Combinatorics I

prof. Sam

HW # 3.

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Total: 50/50

Q1 The complete bipartite graph $K_{n,m}$ is the graph with vertex set $X \cup Y$ where $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$ and with edges $\{x_i, y_j\}$ for all $1 \le i \le n$, $1 \le j \le m$.
Use the Matrix-Tree Thm to show that the number of spaning trees of $K_{n,m}$ is $n^{m-1} m^{n-1}$.

solution:

Using notation I_k for $k \times k$ unit matrix and J_{bc} for $b \times c$ matrix with -1 in its every cell, we order the vertices having first all the X's and then all the Y's and get the Laplacian matrix as

$$L = \begin{pmatrix} E & F \\ G & D \end{pmatrix} \longrightarrow (1)$$

Good

(where $E = mI_n$, $D = nI_m$, $F = G^T = J_{nm}$) $\longrightarrow$ (2)

we use the MTT Theorem, delete the top row and the leftmost column of L and evalute the number of spaning trees as

$$K_{n,m} = \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C)\det(D) \longrightarrow (3)$$

(where $A = mI_{n-1}$, $B = C^T = J_{n-1,m}$) $\longrightarrow$ (4)

Now, we note that $\det(D) = n^m \longrightarrow$ (5)

Next, we note that $D^{-1} = \frac{1}{n} I_m \longrightarrow$ (6)

Then $BD^{-1}C = \frac{1}{n} BC = \frac{m}{n}(-J_{n-1,n-1}) \longrightarrow$ (7)

$A - BD^{-1}C = \frac{m}{n} H \longrightarrow$ (8)

(where $H = (nI_{n-1} + J_{n-1,n-1})$) $\rightarrow$ (9)

And Matrix H has $(n-1)$ in each diagonal ~~cell~~ cell and (-1) in each off-diagonal cell.

If we add to the top row all the rows below it (which does not change det H, each cell of the top row will contain

$$(n-1) + (n-2)(-1) = 1 \rightarrow (10)$$

Now, if we add the top row to each of the other rows, we shall have 0 in all the off-diagonal cells below the top row, and n in all the $(n-2)$ diagonal cells below the top row.

Therefore,

$$\det(H) = n^{n-2} \rightarrow (11)$$

good! (and we computed this det. before in class as well…)

From (11) and (8), we obtain

$$\det(A - BD^{-1}C) = \left(\frac{m}{n}\right)^{n-1} n^{n-2} = m^{n-1} n^{-1} \rightarrow (12)$$

From (12), (6), and (3), we obtain

$$k_{n,m} = m^{n-1} n^{-1} n^{m}$$

$$= m^{n-1} n^{-1} n^{m} = m^{n-1} n^{m-1} \qquad \square$$

Excellent

10/10

Q2| Comput the number of closed walks of length l in the complete bipartite graph $K_{n,m}$. Use this computition, together with the Transfer Matrix Method, to find the eigenvalues of the adjacency matrix of $K_{n,m}$.

~~proof~~:

Sol : using the notations that used in Stanley's book. ~~Let~~

The eigenvalues $\{\mu_k\}$ of the adjacency matrix are the roots of equation

$$\det(A - \mu I) = 0 \longrightarrow (1)$$

For the complete bipartite graph $K_{m,n}$, the matrix is $(m+n)\times(m+n)$

So,

$$\det(A - \mu I) = (-\mu)^{m+n} \det(I - \frac{1}{\mu} A)$$

$$= (-\mu)^{m+n} Q(\frac{1}{\mu}) \longrightarrow (2)$$

$\boxed{\text{Note}}$

Corollary (4.7.3) stats that:

Let $Q(\lambda) = \det(I - \lambda A)$, then $\sum_{n\geq 1} C_D(n) \lambda^n = \frac{\lambda Q'(\lambda)}{Q(\lambda)}$.

Now, we note that the definition of closed walks used by "Stanly book" considers walks that start at different points of the same loop as different.

In a simple bipartite graph a closed walk can only have an even number of edges.

A walk starting in either part has mn option for the 1^{st} edge

The last edge has only 1 option as it has to connect to the start

If $L > 2$, then each step but the last has either m or n options as follows.

If the walk starts in the m part it has m options for the 2^{nd} edge, n options for the 3^{rd} edge, and so on.

So, the total number of option for it is $(mn)^k$, and the same holds for a walk starting in the n part.

Therefore,

$$C(L) = 2 \begin{cases} (mn)^k, & L = 2k. \\ 0, & L = 2k+1 \end{cases}$$ Yes!

[Note]

The special case of Theorem 4.7.2 is paticularly ~~elg~~ elegant.

Let $C_D(n) = \sum_{\Gamma} w(\Gamma)$, where the sum is over all closed walk Γ in D of length n.

Now, we use Corollary 4.7.3:

$$\frac{d \ln Q}{d\lambda} = \frac{-1}{\lambda} \sum_{L \geq 1} C(L)\lambda^L = -\frac{2}{\lambda} \sum_{k \geq 1} (nm)^k \lambda^{2k} = -\frac{2}{\lambda} \frac{nm\lambda^2}{1-nm\lambda^2}$$

$$= \frac{2\lambda}{\lambda^2 - \frac{1}{mm}} = \frac{1}{\lambda - \frac{1}{\sqrt{nm}}} = \frac{d}{d\lambda} \ln\left|\lambda^2 - \frac{1}{nm}\right| \rightarrow (4)$$

From (4), using integration constant c:

$$Q(\lambda) = c\left(\lambda^2 - \frac{1}{nm}\right) \longrightarrow (5)$$

we note that, from it definition,

$$Q(\lambda = 0) = 1 \longrightarrow (6)$$

we note from (6) and (5),

$$Q(\lambda) = 1 - mn\lambda^2 \longrightarrow (7).$$

From (7) and (2),

$$\det(A - \mu I) = (-\mu)^{m+n}\left(1 - \frac{mn}{\mu^2}\right)$$

$$= (-1)^{m+n}\mu^{m+n-2}(\mu^2 - mn) \longrightarrow (8)$$

we see from (8) that all but 2 of the eigenvalues are 0, and two non-zero ones are

$$\mu = \pm\sqrt{mn}$$ Perfect. 10/10

$\square$.

Q3| Recall that for a poset P, $J(P)$ denotes the set of order ideals of P. Find all finite posets P for which

$$\sum_{I \in J(P)} x^{\# I} = (1+x)(1+x^2)(1+x+x^2).$$

Solution:

$$(1+x)(1+x^2)(1+x+x^2) = (1+x)(1+x+2x^2+x^3+x^4)$$

$$= 1+2x+3x^2+3x^3+2x^4+x^5 \longrightarrow \quad (1)$$

The coefficients tell us how many ideals of each size there are.

The empty set is the unique order ideal of size 0.

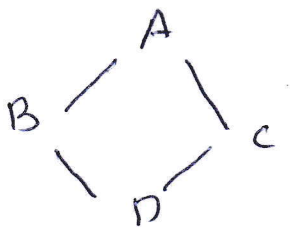
Another unique order ideal is the whole poset, so we deduce that P has 5 elements.

Each of the minimal element ~~from~~ forms an order ideal of size 1, so there are 2 minimal elements. Taking out a maximal element from P leaves an order ideal of size 4, so there are 2 maximal elements.

Besides the maximal and minimal, we also have to make sure that there are 3 order ideals of size 2 and 3 order ideals of size 3.

good!

All the possible such posets, up to isomorphism, are drawn:



$\longrightarrow$ Figur 1. (poset with an isolated element and with the required # of order ideals).

are you missing an E (isolated vertex) here?

Figur1: has a poset with one isolated element which is both a minimum and a maximum.

Its order ideals of size ~~two~~ 2 are BD, CD, and DE.

Its order ideals of size 3 are BCD, BDE and CDE.

Having more than 1 isolated element would create too many minima and maxima.

Removing any of the relations in Figure1 would create either an extra maxima or more minimum.

When there are no isolated elements, the 2 minima and 2 maxima are 4 distinct elements.
The 5^{th} element must have at least one other element larger than it. And at least one smaller to avoid having an extra minimum and maximum.

Now, if it has no other relations, there are two options that can provide for our requirements, up to isomorphism, drawn in the following Figure2 and Figure3.

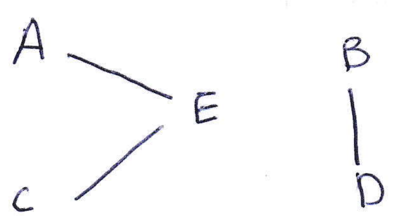


Figure2: poset with equired # of order ideals.

Figur1: has a poset with one isolated element which is both a minimum and a maximum.

Its order ideals of size ~~two~~ ≤ 2 are BD, CD, and DE.

Its order ideals of size 3 are BCD, BDE and CDE.

Having more than 1 isolated element would create too many minima and maxima.

Removing any of the relations in Figure1 would create either an extra maxima or more minimum.

When there are no isolated elements, the 2 minima and 2 maxima are 4 distinct elements.
The 5^{th} element must have at least one other element larger than it. And at least one smaller to avoid having an extra minimum and maximum.

Now, if it has no other relations, there are two options that can provide for our requirements, up to isomorphism, drawn in the following Figure2 and Figure3.

[repeated page in scan]

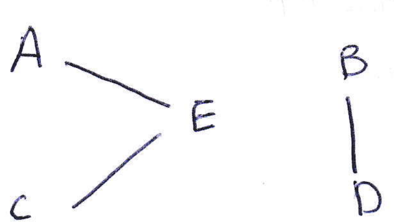


Figure2: poset with required # of order ideals (7)

In Figure 2, order ideals of size 2

are : CD, CE and BD and order idals of size 3

are: ACE, CED and CBD.

If we remove any of the relations in fugure 2, it would add either a maximum or minimum or both.

If we add a relation that is not yet in figure 2, it would chang the number of order ideals of size 2 or 3 or both.

good

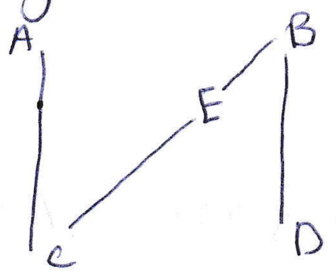


Figure 3: poset with required # of order ideals.

In this figure, the poset of order ideals of size 2 are AC, CE, and CD, and order ideals of size 3 are ACE, ACD and CED.

If we remove any of the relations in Figure 2, it would add either a maximum or a minimum or both.

If we add a relations that is not yet in Figure 2, it will change the number of order ideals of size 2 or 3 or both.

We, therefore, conclude that the poset with Hasse digrams as shown in Figure 1–3 are the only ones, up to isomorphism with the required numbers of order ideals.

very good
10/10

Q4| Let P be a finite poset. An antichain A of P is a subset $A \subseteq P$ of pairwise incomparable elements. Let $A(P)$ denote the set of antichain of P. Define a partial order $\leq$ on $A(P)$ by $A \leq A'$ iff for every $p \in A$, there is some $p' \in A'$ with $p \leq p'$.

Show that $(A(P), \leq)$ is isomorphic to $(J(P), \subseteq)$, the distributive lattice of order ideals of P order by containment.

Proof:

Let us consider the following mapping:

$$f : \mathcal{A}(P) \longrightarrow J(P)$$

$$\forall A \in \mathcal{A}(P), \quad f(A) = \{x \in P \mid x \leq p, \ \forall p \in A\} \longrightarrow \textcircled{B}^{(1)}$$

we see that $f(A)$ is, indeed, an order ideal, as any $x \leq p \in f(A)$ is included in $f(A)$.

we see also that in (1) preserves ordering, because, if $A \leq A'$ then $f(A) \subseteq f(A')$.

We also see that mapping (1) has an inverse:

$$f^{-1} = g : J(P) \longrightarrow A(P)$$

$$\forall I \in J(P), \quad g(I) = \{\text{max. of } A\} \longrightarrow (2)$$

we see that mapping (2), preserves ordering because, if $I \subseteq I'$, If $I \subseteq I'$, then any maximum of I must have a smaller or equal relation with at least one maximum of I'.

Therefore, mappings g and f are isomorphisms $\square$.

Good. 10/10

Q5) Let L be a finite lattice, with minimum element $\hat{0}$. Let $f_L(m)$ be the number of m-tuples $(t_1, \dots, t_m) \in L^m$ such that $t_1 \wedge t_2 \wedge \dots \wedge t_m = \hat{0}$. Use Möbius inversion to show that

$$f_L(m) = \sum_{t \in L} \mu(\hat{0}, t) \cdot (\#\{s \in L : s \geq t\})^m,$$

where μ is the Möbius function of L.

Sol:

Note that $g_L(m, t) = \#\{(t_1, \dots, t_m) \in L^m : t_1 \wedge t_2 \wedge \dots \wedge t_m \geq t\}$

$$= (\#\{s \in L : s \geq t\})^m \longrightarrow (1)$$

could explain more but yes

Now, from (1), we obtain:

$$\sum_{t \in L} \mu(\mu, t)(\#\{s \in L : s \geq t\})^m = \sum_{t \in L} \mu(M, t)\, g_L(m, t)$$

$$= (\mu g_L)(m, M) \longrightarrow (2)$$

on the other hand,

$$g_L(m,t) = \#\{(t_1,\dots,t_m) \in L^m : t_1 \wedge t_2 \wedge \dots \wedge t_m \geq t\}$$

$$= \sum_{s \geq t} f_L(m,s) = (\zeta f_L)(m,t) \longrightarrow (3)$$

yes

From (2) and (3), we obtain

$$\sum_{t \in L} \mu(u,t)(\#\{s \in L : s \geq t\})^m = (\mu \zeta f_L)(m,u)$$
$$= f_L(m,u) \longrightarrow (4)$$

For $u = \hat{0}$ in (4), we obtain:

$$f_L(m) = f_L(m,\hat{0}) = \sum_{t \in L} \mu(\hat{0},t)(\#\{s \in L : s \geq t\})^m \quad \square.$$

Good. 10/10