Cyclic sieving for plane partitions and symmetry

UMN Combinatorics Seminar

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Section 1

Plane partitions and their symmetry classes

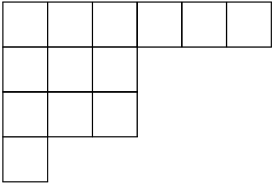
Plane partitions

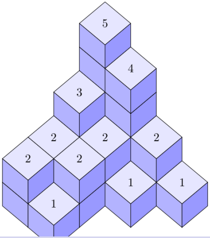
An *(integer) partition* can be viewed as a $1D$-array $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of weakly decreasing positive numbers: e.g., $\lambda = (6, 3, 3, 1)$.

A *plane partition* is a $2D$-array of weakly decreasing positive numbers: e.g.,

$$\pi = \begin{matrix} 5 & 4 & 2 & 1 \\ 3 & 2 & 1 & \\ 2 & 2 & & \\ 2 & 1 & & \end{matrix}$$

Just like partitions have a "visual" representation as Young diagrams, plane partitions also have a representation as a "stack of cubes":

$\lambda =$ 

$\pi =$ 

Plane partitions "in a box"

We'll mostly use the $2D$ representation, and focus on plane partitions that "fit in an $a \times b$ rectangle" (i.e., those with # rows $\leq a$ and # col.'s $\leq b$). Let $\mathrm{PP}^m(a \times b) := \{\text{plane partitions in } a \times b \text{ rect., with entries} \leq m\}$:

$$\begin{array}{|c|c|c|c|} \hline 5 & 2 & 1 & 0 \\ \hline 5 & 1 & 0 & 0 \\ \hline \end{array} \in \mathrm{PP}^5(2 \times 4)$$

By filling in "missing" entries with 0, these are $a \times b$ arrays $\pi = (\pi_{i,j})$ of nonnegative integers $\leq m$ that weakly decrease in rows and columns.

Theorem (MacMahon's formula (c.1900) for plane partitions in a box)

$$\mathrm{Mac}(a,b,m;q) := \sum_{\pi \in \mathrm{PP}^m(a \times b)} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{(1-q^{i+j+m-1})}{(1-q^{i+j-1})},$$

where $|\pi| = \sum_{\substack{1 \leq i \leq a, \\ 1 \leq j \leq b}} \pi_{i,j}$ *is the size of the plane partition* π.

Symmetries of plane partitions

There are various symmetry operators on $\mathrm{PP}^m(a \times b)$:

Complementation: Rotate the $2D$ diagram 180° and replace each entry i by $m - i$. (In $3D$ picture, this is complementation in the $a \times b \times m$ box.)

Transposition (requires $a = b$): Flip π across the $x = y$ diagonal.

Rotation (requires $a = b = m$): In the $3D$ picture, rotate the three axes.

Up to conjugacy, these operators generate 10 symmetry classes.

Remarkably there are **product formulas** for all 10 symmetry classes (and in many cases, q-analogs as well), although there remains **no uniform explanation** for all of these formulas.

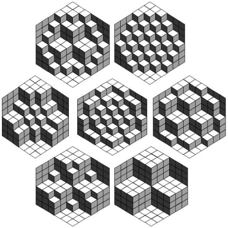
Example: TSSCPPs (and ASMs)

Most symmetric = *totally symmetric, self-complementary plane partitions*.

Theorem (Andrews, 1994)

$$\#\textit{TSSCPPs of size } 2n = \prod_{i=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

This is the same as the number of $n \times n$ *alternating sign matrices*:



$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

$$\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}0&1&0\\1&-1&1\\0&1&0\end{bmatrix} \quad \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$$

$$\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix} \quad \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$

Beginning of a beautiful story w/ connections to statistical mechanics, etc.

The involutive symmetries

From now on we will only consider involutive symmetries:

Complementation $= \mathrm{Co}$, **Transposition** $= \mathrm{Tr}$, ~~**Rotation**~~

Reasons to do this:

- can stick to $2D$ picture;
- a and b play a different role than m;
- these have meaning in representation theory of Lie algebras.

With only involutive symmetries, there are 5 symmetry classes:

$$\text{all } \pi; \qquad \mathrm{Tr}(\pi) = \pi; \qquad \mathrm{Co}(\pi) = \pi;$$

$$\mathrm{Tr} \cdot \mathrm{Co}(\pi) = \pi; \qquad \mathrm{Tr}(\pi) = \pi \text{ and } \mathrm{Co}(\pi) = \pi.$$

Symmetric plane partitions (the "MacMahon conjecture")

Theorem (Conjectured: MacMahon, 1899; Proved: Andrews, 1978, and later: Macdonald, Gordon, Proctor, ...)

$$\mathrm{SymMac}(n,m;q) := \sum_{\substack{\pi\in\mathrm{PP}^m(n\times n),\\ \mathrm{Tr}(\pi)=\pi}} q^{|\pi|} = \prod_{1\leq i<j\leq n} \frac{(1-q^{2(i+j+m-1)})}{(1-q^{2(i+j-1)})} \prod_{i=1}^{n} \frac{(1-q^{2i+m-1})}{(1-q^{2i-1})}.$$

Theorem (Conj: Bender-Knuth, 1972; Equiv to Mac: Andrews, 1977)

$$\mathrm{SymMac}'(n,m;q) := \sum_{\substack{\pi\in\mathrm{PP}^m(n\times n),\\ \mathrm{Tr}(\pi)=\pi}} q^{|\pi|'} = \prod_{1\leq i\leq j\leq n} \frac{(1-q^{i+j+m-1})}{(1-q^{i+j-1})},$$

where $|\pi|' := \sum_{1\leq i\leq j\leq n} \pi_{i,j}$ *is the "size of* π/Tr.*"*

Stembridge's $q = -1$ phenomenon

Stembridge observed that product formulas for the other 3 symmetry classes are compactly expressed via setting $q := -1$ in q-analogs:

Theorem (Proved: Stanley, 1986; "Explained": Stembridge, 1994, Kuperberg, 1994)

$$\#\{\pi \in \mathrm{PP}^m(a \times b) \colon \mathrm{Co}(\pi) = \pi\} = \mathrm{Mac}(a, b, m; q := -1)$$

Theorem (Proved: Proctor, 1990; "Explained": Kuperberg, 1994)

$$\#\{\pi \in \mathrm{PP}^m(n \times n) \colon \mathrm{Tr} \cdot \mathrm{Co}(\pi) = \pi\} = \mathrm{SymMac}(n, m; q := -1)$$

Theorem (Proved: Proctor, 1983; "Explained": Stembridge, 1994)

$$\#\{\pi \in \mathrm{PP}^m(n \times n) \colon \mathrm{Tr}(\pi) = \pi, \mathrm{Co}(\pi) = \pi\} = \mathrm{SymMac}'(n, m; q := -1)$$

Section 2

Promotion and cyclic sieving

Piecewise-linear toggles

Define PL-toggle $\tau_{i,j}\colon \mathrm{PP}^m(a\times b)\to\mathrm{PP}^m(a\times b)$, $1\le i\le a$, $1\le j\le b$, by

$$(\tau_{i,j}\pi)_{k,l}:=\begin{cases}\pi_{k,l} & \text{if } (k,l)\neq(i,j);\\ \min(\pi_{i,j-1},\pi_{i-1,j})+\max(\pi_{i+1,j},\pi_{i,j+1})-\pi_{i,j} & \text{if } (k,l)=(i,j),\end{cases}$$

with $\pi_{0,j}:=\pi_{i,0}:=m$ and $\pi_{a+1,j}:=\pi_{i,b+1}:=0$. The $\tau_{i,j}$ are involutions.

Example ($a=$, $b=3$, $m=5$)

$$\pi=\begin{array}{|c|c|c|}\hline 4 & \boxed{2} & 1\\\hline 2 & 0 & 0\\\hline\end{array}\xrightarrow{\tau_{1,2}}\begin{array}{|c|c|c|}\hline 4 & 3 & 1\\\hline 2 & 0 & 0\\\hline\end{array},$$ since $\min(4,5)+\max(0,1)-2=3$.

These were studied by: Berenstein-Kirillov, 1995; Einstein-Propp, 2013; ...
They generalize toggles of poset order ideals which were studied by: Cameron–Fon-der-Flaass, 1995; Striker, Williams, 2012; ...

Promotion of plane partitions

Promotion $\mathrm{Pro}\colon \mathrm{PP}^m(a \times b) \to \mathrm{PP}^m(a \times b)$ is the invertible operator:

$$\mathrm{Pro} := \text{composition of all toggles } \tau_{i,j} \text{ “from lower-left to upper-right.”}$$

Example ($a = b = 2$, $m = 4$)

Promotion on a $\pi \in \mathrm{PP}^4(2 \times 2)$ looks like:

$$\pi = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & 0 \\ \hline \end{array} \xrightarrow{\tau_{2,1}} \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & 0 \\ \hline \end{array} \xrightarrow{\tau_{2,2}} \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & 1 \\ \hline \end{array} \xrightarrow{\tau_{1,1}} \begin{array}{|c|c|}\hline 4 & 2 \\ \hline 1 & 1 \\ \hline \end{array} \xrightarrow{\tau_{1,2}} \begin{array}{|c|c|}\hline 4 & 3 \\ \hline 1 & 1 \\ \hline \end{array} = \mathrm{Pro}(\pi)$$

Note: $\exists$ bijection from $\mathrm{PP}^m(a \times b)$ to semistandard Young tableaux of shape $a \times m$ with entries in $\{1, 2, \ldots, a + b\}$ (based on GT-patterns) so that Pro corresponds to usual (jeu-de-taquin) promotion on SSYT.

The case $m = 1$ (subsets)

$\exists$ bijection from $\mathrm{PP}^1(a \times b)$ to a-subsets of $\{1, 2, \ldots, a+b\}$ so that Pro corresponds to the action of the *long cycle* $c := (1, 2, \ldots, a+b) \in \mathfrak{S}_{a+b}$ (read 1/0-separating path from up-right to low-left as series of Ls and Ds):

$$\pi = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \mapsto \mathrm{L\underline{D}LL\underline{DD}LL\underline{D}} \mapsto \{2, 5, 6, 9\} = I$$

$$\mathrm{Pro}(\pi) = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 \\ \hline \end{array} \mapsto \mathrm{\underline{D}L\underline{D}LL\underline{DD}LL} \mapsto \{1, 3, 6, 7\} = c(I)$$

Cyclic sieving of plane partitions

The *cyclic sieving phenomenon* of Reiner-Stanton-White, 2001 says that we can often count fixed points of a cyclic action on a combinatorial set by plugging roots of unity into a polynomial related to this set.

One of the most impressive CSPs is:

Theorem (Rhoades, 2010; Shen-Weng, 2018)

For any $k \geq 1$,

$$\#\{\pi \in \mathrm{PP}^m(a \times b) \colon \mathrm{Pro}^k(\pi) = \pi\} = \mathrm{Mac}(a, b, m; q := \zeta^k),$$

where $\zeta := e^{2\pi i/(a+b)}$ is a primitive $(a+b)$th root of unity.

Implies promotion has very regular orbit structure: order of Pro is $a+b$; every symmetry class has a product formula; etc.

Both proofs use the "linear algebra" paradigm (will explain this later) and are **far from elementary**.

Cyclic sieving example (subsets)

Corollary ($m = 1$ case of previous theorem)

For any $k \geq 1$,

$$\#\{I \subseteq \{1,\ldots,a+b\}, \#I = a\colon c^k(I) = I\} = \begin{bmatrix} a+b \\ a \end{bmatrix}_{q:=\zeta^k},$$

where $\zeta := e^{2\pi i/(a+b)}$ *and* $\left[\begin{smallmatrix} a+b \\ a \end{smallmatrix}\right]_q := \prod_{1\leq i\leq a} \frac{(1-q^{a+b+1-i})}{(1-q^i)}$ *is the* q*-binomial.*

This is one of the Ur-CSPs, going back to Reiner-Stanton-White, 2001.

Example ($a = b = 2$)

$\left[\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right]_q = 1 + q + 2q^2 + q^3 + q^4 \Rightarrow \left[\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right]_{q:=1} = 6$, $\left[\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right]_{q:=\pm i} = 0$, $\left[\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right]_{q:=-1} = 2$

$c^0(I) = I$	$c^1(I) = I$	$c^2(I) = I$	$c^3(I) = I$
$\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$	(none)	$\{1,3\}$, $\{2,4\}$	(none)

Section 3

Interaction of promotion and the symmetries

Promotion and complementation (alias evacuation)

Note $\mathrm{Co} \cdot \mathrm{Pro} = \mathrm{Pro}^{-1} \cdot \mathrm{Co}$. A "dihedral sieving" result relates them:

Theorem (Abuzzahab-Korson-Li-Meyer, 2005 and Rhoades, 2010)

For any even $k \in \mathbb{Z}$, we have

$$\#\{\pi \in \mathrm{PP}^m(a \times b)\colon \mathrm{Co} \cdot \mathrm{Pro}^k(\pi) = \pi\} = \mathrm{Mac}(a, b, m; q := -1).$$

For any odd $k \in \mathbb{Z}$, we have

$$\#\left\{ \begin{array}{c} \pi \in \mathrm{PP}^m(a \times b)\colon \\ \mathrm{Co} \cdot \mathrm{Pro}^k(\pi) = \pi \end{array} \right\} = \begin{cases} \mathrm{Mac}(a, b, m; q := -1) & \text{if } a \text{ or } b \text{ is even} \\ (-1)^{\kappa(\lambda)}\, s_\lambda(\overbrace{1, -1, \ldots, -1, 1}^{a+b-1 \text{ values}}, 1) & \text{if } a \text{ and } b \text{ are odd,} \end{cases}$$

where s_λ is a Schur function, $\kappa(\lambda) := 0\lambda_1 + 1\lambda_2 + 2\lambda_3 + \cdots$, and $\lambda := m^a$ is a rectangular partition. (There is a product formula for this evaluation.)

Promotion and transposition

Note $\mathrm{Tr} \cdot \mathrm{Pro} = \mathrm{Pro}^{-1} \cdot \mathrm{Tr}$, and we get another "dihedral sieving" result:

Theorem (H., 2019)

For any $k \in \mathbb{Z}$, we have

$$\#\{\pi \in \mathrm{PP}^m(n \times n) \colon \mathrm{Tr} \cdot \mathrm{Pro}^k(\pi) = \pi\} = \mathrm{SymMac}(n, m; q := (-1)^k).$$

Note: in fact, there is a straightforward bijection from the set

$$\{\pi \in \mathrm{PP}^{2M}(n \times n) \colon \mathrm{Tr} \cdot \mathrm{Pro}(\pi) = \pi\}$$

to a set of "plane partition flavor" that Proctor, 1990 showed is counted by

$$\mathrm{SymMac}(n, 2M; q := -1) = \prod_{1 \leq i \leq j \leq n-1} \frac{i+j+2M}{i+j}.$$

Promotion and transpose-complementation

Note $(\mathrm{Tr}\cdot\mathrm{Co})\cdot\mathrm{Pro} = \mathrm{Pro}\cdot(\mathrm{Tr}\cdot\mathrm{Co})$. We get "bicyclic sieving":

Theorem (H., 2019)

For any $k \in \mathbb{Z}$, we have

$$\#\{\pi \in \mathrm{PP}^m(n\times n)\colon (\mathrm{Tr}\cdot\mathrm{Co})\cdot\mathrm{Pro}^{n+k}(\pi) = \pi\} = \mathrm{SymMac}(n,m;q:=\zeta^k),$$

where $\zeta := e^{\pi i/n}$ is a primitive $(2n)$th root of unity.

This requires a non-elementary argument ("linear algebra" paradigm).

Altogether, have that for any $g \in \langle \mathrm{Pro},\mathrm{Co},\mathrm{Tr}\rangle$, number of plane partitions fixed by g is given by some CSP-like evaluation.

Promotion & transpose-complementation ex. (subsets)

For $I \subseteq \{1,\ldots,a+b\}$, use $\overline{I} := \{1,\ldots,a+b\} \setminus I$ to denote complement.

Corollary ($m = 1$ case of previous theorem)

For any $k \geq 1$,

$$\#\{I \subseteq \{1,\ldots,2n\}, \#I = n\colon c^{n+k}(I) = \overline{I}\} = F(q := \zeta^k),$$

where $\zeta := e^{\pi i/n}$ *and* $F(q) := \prod_{i=1}^{n}(1+q^{2i-1})$.

Example ($n = 2$)

$$F(q) = (1+q)(1+q^3) = 1+q+q^3+q^4 \Rightarrow F(1) = 4, F(\pm i) = 2, F(-1) = 0$$

$c^2(I) = \overline{I}$	$c^3(I) = \overline{I}$	$c^0(I) = \overline{I}$	$c^1(I) = \overline{I}$
$\{1,2\}$, $\{1,4\}$, $\{2,3\}$, $\{3,4\}$	$\{1,3\}$, $\{2,4\}$	(none)	$\{1,3\}$, $\{2,4\}$

Section 4

CSPs and the Grassmannian coordinate ring

"Linear algebra" paradigm for proving CSPs

There is a general technique for proving CSPs using linear algebra: find a vector space V and linear transformation $T\colon V \to V$ such that

- there is a basis of V permuted by T according to the cyclic action (so that $\mathrm{tr}_V(T^k)$ is number of fixed points of kth power of cyclic action);
- there is an eigenbasis of V we understand well enough (to show $\mathrm{tr}_V(T^k)$ is evaluation of polynomial at corresponding root of unity).

Often V has more structure, e.g., it is a GL-representation. Then for the trace computation we can use tools like the Weyl character formula.

Usually the hard part is finding a basis that's permuted in the right way.

The Grassmannian and its coordinate ring

Grassmannian $\mathrm{Gr}(a, a+b) :=$ space of a-dimensional subsets of $\mathbb{C}^{a+b}$.

For $U \in \mathrm{Gr}(a, a+b)$ and $I \subseteq \{1, \ldots, a+b\}, \#I = a$, define the *Plücker coordinate*

$$\Delta_I(U) := \begin{array}{c} \text{maximal minor of matrix whose column span is U} \\ \text{given by selecting columns } I \end{array}$$

Map $\mathrm{Gr}(a, a+b) \to \mathbb{P}^{\binom{a+b}{a}-1}$, $U \mapsto [\Delta_I(U) \colon I \subseteq \{1, \ldots, a+b\}, \#I = a]$ is the well-known *Plücker embedding*.

Define

$$\begin{aligned} R(a, a+b) &:= \text{homogeneous coordinate ring of } \mathrm{Gr}(a, a+b) \subseteq \mathbb{P}^{\binom{a+b}{a}-1} \\ &= \mathbb{C}[\Delta_I \colon I \subseteq \{1, \ldots, a+b\}, \#I = a]/\langle \text{Plücker relations} \rangle, \\ R(a, a+b)_m &:= \text{degree } m \text{ component of } R(a, a+b). \end{aligned}$$

Bases of the coordinate ring

$R(a, a+b)$ (and its components $R(a, a+b)_m$) have many important bases, including:

- **standard monomial basis** (Young, 1928, Hodge, 1943): easiest basis to define; elements consists of certain products $\Delta_{I_1}\Delta_{I_2}\cdots\Delta_{I_m}$ of Plücker coordinates,
- **dual canonical basis** (Lusztig, 1990, Kashiwara, 1993; Du, 1992): coming from structure of $R(a, a+b)$ as a $GL(a+b)$ representation,
- **theta basis** (Gross-Hacking-Keel-Kontsevich, 2018; Rietsch-Williams, 2017, Shen-Weng, 2018): coming from structure of $R(a, a+b)$ as a cluster algebra.

GL action on coordinate ring

Action of $GL(a+b)$ on $\mathbb{C}^{a+b}$ induces action on $\mathrm{Gr}(a, a+b)$ and in turn on the coordinate ring $R(a, a+b)$.

We have (e.g., by Borel-Weil theorem) that $R(a, a+b)_m := V^*(m\omega_a)$, the (dual of) the irreducible $GL(a+b)$ representation w/ highest weight $m\omega_a$, where ω_a is the ath fundamental weight.

Therefore the dimension of $R(a, a+b)_m$ is the number of semistandard Young tableaux of shape $a \times m$ with entries in $\{1, \ldots, a+b\}$; equivalently,

$$\dim(R(a, a+b)_m) = \#\mathrm{PP}^m(a \times b).$$

Hence all the bases of $R(a, a+b)_m$ we mentioned can be viewed as indexed by the plane partitions in $\mathrm{PP}^m(a \times b)$.

Combinatorial operators as linear maps

To carry out the linear algebra paradigm, need to realize the combinatorial operators on plane partitions as linear maps.

Promotion will correspond to the *twisted cyclic shift* $\chi \in GL(a+b)$:

$$\chi := \left(\begin{array}{c|c} 0 & (-1)^{a-1} \\ \hline \mathrm{Id}_{a+b-1} & 0 \end{array} \right) \in GL(a+b).$$

Complementation will correspond to the *twisted reflection* $\overline{w_0} \in GL(a+b)$:

$$\overline{w_0} := i^{(a-1)} \cdot \mathrm{antidiag}(1,1,\ldots,1) \in GL(a+b).$$

Transposition will correspond to the *symplectic orthogonal complement* $\phi\colon \mathrm{Gr}(n,2n) \to \mathrm{Gr}(n,2n)$, where the symplectic form on $\mathbb{C}^{2n}$ is defined by the skew-symmetric matrix $\mathrm{antidiag}(1,-1,1,-1,\ldots,1,-1)$.

Behavior of these maps on the bases

Work of

Stembridge, 1994, Kuperberg, 1994, Berenstein-Zelevinsky, 1996, Stembridge, 1996, Rhoades, 2010, Shen-Weng, 2018, H., 2019

says the following about how these maps act on the bases of $R(a, a+b)_m$:

Basis	χ acts as Pro?	$\overline{w_0}$ acts as Co?	ϕ acts as Tr?
Standard monomial	No! (if $m \neq 1$)	Yes	Yes
Dual canonical	Yes	Yes	Yes
Theta	Yes	Don't know	Think so

The row of Yes's for the dual canonical basis lets us carry out the linear algebra paradigm to prove our CSP-like results.

Cluster algebra people: would love to think about the theta basis with you!

Section 5

Rowmotion and further conjectures

Rowmotion and promotion

There is another operator on plane partitions closely related to promotion called *rowmotion* $\mathrm{Row}\colon \mathrm{PP}^m(a \times b) \to \mathrm{PP}^m(a \times b)$:

$$\mathrm{Row} := \text{composition of all toggles } \tau_{i,j} \text{ "from upper-left to lower-right."}$$

Example ($a = b = 2$, $m = 4$)

Rowmotion on a $\pi \in \mathrm{PP}^4(2 \times 2)$ looks like:

$$\pi = \begin{array}{|c|c|}\hline \boxed{2} & 2 \\ \hline 1 & 0 \\ \hline \end{array} \xrightarrow{\tau_{1,1}} \begin{array}{|c|c|}\hline 4 & \boxed{2} \\ \hline 1 & 0 \\ \hline \end{array} \xrightarrow{\tau_{1,2}} \begin{array}{|c|c|}\hline 4 & 2 \\ \hline \boxed{1} & 0 \\ \hline \end{array} \xrightarrow{\tau_{2,1}} \begin{array}{|c|c|}\hline 4 & 2 \\ \hline 3 & \boxed{0} \\ \hline \end{array} \xrightarrow{\tau_{2,2}} \begin{array}{|c|c|}\hline 4 & 2 \\ \hline 3 & 2 \\ \hline \end{array} = \mathrm{Row}(\pi)$$

Theorem (Striker-Williams, 2012)

Pro *and* Row *are conjugate. In fact, there's an explicit* $D \in \langle \tau_{i,j} \rangle$ *for which* $D \cdot \mathrm{Row} \cdot D^{-1} = \mathrm{Pro}$.

Interaction of conjugating map and symmetries

Straightforward but tedious computations with toggles give:

Lemma

$D \cdot \mathrm{Co} \cdot \mathrm{Row}^{-(a+1)} = \mathrm{Co} \cdot D$ and (if $a = b = n$) $D \cdot \mathrm{Tr} \cdot \mathrm{Row}^{n} = \mathrm{Tr} \cdot \mathrm{Co} \cdot D$.

This means that all results hold for Row; i.e., for any $g \in \langle \mathrm{Row}, \mathrm{Co}, \mathrm{Tr} \rangle$, $\#$ plane partitions fixed by g is given by a CSP-like evaluation. Slight differences (e.g., $\mathrm{Row} \cdot \mathrm{Tr} = \mathrm{Tr} \cdot \mathrm{Row}$) lead to slightly different formulas:

Corollary

For any $k \in \mathbb{Z}$, we have

$$\#\{\pi \in \mathrm{PP}^{m}(n \times n) \colon \mathrm{Tr} \cdot \mathrm{Row}^{k}(\pi) = \pi\} = \mathrm{SymMac}(n, m; q := \zeta^{k}),$$

where $\zeta := e^{\pi i/n}$ is a primitive $(2n)$th root of unity.

Rowmotion of P-partitions

For a finite poset P, a *P-partition of height m* is a weakly order preserving map $\pi \colon P \to \{0, 1, \ldots, m\}$. Denote set of these by $\mathrm{PP}^m(P)$.

PL-toggles τ_p for $p \in P$ defined exactly analogously:

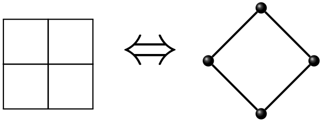
$$(\tau_p \pi)(q) := \begin{cases} \pi(q) & \text{if } p \neq q; \\ \min(\{\pi(r) \colon p \lessdot r\}) + \max(\{\pi(r) \colon r \lessdot p\}) - \pi(p) & \text{if } p = q, \end{cases}$$

with $\min(\varnothing) := m$ and $\max(\varnothing) := 0$.
Rowmotion $\mathrm{Row} \colon \mathrm{PP}^m(P) \to \mathrm{PP}^m(P)$ still makes sense:

$$\mathrm{Row} := \tau_{p_1} \cdot \tau_{p_2} \cdots \tau_{p_n},$$

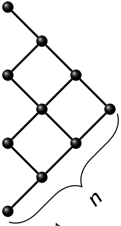
where $p_1, \ldots, p_n$ is any linear extension of P (toggle "top-to-bottom").

Case of plane partitions in a box corresponds to the "rectangle poset":

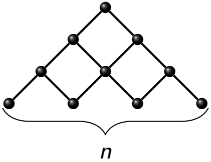


Conjectured CSPs for “triangular” posets

Two triangular posets also (conjecturally) behave very well under rowmotion:



$\rhd_n$



$\triangle_n$

Conjecture (H., 2019)

$\forall k \in \mathbb{Z}, \#\{\pi \in \mathrm{PP}^m(\rhd_n) \colon \mathrm{Row}^k(\pi) = \pi\} = \mathrm{SymMac}'(n, m; q := e^{\pi i k/n})$.

Conjecture (Propp, 2015, H., 2019)

$\forall k \in \mathbb{Z}, \#\{\pi \in \mathrm{PP}^m(\triangle_n) \colon \mathrm{Row}^k(\pi) = \pi\} = \mathrm{Cat}(n, m; q := e^{\pi i k/(n+1)})$, *where* $\mathrm{Cat}(n, m; q) := \prod_{1 \le i \le j \le n} \frac{(1-q^{i+j+2m})}{(1-q^{i+j})}$.

Embedding triangles in the square

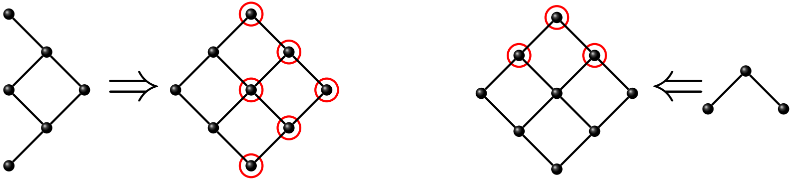
Arguments of Grinberg-Roby, 2015 give the following embeddings:

Lemma

There is a Row*-equivariant bijection between* $\mathrm{PP}^m(\triangleright_n)$ *and the subset of* $\pi \in \mathrm{PP}^m(n \times n)$ *for which* $\mathrm{Tr}(\pi) = \pi$*.*

Lemma

There is a Row*-equivariant bijection between* $\mathrm{PP}^M(\triangle_{n-1})$ *and the subset of* $\pi \in \mathrm{PP}^{2M}(n \times n)$ *for which* $\mathrm{Tr} \cdot \mathrm{Row}^n(\pi) = \pi$*.*



Reformulation of conjectured triangular CSPs

Conjecture (Reformulation of $\triangleright_n$ conjecture)

$$\forall k \in \mathbb{Z}, \#\{\pi \in \mathrm{PP}^m(n \times n) \colon \mathrm{Tr}(\pi) = \pi, \mathrm{Row}^k(\pi) = \pi\}$$
$$= \mathrm{SymMac}'(n, m; q := e^{\pi i k/n})$$

Conjecture (Reformulation of $\triangle_n$ conjecture)

$$\forall k \in \mathbb{Z}, \#\{\pi \in \mathrm{PP}^{2M}(n \times n) \colon \mathrm{Tr} \cdot \mathrm{Row}^n(\pi) = \pi, \mathrm{Row}^k(\pi) = \pi\}$$
$$= \mathrm{Cat}(n-1, M; q := e^{\pi i k/n})$$

Question

Is the number of plane partitions in $\mathrm{PP}^m(n \times n)$ fixed by H, where H is **any** *subgroup of $\langle \mathrm{Row}, \mathrm{Tr} \rangle$, given by a cyclic sieving-like evaluation of a nice polynomial at a root of unity?*

Thank you!

these slides are available on my website
paper: `arXiv:1907.09337`