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More notation for derivatives §2.8

By definition, $f'(x) = \lim_{k \to x} \frac{f(k) - f(x)}{k - x}$

but using $h := k - x$ ("distance to limit point") can rewrite

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

By writing $\Delta x = h$ (thinking of this as "change in x")

and $\Delta f = f(x+h) - f(x)$ ("change in f")

can write again $f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$.

This way of thinking gives another notation for the derivative, sometimes called "differential notation"

$$d/dx\,(f) = \frac{df}{dx} = f'(x) \quad \leftarrow \text{"prime notation"}$$

think of this as an "operator" acting on f.

Or if $y = f(x)$ would also write $f'(x) = \frac{dy}{dx}$.

Multiple derivatives Since $f'(x)$ is a function, we can take the derivative of it again.

"2nd derivative" of $f(x)$ is $f''(x)$, and so on w/ more primes

In differential notation, write

$$d/dx\,(d/dx\,(f)) = \frac{d^2 f}{dx^2} \quad \text{and so on...}$$

Multiple derivatives have important real-world meaning too, e.g.

$f(x)$ = position at time x

$f'(x)$ = velocity at time x

$f''(x)$ = acceleration at time x

Rules for differentiation §3.1

Now we will spend a lot of time learning rules for derivatives.

The simplest derivative is for a constant function:

Thm If $f(x) = c$ for some constant $c \in \mathbb{R}$, then $f'(x) = \underline{\underline{0}}$.

Pf: We could write a limit, or just remember tangent line definition: $\overset{y=f(x)}{\longleftrightarrow}$

If $y = f(x)$ is a line, then the tangent line at any point is $y = f(x)$, which has slope $= 0$ (horizontal) if $f(x) = c$. ▨

Actually, we see the same argument works for any linear function:

Thm If $f(x) = mx + b$ is a linear function, then $f'(x) = m$ (slope of line).

Some other simple rules for derivatives are

Thm
- (sum) $(f+g)'(x) = f'(x) + g'(x)$
- (difference) $(f-g)'(x) = f'(x) - g'(x)$
- (scaling) $(c \cdot f)'(x) = c f'(x)$ for $c \in \mathbb{R}$.

Pf: These all follow from the corresponding limit laws.

E.g., for sum have

$$(f+g)'(x) = \lim_{h \to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x). ▨$$

The first really interesting derivative is for $f(x) = x^n$, a power function. We've seen:

$$\frac{d}{dx}(x^0) = 0, \quad \frac{d}{dx}(x^1) = 1, \quad \frac{d}{dx}(x^2) = 2x$$

Do you see a pattern?

Thm For any nonnegative integer n, if $f(x) = x^n$ then

$$\boxed{f'(x) = n \cdot x^{n-1}}$$

"bring n down" from exponent.

Pf: We can use an algebra trick:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^n - a^n}{x - a}$$

check that this multiplies correctly!

$$= \lim_{x \to a} \frac{(x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1})}{x - a}$$

$$= \lim_{x \to a} (x^{n-1} + ax^{n-2} + \cdots + a^{n-1}) = \underbrace{x^{n-1} + x^{n-1} + \cdots + x^{n-1}}_{n \text{ terms}}$$

$$= n \cdot x^{n-1} \checkmark$$

This is one of the most important formulas in calculus! Remember it!!

E.g. If $f(x) = x^3$, what is $f''(x)$?

Well $f'(x) = 3 \cdot x^2$, so $f''(x) = 3 \cdot 2x = 6x$.

All derivatives of x^n easy to compute this way!

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Derivatives for more kinds of functions § 3.1, 3.3

We will learn rules for derivatives of more common functions.

Thm For any real number n, if $f(x) = x^n$

then $\boxed{f'(x) = n \cdot x^{n-1}}$.

Same as for positive integers! Though proof needs a little more (we'll skip...)

E.g. Q: If $f(x) = \sqrt{x}$, what is $f'(x)$?

A: $f(x) = x^{1/2}$ so $f'(x) = 1/2 \cdot x^{1/2 - 1} = 1/2\, x^{-1/2}$
$= 1/2 \frac{1}{x^{1/2}} = \frac{1}{2\sqrt{x}}$.

Q: If $f(x) = 1/x$, what is $f'(x)$?

A: $f(x) = x^{-1}$ so $f'(x) = -1 \cdot x^{-1-1} = -x^{-2}$
$= \frac{-1}{x^2}$.

The exponential function e^x has a surprisingly simple derivative:

Thm If $f(x) = e^x$ then $f'(x) = e^x$ $(= f(x))$.

Taking derivative of e^x does nothing!

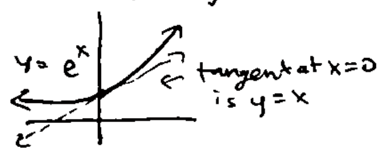
So also $f''(x) = e^x$, $f'''(x) = e^x$, etc...

Pf: We write

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x \cdot e^h - e^x \cdot e^0}{h} = e^x \cdot \underbrace{\lim_{h \to 0} \frac{e^h - e^0}{h}}_{} = e^x \cdot f'(0)$$

So we just need to show that $f'(0) = 1$.



But remember the way we defined e is as unique # $b > 1$ s.t. slope of tangent of b^x at $x=0$ is <u>one</u>! So $f'(0)=1$ by definition! ☐

10/5 So $d/dx(e^x) = e^x$, and for trig. functions d/dx <u>also</u> <u>simple</u>:

<u>Thm</u> $\boxed{d/dx(\sin(x)) = \cos(x),}$
and $\boxed{d/dx(\cos(x)) = -\sin(x)}$

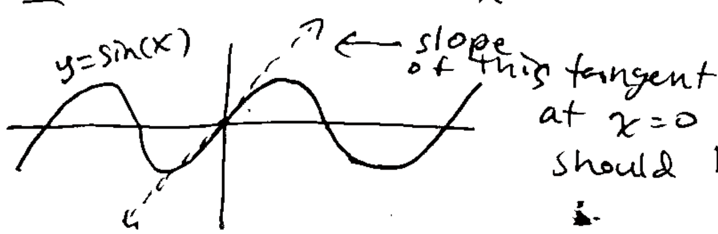
<u>E.g.</u> If $f(x) = \sin(x)$, what is $f''(x)$? Well $f'(x) = \cos(x)$
So $f''(x) = -\sin(x) = -f(x)$. ✓

→ To prove this takes a bit of work, uses some algebra tricks like "sum formula" $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
and other related trig identities.
You can see a full proof in the book (§ 3.3).
Let's focus on the most important lemma:

<u>Lemma</u> If $f(x) = \sin(x)$, then

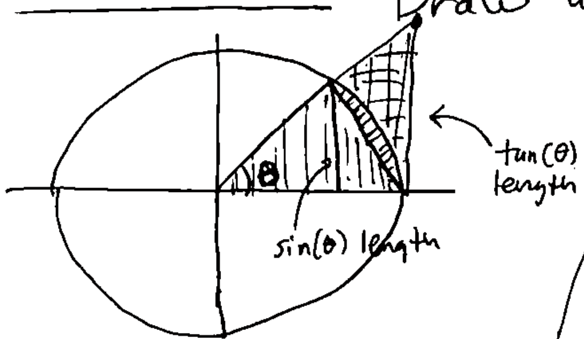
$$f'(0) = \lim_{h\to 0} \frac{\sin(0+h) - \sin(0)}{h}$$

$$= \lim_{x \to 0} \frac{\sin(x)}{x} = 1 = \cos(0).$$



There is a nice "geometric" proof of this lemma:

<u>Pf of lemma:</u> Draw unit circle and θ radian angle:



The triangle has area

$$\frac{1}{2}\cdot 1 \cdot \sin(\theta) = \frac{\sin\theta}{2}$$

The wedge has area

$$\frac{1}{2}\theta$$

and triangle has area

$$\frac{1}{2}\cdot 1 \cdot \tan(\theta) = \frac{1}{2}\,\frac{\sin(\theta)}{\cos\theta}$$

$$\frac{\sin\theta}{2} \le \frac{\theta}{2} \le \frac{\sin\theta}{2\cos\theta}$$

divide by $\frac{\sin(\theta)}{2}$ $\Rightarrow$ $1 \le \frac{\sin(\theta)}{\theta} \le \frac{1}{\cos(\theta)}$ for all $\theta > 0$

$$\Rightarrow \lim_{\theta\to 0} 1 \le \lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} \le \lim_{\theta\to 0} \frac{1}{\cos(\theta)}$$

by the <u>Squeeze Theorem</u>

but clearly $\lim_{\theta\to 0} 1 = 1$ and since $\cos(0) = 1$ also $\lim_{\theta\to 0} \frac{1}{\cos(\theta)} = 1$, so indeed $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$, as claimed! ☒

Good to just remember these basic formulas:

$d/dx\,(x^n) = n\cdot x^{n-1}$	$d/dx\,(\sin(x)) = \cos(x)$
$d/dx\,(e^x) = e^x$	$d/dx\,(\cos(x)) = -\sin(x)$

don't forget negative sign!

10/7 The product and quotient rules §3.2

Suppose we want to take derivative of a product $f \cdot g$ of two functions $f(x)$, $g(x)$ that are differentiable. Might think/hope derivative is product of derivatives, but it's easy to see $(f \cdot g)'(x) \neq f'(x) \cdot g'(x)$

E.g. Let $f(x) = x$, $g(x) = x^2$, then $f'(x) \cdot g'(x) = 1 \cdot 2x = 2x$, but $(f \cdot g)(x) = x^3$ so $(f \cdot g)'(x) = 3x^2$.

Instead we have the product rule:

Thm For two differentiable functions $f(x)$, $g(x)$:

$$\boxed{d/dx\Big((f \cdot g)(x)\Big) = f(x) \cdot \frac{dg}{dx} + g(x) \cdot \frac{df}{dx}}.$$

Pf: Write $u = f(x)$, $v = g(x)$, $\Delta u = f(x+h) - f(x)$, $\Delta v = g(x+h) - g(x)$.

Then
$$\underset{(f \cdot g)(x+h) - (f \cdot g)(x)}{\underbrace{\Delta(uv)}} = (u + \Delta u)(v + \Delta v) - uv$$
$$= \cancel{uv} + u\Delta v + v\Delta u + \Delta u \Delta v - \cancel{uv}$$
$$= u\Delta v + v\Delta u + \Delta u \Delta v$$

So
$$d/dx(uv) = \lim_{\Delta x = h \to 0} \frac{u\Delta v + v\Delta u + \Delta u \Delta v}{\Delta x}$$
$$= u\frac{dv}{dx} + v\frac{du}{dx} + \Big(\underset{0}{\underbrace{\lim_{\Delta x \to 0} \Delta u}}\Big) \cdot \frac{dv}{dx}$$
$$= u\frac{dv}{dx} + v\frac{du}{dx}$$

E.g. w/ $f(x) = x$, $g(x) = x^2$, compute
$$(f \cdot g)'(x) = f(x)\,g'(x) + g(x)\,f'(x) = x \cdot 2x + x^2 \cdot 1$$
$$= 3x^2 = d/dx\,(x^3)$$

E.g $d/dx\,(xe^x) = x\,d/dx(e^x) + e^x\,d/dx(x) = xe^x + e^x$ //

The quotient rule is a bit more complicated:

Thm For two differentiable functions $f(x), g(x)$ (w/ $g(x) \neq 0$)

$$\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot\frac{df}{dx} - f(x)\cdot\frac{dg}{dx}}{g(x)^2}}$$

Looks similar in some ways to product rule, but as said above more complicated. Can be proved same way using Δu and Δv, etc. - see the book. Alternatively, can be proved by combining product rule with the chain rule, another differentiation rule we will learn soon...

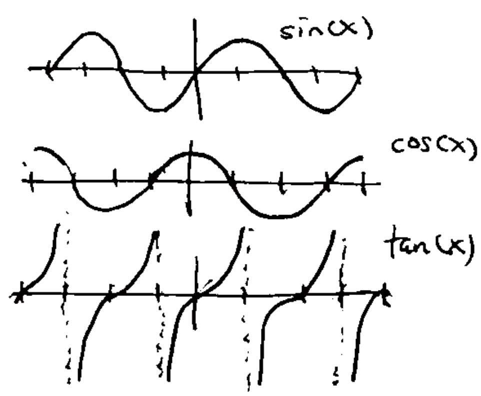
E.g. Let $f(x) = x$, $g(x) = 1-x$ so $\frac{f}{g}(x) = \frac{x}{1-x}$.

Then $\left(\frac{f}{g}\right)'(x) = \frac{g(x)\cdot\frac{df}{dx} - f(x)\cdot\frac{dg}{dx}}{g(x)^2} = \frac{(1-x)\cdot 1 - x\cdot(-1)}{(1-x)^2}$

$= \frac{1-x+x}{(1-x)^2} = \frac{1}{(1-x)^2}$.

Any rational function can be differentiated this way...

E.g. Recall $\tan(x) = \frac{\sin(x)}{\cos(x)}$



Thus, $\boxed{(\tan)'(x)} =$

$$\frac{\cos(x)\cdot(\sin)'(x) - \sin(x)\cdot(\cos)'(x)}{(\cos(x))^2}$$

$$= \frac{\cos(x)\cdot\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \boxed{\frac{1}{\cos^2(x)}}$$ using Pythagorean identity:

This is the one trig identity really worth remembering! $\longrightarrow$ $\boxed{\sin^2(x) + \cos^2(x) = 1}$ for any x.

10/12 <u>Chain</u> <u>rule</u> §3.4

Let $f(x) = \sqrt{x^2+1}$. How do we find $f'(x)$?
So far we don't know how... to do this we need the <u>chain rule</u>, which tells us how to take derivatives of <u>compositions</u> of <u>functions</u>:

<u>Theorem</u> for two differentiable fn's $f(x), g(x)$, the derivative of their composition is $\boxed{(f\circ g)'(x) = f'(g(x)) \cdot g'(x).}$

For a proof, see the textbook. But note that in differential notation the chain rule can be written:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

(where $y = f(g(x))$ and $u = g(x)$), so roughly speaking we just "cancel" the du's (but have to be careful about division by 0).

<u>E.g.</u> for $f(x) = \sqrt{x^2+1}$, write $f(x) = h(g(x))$ where $h(x) = \sqrt{x}$ and $g(x) = x^2+1$. Then by chain rule:

$$f'(x) = h'(g(x)) \cdot g'(x) = \tfrac{1}{2} \cdot (g(x)^{-1/2}) \cdot 2x$$
$$= \frac{x}{\sqrt{x^2+1}}.$$

<u>E.g.</u> Let $f(x) = \sin(x^2)$. Then,

$$f'(x) = \overbrace{\cos(x^2)}^{d/dx(\sin(x)) \text{ plug in } x = x^2} \cdot 2x. \leftarrow d/dx(x^2).$$

<u>E.g.</u> Let $f(x) = \sin^2(x)$, Then,

$$f'(x) = \underbrace{2 \cdot \sin(x)}_{d/dx(x^2) \text{ plug in } x = \sin(x)} \cdot \cos(x) \leftarrow d/dx(\sin(x)).$$

Let's show how to deduce the quotient rule by combining the product and chain rules:

Eg. For $h(x) = \frac{f(x)}{g(x)} = f(x) \cdot (g(x))^{-1}$,

Can compute $h'(x) = f(x) \cdot d/dx\left(\frac{1}{g(x)}\right) + \frac{1}{g(x)} \cdot f'(x)$.

But by the chain rule, w/ $\varphi(x) = \frac{1}{x}$,

$$d/dx\left(\frac{1}{g(x)}\right) = \varphi'(g(x)) \cdot g'(x)$$

and $\varphi'(x) = -1 \cdot x^{-2}$, so

$$d/dx\left(\frac{1}{g(x)}\right) = -\frac{1}{g(x)^2} \cdot g'(x).$$

Thus, $h'(x) = \frac{-f(x) \cdot g'(x)}{g(x)^2} + \frac{1}{g(x)} \cdot f'(x)$

$$= \frac{g(x) f'(x) - f(x) g'(x)}{g(x)^2}.$$

This is exactly the formula we claimed last class.

So upshot is... you "only" need to remember the product rule and chain rule in order to differentiate all combinations of all the kinds of functions we have been studying....

But the quotient rule formula can still be very useful to remember as a faster "short cut."

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§3.4, 3.6

<u>Derivatives</u> <u>of</u> <u>exponentials</u> <u>and</u> <u>logarithms</u>

The chain rule allows us to compute derivatives of arbitrary exponential and logarithmic functions.

Let's start with the exponential $f(x) = b^x$ for some base $b>0$. Recall that

$$b^x = e^{\ln(b)\cdot x} \quad \text{by rules of exponents.}$$

Thus $d/dx(b^x) = d/dx(e^{\ln(b)\cdot x})$

$$= e^{\ln(b)\cdot x}\cdot \ln(b) \quad \text{by chain rule}$$

$$= \ln(b)\cdot b^x.$$

<u>So</u> derivative is very similar to how e^x behaves...

What about logarithms? Recall that

$$x = e^{\ln(x)} \quad \text{(because } e \text{ and } \ln \text{ are inverses)}$$

Taking d/dx of both sides gives:

$$d/dx(x) = d/dx(e^{\ln(x)})$$

$$1 = e^{\ln(x)}\cdot d/dx(\ln(x)) \quad \text{by chain rule}$$

$$1 = x\cdot d/dx(\ln(x))$$

$$\Rightarrow \quad d/dx(\ln(x)) = 1/x.$$

(We are assuming that $\ln(x)$ is differentiable here but it is... don't worry too much about that);

$$\boxed{d/dx(\ln(x)) = 1/x}$$

Notice: You might think there is some $f(x) = a \cdot x^n$ with $f'(x) = 1/x = x^{-1}$, but we would need $n = 0$ and $a = \frac{1}{0}$ for this ($f'(x) = a \cdot n \cdot x^{n-1}$) so not possible!

How about arbitrary logarithms?
If $f(x) = \log_b(x)$ for base $b > 0$

Recall that $\log_b(x) = \dfrac{\ln(x)}{\ln(b)}$ by rules of logs,

So that $f'(x) = \dfrac{1}{\ln(b)} \cdot \dfrac{1}{x}$ (don't even need chain rule for this.)

So now that we know:

- sum, difference, scaling rules
- product and quotient rules
- chain rule
- $d/dx\,(x^n) = n \cdot x^{n-1}$ for any $n \in \mathbb{R}$
- $d/dx\,(e^x) = e^x$
- $d/dx\,(\sin(x)) = \cos(x)$
  and $d/dx\,(\cos(x)) = -\sin(x)$
- $d/dx\,(\ln(x)) = 1/x$

We can compute the derivative of basically any of the kinds of functions we have been studying all semester!