

Symmetry of functions



The function $f(x) = x^2$ is symmetric about the vertical (y-) axis: if I reflect graph across y-axis, I get back same thing

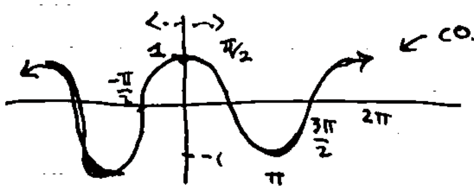


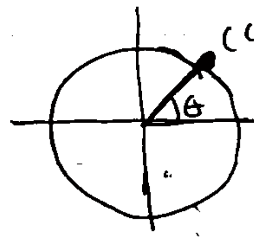
← The function $f(x) = x^3$ is symmetric about $(0,0)$: if I rotate it 180° about $(0,0)$ then I get back same thing.

These two kinds of symmetry are called even and odd for functions

A function $f(x)$ is called <u>even</u> if $f(x)=f(-x)$ for all x.
Same as saying symmetric across y-axis.

Examples of even fn's: x^2, x^2+1, x^4, $|x|$, $\cos(x)$



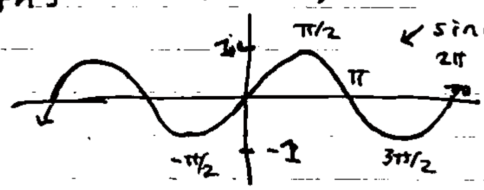


$(\cos\theta, \sin\theta)$

recall $\cos(\theta)$ and $\sin(\theta)$ give x + y coordinate of pt on unit circle at θ radians.

A function $f(x)$ is called <u>odd</u> if $f(-x) = -f(x)$ for all x.
Same as saying 180°-rotationally symmetric about $(0,0)$

Examples of odd fn's: x^3, x, x^5+x^3, $\sin(x)$



Can you guess why we use names "<u>even</u>" and "<u>odd</u>"?

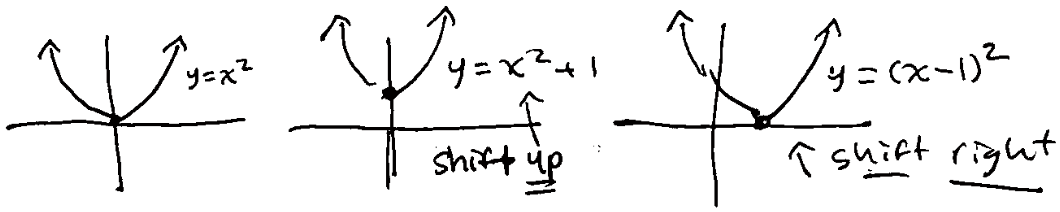
<u>Transformations</u> <u>of</u> <u>functions</u> §1.3

Given $f(x)$ can make new functions by applying various transformations, like <u>translations</u>:

$y = f(x) + c$ — function whose graph is $f(x)$ translated <u>up</u> by c
$y = f(x) - c$ — graph is $f(x)$ translated <u>down</u> by c
$y = f(x - c)$ — graph is $f(x)$ translated <u>right</u> by c
$y = f(x + c)$ — graph is $f(x)$ translated <u>left</u> by c

(for $c > 0$)

E.g.



Can also stretch a function: for $c > 1$

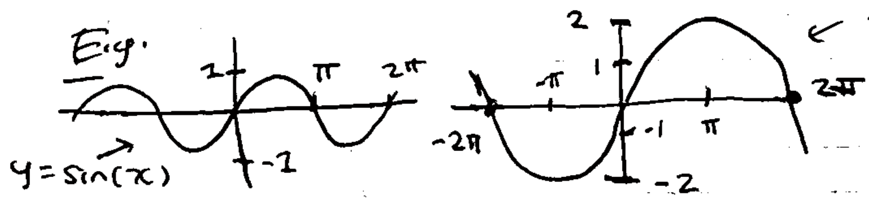
$y = c\,f(x)$ - stretch graph vertically by factor of c

$y = 1/c \cdot f(x)$ - shrink graph vertically by c

$y = f(\frac{x}{c})$ - stretch graph horizontally by c

$y = f(c \cdot x)$ - shrink graph horizontally by c

E.g.



stretch vertically and horizontally by factor of 2.

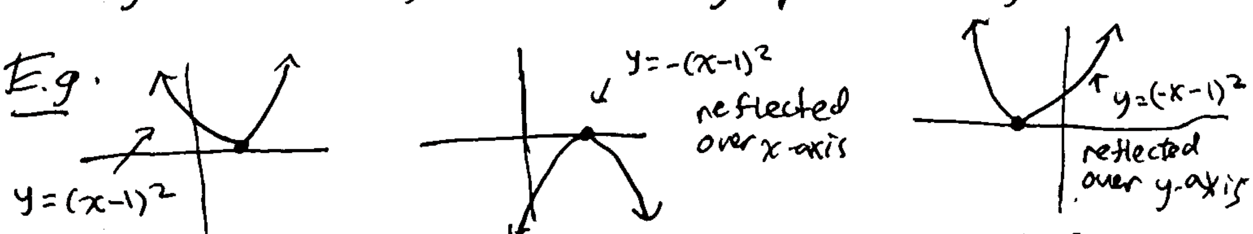
We see in this example how we can combine multiple transformations!

One more geometric transformation: reflection

$y = -f(x)$ - reflect graph about x-axis

$y = f(-x)$ - reflect graph about y-axis

E.g.



Q: What happens w/ reflections for even + odd fns? //

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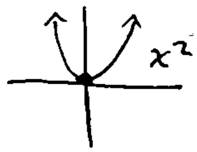
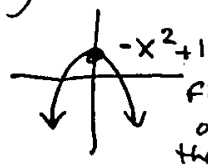
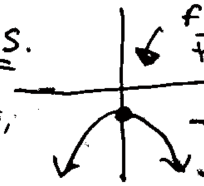
§1.3 When ~~[illegible]~~ applying multiple transformations, order is important!





first reflect across x-axis, then shift up by one.

VS.



first shift up by 1, then, reflect across x-axis

$-(x^2+1) = -x^2-1$

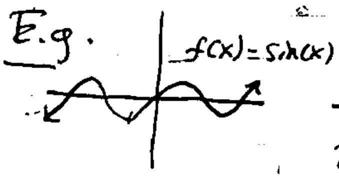
Another way to get new functions from old is by combining functions in different ways.

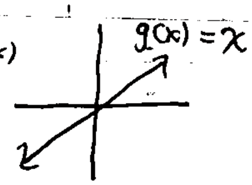
Def'n If f, g are two fn's, we define their sum, difference, product, and quotient by

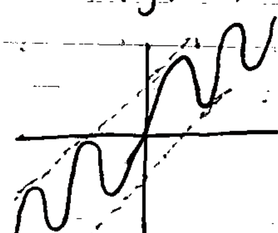
$(f+g)(x) = f(x) + g(x)$ $\quad$ $(f-g)(x) = f(x) - g(x)$

$(f \cdot g)(x) = f(x) \cdot g(x)$ $\quad$ $(f/g)(x) = f(x)/g(x)$

E.g.

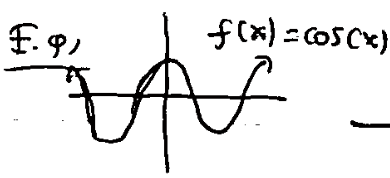


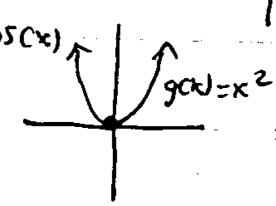


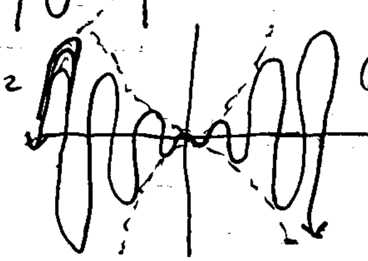


$(f+g)(x) = \sin(x) + x$

E.g.







$(f \cdot g)(x) = \cos(x) \cdot x^2$

E.g. $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ ← not always easy to graph combinations!

The domain of $f+g$, $f-g$, and $f \cdot g$ is the intersection of the domains of $f(x)$ and $g(x)$.

E.g. domain of $1/x + \sqrt{x}$ is $(0, \infty)$.

The domain of f/g is the intersection of the domain of $f(x)$ and set of all x for which $g(x) \neq 0$ (so that we don't divide by zero).

E.g. domain of $\tan(x) = \{x \in \mathbb{R} : x \neq \frac{\pi}{2} + n\pi$ for some $n \in \mathbb{Z}\}$ since $\cos(\frac{\pi}{2} + n \cdot \pi) = 0$ for all $n \in \mathbb{Z}$.

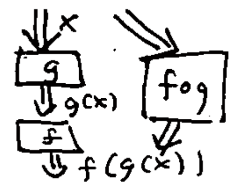
Another very important way to combine functions is composition:

Def'n If f and g are two functions, their composition $f \circ g$ is

$$f \circ g(x) = f(g(x))$$

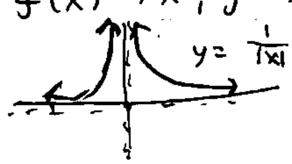
"Do g first, then do f to that!"

"f of g of x"



E.g. $f(x) = x^2$, $g(x) = 2x-1$, $(f \circ g)(x) = (2x-1)^2 = 4x^2 - 4x + 1$

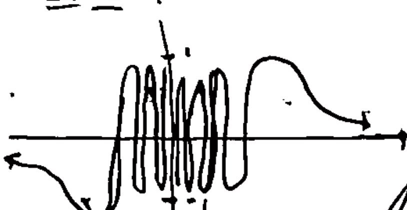
E.g. $f(x) = 1/x$, $g(x) = |x|$, $(f \circ g)(x) = \frac{1}{|x|} = \begin{cases} 1/x & \text{if } x > 0 \\ -1/x & \text{if } x < 0 \end{cases}$



note: $\frac{1}{|x|}$ is even since $\frac{1}{|-x|} = \frac{1}{|x|}$ ✓
and $x = 0$ not in domain.

E.g. $f(x) = \sin(x)$, $g(x) = 1/x$, $(f \circ g)(x) = \sin(1/x)$

What does $\sin(1/x)$ look like? As $x \to \infty$, $1/x$ barely changes, so $\sin(1/x)$ stops oscillating. As $x \to 0$ from the right, $1/x$ changes a lot, so $\sin(1/x)$ oscillates like crazy:



↙ very hard to draw accurately!
and note $x = 0$ not in domain!

8/31 Little more on compositions of functions: §1.3

Domain of $f \circ g$ is set of all x in domain of g such that $g(x)$ is in the domain of f.

E.g. $f(x) = \sqrt{x+1}$ and $g(x) = 1/x$ so that $(f \circ g)(x) = \sqrt{1/x + 1}$ then domain$\left(\sqrt{1/x+1}\right) = [-\infty,-1] \cup (0,\infty)$.

If $(f \circ g)(x) = x$ then we say f is the inverse function of g. f "undoes" what g does!

E.g. $f(x) = \sqrt{x}$, $g(x) = x^2$, $(f \circ g)(x) = \sqrt{x^2} = x$

$\sqrt{x}$ "undoes" x^2 so it is the inverse (we'll be a bit more careful about domain issues later)

Inverses will allow us to define the logarithm from the exponential, which brings us to ...

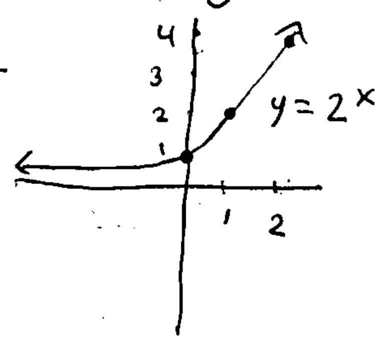
§1.4 Exponential functions

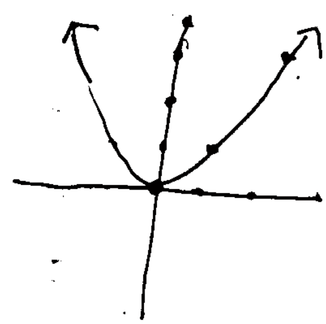
Def'n Fix real number $a > 0$. The exponential function with base a is $f(x) = a^x$.

Do not confuse a^x with power function x^a.

E.g. $f(x) = 2^x$ vs. $g(x) = x^2$

x	$f(x)$	$g(x)$
0	1	0
1	2	1
2	4	4
3	8	9
4	16	16



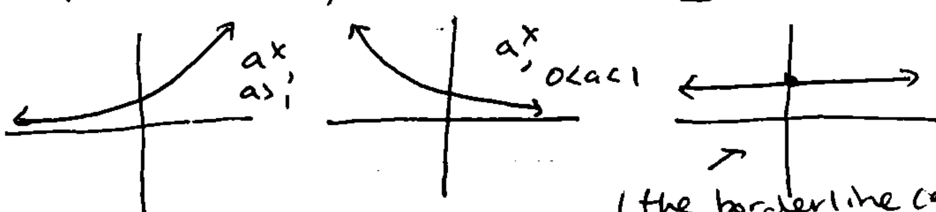


At <u>first</u>, x^2 grows <u>more</u> quickly than 2^x, but this is <u>misleading</u>: eventually, 2^x grows <u>much</u>, <u>much</u> faster than x^2!

In fact, any exponential a^x for $a>1$ (eventually) grows <u>much</u>, <u>much</u> <u>faster</u> than any <u>polynomial</u>.

Recall that a <u>polynomial</u> is a function
$$f(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \cdots + a_1 \cdot x + a_0$$
that is some <u>linear</u> <u>combination</u> of power functions.
We will prove this assertion later (using calculus!).

For $a>1$, a^x represents <u>exponential growth</u>,
for $0<a<1$, a^x represents <u>exponential decay</u>

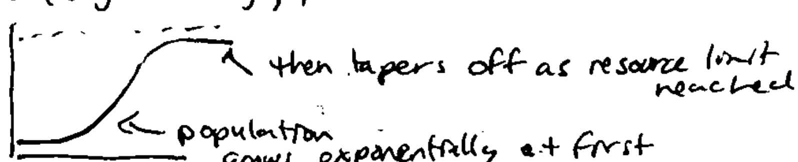


(the borderline case $1^x = 1$ is <u>degenerate</u>.)

Sometimes we also consider
Ca^x for fixed C an exponential function.

In sciences (e.g. biology) often see mix of exponential growth and decay!

<u>e.g.</u>



<u>Remember</u>: <u>fixed</u> <u>exponent</u> (x^a) $\Rightarrow$ <u>power function</u>
<u>Fixed</u> <u>base</u> (a^x) $\Rightarrow$ <u>exponential function</u>.
(So <u>e.g.</u> x^x is neither of these: base + exponent are both variables...)

9/2 The special number e

There is one special base that is "the best":
the number $e \approx 2.71\ldots$ ← irrational number, like π

How to define e precisely? Can use a limit:

$$e = \lim_{n\to\infty} (1+1/n)^n$$

Can explain this formula using compound interest.

Suppose you have an investment that returns 100% per year (that's an incredible investment!).

If you invest \$100, how much will you have after 1 year?

If the interest is only calculated at the end of the year you get $\$100 \cdot (1 + \underset{\text{100\% return}}{1}) = \$200$.

But imagine instead the interest is given every 6 months.

Then after 6 months you get $\$100(1 + \underset{\frac{1}{2}100\% = 50\% \text{ return in } 1/2 \text{ year}}{0.5}) = \$150$,

and after the next 6 months you get $\$150(1+0.5) = \$225$.

We see that compounding more often gives more money in the end, even with the "same rate".

If we earn interest n times in the year, we get

$$\$100 \cdot \underbrace{(1+\tfrac{1}{n})(1+\tfrac{1}{n})\cdots(1+\tfrac{1}{n})}_{n \text{ times}}$$

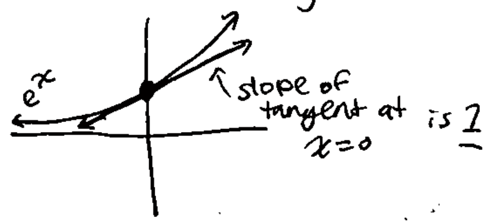
$$= \$100 \cdot (1+\tfrac{1}{n})^n$$ in the end, and

if we "continuously compound the interest" we end with $\$100 \cdot \lim_{n\to\infty}(1+\frac{1}{n})^n = \$100 \cdot e \approx \$271$.

principal rate time

This explains the "Pe^{rt}" formula for compound interest you may have seen before.

There is another geometric way to think about the significance of base e:



Of all the a^x, the one that has a tangent line of slope 1 at $x=0$ is $a=e$.

When we start to talk about derivatives and tangents, we will see why this is such a desirable property.

We mentioned that we define the logarithm as the inverse of the exponential function.

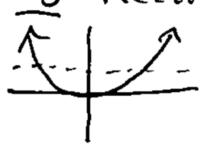
Def'n A function $g(x)$ has an inverse function $f=g^{-1}$ if and only if it is one-to-one. In this case, the inverse function $f=g^{-1}$ is defined by

$f(y)=x$ if y is the unique element in the range of g such that $g(x)=f$.

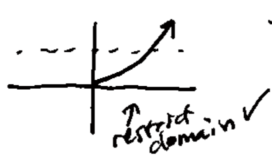
(f "undoes" g so that $(f\circ g)(x)=x$).

E.g. Since $g(x)=x^3$ is one-to-one, it admits an inverse $f=g^{-1}$ which is $f=\sqrt[3]{x}$.

E.g. Recall $g(x)=x^2$ is not one-to-one: it fails the horizontal line test! So it does not have an inverse on all of $\mathbb{R}$.

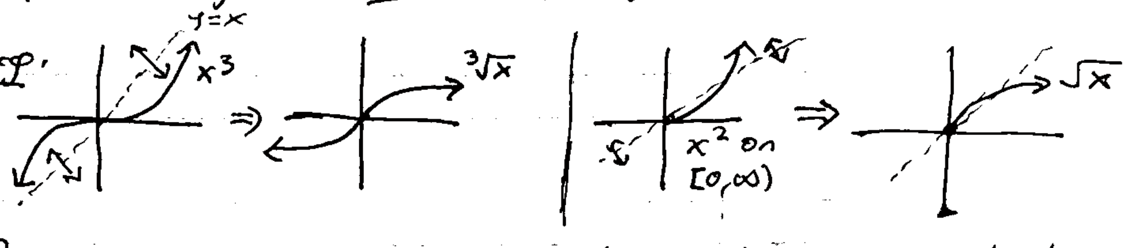


But if we restrict the domain to $[0,\infty)$, then $f(x)=\sqrt{x}$ is its inverse, like we'd expect.



There is a geometric way to think about inverses:
graph of $f = g^{-1}$ is reflection of graph of g over line $y = x$.

E.g.



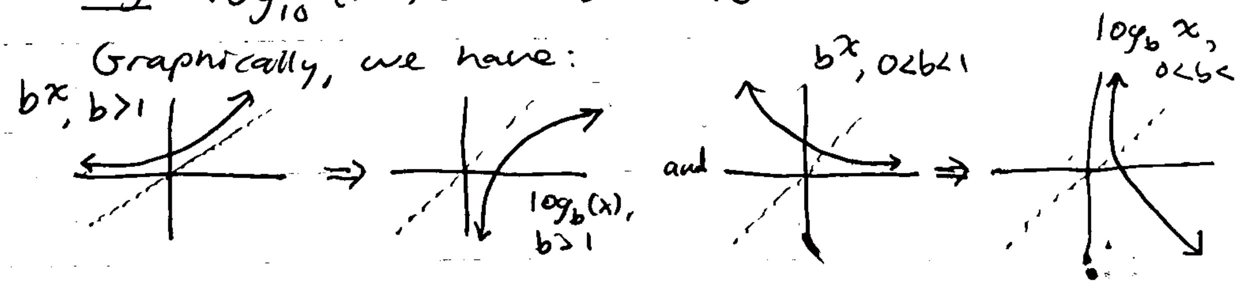
This geometric interpretation also makes clear that
domain of f = range of g and range of f = domain of g
for inverse functions $f = g^{-1}$!

Looking at the graph of b^x for any $b > 0$, $b \neq 1$,
we see it passes the horizontal line test, so
it has an inverse: the base b logarithm.

Def'n $\log_b$, the base b logarithm, is the inverse of b^x
meaning $\boxed{\log_b(y) = x \text{ if and only if } b^x = y}$

E.g. $\log_{10}(100) = 2$ since $10^2 = 100$.

Graphically, we have:



Note that since range (b^x) is $(0,\infty)$ (positive numbers)
domain $(\log_b(10))$ is $(0,\infty)$:
We can only take logarithms of positive numbers!