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Standard Young Tableaux

The product formula for the g.f. of r.p.p.'s of shape λ is nice, but what about counting a finite combinatorial set?

DEF'N A Standard Young Tableau of shape λ is a filling of boxes of λ w/ the numbers $1, 2, \ldots, n := |\lambda|$, each appearing exactly once, so that numbers are strictly increasing along rows & down columns.

eg. The 2 SYT's of shape $\lambda = (2,2)$ are

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

Let $f^\lambda := \#$ SYT's of shape λ. Note that also
$$f^\lambda = [x_1 x_2 \cdots x_n] \, S_\lambda(x_1, x_2, \cdots) \quad \leftarrow \text{"coefficient of } x_1 \cdots x_n \text{ in Schur fn."}$$
since an SYT is the same as a semi standard tableau of content $= (1,1,1,\cdots,1)$.

Thm ("Hook Length Formula", Frame-Robinson-Thrall, 1954)
$$f^\lambda = n! \cdot \prod_{u \in \lambda} \frac{1}{h(u)} \quad \text{for any partition } \lambda \vdash n$$

eg. w/ $\lambda = (2,2)$, hook lengths are: $\begin{array}{|c|c|} \hline 3 & 2 \\ \hline 2 & 1 \\ \hline \end{array}$
So that $f^\lambda = 4!/3 \cdot 2 \cdot 2 \cdot 1 = 2$ ✓

Note that $n! = \#$ ways to fill boxes of λ w/ numbers $1, 2, \cdots, n$ (each used once) w/out any requirement on order of #'s

So ... HLF has a **probabilistic** interpretation:

It says that the probability a random filling is an SYT is exactly $\prod_{u \in \lambda} \frac{1}{h(u)}$.

Bogus probabilistic proof of HLF:

- A filling is an SYT iff each entry is smallest among #'s in its hook.
- In a random filling, the probability that box u has entry smallest in its hook is $1/h(u)$
- So prob. random filling is SYT $= \prod_{u \in \lambda} 1/h(u)$

have to put smallest # here
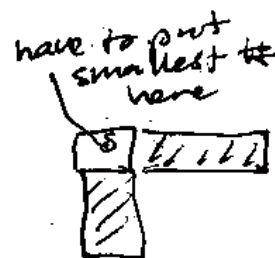
PROBLEM: 1st two bullets are correct, but can only take products for probabilities of **independent** <u>events</u>, and these events are very much <u>not independent</u>!

There is a valid probabilistic proof of HLF based on construction of a random SYT via "<u>hook walk</u>"

- Choose a random box u in λ to start at,
- unless we're at a SE border box, move to another random box in the hook of u
- when we hit a SE border box, put the number n there.

Then we repeat w/ where to put $n-1$, $n-2$, ... etc. down to 1.

e.g. In
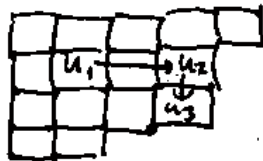
we might start at u_1, then go to u_2, then go to u_3, and put $n = 15$ there

Main thing to show is that this procedure really produces each SYT w/ equal probability $\left(= \frac{\prod h(u)}{n!}\right)$

See Sagan §7.3 for proof of this... We'll give different proof! ...

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Instead, we will deduce HLF for SYTs from g.f. of r.p.p.'s!
Actually, it will be easiest to explain this deduction in the
more general setting of (finite) posets! _{reflexive, antisymmetric, transitive}
Recall a poset is a set with a partial order. We
draw posets using Hasse diagrams:

$P =$ [Hasse diagram: a at bottom, b and c on sides, d at top] $a < b, \ a < c, \ b < d, \ c < d$
                                                    (implied: $a < d$)

DEF'N A linear extension of a poset P is a list
$P_1, P_2, \dots, P_n$ of all elements s.t. $P_i \leq P_j \implies i \leq j$.
We let $\mathcal{L}(P) := \{$lin. ext.'s of $P\}$.

e.g. w/ P as above, $\mathcal{L}(P) = \{abcd, \ acbd\}$

DEF'N We say P is naturally labeled if elt's are $P = \{1, 2, \dots, n\}$
and have $i \leq_P j \implies i \leq j$ (as numbers). In this
case, we treat $\mathcal{L}(P) \subseteq S_n$ as a set of permutations.
e.g. $P = $ [Hasse diagram: 1 at bottom, 2 and 3 on sides, 4 at top] is nat. labeled and $\mathcal{L}(P) = \{1234, 1324\}$

Rmk: Note that the identity $12 \cdots n$ is always in $\mathcal{L}(P)$.

Recall that a descent of a permutation $\sigma = \sigma_1, \sigma_2, \dots, \sigma_n$
is a position $1 \leq i \leq n-1$ s.t. $\sigma_i > \sigma_{i+1}$. Set $D(\sigma) = \{$descents of $\sigma\}$
and recall that the major index of σ is
$$maj(\sigma) := \sum_{i \in D(\sigma)} i.$$

e.g. $D(1234) = \varnothing$ and $D(1324) = \{2\}$ so that
$$\sum_{\sigma \in \mathcal{L}(P)} q^{maj(\sigma)} = 1 + q^2, \text{ where } P \text{ is nat. labeled}$$
poset as above.

<u>DEF^N</u> A <u>P-partition</u> (for poset P) is a function $\pi: P \to \mathbb{N}$ that is <u>order-reversing</u>: i.e., $p \leq q \Rightarrow \pi(p) \geq \pi(q)$.

We use $|\pi| := \sum_{p \in P} \pi(p)$ (like w/ the r.p.p.'s).

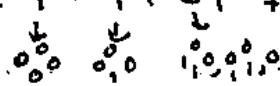e.g. One P-partition is ▢◇▢ ▢ = value of π (as opposed to label)

Thm (G.f. for P-partitions)

For P naturally labeled, with $\#P = n$,
$$\sum_{\pi \text{ P-partition}} q^{|\pi|} = \frac{\sum_{\sigma \in \mathcal{L}(P)} q^{maj(\sigma)}}{(1-q)(1-q^2)\cdots(1-q^n)}$$

<u>e.g.</u> w/ P as before, g.f. is $= \frac{1+q^2}{(1-q)(1-q^2)(1-q^3)(1-q^4)} = 1 + q + 3q^2 + 4q^3 + 7q^4 \cdots$

<u>Rmk:</u> w/ $P = \overset{\bullet}{1} \overset{\bullet}{2} \cdots \overset{\bullet}{n}$ an n-element <u>antichain</u>

thm says $\sum_{\substack{\text{all } f:[n]\to\mathbb{N}}} q^{|f|} = \frac{\sum_{\text{all } \sigma \in S_n} q^{maj(\sigma)}}{(1-q)(1-q^2)\cdots(1-q^n)}$

$\frac{1}{(1-q)^n}$ $\Longleftrightarrow$ $\sum_{\sigma \in S_n} q^{maj(\sigma)} = [n]_q!$,

something we proved last semester (maj and inv ~~equidistributed~~ <u>same distr.</u>).

In fact, proof we give will be same as last semester

<u>Cor</u> for any poset P,
$$\#\mathcal{L}(P) = \lim_{q \to 1} \left(\sum_{\pi \text{ P-partition}} q^{|\pi|} \right) \cdot (1-q)(1-q^2)\cdots(1-q^n)$$

<u>Pf:</u> mult. both sides by $(1-q)(1-q^2)\cdots(1-q^n)$ in thm above, and take limit $q \to 1$ (or just plug in $q=1$) ∴ ▢

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Before we prove the g.f. for P-partitions thm, let's explain how this corollary implies the HLF for SYTs. Basically we just need to match up the various terms. To any partition $\lambda \vdash n$, associate poset P_λ ($\#P_\lambda = n$) where elt's are boxes, and $u \geq v \iff u$ northwest of v.

e.g. $\lambda = \begin{array}{c}\boxed{}\end{array} \iff P_\lambda = $ [poset diagram with vertices a, b, c, d, e]

With this construction, r.p.p.'s of sh $= \lambda$ $=$ P_λ-partitions and $\exists$ bij. between SYTs of sh $= \lambda$ and $\mathcal{L}(P_\lambda)$:

$$T \longmapsto \text{box w/ } n, \text{ box w/ } n-1, \cdots \text{ box w/ } 1$$

e.g. $\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\\hline 3 & 5 \\\cline{1-2}\end{array} \longmapsto d, e, b, c, a \in \mathcal{L}(P_\lambda)$

So by cor have for any $\lambda \vdash n$ that

$$f^\lambda = \#SYTs \text{ of sh} = \lambda = \#\mathcal{L}(P_\lambda) = \lim_{q \to 1} \frac{(1-q)(1-q^2)\cdots(1-q^n) \cdot}{\sum_{\sigma \text{ r.p.p. } \partial f \text{ sh} = \lambda} q^{|\pi|}}$$

$$= \lim_{q \to 1} \frac{(1-q)(1-q^2)\cdots(1-q^n)}{\prod_{u \in \lambda}(1-q^{h(u)})} = n! \cdot \prod_{u \in \lambda} \frac{1}{h(u)}$$

g.f. of r.p.p.'s a.k.a. Hillman–Grassl

L'Hôpital's rule   Proving HLF!

Rmk By choosing a particular natural labeling of P_λ, can also obtain a q-analog of the HLF for SYTs this way. giving maj-g.f. of tableaux ...

So now to finish everything, need to prove P-partition g.f.:

$\underline{Pf}$ (of P-partition g.f. in terms of $\mathcal{L}(P)$):

The idea (which we saw last semester!) is to break $\{$ P-partitions $\}$ into pieces corresponding to lin. ext.'s.

$\underline{Lemma}$ Every $f: [n] \to \mathbb{N}$ has a unique $\sigma \in S_n$ such that f is $\underline{\sigma\text{-compatible}}$ in sense that
- $f_{\sigma_1} \geq f_{\sigma_2} \geq \cdots \geq f_{\sigma_n}$
- if $i \in D(\sigma)$ then $f_{\sigma_i} > f_{\sigma_{i+1}}$

This f is a P-partition $\iff \sigma \in \mathcal{L}(P)$.

$\underline{Pf:}$ Write $f_{\sigma_1} = f_{\sigma_2} = \cdots = f_{\sigma_a} > f_{\sigma_{a+1}} = \cdots = f_{\sigma_b} > f_{\sigma_{b+1}} = \cdots$

So that $\sigma_1 < \sigma_2 < \cdots < \sigma_a$ and $\sigma_{a+1} < \cdots < \sigma_b$ and $\ldots$

e.g. $f = (\overset{1}{2}, \overset{2}{0}, \overset{3}{5}, \overset{4}{0}, \overset{5}{3}, \overset{6}{3}, \overset{7}{2}, \overset{8}{0})$ has $f_3 \geq f_5 = f_6 > f_1 = f_7 > f_2 = f_4 = f_8$ [with numbers $5\ 3\ 3\ 2\ 2\ 0\ 0\ 0$]

So $\sigma = 3, 5, 6, 1, 7, 2, 4, 8$ is unique perm. f is compatible with. The statement about P-partition $\iff$ lin. ext. is clear. $\blacksquare$

Thus,
$$\sum_{\pi \text{ P-partition}} q^{|\pi|} = \sum_{\sigma \in \mathcal{L}(P)} \sum_{\substack{f \text{ } \sigma\text{-compatible} \\ \text{fun. } [n] \to \mathbb{N}}} q^{|f|}$$

$$= \sum_{\sigma \in \mathcal{L}(P)} \sum_{\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)} q^{maj(\sigma) + |\lambda|}$$

$$= \sum_{\sigma \in \mathcal{L}(P)} q^{maj(\sigma)} \cdot \sum_{\lambda : \ell(\lambda) \leq n} q^{|\lambda|}$$

$$= \sum_{\sigma \in \mathcal{L}(P)} q^{maj(\sigma)} \cdot \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)} \checkmark$$

subtract off the smallest σ-compatible $f_0 : [n] \to \mathbb{N}$

[with numbers $3\ 5\ 6\ 1\ 7\ 2\ 4\ 8$]
$(5, 3, 3, 2, 2, 0, 0, 0) = f$
$- (2, 2, 2, 1, 1, 0, 0, 0) = f_0$
$\overline{\quad(3, 1, 1, 1, 1, 0, 0, 0) = \lambda}$

$\underline{NOTE:}$ $\boxed{|f_0| = maj(\sigma)}$ since $i \in D(\sigma)$ $\Rightarrow$ have to increase by one values in 1st i spots to get a strict decrease