Math 210 (Modern Algebra I), HW# 1,

Fall 2025; Instructor: Sam Hopkins; Due: Wednesday, August 27th

In all of these problems, G denotes a group.

1. (a) Prove that G is abelian if and only if $(ab)^2 = a^2b^2$ for all $a, b \in G$.
   (b) Give an example of a group G and elements $a, b \in G$ with $(ab)^2 \neq a^2b^2$.
   (c) Prove that if $a^2 = e$ for all $a \in G$, then G is abelian (where $e \in G$ is the identity).
2. Let $x \in G$. Prove that the cyclic subgroup $\langle x \rangle \subseteq G$ generated by x is infinite if and only if $x^i \neq x^j$ for all $i \neq j \in \mathbb{Z}$.
3. Prove that if G is finite and has even order, it contains an element of order 2.
   Hint: Consider the set $t(G) = \{g \in G \colon g \neq g^{-1}\}$; show that $t(G)$ has an even number of elements and any non-identity element of $G \setminus t(G)$ has order 2.
4. Consider the finite abelian group $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, with the group operation written additively as $+$. Write down the "addition table" for G, i.e., the table whose columns and rows are indexed by the elements of G, and with the entry in row a and column b being $a + b$.
5. Let $\sigma = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12) \in S_{12}$ be a 12-cycle in the symmetric group S_{12}. Write the cycle decomposition of σ^i for each $i = 0, 1, \ldots, 11$. What pattern do you notice? In particular, which powers of σ are also 12-cycles?