Combinatorics.

prof. Sam

HW #1.

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Total score: 51/60

Q.1 Let $P_k(n)$ denote the number of partitions of n into k parts.

prove bijection that:

$$P_0(n) + P_1(n) + P_2(n) + \dots + P_k(n) = P_k(n+k).$$

You are right that p_k(n) can be viewed as the number of ways to place n (indistinguishable) balls into k (indistinguishable) boxes, but importantly subject to the condition that each box gets one ball. Meanwhile p_1(n) + … + p_k(n) is therefore the number of ways to place these n balls into k boxes without the requirement that each box get one ball. So by your observation that we can first put one ball in each box, your argument that p_k(n) = p_1(n-k) + … + p_k(n-k) is correct. At that point, you don't need to do more: just replace n by n+k to finish!

proof:

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First, the identity $P_k(n) = P_k(n-k) + P_{k-1}(n-1)$.

Indeed, we can observe $P_k(n)$ alternatively as the number of different ways to place n objects into k boxes. If we place one object in each of the k boxes, we are left with $n-k$ objects for the k boxes. In fact, we can place this $n-k$ objects in $1, 2, 3, \dots$ or in all k boxes. Thus, $P_k(n) = P_1(n-k) + P_2(n-k) + \dots + P_n(n-k)$.

Now, by repeating the same argument for $n-1$ objects and $k-1$ boxes:

$$P_{k-1}(n-1) = P_1((n-1)-(k-1)) + P_2((n-1)-(k-1)) + \dots + P_{k-1}((n-1)-(k-1))$$

$$= P_1(n-k) + P_2(n-k) + \dots + P_{k-1}(n-k).$$

By combining the last two equalities, we can have that

$$P_k(n) = P_{k-1}(n-1) + P_k(n-k)$$

So, applying this identity for $n+k$ objects and k boxes, we get:

$P_k(n+k) = P_{k-1}(n+k-1) + P_k(n)$. We continue with the identity on the first summand, given n objects and $k-1$ boxes, namely $P_{k-1}(n+k-1) = P_{k-2}(n+k-2) + P_{k-1}(n)$, which substituted above yields

yields $P_k(n+k) = P_k(n) + P_{k-1}(n) + P_{k-2}(n+k-2)$.

Now, it is easy to observe that if we continue to apply the identity reccurently (next on n objects and $k-2$ boxes), we obtain:

$$P_k(n+k) = P_k(n) + P_{k-1}(n) + P_{k-2}(n) + P_{k-3}(n+k-3) = P_k(n) + P_{k-2}(n) + \cdots + P_0(n).$$ ~~approved~~. $\square$.

Q2) Fix natural numbers k, n. Let $[n]$ denote the set $[n] = \{1, 2, \ldots, n\}$. Give a simple formula for the number of ordered k-tuples $(T_1, \ldots, T_k)$ of subsets of $[n]$ satisfying:

- $T_i \cap T_j = \Phi$ for all $i \neq j$
- $\bigcup_{i=1}^{k} T_i = [n]$.

Sol

Let $P(n)$ be the numbers of all possible partition of natural numbers.

Then $P(n) = n(n-1)(n-2) \cdots 2 \cdot 1$. $\square$.

You are thinking of *permutations*, not partitions. But this problem is not asking for permutations, it's asking for the number of ways to place n *distinguishable* balls into k *distinguishable boxes (where boxes may get 0 balls). E.g. the sequence ({1,3}, {}, {4}, {}, {2}) means balls 1 and 3 go into the 1st box, ball 2 goes into the 5th box, and ball 4 goes into the 3rd box. Using this way of thinking about it, it's easy to see the number is k^n: for each of the n balls we can choose one of k boxes, and the choices are independent.

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Q3) Show that:

$$\sum_{n_1,\dots,n_k \geq 0} \min(n_1,\dots,n_k)\, x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} = \frac{x_1 x_2 \cdots x_k}{(1-x_1)(1-x_2)\cdots(1-x_k)(1-x_1x_2\cdots x_k)}.$$

proof:

First, we note that:

$$\frac{1}{(1-x_1)\cdots(1-x_k)} = \sum_{m_1,\dots,m_k \geq 0} x_1^{m_1} \cdots x_k^{n_k} \rightarrow ①$$

Good!

Then from (1), we obtain:

$$\frac{x_1 \cdots x_k}{(1-x_1)\cdots(1-x_k)(1-x_1\cdots x_k)} = \sum_{a=1}^{\infty} \sum_{m_k \geq 0} x_1^{m_1} \cdots x_k^{m_k} (x_1 \cdots x_k)^a \rightarrow ②$$

In ②, we have terms $x_1^{n_1} \cdots x_k^{n_k}$ with every combination of $\{n_1,\dots,n_k\}$. Given some $\{n_1,\dots,n_k\}$ such terms come from

$a=1 \quad m_1 = n_1 - 1, \dots, m_k = n_k - 1$

$a=2 \quad m_1 = n_1 - 2, \dots, n_k = m_k - 2$

So, $a = \min(n_1,\dots,n_k) \quad m_1 = n_1 - a, \dots, n_k = m_k - a$

Then, all of these terms come with coefficient 1, so when we bunch them together we obtain: $\sum_{n_1,\dots,n_k \geq 0} \min(n_1,\dots,n_k)\, x_1^{n_1} \cdots x_k^{n_k}$.

Nice! 10/10

Thus $$\sum_{n} \frac{x_1 \cdots x_k}{(1-x_1)\cdots(1-x_k)(1-x_1x_2\cdots x_k)} = \sum_{n_1,\dots,n_k \geq 0} \min(n_1,\dots,n_k)\, x_1^{n_1}\cdots x_k^{n_k}.$$

Q4) Let $\bar{c}(n,m)$ denote the number of composition of n into parts of size at most m. Show that:

$$\sum_{n\geq 0} \bar{c}(n,m)\, x^n = \frac{1-x}{1-2x+x^{m+1}}$$

proof:

First, note that:

$$1-2x+x^{m+1} = (1-x)\left[1-(x+x^2+\cdots+x^m)\right]$$

Then $$\frac{1-x}{1-2x+x^{m+1}} = \frac{1}{1-(x+x^2+\cdots+x^m)} = \sum_{k=0}^{\infty}(x+x^2+\cdots+x^m)^k$$ Yes.

which is showing that the coefficient of x^n is the number of all possible ordered compositions of n with largest part at most m.

You could say maybe one more word about why this is counting compositions, but okay, correct. 10/10

So $$\frac{1-x}{1-2x+x^{m+1}} = \sum_{k=0}^{\infty}(x+x^2+\cdots+x^m)^k = \sum_{n\geq 0}\bar{c}(n,m)\,x^n \quad \blacksquare$$

Q5) prove that. for any $n \geq 0$

$$4^n = \sum_{k=0}^{n} \binom{2k}{k} \binom{2(n-k)}{n-k} \longrightarrow (1)$$

<u>proof:</u>

First let $f(x) = \sum_{n=0}^{\infty} \binom{2n}{n} x^n$.

As the right hand side is the generating function for the binomial cofficients, which is $\frac{1}{\sqrt{4-x}}$, we have $f(x) = \frac{1}{\sqrt{4-x}}$.

Consider now $f^2(x) = \frac{1}{4-x}$. The coefficient of x^n both in the left and right hand side of the expensions of the two generating functions should be equal.

we know that the cofficient of x^n in $\frac{1}{4-x}$ is 4^n.

on the other hand, the coefficient of x^n in $f^2(x)$ is obtained by the convolution of $f(x)$ with itself, and by defnition, that is exactly the left hand side of (1);

where $a_k = \binom{2k}{k}$ and $a_{n-k} = \binom{2(n-k)}{n-k}$

$$\therefore \quad 4^n = \sum_{k=0}^{n} \binom{2k}{k} \binom{2(n-k)}{n-k}$$

Yes, exactly! 10/10

Q6) Let $n \geq 1$, and let $ODD(n)$ denote the subset of permutations in the symmetric group $\mathfrak{S}_n$ with no cycles of even size. Prove that:

$$\sum_{\sigma \in ODD(n)} 2^{\#\text{cycles}(\sigma)} = 2 \cdot n!$$

proof:

Recall that Touchard Theorem and set $t_i = 2$.

Then

$$\sum_{n \geq 0} \left(\frac{x^n}{n!} \sum_{\sigma \in ODD(n)} 2^{\#\text{cycles}} \right) = e^{2\left(x + \frac{x^2}{2} + \cdots\right)}$$

$$= e^{2(-\log(1-x))}$$

$$= e^{\log(1-x)^{-2}}$$

$$= (1-x)^{-2}$$

$$= \sum_{n \geq 0} \binom{-2}{n} (-x)^n$$

$$= \sum_{n \geq 0} \binom{-2+n-1}{n} x^n$$

$$= 2\, n! \qquad \square$$

No… you don't want to set all t_i = 2 in Touchard's theorem, because we are only summing over the permutations in ODD(n), the subset of permutations with no even cycles. What you want to do is set t_1 = t_3 = t_5 = … = 2 (i.e., all t_i with i odd), while setting t_2 = t_4 = t_6 = … = 0 (i.e., all t_i with i even). This will give a different formal power series than what you have written. [Also, the last part of what you wrote makes no sense because you went from a formal power series in x to just a number.]

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