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### longest increasing subsequences

DEF'N Let  $\sigma = \sigma_1 \sigma_2 \dots \sigma_n \in S_n$  be a permutation.

A subsequence of  $\sigma$  is  $\sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_k}$  for  $i_1 < \dots < i_k$  and is increasing if  $\sigma_{i_1} < \sigma_{i_2} < \dots < \sigma_{i_k}$ .

Let  $\text{lis}(\sigma) := \text{length of longest increasing subsequence}$

e.g. For  $\sigma = \underline{2} \underline{4} 7 \underline{9} \underline{5} \underline{1} \underline{3} \underline{6} \underline{8}$  have  $\text{lis}(\sigma) = 5$   
(with longest ~~subsequence~~ increasing subsequence underlined).

Note: L.I.S. need not be unique:  $\frac{1}{\text{m}} \frac{2}{\text{m}} \frac{4}{\text{m}} \frac{3}{\text{m}}$

Increasing subsequences are a basic kind of permutation pattern (ask Prof. Burstein for more info...)

Studying LIS's is very natural from point of view of statistical analysis of time series data.

There is a close connection between the Robinson-Schensted algorithm and longest increasing subsequences:

Thm Suppose  $\sigma \xrightarrow{\text{RS}} (P, Q)$  w/  $\text{sh}(P) = \lambda = (\lambda_1, \lambda_2, \dots)$ .

Then  $\lambda_1 = \text{lis}(\sigma)$ .

e.g.  $\sigma = \underline{5} \underline{2} \underline{3} \underline{6} \underline{4} \underline{1} \underline{7} \xrightarrow{\text{RS}} (P = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 7 \\ \hline 2 & 6 & & \\ \hline 5 & & & \\ \hline \end{array}, Q = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 & 7 \\ \hline 2 & 5 & & \\ \hline 6 & & & \\ \hline \end{array})$

and indeed  $\lambda_1 = 4 = \text{lis}(\sigma)$ .

But note: 1<sup>st</sup> row of  $P$  ( $= 1347$ ) is not a LIS of  $\sigma$  (just has same length)

Pf of thm: Suppose  $\sigma = P_0, P_1, \dots, P_n = P$  is the sequence of insertion tableau we build up when inserting  $\tau_1, \tau_2, \dots, \tau_n$ .

Claim: When inserting  $\tau_k$  into  $P_{k-1}$ , if it enters in the  $j^{\text{th}}$  column, then the longest increasing subsequence ending at  $\tau_k$  has length  $j$ .

Pf: By induction. The case  $k=1$  is fine. So suppose  $x$  is entry in  $P_{k-1}$  in position  $(1, j-1)$  (i.e., left of  $\tau_k$ ). Then by induction there is a subsequence  $\tau'$  of  $\tau_1, \dots, \tau_{k-1}$  of length  $j-1$  ending at  $x$ , and since  $x < \tau_k$  (or else we would've bumped it), the concatenation  $\tau' \tau_k$  is a length  $j$  increasing subsequence. Similarly, to show there cannot be a longer subsequence, let  $y \in \{\tau_1, \dots, \tau_{k-1}\}$  be s.t.  $y < \tau_k$ . By induction, when we inserted  $y$  we did so at col. with longest subseq. ending at  $y$ , call it  $j'$ . Cannot have  $j' \leq j$ , otherwise we would've inserted  $\tau_k$  into a later column. So  $j' < j$ , and so longest inc. subseq. ending at  $\tau_k$  can have length at most  $j' + 1 \leq j$ .  $\checkmark$

What about the whole shape  $\lambda = (\lambda_1, \lambda_2, \dots)$ ?

Thm (Greene) Suppose  $\sigma \xrightarrow{\text{rs}} (P, Q)$  w/  $\text{sh}(P) = \lambda$ . Then for all  $K$ ,  $\lambda_1 + \lambda_2 + \dots + \lambda_K = \text{length of}$  longest subsequence of  $\sigma$  that is a union of  $K$  increasing subsequences.

E.g. w/  $\tau = \underline{2} \underline{4} \underline{7} \underline{9} \underline{5} \underline{1} \underline{3} \underline{6} \underline{8}$  have  $P = \boxed{\begin{matrix} 2 & 4 & 9 \\ 7 & \end{matrix}}$  and  
 $2 \ 4 \ 7 \ 9 \sqcup 1 \ 3 \ 6 \ 8$  is a union of 2 increasing subsequences.

4/15 Can define decreasing subsequences of perm.  $\pi$  analogously, and let  $l\text{ds}(\pi)$  := length of longest decr. subseq.

Thm If  $f \xrightarrow{\text{rs}} (P, Q)$  w/  $\text{sh}(P) = \lambda$ , then  $\text{lcs}(G) = l(\lambda)$   
 In fact this follows immediately from ...

Then\* for  $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$  let  $\sigma^{\text{rev}} = \sigma_n \sigma_{n-1} \dots \sigma_1$ . Then if  $\sigma \mapsto (P, Q)$  have  $\sigma^{\text{rev}} \mapsto (P', Q')$  where  $P' = P^t \leftarrow \underline{\text{transpose}}$ .

To prove this symmetry property of RS, can use column insertion, which works same as (row) insertion, but where we try to put # into 1<sup>st</sup> column, and bump #'s from  $i^{\text{th}}$  column to  $(i+1)^{\text{st}}$  column, etc.

Key Lemma: Row and column insertions commute, i.e.,

$$T \xleftarrow[\text{row } a]{} \xleftarrow[\text{col } b]{} = T \xleftarrow[\text{col } b]{} \xleftarrow[\text{row } a]{}.$$

PS: See Sagan. B

Pf of thm<sup>x</sup>:  $p' = \sigma_1 \xrightarrow{\text{row}} \dots \xrightarrow{\text{row}} \sigma_n \xrightarrow{\text{row}} \emptyset$  (1st insertion  
 is same w/  
 row or col)

$$\begin{aligned}
 &= \sigma_1 \xrightarrow{\text{row}} \dots \xrightarrow{\text{row}} \sigma_n \xrightarrow{\text{col}} \emptyset \\
 &= \sigma_n \xrightarrow{\text{col}} \sigma_1 \xrightarrow{\text{row}} \dots \xrightarrow{\text{row}} \emptyset \quad (\text{key lemma}) \\
 &= \sigma_n \xrightarrow{\text{col}} \sigma_{n-1} \xrightarrow{\text{col}} \dots \xrightarrow{\text{col}} \sigma_1 \xrightarrow{\text{col}} \emptyset \quad (\text{repeat}) \\
 &= (\sigma_n \xrightarrow{\text{row}} \sigma_{n-1} \xrightarrow{\text{row}} \dots \xrightarrow{\text{row}} \sigma_1 \xrightarrow{\text{row}} \emptyset)^t \quad (\text{transpose of col insert} = \text{row insert}) \\
 &= p^t \quad \checkmark
 \end{aligned}$$

Cor (Erdős-Szekeres Theorem)

for any  $\tau \in S_{(n-1)(m-1)+1}$ , have either  
 $\text{lis}(\tau) \geq n$  or  $\text{lds}(\tau) \geq m$ .

Pf: Best way to minimize width and length of a partition  
is  $\lambda = \frac{1}{m} \begin{matrix} \boxed{\text{#}} \\ \vdots \\ \boxed{\text{#}} \end{matrix}$  but we need one more box ✓

Q: What is the expected length of longest incr. subseq.  
of a random permutation?

Let  $X_n := \text{lis}(\tau)$  for  $\tau \in S_n$  (uniformly) random.

Ulam's Problem: Compute  $\lim_{n \rightarrow \infty} \frac{\mathbb{E} X_n}{\sqrt{n}} = c$ .  
c. 1960's

E-S Thm says for any  $\tau \in S_n$ , have  $\text{lis}(\tau) \geq \sqrt{n}$  or  
 $\text{lis}(\tau^{\text{rev}}) \geq \sqrt{n}$   
so that  $c \geq \frac{1}{2}$ . In fact...

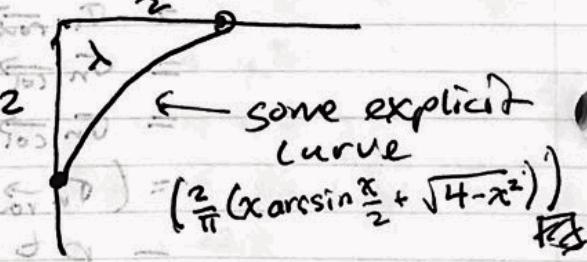
Thm (Logan-Shepp, Kerov-Vershik, 1977)  
Solution to Ulam's Problem is  $c = \frac{2}{\pi}$

Idea of pf: Same as asking for length of  $\lambda$  when we  
insert  $\tau \in S_n$  into RS. In fact, this random  
partition  $\lambda$  has

a precise

limit shape

(rescaling by  $\frac{1}{\sqrt{n}}$ ):



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## Representation Theory of finite Groups:

In the last couple days, I want to explain why ring of Sym. fn.'s is important in algebra.

DEF'N Let  $V$  be an  $n$ -dim'l vector space over  $\mathbb{C}$ .

The general linear group  $GL(V) = \{\text{invertible linear maps } V \rightarrow V\}$ ,  
I.e.,  $GL(V) \cong \{n \times n \mathbb{C}\text{-matrices } M \text{ w/ } \det(M) \neq 0\}$ .

Note:  $GL(V)$  is an infinite group.

Let  $G$  be a finite group. We want to "represent"  $G$  by matrices.

DEF'N A representation of  $G$  is a group homomorphism  
 $\varphi: G \rightarrow GL(V)$  for some v.s.  $V$ . In other words,  
for each  $g \in G$  we have a matrix  $\varphi(g)$ , and:  
•  $\varphi(gh) = \varphi(g) \cdot \varphi(h) \quad \forall g, h \in G$ ,  
•  $\varphi(e) = I_n$  identity matrix.

A representation of  $G$  is very similar to an action,  
except it is linear: we act by matrices, not permutations.

e.g. For any  $V$  and any  $G$ , can set  $\varphi(g)(v) = v \quad \forall v \in V$ , i.e.,  
 $\varphi(g) = I_n$  identity matrix. This is called the trivial representation and is boring.

e.g. Suppose  $G \curvearrowright X$  a finite set. Let  $\mathbb{C}[X] := \left\{ \sum_{x \in X} c_x x : c_x \in \mathbb{C} \right\}$   
be v.s. of formal linear combinations of elements of  $X$ .  
Then  $\mathbb{C}[X]$  is a  $G$  representation where  $\varphi(g)(x) = g \cdot x$   
for all basis vectors  $x \in \mathbb{C}[X]$ . In other words, each  
 $\varphi(g)$  is the permutation matrix of its corresponding permutation.  
This is called a permutation representation.

e.g. Let  $G = \mathbb{Z}/n\mathbb{Z} = \{0, 1, \dots, n-1\}$ . Let  $V = \mathbb{C}$ .  
 We can define a representation  $\varphi: G \rightarrow GL(V)$  by  
 $\varphi(k) = (e^{2\pi i \cdot k/n})$  as matrix  $\forall k = 0, 1, \dots, n-1$ .

e.g. Let  $G = S_n$  symmetric gp. and let  $V = \mathbb{C}$   
 The sign representation  $\varphi: S_n \rightarrow GL(\mathbb{C})$  is  $\varphi(\sigma) = \begin{pmatrix} \text{sgn}(\sigma) & \\ & \text{id}_{(V)} \end{pmatrix}$

e.g. If  $U, V$  are  $G$ -representations, then, direct sum  $U \oplus V$   
 is another representation as matrices  $\begin{pmatrix} \varphi(g)_{U \oplus V} & 0 \\ 0 & \varphi(g)_V \end{pmatrix}$  "block sum".

DEF'N A repr'n  $\varphi: G \rightarrow GL(V)$  is irreducible if we  
 cannot find a nontivial subspace  $U$  (i.e.,  $0 \neq U \neq V$ )  
 s.t.  $g \in U \Rightarrow g \in U, g \in G$  (i.e., invariant under all  $G$ ).

FACT Every representation  $V$  of  $G$  is a direct sum  
 $V = V_1 \oplus V_2 \oplus \dots \oplus V_k$  of irreducible reprns  $V_i$ .

e.g. Let  $V = \mathbb{C}^n$  w/ standard basis  $\{e_1, e_2, \dots, e_n\}$  and  $G = S_n$ ,  
 Let  $\varphi: S_n \rightarrow GL(V)$  be the standard permutation repr'n,  
 i.e.  $\varphi(\sigma) e_i = e_{\sigma(i)}$ .  $\forall \sigma \in S_n, i=1, \dots, n$ :  $V$  is reducible,

since  $U_i = \{c e_1, c_2, \dots, c_n \in V : c \in \mathbb{C}\}$  is a nontrivial invariant subspace.

With  $U_0 = \{(x_1, \dots, x_n) \in V : x_1 + \dots + x_n = 0\}$ , we have

$V = U_1 \oplus U_0$  and  $U_1, U_0$  are irreducible reprns,

trivial reprn

The FACT alone says that to understand all  $G$ -reprns,  
 it's enough to understand the irreducible ones...

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## Characters of representations

Representations  $\varphi: G \rightarrow GL(V)$  are matrix-valued functions, hence complicated to understand. It turns out we can "reduce" to studying "ordinary"  $\mathbb{C}$ -valued fn's  $\chi: G \rightarrow \mathbb{C}$ .

DEF'N Let  $\varphi$  be a representation of finite group  $G$ .

Its character  $\chi_\varphi: G \rightarrow \mathbb{C}$  is the function

$$\chi_\varphi(g) = \text{Tr}(\varphi(g)) \leftarrow \begin{array}{l} \text{trace of} \\ \text{matrix} \end{array} \text{ for all } g \in G.$$

e.g. If  $V$  is 1-dim'l, then  $\varphi$  and  $\chi_\varphi$  are the same thing...

R.-g. If  $\varphi$  is the permutation repr'n of an action  $G \curvearrowright X$

$$\text{then } \chi_\varphi(g) = \#\text{Fix}(g: X \rightarrow X) \leftarrow \begin{array}{l} \text{why? think} \\ \text{abt. perm. matrix..} \end{array}$$

FACT For two  $G$ -repr's  $\varphi_1: G \rightarrow GL(V_1)$ ,  $\varphi_2: G \rightarrow GL(V_2)$

$$\text{have } \chi_{\varphi_1} = \chi_{\varphi_2} \Leftrightarrow \varphi_1 \text{ isomorphic to } \varphi_2$$

( $\varphi_1 \simeq \varphi_2$  means  $\exists$  v.s. iso.  $V_1 \cong V_2$  that commutes w/  $G$ -action)

Upshot: enough to study characters, in fact, since we

$$\text{have } \chi_{\varphi_1 \oplus \varphi_2} = \chi_{\varphi_1} + \chi_{\varphi_2}, \text{ enough to}$$

study characters of irreducible reprns (+ their lin. comb's).

In fact, Characters  $\chi$  are not just any kind of function  $G \rightarrow \mathbb{C}$ ...

DEF'N A conjugacy class of  $G$  is set of the form

$$C = \{ghg^{-1}: g \in G\} \text{ for some } h \in G. \text{ A function}$$

$f: G \rightarrow \mathbb{C}$  is called a class function if it is constant on conjugacy classes, i.e.  $f(h) = f(ghg^{-1}) \forall g, h \in G$ .

Let  $Cl(G) := \text{v.s. of class functions } f: G \rightarrow \mathbb{C}$ .

Prop. Any character  $\chi_\varphi$  is a class function.

PS:  $\chi_\varphi(ghg^{-1}) = \text{Tr}(ghg^{-1}) = \text{Tr}(g^{-1} \cdot gh) = \text{Tr}(h)$

recall  $\text{Tr}(AB) = \text{Tr}(BA)$  for matrices  $A, B$   $\square$

FACT 1.  $\{\chi_{\varphi_1}, \dots, \chi_{\varphi_k}\}$  is a basis of  $\text{Cl}(G)$ , where  $\varphi_1, \dots, \varphi_k$  are the irrep's of  $G$  (up to iso.).

2. With the inner product  $\langle , \rangle : \text{Cl}(G) \times \text{Cl}(G) \rightarrow \mathbb{C}$

given by  $\langle f, f' \rangle := \frac{1}{\#G} \sum_{g \in G} f(g) \overline{f'(g)}$ ,

the basis  $\{\chi_{\varphi_1}, \dots, \chi_{\varphi_k}\}$  is orthonormal.

3. If  $\varphi = \bigoplus_m \varphi_m$  is decomposition of  $\varphi$  into irrep's,

then  $c_m = \langle \chi_\varphi, \chi_{\varphi_m} \rangle$ .

Note in particular that

$$\begin{aligned} \#\text{irreps (irreducible repr's)} &= \dim \text{Cl}(G) \\ &= \#\text{conjugacy classes of } G. \end{aligned}$$

e.g.  $G$  acts on itself by multiplication on the left, and corresponding perm. rep. is called the regular reprn.  $\mathbb{C}[G]$

How does  $\mathbb{C}[G]$  decompose into irrep's?

$$\begin{aligned} \langle \chi_{\mathbb{C}[G]}, \chi_{\varphi_m} \rangle &= \frac{1}{\#G} \sum_{g \in G} \chi_{\mathbb{C}[G]}(g) \overline{\chi_{\varphi_m}(g)} \\ &\stackrel{\# \text{Fix}(g: G \rightarrow G)}{=} \begin{cases} \#G & \text{if } g = e \\ 0 & \text{otherwise} \end{cases} \\ &= \frac{1}{\#G} \cdot \#G \cdot \overline{\chi_{\varphi_m}(e)} = \dim(\varphi_m). \end{aligned}$$

Hence

$$\#G = \dim \mathbb{C}[G] = \dim \left( \bigoplus_m \dim(\varphi_m) \cdot \varphi_m \right) = \sum_m (\dim(\varphi_m))^2$$

looks familiar...

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## Characters of the Symmetric Group.

Finally, by focusing on case  $G = S_n$ , we see symmetric functions.

Prop. Two permutations  $\sigma, \sigma' \in S_n$  belong to same conjugacy class  
 $\Leftrightarrow$  they have the same cycle structure.

Pf: Exercise for you. Pf

So # conj. classes in  $S_n = \#$  cycle structures = # partitions  $\lambda + n$

So #irrep's of  $S_n = \# \lambda + n$ , and in fact there is a standard way to index irrep's by partitions.

e.g. Let  $\text{triv}: S_n \rightarrow GL(\mathbb{C})$  be the trivial rep'n. Then

$$\text{triv} = \bigoplus_{\lambda \vdash n} \mathbb{C}_{(\lambda)} = \mathbb{C}_{(n)}$$

e.g. for  $\text{sgn}: S_n \rightarrow GL(\mathbb{C})$  sign rep'n,  $\text{sgn} = \bigoplus_{\lambda \vdash n} \mathbb{C}_{(\lambda^n)}$ .

e.g. Recall standard perm rep'n  $\mathbb{C}^n = U_1 \oplus U_0$

then  $U_0 = \bigoplus_{\lambda \vdash n-1} \mathbb{C}_{(\lambda^n, 1)}$ .

Write  $\chi_\lambda = \chi_{\rho_\lambda} =$  character of irrep indexed by  $\lambda + n$ .

DEFN The Frobenius characteristic  $\text{Fr}: Cl(S_n) \rightarrow \text{Sym}(n)$

is given by  $\text{Fr}(\delta_\lambda) = p_\lambda \leftarrow \text{power sum}$

where  $\delta_\lambda$  is class function  $\delta_\lambda(\sigma) = \begin{cases} Z_\lambda & \text{if cycle type}(\sigma) = \lambda \\ 0 & \text{otherwise} \end{cases}$

and  $Z_\lambda = \frac{n!}{1^{m_1} m_1! 2^{m_2} m_2! \dots} \stackrel{\text{recall =}}{=} \# \text{perm's in } S_n \text{ w/ cycle type } \lambda = \lambda = (1^{m_1} 2^{m_2} \dots)$ .

Since the  $\delta_\lambda$  are a basis of  $Cl(S_n)$  and  $p_\lambda$  are a basis of  $\text{Sym}(n)$ , this is clearly a v.i.s. isomorphism.

Thm  $\text{Fr}(X_\lambda) = S_\lambda \leftarrow \text{schur function}$ .

This is (one reason) why Schur fn's are so important!

Cor  $\dim \varphi_\lambda = f^\lambda = \# \text{SYT of sh. } \lambda$

Pf: Via Fr, same as coeff. of  $[x_1 x_2 \cdots x_n]$  in  $S_\lambda = f^\lambda \checkmark$

More generally

Cor If  $X_\lambda(\mu) = \text{ch. evaluated at a perm. of cycle type } \mu$ ,

$$\text{then } S_\lambda = \sum_{\mu} X_\lambda(\mu) \cdot z_\mu^{-1} P_\mu.$$

exists combinatorial rule for these coeff's, called  
the Murnaghan-Nakayama rule.

Also note that... by the regular representation, have

$$n! = \# S_n = \sum_{\lambda \vdash n} \dim(\varphi_\lambda)^2 = \sum_{\lambda \vdash n} (f^\lambda)^2,$$

which we saw earlier using R.S. algorithm.

Finally, using something called the induction product  
of representations of  $S_k \times S_{n-k} \rightarrow S_n$ ,

we can get ring structure on  $\text{Sym} = \bigoplus \text{Sym}(n)$ ,

Structure constants  $S_\lambda, S_\mu = \sum_v c_{\lambda\mu}^v S_v$  are  
called Littlewood-Richardson coefficients,  
also very important!