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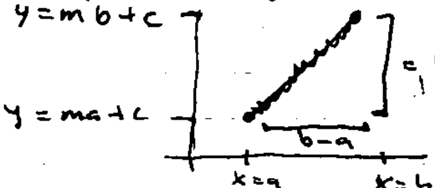
Arc lengths of curves §8.1

Having studied techniques for integration, we return to applications of integrals. We have already used integrals to compute areas (2D measures), and volumes (3D measures), what about lengths (1D measures)?

Specifically, suppose we have a curve $y=f(x)$ from $x=a$ to $x=b$: what is the length of this curve? Of course, if the curve were a line: $y = mx + c$ we could compute its length using pythagorean Theorem:



$= (mb+c) - (ma+c) = m(b-a)$

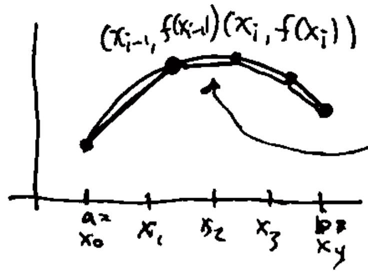
$\Rightarrow$ length of line segment

$= \sqrt{\Delta y^2 + \Delta x^2}$

$= \sqrt{m^2(b-a)^2 + (b-a)^2}$

notice: length depends on slope m of line $\rightarrow = (b-a)\sqrt{m^2+1}$

But what if $y=f(x)$ is not a line? As usual, we break it into small parts where we treat it as approximately linear:



- break $[a,b]$ into n-subintervals of width Δx $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ with $\Delta x = \frac{b-a}{n}$,
- length between $(x_{i-1}, f(x_{i-1}))$ & $(x_i, f(x_i))$ is $\sqrt{(x_i - x_{i-1})^2 + (f(x_{i-1}) - f(x_i))^2}$

$= \sqrt{(\Delta x)^2 + (\Delta y_i)^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}\, \Delta x$

(where $\Delta y_i = f(x_i) - f(x_{i-1})$)

Thus, length of $y=f(x)$ from $x=a$ to $x=b$ $= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}\, \Delta x$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \boxed{\int_a^b \sqrt{1 + (f'(x))^2}\, dx}$$

In limit, $\frac{\Delta y_i}{\Delta x}$ becomes the derivative $\frac{dy}{dx}$

E.g. If $f(x) = mx + c$ is a line then $f'(x) = m$

So length from $x=a$ to $x=b$ is $= \int_a^b \sqrt{1+(f'(x))^2}\,dx = \int_a^b \sqrt{1+m^2}\,dx = (b-a)\sqrt{1+m^2}$ ✓

E.g. Consider curve $y = x^{3/2}$ from $x=0$ to $x=1$.

Length $= \int_0^1 \sqrt{1+(\frac{d}{dx}x^{3/2})^2}\,dx = \int_0^1 \sqrt{1+(\frac{3}{2}x^{1/2})^2}\,dx$

$= \int_0^1 \sqrt{1+\frac{9}{4}x}\,dx$ ← can solve w/ a u-sub.

① indef. integral $\int \sqrt{1+\frac{9}{4}x}\,dx = \int \sqrt{u}\,\frac{4}{9}\,du = \frac{4}{9}\cdot\frac{2}{3}u^{3/2}$

$u = 1+\frac{9}{4}x$

$du = \frac{9}{4}dx$

$= \frac{8}{27}(1+\frac{9}{4}x)^{3/2}$

② Plug in: $\int_0^1 \sqrt{1+\frac{9}{4}x}\,dx = \left[\frac{8}{27}(1+\frac{9}{4}x)^{3/2}\right]_0^1 = \frac{8}{27}\left(\left(\frac{13}{4}\right)^{3/2} - 1\right)$

E.g. Even for curve $y=x^2$ from $x=0$ to $x=1$, integral is nasty:

Length $= \int_0^1 \sqrt{1+(\frac{d}{dx}x^2)^2}\,dx = \int_0^1 \sqrt{1+(2x)^2}\,dx = \int_0^1 \sqrt{1+4x^2}\,dx$

① indef. integral $\int \sqrt{1+4x^2}\,dx$ good idea: try sub! $x = \frac{1}{2}\tan\theta$

$dx = \frac{1}{2}\sec^2\theta\,d\theta$

$= \int \sqrt{1+\tan^2\theta}\;\frac{1}{2}\sec^2\theta\,d\theta = \frac{1}{2}\int \sec^3\theta\,d\theta$

But... $\int \sec^3\theta\,d\theta$ is not easy! Int. by parts helps, but even then you still need to know $\boxed{\int \sec\theta\,d\theta = \ln(\sec\theta + \tan\theta)}$

E.g. Sometimes $(1+(f'(x))^2)$ has a square root:

if $f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln(x)$ then $f'(x) = \frac{1}{2}x - \frac{1}{2x} = \frac{x^2-1}{2x}$

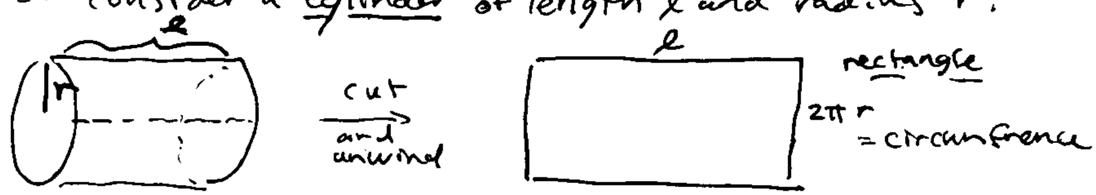
So... $\int \sqrt{1+(f'(x))^2}\,dx = \int \sqrt{1+\left(\frac{x^2-1}{2x}\right)^2}\,dx = \int \sqrt{1+\frac{x^4-2x^2+1}{4x^2}}\,dx$

$= \int \sqrt{\frac{x^4+2x^2+1}{4x^2}}\,dx = \int \sqrt{\frac{(x^2+1)^2}{(2x)^2}}\,dx = \int \frac{x^2+1}{2x}\,dx$ ← doable integral!

2/22 Area of Surface of Revolution § 8.2

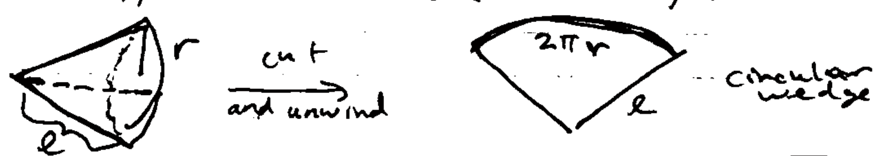
Intuitively, the surface area of a 3D shape is the amount of wrapping paper you would need to wrap it.

As usual, we start our discussion of surface area with simple shapes. First consider a cylinder of length ℓ and radius r:



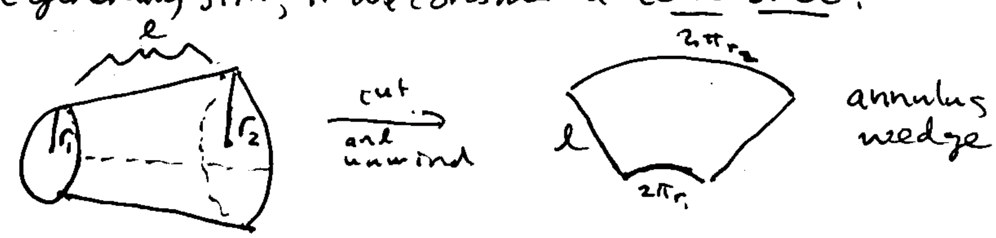
(Note: we do not consider area of left/right end of cylinder: the cyl. is "open").
By cutting the cylinder and unwinding it into a rectangle we see that it has surface area = $\boxed{2\pi r \cdot \ell}$.

Similarly, if we take a cone of slant length ℓ and base radius r:



a simple geometry calculation shows area = $\boxed{\pi r \ell}$
More generally still, if we consider a cone slice:

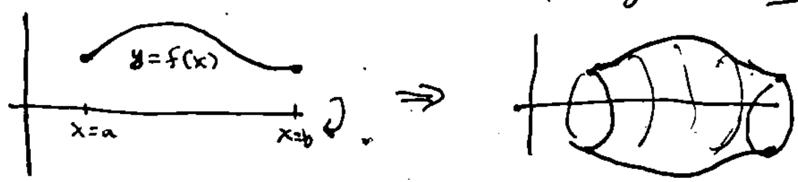
$r = \frac{r_1 + r_2}{2}$



then its surface area is = $\boxed{2\pi r \ell}$ where ℓ = slant length and $\boxed{r = (r_1 + r_2)/2}$ is average of radiuses of the bases.

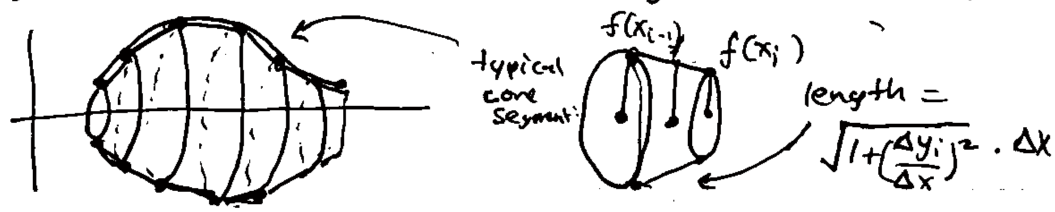
Cylinders, cones, and cone slices are all examples of surfaces of revolution, and we can use calculus to give an integral formula for area of any surface of revolution.

Consider a curve $y=f(x)$ from $x=a$ to $x=b$. By rotating this curve around the x-axis, we get a surface of revolution:



So a surface of revolution is just the (lateral) boundary of the corresponding solid of revolution.

As usual, to find the area of the surface of revolution, we break the curve into short intervals where we approximate it by a linear function, giving cone segments:



We explained last class when talking about arc lengths that the slant length of the i^{th} cone segment is $=\sqrt{1+\left(\frac{\Delta y_i}{\Delta x}\right)^2}\,\Delta x$

Meanwhile, the circumference is $\approx 2\pi f(x_i^*)$ for $x_i^* \in [x_{i-1}, x_i]$.

So the area of the i^{th} segment $\approx 2\pi f(x_i^*)\cdot\sqrt{1+\left(\frac{\Delta y_i}{\Delta x}\right)^2}\,\Delta x$

and total area of surface $\approx \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+\left(\frac{\Delta y_i}{\Delta x}\right)^2}\,\Delta x$.

Taking limit as $n\to\infty$, we get:

$$\boxed{\text{Area of surface of revolution (rotating } y=f(x) \text{ about } x\text{-axis, } x=a \text{ to } x=b) = \int_a^b \underline{2\pi f(x)}\,\underline{\sqrt{1+(f'(x))^2}\,dx}}$$

To remember this formula: think

circumference × length

$$2\pi f(x) \cdot \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

E.g. Consider $y=\sqrt{x}$ from $x=0$ to $x=1$, rotated about x-axis.

Area of surface of revolution $= \int_0^1 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$ where $f(x)=x^{1/2}$, $f'(x)=\frac{1}{2}x^{-1/2}$

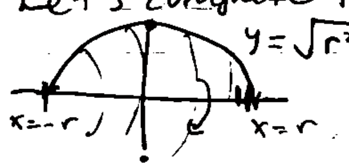
$$= \int_0^1 2\pi\sqrt{x}\cdot\sqrt{1+\left(\frac{1}{2}\frac{1}{\sqrt{x}}\right)^2}\,dx = \int_0^1 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx$$

$$= 2\pi\int_0^1\sqrt{x\cdot\left(1+\frac{1}{4x}\right)}\,dx = 2\pi\int_0^1\sqrt{x+\frac{1}{4}}\,dx.$$

(1) Indef. integral $\int\sqrt{x+\frac{1}{4}}\,dx = \int\sqrt{u}\,du = \frac{2}{3}u^{3/2}$
$u = x+\frac{1}{4}$, $du = dx$
$= \frac{2}{3}\left(x+\frac{1}{4}\right)^{3/2}$

(2) Plug in FTC $\Rightarrow 2\pi\int_0^1\sqrt{x+\frac{1}{4}}\,dx = 2\pi\left[\frac{2}{3}\left(x+\frac{1}{4}\right)^{3/2}\right]_0^1 = \frac{4\pi}{3}\left(\left(\frac{5}{4}\right)^{3/2}-\left(\frac{1}{4}\right)^{3/2}\right)$

2/24 E.g. Let's compute the surface area of a sphere of radius r:



$y=\sqrt{r^2-x^2}$ $\Leftarrow$ sphere = surface of revolution for $y=f(x)=\sqrt{r^2-x^2}$ for $x=-r$ to $x=r$

Area $= \int_{-r}^{r} 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$ $\quad f'(x) = 2x\cdot\frac{1}{2}(r^2-x^2)^{-1/2} = \frac{x}{\sqrt{r^2-x^2}}$

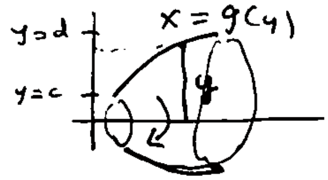
$$= \int_{-r}^{r} 2\pi\sqrt{r^2-x^2}\sqrt{1+\left(\frac{x}{\sqrt{r^2-x^2}}\right)^2}\,dx = \int_{-r}^{r}2\pi\sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx$$

$$= \int_{-r}^{r} 2\pi\sqrt{(r^2-x^2)\left(1+\frac{x^2}{r^2-x^2}\right)}\,dx = \int_{-r}^{r}2\pi\sqrt{(r^2-x^2)+x^2}\,dx$$

$$= \int_{-r}^{r}2\pi\sqrt{r^2}\,dx = 2\pi r\int_{-r}^{r}dx = \boxed{4\pi r^2}$$

Note: If we did $\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$ here instead, we get $\int_a^b 2\pi r\,dx = 2\pi r\,(b-a)$. $\leftarrow$ surface area of sphere segment

It is also possible to compute surface areas by integrating w.r.t. $\underline{\underline{y}}$.
Suppose that $x = g(y)$ for $y = c$ to $y = d$,
and we rotate this curve about the x-axis:



$\Leftarrow$ Same surface of revolution but given $\underline{x}$ $\underline{\text{in}}$ $\underline{\text{terms}}$ of $\underline{y}$

A similar computation shows that:

$$\boxed{\underline{\text{area}} = \int_c^d \underbrace{2\pi y}_{} \cdot \sqrt{1 + (g'(y))^2}\, dy}$$

$\underline{\text{Think}}$: circumfrence $\times$ length

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \sqrt{1+\left(\frac{dy}{dx}\right)}\, dx$$

$\underline{\text{Eg}}$. Consider curve $x = \frac{2}{3} y^{3/2}$ from $y = 0$ to $y = 3$.
Compute surface area of surface got by rotating about $\underline{x\text{-axis}}$:
Since we already have x in terms of y, it is easiest here to use the y-integral formula:

$$\underline{\text{Area}} = \int_c^d 2\pi y \sqrt{1+(g'(y))^2}\, dy \quad \text{where } g(y) = \tfrac{2}{3} y^{3/2}$$

$$\text{so } g'(y) = y^{1/2}$$

$$= \int_0^3 2\pi y \sqrt{1+(y^{1/2})^2}\, dy = \int_0^3 2\pi y \sqrt{1+y}\, dy$$

① $\underline{\text{indef. integral}}$ $\displaystyle\int y\sqrt{1+y}\, dy = \int (u-1)\sqrt{u}\, du$

$u = 1+y \Rightarrow y = u-1$
$du = dy$

$$= \int u^{3/2} - u^{1/2}\, du = \tfrac{2}{5} u^{5/2} - \tfrac{2}{3} u^{3/2}$$

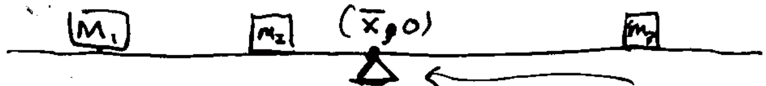
$$= \tfrac{2}{5}(1+y)^{5/2} - \tfrac{2}{3}(1+y)^{3/2}$$

② Plug in FTC $\Rightarrow$ $\displaystyle \text{Area} = 2\pi \int_0^3 y\sqrt{1+y}\, dy = 2\pi\left[\tfrac{2}{5}(1+y)^{5/2} - \tfrac{2}{3}(1+y)^{3/2}\right]_0^3$

$$= 2\pi\left(\left(\tfrac{2}{5}(4^{5/2}) - \tfrac{2}{3}(4^{3/2})\right) - \left(\tfrac{2}{5} - \tfrac{2}{3}\right)\right) = \cdots = \frac{232\pi}{15} \quad //$$

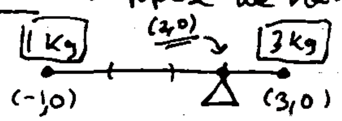
2/27 Center of Mass and Centroid § 8.3

Suppose we have n objects $O_1, O_2, \ldots, O_n$ on a line, where O_i is located at $(x_i, 0)$ and has mass m_i:



At what point should we place the fulcrum of a scale so that the objects will be perfectly balanced? This point is called the center of mass and can be computed by formula $\boxed{\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_i m_i}}$ ← think: weight each point by mass of object there.

E.g. Suppose we have 1 kg object at $(-1,0)$, 3 kg object at $(3,0)$:



$$\bar{x} = \frac{(1 \cdot -1 + 3 \cdot 3)}{1+3} = \frac{8}{4} = 2.$$

So center of mass is at $(2,0)$.

Similarly if objects $O_1, \ldots, O_n$ are on the plane w/ O_i located at (x_i, y_i) (and still having mass m_i), we define the center of mass to be $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{\sum_i x_i m_i}{\sum_i m_i} \quad \text{and} \quad \bar{y} = \frac{\sum_i y_i m_i}{\sum_i m_i}.$$

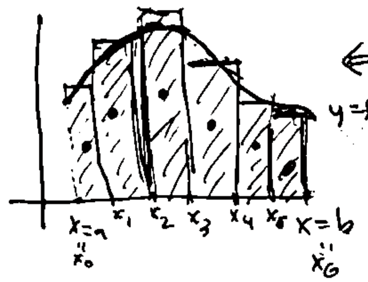
But what if instead of discrete point masses, we have a region of mass? We could still ask for the center of mass

← as the "balancing point," if we imagine the region as a "plate" balancing on a "stick."

For simplicity, assume region has uniform density, (e.g. 1 Kg / unit area), then center of mass called centroid.

Also, let us assume our region R is the region below curve $y = f(x)$ from $x = a$ to $x = b$.

Then, as usual, will approx. region by rectangles and use integrals calculus.



← as usual we let $\Delta x = \frac{b-a}{n}$ and set $x_i = a + i\Delta x$ for $i = 0, 1, \dots, n$

break region up into rectangles:

Let $\bar{x}_i = \frac{x_{i-1} + x_i}{2}$ be midpoint of $[x_{i-1}, x_i]$, and let i^{th} rectangle have width Δx and height $f(\bar{x}_i)$.

Since density is 1kg/unit area, mass of i^{th} rectangle = width × height $= f(\bar{x}_i)\Delta x = m_i$.

Also, centroid of rectangle is just middle: $\left(\bar{x}_i, \frac{f(\bar{x}_i)}{2}\right)$

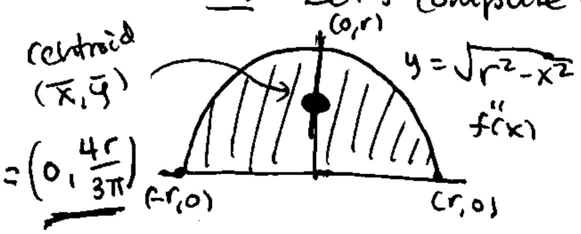
So imagine we had point masses of mass $m_i = f(\bar{x}_i)\Delta x$ and at locations $\left(\bar{x}_i, \frac{f(\bar{x}_i)}{2}\right)$, then center of mass would be

$$\bar{X} \approx \frac{\sum_{i=1}^{n} \bar{x}_i f(\bar{x}_i)\Delta x}{\sum_i f(\bar{x}_i)\Delta x}, \quad \bar{y} \approx \frac{\sum_{i=1}^{n} \frac{f(\bar{x}_i)}{2} f(\bar{x}_i)\Delta x}{\sum_i f(\bar{x}_i)\Delta x}.$$

Letting $n \to \infty$, we get that centroid is $(\bar{x}, \bar{y})$

where $\boxed{\bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx, \ \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}(f(x))^2\,dx, \text{ and } A = \int_a^b f(x)\,dx = \text{area of region } R.}$

E.g. Let's compute the centroid of a semicircle of radius r:



Here we could compute area $A = \int_{-r}^{r} \sqrt{r^2 - x^2}\,dx$ but we know already from geometry that $A = \pi r^2/2$.

Similarly, could compute $\bar{x} = \frac{1}{A}\int_{-r}^{r} x\sqrt{r^2 - x^2}\,dx$, but clear from symmetry that $\bar{x} = 0$ (semicircle symmetric over y-axis).

So we only need to compute $\bar{y} = \frac{1}{A}\int_{-r}^{r} \frac{1}{2}(f(x))^2\,dx = \frac{1}{\pi r^2/2}\int_{-r}^{r} \frac{1}{2}\left(\sqrt{r^2 - x^2}\right)^2 dx$

$$= \frac{1}{\pi r^2}\int_{-r}^{r} r^2 - x^2\,dx = \frac{1}{\pi r^2}\left[r^2 x - \frac{1}{3}x^3\right]_{-r}^{r}$$

$$= \frac{1}{\pi r^2}\left(\left(r^3 - \frac{1}{3}r^3\right) - \left(-r^3 + \frac{1}{3}r^3\right)\right) = \frac{1}{\pi r^2}\left(\frac{2}{3}r^3 + \frac{2}{3}r^3\right) = \boxed{\frac{4r}{3\pi}}$$

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