9/7 <u>The special number $e$</u>

There is one special base that is "the best":
the number $e \approx 2.71\ldots$ ← irrational number, like $\pi$

How to define $e$ precisely? Can use a limit:

$$e = \lim_{n\to\infty} (1 + 1/n)^n$$

Can explain this formula using compound interest.

Suppose you have an investment that returns 100% per year (that's an incredible investment!).

If you invest \$100, how much will you have after 1 year?

If the interest is only calculated at the end of the year you get $\$100 \cdot (1 + \underset{100\%\ \text{return}}{1}) = \$200$.

But imagine instead the interest is given every 6 months.

Then after 6 months you get $\$100(1 + \underset{\frac{1}{2}100\% = 50\%\ \text{return in } 1/2\ \text{year}}{0.5}) = \$150$,

and after the next 6 months you get $\$150(1+0.5) = \$225$.

We see that compounding more often gives more money in the end, even with the "same rate".

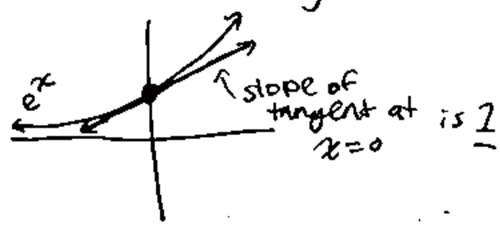
If we earn interest $n$ times in the year, we get

$$\$100 \cdot \underbrace{(1+\tfrac{1}{n})(1+\tfrac{1}{n})\cdots(1+\tfrac{1}{n})}_{n \text{ times}}$$
$$= \$100 \cdot (1+\tfrac{1}{n})^n$$ in the end, and

if we "continuously compound the interest" we end with $\$100 \cdot \lim_{n\to\infty}(1+\frac{1}{n})^n = \$100 \cdot e \approx \$271.$

This explains the "$\overset{\text{principal}}{P}\, e^{\overset{\text{rate}}{r}\overset{\text{time}}{t}}$" formula for compound interest you may have seen before.

There is another geometric way to think about the significance of base $e$:




Of all the $a^x$, the one that has a tangent line of slope $1$ at $x=0$ is $a=e$.

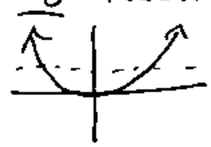
When we start to talk about derivatives and tangents, we will see why this is such a desirable property.

We mentioned that we define the logarithm as the inverse of the exponential function.

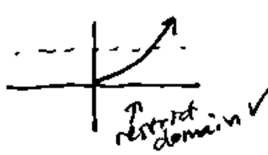
**Def'n** A function $g(x)$ has an inverse function $f=g^{-1}$ if and only if it is one-to-one. In this case, the inverse function $f=g^{-1}$ is defined by $f(y)=x$ if $x$ is the unique element in the domain of $g$ such that $g(x)=y$. ($f$ "undoes" $g$ so that $(f\circ g)(x) = x$).

**E.g.** Since $g(x)=x^3$ is one-to-one, it admits an inverse $f=g^{-1}$ which is $f=\sqrt[3]{x}$.

**E.g.** Recall $g(x)=x^2$ is not one-to-one: it fails the horizontal line test! So it does not have an inverse on all of $\mathbb{R}$.

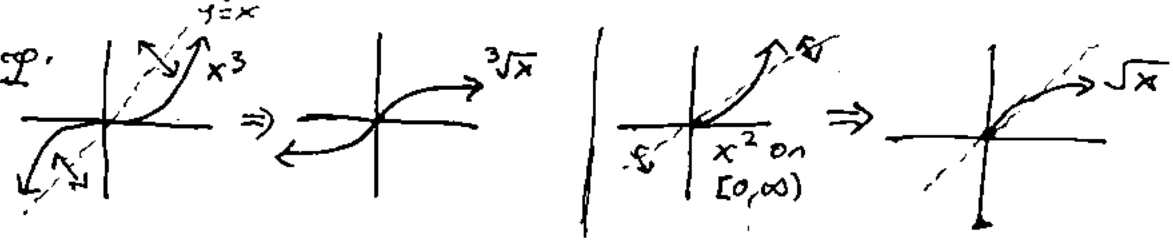


But if we restrict the domain to $[0,\infty)$, then $f(x)=\sqrt{x}$ is its inverse, like we'd expect.



There is a <u>geometric</u> way to think about inverses: graph of $f = g^{-1}$ is <u>reflection</u> of graph of $g$ <u>over line</u> $y=x$.

<u>E.g.</u>



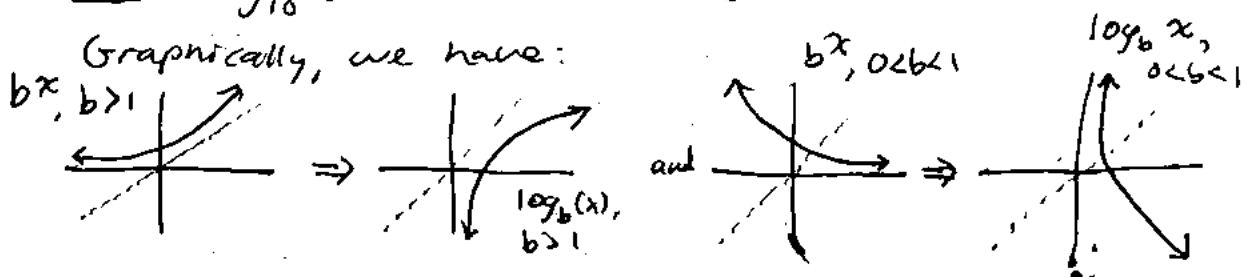

This geometric interpretation also makes clear that <u>domain</u> of $f$ = <u>range</u> of $g$ and <u>range</u> of $f$ = <u>domain</u> of $g$ for inverse functions $f = g^{-1}$!

Looking at the graph of $b^x$ for any $b>0$, $b \neq 1$, we see it passes the horizontal line test, so it has an inverse: the base $b$ <u>logarithm</u>.

<u>Def'n</u> $\log_b$, the base $b$ <u>logarithm</u>, is the inverse of $b^x$ meaning $\boxed{\log_b(y) = x \text{ if and only if } b^x = y}$

<u>E.g.</u> $\log_{10}(100) = 2$ since $10^2 = 100$.

Graphically, we have:




Note that since range $(b^x)$ is $(0,\infty)$ (positive numbers) <u>domain</u> $(\log_b(10))$ is $(0,\infty)$:

We can <u>only</u> take <u>Logarithms</u> of <u>positive</u> numbers!

9/9 [Aside: to find inverse of $g(x)$, write $y=g(x)$ and "solve for $x$".
e.g. $g(x)=x^3-1 \rightsquigarrow y=x^3-1$
$y+1=x^3$
$\sqrt[3]{y+1}=x$ so inverse $f=g^{-1}$ is $f(y)=\sqrt[3]{y+1}$. ✓ ]

## The natural logarithm and properties of exponentials and logarithms

We mentioned that of all exponential functions, the one $e^x$ for special number $e \approx 2.71\ldots$ is most preferred.
Consequently, we define the natural logarithm
$\ln(x) := \log_e(x)$ as the "best logarithm".

It might seem that $e^x$ and $\ln(x)$ are not enough to recover all the exponentials and logarithms, but actually they are: because of basic properties of exponentials and logarithms.

Recall from high school algebra these facts about exponentials:

Prop: 1. $b^{x+y}=b^x b^y$ 2. $b^{x-y}=\frac{b^x}{b^y}$
3. $(b^x)^y=b^{xy}$ 4. $(ab)^x=a^x b^x$

These let us prove that for logarithms:

Prop. 1. $\log_b(xy)=\log_b x+\log_b y$
2. $\log_b(\frac{x}{y})=\log_b(x)-\log_b(y)$ 3. $\log_b(x^r)=r\log_b x$.

Why are these useful? They reduce everything to $e^x$ and $\ln(x)$:

Thm 1. $b^x=e^{x\ln(b)}$
2. $\log_b x=\frac{\ln(x)}{\ln(b)}$.

Pf: For 1., use $e^{x\ln(b)}=(e^{\ln(b)})^x$
$=b^x$. ✓

For 2., let $y = \log_b(x)$, so $b^y = x$.
Take ln of both sides $\ln(b^y) = \ln(x)$
$\Leftrightarrow y \cdot \ln(b) = \ln(x)$
$\Leftrightarrow y = \ln(x)/\ln(b)$.

So from now on we will usually stick to $e^x$ and $\ln(x)$.

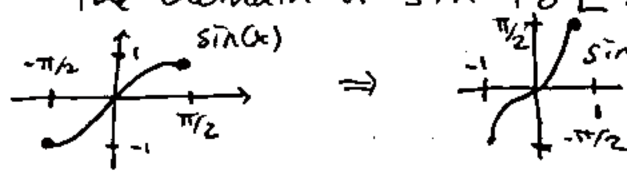
It is thus worth remembering these special values of $e^x$, $\ln(x)$: **Prop.** • $e^0 = 1$ • $\ln(1) = 0$ • $e^1 = e$ • $\ln(e) = 1$

## Inverse trig functions

Since we discussed inverse of $e^x$, you might wonder about inverses of trigonometric fn's like sin and cos.

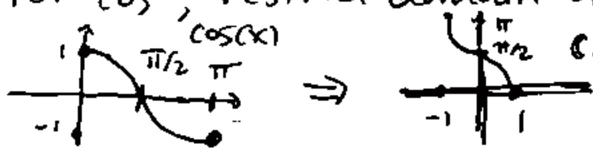
But sin and cos are not one-to-one, so to take inverses, we need to restrict their domains.

**Def'n** To define $\sin^{-1}(x)$ (or $\arcsin(x)$) we restrict the domain of sin to $[-\pi/2, \pi/2]$:




(D: $[-1, 1]$, R: $[-\pi/2, \pi/2]$)

For $\cos^{-1}$, restrict domain of cos to $[0, \pi]$:




(D: $[-1, 1]$, R: $[0, \pi]$)

Inverse trig functions are pretty complicated and we will not work with them in this class! (But it's good to know they exist...)

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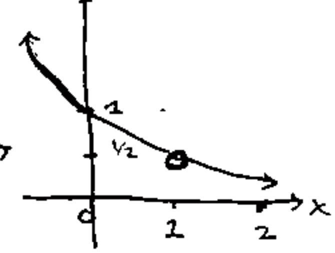
Intro to limits and derivatives §2.2 [2.1+]

So far we have reviewed functions you hopefully saw before in algebra/pre-calculus. Starting today, we will introduce calculus in earnest.

The first important notion in calculus is that of a limit.

Consider the function

$$f(x) = \frac{x-1}{x^2-1}$$




If we graph it near $x=1$, it looks something like

Note the "O" at $x=1$:

this shows $x=1$ is not in the domain of $f$, (because we would divide by zero at $x=1$).

However, it looks like there is a value $f(x)$ "should" take at $x=1$: the value $1/2$.

At $x$ values near 1, $f(x)$ gets close to $1/2$, and gets closer to $1/2$ the nearer to $x=1$ we get.

We express this by $\lim\limits_{x\to 1} \frac{x-1}{x^2-1} = 1/2$

or in words "the limit of $f(x)$ as $x$ goes to 1 is $1/2$."

Def'n (Intuitive definition of a limit)

The limit of $f(x)$ at $x_0$ is $L$, written

$$\lim_{x\to x_0} f(x) = L$$

if we can force $f(x)$ to be as close to $L$ as we want by requiring the input $x$ to be sufficiently close (but not equal!) to $x_0$.

Notice how the definition of the limit does not require $f(x)$ to be defined at $x_0$, or for $f(x_0)$ to equal $\lim_{x \to x_0} f(x)$ if it is defined. But...
if this is the case we say $f(x)$ is continuous at $x_0$.

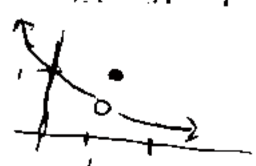
<u>Def'n</u> $f(x)$ is continuous at a point $x_0$ in its domain
$$\text{if } f(x_0) = \lim_{x \to x_0} f(x).$$

Most of the functions we've looked at so far like $x^n$, $\sqrt[n]{x}$, $\sin(x)$, $\cos(x)$, $e^x$, $\ln(x)$, etc. are continuous at all points in their domain.
Very roughly, this means we can "draw the graph without lifting our pencil."

For an example of a function that is <u>not</u> continuous (i.e., <u>discontinuous</u>) at a point in its domain:

<u>E.g.</u> Let $f(x) = \begin{cases} \frac{x-1}{x^2-1} & \text{if } x \neq 1 \ (\text{or } -1) \\ 1 & \text{if } x = 1 \end{cases}$

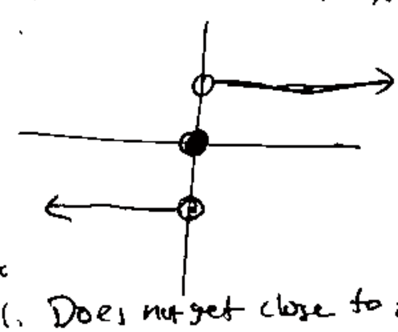
The graph of $f(x)$ is near $x=1$




and

Since $\lim_{x \to 1} f(x) = \frac{1}{2} \neq 1 = f(1)$, it is dis<u>continuous</u> at $x=1$.

<u>E.g.</u> Let $f(x) = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$



Then $\lim_{x \to 0} f(x)$ <u>does</u> <u>not</u> <u>exist</u>,
Because for values of $x$ slightly more than $0$, have $f(x) \simeq 1$, while for values of $x$ slightly less than $0$, have $f(x) = -1$. Does not get close to one number.

This last example relates to the notion of one-sided limits.

Def'n We write $\lim_{x \to x_0^-} f(x) = L$ and say the left-hand limit of $f(x)$ at $x_0$ is $L$ (or "limit as $x$ approaches $x_0$ from the left") if we can make $f(x)$ as close to $L$ as we want by restricting $x$ to be sufficiently close to and less than $x_0$.

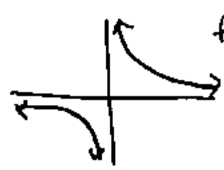
We write $\lim_{x \to x_0^+} f(x) = L$ and say the right-hand limit is $L$ for analogous thing but with values greater than $x_0$.

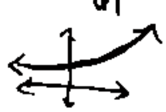
E.g. With $f(x)$ as in last example, we have
$$\lim_{x \to 0^-} f(x) = -1 \quad \text{and} \quad \lim_{x \to 0^+} f(x) = +1.$$

Note $\lim_{x \to x_0} f(x)$ exists and is $L$ iff $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ exist and both equal $L$.

Related to one-sided limits are limits at infinity.

Def'n We write $\lim_{x \to \infty} f(x) = L$ if we can make $f(x)$ arbitrarily close to $L$ by requiring $x$ to be big enough.
We write $\lim_{x \to -\infty} f(x) = L$ if same but with small enough.

E.g.  for $f(x) = 1/x$ have $\lim_{x \to \infty} f(x) = 0 = \lim_{x \to -\infty} f(x)$.

E.g.  for $f(x) = e^x$ have $\lim_{x \to -\infty} f(x) = 0$ (but not $x \to \infty$)

E.g. When we defined $e = \lim_{n \to \infty} (1 + 1/n)^n$ we were using a limit at infinity.

We saw $f(n) = (1+1/n)^n$ has $f(1) = 2$
$f(2) = 2.25$
$\ldots f(100) = 2.7048\ldots$
$f(1000) = 2.7169\ldots$
getting closer and closer to $e$ as we made $n$ bigger and bigger.

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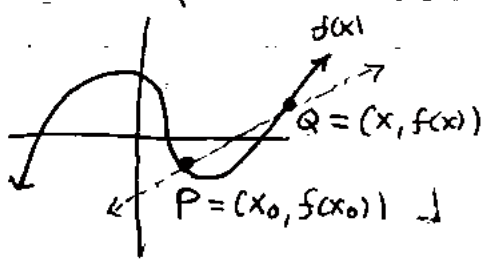
§2.1, 2.7 If most "normal" functions we work with are continuous at all points in their domain, might wonder why we define limits at all, especially for points not in domain.

Reason is we want to define the derivative as a limit, and this naturally involves a limit that is "0/0" (so not defined just by "plugging in values").

Recall our discussions from 1st day of class:




We have a point $P$ on a curve, i.e. graph of function $f(x)$. Assume $P = (x_0, f(x_0))$ is fixed. For another point $Q$ on the curve, w/ $Q = (x, f(x))$:

What is the slope of the secant line from $P$ to $Q$?

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f(x) - f(x_0)}{x - x_0}.$$

Recall that the tangent line of the curve at $P$ is the limit of the secant line as we send $Q$ to $P$. So what is the slope of the tangent line at $P$?

$$\text{slope of tangent} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

This is the derivative of $f(x)$ at $x_0$!

Def'n The derivative of $f(x)$ at a point $a$ in its domain is $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

E.g. Let's compute the derivative of $f(x) = x^2$ at $x = 1$. We need to compute

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

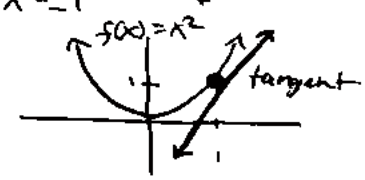
To do this, we use the algebra trick:

$$(x^2 - 1) = (x+1)(x-1)$$

$$\text{So } \lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}} = \lim_{x \to 1} (x+1) = \underline{\underline{2}}.$$

We will justify all these steps later when we talk about rules for computing limits (but it should match $\lim_{x \to 1} \frac{x-1}{x^2-1} = \frac{1}{2}$ from before...)

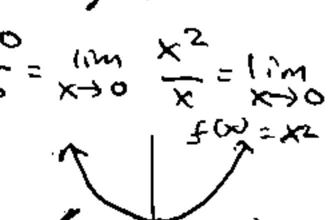
And it looks reasonable that the slope of the tangent at $x = 1$ is 2:




E.g. If instead we compute the derivative of $f(x) = x^2$ at point $x = 0$ we get

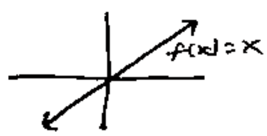
$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 - 0}{x - 0} = \lim_{x \to 0} \frac{x^2}{x} = \lim_{x \to 0} x = 0$$

and again it looks like the slope of tangent at $x = 0$ is zero (horizontal).

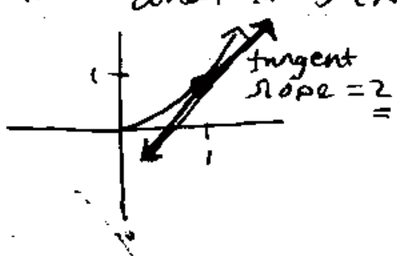


Why do we care about derivatives?
They tell us "instantaneous rate of change"

E.g. Suppose a car's position in meters (away from initial point) after $x$ seconds is given by $f(x)$.
How can we find the speed of the car at time $x=a$?
If $f(x)=x$, so that the car were moving at a constant rate of 1 m/s, then clearly at any time its speed is this 1 m/s.




But what if $f(x)=x^2$ (which is reasonable for an accelerating car)?
To find the speed at time $x=1$, we could measure its position at time $x=1$ and $x=b$ for $b$ a little bit after 1, and compute




$$\frac{f(b)-f(1)}{b-1} \leftarrow \text{rate of growth: } \frac{\text{rise}}{\text{run}}$$

To be super accurate we want $b$ to be very close to 1.
So the best definition of speed at time 1 is
$\lim_{b\to 1}\frac{f(b)-1}{b-1}$, i.e., the derivative of $f(x)$ at $x=1$!
We saw before that for $f(x)=x^2$ this is $\underline{\underline{2}}$, so the accelerating car is going faster at time $x=1$.
But at time $x=0$, its speed is $\lim_{x\to 0}\frac{x^2}{x}=0$, meaning it is just starting to accelerate from speed zero.

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§ 2.3

Rules for limits:

The following rules of limits allow us to compute many limits in practice:

Thm (Limit Laws) Suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[c\cdot f(x)] = c\cdot \lim_{x\to a} f(x)$ for any constant $c\in\mathbb{R}$
4. $\lim_{x\to a}[f(x)\,g(x)] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ as long as $\lim_{x\to a} g(x)\neq 0$.

"Limit of sum is sum of limits, etc."

Together with:

Thm (Base case limits)

$\lim_{x\to a} c = c$ for any constant $c\in\mathbb{R}$

and $\lim_{x\to a} x = a$.

these tell us that

Thm. If $P(x)$ is a polynomial then $\lim_{x\to a} P(x) = P(a)$

· If $\frac{P(x)}{Q(x)}$ is a rational function and $a$ is in its domain, then $\lim_{x\to a}\frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$.

"Can evaluate limits of polynomials/rational functions by plugging in a"

Let's see how we can use these laws to show

Eg. $\lim_{x\to 1} \frac{x-1}{x^2-1} = 1/2$

Pf: $\lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{x-1}{(x-1)(x+1)}$ "difference of squares"

$= \lim_{x\to 1} \frac{1}{x+1} \cdot \lim_{x\to 1} \frac{x-1}{x-1}$ "product of limits"

$= \frac{1}{2} \cdot 1$ ▢

How do we know $\lim_{x\to 1} \frac{x-1}{x-1} = 1$? Notice that $\frac{x-1}{x-1} = 1$ for any $x \neq 1$. We need one more rule:

Thm If $f(x) = g(x)$ for all $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

This makes sense because remember that "the limit at $x=a$ only cares about $f(x)$ near $x=a$, not what happens exactly at $x=a$."

This rule lets us "cancel factors" in a limit!

Also have

Thm (Limits of powers / roots) for any positive integer $n$,

$\lim_{x\to a} [f(x)]^n = \left(\lim_{x\to a} f(x)\right)^n$ and $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$

(wherever the right-hand side is defined.)

These tell us: If $f(x)$ is any "algebraic function" (built out of powers and roots, together with addition/subtraction/multiplication/division) and $a$ is in the domain of $f(x)$, then $\lim_{x\to a} f(x) = f(a)$.

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