

# Math 210 (Modern Algebra I), HW# 4,

Fall 2024; Instructor: Sam Hopkins; Due: Wednesday, October 16th

Recall that all rings  $R$  are assumed to be unital (i.e., have a 1) but not necessarily commutative.

1. A ring  $R$  which satisfies  $a^2 = a$  for all  $a \in R$  is called a *Boolean ring*.
  - (a) Prove that a Boolean ring  $R$  is commutative, and satisfies  $a + a = 0$  for all  $a \in R$ .
  - (b) Let  $U$  be a set and let  $\mathcal{P}(U)$  denote the set of all subsets of  $U$ . For  $A, B \in \mathcal{P}(U)$ , define  $A + B = (A \setminus B) \cup (B \setminus A)$  and  $AB = A \cap B$ . Prove that this gives  $\mathcal{P}(U)$  the structure of a Boolean ring.
2. Let  $G$  be a finite group and  $R = \mathbb{Q}[G]$  be the group algebra of  $G$  over the rational numbers  $\mathbb{Q}$ . Consider the element  $x = \frac{1}{|G|} \sum_{g \in G} g \in R$ . Prove that  $x$  is an *idempotent*, i.e., that  $x^2 = x$ .
3. Recall that the *center* of a (noncommutative) ring  $R$  is  $Z(R) = \{x \in R : xy = yx \text{ for all } y \in R\}$ .

Now let  $R$  be a commutative ring and consider the ring  $M_n(R)$  of  $n \times n$  matrices with entries in  $R$ . What is the center  $Z(M_n(R))$  of this matrix ring?

4. Let  $\mathbb{H}$  denote the quaternions. Recall that an element  $p \in \mathbb{H}$  can be represented as a formal sum  $p = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ , with  $a, b, c, d \in \mathbb{R}$  real numbers, and with  $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ . Define the *norm* of such an element  $p$  to be  $|p| = \sqrt{a^2 + b^2 + c^2 + d^2}$ . Prove that this norm is multiplicative, i.e., that  $|pq| = |p||q|$  for  $p, q \in \mathbb{H}$ .
5. Let  $R$  be a commutative ring.
  - (a) Let  $I$  be an ideal of  $R$  and define its *radical* to be  $\text{Rad}(I) = \{x \in R : x^n \in I \text{ for some } n \geq 0\}$ . Prove that  $\text{Rad}(I)$  is also an ideal of  $R$ . **Hint:** feel free to use the Binomial Theorem.
  - (b) Recall that  $x \in R$  is *nilpotent* if there is some  $n \geq 0$  such that  $x^n = 0$ . Prove that the collection of all nilpotent elements is an ideal of  $R$ . **Hint:** you can use the previous part.