Homework 3 - Combinatorics 2

1. A plane partition is an infinite 2D-array $\pi = (\pi_{i,j})_{i=1,2,\dots}^{j=1,2,\dots}$ of nonnegative integers $\pi_{i,j} \in \mathbb{N}$ such that only finitely many entries are nonzero and the entries are weakly decreasing along rows and down columns in the sense that $\pi_{i,j} \geq \pi_{i',j'}$ if $i \leq i'$ and $j \leq j'$. The size $|\pi|$ of π is the sum of the entries: $|\pi| := \sum_{i,j\geq 1} \pi_{i,j}$. Prove that

$$\sum_{\pi \text{ a plane partition}} q^{|\pi|} = \prod_{i \ge 1} \frac{1}{(1 - q^i)^i}$$
 (1)

Hint: We proved the following product formula for *reverse* plane partitions of shape λ :

$$\sum_{\pi \in \text{RPP}(\lambda)} q^{|\pi|} = \prod_{u \in \lambda} \frac{1}{1 - q^{h(u)}} \tag{2}$$

From the hint, use know that rotating an RPP 180°, gives us the upper left corner of a plane partition.

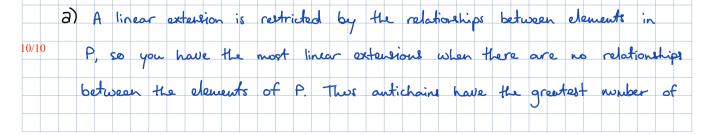
Now consider the formula for RPPs of shape λ : $\pi \in \mathbb{RPP}(x)$ $q^{|\pi|} = \prod_{i \in X} \frac{1}{q^{i}(x)}$.

For a square $n \times n$ RPP, there are i # boxes with book length i, i \(\in n \).

Notice then that for PPs, when you flip an RPP around, you get this $\forall i = x \in \mathbb{RP}(x)$.

I here are i, $\frac{1}{1-q^{i}}$ $s = x \in \mathbb{RP}(x)$ a plane partition $q^{|\pi|} = \prod_{i \geq 1} \left(\frac{1}{1-q^{i}} \right)^{i} = \prod_$

- 2. Recall that a linear extension of a (finite) poset P is a list p_1, \ldots, p_n of all its elements (each appearing once) where $p_i \leq p_j$ implies $i \leq j$. $\mathcal{L}(P)$ denotes the set of linear extensions of P.
 - (a) Among posets P with n elements, which has the greatest number $\#\mathcal{L}(P)$ of linear extensions? Which has the least?
 - (b) The dual P^* of a poset P is the poset with the same elements but the reverse order: $p \leq_P q \Leftrightarrow q \leq_{P^*} p$. Prove that $\#\mathcal{L}(P) = \#\mathcal{L}(P^*)$.
 - (c) The (disjoint) union $P \cup Q$ of two posets P and Q is the poset whose elements are the elements in the union of the two sets, where the order within P and within Q is the same, but all $p \in P$ are incomparable to all $q \in Q$. Give a formula for $\#\mathcal{L}(P \cup Q)$ in terms of $\#\mathcal{L}(P)$, $\#\mathcal{L}(Q)$, and n = #P and m = #Q.



linear extensions. Similarly, the least would be when each element can only be in one order for a linear extension. This is the case with chains Therefore chains the least number of linear actuations. Good! b) Define f: L(P) → L(P*) by f(E) = f(E,, E,,..., E,) = (E,,..., E, E,) for some linear extension E & L(P). This creates a new list with the some relationships as those in E, but reversed. Thus this new list is a linear extension of P*, and since f is reversible, # & (P) = # & (P*) C) P and Q are disjoint, so to form a linear extension of PUQ, we can just combine arbitrary extensions of P and Q. So now we have # 6 (P) . # 6 (a). the ways to merge the lists together without needing up the order of each list. To find this, let's suppose that we want to put $p \in \mathcal{S}(P)$ into $g \in \mathcal{S}(Q)$. Then we can either put each element in p either before an element of q or at the end. This is the same process that we used for stars and bars, so using the some logic, we get that there are ("+") ways to merge the fists tagether while maintaining order. Very good way of putting it ("merging") :. there are # & (P) . # & (a) . (n+m) linear extensions of Pua 3. Recall that f^{λ} denotes the number of Standard Young Tableaux of shape λ . Give a simple formula for f^{λ} in the case of a *hook* shaped partition $\lambda = (k, \overbrace{1, 1, \dots, 1})$ for $1 \le k \le n$. Standard Young Tableauxs are strictly increasing along both rows and 10/10 columns. So we only really have the option to choose in the first row, which gives us k-1 total choices to define the whole filling.

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And if n = 2k+1, then take perawtaions contains the sequences 1,2,..., k+1 and 2k+1, 2k,..., k+1. This results in min(lis(o), lds(o))=k, which is max. And thus this has been described.