

Howard Math 156: Calculus I Fall 2022
Instructor: Sam Hopkins (sam.hopkins@howard.edu)
— ('call me "Sam")

8/22

Logistics

Classes: MTWF, 2:10-3pm, ASB-B #105

Office hrs: Tue 1-2pm, Annex III - #220

Or by appointment (email me!)

Website: samuelhopkins.com/classes/156.html

Text: Calculus, Early Transcendentals by Stewart, 9e

Grading: 40% (in-person) quizzes

40% two (in-person) midterms

20% final exam

There will be 12 in-person quizzes taken on Tuesdays.

(About 20 mins, we will then go over them for rest of class).

Your lowest 2 scores will be dropped (so $\frac{10}{12}$ count).

The 2 midterms will happen in-class, also on Tuesdays.

The final will be during finals week

Beyond that, I may assign additional HW (not graded) and I expect you to **SHOW UP TO CLASS + PARTICIPATE!**

that means.. interrupt me by **ASKING QUESTIONS!**

and please say your names when you ask a question so I learn to put names to faces)

What is calculus about?

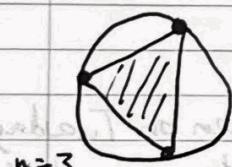
Calculus is different from the math you've seen.
It deals with change, with infinities (and infinitesimals)
and with limiting processes.

It's good to have a preview of all this new stuff.
Let's go over the book's introduction to calculus...

Area of a circle:

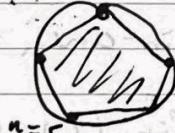
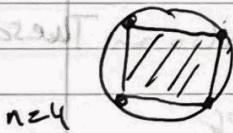
We all know that the area of a circle of radius R
is πR^2 , where $\pi = 3.14159\dots$ is a special number.

But how would you figure this out if you didn't know?



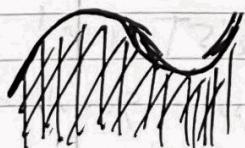
You could try to approximate the area
by using a simpler shape, like a regular triangle
whose area you already know how to compute.

But this clearly leaves some area out... so you
might consider instead regular 4-gon, 5-gon, ...

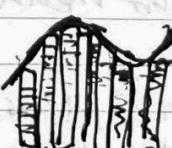


Each inscribed regular n -gon gives a better
and better approximation
to the area of the circle, and the true area can
be calculated by taking a limit as n goes to ∞ .

We won't study this exact problem, but we
will consider the area under a curve:

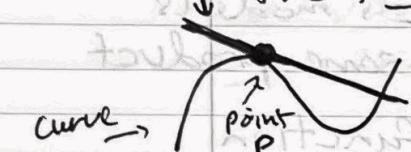


Can also be obtained
by a limit of simpler
shapes.



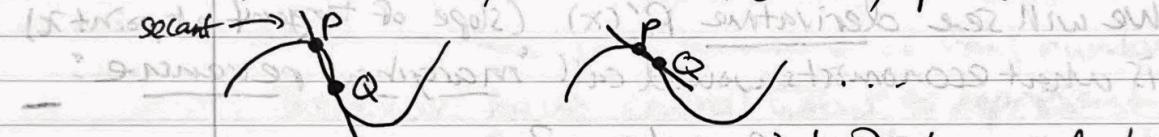
Put thin rectangles under the curve!

(good for 7 to 1.18) 2017 HW p18
Tangent to a curve: How would you find
the tangent line to a curve at a point?



The tangent is the line that
"just touches" the curve at that
point...

Calling the point P , can draw secant line through
Point P and Q , another nearby point on curve:



As we move the ~~nearby~~ point Q closer and closer to P ,
we get a better and better approx. of the tangent.
In the limit, the secant line becomes the tangent line.

Why care about tangents to curves? They tell us
about velocity and acceleration in physics
(and rates of change in sciences in general).

Also, allows us to approximate
whole curve : ("Newton's method" ... used by NASA!)



Big idea of calculus:

Even though the area problem and
the tangent line problem seem pretty different,
they are actually... the same problem
or more precisely... the opposite problems!

This semester, we will learn why (& how)!

8/24 Functions (§1.1 of text book)

functions are the basic thing we will study in calculus.

They are fundamental in all sciences as models

e.g. If we produce x units of some product

our revenue may be given by function

$$R(x) = p \cdot x \text{ where } p = \text{price of product}$$

(Very simple linear model, doesn't take into account costs)

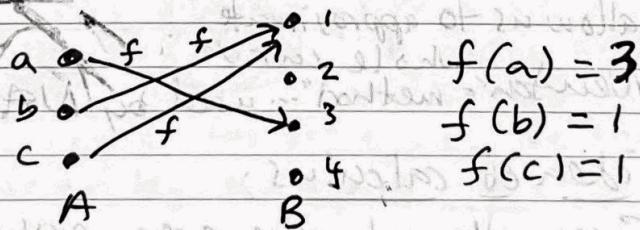
We will see derivative $R'(x)$ (slope of tangent at point x)
is what economists would call 'marginal revenue'

But what is a function?

formally, a function f between two sets A and B
is a relation between the elements of A and B

such that every element of A is related
to a unique element of B

e.g. $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$



The set A is called the domain of f and set B is called the codomain. The range of f is the set of all $f(x)$ for $x \in A$.

E.g. Range for f above is $\{1, 3\}$

The function is called one-to-one if every element in range is related to a unique $x \in A$.

E.g. example f above not one-to-one since $f(b) = 2$ and $f(c) = 2$.

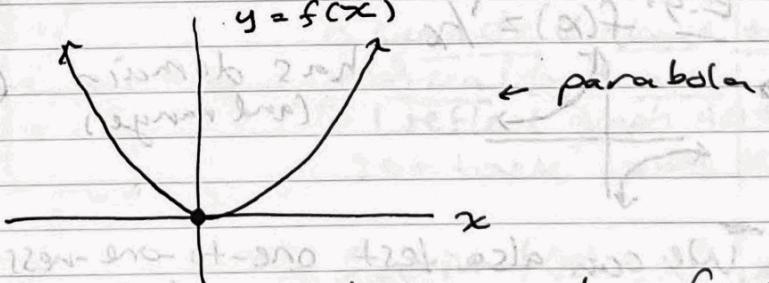
That is the formal mathematical definition of a function, but we will normally work just with functions f whose domain and range are subsets of \mathbb{R} .

Then we have several ways to represent such an f than an "arrow diagram" or chart (and we have to because there are ^{uncountably} _{infinite} # of numbers).

You are probably used to functions defined by an algebraic formula like

$$f(x) = x^2$$

which we can also represent by a graph



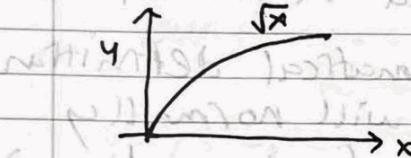
How do we know if a graph represents a function?

"Vertical line test": graph represents a function \Leftrightarrow each vertical line intersects ≤ 1 point

E.g. $x = y^2$ Not a function because vert. line $x=4$ intersects two points!

The domain of $f(x) = x^2$ is all of the real numbers, also denoted $(-\infty, \infty)$. The range is the nonnegative reals, also denoted $[0, \infty)$.

What about $f(x) = \sqrt{x}$?



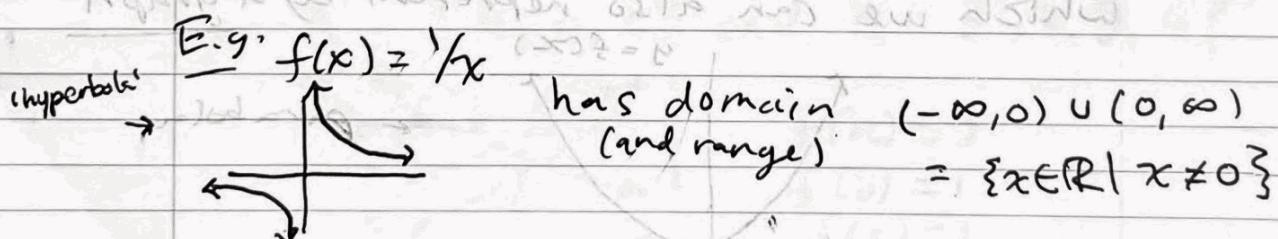
we mean positive square root
when we write this

The domain is $[0, \infty)$ and range is also $[0, \infty)$.

In general to find the domain of a function, you think about what values you're allowed to plug into it. With a square root, need nonneg. #'s.

E.g. domain of $\sqrt{x-1}$ is $\{x \in \mathbb{R} \mid x \geq 1\}$
 $= [1, \infty)$

If you have a denominator, it cannot be zero.



We can also test one-to-one-ness graphically using the "horizontal line" test. If a horizontal line intersects the graph of function f has every horizontal line intersect ≤ 1 point

E.g.

$f(x) = x^2$ is NOT one-to-one.

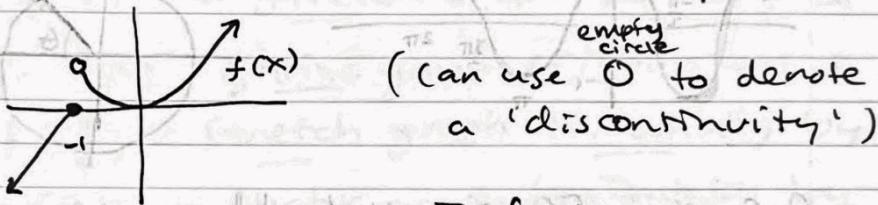
Q: what about $f(x) = x^3$?

8/26

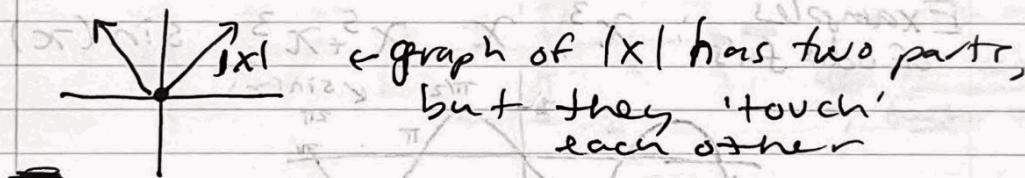
Not every function is determined by a single formula.
We can define a piecewise function like

$$f(x) = \begin{cases} x+1 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

The graph of $y = f(x)$ has two parts:

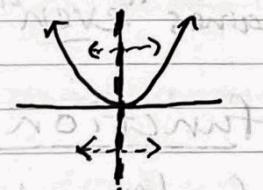


Another important piecewise function is
absolute value $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

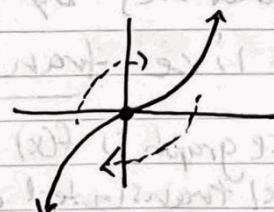


Symmetry of functions

The function $f(x) = x^2$



is symmetric about the vertical (y-) axis:
if I reflect graph across y-axis,
I get back same thing



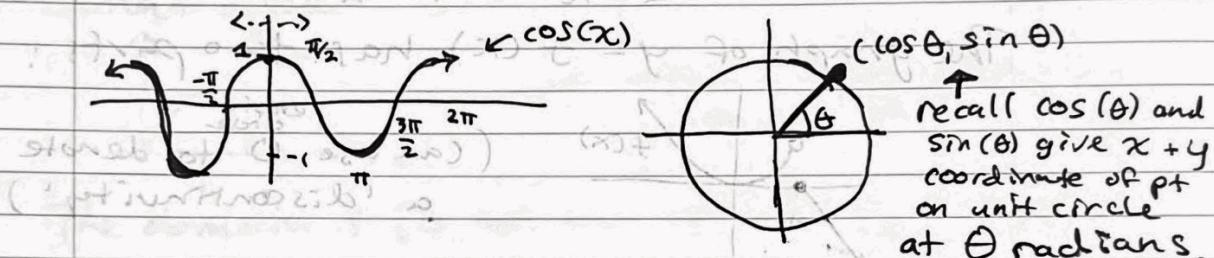
The function $f(x) = x^3$

is symmetric about (0,0):
if I rotate it 180° about (0,0)
then I get back same thing.

These two kinds of symmetry are called even and odd for functions

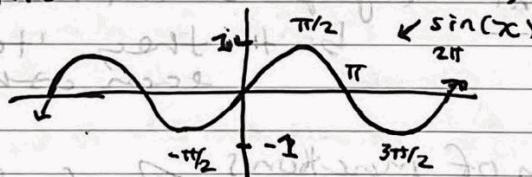
A function $f(x)$ is called even if $f(x) = f(-x)$ for all x .
Same as saying symmetric across y -axis.

Examples of even fn's: $x^2, x^2+1, x^4, |x|, \cos(x)$



A function $f(x)$ is called odd if $f(-x) = -f(x)$ for all x .
Same as saying 180° -rotationally symmetric about $(0,0)$.

Examples of odd fn's: $x^3, x, x^5 + x^3, \sin(x)$



Can you guess why we use names "even" and "odd"?

Transformations of functions § 1.3

Given $f(x)$ can make new functions by applying various transformations, like translations:

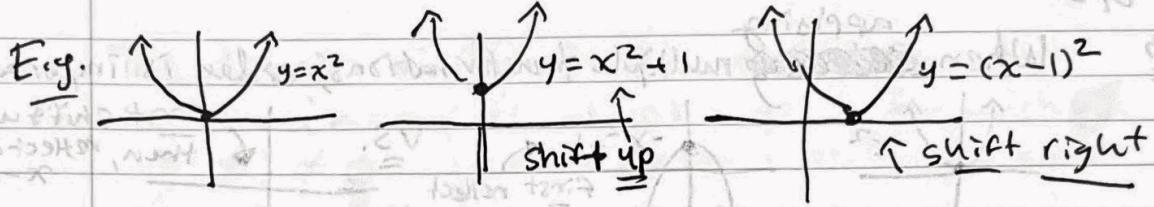
$y = f(x) + c$ - function whose graph is $f(x)$ translated up by c

$y = f(x) - c$ - graph is $f(x)$ translated down by c

$y = f(x - c)$ - graph is $f(x)$ translated right by c

$y = f(x + c)$ - graph is $f(x)$ translated left by c

(for $c > 0$)



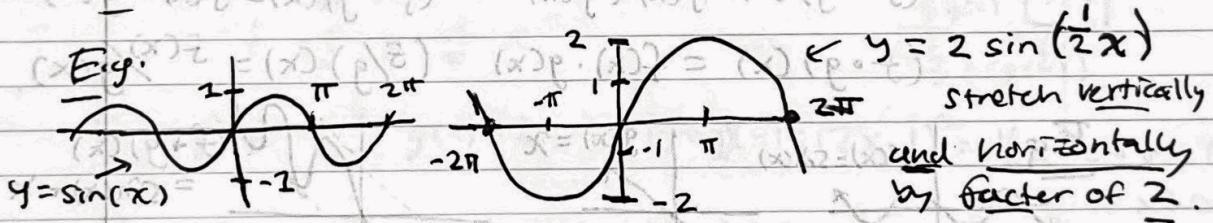
(call also stretch a function: for $c > 1$)

$y = c f(x)$ - stretch graph vertically by c factor of

$y = \frac{1}{c} f(x)$ - shrink graph vertically by c

$y = f(\frac{x}{c})$ - stretch graph horizontally by c

$y = f(c \cdot x)$ - shrink graph horizontally by c

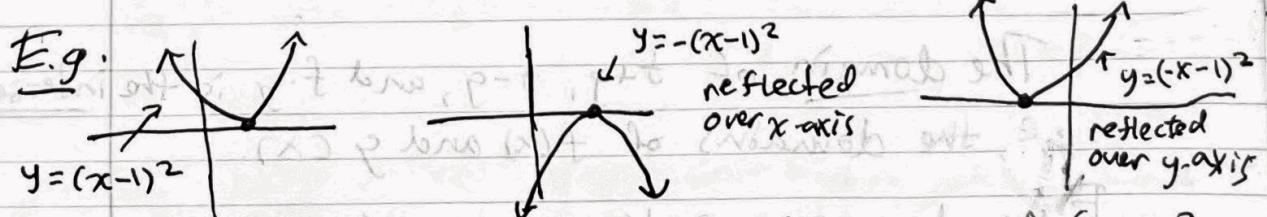


We see in this example how we can combine, multiple transformations!

One more geometric transformation: reflection

$y = -f(x)$ - reflect graph about x -axis

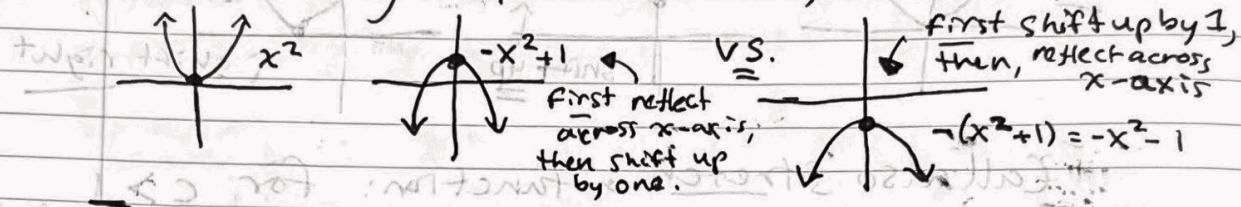
$y = f(-x)$ - reflect graph about y -axis



Q: What happens w/ reflections for even + odd fns?

8/29

3.1.3 When applying multiple transformations, order is important!

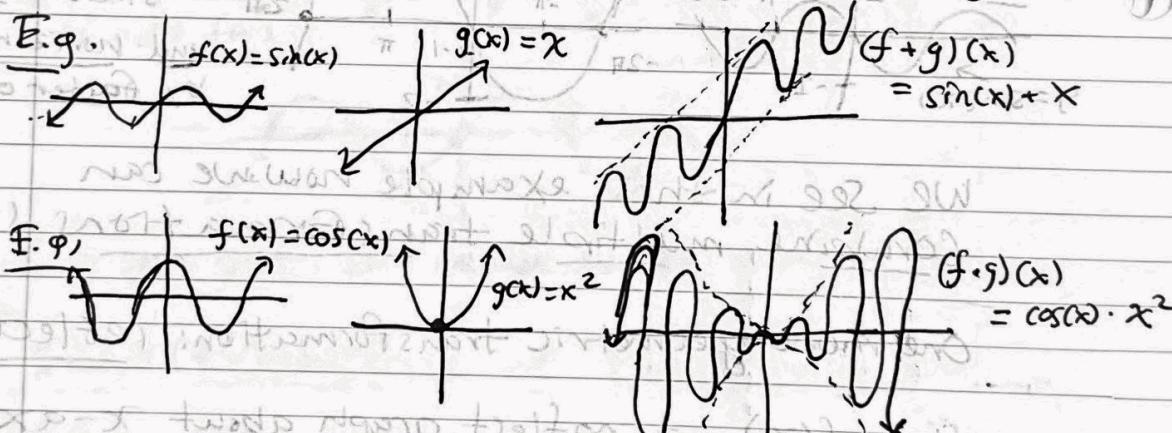


Another way to get new functions from old is by combining functions in different ways.

Def'n If f, g are two fun's, we define their sum, difference, product, and quotient by

$$(f+g)(x) = f(x) + g(x) \quad (f-g)(x) = f(x) - g(x)$$

$$(f \cdot g)(x) = f(x) \cdot g(x) \quad (f/g)(x) = \frac{f(x)}{g(x)}$$



E.g. $\tan(x) = \frac{\sin(x)}{\cos(x)}$ ← not always easy to graph combinations!

The domain of $f+g$, $f-g$, and $f \cdot g$ is the intersection of the domains of $f(x)$ and $g(x)$.

E.g. domain of $\sqrt{x} + \sqrt[3]{x}$ is $(0, \infty)$.

E.12 (pertaining to 2nd figures no errors etc.) 18/8

The domain of f/g is the intersection of the domain of $f(x)$ and set of all x for which $\underline{g(x) \neq 0}$ (so that we don't divide by zero).

E.g. domain of $\tan(x) = \{x \in \mathbb{R} : x \neq \frac{\pi}{2} + n\pi \text{ for some } n \in \mathbb{Z}\}$
since $\cos(\frac{\pi}{2} + n \cdot \pi) = 0$ for all $n \in \mathbb{Z}$.

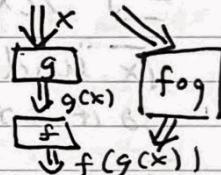
Another very important way to combine functions is composition:

Def'n If f and g are two functions, their composition $f \circ g$ is

$$f \circ g(x) = f(g(x))$$

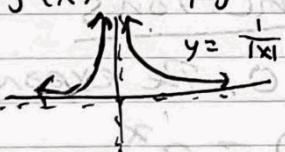
"Do g ~~first~~, then do f to that!"

" f of g of x "



E.g. $f(x) = x^2$, $g(x) = 2x - 1$, $(f \circ g)(x) = (2x - 1)^2 = 4x^2 - 4x + 1$

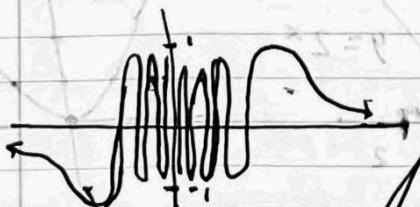
E.g. $f(x) = \frac{1}{x}$, $g(x) = |x|$, $(f \circ g)(x) = \frac{1}{|x|} = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ -\frac{1}{x} & \text{if } x < 0 \end{cases}$



note: $\frac{1}{|x|}$ is even since $\frac{1}{|-x|} = \frac{1}{|x|}$ ✓
and $x=0$ not in domain.

E.g. $f(x) = \sin(x)$, $g(x) = \frac{1}{x}$, $(f \circ g)(x) = \sin(\frac{1}{x})$

What does $\sin(\frac{1}{x})$ look like? As $x \rightarrow \infty$, $\frac{1}{x}$ barely changes, so $\sin(\frac{1}{x})$ stops oscillating. As $x \rightarrow 0$ from the right, $\frac{1}{x}$ changes a lot, so $\sin(\frac{1}{x})$ oscillates like crazy:



Very hard to draw accurately!

and note $x=0$ not in domain!

~~domain of g is $x \neq 0$~~
~~so $x=0$ is not in domain~~
~~but $x=0$ is in domain of f~~
~~so $x=0$ is in domain of f ∘ g~~

8/3 | Little more on compositions of functions: § 1.3

Domain of $f \circ g$ is set of all x in domain of g such that $g(x)$ is in the domain of f .

E.g. $f(x) = \sqrt{x+1}$ and $g(x) = \sqrt{x}$ so that

$$(f \circ g)(x) = \sqrt{\sqrt{x} + 1} \text{ then domain } (\sqrt{\sqrt{x} + 1}) = [-\infty, -1] \cup (0, \infty)$$

If $(f \circ g)(x) = x$ then we say f is the inverse function of g . f "undoes" what g does!

E.g. $f(x) = \sqrt{x}$, $g(x) = x^2$, $(f \circ g)(x) = \sqrt{x^2} = x$

\sqrt{x} "undoes" x^2 (we'll be a b.t more careful so it is the inverse about domain issues later)

Inverses will allow us to define the logarithm from the exponential, which brings us to ...

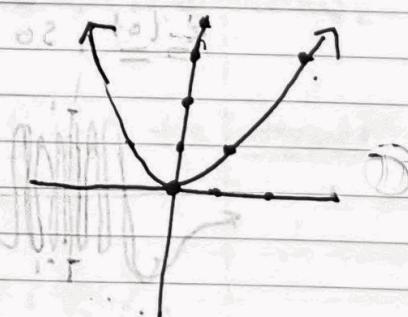
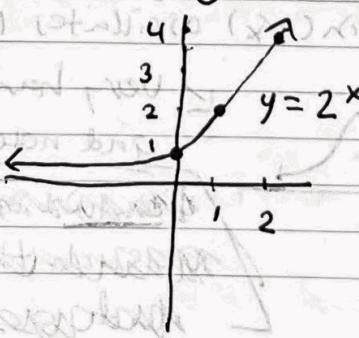
§ 1.4 Exponential functions

Def'n Fix real number $a > 0$. The exponential function with base a is $f(x) = a^x$.

Do not confuse a^x with power function x^a .

E.g.: $f(x) = 2^x$ vs. $g(x) = x^2$

x	$f(x)$	$g(x)$
0	1	0
1	2	1
2	4	4
3	8	9
4	16	16



At first, x^2 grows more quickly than 2^x , but this is misleading: eventually, 2^x grows much, much faster than x^2 !

In fact, any exponential a^x for $a > 1$ (eventually) grows much, much faster than any polynomial.

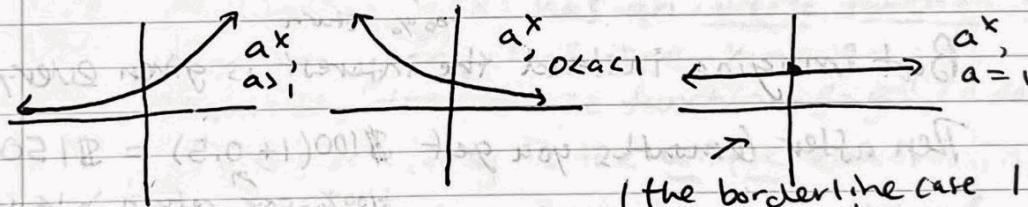
Recall that a polynomial is a function

$$f(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \dots + a_1 \cdot x + a_0$$

that is some linear combination of power functions.

We will prove this assertion later (using calculus!).

For $a > 1$, a^x represents exponential growth,
for $0 < a < 1$, a^x represents exponential decay

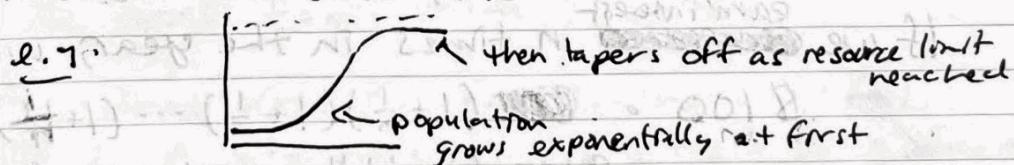


(the borderline case $a^x = 1$
is degenerate.)

Sometimes we also consider

Ca^x for fixed C an exponential function.

In sciences (e.g. biology) often see mix of exponential growth and decay!



Remember: fixed exponent (x^a) \Rightarrow power function
fixed base (a^x) \Rightarrow exponential function.

(So l.e.g. x^x is neither of these: base + exponent are both variables...)