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# Upho lattices and their cores II

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**Abstract.** A poset is called upper homogeneous, or "upho," if all of its principal order filters are isomorphic to the whole poset. In previous work of the first author, it was shown that each (finite type $\mathbb{N}$-graded) upho lattice has associated to it a finite graded lattice, called its core, which determines the rank generating function of the upho lattice. In that prior work the question of which finite graded lattices arise as cores was explored. Here, we study the question of in how many different ways a given finite graded lattice can be realized as the core of an upho lattice. We show that if the finite lattice has no nontrivial automorphisms, then it is the core of finitely many upho lattices. We also show that the number of ways a finite lattice can be realized as a core is unbounded, even when restricting to rank-two lattices.

## 1 Introduction

A poset $\mathcal{P}$ is called *upper homogeneous*, or *upho*, if for every $p \in \mathcal{P}$, the principal order filter $V_p = \{q \in \mathcal{P} \colon q \geq p\}$ is isomorphic to the original poset $\mathcal{P}$. This class of infinite, self-similar posets was introduced a few years ago by Stanley [13, 14], and has subsequently been studied by a number of authors [6, 7, 9, 5].

In [7, 8, 9], the first author explained how each (finite type $\mathbb{N}$-graded) upho *lattice* has associated to it a finite graded lattice which controls many features of the upho lattice. More precisely, for an upho lattice $\mathcal{L}$, its *core* is $L := [\hat{0}, s_1 \vee \cdots \vee s_r]$, the interval from the minimum $\hat{0}$ to the join of its atoms $s_1, \ldots, s_r$. We have $F(\mathcal{L}; x) = \chi^*(L; x)^{-1}$ where $F(\mathcal{L}; x)$ is the rank generating function of the upho lattice $\mathcal{L}$ and $\chi^*(L; x)$ is the (reciprocal) characteristic polynomial of its core $L$. So the core determines how quickly the upho lattice grows.

In [9], the question of which finite graded lattices are cores of upho lattices was explored. It was shown that this is a very subtle question: many well-studied finite lattices are cores, but many also are not. In general, it is unclear how one can determine if a given finite lattice is the core of some upho lattice.

Importantly, the core does not determine the upho lattice, in the sense that the same finite lattice can be the core of multiple upho lattices: e.g., the rank-two Boolean lattice is

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the core of two different upho lattices, as shown in Figure 1. Here we study the question of in how many different ways a given finite lattice can be realized as a core. In a sense, we study whether an upho lattice can be represented by a finite amount of data.

Specifically, for a finite graded lattice $L$, let $\kappa(L)$ be the number of different upho lattices of which $L$ is a core. We are interested in the behavior of the function $\kappa(L)$. We will see that the way this function behaves is also quite subtle: in some ways, this function is "small"; in other ways, it is "big."

Concerning the "smallness" of $\kappa(L)$, our first major results says that if the finite lattice $L$ has no nontrivial automorphisms, then $\kappa(L)$ is finite. This result suggests that $\kappa(L)$ may always finite for all finite lattices $L$. But we currently cannot rule out the possibility that $\kappa(L)$ is infinite, even uncountably infinite, for some finite lattice $L$ with a nontrivial automorphism.

Concerning the "bigness" of $\kappa(L)$, our second major result says that this function is unbounded. Actually, as soon as we know there is one lattice $L$ with $\kappa(L) > 1$, a simple product construction implies that $\kappa(L)$ is unbounded. But we show, what is much less trivial, that $\kappa(L)$ is unbounded even when restricting to lattices $L$ of rank two. There is a unique rank-two graded lattice with $n$ atoms, denoted $M_n$, and we show more precisely that for each $n \geq 2$, $\kappa(M_n) \geq p(n)$, where $p(n)$ is the number of integer partitions of $n$.

In the proof of both of our major results, monoids are used in an essential way. Previous work [6, 9, 5] highlighted the close connection between monoids and upho posets; the key observation is that each (homogeneously finitely generated) left-cancellative monoid gives rise to an upho poset.

An intriguing open question is whether *every* upho lattice (or even every upho poset[1]) comes from a monoid in this way. If so, it would mean that this order-theoretic definition has intrinsic algebraic content. It would also imply the finiteness of $\kappa(L)$ for all finite lattices $L$, and might lead to an algorithm for listing all upho lattices of which $L$ is a core. We discuss some speculations along these lines in the final section.

This is an extended abstract summarizing our work; for a complete version, see [10].


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[1] [6, Remark 5.1] says that "[the monoid construction] is not a method to construct *all* upho posets," but Y. Gao (private communication) has told us he does not know any example to substantiate this claim, so indeed the possibility that all upho posets come from monoids remains open.

## 2 Background on upho posets and monoids

We follow standard terminology for posets as laid out for instance in [12, Chapter 3]. Let $P = (P, \leq)$ be a poset. We use $\hat{0}$ and $\hat{1}$ to denote the minimum and maximum of $P$ when they exist. We use $\lessdot$ for the cover relation of $P$. An *atom* of $P$ is an element $s \in P$ with $\hat{0} \lessdot s$. We routinely work with both finite and infinite posets, so from now on we use the convention that normal script letters (like $P$ and $L$) denote finite posets while calligraphic letters (like $\mathcal{P}$ and $\mathcal{L}$) denote infinite posets.

A finite poset $P$ is *graded* if we can write it as a disjoint union $P = P_0 \sqcup P_1 \sqcup \cdots \sqcup P_n$ such that every maximal chain in $P$ is of the form $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ with $x_i \in P_i$. In this case, we say the poset has *rank* $n$, and the poset's *rank function* $\rho\colon P \to \{0, 1, \ldots, n\}$ is given by $\rho(x) = i$ if $x \in P_i$. The *rank generating* and *(reciprocal) characteristic polynomials* of such a $P$ are then defined to be

$$F(P; x) := \sum_{p \in P} x^{\rho(p)}; \qquad \chi^*(P; x) := \sum_{p \in P} \mu(\hat{0}, p)\, x^{\rho(p)},$$

where as usual $\mu(\cdot, \cdot)$ is the Möbius function of $P$. For example, the *Boolean lattice* $B_n$ of subsets of $[n] := \{1, \ldots, n\}$ is graded with $F(B_n; x) = (1 + x)^n$ and $\chi^*(B_n; x) = (1 - x)^n$.

We use $\mathbb{N} := \{0, 1, 2, \ldots\}$ for the natural numbers. An infinite poset $\mathcal{P}$ is *$\mathbb{N}$-graded* if we can write it as a disjoint union $\mathcal{P} = \mathcal{P}_0 \sqcup \mathcal{P}_1 \sqcup \cdots$ such that every maximal chain in $\mathcal{P}$ is of the form $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots$ with $x_i \in \mathcal{P}_i$. In this case, the poset's *rank function* $\rho\colon \mathcal{P} \to \mathbb{N}$ is given by $\rho(x) = i$ if $x \in \mathcal{P}_i$. We say such a $\mathcal{P}$ is *finite type $\mathbb{N}$-graded* if $\#\mathcal{P}_i < \infty$ for all $i$. The *rank generating* and *characteristic generating functions* of such a $\mathcal{P}$ are then defined to be

$$F(\mathcal{P}; x) := \sum_{p \in \mathcal{P}} x^{\rho(p)}; \qquad \chi^*(\mathcal{P}; x) := \sum_{p \in \mathcal{P}} \mu(\hat{0}, p)\, x^{\rho(p)}.$$

E.g., $\mathbb{N}^n$ is finite type $\mathbb{N}$-graded with $F(\mathbb{N}^n; x) = \frac{1}{(1-x)^n}$ and $\chi^*(\mathbb{N}^n; x) = (1 - x)^n$.

As mentioned in the introduction, a poset $\mathcal{P}$ is *upper homogeneous*, or *upho*, if for each element $p \in \mathcal{P}$, the corresponding *principal order filter* $V_p := \{q \in \mathcal{P} \colon q \geq p\}$ is isomorphic to $\mathcal{P}$. To avoid trivialities, we assume all upho posets have at least two elements; then they must be infinite. In order to able to apply the tools of enumerative and algebraic combinatorics to study them, **all upho posets are assumed finite type $\mathbb{N}$-graded** from now on. For example, $\mathbb{N}^n$ is an upho lattice.

As observed previously by the first author, upho posets have an interesting symmetry regarding their rank and characteristic generating functions.

**Theorem 1** ([7, Theorem 1]). *For an upho poset $\mathcal{P}$, $F(\mathcal{P}; x) = \chi^*(\mathcal{P}; x)^{-1}$.*

For lattices, we can say more. For an upho lattice $\mathcal{L}$, its *core* is $L := [\hat{0}, s_1 \vee \cdots \vee s_r]$, the interval from its minimum $\hat{0}$ to the join of its atoms $s_1, \ldots, s_r$. Notice that the core of an upho lattice is a finite graded lattice. An easy corollary of Theorem 1 is:

**Corollary 2** ([7, Corollary 6]). *For an upho lattice $\mathcal{L}$ with core $L$, $F(\mathcal{L};x) = \chi^*(L;x)^{-1}$.*

So Corollary 2 says the core of an upho lattice determines how quickly it grows. For instance, $\mathbb{N}^n$ is an upho lattice with core $B_n$, and we see $F(\mathbb{N}^n;x) = \frac{1}{(1-x)^n} = \chi^*(B_n;x)^{-1}$.

The core does *not* completely determine the upho lattice, in the sense that a given finite graded lattice can be the core of multiple different upho lattices. For example, there are two different upho lattices with core $B_2$, depicted in Figure 1.

For a finite graded lattice $L$, let $\kappa(L)$ denote the cardinality of the set of different upho lattices with core $L$. Our main interest here is in understanding the function $\kappa(L)$. A priori $\kappa(L)$ could be infinite, even uncountably infinite, for some $L$. But we will provide some reasons to think $\kappa(L)$ should be finite for all $L$.

An important source of upho posets are monoids, as has been observed previously in [6, 9, 5]. Let us review the connection. For basics on monoids, consult, e.g., [4, 3].

Let $M = (M, \cdot)$ be a monoid. Say that $M$ is *left cancellative* if $ab = ac$ implies that $b = c$ for $a, b, c \in M$. Left cancellativity is a version of upper homogeneity. But we also need to enforce our finiteness requirements. Let us say that $M$ is *homogeneously finitely generated* if it has a presentation $M = \langle S \mid R \rangle$ where the set $S$ of generators is finite and where every relation in $R$ is homogeneous, i.e., of the form $w = w'$ with $\ell(w) = \ell(w')$, where $\ell(w)$ denotes the length of the word $w$. Finally, we use $\leq_L$ to denote the *left divisibility* relation on $M$: $a \leq_L b$ for $a, b \in M$ means that $b = ac$ for some $c \in M$. With all this terminology, we have the following result:

**Lemma 3** ([6, Lemma 5.1]; see also [9, 5]). *Let $M$ be a homogeneously finitely generated, left-cancellative monoid. Then $\mathcal{P} := (M, \leq_L)$ is a (finite type $\mathbb{N}$-graded) upho poset.*

For example, the free commutative monoid $\langle x_1, \ldots, x_n \colon x_i x_j = x_j x_i \rangle$ satisfies the conditions of Lemma 3, and gives us the upho lattice $\mathbb{N}^n$.

# 3 Colorable upho posets, automorphisms, and finiteness

In this section we focus on proving that $\kappa(L)$ is finite. Our major result says that $\kappa(L)$ is finite when the finite lattice $L$ has no nontrivial automorphisms. We use the connection to monoids to prove this result.

It is helpful to recast the monoid construction of upho posets in a slightly more combinatorial framework (c.f. [5, §3]). An *upho coloring* of a finite type $\mathbb{N}$-graded poset $\mathcal{P}$ is a function $c$ that maps each cover relation of $\mathcal{P}$ to an atom of $\mathcal{P}$, such that:

- $c(\hat{0} \lessdot s) = s$ for every atom $s \in \mathcal{P}$, and
- for each $p \in \mathcal{P}$, there is an isomorphism $\varphi_p \colon V_p \to \mathcal{P}$ which preserves the coloring in the sense that $c(\varphi_p(x) \lessdot \varphi_p(y)) = c(x \lessdot y)$ for all $x \lessdot y \in V_p$.

Of course, if $\mathcal{P}$ has an upho coloring $c$, then it is an upho poset. Let us call a $\mathcal{P}$ together with such a $c$ a *colored upho poset*, and call an upho poset $\mathcal{P}$ *colorable* if it admits such a $c$.

The colorable upho posets are exactly those coming from monoids.

**Lemma 4** ([5, Corollary 3.6])**.** *There is a bijective correspondence between homogeneously finitely generated, left-cancellative monoids $M$ and colored upho posets $\mathcal{P}$. Given such a monoid $M$, we let $\mathcal{P} = (M, \leq_L)$ as in Lemma 3, with the coloring given by $c(x \lessdot y) = s$ if $y = xs$. Conversely, given such a colored upho poset $\mathcal{P}$, the associated monoid $M$ has presentation*

$$M = \left\langle s_1, \ldots, s_r \colon \begin{array}{c} c(\hat{0} = x_0 \lessdot x_1)c(x_1 \lessdot x_2) \cdots c(x_{k-1} \lessdot x_k = p) = \\ c(\hat{0} = y_0 \lessdot y_1)c(y_1 \lessdot y_2) \cdots c(y_{k-1} \lessdot y_k = p) \end{array} \right\rangle$$

*where the generators are the atoms $s_1, \ldots, s_r$ of $\mathcal{P}$, and the relations correspond to all pairs of saturated chains $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = p$, $\hat{0} = y_0 \lessdot y_1 \lessdot \cdots \lessdot y_k = p$ from $\hat{0}$ to any $p \in \mathcal{P}$.*

In the case of lattices, we can substantially reduce the set of relations we need.

**Lemma 5.** *Let $\mathcal{L}$ be a colored upho lattice. Then the monoid $M$ associated to it by Lemma 4 is*

$$M = \left\langle s_1, \ldots, s_r \colon \begin{array}{c} c(\hat{0} = x_0 \lessdot x_1)c(x_1 \lessdot x_2) \cdots c(x_{k-1} \lessdot x_k = x_1 \vee y_1) = \\ c(\hat{0} = y_0 \lessdot y_1)c(y_1 \lessdot y_2) \cdots c(y_{k-1} \lessdot y_k = x_1 \vee y_1) \end{array} \right\rangle$$

*where the generators are the atoms $s_1, \ldots, s_r$ of $\mathcal{L}$, and the relations correspond to all pairs of saturated chains $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = x_1 \vee y_1$, $\hat{0} = y_0 \lessdot y_1 \lessdot \cdots \lessdot y_k = x_1 \vee y_1$ from $\hat{0}$ to the join $x_1 \vee y_1$ of two atoms $x_1, y_1$ of $\mathcal{L}$.*

Lemma 5 says that the coloring of a colored upho lattice $\mathcal{L}$ is completely determined by the way that its core $L$ is colored. From this we immediately conclude:

**Corollary 6.** *Each finite graded lattice $L$ is the core of finitely many colorable upho lattices $\mathcal{L}$.*

But how to construct upho colorings? Let $\mathcal{P}$ be an upho poset. A *system of isomorphisms* for $\mathcal{P}$ is a collection of isomorphisms $\varphi_p \colon V_p \to \mathcal{P}$ for each $p \in \mathcal{P}$. Note that, by definition of upper homogeneity, a system of isomorphisms for $\mathcal{P}$ exists. Let us say that a system of isomorphisms is *compatible* if

$$\varphi_q = \varphi_{\varphi_p(q)} \circ (\varphi_p|_{V_q}) \tag{3.1}$$

for each $p \leq q \in \mathcal{P}$. Compatible systems of isomorphisms give us upho colorings.

**Lemma 7.** *An upho poset $\mathcal{P}$ is colorable if and only if it has a compatible system of isomorphisms.*

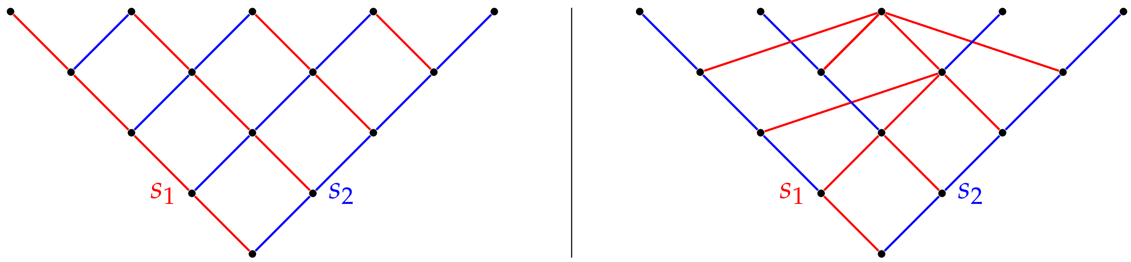


**Figure 1:** Two different (colored) upho lattices with core $B_2$: see Examples 8 and 9.

*Proof.* Given a colored upho poset $\mathcal{P}$, the system of isomorphisms certifying that its coloring $c$ is upho must be compatible. Conversely, given a compatible system of isomorphisms $\varphi_p$ for $\mathcal{P}$, the coloring $c(x \lessdot y) := \varphi_x(y)$ is upho. $\square$

**Example 8.** Let $n \geq 1$ and let $\mathcal{L} = \mathbb{N}^n$. Then $\mathcal{L}$ is an upho lattice, with core $B_n$, and a compatible system of isomorphisms is $\varphi_{(x_1,\ldots,x_n)}(y_1,\ldots,y_n) = (y_1 - x_1, \ldots, y_n - x_n)$. The corresponding monoid is $M = \langle s_1, \ldots, s_n \colon s_i s_j = s_j s_i \text{ for } 1 \leq i < j \leq n \rangle$. For $n = 2$, the colored upho lattice $\mathcal{L}$ is depicted on the left in Figure 1.

**Example 9.** Let $n \geq 1$ and let $\mathcal{L} = \{\text{finite } S \subseteq \{1,2,\ldots\} \colon \max(S) < \#S + n\}$ (with the convention $\max(\varnothing) = 0$), partially ordered by inclusion. Then $\mathcal{L}$ is another upho lattice with core $B_n$ (see [7, Remark 8] and [9, §3.5.1]). A compatible system of isomorphisms is $\varphi_S(T) = f_S(T \setminus S)$ where $f_S \colon \{1,2,\ldots\} \setminus S \to \{1,2,\ldots\}$ is the order-preserving bijection. The corresponding monoid is $M = \langle s_1, \ldots, s_n \colon s_i s_{j-1} = s_j s_i \text{ for } 1 \leq i < j \leq n \rangle$. For $n = 2$, the colored upho lattice $\mathcal{L}$ is depicted on the right in Figure 1.

**Remark 10.** The upho lattices from Example 9 belong to a more general construction for any *uniform sequence of supersolvable geometric lattices*; see [9, §3]. These sequences include not just Boolean lattices, but also subspace lattices, partition lattices, etc. The conditions which go into the definition of "uniform sequence" imply the resulting systems of isomorphisms are compatible. So, all the upho lattices in [9, §3] come from monoids.

When can we guarantee that an upho poset $\mathcal{P}$ has a *compatible* system of isomorphisms? Notice that if (3.1) fails for some $p \leq q \in \mathcal{P}$, then the left-hand and right-hand sides are two different isomorphisms $V_q \to \mathcal{P}$. Composing one of these isomorphisms with the inverse of the other would then yield a *nontrivial* automorphism $\mathcal{P} \to \mathcal{P}$. So:

**Corollary 11.** *If an upho poset $\mathcal{P}$ has no nontrivial automorphisms, then it is colorable.*

To finish the proof of our first main result, we need the following observation.

**Lemma 12.** *If a finite graded lattice $L$ has no nontrivial automorphisms, then any upho lattice $\mathcal{L}$ of which it is the core also has no nontrivial automorphisms.*

Putting everything together, we have our main result of this section.

**Theorem 13.** *Let $L$ be a finite graded lattice which has no nontrivial automorphisms. Then $L$ is the core of finitely many upho lattices, i.e., $\kappa(L)$ is finite.*

**Remark 14.** For many classes of finite, combinatorial structures, "most" members have no nontrivial automorphisms, in the sense that the proportion of such structures on $[n]$ with a nontrivial automorphism goes to 0 as $n \to \infty$. This is known to be true for finite graphs, and for finite posets, and we suspect it is true for finite lattices as well. Hence, Theorem 13 should apply to most lattices $L$. But on the other hand, we also expect $\kappa(L) = 0$ for most $L$. So it is reasonable to ask whether there are $L$ to which Theorem 13 applies but for which we also know $\kappa(L) > 0$. We provide an infinite sequence of such $L$, with ranks going to infinity, in the following Example 15.

**Example 15.** In [9, §4.3.1] it is explained that the weak order of any finite Coxeter group $W$ is the core of an upho lattice, the *classical braid monoid* (which is an upho lattice because it is a Garside monoid [4, 3]). For any Dynkin diagram that has no "$\infty$" labels, the automorphisms of weak order are exactly the Dynkin diagram automorphisms (see [1, Corollary 3.2.6]). Thus, with $L$ the weak order of the type $B_n$ Coxeter group for $n > 2$, $L$ has no nontrivial automorphisms and $\kappa(L) > 0$.

# 4 Rank-two cores

In this section we explore rank-two cores. Rank-one cores are trivial: the only rank-one lattice is the two element chain, and it is the core of a unique upho lattice, $\mathbb{N}$. But already rank-two cores are quite interesting. For each $n \geq 1$, there is a unique rank-two lattice with $n$ atoms, denoted[2] $M_n$. (In particular, $M_2 = B_2$.) Our major result in this section is that $\kappa(M_n) \geq p(n)$, the number of integer partitions of $n$, for each $n \geq 2$. Along the way we also show that $\kappa(M_2) = 2$; in fact, this is the only nontrivial finite lattice where we completely understand all the ways it can be realized as a core of an upho lattice.

Fix $n \geq 2$. We start by describing two different recursive constructions that produce upho lattices $\mathcal{L}$ with core $M_n$. In both constructions we will build up a sequence of finite posets $P_0 \subseteq P_1 \subseteq \cdots$, with $P_i$ of rank $i$, and then set $\mathcal{L} := \lim_{i \to \infty} P_i$.

The first construction we call the *dominating vertex construction*. We start by setting $P_0$ to be the one-element poset. Then, for each $i \geq 1$, we obtain $P_i$ from $P_{i-1}$ by doing the following:

- we append a new element which covers all the elements of rank $i-1$;
- then for each element $p$ of rank $i-1$, we also append $n-1$ additional new elements covering only $p$.

[2] Please do not associate the "M" in the lattice $M_n$ with "monoid;" it stands rather for "modular."

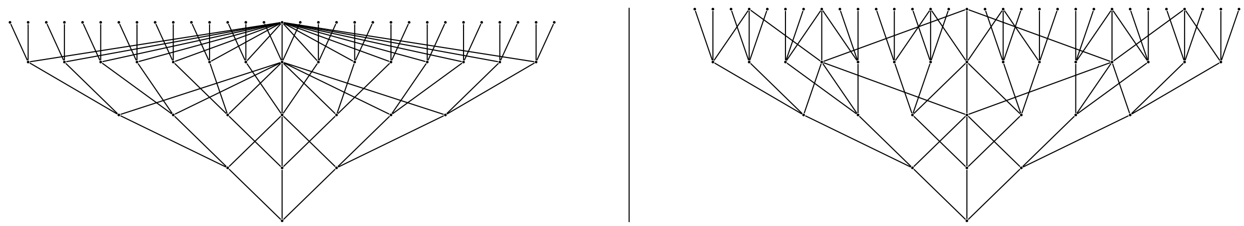

**Figure 2:** On the left, the dominating vertex construction of an upho lattice with core $M_3$; and on the right, the flip construction of an upho lattice with core $M_3$.

We call this the dominating vertex construction because of the element of rank $i$ which covers ("dominates") all elements of rank $i-1$. It is clear that $P_0 \subsetneq P_1 \subsetneq \cdots$, with $P_i$ of rank $i$, so that we can define $\mathcal{L} := \lim_{i\to\infty} P_i$ to be the result of this construction. For example, the left side of Figure 2 depicts this result in the case of $n = 3$.

The second construction we call the *flip construction*. We start by setting $P_0$ to be the one-element poset and $P_1$ to be the "claw" poset with minimum $\hat{0}$, $n$ atoms, and no other elements. Then, for each $i \geq 2$, we obtain $P_i$ from $P_{i-1}$ by doing the following:

- for each element $p$ of rank $i-2$, letting $q_1, \ldots, q_n$ be the elements of rank $i-1$ covering $p$, we append a new element which covers exactly $q_1, \ldots, q_n$;
- then for each element $p$ of rank $i-1$, we also append enough additional new elements covering only $p$ to make $p$ be covered by exactly $n$ elements.

We call this the flip construction because the first step can be seen as taking the portion of the Hasse diagram between ranks $i-2$ and $i-1$ and placing a reflected copy above it. It is again clear that $P_0 \subsetneq P_1 \subsetneq \cdots$, with $P_i$ of rank $i$, so that we can define $\mathcal{L} := \lim_{i\to\infty} P_i$ to be the result of this construction. For example, the right side of Figure 2 depicts this result in the case of $n = 3$.

**Theorem 16.** *For any $n \geq 2$, the dominating vertex and flip constructions produce two different upho lattices with core $M_n$.*

We also are able to show, essentially by a counting argument, that if an upho lattice agrees with either of these constructions up to rank three, it agrees forever.

**Theorem 17.** *Let $\mathcal{L}$ be an upho lattice with core $M_n$ for some $n \geq 2$. Then if $\mathcal{L}$ agrees with the result of the dominating vertex construction up to rank three, in fact $\mathcal{L}$ must be the result of the dominating vertex construction, and similarly for the flip construction.*

**Remark 18.** From Theorem 17 one can deduce that, for any $n \geq 2$, the flip construction produces the only *modular* upho lattice with core $M_n$.

These two constructions yield all possible upho lattices with core $M_2$.

**Corollary 19.** *The only upho lattices with core $M_2$ are those produced by the dominating vertex and flip constructions; they are depicted in Figure 1. Hence, $\kappa(M_2) = 2$.*

However, for $n \geq 3$, there are more possibilities beyond these two for upho lattices with core $M_n$. In order to understand these possibilities, we need to return to monoids. The following theorem gives us a rich source of upho lattices with core $M_n$.

**Theorem 20.** *Let $n \geq 2$ and let $f\colon [n] \to [n]$ be any function. Define the homogeneously finitely generated monoid $M(f)$ by*

$$M(f) = \langle x_1, \dots, x_n \colon x_1 x_{f(1)} = x_2 x_{f(2)} = \cdots = x_n x_{f(n)} \rangle.$$

*Then $M(f)$ is left cancellative, and any two elements in $M(f)$ have a least common right multiple. Hence, $\mathcal{L}(f) := (M(f), \leq_L)$ is an upho lattice, with core $M_n$.*

**Remark 21.** In Theorem 20, choosing an $f\colon [n] \to [n]$ which is a bijection yields the result of the flip construction for $\mathcal{L}(f)$, while choosing an $f$ which has image of size one yields the result of the dominating vertex construction for $\mathcal{L}(f)$. These are the "extreme" cases.

To prove that the monoid $M(f)$ in Theorem 20 is left cancellative, we use the following simple lemma.

**Lemma 22.** *Let $M = \langle S \mid R \rangle$ be a homogeneously finitely generated monoid such that for each generator $s \in S$, there is at most one word $w$ that begins with $s$ which appears in the relations in $R$. Then $M$ is left cancellative.*

To prove that the monoid $M(f)$ in Theorem 20 has least common multiples, we use a variant of [2, Lemma 2.1], which says that this lattice property can be checked locally.

To finish the proof of our main result in this section, we need to think about when different functions $f\colon [n] \to [n]$ yield different upho lattices $\mathcal{L}$. As we have seen in Remark 21, what matters are the fiber sizes of $f$. More precisely, we have the following.

**Lemma 23.** *Let $n \geq 2$. For an integer partition $\lambda \vdash n$, define $f_\lambda\colon [n] \to [n]$ by letting*

$$\begin{aligned} f_\lambda(1) = f_\lambda(2) = \cdots = f_\lambda(\lambda_1) &= \lambda_1, \\ f_\lambda(\lambda_1+1) = f_\lambda(\lambda_1+2) = \cdots = f_\lambda(\lambda_1+\lambda_2) &= \lambda_1+\lambda_2, \\ f_\lambda(\lambda_1+\lambda_2+1) = f_\lambda(\lambda_1+\lambda_2+2) = \cdots = f_\lambda(\lambda_1+\lambda_2+\lambda_3) &= \lambda_1+\lambda_2+\lambda_3, \end{aligned}$$

*and so on. (So $f_\lambda$ is idempotent, and $\lambda$ is the partition of the fiber sizes.) Then, with the notation of Theorem 20, $\mathcal{L}(f_\lambda)$ is isomorphic to $\mathcal{L}(f_\nu)$ for two partitions $\lambda, \nu \vdash n$ if and only if $\lambda = \nu$.*

Theorem 20 and Lemma 23 together establish our main result of this section.

**Theorem 24.** *For any $n \geq 2$, we have $\kappa(M_n) \geq p(n)$, the number of integer partitions of $n$.*

**Remark 25.** While we have seen that $\kappa(M_2) = 2$, it remains possible that $\kappa(M_n)$ is infinite for some $n \geq 3$, even possibly for $n = 3$. Indeed, the $M_n$ have nontrivial automorphisms, so Theorem 13 does not apply.

# 5 Listing all the ways a finite lattice can be a core

By now we see that the following question is central to understanding all the ways a given finite lattice can be realized as a core.

**Question 26.** *Does every upho lattice come from a monoid; i.e., in the language of Section 3, is every upho lattice colorable?*

We are not sure whether we should expect a positive answer to Question 26, but suppose for the moment that it had one. Then from Corollary 6 it would immediately follow that $\kappa(L)$ is finite for all finite graded lattices $L$. But it would still not be clear how to list all the upho lattices of which $L$ is a core. In this section, we speculate about how one could devise an algorithm for listing all the (colored) upho lattices of which a given finite lattice is a core.

Let $L$ be a finite graded lattice for which $\hat{1}$ is the join of the atoms. A *pre-upho coloring* of $L$ is a function $c$ that maps each cover relation of $L$ to an atom of $L$, such that:

- $c(\hat{0} \lessdot s) = s$ for every atom $s \in L$;
- for each $x \in L \setminus \{\hat{0}, \hat{1}\}$, letting $y_1, \ldots, y_k$ be the elements covering $x$, there is a rank- and color-preserving embedding of the interval $[x, y_1 \vee \cdots \vee y_k]$ into $L$.

(By an embedding of $P$ into $Q$ we mean a map $\varphi\colon P \to Q$ which is an isomorphism onto its image; that it is rank-preserving means $\rho(\varphi(p)) = \rho(p)$ for all $p \in P$; that it is color-preserving means $c(\varphi(x) \lessdot \varphi(y)) = c(x \lessdot y)$ for all $x \lessdot y \in P$.)

The following is a colored version of [9, Lemma 5.11].

**Lemma 27.** *Let $\mathcal{L}$ be an upho lattice with core $L$. Let $c$ be an upho coloring of $\mathcal{L}$. Then $c$ restricts to a pre-upho coloring of $L$.*

So in order for $L$ to be the core of some colored upho lattice, it must have a pre-upho coloring. What about the converse? Does a pre-uhpo coloring of $L$ give us a colored upho lattice of which $L$ is the core? We speculate about this in the following conjecture.

**Conjecture 28.** *Let $L$ be a finite graded lattice for which $\hat{1}$ is the join of the atoms. Let $c$ be a pre-upho coloring of $L$. Define the monoid $M$ by*

$$M = \left\langle s_1, \ldots, s_r \colon \begin{array}{c} c(\hat{0} = x_0 \lessdot x_1) c(x_1 \lessdot x_2) \cdots c(x_{k-1} \lessdot x_k = x_1 \vee y_1) = \\ c(\hat{0} = y_0 \lessdot y_1) c(y_1 \lessdot y_2) \cdots c(y_{k-1} \lessdot y_k = x_1 \vee y_1) \end{array} \right\rangle$$

*where the generators are the atoms $s_1, \ldots, s_r$ of $L$, and the relations correspond to all pairs of saturated chains $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = x_1 \vee y_1$, $\hat{0} = y_0 \lessdot y_1 \lessdot \cdots \lessdot y_k = x_1 \vee y_1$ from $\hat{0}$ to the join $x_1 \vee y_1$ of two atoms $x_1, y_1$ of $L$. Then $M$ is left cancellative, and every pair of elements in $M$ has a greatest common left divisor. Hence $\mathcal{L} := (M, \leq_L)$ is an upho meet semilattice, and there is a rank-preserving embedding of $L$ into $\mathcal{L}$.*

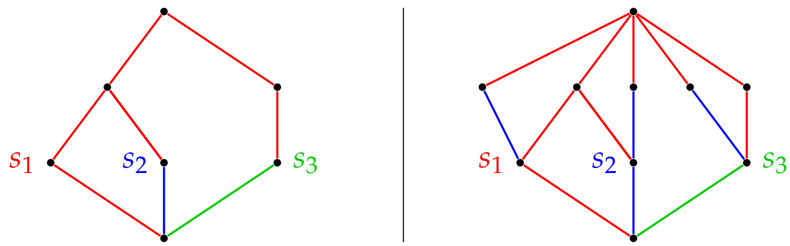


**Figure 3:** Inputting the coloring of the lattice $L$ on the left into Conjecture 28 yields an upho lattice $\mathcal{L}$ whose core, depicted on the right, is bigger than $L$.

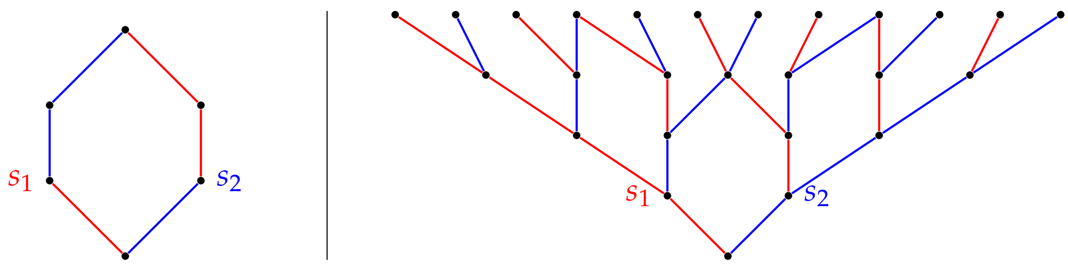


**Figure 4:** Inputting the coloring of the hexagon lattice $L$ on the left into Conjecture 28 yields the upho meet semilattice $\mathcal{L}$ on the right, which is not a lattice (c.f. [14, Figure 1]).

Verifying that a coloring of a finite lattice $L$ is pre-upho is clearly a finite check. Hence, Conjecture 28, if correct, would almost give an algorithm for listing all the colored upho lattices of which a given finite lattice $L$ is a core. However, there are a few deficiencies in this conjecture, as we now explain.

First of all, the conjecture only guarantees that there is a rank-preserving embedding of $L$ into $\mathcal{L}$. This means that $L$ sits inside the core of $\mathcal{L}$, but the core could potentially be bigger than just $L$. Indeed, Figure 3 shows an example where the core of the output $\mathcal{L}$ is strictly bigger than the input $L$. But this is not such a serious problem for our desired algorithm, as we can check that the core of any output $\mathcal{L}$ is really $L$ just by looking at a finite portion of $\mathcal{L}$.

Another issue with Conjecture 28 is that it only guarantees that $\mathcal{L}$ is a meet semilattice: it may fail to be a lattice because pairs of elements may fail to have upper bounds. Indeed, Figure 4 shows an example where the output $\mathcal{L}$ is not a lattice. This is a more serious problem, because checking that upper bounds exist is a priori an infinite check.

Finally, different monoids $M$ may lead to isomorphic upho lattices $\mathcal{L}$, so if we wanted our algorithm to list each upho lattice only once, we would have to check for isomorphisms, which again is a priori an infinite check.

To conclude, we note that the construction in Conjecture 28 feels spiritually similar to a known construction of Garside monoids from colored finite lattices discussed in [11]. Hence, techniques from Garside theory [4, 3] might be useful for proving this conjecture.

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