Math 210 (Modern Algebra I), Final Exam: Part 2,

Fall 2025; Instructor: Sam Hopkins; Taken on: Wednesday, December 3rd

This is the in-person part of the final, which you have 80 minutes to do. Partial credit will be given generously, so write as much as you know for each problem. Each problem is worth 10 points.

1. (a) Consider the permutation $\sigma = (1,2,3,4,5,6) \in S_6$, i.e., a six-cycle in the symmetric group on six letters. Write down the cycle notation of σ^i for $i = 0, 1, \ldots, 5$.
   (b) For general $n \geq 1$, if $\sigma \in S_n$ is an n-cycle, for which $i = 0, 1, \ldots, n-1$ will we have that σ^i is also an n-cycle? Explain.
2. (a) How many Sylow 2-subgroups of the symmetric group S_3 are there? List them all.
   (b) How many Sylow 3-subgroups of the symmetric group S_3 are there? List them all.
   (c) Verify that the subgroup counts you found in a) and b) agree with the Sylow theorems.
3. For all of the following, explain why your examples work.
   (a) Give an example of a proper ideal of the integers $\mathbb{Z}$ which is *not* a *prime* ideal.
   (b) Give an example of a nonzero, proper ideal of the Gaussian integers $\mathbb{Z}[i]$ which *is prime.*
   (c) Give an example of a proper ideal of the polynomial ring $\mathbb{Q}[x]$ over the rationals which is *not prime.*
   (d) Give an example of an ideal of the polynomial ring $\mathbb{Z}[x]$ over the integers which is *not* a *principal* ideal.
4. Give a specific example of a commutative ring R and a short exact sequence of R-modules which is not split exact. Explain what this means, and why your example works.