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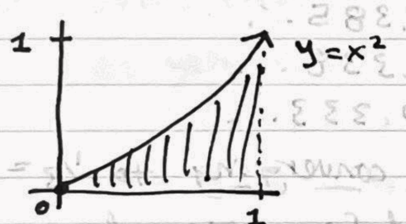
Area under a curve § 5.1

On the 1st day of class, we briefly discussed two problems that calculus solves: the tangent to a curve, and the area under a curve.

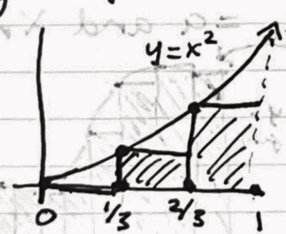
We've spent many weeks discussing the tangent and its relation to the derivative. We end the semester discussing area under a curve and the integral.

Let $f(x) = x^2$ and consider curve $y = f(x)$. What's the area between this curve and the x-axis, for $0 \le x \le 1$?

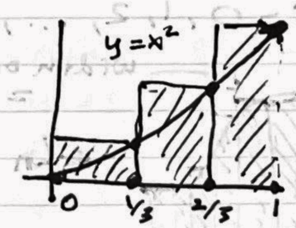


⇐ A = area of shaded region

In geometry we learn formulas for areas of shapes like triangles, rectangles, circles, ... but this shape is different. However, we could approximate the area A by using shapes like rectangles which are easy to work with:



L_3 = area of ← ▨



R_3 = area of ← ▨

On the left we drew 3 rectangles of width $1/3$ where the left vertex of the top of each rectangle touches $y = f(x)$, and on the right we drew 3 rectangles of width $1/3$ where the right vertex of the top of each rect. touches $y = f(x)$.

We see that $L_3 < A < R_3$.

width of 1st rectangle, height of 1st rect., width of 2nd rect., height of 2nd rect.

We can compute $L_3 = \underbrace{(\tfrac{1}{3})}_{\text{width of 1st rectangle}}\cdot \underbrace{0^2}_{\text{height of 1st rect.}} + \underbrace{(\tfrac{1}{3})}_{\text{width of 2nd rect.}}\underbrace{(\tfrac{1}{3})^2}_{\text{height of 2nd rect.}} + (\tfrac{1}{3})(\tfrac{2}{3})^2 = \frac{11}{81}$

and $R_3 = (\tfrac{1}{3})(\tfrac{1}{3})^2 + (\tfrac{1}{3})(\tfrac{2}{3})^2 + (\tfrac{1}{3})\,1^2 = \frac{42}{81}$

so that $0.1358\ldots = \frac{11}{81} < A < \frac{42}{81} = 0.5185\ldots$

If we let L_n and R_n denote the analagous areas of rectangles but where we use n rectangles of width $1/n$ (touching curve at left and right top vertices, resp.) then we always have $L_n < A < R_n$ and larger values of n give better approximations!

e.g. $n = 10 \Rightarrow 0.285\ldots < A < 0.385\ldots$

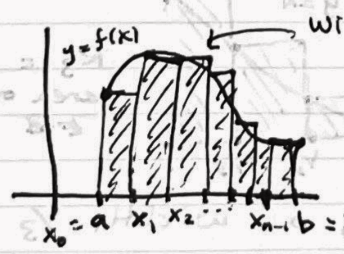
$n = 100 \Rightarrow 0.328\ldots < A < 0.338\ldots$

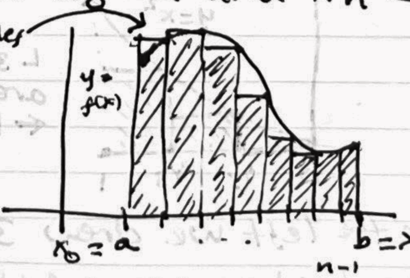
$n = 1000 \Rightarrow 0.332\ldots < A < 0.333\ldots$

It looks like the bounds are converging to $1/3 = 0.\overline{3}$

This is true! Suggests we can define area under curve as a limit:

Def'n Let $f(x)$ be defined on a closed interval $[a,b]$. Fix n, and let $\Delta x = \frac{b-a}{n}$, and let $x_i = a + i \cdot \Delta x$ for all $i = 0, 1, 2, \ldots, n$ (so $x_0 = a$ and $x_n = b$).





Σ is shorthand for sum

Let $L_n = \Delta x \cdot f(x_0) + \Delta x \cdot f(x_1) + \ldots + \Delta x\, f(x_{n-1}) = \sum_{i=0}^{n-1} \Delta x \cdot f(x_i)$

and $R_n = \Delta x \cdot f(x_1) + \Delta x \cdot f(x_2) + \cdots + \Delta x\, f(x_n) = \sum_{i=1}^{n} \Delta x \cdot f(x_i)$.

Then, as long as $f(x)$ is continuous, the limits of the areas $\lim_{n\to\infty} L_n$ and $\lim_{n\to\infty} R_n$ exist and are equal, so we define

A = area under the curve $= \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n$.

E.g.: Let us return to $f(x) = x^2$ defined on $[0,1]$.

Then $R_n = \frac{1}{n} \cdot f(\frac{1}{n}) + \frac{1}{n} \cdot f(\frac{2}{n}) + \cdots + \frac{1}{n} f(\frac{n}{n})$

$$= \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$

$$= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right).$$

Proposition $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

E.g. $1^2 = 1 = \frac{1(1+1)(2+1)}{6}$, $1^2 + 2^2 = 5 = \frac{2(2+1)(4+1)}{6}$, ...

Proof: This can be proved by mathematical induction.

Maybe you have seen the similar formula:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

The n^2 one is slightly more complicated, but basically same. $\boxtimes$

So $R_n = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{2n^3 + 3n^2 + n}{6n^3}$

Thus $A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{2n^3+3n^2+n}{6n^3} = \frac{2}{6} = \frac{1}{3}$.

This definition of area under the curve in terms of limits of rectangle sums is conceptually clear, but difficult to compute with: we have to find formulas like $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

One of the main insights of calculus is that there is another way to find these areas using anti-derivatives of functions, which is much more computationally easy!

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The Definite Integral § 5.2

Area under the curve is so important that we give it a special name and notation.

Def'n Let $f(x)$ be a continuous function defined on $[a,b]$. The (definite) integral of $f(x)$ from a to b is

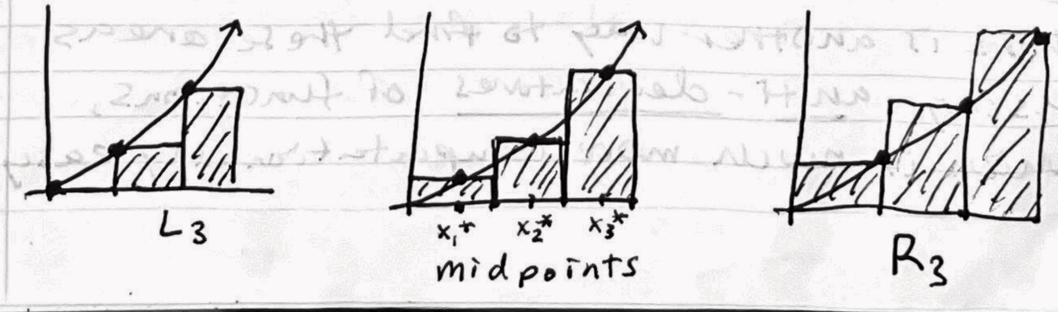
$$\int_a^b f(x)\,dx = \text{area under curve } y=f(x) \text{ from } x=a \text{ to } x=b.$$

More precisely, fix n and let $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\cdot\Delta x$ for $i = 0,1,\dots,n$. Choose a point $x_i^* \in [x_{i-1}, x_i]$ for each $i = 1,2,\dots,n$. Then define

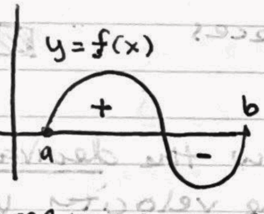
$$A_n = \sum_{i=1}^{n} \Delta x \cdot f(x_i^*)$$

and finally $\int_a^b f(x)\,dx = \lim_{n\to\infty} A_n$.

Note: If we choose $x_i^* = x_{i-1}$ for all i, then $A_n = L_n$. If we choose $x_i^* = x_i$ for all i, then $A_n = R_n$. But no matter which point we choose to determine the height of the thin rectangles in our approximation of the area under the curve, in the limit all give the same value. However, for some fixed n, the approximations will be different, and often the best choice is to use midpoints $x_i^* = \frac{x_{i-1}+x_i}{2}$.



For $f(x)$ always above x-axis, $\int_a^b f(x)\,dx$ really is the area under the curve, but for $f(x)$ that goes below the x-axis, we have to subtract that area:



$$\int_a^b f(x)\,dx = +\,(\text{area above x-axis and below } y=f(x)) - (\text{area below x-axis and above } y=f(x))$$

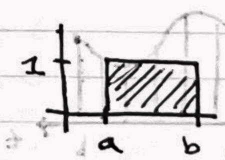
Some more properties of the integral:

Thm $\int_a^b (c\cdot f(x) + d\cdot g(x))\,dx = c\cdot\int_a^b f(x)\,dx + d\cdot\int_a^b g(x)\,dx$ for $c, d \in \mathbb{R}$ constants.

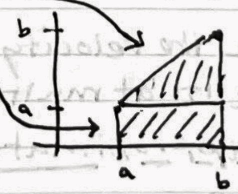
In other words, the integral is linear (just like the derivative).

Pf: $\sum_{i=1}^{n} \Delta x (c f(x_i) + d g(x_i)) = c \cdot \sum_{i=1}^{n} f(x_i) + d\cdot \sum_{i=1} g(x_i)$. ■

E.g. $\int_a^b 1\,dx = (b-a)$ since just have a rectangle →



$$\int_a^b x\,dx = \underbrace{a\cdot(b-a)}_{\text{rectangle}} + \underbrace{\tfrac{1}{2}(b-a)(b-a)}_{\text{triangle}}$$
$$= \tfrac{1}{2}(a+b)(b-a) = \tfrac{1}{2}(b^2-a^2)$$



So that $\int_a^b (mx + c)\,dx = \frac{m}{2}(b^2 - a^2) + c(b-a)$ and we now know the integral of any linear function.

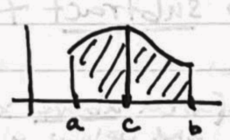
Even though we only defined $\int_a^b f(x)\,dx$ when $a \le b$ it also makes sense to let $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$, i.e. swapping endpoints of integral negates it.

Notice in particular that $\int_a^a f(x)\,dx = 0$.

Also... Proposition For any $c \in [a,b]$, we have

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$$

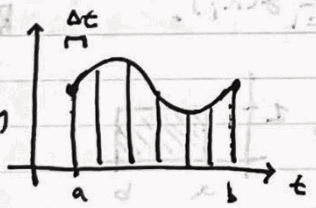
Pf: Picture:



$\Leftarrow$ split the area into two pieces

Position from velocity: We explained how the derivative (slope of tangent) lets us compute the velocity $v(t)$ of a car at time t if all we know is its position function $p(t)$. The integral does the opposite!

Specifically, suppose we know $v(t)$, velocity of a car as function of time t, on some interval $[a,b]$.



If $v(t)$ were constantly = fixed v, then the distance the car travels from time a to time b would just be $= v \cdot (b-a)$ ← elapsed time

But since the velocity is changing, we need to measure it at multiple times in the interval $[a,b]$. We can approximate the distance traveled by letting $\Delta t = \frac{b-a}{n}$ and $t_i = a + i \cdot \Delta t$ for $i = 0, 1, \ldots, n$.

Then distance traveled $\approx \sum_{i=1}^{n} \Delta t \cdot v(t_i)$

since on each short time interval $[t_{i-1}, t_i]$ the velocity is approximately constant.

And in the limit, we have exactly that:

$$\text{distance car travels in time } a \text{ to time } b = \int_a^b v(t)\,dt,$$ the integral!

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The Fundamental Theorem of Calculus §5.3

The following theorem gives a way to compute integrals:

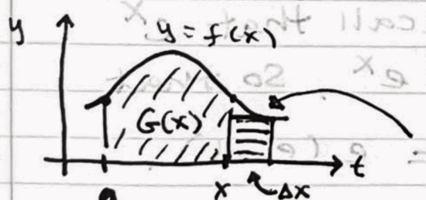
Theorem Let $f(x)$ be a continuous function.

1) Define the function $G(x) = \int_a^x f(t)\,dt$ (for a fixed $a \in \mathbb{R}$).
   Then $G'(x) = f(x)$.

2) Suppose that $F(x)$ is any anti-derivative of $f(x)$.
   Then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Pf: This is just a proof sketch, see book for details.

For 1) The function $G(x)$ computes area under the curve $y = f(t)$ for $t = a$ to $t = x$:



If we increase x by Δx, then how does $G(x)$ change? Well, since $f(x)$ is continuous, we roughly add $\Delta x \cdot f(x)$ to $G(x)$.

Thus, $\Delta G \approx \Delta x \cdot f(x)$, i.e., $f(x) \approx \frac{\Delta G}{\Delta x}$.

As $\Delta x \to 0$, we get exactly that $\frac{dG}{dx} = f(x)$. ✓

For 2): We know from 1) that $G(x)$ is one anti-derivative of $f(x)$ (since $G'(x) = f(x)$).

So there is some constant $c \in \mathbb{R}$ such that $G(x) = F(x) + c$.

Now, $G(a) = \int_a^a f(x)\,dx = 0$, so $c = -F(a)$.

Thus, $\int_a^b f(x)\,dx = G(b) = F(b) - F(a)$. ✓

For us the point of the Fund. Thm. of Calculus is that it lets us evaluate integrals by computing anti-derivatives.

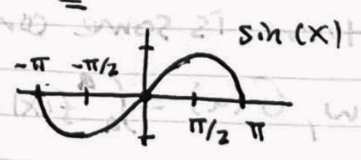
E.g.: We saw before that $\int_0^1 x^2 = 1/3$.
Let's do this again, faster. Recall that $F(x) = 1/3\, x^3$ is one anti-derivative of $f(x) = x^2$ since $F'(x) = f(x)$.
Thus, by F.T.C., $\int_0^1 x^2 dx = F(1) - F(0) = 1/3 (1)^3 - 1/3 (0)^3 = 1/3$.

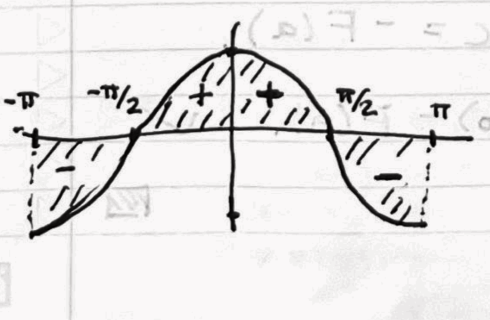
Since we so often want to compute $F(b) - F(a)$, we use the shorthand notation $F(x)\big]_a^b = F(b) - F(a)$.
Thus, F.T.C. says that $\int_a^b f(x)\,dx = F(x)\big]_a^b$.

E.g. To compute $\int_1^2 e^x dx$, we recall that e^x is the anti-derivative of e^x, so that
$$\int_1^2 e^x dx = e^x\big]_1^2 = e^2 - e^1 = e(e-1).$$

E.g.: $\sin(x)$ is an anti-derivative of $\cos(x)$, so
$$\int_{-\pi}^{\pi} \cos(x)\,dx = \sin(x)\big]_{-\pi}^{\pi} = \sin(\pi) - \sin(-\pi) = 0 - 0 = 0.$$

This makes sense, since:





← areas above and below curve $y = \cos(x)$ from $x = -\pi$ to $x = \pi$ cancel out, leaving 0 overall.