

# Howard Math 274, HW# 2,

Spring 2022; Instructor: Sam Hopkins; Due: Friday, March 25th

1. Let  $\lambda = (\lambda_1, \lambda_2, \dots), \mu = (\mu_1, \mu_2, \dots) \vdash n$  be partitions of  $n$ . Recall that the *lexicographic order*  $\prec$  on partitions of  $n$  is given by  $\mu \prec \lambda$  iff there is some  $j$  such that  $\mu_i = \lambda_i$  for all  $i < j$  and  $\mu_j < \lambda_j$ . It is a total order: we either have  $\mu \prec \lambda$  or  $\lambda \prec \mu$  or  $\lambda = \mu$ .

A different order on partitions of  $n$  is the dominance order. The *dominance order*  $\leq$  is defined by  $\mu \leq \lambda$  iff  $\mu_1 + \mu_2 + \dots + \mu_j \leq \lambda_1 + \lambda_2 + \dots + \lambda_j$  for all  $j$ . The dominance order is only partial order: we might have neither  $\mu \leq \lambda$  nor  $\lambda \leq \mu$ .

Show that the lexicographic order *extends* the dominance order in the sense that if  $\mu \leq \lambda$  and  $\mu \neq \lambda$  then necessarily  $\mu \prec \lambda$ .

**Bonus problem, just to think about:** Recall from the previous semester that a *lattice* is a partial order where every pair of elements has a least upper bound and a greatest lower bound. Show that dominance order on partitions of  $n$  is a lattice.

2. Show that we could've used dominance order instead of lexicographic order in our arguments about the triangularity of the transition matrices from  $p_\lambda$  or  $e_\lambda$  to  $m_\mu$ . That is, show that

$$p_\lambda = \sum_{\lambda \leq \mu} \alpha_\mu m_\mu \quad \text{and} \quad e_\lambda = \sum_{\mu \leq \lambda^t} \beta_\mu m_\mu \quad \text{for coefficients } \alpha_\mu, \beta_\mu \in \mathbb{C}$$

for any  $\lambda \vdash n$ , where  $\leq$  is dominance order and  $\lambda^t$  is the transpose (a.k.a. conjugate) of  $\lambda$ .

3. Let  $\lambda \vdash n$  and define  $f^\lambda$  to be the coefficient of  $x_1 x_2 \dots x_n$  in the Schur function  $s_\lambda(x_1, x_2, \dots)$ . Explain why  $f^\lambda = f^{\lambda^t}$ . Give an example showing that this is not true for other coefficients of Schur functions, i.e., that  $s_\lambda \neq s_{\lambda^t}$  in general.
4. The Cauchy–Binet formula says that if  $A = (A_{i,j})$  is an  $m \times n$  matrix and  $B = (B_{i,j})$  is an  $n \times m$  matrix, then the determinant of the  $m \times m$  matrix  $AB$  can be computed by

$$\det(AB) = \sum_{I \subseteq [n], \#I=m} \det(A|_{\text{cols}=I}) \det(B|_{\text{rows}=I}).$$

Here, as always,  $[n] := \{1, 2, \dots, n\}$ , and  $A|_{\text{cols}=I}$  (resp.,  $B|_{\text{rows}=I}$ ) means the  $m \times m$  matrix we get by restricting  $A$  to the columns in  $I$  (resp., by restricting  $B$  to the rows in  $I$ ).

Deduce the Cauchy–Binet formula from the Lindström–Gessel–Viennot formula.

**Hint:** Consider the network with source vertices  $s_1, \dots, s_m$ , target vertices  $t_1, \dots, t_m$ , and internal vertices  $k_1, \dots, k_n$ , and edges  $s_i \rightarrow k_j$  with weight  $A_{i,j}$  and  $k_i \rightarrow t_j$  with weight  $B_{i,j}$ .

5. Let  $\lambda = (\lambda_1, \lambda_2, \dots)$  be a partition and  $k$  an integer. Give a formula for  $m_\lambda(\overbrace{1, 1, \dots, 1}^{k \text{ 1's}})$ .

**Hint:** Your formula can use the *length*  $\ell(\lambda) := \max\{i: \lambda_i > 0\}$  of the partition, as well as the *multiplicities*  $m_i(\lambda) := \{j: \lambda_j = i\}$  for  $i \geq 1$ .