11/9

# Permutations and Combinations §6.2

Def'n A permutation of $n$ distinct elements $x_1, x_2, \ldots, x_n$ is an ordering of the elements, i.e., a list, where each $x_i$ appears exactly once.

E.g. There are 6 permutations of A, B, C:

ABC ACB BAC BCA CAB CBA

It also makes sense to define $\boxed{0! = 1}$

Recall that for a positive integer $n$, we defined $n$ factorial as $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$

Theorem The # of permutations of $n$ elements is $n!$

Pf: Imagine creating the permutation by choosing $1^{st}$ element, then $2^{nd}$, ..., up to the $n^{th}$. There are $n$ choices for $1^{st}$, but then only $n-1$ for the $2^{nd}$ (since we cannot choose what we chose for $1^{st}$), $(n-2)$ for $3^{rd}$, etc. down to 1 choice (whatever is left) for the last. By mult. principle $\Rightarrow n \times (n-1) \times \cdots \times 1$ total! ☒

We can even do a slightly more general thing:

Def'n An $r$-permutation of $x_1, \ldots, x_n$ is a length $r$ list of elements in $x_1, \ldots, x_n$ where each appear at most once. (We need $r \le n$ for such a list to exist.)

Eg. There are 12 2-permutations of A, B, C, D:

AB AC AD BA BC BD CA CB CD DA DB DC

We use $P(n,r) :=$ # of $r$-permutations of an $n$-element set.

Thm $P(n,r) = n \times (n-1) \times \cdots \times (n-r+1) = \dfrac{n!}{(n-r)!}$

Pf: Same as proof for usual permutations, we just stop after the $r^{th}$ step of the process. ☒

We also often want to count unordered collections of given size.

Def'n An r-combination of $x_1, \dots, x_n$ is a length $r$ unordered collection of elements in $x_1, \dots, x_n$, i.e., a size $r$ subset of $\{x_1, \dots, x_n\}$.

E.g. There are 6 2-combinations of A, B, C, D:
$\{A, B\}$ $\{A, C\}$ $\{A, D\}$ $\{B, C\}$ $\{B, D\}$ $\{C, D\}$.

How can we count the r-combinations of an n element set?
Let $C(n,r)$ = # r-combinations of n element set.
We will also use the notation $\binom{n}{r} = C(n,r)$ later
read this as "n choose r".

We can create an r-permutation of $\{x_1, \dots, x_n\}$ as follows:
1. Pick one of the $C(n,r)$ r-combinations, call it $\{y_1, \dots, y_r\} \subseteq \{x_1, \dots, x_n\}$
2. Choose any of the $r!$ permutations of $y_1, \dots, y_r$.

E.g. To make an 2-permutation of A, B, C, D, we first pick one of the 6 2-combinations, and then choose one of the $2! = 2$ ways to permute its letters:

$$\begin{array}{cccccc} \{A,B\} & \{A,C\} & \{A,D\} & \{B,C\} & \{B,D\} & \{C,D\} \\ \swarrow\ \searrow & \swarrow\ \searrow & \swarrow\ \searrow & \swarrow\ \searrow & \swarrow\ \searrow & \swarrow\ \searrow \\ AB\ \ BA & AC\ \ CA & AD\ \ DA & BC\ \ CB & BD\ \ DB & CD\ \ DC \end{array}$$

By the multiplication principle, this means:

# ways to make an r-permutation of $x_1, \dots, x_n$ = # of ways to make r-comb. of $x_1, \dots, x_n$ $\times$ # of permutations of r things

i.e., $P(n,r) = C(n,r) \times r!$

But now the trick is: we can use this to get a formula for $C(n,r)$, since we know $P(n,r) = \frac{n!}{(n-r)!}$

<u>Theorem</u> $C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!\,r!}$

<u>Pf</u>: We just explained why $P(n,r) = C(n,r) \cdot r!$. ☒

<u>E.g.</u> We saw that there were 6 2-combinations of $A, B, C, D$ and $C(4,2) = \frac{4!}{2!2!} = \frac{4\times 3\times 2\times 1}{2\times 1\times 2\times 1} = 6$ ✓

As mentioned, we also write $\binom{n}{r} = C(n,r)$, and will have a lot to say about these $\binom{n}{r}$ ("binomial coefficients") in a little bit, but here is just a taste:

<u>Exercise</u>: Show $\sum_{r=0}^{n} C(n,r) = 2^n$.

Hint: Imagine choosing an arbitrary subset of $\{1,2,\ldots,n\}$ by first choosing the <u>size</u> of the subset.

<u>Q</u>: A <u>standard deck</u> of <u>cards</u> has 52 cards in it:

- there are 4 different <u>suits</u>: spades, hearts, clubs, diamonds
- there are 13 different <u>ranks</u>: 2–10 and A K Q J

For a total of $4 \times 13 = 52$ different cards.

A poker hand is 5 of these cards.

1) How many poker hands are there?
2) How many hands have cards of all the same suit (called a "<u>flush</u>")?

1): $C(52,5) = 2{,}598{,}960$

2): $4 \times C(13,5) = 5{,}148$ (imagine first picking the suit, then choosing the 5 of 13 ranks of that suit).

This means $\approx 0.2\%$ of hands are flushes (<u>very</u> rare!)

## 11/18 Generalized Permutations and Combinations §6.3

There are $n!$ permutations of $n$ distinct letters:
e.g. ABC ACB BAC BCA CAB CBA

But what if we try to permute the letters of a word that has repeated letters.

E.g. How many ways are there to ~~permute~~ rearrange the letters in
MISSISSIPPI?

Some of the 11! permutations will be "the same"
So the answer is something less than 11!.

Let's start with something easier: what if we want to count rearrangements of AAABBBBB.

A rearrangement is 8 letters, 3 of them A's, 5 B's:

_ _ _ _ _ _ _ _

Think of 8 positions for our letters. We can choose any 3 of these for the A's, then the B's go in the other spots

B B A B A B A B

So, we are choosing 3 spots out of 8, which gives
$C(8,3) = 8!/(3!\cdot 5!)$ total rearrangements.

For MISSISSIPPI we can do something similar, but in more steps. We have 11 spots: choose 4 of them for where the I's go:

_ I _ I I _ I _ _ _ _ $C(11,4)$

Then choose from the remaining 7 spots, 4 for the S's:

_ I S I I _ I S S _ S $C(7,4)$

Then from the remaining 3 spots, choose 2 for the P's

P I S I I _ I S S P S $C(3,2)$

Then M goes in the remaining spot in $C(1,1)$ ways.

Altogether there are $C(11,4) \cdot C(7,4) \cdot C(3,2) \cdot C(1,1)$
$= \frac{11!}{4!\,7!} \cdot \frac{7!}{4!\,3!} \cdot \frac{3!}{2!\,1!} \cdot \frac{1!}{1!} = \frac{11!}{4!\,4!\,2!\,1!}$
rearrangements of MISSISSIPPI.

Theorem For a word which has $m$ different kinds of letters, with $n_1$ of the 1st letter, $n_2$ of the second letter, ... and $n_m$ of the $m^{th}$ letter, with $n = n_1 + n_2 + \cdots + n_m$ total letters, # of rearrangements is $n! / (n_1! \cdot n_2! \cdot n_3! \cdots n_m!)$.

Pf: Same as what we just explained. ∎

E.g. For MISSISSIPPI, $n=11$, $n_1 = 4$ I's, $n_2 = 4$ P's, $n_3 = 2$ S's, $n_4 = 1$ M
So that # rearrangements $= 11!/(4! \cdot 4! \cdot 2! \cdot 1!)$

If all letters are distinct, we get the usual $n!/(1! \cdot 1! \cdots 1!) = n!$ permutations and the more repeated letters we have, the more we have to divide $n!$ by to account for the repeats.

11/21 Those were generalized permutations (w/ repeated entries), what about generalized combinations (allowing repeats)?

Imagine you go to a bagel store. They have 4 different kinds of bagels: plain, everything, sesame, cinnamon raisin.

You want to buy 13 bagels (= a baker's dozen). How many ways are there to do this? It would be a $C(n,k)$ type problem if we had to pick all distinct flavors of bagels, but we can repeat flavors.

We can represent a selection of bagels like this:

$$\underset{\text{plain}}{***} \mid \underset{\text{everything}}{*****} \mid \underset{\text{sesame}}{**} \mid \underset{\text{cinnamon raisin}}{***}$$

This means we pick 3 plain bagels, 5 everything, 2 sesame, and 3 cinnamon raisin.
Every picture of 13 $*$'s ("stars") and 3 $|$'s ("bars") gives a unique bagel flavor selection.

How many such pictures are there? # of rearrangements of 13 $*$'s and 3 $|$'s $= C(16, 13)$ (as we saw before with letters A and B).

Theorem The number of ways to select $k$ things from $m$ options, possibly allowing selecting an option multiple times, is $C(k+m-1, k) = C(k+m-1, m-1)$.

[The second $=$ is because $C(n,k) = \frac{n!}{k!(n-k)!} = C(n, n-k)$]

E.g. You have 11 candies (all the same kind) and 3 little children to give them to. How many ways can you distribute the candies?

A: Represent a candy distribution like

$$\underset{\text{1st child}}{\overset{\text{4 candies}}{****}} \mid \underset{\text{2nd child}}{\overset{\text{2 candies}}{**}} \mid \underset{\text{3rd child}}{\overset{\text{5 candies}}{*****}}$$

This shows there are $C(11+2, 11)$ ways to distribute the candies (and shows this problem is the same as the bagel picking one). //

§6.7

11/23 Binomial Coefficients and the Binomial Theorem

Let's start with an algebra exercise:

$(a+b)^3 = (a+b)(a+b)(a+b)$

$= aaa + aab + aba + abb$
$+ baa + bab + bba + bbb$

$= a^3 + 3a^2b + 3ab^2 + b^3$

What is the significance of this sequence 1, 3, 3, 1?

If we did

$(a+b)^4 = \cdots = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

we get the sequence of coefficients 1, 4, 6, 4, 1.

And in general...

Theorem (Binomial Theorem)

$$(a+b)^n = \sum_{k=0}^{n} C(n,k)\, a^{n-k} b^k$$

Pf: Imagine expanding out $(a+b)^n$:

$(a+\underline{b})(\underline{a}+b) \cdots (a+\underline{b})$ ← n terms total

If we want to make a term of $a^{n-k} b^k$ from these multiplications, we have to choose the "b" part from exactly k of the $(a+b)$'s, and the "a" from the other $n-k$ of the $(a+b)$'s. So the number of ways to do this is the # of ways to choose exactly k positions from n, which by definition is $C(n,k) = \frac{n!}{k!(n-k)!}$ ▨

Note: In the context of the binomial theorem it is common to use the notation $\binom{n}{k} = C(n,k)$ for the "n choose k" numbers: $\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = (a+b)^n$.
The $\binom{n}{k}$ are also called binomial coefficients.

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With the binomial theorem we can give short proofs of some identities we've already seen, like:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Pf: We know $\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = (a+b)^n$ by Bin. Thm.
Let $a=1$ and $b=1 \Rightarrow \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = (1+1)^n$

$$\therefore \sum_{k=0}^{n} \binom{n}{k} = 2^n. \quad \square$$

What about the alternating sum of the $\binom{n}{k}$'s?
i.e. $C(3,0) - C(3,1) + C(3,2) - C(3,3)$
$= 1 - 3 + 3 - 1 = 0$
or $C(4,0) - C(4,1) + C(4,2) - C(4,3) + C(4,4)$
$= 1 - 4 + 6 - 4 + 1 = 0$

Thm For $n \geq 1$, $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

Pf: Let $b=-1$ and $a=1$ in the binomial theorem:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = (1-1)^n = 0^n = 0. \quad \square$$

WARNING: $C(0,0) = \frac{0!}{0!\,0!} = 1$, so
for $n=0$ we have $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 1$.
Means $0^0$ should be interpreted as $0^0 = 1$.

## Pascal's Triangle §6.7

The Binomial Theorem $(x+y)^n = \sum_{k=0}^{n} C(n,k)\, x^k y^{n-k}$ suggests that we should view the sequence $C(n,0), C(n,1), C(n,2), \ldots, C(n,n)$ in a "row". Actually, we can put all of these rows together into an infinite triangular array:

$$\begin{array}{ccccccc} & & & C(0,0) & & & \\ & & C(1,0) & & C(1,1) & & \\ & C(2,0) & & C(2,1) & & C(2,2) & \\ C(3,0) & & C(3,1) & & C(3,2) & & C(3,3) \\ & & & \vdots & & & \end{array}$$

Notice how we put each row a half step to the left of the row above it, so the "centers" are the same. The numbers for these $C(n,k)$ give:

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 1 & & 1 & & & & & \\ & & & & 1 & & 2 & & 1 & & & & \\ & & & 1 & & 3 & & 3 & & 1 & & & \\ & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\ & & & & & & \vdots & & & & & & \end{array}$$

This array of binomial coefficients is called Pascal's Triangle.

We can view many of the results about the $C(n,k)$ we have already seen using Pascal's triangle:

- $\sum_{k=0}^{n} C(n,k) = 2^n$ means that the sum of the $n^{th}$ row of Pascal's triangle
- $C(n,k) = C(n,n-k)$ means that Pascal's triangle is symmetric about its central vertical axis.

It is easy to fill out Pascal's triangle, because of the following recurrence for the $C(n,k)$:

**Theorem** (Pascal's identity)

$C(n+1,k) = C(n,k) + C(n,k-1)$ for all $1 \le k \le n$.

**Note**: This means each entry in Pascal's triangle is the sum of the two above it,

e.g. $\begin{matrix} 5 & & 10 \\ & 15 & \end{matrix}$

Together with $C(n,0) = C(n,n) = 1$ on outside this lets us repeatedly fill in all of triangle.

**Pf of Pascal's identity**: We will give a combinatorial proof. $C(n+1,k)$ is the number of size $k$ subsets of $\{1,2,\ldots,n+1\}$. Let us show that $C(n,k)+C(n,k-1)$ is also the number of such subsets.

Let $S$ be a size $k$ subset of $\{1,2,\ldots,n+1\}$. If $n+1 \notin S$, then $S$ is also a size $k$ subset of $\{1,2,\ldots,n\}$, counted by $C(n,k)$. If $n+1 \in S$, then $S \setminus \{n+1\}$ is a size $k-1$ subset of $\{1,2,\ldots,n\}$. So there is a bijective correspondence between size $k$ subs of $\{1,\ldots,n+1\}$ and size $k$ or $k-1$ subsets of $\{1,\ldots,n\}$, with the later being counted by $C(n,k) + C(n,k-1)$ by the addition principle. $\square$

**Remark**: Exercise is to prove Pascal's identity by taking Binomial Theorem $(x+y)^n = \sum_{k=0}^{n} C(n,k) x^k y^{n-k}$ and multiplying both sides by $(x+y)$...

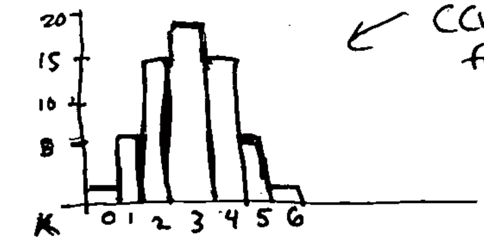
see HW problem
on odd vs. even
values!

## Pascal's triangle and Bell curve shape

There are many interesting patterns in Pascal's triangle.
One very important pattern: what does $n^{th}$ row of Pascal's triangle "roughly look like"?

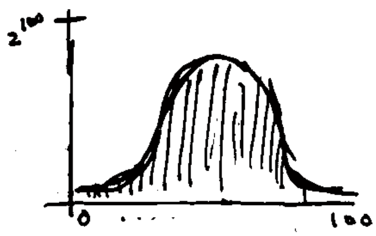
Consider plotting $n^{th}$ row as a histogram:

E.g. $n=6$




$\leftarrow$ $C(n,k)$ for $k=0$ to $n$

$n=100$




might be hard to see for small values of $n$, but for big values of $n$, a distinctive shape emerges: the "Bell Curve" shape

Note: $\dfrac{C(n,k)}{2^n}$ is the probability of getting exactly $k$ heads if you flip a coin $n$ times.

This limiting Bell curve shape describes not just the behavior of coin-flippings but also all kinds of natural processes in e.g. physics, biology, economics, etc.
Also called "normal distribution".

WARNING: Do not overinterpret the Bell Curve. It leads to some very bad (& even evil) pop/pseudo-science. For one classic error with the Bell curve, read/watch "Jurassic Park."

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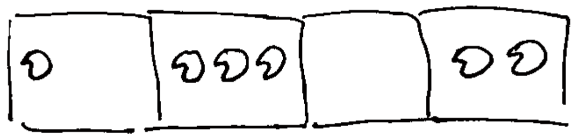
## The Pigeonhole Principle §6.8

Sometimes, rather than count the # of some kind of discrete object, we just want to show that at least one exists. The Pigeonhole Principle is useful for this . .

**Theorem** If you put $n$ pigeons into $k$ holes, and $k < n$, then at least one hole has at least 2 pigeons.

E.g.



← 6 pigeons into 4 holes, at least one hole has at least two pigeons.

The trick when using the pigeonhole principle is to figure out what should be the "pigeons" and what the "holes".

E.g. If there are at least 367 people in a room, then there must be at least two who have the same birthday.

Here the "holes" are the calendar dates and the "pigeons" are the people in the room.

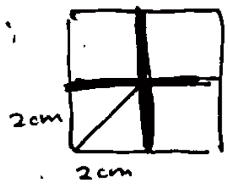
There are only 366 different holes (remember: leap day) so there must be a "collision" of birthdays.

NOTE: The Pigeonhole Principle doesn't tell us which hole has 2 pigeons, e.g. don't know which people have same birthday. It is "non-constructive".

Remark: With only 23 people, > 50% chance two people have same birthday. With 50 people, > 97% chance of two people sharing a birthday.

E.g. Show that if you put 5 dots on a 4cm x 4cm square, at least 2 dots are within 3cm of each other.

Idea:




← Break 4cm x 4cm square into four 2cm x 2cm squares.
Then by Pigeonhole Principle, at least 2 dots are in the same smaller square.

And the maximum distance of 2 dots in a smaller square = length of the diagonal
$= 2 \cdot \sqrt{2}$ cm $\approx 2 \times 1.4..$cm $< 3$cm ✓

E.g. Two numbers are coprime if they have no common factor bigger than 1.

E.g. 2 and 6 are not coprime since both divisible by 2
9 and 15 are not coprime since both divisible by 3
But 2 and 15 are coprime since no common factors.

Thm If $S$ is a subset of $\{1, 2, \dots, 20\}$ of size $\geq 11$ then there are two numbers $a$ and $b$ in $S$ such that $a$ and $b$ are coprime.

Note: Not true for size of $S = 10$ since
$\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$
has all #'s with two as a factor,
So no two in $S$ are coprime . . .

Pf: We first need the following Lemma:

Lemma The numbers $n$ and $n+1$ are always coprime, for any integer $n$.

Pf: Suppose $r>1$ is a factor (divisor) of $n$. Then $n+1 \equiv 1 \mod r$, meaning the remainder when dividing $n+1$ by $r = 1$. So $n+1$ not divisible by $r$, so $n$ and $n+1$ have no common factors. $\boxtimes$

Next, we use the pigeonhole principle:
Let the "holes" be pairs of consecutive #'s:
$\{1,2\}, \{3,4\}, \{5,6\}, \ldots, \{19,20\}$.
These are 10 holes. So if $S$ has size 11, it has at least 2 #'s in the same hole, and by the previous lemma those #'s are coprime. $\boxtimes$

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As you can see, even though the statement of the pigeonhole principle is very simple, figuring out how to apply it to a given problem can take a lot of creativity.