Math 157 – Calculus II Final Exam – Spring 2023

May 2, 2023

SHOW ALL WORK. Justify your answers! Simplify your answers.
Show details to earn full points. Give EXACT answers whenever possible.
Solve all parts of any 10 out of the 15 problems below.
Each of the 15 problems = 10 points. Exam total = 100 points.

1. (a) Sketch the region enclosed by the curves $y = x^3$ and $y = x$ and find its area.
   (b) Find the average value f_{avg} of the function $f(x) = x^2$ on the interval $[-1, 2]$.

2. Let R be the region in the first quadrant below the curve $y = x^3$ from $x = 1$ to $x = 2$. Compute the volume of the solid obtained by rotating R:

   (a) about the x-axis; (b) about the y-axis.

3. Evaluate the integrals:

   (a) $\displaystyle\int \frac{\ln x}{x^2}\, dx$; (b) $\displaystyle\int \frac{x}{e^{4x}}\, dx$.

4. Evaluate the integrals:

   (a) $\displaystyle\int_0^\infty \frac{x}{(x^2+1)^3}\, dx$; (b) $\displaystyle\int_0^{\pi/4} \sin^2 x \cos^3 x\, dx$.

5. Evaluate the integrals:

   (a) $\displaystyle\int \tan^5 x \sec^3 x\, dx$; (b) $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\, dx$.

6. Evaluate the integrals:

   (a) $\displaystyle\int \frac{\sqrt{16-9x^2}}{x^2}\, dx$; (b) $\displaystyle\int \frac{4x}{(x+1)(x^2+1)}\, dx$.

7. Compute the area of the surface obtained by rotating the curve given by $y = 2\sqrt{x}$ for $x = 0$ to $x = 1$ about the x-axis.

8. (a) Find an equation of the line tangent to the curve given by $x = 2 + \ln t$, $y = t^2 - 3$ at the point $(2, -2)$.
   (b) Find the length of the curve defined by $x = -\sin^3 t, y = -\cos^3 t$ over the interval $0 \le t \le \frac{\pi}{2}$.

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9. Determine whether each of the following sequences $(a_n)_{n\geq 1}$ converges, and if so, find its limit. Remember to justify your answers.

   (a) $a_n = \dfrac{\sin n}{n}$;

   (b) $a_n = \arctan(n^2 - 1)$;

   (c) $a_n = n\cos\left(\dfrac{1}{n}\right)$;

   (d) $a_n = \left(\left(1+\dfrac{1}{n}\right)^n\right)^n$.

10. Determine whether each of the following series converges conditionally, converges absolutely, or diverges. Remember to justify your answers.

    (a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos\sqrt{n}}{n^3}$;

    (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+1}{3n^2+10n+2}$;

    (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$;

    (d) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n^3}{n!}$.

11. (a) Plot the point whose polar coordinates $\left(2, \frac{\pi}{4}\right)$ are given. Then find two other pairs of polar coordinates of this point, one with $r > 0$ and one with $r < 0$.

    (b) Compute the area of one loop of the polar curve given by $r = 4\cos 3\theta$. *Hint:* Start by determining the values of θ where the loop begins and ends.

12. Find the interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-3)^n}{5^n\sqrt{n}}$.

13. A cable 30 meters long weighs 600 Newtons and hangs from the top of a building that is 40 meters above the ground. Find the work needed to pull the upper two thirds of the cable to the top of the building.

14. Consider the function $f(x) = \sin x$.

    (a) Write the degree three Taylor polynomial $T_3(x)$, centered at $x = 0$, for this $f(x)$.

    (b) Use your answer in part (a) to give an estimate for the value of $f(\frac{1}{2})$.

    (c) Give an upper bound on the error for your estimate from part (b). *Hint*: Recall that the Taylor series for $\sin x$ is alternating.

15. The interval $[-1, 5]$ is partitioned into n subintervals $[x_{k-1}, x_k]$ for $k = 1, ..., n$ each of width $\Delta x = x_k - x_{k-1}$. Choose any x_k^* such that $x_{k-1} \leq x_k^* \leq x_k$. Let the function f be continuous over $[-1, 5]$. Do the following.

    (a) State the limit definition of $\displaystyle\int_{-1}^{5} f(x)\,dx$.

    (b) For $n = 3$, write the midpoint approximation for the integral in part (a) in terms of f.

    (c) For $f(x) = x^2 + 2$, use the expression in part (b) to estimate the value of the integral in part (a).

    (d) What is the error of your approximation compared to the true value of this definite integral?