UPHO LATTICES

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Dedicated to Richard Stanley, in celebration of his 80 years of life

Abstract. A poset is called upper homogeneous, or "upho," if every principal order filter of the poset is isomorphic to the whole poset. We study (finite type $\mathbb{N}$-graded) upho lattices, with an eye towards their classification.

Any upho lattice has associated to it a finite graded lattice called its core, which determines its rank generating function. We investigate which finite graded lattices arise as cores of upho lattices, providing both positive and negative results. On the one hand, we show that many well-studied finite lattices do arise as cores, and we present combinatorial and algebraic constructions of the upho lattices into which they embed. On the other hand, we show there are obstructions which prevent many finite lattices from being cores.

We also investigate some important subvarieties of lattices: namely, distributive and modular lattices. It is easy to show that the only upho distributive lattices are $\mathbb{N}^n$. Upho modular lattices are more interesting. Stanley observed that the poset of full rank submodules of a free module over a discrete valuation ring with finite residue field gives an upho modular lattice. He furthermore conjectured that essentially all upho modular lattices are of this form. We produce an exceptional example of an upho modular lattice whose core is a non-Desarguesian finite projective plane, which means that Stanley's conjecture must be modified.

1. Introduction

Symmetry is a fundamental theme in mathematics. A close cousin of symmetry is self-similarity, where a part resembles the whole. In this paper, we study certain partially ordered sets that are self-similar in a precise sense. Namely, a poset is called *upper homogeneous*, or "*upho*," if every principal order filter of the poset is isomorphic to the whole poset. In other words, a poset $\mathcal{P}$ is upho if, looking up from each element $p \in \mathcal{P}$, we see another copy of $\mathcal{P}$. Upho posets were introduced recently by Stanley [17, 19]. We believe they are a natural and rich class of posets which deserve further attention.

Upho posets are infinite. In order to be able to apply the tools of enumerative and algebraic combinatorics, we need to impose some finiteness condition on the posets we consider. Thus, we restrict our attention to finite type $\mathbb{N}$-graded posets. These are the infinite posets $\mathcal{P}$ that possess a rank function $\rho\colon \mathcal{P} \to \mathbb{N}$ for which we can form the rank generating function

$$F(\mathcal{P};x) := \sum_{p \in \mathcal{P}} x^{\rho(p)}.$$

Henceforth, upho posets are assumed finite type $\mathbb{N}$-graded unless otherwise specified.

The main problem we pursue in this paper is the following.

Problem 1.1. Classify upho lattices.

Problem 1.1 is likely a hard problem, perhaps even impossible. But let us explain why there is some hope of making progress on this problem. It was shown by Gao et al. [6] that there are uncountably many different rank generating functions of (finite type $\mathbb{N}$-graded) upho posets. This prevents us from being able to say much about upho posets in general. However, the situation is different for *lattices*: in [9] we showed that the rank generating function of a upho lattice is the inverse of a polynomial with integer coefficients.

More precisely, we made the following observation about the rank generating function of an upho lattice. Let $\mathcal{L}$ be an upho lattice, and let $L := [\hat{0}, s_1 \vee \cdots \vee s_r] \subseteq \mathcal{L}$ denote the interval in $\mathcal{L}$ from its minimum $\hat{0}$ to the join of its atoms $s_1, \ldots, s_r$. We refer to the finite graded lattice L as the *core* of the upho lattice $\mathcal{L}$. We showed in [9] that

$$F(\mathcal{L}; x) = \chi^*(L; x)^{-1}, \tag{1.1}$$

where $\chi^*(L; x) = \sum_{p \in L} \mu(\hat{0}, p) x^{\rho(p)}$ is the (reciprocal) characteristic polynomial of L. In this way, the core of an upho lattice determines its rank generating function.[1]

The core does not determine the upho lattice completely. That is, there are different upho lattices with the same core. Nevertheless, to resolve Problem 1.1 we would certainly need to answer the following question.

Question 1.2. Which finite graded lattices are cores of upho lattices?

Question 1.2 can be thought of as a kind of tiling problem: our goal is to tile an infinite, fractal lattice $\mathcal{L}$ using copies of some fixed finite lattice L, or show that no such tiling is possible. In addressing Question 1.2 here, we provide both positive and negative results.

On the positive side, we show that many well-studied families of finite graded lattices are cores of upho lattices. Our first major result is the following.

Theorem 1.3. *Any member of a uniform sequence of supersolvable geometric lattices is the core of some upho lattice.*

Supersolvable lattices were introduced by Stanley in [14], and uniform sequences of lattices were introduced by Dowling in [4]. These two notions represent two different ways that a finite lattice can have a recursive structure. Examples of uniform sequences of supersolvable geometric lattices include:

- the finite Boolean lattices B_n, i.e., the lattices of subsets of $\{1, 2, \ldots, n\}$;
- the q-analogues $B_n(q)$ of B_n, i.e., the lattices of $\mathbb{F}_q$-subspaces of $\mathbb{F}_q^n$;
- the partition lattices Π_n, i.e., the lattices of set partitions of $\{1, 2, \ldots, n\}$;
- the Type B partition lattices Π_n^B, i.e., the intersection lattices of the Type B_n Coxeter hyperplane arrangements;
- (generalizing the previous two items) the "Dowling lattices" [5, 4] $Q_n(G)$ associated to any finite group G.

[1] In fact, since the flag f-vector of any upho poset is determined by its rank generating function (see [19, §3]), the core of an upho lattice determines its entire flag f-vector.

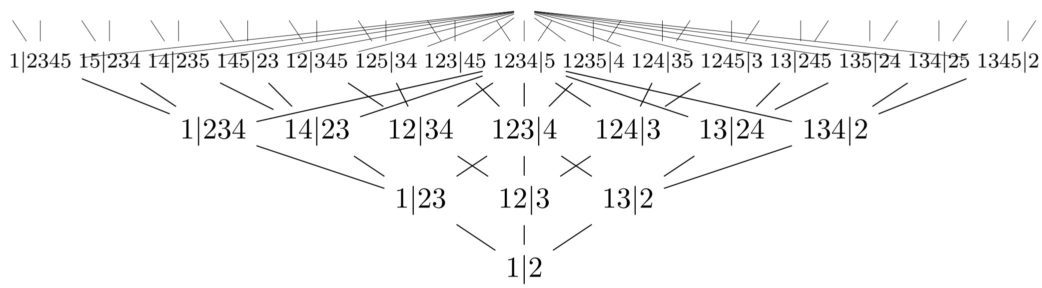

FIGURE 1. Partitions of sets of the form $\{1, 2, \ldots, n\}$ into 2 blocks, ordered by refinement. This is an upho lattice with core Π_3.

Hence, these are all cores of upho lattices. We discuss these examples in detail, providing explicit descriptions of the upho lattices for which they are cores. Figure 1 depicts one upho lattice produced via this construction.

In addition to combinatorial constructions, we also explore algebraic constructions of upho lattices. Monoids provide one algebraic source of upho lattices, as the following lemma explains.

Lemma 1.4. *(c.f.* [6, Lemma 5.1]*) Let M be a finitely generated monoid whose defining relations are homogeneous. If M is left cancellative and every pair of elements in M have a least common right multiple, then $(M, \leq_L)$ is an upho lattice, where $\leq_L$ denotes the partial order of left divisibility.*

A class of monoids satisfying the conditions of Lemma 1.4 are the (homogeneous) Garside monoids [2]. The core of a Garside monoid consists of its simple elements. Examples of lattices of simple elements in Garside monoids include:

- the weak order of a finite Coxeter group W;
- the noncrossing partition lattice of a finite Coxeter group W.

Hence, these are also cores of upho lattices. We review these examples coming from Garside and Coxeter theory in detail.

On the negative side, we show that there are various obstructions which prevent arbitrary finite graded lattices from being realized as cores of upho lattices. There are restrictions on the characteristic polynomial of the lattice coming from the equation (1.1). There are also some structural obstructions, requiring the lattice to be partly self-similar. These obstructions allow us to show, for instance, that the following plausible candidates cannot in fact be realized as cores:

- the face lattice of the n-dimensional cross polytope, for $n \geq 3$;
- the bond lattice of the cycle graph C_n, for $n \geq 4$;
- (generalizing the previous item) the lattice of flats of the uniform matroid $U(n, k)$, for $2 < k < n$.

The upshot is that Question 1.2 is quite subtle: it can be difficult to recognize when a given finite graded lattice is the core of an upho lattice. Many well-behaved finite lattices are cores of upho lattices, but many too are not.

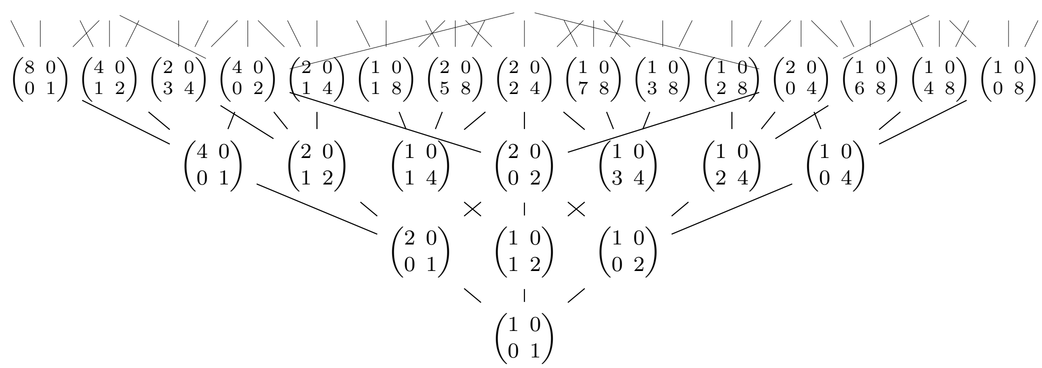

FIGURE 2. Subgroups of $\mathbb{Z}^2$ of index a power of 2, ordered by reverse inclusion. This is an upho modular lattice with core $B_2(2)$.

In the absence of a definitive resolution of Problem 1.1, we might instead try to classify subvarieties of upho lattices. Two of the most important subvarities of lattices are the distributive lattices and the modular lattices. Upho distributive lattices are not very interesting. Their classification, which we state in the following theorem, is a simple consequence of the representation theorem for locally finite distributive lattices (see [16, Proposition 3.4.3]).

Theorem 1.5. *The only upho distributive lattices are* $\mathbb{N}^n$*.*

Upho modular lattices, on the other hand, are quite interesting. Stanley observed the following source of upho modular lattices, which we review in detail.

Theorem 1.6 (Stanley)**.** *Let R be a discrete valuation ring (DVR) with residue field $\mathbb{F}_q$. Then the poset of finite colength R-submodules of R^n ordered by reverse inclusion is an upho modular lattice.*

Note that the core of $\mathbb{N}^n$ is B_n, while the core of the upho modular lattice of finite colength submodules of R^n, for R a DVR with residue field $\mathbb{F}_q$, is $B_n(q)$. Figure 2 depicts an upho modular lattice of this form. Stanley conjectured, moreover, that these examples give essentially all the upho modular lattices (see [6, Conjecture 1.1]). We disprove Stanley's conjecture by producing the following exceptional example of an upho modular lattice, which cannot come from a DVR because its core is different from $B_n(q)$.

Theorem 1.7. *There is an upho modular lattice whose core is (the rank three modular lattice associated to) a non-Desarguesian finite projective plane.*

Specifically, following [7], we explain a way to produce an upho modular lattice from any (sufficiently symmetric) affine building. Then we appeal to [10], where a $\widetilde{A}_2$-building whose residue planes are all non-Desarguesian projective planes is constructed. These exceptional examples can only exist in rank three, so it remains possible that Stanley's conjecture on upho modular lattices is true in higher rank.

To conclude this introduction, we remark that the following questions, which we do not pursue here, are also naturally suggested by our work.

Question 1.8.

(1) For a finite graded lattice L, let $\kappa(L)$ denote the cardinality of the collection of (isomorphism classes of) upho lattices $\mathcal{L}$ with core L. How does $\kappa(L)$ behave? For example, is $\kappa(L)$: finite for each L; infinite for some L, but always countable; or uncountably infinite for some L?
(2) For a finite modular lattice L, let $\kappa_{\mathrm{mod}}(L)$ denote the cardinality of the collection of upho modular lattices $\mathcal{L}$ with core L. How does $\kappa_{\mathrm{mod}}(L)$ behave?

In essence, part (1) asks how big is the difference between answering Question 1.2 and resolving Problem 1.1, and part (2) asks the same but for modular lattices.

The rest of the paper is structured as follows. In Section 2, we go over some definitions and preliminary results. In Section 3, we construct upho lattices from uniform sequences of supersolvable lattices. In Section 4, we explain how monoids give rise to upho lattices. In Section 5, we discuss obstructions to realizing a finite graded lattice as the core of an upho lattice. Finally, in Section 6, we explore distributive and modular upho lattices.

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2. Preliminaries

In this section, we review some basics regarding posets and upho posets. We generally stick to standard notation for posets, as laid out of instance in [16, §3]. We use $\mathbb{N} := \{0, 1, \ldots\}$ to denote the natural numbers, $\mathbb{Z}$ to denote the integers, $\mathbb{Q}$ the rationals, and $\mathbb{R}$ the reals.

2.1. Poset basics.

Let $P = (P, \leq)$ be a poset. When working with multiple posets we write $\leq_P$ for clarity. We use standard conventions like writing $y \geq x$ to mean $x \leq y$, writing $x < y$ to mean $x \leq y$ and $x \neq y$, and so on. We also routinely identify any subset $S \subseteq P$ with the corresponding induced subposet $S = (S, \leq)$.

2.1.1. *Basic terminology.* An *interval* of P is a subset $[x, y] := \{z \in P \colon x \leq z \leq y\}$ for $x \leq y \in P$. The poset P is *locally finite* if every interval of P is finite.

For $x, y \in P$, we say x is *covered* by y, written $x \lessdot y$, if $x < y$ and there is no $z \in P$ with $x < z < y$. The *Hasse diagram* of P is the directed graph whose vertices are the elements of P with an edge from x to y when $x \lessdot y$. We draw the Hasse diagram of P in the plane, with x below y if there is an edge from x to y (and therefore we do not draw the arrows on the edges). Any locally finite poset is determined by its Hasse diagram.

A *chain* of P is a totally ordered subset, i.e., a subset $C \subseteq P$ for which any two elements in C are comparable. An *antichain* of P is a subset $A \subseteq P$ for which any two elements in A are incomparable. We say a chain is *maximal* if it is maximal by inclusion among chains, and similarly for antichains.

An *order filter* is an upwards-closed subset, i.e., a subset $F \subseteq P$ such that if $x \in I$ and $x \leq y$ then $y \in I$. Dually, an *order ideal* of P is a downwards-closed subset, i.e., a subset $I \subseteq P$ such that if $y \in I$ and $x \leq y$ then $x \in I$. An order filter (respectively, order ideal) is *principal* if it is of the form $V_p \coloneqq \{q \in P \colon p \leq q\}$ (resp., of the form $\Lambda_p \coloneqq \{q \in P \colon q \leq p\}$) for some $p \in P$.

A *minimum* of P, which we always denote by $\hat{0} \in P$, is an element with $\hat{0} \leq x$ for all $x \in P$. Dually, a *maximum*, denoted $\hat{1} \in P$, is an element with $x \leq \hat{1}$ for all $x \in P$. Clearly, minimums and maximums are unique if they exist. If P has a minimum $\hat{0}$, then we call $s \in P$ an *atom* if $\hat{0} \lessdot s$. Dually, if P has a maximum $\hat{1}$, then we call $t \in P$ a *coatom* if $t \lessdot \hat{1}$.

2.1.2. *New posets from old.* The *dual poset* P^* of P is the poset with the same set of elements but with the opposite order, i.e., $x \leq_{P^*} y$ if and only if $y \leq_P x$. We say P is *self-dual* if it is isomorphic to its dual poset.

Now let Q be another poset. The *direct sum* $P+Q$ of P and Q is the poset whose set of elements is the (disjoint) union $P \sqcup Q$, with $x \leq_{P+Q} y$ if either $x, y \in P$ and $x \leq_P y$, or $x, y \in Q$ and $x \leq_Q y$. We say P is *connected* if it cannot be written as a nontrivial direct sum. For a positive integer $n \geq 1$, we denote the direct sum of n copies of P by nP.

The *direct product* $P \times Q$ of P and Q is the poset whose set of elements is the (Cartesian) product $P \times Q$, with $(p_1, q_1) \leq_{P \times Q} (p_2, q_2)$ if $p_1 \leq_P p_2$ and $q_1 \leq_Q q_2$. We say P is *indecomposable* if it cannot be written as a nontrivial direct product. For a positive integer $n \geq 1$, we denote the direct product of n copies of P by P^n.

2.1.3. *Möbius functions.* Suppose for the moment that P is locally finite, and let $\mathrm{Int}(P)$ denote the set of intervals of P. The *Möbius function* $\mu \colon \mathrm{Int}(P) \to \mathbb{Z}$ of P is defined recursively by

$$\mu(x, x) = 1 \text{ for all } x \in P; \qquad \mu(x, y) = -\sum_{x \leq z < y} \mu(x, z) \text{ for all } x < y \in P,$$

where we use the standard notational shorthand $\mu(x, y) = \mu([x, y])$. The most important application of Möbius functions is the Möbius inversion formula (see [16, §3.7]), a kind of generalization of the principle of inclusion-exclusion to any poset.

The Möbius function of a product of posets decomposes as a product of Möbius functions. In other words, we have

$$\mu_{P \times Q}((p_1, q_1), (p_2, q_2)) = \mu_P(p_1, p_2) \cdot \mu_Q(q_1, q_2),$$

for all $(p_1, q_1) \leq (p_2, q_2) \in P \times Q$ (see [16, Proposition 3.8.2]).

2.1.4. *Lattices.* For $x, y \in P$, an *upper bound* of x and y is a $z \in P$ with $x \leq z$ and $y \leq z$, and the *join* (or *least upper bound*) of x and y, denoted $x \vee y$, is the minimum among all upper bounds of x and y, if such a minimum exists. Dually, a *lower bound* of x and y is a $z \in P$ with $z \leq x$ and $z \leq y$, and the *meet* (or *greatest lower bound*) of x and y, denoted $x \wedge y$, is the maximum among all lower bounds of x and y, if such a maximum exists. If $x \vee y$ exists for every $x, y \in P$, then P is called a *join semilattice*. Dually, if $x \wedge y$ exists for every $x, y \in P$, then P is called a *meet semilattice*. The poset P is a *lattice* if it is both a join and meet semilattice.

Now let L be a lattice. The operations of $\vee$ and $\wedge$ are associative and commutative, and therefore for any finite, nonempty subset $S = \{x_1, \dots, x_n\} \subseteq L$ we can set $\bigvee S \coloneqq x_1 \vee \cdots \vee x_n$ and $\bigwedge S \coloneqq x_1 \wedge \cdots \wedge x_n$. If L has a minimum $\hat{0}$ then by convention we set $\bigvee \emptyset \coloneqq \hat{0}$, and dually if L has a maximum $\hat{1}$ we set $\bigwedge \emptyset \coloneqq \hat{1}$. A *finite* lattice L always has a minimum $\hat{0} = \bigwedge L$ and a maximum $\hat{1} = \bigvee L$.

If L has a minimum $\hat{0}$ and a maximum $\hat{1}$, then a *complement* of an element $x \in L$ is a $y \in L$ with $x \wedge y = \hat{0}$ and $x \vee y = \hat{1}$. The lattice L is called *complemented* if every element has a complement.

The lattice L is *distributive* if the operation of meet distributes over that of join, i.e., $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in L$. It is well-known that this is also equivalent to join distributing over meet, i.e., $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in L$.

Example 2.1. For any poset P, we use $J(P)$ to denote the poset of order ideals of P, ordered by inclusion. Then, $J(P)$ is always a distributive lattice, where the operations of join and meet are union and intersection, respectively. As a variant of this construction, we use $J_{\mathrm{fin}}(P)$ to denote the finite order ideals of P, which again always gives a distributive lattice. We note that $J(P + Q) = J(P) \times J(Q)$ and similarly $J_{\mathrm{fin}}(P + Q) = J_{\mathrm{fin}}(P) \times J_{\mathrm{fin}}(Q)$.

The lattice L is *modular* if whenever $x \leq y \in L$, we have $x \vee (z \wedge y) = (x \vee z) \wedge y$ for all $z \in L$. Observe that modularity is a weaker condition than distributivity: all distributive lattices are modular, but most modular lattices are not distributive.

We note that the product $L_1 \times L_2$ of two lattices L_1 and L_2 remains a lattice, and similarly if we append the adjectives "distributive" or "modular." Also, any interval in a lattice is a lattice, and similarly if we append the adjectives "distributive" or "modular." These properties follow from the fact that lattices, distributive lattices, and modular lattices, are varieties in the sense of universal algebra.

2.1.5. *Convention for finite versus infinite posets.* We will routinely work with both finite and infinite posets, although the posets will always be at least locally finite. For clarity, we will from now on use the following convention: normal script letters (like P and L) will denote finite posets, while caligraphic letters (like $\mathcal{P}$ and $\mathcal{L}$) will denote infinite posets.

2.2. **Finite graded posets.** Let P be a finite poset. For a nonnegative integer $n \geq 0$, we say that P is *n-graded* if P has a minimum $\hat{0}$, a maximum $\hat{1}$, and every maximal chain of P is of the form $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$. In this case, the *rank function* $\rho\colon P \to \mathbb{N}$ is defined by setting $\rho(x_i) \coloneqq i$ for each element x_i of a maximal chain $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n$. Equivalently, the rank function is determined by the requirements that $\rho(\hat{0}) = 0$ and $\rho(y) = \rho(x) + 1$ whenever $x \lessdot y \in P$.

Example 2.2. For any positive integer $n \geq 1$, we let $[n] \coloneqq \{1, 2, \dots, n\}$. We view $[n]$ as a poset, with the usual total order. This chain poset $[n]$ is the most basic example of a finite $(n-1)$-graded poset.

We say that the finite poset P is *graded* if it is n-graded for some n. In this case, we say that the *rank* of P is n and, slightly abusing notation, write $\rho(P) \coloneqq n$. If P and Q are two finite graded posets, then their product $P \times Q$ is also graded of rank $\rho(P \times Q) = \rho(P) + \rho(Q)$. Also, any interval in a finite graded poset is graded.

2.2.1. *Generating polynomials for finite graded posets.* Now assume that P is graded. The *rank generating polynomial* of P is

$$F(P;x) \coloneqq \sum_{p \in P} x^{\rho(p)}.$$

The *characteristic polynomial* of P is

$$\chi(P;x) \coloneqq \sum_{p \in P} \mu(\hat{0},p)\, x^{\rho(P)-\rho(p)}.$$

The exponent of x in each term of the characteristic polynomial $\chi(P;x)$ records the corank $\rho(P) - \rho(p)$ of the element $p \in P$. Using the corank in the characteristic polynomial is very standard, but, for reasons that will become clear soon, we need a version of the characteristic polynomial where the exponent records the usual rank instead. Hence, we define the *reciprocal characteristic polynomial* of P to be

$$\chi^*(P;x) \coloneqq \sum_{p \in P} \mu(\hat{0},p)\, x^{\rho(p)}.$$

Observe that $\chi^*(P;x) = x^{\rho(P)} \cdot \chi(P;x^{-1})$.

These invariants of finite graded posets all play nicely with products. Namely, $F(P \times Q;x) = F(P;x) \cdot F(Q;x)$ and $\chi^*(P \times Q;x) = \chi^*(P;x) \cdot \chi^*(Q;x)$.

2.2.2. *Finite graded lattices.* Here we are most interested in finite graded lattices.

Example 2.3. The rank n *(finite) Boolean lattice* B_n is the poset of subsets of $[n]$, ordered by inclusion. B_n is a finite graded lattice, with $\rho(S) = \#S$ for all $S \in B_n$. Its Möbius function is given by $\mu(S,T) = (-1)^{\#T \setminus S}$ for all $S \leq T \in B_n$. Hence, $F(B_n;x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$ and $\chi^*(B_n;x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k = (1-x)^n$. These formulas can also be seen from the fact that $B_n = J(n\,[1]) = [2]^n$.

Example 2.4. Birkhoff's representation theorem for finite distributive lattices says that every finite distributive lattice L has the form $L = J(P)$ for a unique finite poset P (the subposet of *join-irreducible* elements of L); see [1, §3.3] or [16, §3.4]. So let $L = J(P)$ be a finite distributive lattice. Then L is graded, with $\rho(I) = \#I$ for $I \in J(P)$. Its Möbius function is given by

$$\mu(I,I') = \begin{cases} (-1)^{\#I' \setminus I} & \text{if } I' \setminus I \text{ is an antichain of } P; \\ 0 & \text{otherwise,} \end{cases}$$

for $I \leq I' \in J(P)$ (see [16, Example 3.9.6]). Hence, $F(L;x) = \sum_{I \in J(P)} x^{\#I}$ and $\chi^*(L;x) = \sum_{I \subseteq \min(P)} (-x)^{\#I} = (1-x)^{\#\min(P)}$, where $\min(P)$ is the set of minimal elements of P. Observe how this example generalizes Example 2.3.

Example 2.4 explains that all finite distributive lattices are graded. More generally, all finite modular lattices are graded. In fact, a finite lattice L is modular if and only if L is graded and $\rho(p)+\rho(q)=\rho(p\vee q)+\rho(p\wedge q)$ for all $p,q\in L$ (see, e.g., [1, §2.8] or [16, §3.3]).

Example 2.5. Let q be a prime power, and $\mathbb{F}_q$ the finite field with q elements. We denote by $B_n(q)$ the poset of $\mathbb{F}_q$-subspaces of the vector space $\mathbb{F}_q^n$, ordered by inclusion. This *subspace lattice* $B_n(q)$, also known as the *q-analogue* of B_n, is a finite modular lattice. Its rank function is $\rho(U)=\dim(U)$ for $U\in B_n(q)$, and its Möbius function is $\mu(U,V)=(-1)^k q^{\binom{k}{2}}$, where $k=\dim(V)-\dim(U)$, for $U\leq V\in B_n(q)$ (see [16, Example 3.10.2]). Hence, $F(B_n(q);x)=\sum_{k=0}^n \begin{bmatrix} n\\ k\end{bmatrix}_q x^k$ and $\chi^*(B_n(q);x)=\sum_{k=0}^n(-1)^k q^{\binom{k}{2}}\begin{bmatrix} n\\ k\end{bmatrix}_q x^k=(1-x)(1-qx)(1-q^2x)\cdots(1-q^{n-1}x)$. Here we used the standard notation for the *q-binomial coefficient* $\begin{bmatrix} n\\ k\end{bmatrix}_q:=\frac{[n]_q!}{[k]_q![n-k]_q!}$, where $[n]_q:=\frac{(1-q^n)}{(1-q)}=1+q+q^2+\cdots+q^{n-1}$ and $[n]_q!:=[n]_q\cdot[n-1]_q\cdots[1]_q$.

There are variants of the modular property for finite lattices that are very interesting from a combinatorial point of view. A finite lattice L is called *(upper) semimodular* if it is graded and satisfies $\rho(p)+\rho(q)\geq\rho(p\vee q)+\rho(p\wedge q)$ for all $p,q\in L$. The finite lattice L is called *atomic* if every element is a join of atoms. Finally, L is *geometric* if it is both semimodular and atomic. For example, the modular lattices B_n and $B_n(q)$ are geometric, since they are atomic. However, most geometric lattices are not modular. Geometric lattices are intensely studied, because they are precisely the lattices of flats of matroids (see [16, §3.3]). Any interval in a semimodular lattice is semimodular, and, less obviously, any interval in a geometric lattice is geometric (see, e.g., [1, §8.9, Exercise 7] and [16, Proposition 3.3.3]).

Example 2.6. Recall that a *partition* of a set X is a collection $\pi=\{B_1,\ldots,B_k\}$ of nonempty subsets of X ($\emptyset\subsetneq B_1,\ldots,B_k\subseteq X$) that are pairwise disjoint ($B_i\cap B_j=\emptyset$ for $i\neq j$) and whose union is X ($\cup_{i=1}^k B_i=X$). The sets $B_i\in\pi$ are called the *blocks* of π. The *partition lattice* Π_n is the poset of partitions of $[n]$, ordered by refinement. In other words, for two partitions π and π' of $[n]$, we have $\pi\leq\pi'$ if every block $B\in\pi$ satisfies $B\subseteq B'$ for some block $B'\in\pi'$. The partition lattice is a geometric lattice of rank $\rho(\Pi_n)=n-1$, with $\rho(\pi)=n-\#\pi$ for $\pi\in\Pi_n$. Hence, $F(\Pi_n;x)=\sum_{k=0}^n S(n,n-k)x^k$, where $S(n,k)$ are the *Stirling number of the second kind*. Furthermore, it is well-known that $\chi^*(\Pi_n;x)=\sum_{k=0}^n s(n,n-k)x^k=(1-x)(1-2x)\cdots(1-(n-1)x)$, where $s(n,k)$ are the *(signed) Stirling number of the first kind* (see [16, Example 3.10.4]).

2.3. **Infinite graded posets.** Let $\mathcal{P}$ be an infinite poset. We say $\mathcal{P}$ is *$\mathbb{N}$-graded* if $\mathcal{P}$ has a minimum $\hat{0}$ and every maximal chain of P is of the form $\hat{0}=x_0\lessdot x_1\lessdot x_2\lessdot\cdots$. In this case, the *rank function* $\rho\colon\mathcal{P}\to\mathbb{N}$ is defined by setting $\rho(x_i):=i$ for each element x_i of a maximal chain $x_0\lessdot x_1\lessdot x_2\lessdot\cdots$. Equivalently, the rank function is determined by the requirements that $\rho(\hat{0})=0$ and $\rho(y)=\rho(x)+1$ whenever $x\lessdot y\in\mathcal{P}$.

If $\mathcal{P}$ and $\mathcal{Q}$ are $\mathbb{N}$-graded, then $\mathcal{P}\times\mathcal{Q}$ is $\mathbb{N}$-graded. Also, any interval in a *locally finite* $\mathbb{N}$-graded poset is a finite graded poset.

2.3.1. *Generating functions for infinite graded posets.* Let $\mathcal{P}$ be an $\mathbb{N}$-graded poset. In order to define sensible analogs of the rank generating and characteristic polynomials for $\mathcal{P}$, we need to make a further finiteness assumption. So let us say that $\mathcal{P}$ is *finite type* if $\{p \in \mathcal{P} \colon \rho(p) = i\}$ is finite for each $i \in \mathbb{N}$. Observe that a finite type $\mathbb{N}$-graded poset is locally finite (but finite type is a stronger requirement than just being locally finite).

So now assume that $\mathcal{P}$ is a finite type $\mathbb{N}$-graded poset. Then we define the *rank generating function* of $\mathcal{P}$ to be

$$F(\mathcal{P};x) = \sum_{p \in \mathcal{P}} x^{\rho(p)},$$

a formal power series in the variable x. And we define the *characteristic generating function* of $\mathcal{P}$ to be

$$\chi^*(\mathcal{P};x) = \sum_{p \in \mathcal{P}} \mu(\hat{0},p)\, x^{\rho(p)},$$

again, a formal power series. We write $\chi^*(\mathcal{P};x)$ with an asterisk to emphasize that the characteristic generating function of an infinite poset $\mathcal{P}$ uses rank in the exponent, like the *reciprocal* characteristic polynomial $\chi^*(P;x)$ of a finite poset P.

Again, these invariants play nicely with products: $F(\mathcal{P}\times\mathcal{Q};x) = F(\mathcal{P};x)\cdot F(\mathcal{Q};x)$ and $\chi^*(\mathcal{P}\times\mathcal{Q};x) = \chi^*(\mathcal{P};x)\cdot\chi^*(\mathcal{Q};x)$.

Example 2.7. The set of natural numbers $\mathbb{N} = \{0,1,\ldots\}$, with their usual total order, is the most basic example of an $\mathbb{N}$-graded poset. In fact, $\mathbb{N}$ is a finite type $\mathbb{N}$-graded lattice with $F(\mathbb{N};x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ and $\chi^*(\mathbb{N};x) = 1-x$. Hence, for any positive integer $n \geq 1$, $\mathbb{N}^n$ is a finite type $\mathbb{N}$-graded lattice with $F(\mathbb{N}^n;x) = \frac{1}{(1-x)^n}$ and $\chi^*(\mathbb{N}^n;x) = (1-x)^n$

2.4. **Upho posets.** Let $\mathcal{P}$ be an infinite poset. $\mathcal{P}$ is *upper homogeneous*, or "*upho*," if for every $p \in \mathcal{P}$, the principal order filter $V_p = \{q \in \mathcal{P} \colon q \geq p\}$ is isomorphic to $\mathcal{P}$. To avoid trivialities, let us also require that $\mathcal{P}$ has at least two elements.

Example 2.8. The natural numbers $\mathbb{N}$ form an upho poset. Similarly, the nonnegative rational numbers $\{x \in \mathbb{Q} \colon x \geq 0\}$, with their usual order, form an upho poset. And ditto for the nonnegative real numbers $\{x \in \mathbb{R} \colon x \geq 0\}$.

Example 2.9. Let X be any infinite set. Then the poset of finite subsets of X, ordered by inclusion, is upho.

In this paper we are primarily concerned with upho posets (in fact, upho lattices). In order to be able to apply the tools of enumerative and algebraic combinatorics to study these posets, we must impose some finiteness conditions on them. Hence, from now on, *all upho posets are assumed finite type* $\mathbb{N}$*-graded* unless otherwise specified. Of the preceding examples, only $\mathbb{N}$ is finite type $\mathbb{N}$-graded.

The product $\mathcal{P}\times\mathcal{Q}$ of two upho posets $\mathcal{P}$ and $\mathcal{Q}$ remains upho. So, for instance, $\mathbb{N}^n$ is an upho lattice for any $n \geq 1$. We will soon see many more examples of upho lattices, but for now $\mathbb{N}^n$ is a good prototypical example to have in mind.

Remark 2.10. Upho posets were introduced by Stanley [17, 19]. Stanley was mainly interested in *planar* upho posets (i.e., those with planar Hasse diagrams). In particular, he was interested in counting the maximal chains in $[\hat{0}, p]$ for all $p \in \mathcal{P}$, for various planar upho posets $\mathcal{P}$. When $\mathcal{P} = \mathbb{N}^2$, these numbers form Pascal's triangle. Thus, Stanley used these chain counts for other planar upho posets to produce analogues of Pascal's triangle [18, 19]. We note that planar upho posets have a rather simple structure, as described in [6]. All planar upho posets are meet semilattices, but most are not lattices. We will see that upho lattices can have a very intricate structure.

2.4.1. *Rank and characteristic generating functions of upho posets.* The following important result on rank generating functions of upho posets can be proved by a simple application of Möbius inversion.

Theorem 2.11 ([9, Theorem 1])**.** *For any upho poset $\mathcal{P}$, $F(\mathcal{P}; x) = \chi^*(\mathcal{P}; x)^{-1}$.*

2.4.2. *Upho lattices and their cores.* Now suppose that $\mathcal{L}$ is an upho lattice. Then we define the *core* of $\mathcal{L}$ to be $L \coloneqq [\hat{0}, s_1 \vee s_2 \vee \cdots \vee s_r] \subseteq \mathcal{L}$, where $s_1, \ldots, s_r$ are the atoms of $\mathcal{L}$. Evidently, the core of an upho lattice is a finite graded lattice. The point of the core is the following corollary, which can be proved for instance using Rota's cross-cut theorem (see [16, Corollary 3.9.4]).

Corollary 2.12 ([9, Corollary 6])**.** *Let $\mathcal{L}$ be an upho lattice with core L. Then $\chi^*(\mathcal{L}; x) = \chi^*(L; x)$. Hence, from Theorem 2.11, we conclude $F(\mathcal{L}; x) = \chi^*(L; x)^{-1}$.*

Note that Corollary 2.12 was stated as equation (1.1) in the introduction.

Example 2.13. For any $n \geq 1$, $\mathbb{N}^n$ is an upho lattice with core B_n, and indeed $F(\mathbb{N}^n; x) = \frac{1}{(1-x)^n} = \chi^*(B_n; x)^{-1}$.

With all the terminology and preliminary results fully explained, we now return to Question 1.2: which finite graded lattices L arise as cores of upho lattices $\mathcal{L}$? For example, we just saw that the Boolean lattices B_n do. We will explore this question in the next three sections. But before we do that, we conclude this section with a simple lemma that is useful for constructing upho lattices.

Lemma 2.14. *Let $\mathcal{L}$ be an upho join semilattice. Then $\mathcal{L}$ is an upho lattice.*

Proof. Let $p, q \in \mathcal{L}$. We know $p \vee q$ exists. We need to show $p \wedge q$ exists. So let $X = \{x \colon x \leq p, x \leq q\}$ be the set of lower bounds of p and q. Then X is nonempty since $\hat{0} \in X$, and X is finite since $\mathcal{P}$ is finite type $\mathbb{N}$-graded. Hence, we have $p \wedge q = \bigvee X$. □

3. Upho lattices from sequences of finite lattices

In this section, we will develop a method for producing upho lattices from limits of sequences of finite graded lattices which are appropriately embedded in one another. In order to make "appropriately embedded in one another" precise, we will need two notions from the theory of finite lattices: supersolvability, as introduced by Stanley in [14], and uniformity, as introduced by Dowling in [4].

3.1. **Supersolvable semimodular lattices.** Let L be a (finite) semimodular lattice. An element $p \in L$ is called *modular* if $\rho(p) + \rho(q) = \rho(p \vee q) + \rho(p \wedge q)$ for all $q \in L$. For example, if L is modular, then every element is modular. An important property of modularity is that it is transitive on "lower" intervals:

Proposition 3.1 (See [13] or [15, Proposition 4.10(b)])**.** *Let L be a semimodular lattice. If x is modular in L and y is modular in $[\hat{0}, x]$, then y is modular in L.*

The semimodular lattice L is called *supersolvable* if it possesses a maximal chain $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ of modular elements. As suggested by Proposition 3.1, we can build up a maximal chain of modular elements from the top down, coatom-by-coatom. For this reason, supersolvable lattices have a recursive structure. Moreover, as shown by Stanley [14], they enjoy many remarkable enumerative properties, the most prominent being that their characteristic polynomials factor.

Theorem 3.2 (Stanley [13] [14, Theorem 4.1])**.** *Let L be a supersolvable semimodular lattice with maximal chain of modular elements $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n$. For $i = 1, \ldots, n$, set*

$$a_i := \#\{\text{atoms } s \in L \colon s \leq x_i, s \not\leq x_{i-1}\}.$$

Then $\chi^*(L; x) = (1 - a_1 x)(1 - a_2 x) \cdots (1 - a_n x)$.

Example 3.3. The partition lattice Π_n is a supersolvable geometric lattice. Indeed, $\pi_0 \lessdot \pi_1 \lessdot \cdots \lessdot \pi_{n-1} \in \Pi_n$ is a maximal chain of modular elements, where

$$\pi_i := \{\{1, 2, \ldots, i+1\}, \{i+2\}, \{i+3\}, \cdots, \{n\}\}$$

for $i = 0, 1, \ldots, n-1$. Here $a_i = i$ for $i = 1, \ldots, n-1$, so Theorem 3.2 gives that $\chi^*(\Pi_n; x) = (1-x)(1-2x)\cdots(1-(n-1)x)$, in agreement with what we said earlier.

Remark 3.4. In [14], Stanley defined the notion of supersolvability more generally for any finite lattice, not necessarily semimodular. It is possible that the results in this section could be extended to this broader class of (not necessarily semimodular) supersolvable lattices. However, the examples that we know all involve geometric lattices, so it is unclear, at the moment, what this greater generality would buy us.

Next, we describe a procedure for "trimming" a supersolvable semimodular lattice to produce another one. We will need this trimming procedure later to guarantee that a certain limit poset is finite type $\mathbb{N}$-graded.

So let L be a supersolvable semimodular lattice with a fixed maximal chain of modular elements $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n$. Relative to this chain, we define $\mu \colon L \to \mathbb{N}$ by $\mu(x) := \min\{i \colon x \leq x_i\}$ for all $x \in L$. And then for a positive integer $k \geq 1$, we define

$$L^{(k)} := \{x \in L \colon \mu(x) - \rho(x) < k\}.$$

For instance, $L^{(1)} = \{x_0, x_1, \ldots, x_n\}$ and $L^{(n)} = L$. Note that $L^{(i)} \subseteq L^{(j)}$ for $i \leq j$.

Lemma 3.5. *For any $k \geq 1$, the subposet $L^{(k)}$ of L is again a supersolvable semimodular lattice, with maximal chain of modular elements $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n$.*

The rank of an element in $L^{(k)}$ is the same as its rank in L. And the join of two elements in $L^{(k)}$ is the same as their join in L. But the meet of two elements in $L^{(k)}$ may be different than their meet in L, so $L^{(k)}$ is not in general a sublattice of L.

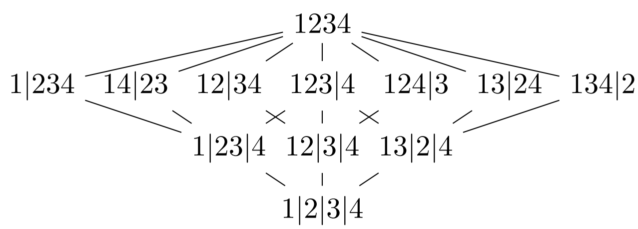

FIGURE 3. The "trimmed" partition lattice $\Pi_4^{(2)}$.

Example 3.6. Again consider the partition lattice Π_n, with maximal chain of modular elements $\pi_0 \lessdot \pi_1 \lessdot \cdots \lessdot \pi_{n-1}$ as described above. Then, for any $k \geq 1$,

$$\Pi_n^{(k)} = \{\pi \in \Pi_n \colon \max\{i \colon i \text{ is in a non-singleton block of } \pi\} < n + k - \#\pi\}.$$

(Recall that a block is called a *singleton* if it has one element. Also, above we use the convention $\max(\emptyset) := 0$.) For instance, Figure 3 depicts $\Pi_4^{(2)}$. In this figure we use the shorthand of writing a partition $\pi = \{B_1, \ldots, B_k\}$ as $B_1 \mid B_2 \mid \cdots \mid B_k$. The perceptive reader may compare this figure to Figure 1.

Proof of Lemma 3.5. We start with two important claims.

Claim 1: For $x, y \in L^{(k)}$, the join $x \vee y$, taken in L, is in $L^{(k)}$.

Claim 2: For $x \in L^{(k)}$ and $i = 0, 1, \ldots, n$, the meet $x \wedge x_i$, taken in L, is in $L^{(k)}$.

Proof of Claim 1: Let $x, y \in L^{(k)}$. Set $c := \max(\mu(x), \mu(y))$, $d := \max(\rho(x), \rho(y))$. Since $\mu(x) - \rho(x) < k$ and $\mu(y) - \rho(y) < k$, we have $c + d < k$. And since x_c is an upper bound for x and y, we have $\mu(x \vee y) \leq c$. Similarly, since $x \vee y$ is greater than or equal to x and to y, we have $\rho(x \vee y) \geq d$. Thus, $\mu(x \vee y) - \rho(x \vee y) \leq c - d < k$.

Proof of Claim 2: Let $x \in L^{(k)}$ and $i \in \{0, 1, \ldots, n-1\}$. If $x \leq x_i$, then $x \wedge x_i = x$ and the claim is clear. So suppose that $x \not\leq x_i$. Since $x_{\mu(x)}$ is an upper bound for x and x_i, we have $\rho(x \vee x_i) \leq \mu(x)$. And since x_i is modular, we have $\rho(x) + \rho(x_i) = \rho(x \vee x_i) + \rho(x \wedge x_i)$, i.e., $\rho(x \wedge x_i) = \rho(x) + \rho(x_i) - \rho(x \vee x_i) \geq \rho(x) + i - \mu(x)$. Meanwhile, since x_i is an upper bound for $x \wedge x_i$, we have $\mu(x \wedge x_i) \leq i$. Thus, $\mu(x \wedge x_i) - \rho(x \wedge x_i) \leq \mu(x) - \rho(x) < k$.

We proceed to prove the lemma. First, let us explain why $L^{(k)}$ is a lattice. Notice Claim 1 already implies that $L^{(k)}$ is a sub-join semilattice of L. Then, we can use the same trick as in the proof of Lemma 2.14: since $L^{(k)}$ is a finite join semilattice with a $\hat{0}$, it is a lattice.

Next, let us explain why $L^{(k)}$ is graded. Let $y_0 < y_1 < y_2 < \cdots < y_m \in L^{(k)}$ be any chain. We will show that we can extend this chain to a chain of length n. To that end, let L' denote the sublattice of L generated by $\{x_0, \ldots, x_n, y_0, \ldots, y_m\}$. An important result of Stanley [14, Proposition 2.1] says that L' is a distributive lattice (in fact, this leads to an alternative characterization of supersolvable lattices). Since L' is distributive, every element in L' can be written as a join of elements of the form $x_i \wedge y_j$. Hence, Claims 1 and 2 combine to imply that $L' \subseteq L^{(k)}$. Then, again since L' is distributive and hence graded, the chain $y_0 < y_1 < y_2 < \cdots < y_m$ can be extended in L' (and thus in $L^{(k)}$) to a maximal chain of length n. So indeed $L^{(k)}$ is graded, and in fact each element in $L^{(k)}$ has the same rank as it does in L.

Next, we show that $L^{(k)}$ is semimodular. Let $x, y \in L^{(k)}$. As mentioned, the join of x and y in $L^{(k)}$ is the same as its join in L. On the other hand, the meet of x and y in $L^{(k)}$ can only be lower than in L. So, the quantity $\rho(x \vee y) + \rho(x \wedge y)$ is smaller when computed in $L^{(k)}$ versus L. Thus, the inequality $\rho(x)+\rho(y) \geq \rho(x\vee y)+\rho(x\wedge y)$ remains true in $L^{(k)}$, and $L^{(k)}$ is semimodular.

Finally, we explain why $L^{(k)}$ is supersolvable, with maximal chain of modular elements $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n$. Let $y \in L^{(k)}$ and $i \in \{0, 1, \ldots, n-1\}$. Claims 1 and 2 imply that $x_i \vee y$ and $x_i \wedge y$ are the same in $L^{(k)}$ as they are in L, so $\rho(x_i) + \rho(y) = \rho(x_i \vee y) + \rho(x_i \wedge y)$ remains true, and thus x_i is modular in $L^{(k)}$. $\square$

3.2. **Uniform sequences of geometric lattices.** A sequence of finite posets $P_0, P_1, P_2, \ldots$ is called a *uniform sequence* if (for each $n = 0, 1, \ldots$):

- P_n is n-graded;
- $[a, \hat{1}_{P_n}]$ is isomorphic to P_{n-1} for all atoms $a \in P_n$.

From now on in this section, $L_0, L_1, \ldots$ is a uniform sequence of geometric lattices.

Uniform sequences also enjoy many remarkable enumerative properties, as first observed by Dowling [4]. For $0 \leq j \leq i$, define the numbers $V(i,j)$ and $v(i,j)$ by

$$\sum_{j=0}^{i} V(i,j)x^{i-j} = F(L_i;x), \qquad \sum_{j=0}^{i} v(i,j)x^{i-j} = \chi^*(L_i;x).$$

These numbers $V(i,j)$ and $v(i,j)$ are called the *Whitney numbers of the second and first kind*, respectively, for our sequence of lattices L_n. By convention, let us also declare $V(i,j) \coloneqq v(i,j) \coloneqq 0$ for $j > i$.

Theorem 3.7 (Dowling [4, Theorem 6]; see also [16, Exercise 3.130])**.** *The infinite, lower unitriangular matrices $[V(i,j)]_{0\leq i,j\leq\infty}$ and $[v(i,j)]_{0\leq i,j\leq\infty}$ are inverses.*

Example 3.8. Taking $L_n = \Pi_{n+1}$ gives a uniform sequence of geometric lattices. In this case, we have $V(i,j) = S(i+1, j+1)$ and $v(i,j) = s(i+1, j+1)$, the Stirling numbers of the second and first kind. Thus, the two infinite, lower unitriangular in Theorem 3.7 are:

$$[S(n,k)]_{1\leq n,k\leq\infty} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & \cdots \\ 1 & 7 & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} [s(n,k)]_{1\leq n,k\leq\infty} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 2 & -3 & 1 & 0 & \cdots \\ -6 & 11 & -6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

It is a classical result that these matrices of Stirling numbers are inverses.

When the geometric lattices L_n in our uniform sequence are also supersolvable, Theorems 3.2 and 3.7 combine to yield a very strong enumerative corollary, as we now explain. First, we need a preparatory result about symmetric polynomials.

Recall that the *complete homogeneous* and *elementary symmetric polynomials* in variables $x_1, \ldots, x_n$ are, respectively, given by

$$h_k(x_1, \ldots, x_n) \coloneqq \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k};$$
$$e_k(x_1, \ldots, x_n) \coloneqq \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k},$$

for $k > 0$. We also use the conventions $h_0(x_1, \ldots, x_n) \coloneqq e_0(x_1, \ldots, x_n) \coloneqq 1$ and $h_k(x_1, \ldots, x_n) \coloneqq e_k(x_1, \ldots, x_n) \coloneqq 0$ for $k < 0$.

Proposition 3.9. *Define infinite, lower unitriangular matrices $A = [a_{i,j}]_{0 \leq i,j \leq \infty}$ and $B = [b_{i,j}]_{0 \leq i,j \leq \infty}$ by letting*

$$a_{i,j} \coloneqq h_{i-j}(x_1, \ldots, x_{j+1});$$
$$b_{i,j} \coloneqq (-1)^{i-j} e_{i-j}(x_1, \ldots, x_i),$$

for all $0 \leq i, j$. Then A and B are inverses.

Proof. Define the matrix $C = [c_{i,j}]_{0 \leq i,j \leq \infty}$ by $C = BA$. Clearly, $c_{n,m} = 0$ for $m > n$. For $m \leq n$,

$$c_{n,m} = \sum_{i=0}^{n-m} (-1)^i e_i(x_1, \ldots, x_n) h_{n-m-i}(x_1, \ldots, x_{m+1}).$$

If $m = n$, this certainly equals 1. So now suppose $m < n$. Then $c_{n,m}$ is the coefficient of z^{n-m} in

$$\left(\sum_{k=0}^{\infty} (-1)^k e_k(x_1, \ldots, x_n) z^k \right) \left(\sum_{k=0}^{\infty} h_k(x_1, \ldots, x_{m+1}) z^k \right) = \prod_{i=1}^{n} (1 - x_i z) \prod_{i=1}^{m+1} \frac{1}{1 - x_i z},$$

which is 0 (since the right-hand side is a polynomial in z of degree $n - m - 1$). Therefore, C is the identity matrix, and A and B are inverses. □

We return to our uniform sequence of geometric lattices $L_0, L_1, \ldots$. Set

$$a_n \coloneqq \#\{\text{atoms } s \in L_n\} - \#\{\text{atoms } s \in L_{n-1}\},$$

for $n = 1, 2, \ldots$.

Corollary 3.10. *Suppose that all the geometric lattices L_n in our uniform sequence are supersolvable. Then, their Whitney numbers of the second and first kind are*

$$V(i, j) = h_{i-j}(a_1, \ldots, a_{j+1});$$
$$v(i, j) = (-1)^{i-j} e_{i-j}(a_1, \ldots, a_i),$$

for $0 \leq j \leq i$.

Proof. Fix $n \geq 1$. Our first goal is to show $\chi^*(L_n; x) = (1 - a_1 x)(1 - a_2 x) \cdots (1 - a_n x)$. So let $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ be a maximal chain of modular elements in L_n. Consider the modular coatom x_{n-1}. Because $\hat{1}$ is the join of the atoms in L_n (since L_n is atomic), there must be some atom $s \in L_n$ with $s \not\leq x_{n-1}$. This s is a complement of x_{n-1}, i.e., $s \vee x_{n-1} = \hat{1}$ and $s \wedge x_{n-1} = \hat{0}$. Since x_{n-1} is modular, there is thus a canonical isomorphism from $[s,\hat{1}]$ to $[0,x_{n-1}]$ given by $x \mapsto x \wedge x_{n-1}$ (see [1, §4.2]). But, by assumption, $[s,\hat{1}]$ is isomorphic to L_{n-1}, so $[0,x_{n-1}]$ is isomorphic to L_{n-1} as well. Then by induction, each $[0,x_i]$ is isomorphic to L_i. Thus, $\#\{\text{atoms } s \in L_n \colon s \leq x_i, s \not\leq x_{i-1}\} = a_i$. So by Theorem 3.2, we indeed have $\chi^*(L_n;x) = (1-a_1x)(1-a_2x)\cdots(1-a_nx)$.

The previous paragraph tells us that $v(i,j) = (-1)^{i-j}e_{i-j}(a_1,\ldots,a_i)$ for all i,j. We know from Theorem 3.7 that the matrices $[V(i,j)]$ and $[v(i,j)]$ are inverses. But we also know from Proposition 3.9 that the inverse of the matrix $[v(i,j)]$ is the matrix $[h_{i-j}(a_1,\ldots,a_{j+1})]$. We conclude $V(i,j) = h_{i-j}(a_1,\ldots,a_{j+1})$ for all i,j. $\square$

Example 3.11. Let us continue our running example with $L_n = \Pi_{n+1}$. As we saw earlier, these partition lattices are supersolvable and we have $a_n = n$ in this case. So Corollary 3.10 tells us that

$$\begin{aligned} S(n,k) &= h_{n-k}(1,2,\ldots,k); \\ s(n,k) &= (-1)^{n-k}e_{n-k}(1,2,\ldots,n-1). \end{aligned}$$

These are classical formulas for the Stirling numbers.

3.3. **Uniform sequences of supersolvable geometric lattices.** Continue to fix a uniform sequence of geometric lattices $L_0, L_1, \ldots$. Observe that, by induction, the interval $[x,\hat{1}] \subseteq L_n$ is isomorphic to $L_{n-\rho(x)}$ for all $x \in L_n$. So, in a sense, L_n is as close to being upho as a finite graded lattice can be: the principal order filters for all elements *of the same rank* are isomorphic. It is therefore reasonable to try to build an upho lattice by taking a limit of the L_n in some way. This is indeed what we will do, but to do it correctly requires some technical precision.

As we have already hinted, in order to take a limit of the L_n we will need to combine the notion of uniformity with that of supersolvability. But we will also need to make sure that the *way* our lattices are supersolvable is compatible with the way they form a uniform sequence. We saw in the proof of Corollary 3.10 that when the L_n are supersolvable there are many isomorphic copies of L_{n-1} sitting inside of L_n: for each atom $s \in L_n$, the "upper" interval $[s,\hat{1}_{L_n}]$ is isomorphic to L_{n-1}, and also for each modular coatom $t \in L_n$, the "lower" interval $[\hat{0}_{L_n},t]$ is isomorphic to L_{n-1}. We will need to *fix* all of these isomorphisms and make sure they are compatible with one another.

So, abusing terminology, by a *uniform sequence of supersolvable geometric lattices* we will mean a uniform sequence of geometric lattices $L_0, L_1, \ldots$ together with (for each $n = 0,1,\ldots$):

- an isomorphism $\theta_s \colon [s,\hat{1}_{L_n}] \to L_{n-1}$ for each atom $s \in L_n$;
- an *embedding* (isomorphism onto its image) $\iota_n \colon L_n \to L_{n+1}$,

satisfying (for each $n = 0,1,\ldots$):

- $(\iota_{n-1} \circ \theta_s)(x) = (\theta_{\iota_n(s)} \circ \iota_n)(x)$ for each atom $s \in L_n$ and all $x \in [s,\hat{1}_{L_n}]$;
- the image of ι_n is $[\hat{0}_{L_{n+1}},t_n]$, where $t_n \in L_{n+1}$ is a modular coatom.

The requirement that the image of ι_n is $[\hat{0}_{L_{n+1}},t_n]$ implies in particular that all the embeddings ι_n are *rank preserving*, i.e., $\rho(\iota_n(x)) = \rho(x)$ for all $x \in L_n$.

Remark 3.12. Notice how in the definition of uniform sequence we required the "upper" intervals $[s, \hat{1}_{L_n}]$ to be isomorphic to L_{n-1} for *all* atoms $s \in L_n$, whereas here we only require that there be *some* (modular) coatom $t_{n-1} \in L_n$ for which the "lower" interval $[\hat{0}_{L_n}, t_{n-1}]$ is isomorphic to L_{n-1}. This is a crucial distinction!

Example 3.13. Let us continue to examine the case $L_n = \Pi_{n+1}$. We can upgrade this sequence to a uniform sequence of supersolvable geometric lattices by defining embeddings $\iota_n \colon \Pi_{n+1} \to \Pi_{n+2}$ and isomorphisms $\theta_s \colon [s, \hat{1}_{\Pi_{n+1}}] \to \Pi_n$ as follows. First of all, we set $\iota_n(\pi) \coloneqq \pi \cup \{\{n+2\}\}$ for all $\pi \in \Pi_{n+1}$. Next, note that any atom $s \in \Pi_{n+1}$ has a unique non-singleton block, of the form $\{i, j\}$ for $1 \leq i < j \leq n+1$. Let us denote this atom by $s_{i,j}$. Then for $\pi = \{B_1, \ldots, B_m\} \in [s_{i,j}, \hat{1}_{\Pi_{n+1}}]$ we set $\theta_{s_{i,j}}(\pi) \coloneqq \bigcup_{\ell=1}^{m} \{f(k) \colon k \in B_\ell \setminus \{j\}\}$, where $f(k)$ is k if $k < j$ and $k-1$ if $k > j$. In other words, to obtain $\theta_{s_{i,j}}(\pi)$ from π, we delete j from whichever block it appears in (necessarily together with i) and then re-index by subtracting one from all numbers greater than j. It is straightforward to verify that these ι_n and θ_s satisfy the requirements listed above.

From now on in this section, let us assume moreover that our sequence $L_0, L_1, \ldots$ is a uniform sequence of *supersolvable* geometric lattices. So we now also have fixed embeddings $\iota_n \colon L_n \to L_{n+1}$, and isomorphisms $\theta_s \colon [s, \hat{1}_{L_n}] \to L_{n-1}$ for each atom $s \in L_n$. First of all, let us justify our terminology by explaining how this additional structure indeed forces the L_n to be supersolvable geometric lattices.

Proposition 3.14. *For $n = 0, 1, \ldots$, the geometric lattice L_n is supersolvable.*

Proof. Let $x_i \coloneqq \iota_{n-1} \circ \cdots \circ \iota_{i+1}(t_i) \in L_n$ for $i = 0, 1, \ldots, n-1$, and $x_n \coloneqq \hat{1}_{L_n}$, where $t_m \in L_{m+1}$ is the distinguished modular coatom for which the image of ι_m is $[\hat{0}_{L_{m+1}}, t_m]$. Since the t_m are modular, and the embeddings ι_m are rank preserving, repeated application of Proposition 3.1 shows that $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n \in L_n$ is indeed a maximal chain of modular elements. □

Henceforth, each supersolvable geometric lattice L_n comes together with the maximal chain of modular elements $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n$ from the proof of Proposition 3.14. This lets us, for instance, speak of $\mu \colon L_n \to \mathbb{N}$ and $L_n^{(k)}$, as defined in Section 3.1. Notice that in addition to preserving rank, the embeddings ι_n preserve the μ statistic, i.e., $\mu(\iota_n(x)) = \mu(x)$ for all $x \in L_n$. Therefore, the embeddings ι_n restrict to embeddings $\iota_n \colon L_n^{(k)} \to L_{n+1}^{(k)}$ for any $k \geq 1$.

3.4. **Limits of uniform sequences of supersolvable geometric lattices.** Continue to fix a uniform sequence of supersolvable geometric lattices $L_0, L_1, \ldots$. We will now take the limit of this sequence, which we will denote by $\mathcal{L}_\infty$. Since we have distinguished embeddings $\iota_n \colon L_n \to L_{n+1}$, it makes sense to define $\mathcal{L}_\infty \coloneqq \bigcup_{n=1}^{\infty} L_n$.

More precisely, we define $\mathcal{L}_\infty \coloneqq \varinjlim L_n$, the direct limit of the directed system formed by the L_n together with the ι_n. That is, we let $\mathcal{L}_\infty \coloneqq \bigsqcup_{n=1}^{\infty} L_n / \sim$, the disjoint union of all the finite lattices L_n modulo the equivalence relation $\sim$ generated by $x \sim \iota_n(x)$ for all $x \in L_n$ and all $n = 0, 1, \ldots$. Denoting the equivalence class of an element $x \in \bigsqcup_{n=1}^{\infty} L_n$ by $[x]$, the partial order on $\mathcal{L}_\infty$ is $[x] \leq [y]$ if $x' \leq y'$ in some L_n for some $x' \in [x], y' \in [y]$.

Because the L_n are graded and the embeddings ι_n are rank preserving, their limit $\mathcal{L}_\infty$ is $\mathbb{N}$-graded. Similarly, since each L_n is a lattice, their limit $\mathcal{L}_\infty$ is a lattice. And the uniformity of the sequence L_n can be used to show that every principal order filter in $\mathcal{L}_\infty$ is isomorphic to $\mathcal{L}_\infty$. But $\mathcal{L}_\infty$ is *not* finite type $\mathbb{N}$-graded: for instance, it has infinitely many atoms. To summarize, the limit poset $\mathcal{L}_\infty$ is an upho lattice, except for the fact that it is not finite type $\mathbb{N}$-graded.

To resolve this final wrinkle, we need to make a "thinner" poset out of the "wide" poset $\mathcal{L}_\infty$. This is where the trimming procedure described in Section 3.1 comes into play. So, define $\mu\colon \mathcal{L}_\infty \to \mathbb{N}$ by $\mu([x]) \coloneqq \min\{n\colon x' \in L_n \text{ for some } x' \in [x]\}$ for all $[x] \in \mathcal{L}_\infty$. And then for any positive integer $k \geq 1$, define

$$\mathcal{L}_\infty^{(k)} \coloneqq \{[x] \in \mathcal{L}_\infty \colon \mu([x]) - \rho([x]) < k\}.$$

Since $\rho([x]) = \rho(x)$ and $\mu([x]) = \mu(x)$ for any $[x] \in \mathcal{L}_\infty$, we equivalently have that $\mathcal{L}_\infty^{(k)} = \bigcup_{n=1}^{\infty} L_n^{(k)}$, the direct limit of the trimmed finite lattices $L_n^{(k)}$ with respect to the embeddings $\iota_n\colon L_n^{(k)} \to L_{n+1}^{(k)}$. Yet another way to think of $\mathcal{L}_\infty^{(k)}$ is that it consists of all the elements of rank $n-k+1$ in each L_n, for $n \geq k-1$. It is this $\mathcal{L}_\infty^{(k)}$ which is a proper (i.e., finite type $\mathbb{N}$-graded) upho lattice.

Example 3.15. We continue with the example of $L_n = \Pi_{n+1}$. Then $\mathcal{L}_\infty$ consists of all partitions of the set $\{1, 2, \ldots\}$ for which all but finitely many blocks are singletons, partially ordered by refinement. This poset $\mathcal{L}_\infty$ is $\mathbb{N}$-graded: we have $\rho(\pi) = \sum_{B\in\pi}(\#B - 1)$. But $\mathcal{L}_\infty$ is *not* finite type $\mathbb{N}$-graded. Meanwhile, $\mathcal{L}_\infty^{(k)}$ can be viewed as the collection of all partitions of a set of the form $[n] = \{1, 2, \ldots, n\}$ (for some $n \geq k$) into k blocks. The partitions in $\mathcal{L}_\infty^{(k)}$ are not all partitions of the same set, but the partial order can be described in the same way: for $\pi, \pi' \in \mathcal{L}_\infty^{(k)}$, we have $\pi \leq \pi'$ if for every $B \in \pi$ there exists a $B' \in \pi'$ with $B \subseteq B'$. And now $\mathcal{L}_\infty^{(k)}$ *is* finite type $\mathbb{N}$-graded: we have $\rho(\pi) = n - k$ if π is a partition of $[n]$, so there are only finitely many elements of each rank. Figure 1 depicts $\mathcal{L}_\infty^{(2)}$ for this example.

The following theorem is the main result of this section. (In the introduction, it was stated, less precisely, as Theorem 1.3.)

Theorem 3.16. *For any $k \geq 1$, $\mathcal{L}_\infty^{(k)}$ is an upho lattice whose core is L_k.*

Proof. Let us first explain why $\mathcal{L}_\infty^{(k)}$ is a finite type $\mathbb{N}$-graded lattice. As mentioned above, $\mathcal{L}_\infty^{(k)} = \bigcup_{n=1}^{\infty} L_n^{(k)}$. We know from Lemma 3.5 that the $L_n^{(k)}$ are graded posets, and the embeddings ι_n are rank preserving, so the limit $\mathcal{L}_\infty^{(k)}$ is $\mathbb{N}$-graded. Similarly, we know from Lemma 3.5 that the $L_n^{(k)}$ are lattices, so the limit $\mathcal{L}_\infty^{(k)}$ is a lattice as well. Finally, as mentioned, for each $n \geq k-1$, the elements of rank $n-k+1$ in $\mathcal{L}_\infty^{(k)}$ are precisely $[x]$ for $x \in L_n$ with $\rho(x) = n-k+1$. Hence, there are only finitely many elements of each rank in $\mathcal{L}_\infty^{(k)}$, i.e., $\mathcal{L}_\infty^{(k)}$ is finite type $\mathbb{N}$-graded.

Next, let us explain why $\mathcal{L}_\infty^{(k)}$ is upho. Actually, it is convenient to first work with $\mathcal{L}_\infty$. We will show that every principal order filter in $\mathcal{L}_\infty$ is isomorphic to $\mathcal{L}_\infty$. It clearly suffices to do this for principal order filters $V_{[s]} \subseteq \mathcal{L}_\infty$ corresponding to atoms $[s] \in \mathcal{L}_\infty$. So let $[s] \in \mathcal{L}_\infty$ be an atom. Recall the distinguished modular coatoms $t_n \in L_{n+1}$ for which $\iota_n(L_n) = [\hat{0}_{L_{n+1}}, t_n]$. Let $n_0 \coloneqq \mu([s])$ and notice that n_0 is also the smallest n_0 for which $[s] \leq [t_{n_0}]$. We will define a series of isomorphisms $\eta_n \colon [[s],[t_n]] \to [\hat{0},[t_{n-1}]]$, for $n \geq n_0$, with isomorphism $\eta\colon V_{[s]} \to \mathcal{L}_\infty$ then obtained as the limit of η_n.

To define η_n, assume we have chosen the representative s of the equivalence class $[s]$ so that $s \in L_n$. Then the elements of $[[s],[t_n]]$ are $[x]$ for $x \in [s, \hat{1}_{L_n}]$. By supposition, we have an isomorphism $\theta_s \colon [s, \hat{1}_{L_n}] \to L_{n-1}$. But also, the elements of $[\hat{0},[t_{n-1}]]$ are $[x]$ for $x \in L_{n-1}$. The isomorphism $\eta_n \colon [[s],[t_n]] \to [\hat{0},[t_{n-1}]]$ is thus defined by composing θ_s with the identifications of $[[s],[t_n]]$ and $[s, \hat{1}_{L_n}]$, and of L_{n-1} and $[\hat{0},[t_{n-1}]]$. Crucially, the compatibility requirement $\iota_{n-1} \circ \theta_s = \theta_{\iota_n(s)} \circ \iota_n$ implies that the restriction of η_{n+1} to the domain of η_n agrees with η_n.

So we get a sequence of isomorphisms $\eta_n \colon [[s],[t_n]] \to [\hat{0},[t_{n-1}]]$, for $n \geq n_0$, which have the property that the restriction of η_n to the domain of η_m agrees with η_m for all $m \leq n$. Moreover, each $[x] \in V_{[s]}$ (respectively, $[x] \in \mathcal{L}_\infty$) belongs to $[[s],[t_n]]$ (resp., $[\hat{0},[t_{n-1}]]$) for sufficiently large n. It therefore makes sense to define the isomorphism $\eta\colon V_{[s]} \to \mathcal{L}_\infty$ by $\eta \coloneqq \bigcup_{n=n_0}^{\infty} \eta_n$, i.e., $\eta([x]) \coloneqq \eta_n([x])$ if $[x]$ belongs to the domain of η_n.

Now we return to $\mathcal{L}_\infty^{(k)}$. We want to show that every principal order filter in $\mathcal{L}_\infty^{(k)}$ is isomorphic to $\mathcal{L}_\infty^{(k)}$, and again it is sufficient to only consider the principal order filters for atoms. So let $[s] \in \mathcal{L}_\infty^{(k)}$ be an atom, and consider the corresponding principal order filter, which for clarity we denote by $V_{[s]}^{(k)} \subseteq \mathcal{L}_\infty^{(k)}$. We claim that the isomorphism $\eta\colon V_{[s]} \to \mathcal{L}_\infty$ above restricts to an isomorphism $\eta\colon V_{[s]}^{(k)} \to \mathcal{L}_\infty^{(k)}$. To see this, observe that, for each $n \geq k-1$, the elements of rank $n-k+1$ in $V_{[s]}^{(k)}$ are the elements of rank $n-k+1$ in $[[s],[t_{n+1}]]$. By construction, η_{n+1} maps these elements to the elements of rank $n-k+1$ in $\Lambda_{[t_n]}$, which are precisely the elements of rank $n-k+1$ in $\mathcal{L}_\infty^{(k)}$. So indeed, η restricts to an isomorphism $\eta\colon V_{[s]}^{(k)} \to \mathcal{L}_\infty^{(k)}$.

Finally, to see why the core of $\mathcal{L}_\infty^{(k)}$ is L_k, notice that the atoms of $\mathcal{L}_\infty^{(k)}$ are $[s]$ for atoms $s \in L_k$. Joins in $\mathcal{L}_\infty^{(k)}$ are joins of the representative elements in the L_n. So the join of the atoms of $\mathcal{L}_\infty^{(k)}$ is $[\hat{1}_{L_k}]$, and its core is $[\hat{0},[\hat{1}_{L_k}]]$, i.e., a copy of L_k. ☐

Corollary 3.17. *For any $k \geq 1$, we have*

$$F(\mathcal{L}_\infty^{(k)};x) = \frac{1}{(1-a_1x)(1-a_2x)\cdots(1-a_kx)}.$$

Proof. We know that $\chi^*(L_k;x) = (1-a_1x)(1-a_2x)\cdots(1-a_kx)$ (see Corollary 3.10). The corollary then follows from Theorem 3.16 and Corollary 2.12. ☐

Remark 3.18. I thank David Speyer and Gjergji Zaimi for explaining the following to me in answers to a question I posted to MathOverflow [8]. Suppose that $A = [a_{ij}]_{0\leq i,j\leq\infty}$ and $B = [b_{ij}]_{0\leq i,j\leq\infty}$ are two infinite, lower unitriangular matrices satisfying:

- A and B are inverses;
- $\displaystyle\sum_{i=k}^{\infty} a_{i,k}\, x^{i-k} = \left(\sum_{i=0}^{k+1} b_{k+1,k+1-i}\, x^i\right)^{-1}$ for all $k \geq 0$.

Let $a_k \coloneqq a_{k,k-1} - a_{k-1,k-2}$ for all $k \geq 1$ (where by convention $a_{0,-1} \coloneqq 0$). Then the entries of A and B are determined by this sequence $a_1, a_2, \ldots$. Specifically, we must have that

$$\begin{aligned} a_{i,j} &= h_{i-j}(a_1, \ldots, a_{j+1}); \\ b_{i,j} &= (-1)^{i-j} e_{i-j}(a_1, \ldots, a_i). \end{aligned}$$

Because of this, we could alternatively have deduced Corollary 3.17 by combining Theorem 3.16 and Corollary 2.12 with Dowling's Theorem 3.7, without any appeal to Stanley's Theorem 3.2.

3.5. Examples of uniform sequences of supersolvable geometric lattices.

The construction presented above in this section would not be interesting if there were not any interesting examples of uniform sequences of supersolvable geometric lattices. Fortunately, there are many interesting examples. We now review all examples that we know of.

3.5.1. *Boolean lattices.* Let $L_n = B_n$ be the sequence of Boolean lattices.

We define the auxiliary data $\iota_n\colon B_n \to B_{n+1}$ and $\theta_s\colon [s, \hat{1}_{B_n}] \to B_{n-1}$ as follows. First, the embedding ι_n comes from the inclusion of sets $[n] \subseteq [n+1]$. That is, we set $\iota_n(S) \coloneqq S$ for all $S \in B_n$. Next, noting that any atom in B_n has the form $\{i\}$ for $1 \leq i \leq n$, we set $\theta_{\{i\}}(S) \coloneqq \{f(j)\colon j \in S \setminus \{i\}\}$ for all $S \in [\{i\}, \hat{1}_{B_n}]$, where $f(j)$ is j if $j < i$ and $j-1$ if $j > i$. In other words, to obtain $\theta_{\{i\}}(S)$ from S, we delete i and then re-index by subtracting one from all numbers greater than i.

It is straightforward to verify that this gives a uniform sequence of supersolvable geometric lattices. Here $a_n = 1$ for all n, and $V(i,j) = \binom{i}{j}$ and $v(i,j) = (-1)^{i-j}\binom{i}{j}$.

Let us use the notation $\mathcal{B}_\infty \coloneqq \mathcal{L}_\infty$ and $\mathcal{B}_\infty^{(k)} \coloneqq \mathcal{L}_\infty^{(k)}$ for this sequence $L_n = B_n$. Then, $\mathcal{B}_\infty$ is the poset of all finite subsets of $\{1, 2, \ldots\}$, ordered by inclusion. And for any $k \geq 1$, $\mathcal{B}_\infty^{(k)} = \{\text{finite } S \subseteq \{1,2,\ldots\}\colon \max(S) < \#S + k\}$, ordered by inclusion. Equivalently, $\mathcal{B}_\infty^{(k)} = \{\text{finite } S \subseteq \{1,2,\ldots\}\colon S \subseteq [\#S + k - 1]\}$. Since the core of this upho lattice $\mathcal{B}_\infty^{(k)}$ is B_k, we have $F(\mathcal{B}_\infty^{(k)}; x) = 1/(1-x)^k$.

Although the core of $\mathcal{B}_\infty^{(k)}$ is B_k, we note that $\mathcal{B}_\infty^{(k)}$ is *not* isomorphic to $\mathbb{N}^k$ (for any $k \geq 2$). Indeed, the element $\{1, 2, \ldots, n\}$ in $\mathcal{B}_\infty^{(k)}$ covers n elements. But in $\mathbb{N}^k$, each element covers at most k elements. This gives the simplest example showing that an upho lattice is not determined by its core.

3.5.2. *Subspace lattices.* Now fix a prime power q, and let $L_n = B_n(q)$ be the sequence of subspace lattices over $\mathbb{F}_q$. In order to define the various maps between the $B_n(q)$, it will be necessary to concretely represent elements of the vector spaces $\mathbb{F}_q^n$. Thus, let each $\mathbb{F}_q^n$ come with a distinguished basis $\{e_1, \ldots, e_n\}$. This gives a canonical inclusion $\mathbb{F}_q^n \subseteq \mathbb{F}_q^{n+1}$. It also means that each subspace $U \in B_n(q)$ can be represented uniquely by the matrix in reduced column echelon form whose column space is U. Similarly, elements $g \in \mathrm{GL}(\mathbb{F}_q^n)$ of the general linear group of $\mathbb{F}_q^n$ can also now be represented by matrices. These matrix representations are useful because we can order matrices lexically, by reading them row-by-row.

We define the auxiliary data $\iota_n \colon B_n(q) \to B_{n+1}(q)$ and $\theta_s \colon [s, \hat{1}_{B_n(q)}] \to B_{n-1}(q)$ as follows. First, ι_n comes from the inclusion of vector spaces $\mathbb{F}_q^n \subseteq \mathbb{F}_q^{n+1}$. That is, to get the representing matrix of $\iota_n(U)$ from that of $U \in B_n(q)$, we append a row of zeros. Next, noting that any atom (1-dimensional subspace) $S \in B_n(q)$ has several complementary $(n-1)$-dimensional subspaces, we choose the complement $T \in B_n(q)$ of S whose representing matrix is first in lexical order. Similarly, there are several $g \in \mathrm{GL}(\mathbb{F}_q^n)$ with $g \cdot T = \mathrm{Span}\{e_1, \ldots, e_{n-1}\}$, so we choose the g whose representing matrix is first in lexical order. Then, set $\theta_S(U) := \iota_{n-1}^{-1}(g(T \cap U))$ for $U \in [S, \hat{1}_{B_n(q)}]$.

It is again an straightforward check that we get a uniform sequence of supersolvable geometric lattices. Here $a_n = q^n$, $V(i,j) = \left[{i \atop j}\right]_q$ and $v(i,j) = (-1)^{i-j} q^{\binom{i-j}{2}} \left[{i \atop j}\right]_q$.

We use notation $\mathcal{B}_\infty(q) := \mathcal{L}_\infty$ and $\mathcal{B}_\infty^{(k)}(q) := \mathcal{L}_\infty^{(k)}$ for this sequence $L_n = B_n(q)$. Let $\mathbb{F}_q^\infty$ denote the (infinite-dimensional) $\mathbb{F}_q$-vector space with basis $\{e_1, e_2, \ldots\}$. Then, $\mathcal{B}_\infty(q)$ is the poset of all finite-dimensional subspaces of $\mathbb{F}_q^\infty$, ordered by inclusion. And for any $k \geq 1$,

$$\mathcal{B}_\infty^{(k)}(q) = \{\text{finite-dimensional } U \subseteq \mathbb{F}_q^\infty \colon U \subseteq \mathrm{Span}(\{e_1, e_2, \ldots, e_{\dim(U)+k-1}\})\}.$$

We have $F(\mathcal{B}_\infty^{(k)}(q); x) = 1/((1-x)(1-qx)\cdots(1-q^{n-1}x))$, since the core of $\mathcal{B}_\infty^{(k)}(q)$ is $B_k(q)$. Evidently, $\mathcal{B}_\infty^{(k)}(q)$ is a q-analogue of $\mathcal{B}_\infty^{(k)}$.

3.5.3. *Partition lattices.* The partition lattices were our running example above, but for completeness we repeat everything here. It is well-known that the partition lattices Π_n are supersolvable geometric lattices, and taking $L_n = \Pi_{n+1}$ gives a uniform sequence. The appropriate auxiliary data ι_n and θ_s were defined in Example 3.13. As a reminder, $\iota_n \colon \Pi_{n+1} \to \Pi_{n+2}$ is given by $\iota_n(\pi) = \pi \cup \{\{n+2\}\}$. Here $a_n = n$, and $V(i,j) = S(i+1, j+1)$ and $v(i,j) = s(i+1, j+1)$, the Stirling numbers of the second and first kind.

Let us use the notation $\prod_\infty := \mathcal{L}_\infty$ and $\prod_\infty^{(k)} := \mathcal{L}_\infty^{(k)}$ for this sequence $L_n = \Pi_{n+1}$. Then, $\prod_\infty$ is the poset of all partitions of the set $\{1, 2, \ldots\}$ for which all but finitely many blocks are singletons, ordered by refinement. And for any $k \geq 1$, $\prod_\infty^{(k)}$ can be identified with the collection of partitions of $[n]$ into k blocks, for some $n \geq k$. The partial order on $\prod_\infty^{(k)}$ is still refinement in the sense that for $\pi, \pi' \in \prod_\infty^{(k)}$ we have $\pi \leq \pi'$ if for every $B \in \pi$ there exists $B' \in \pi'$ with $B \subseteq B'$. We have $F(\prod_\infty^{(k)}; x) = 1/((1-x)(1-2x)\cdots(1-kx))$, since the core of $\prod_\infty^{(k)}$ is Π_{k+1}. As mentioned above, Figure 1 depicts $\prod_\infty^{(2)}$.

3.5.4. *Type B partition lattices.* We now describe a Type B variant of the previous example. For an integer $i \in \mathbb{Z}$, let us use the shorthand $\bar{i} := -i$, and for a subset of integers $S \subseteq \mathbb{Z}$, let $\overline{S} := \{\bar{i} \colon i \in S\}$. For $n \geq 1$, a partition π of the set $[n] \cup \overline{[n]}$ is called a *Type B partition* if:

- for every block $B \in \pi$, we also have $\overline{B} \in \pi$;
- there is at most one block $B \in \pi$ (called the *zero block*) with $B = \overline{B}$.

Let the *Type B partition lattice* Π_n^B be the poset of Type B partitions of $[n] \cup \overline{[n]}$, ordered by refinement.

It is well-known that Π_n^B is a geometric lattice. In fact, just as Π_{n+1} is the lattice of flats of the Coxeter arrangement of Type A_n, Π_n^B is the lattice of flats of the Coxeter arrangement of Type B_n (see [21, 11, 15]). Moreover, it is known that Π_n^B is a *supersolvable* geometric lattice, and taking $L_n = \Pi_n^B$ gives a uniform sequence. (This also follows from a more general result of Dowling [4] discussed below.) Note that the rank function on Π_n^B is given by $\rho(\pi) = n - k$ if π has $2k$ non-zero blocks.

We define $\iota_n \colon \Pi_n^B \to \Pi_{n+1}^B$ and $\theta_s \colon [s, \hat{1}_{\Pi_n^B}] \to \Pi_{n-1}^B$ for this sequence as follows. First, we set $\iota_n(\pi) := \pi \cup \{\{n+1\}, \{\overline{n+1}\}\}$ for all $\pi \in \Pi_n^B$. Next, consider an atom $s \in \Pi_n^B$. The atom s could have a single non-singleton block of the form $\{i, \bar{i}\}$ for $1 \leq i \leq n$; denote this kind of atom by s_i. Or, the atom s could have two non-singleton blocks of the form $\{i, j\}, \{\bar{i}, \bar{j}\}$ (respectively, $\{i, \bar{j}\}, \{\bar{i}, j\}$) for $1 \leq i < j \leq n$; denote this kind of atom by $s_{i,j}$ (resp., $s_{i,\bar{j}}$). In order to not be bogged down by notation, let us describe how to obtain $\theta_s(\pi)$ from π in words only. In the case where $s = s_i$, we obtain $\theta_s(\pi)$ from π by deleting i and $\bar{i}$ from π, and re-indexing by decreasing by one the absolute value of numbers greater in absolute value than i. In the case where $s = s_{i,j}$ or $s_{i,\bar{j}}$, we obtain $\theta_s(\pi)$ from π by deleting j and $\bar{j}$ from π, and re-indexing by decreasing by one the absolute value of numbers greater in absolute value than j. These are slight variations of the maps we used for the Type A partition lattices, and it is again a straightforward, albeit tedious, check that they satisfy the requirements.

For this sequence, $a_n = 2n-1$, and $V(i,j) = S_B(i,j)$ and $v(i,j) = s_B(i,j)$, where $S_B(n,k)$ and $s_B(n,k)$ are the *Type B Stirling numbers of the second and first kind*. These Type B Stirling numbers are defined by

$$\begin{aligned} S_B(n,k) &:= h_{n-k}(1, 3, \ldots, 2k+1); \\ s_B(n,k) &:= (-1)^{n-k} e_{n-k}(1, 3, \ldots, 2n-1). \end{aligned}$$

(See the recent paper [12] for more about the Type B Stirling numbers.) In particular, we have that $\chi^*(\Pi_n^B; x) = (1-x)(1-3x)\cdots(1-(2n-1)x)$.

We use the notation $\prod_\infty^B := \mathcal{L}_\infty$ and $\prod_\infty^{B,(k)} := \mathcal{L}_\infty^{(k)}$ for this sequence $L_n = \Pi_n^B$. Let us call a partition of the set $\mathbb{Z} \setminus \{0\}$ a Type B partition if it satisfies the same two conditions in the bulleted list above. Then $\prod_\infty^B$ is the poset of all Type B partitions of $\mathbb{Z} \setminus \{0\}$ where all but finitely many blocks are singletons, ordered by refinement. And for any $k \geq 1$, we can view $\prod_\infty^{B,(k)}$ as the collection of Type B partitions of a set of the form $[n] \cup \overline{[n]}$ (for $n \geq k-1$) which have $2(k-1)$ non-zero blocks.

The partial order on $\prod_\infty^{B,(k)}$ is still refinement in the sense that for $\pi, \pi' \in \prod_\infty^{B,(k)}$ we have $\pi \leq \pi'$ if for every $B \in \pi$ there exists $B' \in \pi'$ with $B \subseteq B'$. We have $F(\prod_\infty^{B,(k)}; x) = 1/((1-x)(1-3x)\cdots(1-(2k-1)x))$, since the core of $\prod_\infty^{B,(k)}$ is Π_k^B.

3.5.5. *Dowling lattices.* The most sophisticated example of a uniform sequence of supersolvable geometric lattices is due to Dowling [5, 4]. The *"Dowling lattice"* $Q_n(G)$ depends on the choice of a finite group G. Choosing G to be the trivial group gives $Q_n(G) = \Pi_{n+1}$, and choosing $G = \mathbb{Z}/2\mathbb{Z}$ gives $Q_n(G) = \Pi_n^B$. In this way, the Dowling lattices recover the previous two examples as special cases. Dowling first defined $Q_n(G)$ in [5] for G the multiplicative group of a finite field, and then in [4] for any finite group G. See also [3, §5.3] for a graphical description of $Q_n(G)$.

We now review the construction of Dowling lattices. Thus, fix a finite group G, say with m elements. The construction of $Q_n(G)$ requires several technical definitions, so please bear with us.

A *partial partition* of a set X is a collection $\pi = \{B_1, \ldots, B_k\}$ of nonempty subsets of X ($\emptyset \subsetneq B_1, \ldots, B_k \subseteq X$) that are pairwise disjoint ($B_i \cap B_j = \emptyset$ for $i \neq j$). In other words, a partial partition of X is a partition of a subset of X. Note that we allow $k = 0$, i.e., the partition with no blocks. There is a canonical bijection between the partial partitions of $[n]$ and the (usual) partitions of $[n+1]$ which takes the partial partition $\pi = \{B_1, \ldots, B_k\}$ to $\pi' := \pi \cup \{[n+1] \setminus \cup_{i=1}^k B_i\}$.

A *G-labeled set* is a map $\alpha\colon A \to G$ from a set A to G. We also denote such a G-labeled set by the pair (α, A). We say that two G-labeled sets (α, A) and (β, B) are equivalent if $A = B$ and there is $g \in G$ such that $\alpha(x) = g \cdot \beta(x)$ for all $x \in A$. We denote the equivalence class of (α, A) by $[\alpha, A]$.

A *partial G-partition* of a set X is a collection $\alpha = \{[\alpha_1, A_1], [\alpha_2, A_2], \ldots, [\alpha_k, A_k]\}$ of equivalence classes of G-labeled sets for which the underlying sets $\{A_1, \ldots, A_k\}$ form a partial partition of X. We continue to refer to the A_i as the blocks of α.

We can now define $Q_n(G)$. The elements of $Q_n(G)$ are are all partial G-partitions of $[n]$. And the partial order is: for partial G-partitions $\alpha = \{[\alpha_1, A_1], \ldots, [\alpha_k, A_k]\}$ and $\beta = \{[\beta_1, B_1], \ldots, [\beta_\ell, B_\ell]\}$, we have $\alpha \leq \beta$ if

- each block B_j in β is a union $B_j = A_{i_1} \cup \cdots \cup A_{i_r}$ of blocks $A_{i_1}, \ldots, A_{i_r}$ in α;
- for any block A_i in α with $A_i \subseteq B_j$ for some block B_j in β, we have that the restriction $\beta_j \mid_{A_i}\colon A_i \to G$ of the G-labeled set $\beta_j\colon B_j \to G$ to A_i is equivalent to the G-labeled set $\alpha_i\colon A_i \to G$.

For instance, the maximum element of $Q_n(G)$ is the partial G-partition $\alpha = \emptyset$ with no blocks. And the minimum element of $Q_n(G)$ is $\alpha = \{[*, \{1\}], [*, \{2\}], \ldots, [*, \{n\}]\}$, where $*$ denotes the map to G which is constantly equal to the identity $e \in G$.

Dowling [4] proved that $Q_n(G)$ is a supersolvable geometric lattice of rank n, and that taking $L_n = Q_n(G)$ gives a uniform sequence (see also [16, Exercise 3.131]). Note that the rank function on $Q_n(G)$ is given by $\rho(\alpha) = n - k$ if α has k blocks.

We define $\iota_n\colon Q_n(G) \to Q_{n+1}(G)$ and $\theta_s\colon [s, \hat{1}_{Q_n(G)}] \to Q_{n-1}(G)$ for this sequence as follows. First, we set $\iota_n(\alpha) := \alpha \cup \{[n+1, *]\}$ for all $\alpha \in Q_n(G)$. Next, consider an atom $s \in Q_n(G)$. This atom s could have all singleton blocks, being $s = \{[*, \{1\}], \ldots, [*, \{n\}]\} \setminus \{[*, i]\}$ for some $1 \leq i \leq n$; denote such an atom by s_i. Or, the s could have a single non-singleton block of the form $\{i,j\}$ for some $1 \leq i < j \leq n$; denote such an atom by $s_{i,j}$.[2] We describe how to obtain $\theta_s(\alpha)$ from α in words. In the case where $s = s_i$, to obtain $\theta_s(\alpha)$ from α we re-index by subtracting one from all numbers greater than i. In the case where $s = s_{i,j}$, to obtain $\theta_s(\alpha)$ from α we delete j and re-index by subtracting one from all numbers greater than j. It is again a straightforward, albeit tedious, check that these satisfy the requirements.

For this sequence we have $a_n = 1 + (n-1)m$, and hence

$$V(i,j) = h_{i-j}(1, 1+m, 1+2m, \ldots, 1+jm);$$
$$v(i,j) = (-1)^{i-j} e_{i-j}(1, 1+m, 1+2m, \ldots, 1+(i-1)m).$$

In particular, $\chi^*(Q_n(G); x) = (1-x)(1-(1+m)x)\cdots(1-(1+(n-1)m)x)$.

Let us use $\mathcal{Q}_\infty(G) \coloneqq \mathcal{L}_\infty$ and $\mathcal{Q}^{(k)}_\infty(G) \coloneqq \mathcal{L}^{(k)}_\infty$ for this sequence $L_n = Q_n(G)$. Then, $\mathcal{Q}_\infty(G)$ is the poset of partial G-partitions $\alpha = \{[\alpha_1, A_1], [\alpha_2, A_2], \ldots\}$ of the set $\{1, 2, \ldots\}$ for which:

- all but finitely many blocks $A_1, A_2, \ldots$ are singletons;
- the union $A_1 \cup A_2 \cup \cdots$ is cofinite, i.e., $\{1, 2, \ldots\} \setminus \cup_{i=1}^{\infty} A_i$ is finite.

The partial order $\alpha \leq \beta$ for such partial G-partitions is exactly as described above.

Unfortunately, the best description of $\mathcal{Q}^{(k)}_\infty(G)$ we have is slightly ugly. For $k \geq 1$, we can represent the elements of $\mathcal{Q}^{(k)}_\infty(G)$ by pairs (α, n) where $n \in \mathbb{N}$ is a nonnegative integer and α is a partial G-partition of $[n]$ into $k-1$ blocks. The partial order is $(\alpha, i) \leq (\beta, j)$ if $i \leq j$ and $\alpha \cup \{[*, \{i+1\}], \ldots, [*, \{j\}]\} \leq \beta$ according to the partial order on partial G-partitions described above. Alternatively, we can say that $(\alpha, i) \leq (\beta, j)$ if $i \leq j$ and $\alpha \leq \beta'$, where β' is the result of deleting $i+1, \ldots, j$ from β. We have $F(\mathcal{Q}^{(k)}_\infty(G); x) = 1/((1-x)(1-(1+m)x)\cdots(1-(1+(k-1)m)x))$, since the core of $\mathcal{Q}^{(k)}_\infty(G)$ is $Q_k(G)$.

4. Upho lattices from monoids

4.1. Monoids, free monoids, and monoid presentations.

4.2. Upho posets from cancellative monoids.

4.3. A monoid for rank two cores.

4.4. Garside monoids and Coxeter groups.

4.4.1. *The classical positive braid monoid and weak order.*

4.4.2. *The dual braid monoid and noncrossing partition lattice.*

5. Obstructions for cores of upho lattices

5.1. Characteristic polynomial obstructions.

5.1.1. *The face lattice of the octahedron.*

[2] In this notation, we suppress the choice of the G-labelling, but the G-labelling is irrelevant.

5.1.2. *The bond lattice of the four cycle.*

5.2. **Structural obstructions.**

5.2.1. *The face lattice of the cross polytope.*

5.2.2. *The lattice of flats of the uniform matroid.*

6. Distributive and modular upho lattices

6.1. **Classification of upho distributive lattices.**

6.2. **Upho modular lattices with rank two cores.**

6.3. **Upho modular lattices from DVRs with finite residue fields.**

6.4. **Stanley's conjectural classification of upho modular lattices.**

6.5. **Upho modular lattices from affine buildings.**

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