Math 210 (Modern Algebra I), HW# 4,

Fall 2024; Instructor: Sam Hopkins; Due: Wednesday, October 16th

Recall that all rings R are assumed to be unital (i.e., have a 1) but not necessarily commutative.

1. A ring R which satisfies $a^2 = a$ for all $a \in R$ is called a *Boolean ring*.
   (a) Prove that a Boolean ring R is commutative, and satisfies $a + a = 0$ for all $a \in R$.
   (b) Let U be a set and let $\mathcal{P}(U)$ denote the set of all subsets of U. For $A, B \in \mathcal{P}(U)$, define $A + B = (A \setminus B) \cup (B \setminus A)$ and $AB = A \cap B$. Prove that this gives $\mathcal{P}(U)$ the structure of a Boolean ring.
2. Let G be a finite group and $R = \mathbb{Q}[G]$ be the group algebra of G over the rational numbers $\mathbb{Q}$. Consider the element $x = \dfrac{1}{|G|} \sum_{g \in G} g \in R$. Prove that x is an *idempotent*, i.e., that $x^2 = x$.
3. Recall that the *center* of a (noncommutative) ring R is $Z(R) = \{x \in R \colon xy = yx \text{ for all } y \in R\}$.

   Now let R be a commutative ring and consider the ring $M_n(R)$ of $n \times n$ matrices with entries in R. What is the center $Z(M_n(R))$ of this matrix ring?
4. Let $\mathbb{H}$ denote the quaternions. Recall that an element $p \in \mathbb{H}$ can be represented as a formal sum $p = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, with $a, b, c, d \in \mathbb{R}$ real numbers, and with $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. Define the *norm* of such an element p to be $|p| = \sqrt{a^2 + b^2 + c^2 + d^2}$. Prove that this norm is mulpticative, i.e., that $|pq| = |p|\,|q|$ for $p, q \in \mathbb{H}$.
5. Let R be a commutative ring.
   (a) Let I be an ideal of R and define its *radical* to be $\mathrm{Rad}(I) = \{x \in R \colon x^n \in I \text{ for some } n \geq 0\}$. Prove that $\mathrm{Rad}(I)$ is also an ideal of R.
   (b) Recall that $x \in R$ is *nilpotent* if there is some $n \geq 0$ such that $x^n = 0$. Prove that the collection of all nilpotent elements is an ideal of R. **Hint**: you can use the previous part.