SYMMETRY OF NARAYANA NUMBERS AND ROWVACUATION OF ROOT POSETS

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Abstract. This is a note for the 2020 BIRS online workshop on Dynamical Algebraic Combinatorics explaining how ideas from DAC can (conjecturally) be used to resolve (most of) a conjecture of Panyushev about duality for root poset antichains, and symmetry of Narayana numbers.

A *Dyck path of semilength n* is a lattice path in $\mathbb{Z}^2$ with steps of the form $(1,1)$ (*up steps*) and $(1,-1)$ (*down steps*) from $(0,0)$ to $(2n,0)$ which never goes below the x-axis. The Dyck paths of semilength n are of course counted by the ubiquitous *Catalan numbers* $\mathrm{Cat}(n) := \frac{1}{n+1}\binom{2n}{n}$.

A *valley* in a Dyck path is a down step immediately followed by an up step. Let $\mathrm{Nar}(n,k)$ denote the number of Dyck paths of semilength n with exactly k valleys. These $\mathrm{Nar}(n,k)$ are called the *Narayana numbers*. (Warning: often the Narayana numbers are indexed slightly differently, by counting *peaks* instead of valleys.) Evidently $\mathrm{Nar}(n,k) = 0$ if $k \geq n$, and $\mathrm{Cat}(n) = \sum_{k=0}^{n-1} \mathrm{Nar}(n,k)$. There is also an explicit formula for these numbers: namely, $\mathrm{Nar}(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$. But the property of Narayana numbers that will chiefly concern us is the symmetry $\mathrm{Nar}(n,k) = \mathrm{Nar}(n, n-1-k)$ for all $0 \leq k \leq n-1$. This symmetry is not at all obvious from the definition we have given. The *Lalanne–Kreweras involution* [19, 21] is an involution on the set of Dyck paths of semilength n which sends a Dyck path with k valleys to one with $n-1-k$ valleys, and hence combinatorially demonstrates the symmetry of Narayana numbers.

We prefer to describe the Lalanne–Kreweras involution in terms of antichains of the root poset of Type A_{n-1}. The root poset of Type A_{n-1}, $\Phi^+(A_{n-1})$, can be identified with the set of intervals $\{[i,j] \colon i < j \in [n] := \{1,2,\ldots,n\}\}$ ordered by containment. We use $\mathcal{A}(P)$ to denote the set of *antichains* of a poset P. There is a natural bijection between Dyck paths of semilength n and $\mathcal{A}(\Phi^+(A_{n-1}))$, as depicted in Figure 1. Under this bijection, the number of valleys becomes the cardinality of the antichain. Hence, $\mathrm{Nar}(n,k)$ is the number of elements of $\mathcal{A}(\Phi^+(A_{n-1}))$ of cardinality k.

Any antichain in $\mathcal{A}(\Phi^+(A_{n-1}))$ is of the form $\{[i_1,j_1],[i_2,j_2],\ldots,[i_k,j_k]\}$ with $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$ (in other words, it is *nonnesting* when viewed as an arc diagram). The *Lalanne–Kreweras involution*, thought of as an involution $\mathcal{A}(\Phi^+(A_{n-1})) \to \mathcal{A}(\Phi^+(A_{n-1}))$, is the map which sends such an

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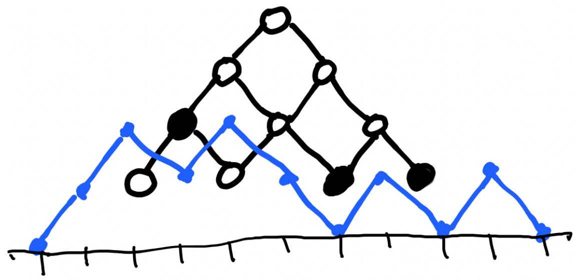

FIGURE 1. The bijection between Dyck paths and antichains of the Type A root poset.

antichain to $\{[i'_1, j'_1], [i'_2, j'_2], \ldots, [i'_{n-1-k}, j'_{n-1-k}]\}$ where

$$\{i'_1 < i'_2 < \cdots < i'_{n-1-k}\} = \{1, 2, \ldots, n-1\} \setminus \{j_1 - 1, j_2 - 1, \ldots, j_k - 1\},$$
$$\{j'_1 < j'_2 < \cdots < j'_{n-1-k}\} = \{2, 3, \ldots, n\} \setminus \{i_1 + 1, i_2 + 1, \ldots, i_k + 1\}.$$

It is easily checked that the resulting subset of $\Phi^+(A_{n-1})$ is in fact an antichain, that the map is involutive, and that an antichain of size k is sent to one of size $n-1-k$. So indeed the Lalanne–Kreweras involution combinatorially demonstrates the symmetry of Narayana numbers.

We are interested in extending the above discussion to other root systems. Let Φ be an irreducible, crystallographic *root system* of rank r. Assume we have chosen a system of *simple roots* $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$, and hence a set of *positive roots* Φ^+, etc. The *root poset* Φ^+ is the poset of positive roots of Φ with $\alpha \leq \beta$ iff $\beta - \alpha$ is a nonnegative sum of simple roots. The *Φ-Catalan number*, $\mathrm{Cat}(\Phi)$, can be defined as the number of antichains of Φ^+. Similarly, we can define the *Φ-Narayana numbers*, $\mathrm{Nar}(\Phi, k)$, to be the number of antichains of Φ^+ of cardinality k. As before, we have $\mathrm{Nar}(\Phi, k) = 0$ if $k > r$ and $\mathrm{Cat}(\Phi) = \sum_{k=0}^{r} \mathrm{Nar}(\Phi, k)$. There is an explicit, case-by-case, formula for the numbers $\mathrm{Nar}(\Phi, k)$ (see [12, Figure 5.12]). But most remarkably, the Φ-Narayana numbers continue to exhibit the symmetry $\mathrm{Nar}(\Phi, k) = \mathrm{Nar}(\Phi, r-k)$ for $0 \leq k \leq r$. As before, this symmetry is far from obvious from this definition of the Φ-Narayana numbers. (However, in Remark 10 we discuss other possible definitions of the Narayana numbers from which the symmetry is easier to see.)

The antichains of Φ^+ have a Lie algebraic meaning, as we now explain. Let $\mathfrak{g}$ be the *simple Lie algebra* associated to Φ, and $\mathfrak{b}$ the *Borel subalgebra* of $\mathfrak{g}$ coming from the choice of positive roots Φ^+. An ideal of $\mathfrak{b}$ is called ad-*nilpotent* if it belongs to $[\mathfrak{b}, \mathfrak{b}]$. There is a simple bijection between $\mathcal{A}(\Phi^+)$ and the set of ad-nilpotent ideals of $\mathfrak{b}$, which associates to an antichain A the ideal $\bigoplus_{\alpha \in F} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the *root space* of $\mathfrak{g}$ of *weight* α, and $F = \{\alpha \in \Phi^+ \colon \alpha \geq \beta$ for some $\beta \in A\}$ is the *order filter* in Φ^+ generated by A. Note that under this bijection, the cardinality of the antichain becomes the number of *generators* of the ad-nilpotent ideal.

This "ad-nilpotent ideals of a Borel subalgebra" perspective was beautifully used by Cellini and Papi [9] to give a uniform proof of the following product formula for the Φ-Catalan numbers:

$$\mathrm{Cat}(\Phi) = \prod_{i=1}^{r} \frac{d_i + h}{d_i},$$

where $d_1, \ldots, d_r$ are the *degrees* and h the *Coxeter number* of the *Weyl group* W of Φ. Specifically, they showed that ad-nilpotent ideals of $\mathfrak{b}$ are in bijection with the W-orbits of the *finite torus* $Q^\vee/(h+1)Q^\vee$, where $Q^\vee$ is the *coroot lattice* of Φ. Haiman [15] had previously shown that the number of W-orbits $Q^\vee/(h+1)Q^\vee$ is given by the above product formula.

In [23], Panyushev continued the investigation of the combinatorics of ad-nilpotent ideals. In particular he was very interested in the symmetry property of Narayana numbers. This symmetry inspired him to conjecture:

Conjecture 1 (Panyushev [23])**.** *There's a natural involution $\mathfrak{P}\colon \mathcal{A}(\Phi^+) \to \mathcal{A}(\Phi^+)$ for which $\#A + \#\mathfrak{P}(A) = r$ for all $A \in \mathcal{A}(\Phi^+)$.*

In terms of "naturality," Panyushev listed various specific desiderata (see [23, Conjecture 6.1]). He described his conjectural $\mathfrak{P}$ as a *"duality"* for ad-nilpotent ideals, which should send an ideal with k generators to one with $r-k$ generators.

Panyushev was unable to define this duality for arbitrary Φ, but he did come up with a definition for Φ of Type A. Although he was unaware of the prior work of Lalanne and Kreweras, the duality $\mathfrak{P}\colon \mathcal{A}(\Phi^+(A_{n-1})) \to \mathcal{A}(\Phi^+(A_{n-1}))$ Panyushev defined is nothing other than the Lalanne–Kreweras involution as we have described it above.[1]

It is also easy to obtain the Types B and C duality maps from the Type A duality map via "folding." That is to say, the antichains of $\Phi^+(B_n)$ are in bijection with the subset of antichains of $\Phi^+(A_{2n-1})$ which are invariant under the horizontal reflection symmetry. The Lalanne–Kreweras involution respects this symmetry, and hence descends to an involution on $\mathcal{A}(\Phi^+(B_n))$. Panyushev showed that this restriction of the Lalanne–Kreweras involution to $\mathcal{A}(\Phi^+(B_n))$ yields his desired duality map $\mathfrak{P}$ in this case as well. Since $\Phi^+(B_n) \simeq \Phi^+(C_n)$, this also gives the desired duality map for Type C.

Thus the remaining parts of Conjecture 1 concern Type D and the exceptional types (except G_2, which is easily dealt with by hand).

Although Panyushev was unable to define $\mathfrak{P}$ for arbitrary Φ in [23], in a later paper he proposed a strategy for doing something similar. Namely, observe that defining an involution $\mathfrak{P}$ as in Conjecture 1 is the same as partitioning $\mathcal{A}(\Phi^+)$ into blocks of size dividing 2, such that in each block the average cardinality of its members is $r/2$. In [24] Panyushev conjectured a way to partition $\mathcal{A}(\Phi^+)$ into blocks of size dividing $2h$, such that in each block the average cardinality of its members is $r/2$. His proposal for such a partitioning used the rowmotion operator.

[1] I learned that Panyushev's duality was the same as the Lalanne–Kreweras involution at a talk (`http://fpsac2019.fmf.uni-lj.si/resources/Slides/205slides.pdf`) given by Martin Rubey at FPSAC 2019 about the FindStat project (`http://www.findstat.org/`).

Let P be a poset. *(Antichain) rowmotion*, denoted $\mathrm{Row}\colon \mathcal{A}(P) \to \mathcal{A}(P)$, is given by

$$\mathrm{Row}(A) := \min(\{x \in P \colon x \not\leq y \text{ for any } y \in A\}),$$

for all $A \in \mathcal{A}(P)$, where $\min(X)$ means the minimal elements of a subset $X \subseteq P$. Rowmotion is invertible, i.e., a bijection. Before Panyushev, rowmotion was studied by Brouwer and Schrijver [7] and Cameron and Fon-der-Flaass [8], among others.

In [24], Panyushev conjectured the following about $\mathrm{Row}\colon \mathcal{A}(\Phi^+) \to \mathcal{A}(\Phi^+)$:

- $\mathrm{Row}^h = \delta$, where δ is the automorphism of Φ^+ given by $\delta\colon \alpha \mapsto -w_0(\alpha)$, with $w_0 \in W$ the *longest element* (so in particular Row^{2h} is the identity);
- for any Row-orbit $\mathcal{O} \subseteq \mathcal{A}(\Phi^+)$, $\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = r/2$.

Thus, the rowmotion orbits would give the partitioning of $\mathcal{A}(\Phi^+)$ into blocks of size dividing $2h$ mentioned above. We note that this second bulleted item is what is nowadays called an instance of *homomesy* [25]. In fact, it was the motivating example for the introduction of the notion of homomesy.

These rowmotion conjectures of Panyushev were proved by Armstrong, Stump, and Thomas [2], albeit in a case-by-case fashion. Subsequently, rowmotion has received a lot of attention, especially in the context of the growing subfield of *dynamical algebraic combinatorics* [27, 31]. We will now explain a (conjectural) way that ideas from dynamical algebraic combinatorics can be used resolve (most of) Conjecture 1.

Following Cameron and Fon-der-Flaass [8] and Striker and Williams [32], rowmotion is now commonly defined and studied in terms of toggles. These toggles are the building blocks of the order ideal version of rowmotion, which we will now describe.

Let P be a poset. We denote the set of *order ideals* of P by $\mathcal{J}(P)$. There's a canonical bijection $\Psi\colon \mathcal{J}(P) \xrightarrow{\sim} \mathcal{A}(P)$ with $\Psi\colon I \mapsto \min(P \setminus I)$ for all $I \in \mathcal{J}(P)$. Its inverse is $\Psi^{-1}\colon A \mapsto \{x \in P \colon x \not\geq y \text{ for any } y \in A\}$. (The bijection Ψ is is depicted in Figure 1.) *(Order ideal) rowmotion*, denoted $\mathrm{Row}\colon \mathcal{J}(P) \to \mathcal{J}(P)$, is given by

$$\mathrm{Row}(I) := \{x \in P \colon x \leq y \text{ for some } y \in \min(P \setminus I)\},$$

for all $I \in \mathcal{J}(P)$. It is again invertible. In fact, we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{J}(P) \arrow[r, "\mathrm{Row}"] \arrow[d, "\Psi"] & \mathcal{J}(P) \arrow[d, "\Psi"] \\ \mathcal{A}(P) \arrow[r, "\mathrm{Row}"] & \mathcal{A}(P) \end{tikzcd}$$

Hence, via Ψ, order ideal and antichain rowmotion "are the same". Moreover, order ideal rowmotion admits the following description in terms of toggles. For $p \in P$, the *(order ideal) toggle* at p is the involution $t_p\colon \mathcal{J}(P) \to \mathcal{J}(P)$ defined by

$$t_p(I) := \begin{cases} I \Delta \{p\} & \text{if } I \Delta \{p\} \in \mathcal{J}(P); \\ I & \text{otherwise,} \end{cases}$$

for all $I \in \mathcal{J}(P)$, where Δ denotes symmetric difference. Cameron–Fon-der-Flaass [8], showed that

$$\mathrm{Row} = t_{p_1} \cdot t_{p_2} \cdots t_{p_{n-1}} \cdot t_{p_n},$$

where $p_1, p_2, \ldots, p_n$ is any linear extension of P.

Rowmotion only ever has good behavior when P is *graded*, i.e., all maximal chains in P have the same length $\mathrm{rk}(P)$.[2] So from now on let us assume P is graded (for instance, root posets are always graded). This means that P comes with a *rank function* $\mathrm{rk}\colon P \to \mathbb{N}$ for which:

- all minimal elements $p \in P$ have $\mathrm{rk}(p) = 0$;
- if $p \lessdot q \in P$ then $\mathrm{rk}(q) = \mathrm{rk}(p) + 1$;
- all maximal elements $p \in P$ have $\mathrm{rk}(p) = \mathrm{rk}(P)$.

Then we can define $t_i := \prod_{p \in P, \mathrm{rk}(p)=i} t_p$ for each $i = 0, 1, \ldots, \mathrm{rk}(P)$ (this composition makes sense because all toggles at a given rank commute). Observe that

$$\mathrm{Row} = t_0 \cdot t_1 \cdots t_{\mathrm{rk}(P)-1} \cdot t_{\mathrm{rk}(P)}.$$

This description of rowmotion explains its name: we carry out toggles "row-by-row," i.e., rank-by-rank.

We now define a bijection (in fact, involution) which is closely related to rowmotion that we call *rowvacuation*, and denote $\mathrm{Rvac}\colon \mathcal{J}(P) \to \mathcal{J}(P)$. Rowvacuation is the following composition of the rank toggles:

$$\mathrm{Rvac} = (t_{\mathrm{rk}(P)}) \cdot (t_{\mathrm{rk}(P)-1} \cdot t_{\mathrm{rk}(P)}) \cdots (t_1 \cdot t_2 \cdots t_{\mathrm{rk}(P)}) \cdot (t_0 \cdot t_1 \cdots t_{\mathrm{rk}(P)-1} \cdot t_{\mathrm{rk}(P)}).$$

Equivalently, we can say that for any $I \in \mathcal{J}(P)$ and $p \in P$ with $\mathrm{rk}(p) = i$, we have $p \in \mathrm{Rvac}(I)$ if and only if $p \in \mathrm{Row}^{i+1}(I)$.

The name "rowvacuation" comes from "evacuation." We recall that *promotion* and *evacuation* are two related invertible operators acting on the set of linear extensions of any poset P, which were first studied by Schützenberger [29]. As explained by Haiman [14] (see also Stanley [30]), they can be defined as compositions of involutions, exactly analogous to the definitions of rowmotion and rowvacuation above. The same basic results connecting promotion and evacuation hold as well for rowmotion and rowvacuation. In order to state these, we need to also define dual rowvacuation. There is a canonical bijection $\mathcal{J}(P) \to \mathcal{J}(P^*)$, where P^* is the dual of P: $I \mapsto I^* := P \setminus I$. *Dual rowvacuation*, denoted $\mathrm{Rvac}^*\colon \mathcal{J}(P) \to \mathcal{J}(P)$, is given by $\mathrm{Rvac}^*(I) := \mathrm{Rvac}(I^*)^*$ for all $I \in \mathcal{J}(P)$. In terms of toggles we have:

$$\mathrm{Rvac}^* = (t_0) \cdot (t_1 \cdot t_0) \cdots (t_{\mathrm{rk}(P)-1} \cdot t_{\mathrm{rk}(P)-2} \cdots t_0) \cdot (t_{\mathrm{rk}(P)} \cdot t_{\mathrm{rk}(P)-1} \cdots t_1 \cdot t_0).$$

Here are the basic results connecting rowmotion, rowvacuation, and dual rowvacuation (which follow just from the facts that the t_i's are involutions and that t_i and t_j commute if $|i - j| \geq 2$):

Proposition 2 (See [16, Proposition 5.1] and [30])**.** *For any graded poset P,*

- Rvac *and* Rvac^* *are both involutions;*
- $\mathrm{Rvac} \cdot \mathrm{Row} = \mathrm{Row}^{-1} \cdot \mathrm{Rvac}$*;*
- $\mathrm{Row}^{\mathrm{rk}(P)+2} = \mathrm{Rvac}^* \cdot \mathrm{Rvac}$.

[2] Note $\mathrm{rk}(\Phi^+)$ is not the rank of the root system Φ: rather, $\mathrm{rk}(\Phi^+) = h - 2$.

Rowvacuation was first formally defined in [16, §5.1], but was essentially considered earlier in the context of *"reciprocity"* results for rowmotion. The other major class of posets which exhibit good behavior of rowmotion, beyond root posets, are the *minuscule posets* [28]. These also come from the representation theory of Lie algebras, with the most prominent example of a minuscule poset being the *rectangle poset*, i.e., the product of two chains. Rowmotion reciprocity results established by Grinberg and Roby [13] (for the rectangle) and Okada [22] (for the other minuscule posets) are equivalent to the assertion that rowvacuation has a very simple description for a minuscule poset P: $\mathrm{Rvac}(I) = \iota(I)^*$, where $\iota\colon P \to P^*$ is the canonical involutive anti-automorphism of P induced by multiplication by w_0; see [16, Theorem 5.2]. (Actually, these results hold at the biational level; we discuss birational dynamics in Remark 13.)

For the root posets, on the other hand, even though rowmotion is well-understood, it is not so clear what rowvacuation does. In fact, we conjecture that rowvacuation (mostly) gives Panyushev's desired duality map from Conjecture 1. In order to directly relate rowvacuation to Panyushev's conjecture, we should use the antichain version of rowvacuation.

We define *(antichain) rowvacuation* $\mathrm{Rvac}\colon \mathcal{A}(P) \to \mathcal{A}(P)$ so that the following diagram commutes:

$$\begin{tikzcd} \mathcal{J}(P) \arrow[r, "\mathrm{Rvac}"] \arrow[d, "\Psi"] & \mathcal{J}(P) \arrow[d, "\Psi"] \\ \mathcal{A}(P) \arrow[r, "\mathrm{Rvac}"] & \mathcal{A}(P) \end{tikzcd}$$

Joseph [18] introduced a version of toggling for antichains and explained how rowmotion can be defined in terms of these antichain toggles. He subsequently also showed how to write rowvacuation using antichain toggles. Let's review these descriptions. For $p \in P$, the *(antichain) toggle* at p is the involution $\tau_p\colon \mathcal{A}(P) \to \mathcal{A}(P)$ defined by

$$\tau_p(A) := \begin{cases} A\Delta\{p\} & \text{if } A\Delta\{p\} \in \mathcal{A}(P); \\ A & \text{otherwise,} \end{cases}$$

for all $A \in \mathcal{A}(P)$, where Δ denotes symmetric difference. Define $\tau_i := \prod_{p\in P, \mathrm{rk}(p)=i} \tau_p$ for each $i = 0, 1, \ldots, \mathrm{rk}(P)$ to be the product of antichain toggles at each rank (these toggles commute so the product makes sense). Then we have:

Lemma 3 (Joseph [18], c.f. [17])**.** *Antichain rowmotion* $\mathrm{Row}\colon \mathcal{A}(P) \to \mathcal{A}(P)$ *and rowvacuation* $\mathrm{Rvac}\colon \mathcal{A}(P) \to \mathcal{A}(P)$ *can be written in terms of antichain toggles as*

$$\begin{aligned} \mathrm{Row} &= \tau_{\mathrm{rk}(P)} \cdot \tau_{\mathrm{rk}(P)-1} \cdots \tau_1 \cdot \tau_0, \\ \mathrm{Rvac} &= (\tau_{\mathrm{rk}(P)}) \cdot (\tau_{\mathrm{rk}(P)} \cdot \tau_{\mathrm{rk}(P)-1}) \cdots (\tau_{\mathrm{rk}(P)} \cdot \tau_{\mathrm{rk}(P)-1} \cdots \tau_1) \cdot (\tau_{\mathrm{rk}(P)} \cdot \tau_{\mathrm{rk}(P)-1} \cdots \tau_1 \cdot \tau_0). \end{aligned}$$

Now having defined antichain rowvacuation, we can relate it to Panyushev's conjectured duality:

Conjecture 4. *Suppose Φ is of the classical Types A, B, C, or D. Then the natural involution $\mathfrak{P}\colon \mathcal{A}(\Phi^+) \to \mathcal{A}(\Phi^+)$ with $\#A + \#\mathfrak{P}(A) = r$ for all $A \in \mathcal{A}(\Phi^+)$ from Conjecture 1 is exactly rowvacuation* $\mathrm{Rvac}\colon \mathcal{A}(\Phi^+) \to \mathcal{A}(\Phi^+)$.

In fact, for Types A, B, and C Conjecture 4 is definitely true, thanks to:

Theorem 5 (Hopkins–Joseph [17])**.** $\mathrm{Rvac}\colon \mathcal{A}(\Phi^+(A_{n-1})) \to \mathcal{A}(\Phi^+(A_{n-1}))$ *is the same as the Lalanne–Kreweras involution.*

Since rowvacuation clearly commutes with the symmetry of the Type A root poset, it follows from Theorem 5 and Panyushev's work that indeed rowvacuation is the antichain duality in Types A, B, and C. But because Conjecture 4 addresses Type D as well, it goes beyond what Panyushev was able to do, and gives a unified construction of the duality, at least for the classical types. I have checked by computer for $\Phi = D_r$ with $r \leq 7$ that $\#A + \#\,\mathrm{Rvac}(A) = r$ for all $A \in \mathcal{A}(\Phi^+)$.

If Conjecture 4 were true, the only remaining cases of Conjecture 1 would be the exceptional types. For $\Phi = G_2$, it is easy to check by hand that there is a unique choice of duality meeting Panyushev's specific requested properties, and that rowvacuation is this map. Unfortunately, I have discovered via computer that for $\Phi = F_4$ there are some $A \in \mathcal{A}(\Phi^+)$ with $\#A + \#\mathrm{Rvac}(A) = 3$ (and also some order ideals where this quantity is 5). Hence for F_4, rowvacuation fails to give the desired duality map. Similarly, rowvacuation fails for $\Phi = E_6, E_7, E_8$. Data on rowvacuation of root posets is collected in Table 1.

It would be interesting to see what changes could possibly be made to rowvacuation to give a unified construction of Panyushev's conjectured duality map in all types, including the exceptional ones.

In order to prove Conjecture 4, it might be helpful to revisit the approach that Armstrong–Stump–Thomas [2] used to solve Panyushev's rowmotion conjectures. Their basic approach was to relate the "nonnesting" and "noncrossing" worlds, as we will now explain.

The set $\mathcal{A}(\Phi^+)$ of antichains of the root poset is often called the set of *nonnesting partitions*[3] of Φ (recall that in Type A these antichains were the nonnesting arc diagrams). Meanwhile, the noncrossing partitions of Φ are defined in terms of absolute order on the Weyl group. Let $T \subseteq W$ denote the set of *reflections* in W. For $w \in W$, the *absolute length* of w, denoted $\ell_T(w)$, is the minimum length of an expression for w as a product of elements of T. *Absolute order* on W is the partial order where $u \leq w$ if and only if $\ell_T(w) = \ell_T(u) + \ell_T(u^{-1}w)$. Fix a choice of *Coxeter element* $c \in W$, i.e., a product of the *simple reflections* s_i, $i \in [r]$ in some order. Then the lattice of *noncrossing partitions* $\mathrm{NC}(W, c)$ is defined to be the interval $[e, w] \subseteq W$ in absolute order between the *identity element* e and the Coxeter element c. Since all Coxeter elements are conjugate, all $\mathrm{NC}(W, c)$ are isomorphic, and we often use $\mathrm{NC}(\Phi)$ to denote this poset as well. $\mathrm{NC}(\Phi)$ is always a graded lattice. In Type A_n, this lattice is (isomorphic to) the classical lattice of noncrossing set

[3]This terminology goes back to a suggestion of Alex Postnikov; see [26, Remark 2].

Φ	$\mathrm{Cat}(\Phi) = \#\mathcal{A}(\Phi^+) = \#\mathrm{NC}(\Phi)$	$\#\{A \in \mathcal{A}(\Phi^+)\colon \#A + \#\mathrm{Rvac}(A) \neq r\}$	$\#\mathcal{A}(\Phi^+)^{\mathrm{Rvac}} = \#\mathrm{NC}(\Phi)^{\mathrm{Krew}\cdot\mathrm{Flip}}$	$\#\mathcal{A}(\Phi^+)^{\mathrm{Row}^{-1}\cdot\mathrm{Rvac}} = \#\mathrm{NC}(\Phi)^{\mathrm{Flip}}$
A_r	$\mathrm{Cat}(r+1)$	0	$\begin{cases} 0 & r \text{ odd} \\ \mathrm{Cat}(r/2) & r \text{ even} \end{cases}$	$\binom{r+1}{\lfloor r+1/2 \rfloor}$
B_r	$\binom{2r}{r}$	0	0	2^r
D_r	$\binom{2r}{r} - \binom{2r-2}{n-1}$	0	0	$\begin{cases} 2^r - \binom{2r-4}{r-2} & r \text{ odd} \\ 2^r & r \text{ even} \end{cases}$
G_2	8	0	0	4
F_4	105	8	1	17
E_6	833	24	5	47
E_7	4160	354	0	132
E_8	25080	4582	14	278

TABLE 1. Data on rowvacuation of root posets. The classical types A_r, B_r, D_r have been checked for $r \leq 8$. The Sage code I used to produce this data is publicly available online at `https://cocalc.com/share/38036fd67aa9ddd0da40288acb82856ac7bda331/rowvacuation_root_posets.sagews`.

partitions of $[n]$ [20], thus explaining the terminology. This definition of noncrossing partitions for arbitrary type is due independently to Brady and Watt [6] and Bessis [4].

It is known that $\#\mathrm{NC}(\Phi) = \mathrm{Cat}(\Phi)$. However, it seems very difficult to directly relate the nonnesting and noncrossing partitions. Indeed, the exact connection between the nonnesting and noncrossing worlds remains an enduring mystery in *Coxeter–Catalan combinatorics* [1].

The lattice $\mathrm{NC}(\Phi)$ is self-dual, and one anti-automorphism is of particular interest: the *Kreweras complement* (not to be confused with the Lalanne–Kreweras involution), denoted $\mathrm{Krew}\colon \mathrm{NC}(W,c) \to \mathrm{NC}(W,c)$ and defined by $\mathrm{Krew}(w) := cw^{-1}$. In Type A this is the classical Kreweras complement of set partitions [20]. Note that the Kreweras complement is *not* an involution: it has order (dividing) $2h$. For instance, in Type A Krew^2 is rotation of noncrossing set partitions. Hence, one might wonder about the orbit structure of Krew. This is where the connection to Panyushev's work arises. After Panyushev had experimentally exhibited the remarkable properties of rowmotion acting on the root poset in [24], Bessis and Reiner [5] conjectured that the orbit structure of Row acting on $\mathcal{A}(P)$ is the same as the orbit structure of Krew acting on $\mathrm{NC}(W,c)$. Armstrong–Stump–Thomas proved this conjecture of Bessis–Reiner, and in fact they defined an explicit bijection between $\mathcal{A}(P)$ and $\mathrm{NC}(W,c)$ which equivariantly maps rowmotion to Kreweras complement.

In order to describe their bijection, we need to assume that our Coxeter element is *bipartite*, i.e., that $c = c_L c_R$, where $L \sqcup R = [r]$ is a bipartition of the nodes of the *Dynkin diagram* of Φ and $c_X := \prod_{i \in X} s_i$ (the products c_L and c_R are well-defined since the bipartite assumption guarantees that these simple reflections commute). Since the Dynkin diagram of Φ is always a tree, a bipartite Coxeter element exists. Because all Coxeter elements are conjugate, there is no loss of generality in assuming bipartiteness.

Before we can describe the Armstrong–Stump–Thomas bijection we also need to talk about parabolic induction. For $J \subseteq [r]$, we use W_J to denote the corresponding *parabolic subgroup*, i.e., the subgroup of W generated by the simple reflections s_i for $i \in J$. If $c_L c_R$ is a bipartite Coxeter element of W, then $c_{L'} c_{R'}$ is a bipartite Coxeter element of W_J, where $L' := L \cap J$ and $R' := R \cap J$; and we have a natural inclusion $\mathrm{NC}(W_J, c_{L'} c_{R'}) \subseteq \mathrm{NC}(W, c_L c_R)$. Meanwhile, we use Φ^+_J to denote the *parabolic root poset*, which is $\Phi^+_J := \{\alpha \in \Phi^+ \colon \alpha \not\geq \alpha_i \text{ for any } i \in [r] \setminus J\}$. For an antichain $A \in \mathcal{A}(\Phi^+)$, we define its *support* to be

$$\mathrm{supp}(A) := \{i \in [r] \colon \alpha_i \leq \alpha \text{ for some } \alpha \in A\}.$$

We can view any antichain $A \in \mathcal{A}(\Phi^+)$ as also an antichain in $\mathcal{A}(\Phi^+_{\mathrm{supp}(A)})$.

With all this notation in hand, we can now describe the Armstrong–Stump–Thomas nonnesting-to-noncrossing bijection which sends rowmotion to Kreweras complement:[4]

Theorem 6 (Armstrong–Stump–Thomas [2])**.** *For a fixed bipartition $L \sqcup R = [r]$, there is a unique bijection $\Theta_W \colon \mathcal{A}(\Phi^+) \xrightarrow{\sim} \mathrm{NC}(W, c_L c_R)$ for which:*

- $\Theta_W(\{\alpha_i \colon i \in L\}) = e$*;*
- $\Theta_W \cdot \mathrm{Row} = \mathrm{Krew} \cdot \Theta_W$*;*
- $\Theta_W(A) = \prod_{i \in L \setminus \mathrm{supp}(A)} s_i \cdot \Theta_{W_{\mathrm{supp}(A)}}(A)$ *for all* $A \in \mathcal{A}(\Phi^+)$ *(where we view* $\Theta_{W_{\mathrm{supp}(A)}}(A) \in \mathrm{NC}(W_{\mathrm{supp}(A)}, c_{L \cap \mathrm{supp}(A)} c_{R \cap \mathrm{supp}(A)}) \subseteq \mathrm{NC}(W, c_L c_R)$*).*

Now the problem is to bring rowvacuation into this story. In other words, how do we find an involution on the set of noncrossing partitions of Φ which plays the role that rowvacuation plays for the nonnesting partitions. Since Kreweras complement is a kind of "rotation," and this hypothetical involution on noncrossing partitions ought to generate a dihedral action with Kreweras complement, we will refer to it as the *flip*[5]. After I came up with an ad hoc definition for the flip in the classical types, Nathan Williams explained to me what the right candidate should be in general: we define the *flip* $\mathrm{Flip} \colon \mathrm{NC}(W, c_L c_R) \to \mathrm{NC}(W, c_L c_R)$ by $\mathrm{Flip}(w) := c_L w^{-1} c_L^{-1}$. Note that since c_L is an involution, Flip is an involution. It fixes $c = c_L c_R$ and permutes the set of reflections, and so is an automorphism of $\mathrm{NC}(W, c_L c_R)$. It's also easily seen that $\mathrm{Flip} \cdot \mathrm{Krew} = \mathrm{Krew}^{-1} \cdot \mathrm{Flip}$.

[4]For an implementation of this bijection in Sage, see `https://cocalc.com/share/b5b8634864c78fe641ecd89f7e046fd6319339ae/ast_bijection.sagews`.

[5]I was going to call it "reflection" but that word has a meaning in the context of Coxeter groups.

With this definition of the flip, we can upgrade the equivalence of cyclic actions in Theorem 6 to an equivalence of dihedral actions essentially "for free," using just the general properties of the bijection listed in that theorem. To be precise:

Theorem 7. *For the bijection* $\Theta_W\colon \mathcal{A}(\Phi^+) \xrightarrow{\sim} \mathrm{NC}(W, c_L c_R)$ *from Theorem 6 we have* $\Theta_W \cdot (\mathrm{Row}^{-1} \cdot \mathrm{Rvac}) = \mathrm{Flip} \cdot \Theta_W$*.*

The reason that we need to use $\mathrm{Row}^{-1} \cdot \mathrm{Rvac}$ instead of just Rvac in Theorem 7 is because sometimes there are no $A \in \mathcal{A}(\Phi^+)$ fixed by Rvac (e.g., if we believe Conjecture 4 then when Φ is of classical type and r is odd there cannot be any fixed points of Rvac), but there will always be some fixed points of Flip (e.g., e and c).

In order to prove Theorem 7 we need to show that $(\mathrm{Row}^{-1} \cdot \mathrm{Rvac})$ behaves well with respect to parabolic induction. In fact, this is true for any poset P. Namely, for an antichain $A \in \mathcal{A}(P)$, slightly abusing notation, let us define its *support* to be $\mathrm{supp}(A) := \{p \in \min(P)\colon p \leq q \text{ for some } q \in A\}$. For a subset $X \subseteq \min(P)$, define $P_X := \{q \in P\colon q \not\geq p \text{ for any } p \in \min(P) \setminus X\}$. We can view any antichain $A \in \mathcal{A}(P)$ as an antichain in $\mathcal{A}(P_{\mathrm{supp}(A)})$. Then we have:

Lemma 8. *For any graded poset* P *and any* $A \in \mathcal{A}(P)$*, we have:*

- $\mathrm{supp}((\mathrm{Row}^{-1} \cdot \mathrm{Rvac})(A)) = \mathrm{supp}(A)$*;*
- $\mathrm{Row}^{-1} \cdot \mathrm{Rvac}(A) = (\mathrm{Row}^{-1} \cdot \mathrm{Rvac})_{\mathrm{supp}(A)}(A)$ *(where by* $(\mathrm{Row}^{-1} \cdot \mathrm{Rvac})_{\mathrm{supp}(A)}(A)$ *we mean the image of* A *under* $\mathrm{Row}^{-1} \cdot \mathrm{Rvac}\colon \mathcal{A}(P_{\mathrm{supp}(A)}) \to \mathcal{A}(P_{\mathrm{supp}(A)})$*).*

Proof. Let's prove the second bulleted item first. To do this, we use the antichain toggle descriptions from Lemma 3 to write:

$$\mathrm{Row}^{-1} \cdot \mathrm{Rvac} = (\tau_0 \cdot \tau_1 \cdots \tau_{\mathrm{rk}(P)-1} \cdot \tau_{\mathrm{rk}(P)}) \cdot (\tau_{\mathrm{rk}(P)}) \cdot (\tau_{\mathrm{rk}(P)} \cdot \tau_{\mathrm{rk}(P)-1}) \cdots (\tau_{\mathrm{rk}(P)} \cdot \tau_{\mathrm{rk}(P)-1} \cdots \tau_1 \cdot \tau_0).$$

When we apply the first τ_0 to A, it will add to our antichain all of $\min(P) \setminus \mathrm{supp}(A)$, and until those elements are removed by the τ_0 at the end of this sequence of toggles (which they will be), no element of $P \setminus P_X$ can be toggled into our antichain. Meanwhile, the status of elements of $\min(P) \setminus \mathrm{supp}(A)$ has no effect on the toggles τ_p for $p \in P_X$. So indeed applying the above sequence of toggles to A will have the same effect if we only apply the toggles τ_p for $p \in P_X$. This proves the second bulleted item.

It follows from the previous paragraph that $\mathrm{supp}((\mathrm{Row}^{-1} \cdot \mathrm{Rvac})(A)) \subseteq \mathrm{supp}(A)$. But since $(\mathrm{Row}^{-1} \cdot \mathrm{Rvac})^2$ is the identity, we also $\mathrm{supp}(A) \subseteq \mathrm{supp}((\mathrm{Row}^{-1} \cdot \mathrm{Rvac})(A))$, thus proving the first bulleted item as well. $\square$

Proof of Theorem 7. In showing that the bijection $\Theta_W\colon \mathcal{A}(\Phi^+) \xrightarrow{\sim} \mathrm{NC}(W, c_L c_R)$ is uniquely defined by the listed properties, Armstrong–Stump–Thomas [2] explained that for every $A \in \mathcal{A}(\Phi^+)$, there is a $k \geq 0$ so that $\mathrm{supp}(\mathrm{Row}^k(A)) \subsetneq [r]$. Hence, the bijection can be computed inductively as follows: we let $A' := \mathrm{Row}^k(A)$ where $k \geq 0$ is minimal so that $\mathrm{supp}(A') \subsetneq [r]$; then we inductively compute $w' := \Theta_{W_{\mathrm{supp}(A')}}(A')$; finally, we set $\Theta_W(A) := \mathrm{Krew}^{-k}(\prod_{i \in L \setminus \mathrm{supp}(A')} s_i \cdot w')$. The base case of this induction is where we use the condition $\Theta_W(\{\alpha_i\colon i \in L\}) = e$.

Thus, to prove this theorem, it suffices to show that:

- as a base case $\Theta_W((\mathrm{Row}^{-1} \cdot \mathrm{Row})(\{\alpha_i\colon i \in L\})) = \mathrm{Flip}(e)$;

- if $\Theta_W((\mathrm{Row}^{-1}\cdot\mathrm{Rvac})(A)) = \mathrm{Flip}(\Theta_W(A))$ then $\Theta_W((\mathrm{Row}^{-1}\cdot\mathrm{Rvac})(\mathrm{Row}(A))) = \mathrm{Flip}(\mathrm{Krew}\cdot\Theta_W(A))$;
- if $\Theta_{W_{\mathrm{supp}(A)}}((\mathrm{Row}^{-1}\cdot\mathrm{Rvac})_{\mathrm{supp}(A)}(A)) = \mathrm{Flip}_{W_{\mathrm{supp}(A)}}(\Theta_{W_{\mathrm{supp}(A)}}(A))$ then

$$\Theta_W((\mathrm{Row}^{-1}\cdot\mathrm{Rvac})(A)) = \mathrm{Flip}(\prod_{i\in L\setminus\mathrm{supp}(A)} s_i\cdot\Theta_{W_{\mathrm{supp}(A)}}(A)).$$

The first bulleted item is clear since $(\mathrm{Row}^{-1}\cdot\mathrm{Row})(\{\alpha_i\colon i\in L\}) = \{\alpha_i\colon i\in L\}$ and $\mathrm{Flip}(e)=e$.

For the second bulleted item, we have

$$\begin{aligned}\Theta_W((\mathrm{Row}^{-1}\cdot\mathrm{Row})(\mathrm{Row}(A)) &= \Theta_W((\mathrm{Row}^{-2}\cdot\mathrm{Row})(A))\\ &= \mathrm{Krew}^{-1}\cdot\Theta_W((\mathrm{Row}^{-1}\cdot\mathrm{Row})(A))\\ &= \mathrm{Krew}^{-1}\cdot\mathrm{Flip}(\Theta_W(A))\\ &= \mathrm{Flip}(\mathrm{Krew}(\Theta_W(A))),\end{aligned}$$

where we use Proposition 2, that $\Theta_W\cdot\mathrm{Row}^{-1} = \mathrm{Krew}^{-1}\cdot\Theta_W$, that $\mathrm{Flip}\cdot\mathrm{Krew} = \mathrm{Krew}^{-1}\cdot\mathrm{Flip}$, etc.

For the third bulleted item, we have

$$\begin{aligned}\Theta_W((\mathrm{Row}^{-1}\cdot\mathrm{Rvac})(A)) &= \Theta_W((\mathrm{Row}^{-1}\cdot\mathrm{Rvac})_{\mathrm{supp}(A)}(A))\\ &= \prod_{i\in L\setminus\mathrm{supp}(A)} s_i\cdot\Theta_{W_{\mathrm{supp}(A)}}((\mathrm{Row}^{-1}\cdot\mathrm{Rvac})_{\mathrm{supp}(A)}(A))\\ &= \prod_{i\in L\setminus\mathrm{supp}(A)} s_i\cdot\mathrm{Flip}_{W_{\mathrm{supp}(A)}}(\Theta_{W_{\mathrm{supp}(A)}}(A))\\ &= \prod_{i\in L\setminus\mathrm{supp}(A)} s_i\cdot\mathrm{Flip}_{W_{\mathrm{supp}(A)}}(\prod_{i\in L\setminus\mathrm{supp}(A)} s_i\cdot\Theta_W(A))\\ &= \prod_{i\in L\setminus\mathrm{supp}(A)} s_i\cdot c_{L\cap\mathrm{supp}(A)}\cdot\Theta_W(A)^{-1}\cdot\prod_{i\in L\setminus\mathrm{supp}(A)} s_i\cdot c_{L\cap\mathrm{supp}(A)}\\ &= c_L(\Theta_W(A)^{-1}c_L\\ &= \mathrm{Flip}(\Theta_W(A)),\end{aligned}$$

where we heavily use Lemma 8, and that $\Theta_{W_{\mathrm{supp}(A)}}(A) = \prod_{i\in L\setminus\mathrm{supp}(A)} s_i\cdot\Theta_W(A)$, etc. $\square$

Now, why might Theorem 7 be useful for resolving Conjecture 4? The reason is that, in the classical types, there are diagrammatic models for the noncrossing partitions. For example, we have already asserted that in Type A, $\mathrm{NC}(\Phi)$ really is the lattice of noncrossing set partitions. Meanwhile, in Type B, $\mathrm{NC}(\Phi)$ can be realized as the 180° rotationally invariant noncrossing set partitions, as explained by Reiner [26]. And in Type D there is also a (rather involved) noncrossing set partitions model for $\mathrm{NC}(\Phi)$ due to Athanasiadis and Reiner [3]. Armstrong–Stump–Thomas explain exactly how the bijection from Theorem 6 maps root poset antichains to these Type A, B, and D noncrossing set partitions (and this explanation constitutes the majority of the proof of Theorem 6). Actually, they work with *noncrossing handshake configurations*, which are a simple variant of the noncrossing set partitions. The advantage of noncrossing handshake configurations is that Kreweras complement literally rotates them. Similarly, it should be possible to show that Flip literally flips the noncrossing handshake configurations across a diameter. Thus, we can understand the way that rowvacuation acts on the antichains of the root poset in the classical types by studying the "simpler" action of the flip on noncrossing handshake configurations for these types. Indeed, this is precisely what Armstrong–Stump–Thomas did to resolve Panyushev's conjectures about root poset rowmotion. The difficulty lies in tracking the statistic of antichain cardinality through the bijection (which is especially hairy in Type D).

We end this note with a few more contextual remarks.

Remark 9. The *q-Φ-Catalan number* is $\mathrm{Cat}(\Phi;q) := \prod_{i=1}^{r} \frac{[h+d_i]_q}{[d_i]_q}$, where we use the standard q-number notation $[n]_q := (1-q^n)/(1-q)$. Affirming a conjecture of Bessis–Reiner [5], Armstrong–Stump–Thomas [2] showed that $\mathrm{Cat}(\Phi;q)$ is a *cyclic sieving* polynomial for the action of Row on $\mathcal{A}(\Phi^+)$ (equivalently, by Theorem 6, for the action of Krew on $\mathrm{NC}(\Phi)$). Since $\mathrm{Cat}(\Phi;q)$ has a product formula, this means that all of the fixed point counts $\#\mathcal{A}(\Phi^+)^{\mathrm{Row}^k}$ have product formulas as well. It makes sense to ask about the fixed point counts $\#\mathcal{A}(\Phi^+)^{\mathrm{Rvac}}$ and $\#\mathcal{A}(\Phi^+)^{\mathrm{Row}^{-1}\cdot\mathrm{Rvac}}$. These counts are recorded in Table 1. Note, interestingly, that there are some large primes appearing in these counts: e.g., 17 in F_4, 47 in E_6, and 139 in E_8. This suggests there is no product formula for these numbers in general, and hence no similar CSP to the aforementioned rowmotion/$\mathrm{Cat}(\Phi;q)$ one.

Remark 10. There are several other alternate definitions of the Φ-Narayana numbers, from which their symmetry is more clear. Let us discuss two of these; see for [12, §5.2] for the references and more details.

First of all, there is an interpretation of the Naryana numbers in terms of cluster combinatorics. The Φ-Narayana numbers are known to be the coefficients of the h-vector of the (simple polytope dual to) the *cluster complex* of Φ. In Type A, this polytope is the classical *associahedron*. From this perspective, symmetry of the Narayana numbers is a consequence of the *Dehn–Sommerville relations*.

Secondly, there is an interpretation of the Narayana numbers in terms of noncrossing partitions. The Φ-Narayana numbers $\mathrm{Nar}(\Phi,k)$ are known to be the number of elements of $\mathrm{NC}(\Phi)$ of rank k. From this perspective, symmetry of the Narayana numbers is a consequence of the fact that the lattice of noncrossing partitions is self-dual. Hence, the Kreweras complement combinatorially demonstrates the symmetry of the Narayana numbers. In other words: when studying the symmetry of the Narayana numbers on the nonnesting side of things, Panyushev devised an operator which is "equivalent" to an operator that evidently demonstrates the symmetry of the Narayana numbers on the noncrossing side of things. However, I do not believe Panyushev was at all aware of the relation between rowmotion and the Kreweras complement at the time he was studying rowmotion.

Remark 11. There are a couple of other maps on order ideals that people have looked at which bear some similarities to rowvacuation. Let's review these.

Recall that the facets of the cluster complex of Φ are another set enumerated by $\mathrm{Cat}(\Phi)$. There is a cyclic action on these facets, which is variously called *Cambrian rotation*, the action of the *deformed* or *tropical Coxeter element*, or the *Auslander–Reiten translate*. This action is different from the Kreweras complement: it has order (dividing) $h+2$, instead of $2h$. In Type A, the facets of the cluster complex correspond to triangulations of a polygon, and the Cambrian rotation is rotation of triangulations. The work of Armstrong–Stump–Thomas says that the Kreweras complement "is" rowmotion in the nonnesting world. Williams considered what the Cambrian rotation ought to be in the nonnesting world. In [33] he proposed a definition for nonnesting Cambrian rotation, acting on $\mathcal{J}(\Phi)$, in terms of toggles: first do all of the toggles, then do all of the toggles except at the minimal elements. This composition of toggles is superficially similar to rowvacuation. However, the definition of rowvacuation involves a lot more toggles than Williams's nonnesting Cambrian rotation. Another difference is that the order of the toggles in nonnesting Cambrian rotation is not the "top-to-bottom" toggling of rowmotion (in Type A it is the "left-to-right" toggling order that Stiker–Williams [32] call "promotion").

As mentioned, for any $I \in \mathcal{J}(P)$ and $p \in P$ with $\mathrm{rk}(p) = i$, we have $p \in \mathrm{Rvac}(I)$ if and only if $p \in \mathrm{Row}^{i+1}(I)$. In other words, we can construct $\mathrm{Rvac}(I)$ by "sewing together" different parts of the future rowmotion iterates of I. In [11, §6], Einstein and Propp define a *recombination map* which they use to relate two toggling operators acting on the order ideals of the rectangle poset: rowmotion and the left-to-right "promotion" operator. Recombination similarly sews together parts of the future rowmotion iterates of an order ideal. However, the parts that Einstein–Propp are sewing together are not ranks- they are rather "fibers."

In [13, §10], Grinberg and Roby explain how to embed the rowmotion dynamics of the Type A root poset into the rowmotion dynamics of the square poset. Their construction again uses parts of future rowmotion iterates. This time the parts *are* ranks. In fact, their embedding essentially takes an order ideal $I \in \mathcal{J}(\Phi^+(A_{n-1}))$ and glues it to an inverted copy of $\mathrm{Rvac}(I)$. (This embedding is closely related to the aforementioned reciprocity property of rowmotion on the square.) Note that Grinberg and Roby describe this embedding at the birational level. Their embedding is discussed in much more detail in [17].

Remark 12. The *major index* of a Dyck path is the sum of the positions of its valleys. The *q-Catalan number* is $\mathrm{Cat}(n;q) := \frac{1}{[n+1]_q}\begin{bmatrix}2n\\ n\end{bmatrix}_q$, where we use the standard q-number notations $[k]_q! := [k]_q[k-1]_q\cdots[1]_q$ and $\begin{bmatrix}a\\ b\end{bmatrix}_q := \frac{[a]_q!}{[b]_q![a-b]_q!}$. $\mathrm{Cat}(n;q)$ is a polynomial in q with nonnegative integer coefficients: it is the major index generating function of Dyck paths of semilength n. The Lalanne–Kreweras involution on Dyck paths of semilength n sends such a path with major index k to one with major index $n(n-1)-k$, and hence combinatorially demonstrates the symmetry of the coefficients of the q-Catalan number.

As mentioned, the q-Φ-Catalan number is $\mathrm{Cat}(\Phi;q) = \prod_{i=1}^{r} \frac{[h+d_i]_q}{[d_i]_q}$. It is known uniformly that $\mathrm{Cat}(\Phi;q)$ is a polynomial in q with nonnegative integer coefficients (see [5]). Nevertheless, there is no known statistic on antichains of Φ^+ of which $\mathrm{Cat}(\Phi;q)$ is the generating function. Thus, in terms of discovering other statistics that rowvacuation might interact nicely with, I am doubtful that there could be an analog of major index in other types. Indeed, even in Type A major index is a little weird in that it does not respect the automorphisms of the root poset.

Remark 13. In 2013, Einstein and Propp [11] introduced *piecewise-linear* and *birational* liftings of rowmotion. Their main idea was to first define piecewise-linear and birational lifts of the toggles, and then define rowmotion (at any level) as the appropriate composition of the toggles. Since the work of Einstein–Propp, there has been a lot of interest in piecewise-linear and birational rowmotion.

With Mike Joseph [17], we have been studying piecewise-linear and birational lifts of the Lalanne–Kreweras involution on the antichains of the Type A root poset. Indeed, this investigation is what prompted me to think that rowvacuation could be the relevant map in all types for resolving Panyushev's Conjecture 1. Since rowvacuation is defined as a composition of toggles, there is a natural way to lift it to the piecewise-linear and birational levels (in fact, in [16] rowvacuation is introduced at the piecewise-linear level). Moreover, Proposition 2 continues to hold at the piecewise-linear and birational levels because it follows from formal algebraic properties of the toggles.

However, we remark that piecewise-linear and birational rowmotion do not behave well for all root posets, only for those *of coincidental type*. For the crystallographic cases this is just Type A and Type B/C. Since Type B/C results are often easily obtained from their Type A counterparts via "folding," it mostly makes sense to study piecewise-linear and birational rowvacuation for the Type A root poset. This is what we have been doing in [17].

Remark 14. The non-crystallographic root systems are $I_2(\ell)$, H_3 and H_4. For $I_2(\ell)$ and H_3 (which are of coincidental type), Armstrong [1] gave ad hoc definitions of root posets which continue to exhibit many of the same remarkable combinatorial properties of the Φ^+ for crystallographic Φ we have been discussing (see, e.g., [10]). Theorem 7 and Conjecture 4 can easily be extended to $I_2(\ell)$ and H_3.

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