# Parametric Equations § 10.1

The 1st half of the semester for Calc II focused on integration.
In 2nd half we explore other topics, starting with Chapter 10
__ on parametric equations & polar coordinates.

Up until now we have considered curves of the
form $y = f(x)$ (or more rarely, $f(x, y) = 0$).
A parameterized curve is defined by two equations:
$$x = f(t) \quad \text{and} \quad y = g(t)$$
where t is an auxilliary variable. Often we think of t
as time, so the curve describes motion of a particle
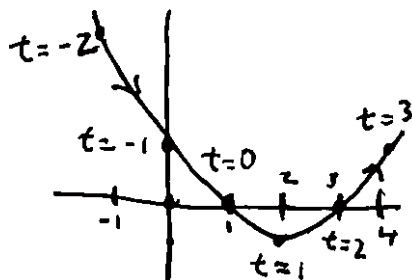where at time t particle is at position $(f(t), g(t))$:

$(f(5), g(5))$
$(f(2), g(2))$
$(f(3), g(3))$
$(f(1), g(1))$
$(f(0), g(0))$
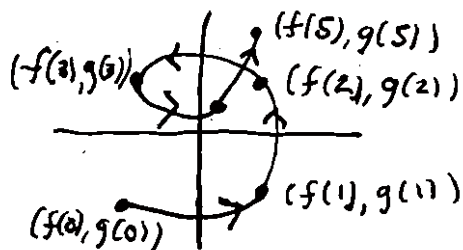
← In this picture the
arrows → show
movement of particle over time

E.g. Consider parametrized curve $\boxed{x = t+1, \quad y = t^2 - 2t}$
We can make a chart with various values of t:

t	x	y
-2	-1	8
-1	0	3
0	1	0
1	2	-1
2	3	0
3	4	3

$\Rightarrow$

← plot of points
$(f(t), g(t))$ for
$t = -1, 0, 1, \ldots, 4$
looks like a parabola

In this case, we can eliminate the variable t:
$$x = t+1 \Rightarrow t = x - 1$$
$$y = t^2 - 2t \Rightarrow y = (x-1)^2 - 2(x-1) = x^2 - 4x + 3$$
So this parametrized curve is just $\boxed{y = x^2 - 4x + 3}$

E.g: Consider the parametric curve:
$$X = \cos(t), \quad y = \sin(t) \quad \text{for} \quad 0 \le t \le 2\pi$$
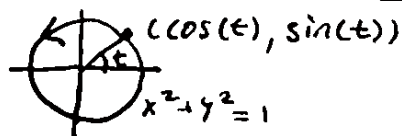How can we visualize this curve?
Notice that $x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$,
So this parametrizes a <u>circle</u> $x^2 + y^2 = 1$.

← here t = angle (in radians) of point $(\cos(t), \sin(t))$ on unit circle

E.g: What about $x = \cos(2t)$, $y = \sin(2t)$, $0 \le t \le 2\pi$?
Notice we still have $x^2 + y^2 = \cos^2(2t) + \sin^2(2t) = 1$,
so the parametrized curve still traces a <u>circle</u>:

← But now the parametrized curve traces the circle <u>twice</u>:
once for $0 \le t \le \pi$
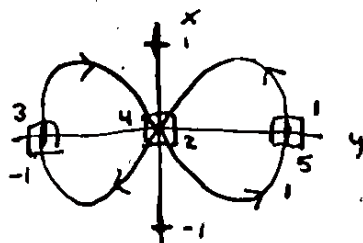and once for $\pi \le t \le 2\pi$

Can think of this particle as moving "faster" than the last one.
We see same curve can be <u>parametrized</u> in <u>different ways</u>!

E.g: Consider the curve $X = \cos(t)$, $y = \sin(2t)$.
It's possible to eliminate t to get $y^2 = 4x^2 - 4x^4$,
but that equation is hard to visualize.
Instead, graph $x = f(t)$ and $y = g(t)$ <u>separately</u>:
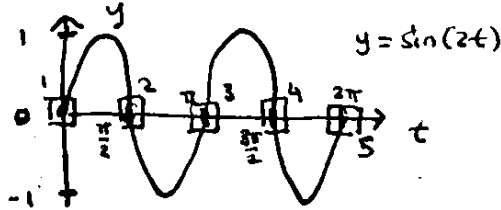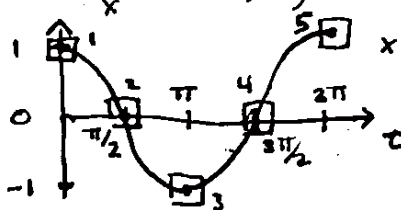
Then <u>combine</u> into one <u>picture</u> showing $(f(t), g(t))$: $\Rightarrow$

← are "snapshots" of the particle as it traces the curve

Calculus with parametrized curves §10.2

Much of what we have done with curves of form $y = f(x)$ in calculus can also be done for parametrized curves:

__Tangent vectors__: Let $(x,y) = (f(t), g(t))$ be a curve. Then, at time t, the slope of tangent vector is given by:

$$\frac{dy}{dx} \underset{\text{chain rule}}{=} \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)} \quad (\text{if } f'(t) \neq 0)$$

If $dy/dt = 0$ (and $dx/dt \neq 0$) $\Rightarrow$ horizontal tangent
If $dx/dt = 0$ (and $dy/dt \neq 0$) $\Rightarrow$ vertical tangent

__E.g.__ Consider curve $x = t^2$, $y = t^3 - 3t$.
First, notice that when $t = \pm\sqrt{3}$ we have

$$x = t^2 = 3 \quad \text{and} \quad y = t^3 - 3t = t(t^2 - 3) = 0,$$

so curve passes thru $(3,0)$ at __two__ times $t = \sqrt{3}$ and $t = -\sqrt{3}$.
We then compute that:

$$dy/dx = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t}$$

$\longrightarrow = -6/2\sqrt{3} = -\sqrt{3}$ at $t = -\sqrt{3}$
$\longrightarrow = 6/2\sqrt{3} = \sqrt{3}$ at $t = \sqrt{3}$

So two tangent lines, of slopes $\pm\sqrt{3}$, for curve at $(3,0)$.
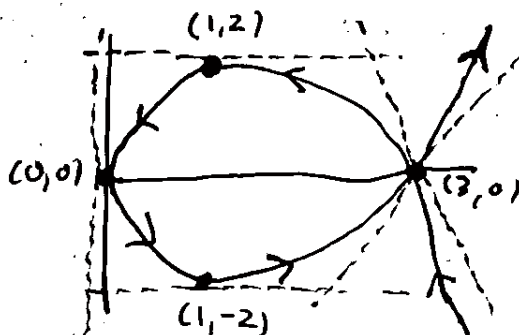
When is the tangent horizontal? When $dy/dt = 3t^2 - 3 = 0$ which is for $t = \pm 1$, at points $(1,2)$ and $(1,-2)$.
When is the tangent vertical? When $dx/dt = 2t = 0$, which is for $t = 0$, at point $(0,0)$.

Putting all of this information together, we can produce a pretty good sketch of the curve $\Rightarrow$

<u>Arc lengths</u>: We saw several times how to find lengths of curves by breaking into line segments:
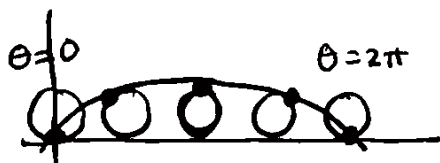

← recall length of each small segment
$$= \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

For a parametrized curve $(x, y) = (f(t), g(t))$ with $\alpha \leq t \leq \beta$ we get length that of curve $= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \boxed{\int_\alpha^\beta \sqrt{f'(t)^2 + g'(t)^2} \, dt.}$

<u>Exercise</u>: Using parametrization $x = \cos(t), y = \sin(t), 0 \leq t \leq 2\pi$, show circumfrence of unit circle $= 2\pi$ using this formula.

<u>E.g.</u> The <u>cycloid</u> is the path a point on unit circle traces as the circle rolls:

← think of this as an animation of a rolling circle, with point • marked where angle θ = "time"

The cycloid is parametrized by:
$$x = \theta - \sin\theta, \quad y = 1 - \cos\theta \quad \text{for } 0 \leq \theta \leq 2\pi$$

Q: What is the arclength of the cycloid?

A: We compute $\frac{dx}{d\theta} = 1 - \cos\theta, \quad \frac{dy}{dx} = \sin\theta$ so that

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2} = \sqrt{2(1-\cos\theta)}$$

$\left(\text{using trig identity } \frac{1}{2}(1 - \cos 2x) = \sin^2 x\right)$ → $= \sqrt{4\sin^2(\theta/2)}$
$= 2\sin(\theta/2)$

$\Rightarrow$ length of cycloid $= \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta$

$= \int_0^{2\pi} 2\sin\left(\frac{\theta}{2}\right) d\theta = \left[-4\cos\left(\frac{\theta}{2}\right)\right]_0^{2\pi}$

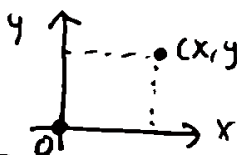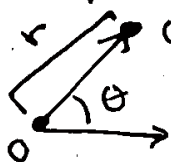$\rightarrow ((-4 \cdot -1) - (-4 \cdot 1)) = \underline{8}.$

11

Polar Coordinates §10.3

We are used to working with the "Cartesian" coordinate system where a point on the plane is represented by (x, y) telling us how far to move along two orthogonal axes to reach that point.
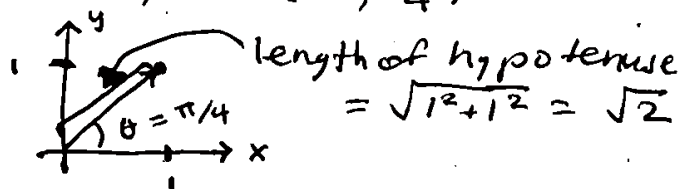
The polar coordinate system is a different way to represent points on the plane by a pair (r, θ):

Here we have a fixed axis ray $\longrightarrow$ emanating from origin 0, and we reach a point (r, θ) by making an angle of θ radians and point out a distance of r.

E.g. The point $(x, y) = (1, 1)$ in Cartesian coord's is the same as $(r, \theta) = (\sqrt{2}, \frac{\pi}{4})$ in polar coord's.
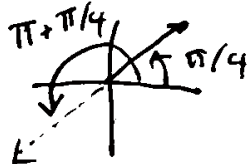
length of hypotenuse $= \sqrt{1^2 + 1^2} = \sqrt{2}$

$\theta = \pi/4$

Notice: There are multiple ways to represent any point in polar coord's because we can add 2π to θ:

$(r, \theta) = (\sqrt{2}, \pi/4)$ same as $(r, \theta) = (\sqrt{2}, 2\pi + \pi/4)$

$\pi/4$
$2\pi + \pi/4$
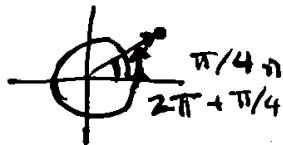
Also··· can add π to θ and replace r by $-r$:

$(r, \theta) = (\sqrt{2}, \pi/4)$ same as $(r, \theta) = (-\sqrt{2}, \pi + \pi/4)$

$\pi + \pi/4$
$\pi/4$

Negative value of r means go backwards that distance along ray.

Question: How to convert between Cartesian & polar coords?
Let's draw a right triangle to help us:
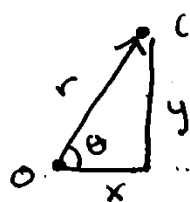
← From this picture we see that

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta$$

which gives (x,y) in terms of (r, θ)
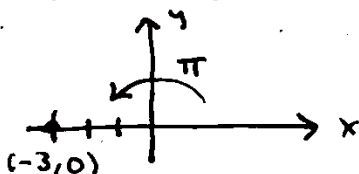
We also have that:

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan \theta = \frac{y}{x}$$

which gives us (r, θ) in terms of (x,y):
specifically, $r = \pm\sqrt{x^2 + y^2}$ and $\theta = \arctan\left(\frac{y}{x}\right)$.

E.g. Find the polar coordinates of $(x,y) = (-3, 0)$.
To solve this problem, it's easiest to just draw the point.

we see this point is at
angle $\theta = \pi$ and
radius $r = 3$.
Check: $3^2 = r^2 = x^2 + y^2 = (-3)^2 + (0)^2$
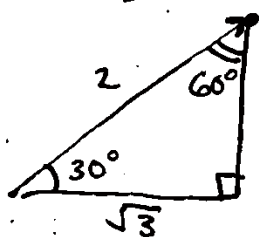and $\theta = \tan(\theta) = \frac{y}{x} = \frac{0}{-3}$.

Could have also chosen $(r, \theta) = (-3, 0)$ here ...

E.g. Find the Cartesian coordinates of $(r, \theta) = (2, \frac{\pi}{6})$.

Here we have $x = r \cos \theta = 2 \cos(\pi/6) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$
and $y = r \sin \theta = 2 \sin(\pi/6) = 2 \cdot \frac{1}{2} = 1$

Can also draw the right triangle to check:

← recall that $\theta = \pi/6$ radians
$= 30°$
corresponds to a special
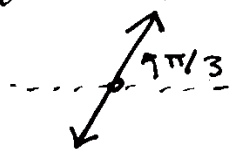$30 - 60 - 90$ triangle

Polar equations and curves:

Just like we draw curves $f(x,y) = 0$ in Cartesian coord's, we can draw curves $f(r, \theta) = 0$ in Polar coord's.

E.g. The equation $r = 2$ gives <u>circle</u> of radius 2, centered at origin:

← circle = all points at radial distance 2 from origin O

E.g. The equation $\theta = \pi/3$ gives <u>line</u> at angle $\pi/3$ thru origin:

$\pi/3$ ← line thru origin = all points at given angle

E.g. What about equation $r = 2 \cos\theta$?
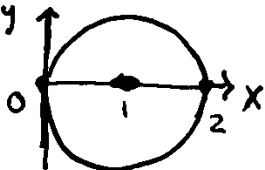
Here it's easiest to switch to Cartesian coord's:

multiplying by r gives

$$r^2 = 2r\cos\theta$$
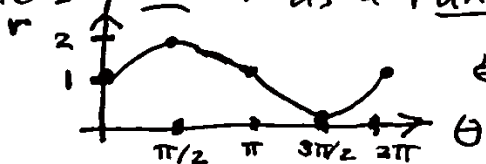$$\Leftrightarrow x^2 + y^2 = 2x$$
$$\Leftrightarrow (x-1)^2 + y^2 = 1$$

which is a <u>circle</u> of radius 1 centered at $(x,y) = (1,0)$:
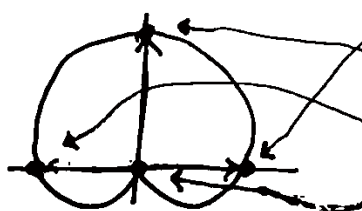
← $(x-1)^2 + y^2 = 1$
   a.k.a. $r = 2\cos\theta$

E.g. What about $r = 1 + \sin(\theta)$?

First let's <u>plot</u> r as a <u>function</u> of θ (in Cartesian coord's):

← shows us how radius of figure changes with <u>angle</u>

"cardiod" ⇒ this "heart-shaped" curve is polar curve $r = 1 + \sin(\theta)$

① at angle $\theta = 0$, $r = 1$
② at angle $\theta = \pi/2$, $r = 2$ so we move out to this point
③ at $\theta = \pi$, back to $r = 1$
④ at $\theta = \frac{3\pi}{2}$, radius shrinks to $r = 0$