Enumeration of barely set-valued tableaux and plane partitions

George Washington University Combinatorics & Algebra Seminar

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Section 1

Tableaux and plane partitions

Standard Young tableaux

The **Young diagram** of a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is left-justified array of boxes with λ_i boxes in ith row:

$$(4,3,1) = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \multicolumn{1}{c}{} \\ \cline{1-3} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

We will care most about the **rectangle shape** $a \times b := (\overbrace{b, b, \ldots, b}^{a \text{ times}})$.

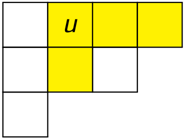
A **standard Young tableau** of shape λ is a filling of the Young diagram with numbers $1, 2, \ldots, n := |\lambda|$, each appearing once, which is increasing along rows and down columns.

Let $\mathcal{SYT}(\lambda) := \{\text{SYTs of shape } \lambda\}$.

$$\mathcal{SYT}(2 \times 2) = \left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \right\}$$

The Hook Length Formula

The **hook** of box u of a Young diagram is all boxes weakly left or below u:



Hook length $h(u) :=$ number of boxes in hook.

Theorem (Hook Length Formula; Frame–Robinson–Thrall, 1954)

$$\#\mathcal{SYT}(\lambda) = \frac{n!}{\prod_{u\in\lambda} h(u)}$$

For example, $\#\mathcal{SYT}(2\times 2) = \dfrac{4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 2\cdot 1} = 2.$

Set-valued tableaux

A **standard set-valued tableau** of shape λ is a filling of Young diagram with numbers $1, 2, \ldots, n+k$ for some $k \geq 0$, each appearing once, but where multiple numbers can be in the same box.

(Each box must get at least one number, and still needs to be increasing.)

Let $\mathcal{SYT}^{+k}(\lambda)$ be the set of these tableaux. So $\mathcal{SYT}^{+0}(\lambda) = \mathcal{SYT}(\lambda)$.

Our focus is on **barely set-valued tableaux** $\mathcal{SYT}^{+1}(\lambda)$.

For example, there are 10 tableaux in $\mathcal{SYT}^{+1}(2 \times 2)$:

$$
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4,5 \\ \hline \end{array} \quad
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4,5 \\ \hline \end{array} \quad
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3,4 & 5 \\ \hline \end{array} \quad
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2,4 & 5 \\ \hline \end{array} \quad
\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2,3 & 5 \\ \hline \end{array}
$$

$$
\begin{array}{|c|c|}\hline 1 & 2,3 \\ \hline 4 & 5 \\ \hline \end{array} \quad
\begin{array}{|c|c|}\hline 1 & 2,4 \\ \hline 3 & 5 \\ \hline \end{array} \quad
\begin{array}{|c|c|}\hline 1 & 3,4 \\ \hline 2 & 5 \\ \hline \end{array} \quad
\begin{array}{|c|c|}\hline 1,2 & 3 \\ \hline 4 & 5 \\ \hline \end{array} \quad
\begin{array}{|c|c|}\hline 1,2 & 4 \\ \hline 3 & 5 \\ \hline \end{array}
$$

Aside: Schur & Grothendieck polynomials

The **Schur function**

$$s_\lambda(x_1, x_2, \ldots) = \sum_{\substack{\text{SSYT } T, \\ \text{shape}(T)=\lambda}} \mathbf{x}^{\text{content}(T)}$$

is the generating function for **semistandard tableaux** (I won't define). Schur functions have many algebraic/geometric guises; one is that they represent Schubert cycles in the cohomology of the Grassmannian.

Similarly, the **(stable) Grothendieck polynomials**

$$G_\lambda(x_1, x_2, \ldots) = \sum_{\substack{\text{set-valued SSYT } T, \\ \text{shape}(T)=\lambda}} (-1)^{|T|-|\lambda|} \mathbf{x}^{\text{content}(T)}$$

represent Schubert cycles in K-theory of the Grassmannian (Buch, 2002).

Plane partitions

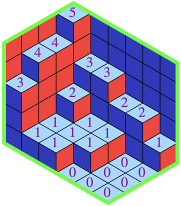
An **plane partition** of shape λ is filling of the Young diagram with nonnegative integers, weakly decreasing in rows and columns.

Let $\mathcal{PP}_m(\lambda) :=$ plane partitions of shape λ with entries in $\{0, 1, \ldots, m\}$.

There is a beautiful 3D representation of plane partitions:

5	3	3	2	2	1
4	2	1	1	1	0
4	1	1	1	0	0
3	1	1	0	0	0

$\Leftrightarrow$



$\in \mathcal{PP}_5(4 \times 6)$

Theorem (MacMahon, c. 1915)

$$\#\mathcal{PP}_m(a \times b) = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{m+i+j-1}{i+j-1}$$

Section 2

Motivation from algebraic geometry

Brill–Noether theory

Let C be a "general" curve of genus g. The **Brill–Noether space** $G^r_d(C)$ is moduli space of maps from C to r-dim'l projective space $\mathbb{P}^r$ of degree d:

$$G^r_d(C) = \Bigg\{$$

$$\xrightarrow[\mathrm{degree}(\phi)=d]{\phi} \mathbb{P}^r \Bigg\}$$

Define the **Brill–Noether number** $\rho = \rho(g, d, r)$ as

$$\rho := g - (r+1)(g - d + r)$$

Theorem (Brill–Noether Theorem; Griffiths–Harris, 1980)

$G^r_d(C)$ is nonempty iff $\rho \geq 0$, and in that case $\dim(G^r_d(C)) = \rho$.

Finer invariants of moduli spaces

We could ask for finer information about $G^r_d(C)$ than just its dimension.

For example, when $\rho = 0$, $G^r_d(C)$ is a finite set of points, and the number of points is known to be

$$\#G^r_d(C) = g! \cdot \prod_{i=0}^{r} \frac{i!}{(g-d+r+i)!}$$

Or when $\rho = 1$, $G^r_d(C)$ is itself a smooth curve, and the genus of this curve is known to be

$$\mathrm{genus}(G^r_d(C)) = 1 + \frac{(r+1)(g-d+r)}{g-d+2r+1} \cdot g! \cdot \prod_{i=0}^{r} \frac{i!}{(g-d+r+i)!}$$

Interesting product formulas...

Euler characteristics via tableaux

Comparing to the Hook Length Formula we see that when $\rho = 0$,

$$\#G^r_d(C) = \#\mathcal{SYT}((r+1) \times (g-d+r))$$

Chan–López-Martín–Pflueger–Teixidor i Bigas (2018) showed when $\rho = 1$,

$$\mathrm{genus}(G^r_d(C)) = 1 + \#\mathcal{SYT}^{+1}((r+1) \times (g-d+r))$$

Corollary (Chan–López-Martín–Pflueger–Teixidor i Bigas, 2018)

$$\#\mathcal{SYT}^{+1}(a \times b) = (ab+1) \cdot \frac{ab}{a+b} \cdot \#\mathcal{SYT}(a \times b)$$

For example, $\#\mathcal{SYT}^{+1}(2 \times 2) = 5 \cdot \frac{4}{4} \cdot \#\mathcal{SYT}(2 \times 2) = 5 \cdot 1 \cdot 2 = 10$.

Chan–Pflueger (2021) showed more generally that for any $\rho \geq 0$, the Euler characteristic of $G^r_d(C)$ is $(-1)^\rho$ times $\#\mathcal{SYT}^{+\rho}((r+1) \times (g-d+r))$. But apparently no product formulas for $\rho > 1$!

Section 3

Down-degree expectations

Decomposing barely set-valued tableaux

A barely set-valued tableau $T' \in \mathcal{SYT}^{+1}(\lambda)$ has a rather simple structure: one special box has two numbers, while all others have a single number.

This leads to a decomposition of T' into a triple (T, i, u) where:

- $T \in \mathcal{SYT}(\lambda)$ is a usual standard tableau;
- $i \in \{0, 1, \ldots, n\}$ is some number;
- u is a **removable box** of the **subshape** $T^{-1}(\{1, 2, \ldots, i\})$.

(A **subshape** *of λ is a Young diagram σ with $\sigma \subseteq \lambda$. A* **removable box** *of a subshape $\sigma \subseteq \lambda$ is a box whose removal gives another subshape.)*

$$T' = \begin{array}{|c|c|c|} \hline 1 & 2 & 4,7 \\ \hline 3 & 5 & 8 \\ \hline 6 & 9 & 10 \\ \hline \end{array} \quad \Leftrightarrow \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & 7 \\ \hline 6 & 8 & 9 \\ \hline \end{array} = T$$

$i = 6$, $u =$ circled box,
$T^{-1}(\{1 \ldots, i\}) =$ yellow

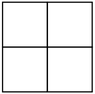
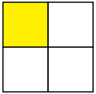
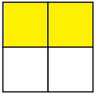
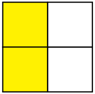
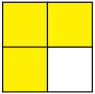
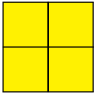
Distributions on subshapes

The decomposition of barely set-valued tableaux T' motivates us to consider the following probability distribution on subshapes of λ:

- choose $T \in \mathcal{SYT}(\lambda)$ uniformly at random;
- choose $i \in \{0, 1, \ldots, n\}$ uniformly at random;
- select the subshape $T^{-1}(\{1, 2, \ldots, i\})$.

Call this distrubtion on subshapes μ_{SYT}. Also, denote the number of removable boxes of a subshape σ by $\mathrm{ddeg}(\sigma)$, the **down-degree** of σ.

For example, with $\lambda = 2 \times 2$:

σ						
$\mu_{\mathrm{SYT}}(\sigma)$	$1/5$	$1/5$	$1/10$	$1/10$	$1/5$	$1/5$
$\mathrm{ddeg}(\sigma)$	0	1	1	1	2	1

Down-degree expectations

The decomposition of barely set-valued T' can be restated in terms of **expected down-degrees** as follows:

Proposition

$$\mathbb{E}_{\mu_{\mathrm{SYT}}}(\mathrm{ddeg}) = \frac{\#\mathcal{SYT}^{+1}(\lambda)}{(n+1)\cdot\#\mathcal{SYT}(\lambda)}$$

For example, with $\lambda = 2 \times 2$:

$$\mathbb{E}_{\mu_{\mathrm{SYT}}}(\mathrm{ddeg}) = (0\cdot\frac{1}{5} + 1\cdot\frac{1}{5} + 1\cdot\frac{1}{10} + 1\cdot\frac{1}{10} + 2\cdot\frac{1}{5} + 1\cdot\frac{1}{5}) = 1 = \frac{10}{5\cdot 2}.$$

"Expected down-degrees" terminology due to Reiner–Tenner–Yong (2018).

Barely set-valued plane partitions

For any $m \geq 1$, we can define distribution μ_{PP_m} on subshapes by:

- choose $\pi \in \mathcal{PP}_m(\lambda)$ uniformly at random;
- choose $i \in \{0, 1, \ldots, m-1\}$ uniformly at random;
- select the subshape $\pi^{-1}(\{0, 1, \ldots, i\})$.

Note: $\mu_{\mathrm{SYT}} = \lim_{m \to \infty} \mu_{\mathrm{PP}_m}$ and $\mu_{\mathrm{PP}_1} =$ **uniform** distribution.

Proposition

$$\mathbb{E}_{\mu_{\mathrm{PP}_m}}(\mathrm{ddeg}) = \frac{\#\mathcal{PP}_m^{+1}(\lambda)}{m \cdot \#\mathcal{PP}_m(\lambda)}$$

Here $\mathcal{PP}_m^{+1}(\lambda)$ is the set of **"barely set-valued plane partitions"** which look like what you'd expect:

$$\begin{array}{|c|c|c|} \hline 2 & 2 & 2,0 \\ \hline 2 & 1 & 0 \\ \hline \end{array} \in \mathcal{PP}_2^{+1}(2 \times 3).$$

Section 4

Toggles, toggle-symmetry, and rooks

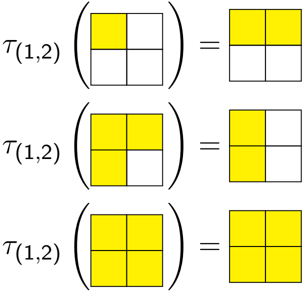
Toggling subshapes

Let $u \in \lambda$ be a box & $\sigma \subseteq \lambda$ a subshape. Define the **toggle** $\tau_u(\sigma)$ to be

$$\tau_u(\sigma) := \begin{cases} \sigma \setminus u & \text{if } u \text{ is a removable from } \sigma; \\ \sigma \cup u & \text{if } u \text{ is addable to } \sigma; \\ \sigma & \text{otherwise.} \end{cases}$$

(Here u being **addable** *means we can add u to σ and get a subshape.)*

For example,



Toggle-symmetric distributions

For box $u \in \lambda$, define toggleability statistics $\mathcal{T}_u^+, \mathcal{T}_u^-, \mathcal{T}_u$ on subshapes by

$$\mathcal{T}_u^+(\sigma) := \begin{cases} 1 & \text{if } u \text{ is addable to } \sigma; \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{T}_u^-(\sigma) := \begin{cases} 1 & \text{if } u \text{ is removable from } \sigma; \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{T}_u(\sigma) := \mathcal{T}_u^+(\sigma) - \mathcal{T}_u^-(\sigma).$$

Definition

A probability distribution μ on subshapes is called **toggle-symmetric** if we have $\mathbb{E}_\mu(\mathcal{T}_u) = 0$ for all boxes $u \in \lambda$.

In other words, we are as likely to be able to toggle u in as toggle it out.

SYT & plane partition distributions are toggle-symmetric

Lemma (Chan–Haddadan–H.–Moci, 2017)

- *The distribution* μ_{SYT} *is toggle-symmetric.*
- *For any* $m \geq 1$*, the distribution* μ_{PP_m} *is toggle-symmetric.*

Proof sketch: For μ_{SYT}: use $\mu_{\mathrm{SYT}} = \lim_{m\to\infty} \mu_{\mathrm{PP}_m}$.

For μ_{PP_m}: for any $\pi \in \mathcal{PP}_m(\lambda)$, the contribution of π to $\mathbb{E}_{\mu_{\mathrm{PP}_m}}(\mathcal{T}_u)$ is negative the contribution of $\tau_u(\pi)$, where the **(piecewise-linear) plane partition toggle** $\tau_u(\pi)$ is defined by the formula

$$\pi = \begin{array}{|c|c|c|}\hline & x & \\ \hline w & u & z \\ \hline & y & \\ \hline\end{array} \mapsto \begin{array}{|c|c|c|}\hline & x & \\ \hline w & u' & x \\ \hline & y & \\ \hline\end{array} = \tau_u(\pi)$$

with $u' := \min(w,x) + \max(y,z) - u$. $\square$

Down-degree as sum of toggleability statistics

What's the point? We can sometimes write down-degree in a clever way...

Theorem (Chan–Haddadan–H.–Moci, 2017)

For the rectangle $\lambda = a \times b$, there are coefficients $c_u \in \mathbb{Q}$, $u \in \lambda$ for which

$$\mathrm{ddeg} = \frac{ab}{a+b} + \sum_{u \in \lambda} c_u \mathcal{T}_u$$

By linearity of expectation we obtain enumerative corollaries:

Corollary

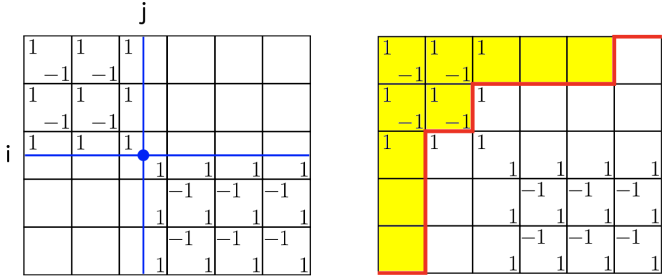
$$\frac{\#\mathcal{SYT}^{+1}(a \times b)}{(n+1) \cdot \#\mathcal{SYT}(a \times b)} = \mathbb{E}_{\mu_{\mathrm{SYT}}}(\mathrm{ddeg}) = \frac{ab}{a+b}$$

$$\frac{\#\mathcal{PP}_m^{+1}(a \times b)}{m \cdot \#\mathcal{PP}_m(a \times b)} = \mathbb{E}_{\mu_{\mathrm{PP}_m}}(\mathrm{ddeg}) = \frac{ab}{a+b}$$

Key technical tool: “rooks”

How to write down-degree as a sum of the $\mathcal{T}_u$? Note $\mathrm{ddeg} = \sum_{u \in \lambda} \mathcal{T}_u^-$.
So the key is to find relations among the toggleability statistics.

The “building block” of toggleability statistics relations is the **rook** R_{ij}:

$R_{ij} =$ 

Lemma

We have $R_{ij}(\sigma) = 1$ for any subshape $\sigma \subseteq a \times b$.

Section 5

q-analogs

Comajor index for SYTs

Let $T \in \mathcal{SYT}(\lambda)$ be a standard tableau. A **descent*** of T is an entry i such that $i+1$ is in a higher row than i. Denote set of descents by $D(T)$. The **comajor index** of T is $\mathrm{comaj}(T) := \sum_{i \in D(T)} (n-i)$.

Theorem (q-Hook-Length-Theorem; Stanley, c. 1970?)

$$\sum_{T \in \mathcal{SYT}(\lambda)} q^{\mathrm{comaj}(T)} = \frac{[n]_q \cdot [n-1]_q \cdots [2]_q \cdot [1]_q}{\prod_{u \in \lambda} [h(u)]_q}$$

We use standard q-notation $[n]_q := 1 + q + \cdots + q^{n-1} = (1-q^n)/(1-q)$.

T	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$
$D(T)$	$\varnothing$	$\{2\}$
$\mathrm{comaj}(T)$	0	2

$$\sum_{T \in \mathcal{SYT}(2\times 2)} q^{\mathrm{comaj}(T)} = q^2 + 1$$

$$= \frac{[4]_q[3]_q[2]_q[1]_q}{[3]_q[2]_q[2]_q[1]_q}$$

Comajor index for barely set-valued SYTs

Let $T \in \mathcal{SYT}^{+1}(\lambda)$ be a barely set-valued tableau. Let $i_*(T)$ denote the bigger number in the special box that has two numbers. A **descent** of T is an entry i such that $i+1$ is in a higher row than i, except that:

- $i_*(T) - 1$ is never a descent;
- $i_*(T)$ is always a descent.

Denote set of descents by $D(T)$. Let $\mathrm{comaj}(T) := \sum_{i \in D(T)} (n+1-i)$.

Theorem (H.–Lazar–Linusson, 2021)

$$\sum_{T \in \mathcal{SYT}^{+1}(a \times b)} q^{\mathrm{comaj}(T)} = [ab+1]_q \cdot \frac{[a]_q [b]_q}{[a+b]_q} \cdot \sum_{T \in \mathcal{SYT}(\lambda)} q^{\mathrm{comaj}(T)}$$

Comajor index generating functions: example

T	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 & 4,5 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 & 4,5 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3,4 & 5 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2,4 & 5 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 4 \\ \hline 2,3 & 5 \\ \hline\end{array}$
$D(T)$	$\{5\}$	$\{2,5\}$	$\{4\}$	$\{2,4\}$	$\{3\}$
$\mathrm{comaj}(T)$	0	3	1	4	2
T	$\begin{array}{\|c\|c\|}\hline 1 & 2,3 \\ \hline 4 & 5 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 2,4 \\ \hline 3 & 5 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 3,4 \\ \hline 2 & 5 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1,2 & 3 \\ \hline 4 & 5 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1,2 & 4 \\ \hline 3 & 5 \\ \hline\end{array}$
$D(T)$	$\{3\}$	$\{4\}$	$\{2,4\}$	$\{2\}$	$\{2,3\}$
$\mathrm{comaj}(T)$	2	1	4	3	5

$$\sum_{T \in \mathcal{SYT}^{+1}(2\times 2)} q^{\mathrm{comaj}(T)} = q^5 + 2q^4 + 2q^3 + 2q^2 + 2q + 1$$

$$= [5]_q \cdot \frac{[2]_q[2]_q}{[4]_q} \cdot (q^2+1)$$

Size generating functions for plane partitions

The **size** $|\pi|$ of a plane partition $\pi \in \mathcal{PP}_m(\lambda)$ is the sum of its entries.

Theorem (MacMahon, c. 1915)

$$\sum_{\pi \in \mathcal{PP}_m(a \times b)} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{[m+i+j-1]_q}{[i+j-1]_q}$$

Define size for barely set-valued plane partitions similarly.

Theorem (H.–Lazar–Linusson, 2021)

$$\sum_{\pi \in \mathcal{PP}_m^{+1}(a \times b)} q^{|\pi|-1} = [m]_q \cdot \frac{[a]_q [b]_q}{[a+b]_q} \cdot \sum_{\pi \in \mathcal{PP}_m(a \times b)} q^{|\pi|}$$

Size generating functions: example

π	$\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$	$\begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}$	$\begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}$	$\begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix}$	$\begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}$	$\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}$
$\lvert\pi\rvert$	0	1	2	2	3	4

$$\sum_{\pi \in \mathcal{PP}_1(2\times 2)} q^{|\pi|} = q^4 + q^3 + 2q^2 + q + 1 = \frac{[4]_q[3]_q[3]_q[2]_q}{[3]_q[2]_q[2]_q[1]_q}$$

π	$\begin{matrix} 1,0 & 0 \\ 0 & 0 \end{matrix}$	$\begin{matrix} 1 & 1,0 \\ 0 & 0 \end{matrix}$	$\begin{matrix} 1 & 0 \\ 1,0 & 0 \end{matrix}$	$\begin{matrix} 1 & 1,0 \\ 1 & 0 \end{matrix}$	$\begin{matrix} 1 & 1 \\ 1,0 & 0 \end{matrix}$	$\begin{matrix} 1 & 1 \\ 1 & 1,0 \end{matrix}$
$\lvert\pi\rvert$	1	2	2	3	3	4

$$\sum_{\pi \in \mathcal{PP}_1^{+1}(2\times 2)} q^{|\pi|-1} = q^3 + 2q^2 + 2q + 1 = [1]_q \cdot \frac{[2]_q[2]_q}{[4]_q} \cdot (q^4 + q^3 + 2q^2 + q + 1)$$

Proofs of q-analogs: q-toggle-symmetry

The basic outline of proofs for q-analogs is same as in case $q = 1$.

For a box u of λ, set $\mathcal{T}_u^q := \mathcal{T}_u^+ - q\mathcal{T}_u^-$. Call a probability distribution μ on subshapes q-**toggle-symmetric** if $\mathbb{E}_\mu(\mathcal{T}_u^q) = 0$ for all $u \in \lambda$.

We define appropriate q-analogs of distributions μ^q_{SYT} and $\mu^q_{\mathrm{PP}_m}$ and show:

Lemma (H.–Lazar–Linusson, 2021)

The distributions μ^q_{SYT} and $\mu^q_{\mathrm{PP}_m}$ are q-toggle-symmetric.

The other ingredient of the proof is:

Theorem (Defant–H.–Poznanović–Propp, 2021)

For $\lambda = a \times b$, there are coefficients $c_u(q) \in \mathbb{Q}(q), u \in \lambda$ for which

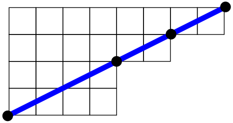
$$\mathrm{ddeg} = \frac{[a]_q[b]_q}{[a+b]_q} + \sum_{u \in \lambda} c_u(q)\mathcal{T}_u^q$$

Section 6

Concluding remarks

Concluding remarks

- Not all shapes λ have product formulas for $\#\mathcal{SYT}^{+1}(\lambda)$, but the rook technique does work for a broader class of **"balanced" shapes**:



- Can also look at **shifted shapes** (see Kim–Schlosser–Yoo (2020)), other posets, etc. In fact the q-analogs hold for all **minuscule posets**.
- Beyond distributive lattices: "**nearly reduced words**," ... (see RTY).
- There are interesting toggle-symmetric distributions not coming from tableaux/plane partitions. For instance, some come from **dynamics** on subshapes. Related to study of **homomesy** for these dynamics.

Thank you!

these slides are available on my website
and papers are on the arXiv:

- Chan, Haddadan, Hopkins, and Moci. "The expected jaggedness of order ideals." `arXiv:1507.00249`
- Reiner, Tenner, and Yong. "Poset edge densities, nearly reduced words, and barely set-valued tableaux." `arXiv:1603.09589`.
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