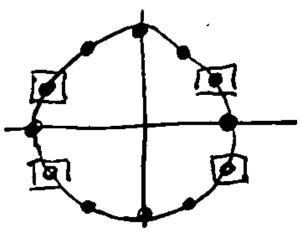
3/10 ~~Cyclotomic~~ Cyclotomic Extensions §5.8

Our goal now is to study finite extensions of $\mathbb{Q}$ of specific forms, leading up to a treatment of the problem which motivated the development of Galois theory: the solvability of polynomials by radicals.

Def'n Recall that a number $u \in \mathbb{C}$ is called an n^{th} root of unity, for some $n \geq 1$, if $u^n = 1$, i.e., if u is a root of $x^n - 1 \in \mathbb{Q}[x]$. If u is an n^{th} root of unity, it is also a $(mn)^{th}$ root of unity for any $m \geq 1$. We say u is a primitive n^{th} root of unity if it is an n^{th} root of unity but not a k^{th} root of unity for any $k < n$.

Prop. The n^{th} roots of unity are $e^{\frac{2\pi i}{n} \cdot j}$ for $j = 0, 1, \ldots, n-1$. The primitive n^{th} roots of unity are those $e^{\frac{2\pi i}{n} \cdot j}$ with $\gcd(j, n) = 1$.

E.g. We've seen before how the n^{th} roots of unity are equally spaced on the unit circle, for instance for $n = 12$ we get



⇐ the primitive 12^{th} roots of unity are circled: they are $e^{\frac{2\pi i}{12} \cdot j}$ for $j = 1, 5, 7, 11$, the integers coprime to 12.

Pf sketch of prop: That the $e^{\frac{2\pi i}{n} j}$ for $j = 0, 1, 2, \ldots, n-1$ are the n^{th} roots of unity follows from the fact that $e^{\frac{2\pi i}{n} \cdot j} \cdot e^{\frac{2\pi i}{n} \cdot k} = e^{\frac{2\pi i}{n} (j + k \bmod n)}$ (phases of complex #'s add when multiplied).

That the primitive ones are the coprime j's then follow from $e^{\frac{2\pi i}{n} \cdot j}$ is a primitive n^{th} root of unity $\Leftrightarrow$ j is a generator of $(\mathbb{Z}/n\mathbb{Z}, +)$ $\Leftrightarrow$ j is a unit in the ring $\mathbb{Z}/n\mathbb{Z}$ $\Leftrightarrow$ j is coprime to n. You will flesh out this argument on your next HW assignment. ∎

Notice: $\zeta_n = e^{\frac{2\pi i}{n}}$ is always a primitive n^{th} root of unity, and all n^{th} roots of unity are powers of this ζ_n.

Def'n Let $n \geq 1$. The n^{th} cyclotomic polynomial $\Phi_n(x) \in \mathbb{C}[x]$ is $\Phi_n(x) = \prod_{\omega \text{ a primitive } n^{\text{th}} \text{ root of unity}} (x - \omega)$ (The book uses $g_n(x)$.)

E.g. The primitive 3^{rd} roots of unity are $\omega = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\omega^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$; so $\Phi_3(x) = (x-\omega)(x-\omega^2) = x^2 + x + 1$.

In fact, the first 6 cyclotomic polynomials are:

$\Phi_1(x) = x-1$, $\Phi_2(x) = x+1$, $\Phi_3(x) = x^2+x+1$, $\Phi_4(x) = x^2+1$

$\Phi_5(x) = x^4+x^3+x^2+x+1$, $\Phi_6(x) = x^2 - x + 1$.

Thm $x^n - 1 = \prod_{d \mid n} \Phi_d(x)$.

Pf: Every root of $x^n - 1$ is an n^{th} root of unity, which is a primitive d^{th} root of unity for some $d \mid n$. $\boxtimes$

Note: Even though $\Phi_d(n)$ is a priori defined as an element of $\mathbb{C}[x]$, looks like it belongs to $\mathbb{Q}[x]$. This is true and we'll prove it! In fact the coefficients are integers, which can get arbitrarily big, but take a while ($\Phi_{105}(x)$ is first with a coeff. not in $\{1,-1\}$).

The way we will show cyclotomic polynomials are rational is by studying the extensions of $\mathbb{Q}$ we get by adjoining their roots.

Def'n The n^{th} cyclotomic extension of $\mathbb{Q}$ is the splitting field of $x^n - 1$. Equivalently, ...

Thm The n^{th} cyclotomic extension is $\mathbb{Q}(\zeta_n)$, where ζ_n is a primitive n^{th} root of unity.

Pf: Since ξ_n is an n^{th} root of unity, it certainly belongs to splitting field of $x^n - 1$. But on the other hand, every root of unity is a power of ξ_n. $\boxtimes$

Thm $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\xi_n)) = \{\psi_k : 1 \le k \le n,\ \gcd(n,k) = 1\}$ where $\psi_k(\xi_n) = \xi_n^{\,k}$. This shows $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\xi_n)) \simeq (\mathbb{Z}/n\mathbb{Z})^{\times}$ the group of units of $\mathbb{Z}/n\mathbb{Z}$, with isomorphism $\psi_k \mapsto k \in \mathbb{Z}/n\mathbb{Z}$.

Pf: Exercise. Point is that primitive n^{th} roots of unity generate $\mathbb{Q}(\xi_n)$, and cannot send ξ_n to a non-primitive n^{th} root of unity because then it would satisfy $x^m - 1$ for some $m < n$. $\boxtimes$

Cor $\Phi_n(x) \in \mathbb{Q}[x]$. In fact, $\Phi_n(x)$ is min. poly. of ξ_n.

Pf: $\mathbb{Q}(\xi_n)$ is a Galois extension of $\mathbb{Q}$ (since it's a splitting field) and every $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\xi_n))$ fixes $\Phi_n(x)$ (since permutes roots), so indeed the coefficients of $\Phi_n(x)$ must be rational. $\boxtimes$

Remark: Notice that $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\xi_n)) \simeq (\mathbb{Z}/n\mathbb{Z})^{\times}$ is always an abelian group, hence every cyclotomic extension is an "abelian extension" (= Galois ext. w/ abelian Galois grp). The order of $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\xi_n))$ is $\varphi(n) = \#\{k \le n : \gcd(n,k)\}$, Euler's totient function. If $\varphi(n) = p$ is prime, then we have seen that $(\mathbb{Z}/p\mathbb{Z})^{\times} = \mathbb{F}_p^{\times} = \mathbb{Z}/(p-1)\mathbb{Z}$ is a cyclic group, so $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\xi_n))$ is cyclic in this case. But in general it is not cyclic, just abelian, e.g.: $(\mathbb{Z}/8\mathbb{Z})^{\times} \simeq (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$. There is a description of $(\mathbb{Z}/n\mathbb{Z})^{\times}$ in general, but it is slightly messy (it's an exercise in the textbook...) //