Parametric Equations § 10.1

The 1st half of the semester for Calc II focused on integration.
In 2nd half we explore other topics, starting with Chapter 10 on parametric equations & polar coordinates.

Up until now we have considered curves of the form $y = f(x)$ (or more rarely, $f(x, y) = 0$).
A parametrized curve is defined by two equations:
$$x = f(t) \quad \text{and} \quad y = g(t)$$
where t is an auxilliary variable. Often we think of t as time, so the curve describes motion of a particle where at time t particle is at position $(f(t), g(t))$:
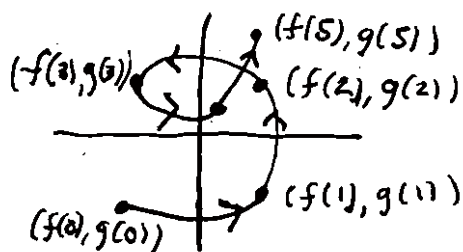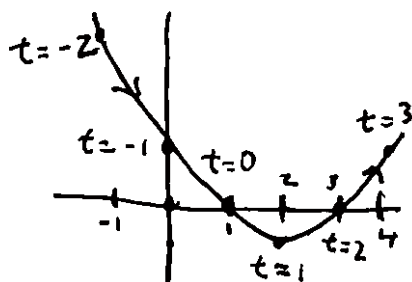

In this picture the arrows → show movement of particle over time

E.g. Consider parametrized curve $\boxed{x = t+1, \quad y = t^2 - 2t}$
We can make a chart with various values of t:

t	x	y
-2	-1	8
-1	0	3
0	1	0
1	2	-1
2	3	0
3	4	3

$\Rightarrow$

plot of points $(f(t), g(t))$ for $t = -1, 0, 1, \ldots, 4$ looks like a parabola

In this case, we can eliminate the variable t:
$$x = t+1 \Rightarrow t = x - 1$$
$$y = t^2 - 2t \Rightarrow y = (x-1)^2 - 2(x-1) = x^2 - 4x + 3$$
So this parametrized curve is just $\boxed{y = x^2 - 4x + 3}$

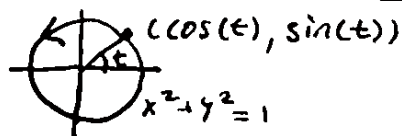E.g: Consider the parametric curve:
$$X = \cos(t), \quad y = \sin(t) \quad \text{for} \quad 0 \le t \le 2\pi$$
How can we visualize this curve?
Notice that $x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$,
So this parametrizes a <u>circle</u> $x^2 + y^2 = 1$.

← here t = angle (in radians) of point $(\cos(t), \sin(t))$ on unit circle

E.g: What about $x = \cos(2t), \ y = \sin(2t), \ 0 \le t \le 2\pi$?
Notice we still have $x^2 + y^2 = \cos^2(2t) + \sin^2(2t) = 1$,
So the parametrized curve still traces a <u>circle</u>:

But now the parametrized curve
← traces the circle <u>twice</u>:
  once for $0 \le t \le \pi$
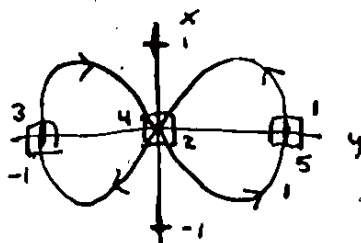 and once for $\pi \le t \le 2\pi$

Can think of this particle as moving "faster" than the last one.
We see same curve can be <u>parametrized</u> in <u>different</u> ways!

E.g: Consider the curve $X = \cos(t), \ y = \sin(2t)$.
It's possible to eliminate t to get $y^2 = 4x^2 - 4x^4$,
but that equation is hard to visualize.
Instead, graph $x = f(t)$ and $y = g(t)$ <u>separately</u>:

$x = \cos(t)$

$y = \sin(2t)$
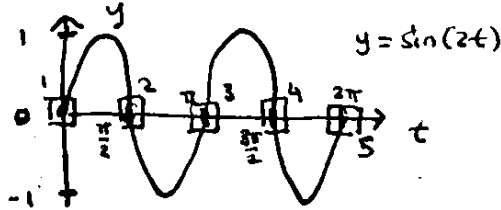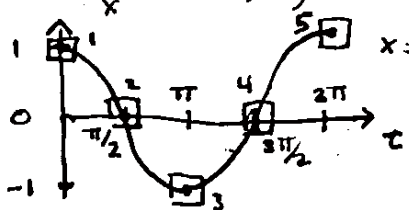
Then <u>combine</u>
into one <u>picture</u> $\Rightarrow$
showing $(f(t), g(t))$:

← are "snapshots"
of the particle
as it traces the curve

4

Calculus with parametrized curves §10.2

Much of what we have done with curves of form $y = f(x)$ in calculus can also be done for parametrized curves:

Tangent vectors: Let $(x, y) = (f(t), g(t))$ be a curve. Then, at time t, the slope of tangent vector is given by:

$$\frac{dy}{dx} \underset{\text{chain rule}}{=} \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)} \qquad (\text{if } f'(t) \neq 0)$$

If $dy/dt = 0$ (and $dx/dt \neq 0$) $\Rightarrow$ horizontal tangent
If $dx/dt = 0$ (and $dy/dt \neq 0$) $\Rightarrow$ vertical tangent

E.g. Consider curve $x = t^2$, $y = t^3 - 3t$.
First, notice that when $t = \pm\sqrt{3}$ we have

$$x = t^2 = 3 \qquad \text{and} \qquad y = t^3 - 3t = t(t^2 - 3) = 0,$$

so curve passes thru $(3, 0)$ at two times $t = \sqrt{3}$ and $t = -\sqrt{3}$.
We then compute that:

$$dy/dx = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t} \quad \begin{array}{l} \nearrow = -6/2\sqrt{3} = -\sqrt{3} \text{ at } t = -\sqrt{3} \\ \searrow = 6/2\sqrt{3} = \sqrt{3} \text{ at } t = \sqrt{3} \end{array}$$

So two tangent lines, of slopes $\pm\sqrt{3}$, for curve at $(3, 0)$.

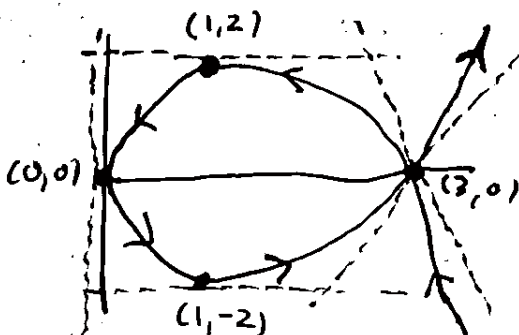When is the tangent horizontal? When $dy/dt = 3t^2 - 3 = 0$ which is for $t = \pm 1$, at points $(1, 2)$ and $(1, -2)$.
When is the tangent vertical? When $dx/dt = 2t = 0$, which is for $t = 0$, at point $(0, 0)$.

Putting all of this information together, we can produce a pretty good sketch of the curve $\Rightarrow$

Arc lengths: We saw several times how to find lengths of curves by breaking into line segments:

← recall length of each small segment
$$= \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

For a parametrized curve $(x,y) = (f(t), g(t))$ with $\alpha \le t \le \beta$ we get that the length of curve $= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \boxed{\int_\alpha^\beta \sqrt{f'(t)^2 + g'(t)^2} \, dt.}$

Exercise: Using parametrization $x = \cos(t), y = \sin(t), 0 \le t \le 2\pi$, show circumfrence of unit circle $= 2\pi$ using this formula.

E.g. The cycloid is the path a point on unit circle traces as the circle rolls!
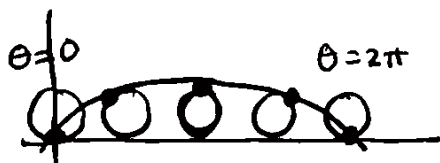
$\theta = 0$ $\theta = 2\pi$ ← think of this as an animation of a rolling circle, with point • marked where angle $\theta = $ "time"

The cycloid is parametrized by:
$$x = \theta - \sin\theta, \quad y = 1 - \cos\theta \quad \text{for } 0 \le \theta \le 2\pi$$

Q: What is the arclength of the cycloid?

A: We compute $\frac{dx}{d\theta} = 1 - \cos\theta, \quad \frac{dy}{dx} = \sin\theta$ so that

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2} = \sqrt{2(1-\cos\theta)}$$

using trig identity
$\frac{1}{2}(1 - \cos 2x) = \sin^2 x$
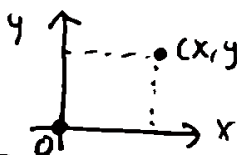
$$= \sqrt{4\sin^2(\theta/2)}$$
$$= 2\sin(\theta/2)$$

$\Rightarrow$ length of cycloid $= \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta$

$$= \int_0^{2\pi} 2\sin\left(\frac{\theta}{2}\right) d\theta = \left[-4\cos\left(\frac{\theta}{2}\right)\right]_0^{2\pi}$$

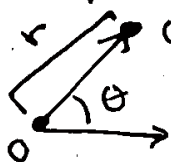$$\to ((-4 \cdot -1) - (-4 \cdot 1)) = \underline{8}.$$

11

Polar Coordinates §10.3

We are used to working with the "Cartesian" coordinate system where a point on the plane is represented by (x, y) telling us how far to move along two orthogonal axes to reach that point.
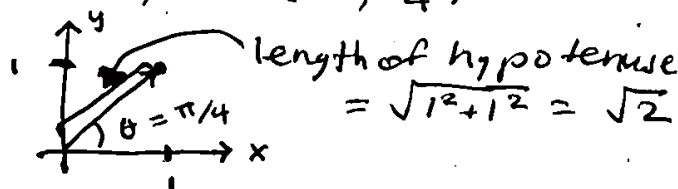
The polar coordinate system is a different way to represent points on the plane by a pair (r, θ):

Here we have a fixed axis ray $\longrightarrow$ emanating from origin O, and we reach a point (r, θ) by making an angle of θ radians and point out a distance of r.

E.g. The point $(x, y) = (1, 1)$ in Cartesian coord's is the same as $(r, \theta) = (\sqrt{2}, \frac{\pi}{4})$ in polar coord's:
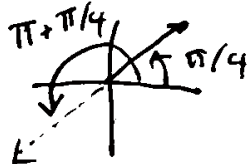
length of hypotenuse
$$= \sqrt{1^2 + 1^2} = \sqrt{2}$$

$\theta = \pi/4$

<u>Notice</u>: There are multiple ways to represent any point in polar coord's because we can <u>add</u> 2π to θ:

$(r, \theta) = (\sqrt{2}, \pi/4)$ same as $(r, \theta) = (\sqrt{2}, 2\pi + \pi/4)$

$\pi/4$
$2\pi + \pi/4$
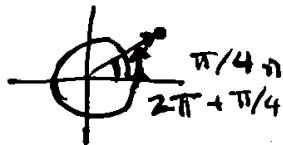
Also... Can add π to θ and replace r by $-r$:

$(r, \theta) = (\sqrt{2}, \pi/4)$ same as $(r, \theta) = (-\sqrt{2}, \pi + \pi/4)$

$\pi + \pi/4$
$\pi/4$

<u>Negative</u> value of r means go <u>backwards</u> that distance along ray.

Question: How to convert between Cartesian & polar coords?
Let's draw a right triangle to help us:
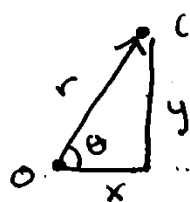
$(x,y) = (r,\theta)$

⇐ From this picture we see that

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

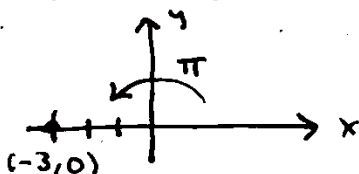which gives (x,y) in terms of (r,θ)

We also have that:

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x}$$

which gives us (r,θ) in terms of (x,y):
specifically, $r = \pm\sqrt{x^2 + y^2}$ and $\theta = \arctan\left(\frac{y}{x}\right)$.

___

E.g. Find the polar coordinates of $(x,y) = (-3, 0)$.
To solve this problem, it's easiest to just draw the point.

we see this point is at
angle $\theta = \pi$ and
radius $r = 3$.
Check: $3^2 = r^2 = x^2 + y^2 = (-3)^2 + (0)^2$
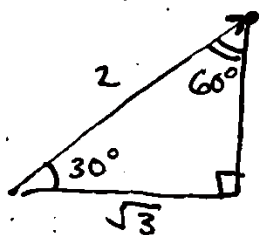and $0 = \tan(\theta) = \frac{y}{x} = \frac{0}{-3}$.

Could have also chosen $(r,\theta) = (-3, 0)$ here ...

___

E.g. Find the Cartesian coordinates of $(r,\theta) = (2, \frac{\pi}{6})$.

Here we have: $x = r\cos\theta = 2\cos(\pi/6) = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$
and $y = r\sin\theta = 2\sin(\pi/6) = 2\cdot\frac{1}{2} = 1$

Can also draw the right triangle to check:

$(\sqrt{3}, 1) = (x,y)$

⇐ recall that $\theta = \pi/6$ radians
$= 30°$
corresponds to a special
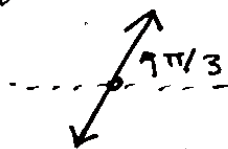"30-60-90" triangle

3/18

Polar equations and curves:

Just like we draw curves $f(x,y) = 0$ in Cartesian coord's, we can draw curves $f(r, \theta) = 0$ in Polar coord's.

Eg: The equation $r = 2$ gives circle of radius 2, centered at origin:

← circle = all points at radial distance 2 from origin O

E.g: The equation $\theta = \pi/3$ gives line at angle $\pi/3$ thru origin:

← line thru origin = all points at given angle

E.g. What about equation $r = 2 \cos \theta$?

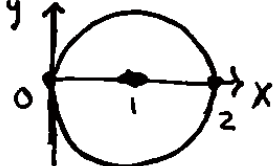Here it's easiest to switch to Cartesian coord's:

multiplying by r gives

$$r^2 = 2r \cos \theta$$
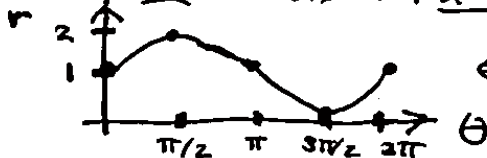$$\Leftrightarrow x^2 + y^2 = 2x$$
$$\Leftrightarrow (x-1)^2 + y^2 = 1$$

which is a circle of radius 1 centered at $(x,y) = (1,0)$:

← $(x-1)^2 + y^2 = 1$
a.k.a. $r = 2 \cos \theta$

E.g. What about $r = 1 + \sin(\theta)$?

First let's plot r as a function of θ (in Cartesian coord's):

← shows us how radius of figure changes with angle

"cardiod" ⟹
this "heart-shaped" curve is polar curve $r = 1 + \sin(\theta)$
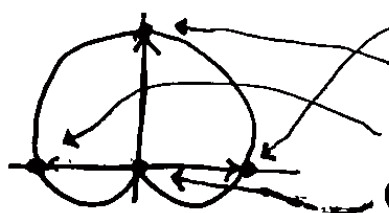
① at angle $\theta = 0$, $r = 1$
② at angle $\theta = \pi/2$, $r = 2$
so we move out to this point
③ at $\theta = \pi$, back to $r = 1$
④ at $\theta = \frac{3\pi}{2}$, radius shrinks to $r = 0$

3/20

Calculus in Polar coordinates §10.4

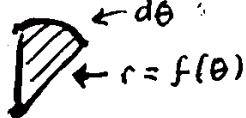We can do all types of calculus stuff in polar coord's too...

<u>Areas</u>: How to compute area "inside" polar curve $r = f(\theta)$?
where $a \le \theta \le b$

The polar curve looks something like this:

For a small change $d\theta$ in θ we get roughly a "pie slice":

$$\underline{area} = \pi r^2 \cdot \frac{d\theta}{2\pi} \Rightarrow$$

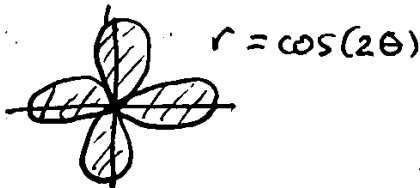$$= \frac{1}{2}(f(\theta))^2 \, d\theta$$

As usual, breaking up area into many small pie slices and summing up area gives an <u>integral</u> in limit:

$$\text{area inside polar curve} = \boxed{\int_a^b \frac{1}{2}(f(\theta))^2 \, d\theta}$$

E.g. Let's look at the curve $r = \cos(2\theta)$ for $0 \le \theta \le 2\pi$.

"four-leaf clover" shape $\Rightarrow$ $r = \cos(2\theta)$

What is area inside this curve? Using formula...

$$\text{Area} = \int_0^{2\pi} \frac{1}{2}(f(\theta))^2 \, d\theta = \int_0^{2\pi} \frac{1}{2}\cos^2 2\theta \, d\theta$$

We've seen before (using int. by parts) that $\int \cos^2 x \, dx = \frac{1}{2}(x + \sin(x)\cos(x))$

So w/ a simple u-sub $\int \frac{1}{2}\cos^2 2\theta \, d\theta = \frac{1}{4}\theta + \frac{1}{8}\sin(2\theta)\cos(2\theta)$

Thus, area $= \int_0^{2\pi} \frac{1}{2}\cos^2 2\theta \, d\theta = \left[\frac{1}{4}\theta + \frac{1}{8}\sin(2\theta)\cos(2\theta)\right]_0^{2\pi}$

$$= \left(\left(\frac{1}{4}\cdot 2\pi + \frac{1}{8}\sin(4\pi)\cos(4\pi)\right) - \left(\frac{1}{4}\cdot 0 + \frac{1}{8}\sin\cos\cos(0)\right)\right) = \boxed{\frac{\pi}{2}}$$

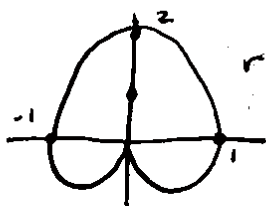**Tangents:** How to find slope of tangent to polar curve $r = f(\theta)$?

Recall $x = r \cos\theta$ and $y = r \sin\theta$ in Cartesian coord's.

So using the product rule we get:

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \quad \text{and} \quad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta}}$$

E.g. Consider the cardiod $r = 1 + \sin\theta$:

$r = 1 + \sin\theta$

$\Rightarrow dr/d\theta = \cos\theta$

Here $\frac{dy}{dx} = \frac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta}$

$= \frac{\cos\theta(1 + 2\sin\theta)}{1 - 2\sin^2\theta - \sin\theta} = \frac{\cos\theta(1 + 2\sin\theta)}{(1+\sin\theta)(1 - 2\sin\theta)}$

So at $\theta = \frac{\pi}{2}$ get $\frac{dy}{dx} = \frac{\cos(\pi/2)(1 + 2\sin(\pi/2))}{(1 + \sin(\pi/2))(1 - 2\sin(\pi/2))}$

$= \frac{0(1+2)}{(1+1)(1-2)} = \frac{0}{-2} = 0 \rightarrow$ horizontal tangent at $\theta = \pi/2$

$(r, \theta) = (2, \pi/2)$

And at $\theta = \pi/3$ get $\frac{dy}{dx} = \frac{\cos(\pi/3)(1 + 2\sin(\pi/3))}{(1 + \sin(\pi/3))(1 - 2\sin(\pi/3))}$

$= \frac{(1/2)(1 + 2\frac{\sqrt{3}}{2})}{(1 + \frac{\sqrt{3}}{2})(1 - 2\frac{\sqrt{3}}{2})} = \frac{1+\sqrt{3}}{(2+\sqrt{3})(1-\sqrt{3})} = \frac{1+\sqrt{3}}{-1-\sqrt{3}} = -1$

$\leftarrow$ tangent slope $= -1$ at $\theta = \pi/3$

Arc lengths: How to compute length of polar curve $r = f(\theta)$?

As before, from $x = r\cos\theta$ and $y = r\sin\theta$ we get

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \quad \text{and} \quad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

so that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta$$
$$+ \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta$$
$$= \left(\frac{dr}{d\theta}\right)^2 + r^2 \quad (\text{using } \sin^2\theta + \cos^2\theta = 1)$$

If we think of (x,y) as parametrized by θ, then

$$\text{length of curve} = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

which in terms of r and θ is then

$$\boxed{\text{length} = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

E.g. For a circle $r = m$ centered at origin,

this formula gives us
$$\text{length} = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{m^2 + 0^2}\, d\theta$$
$$= \int_0^{2\pi} m\, d\theta = 2\pi m,$$

which is correct circumfrence!

E.g. We saw before that $r = 2\cos\theta$, $0 \le \theta \le \pi$ gives a circle of radius 1 centered at $(x,y) = (1,0)$

Here $dr/d\theta = -2\sin\theta$, so the formula gives...

$$\text{arc length} = \int_0^{\pi} \sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta = \int_0^{\pi} 2\, d\theta = 2\pi. \checkmark$$

//