Upho lattices and their cores

Sam Hopkins[*1]

[1]*Department of Mathematics, Howard University, Washington, DC, USA*

Abstract. A poset is called upper homogeneous, or "upho," if every principal order filter is isomorphic to the original poset. We study enumerative and structural properties of (finite type $\mathbb{N}$-graded) upho posets. The first important observation we make about upho posets is that their rank generating functions and characteristic generating functions are multiplicative inverses of one another. This means that each upho lattice has associated to it a finite graded lattice, called its core, which determines its rank generating function. We investigate which finite graded lattices arise as cores of upho lattices, providing both positive and negative results. On the one hand, we show that many well-studied finite lattices do arise as cores, and we present combinatorial and algebraic constructions of the upho lattices into which they embed. On the other hand, we show there are obstructions which prevent many finite lattices from being cores.

1 Introduction

Symmetry is a fundamental theme in mathematics. A close cousin of symmetry is self-similarity, where a part resembles the whole. Here we study certain partially ordered sets that are self-similar in a precise sense. Namely, a poset is called *upper homogeneous*, or "*upho*," if every principal order filter of the poset is isomorphic to the whole poset. In other words, a poset $\mathcal{P}$ is upho if, looking up from each element $p \in \mathcal{P}$, we see another copy of $\mathcal{P}$. Upho posets were introduced recently by Stanley [12, 13]. We believe they are a natural and rich class of posets which deserve further attention.

Upho posets are infinite. In order to be able to apply the tools of enumerative and algebraic combinatorics, we need to impose some finiteness condition on the posets we consider. Thus, we restrict our attention to finite type $\mathbb{N}$-graded posets. These are the infinite posets $\mathcal{P}$ that possess a rank function $\rho \colon \mathcal{P} \to \mathbb{N}$ for which we can form the *rank generating function*

$$F(\mathcal{P}; x) := \sum_{p \in \mathcal{P}} x^{\rho(p)}.$$

Henceforth, upho posets are assumed finite type $\mathbb{N}$-graded unless otherwise specified.

The first important observation we make about (finite type $\mathbb{N}$-graded) upho posets is that their rank generating functions are related in a nice way to the values of their

---

[*]samuelfhopkins@gmail.com

Möbius functions. Specifically, if we define the *characteristic generating function* of an upho poset $\mathcal{P}$ to be

$$\chi^*(\mathcal{P};x) := \sum_{p \in \mathcal{P}} \mu(\hat{0},p)x^{\rho(p)}$$

then

$$F(\mathcal{P};x) = \chi^*(\mathcal{P};x)^{-1} \tag{1.1}$$

i.e., $F(\mathcal{P};x)$ and $\chi^*(\mathcal{P};x)$ are multiplicative inverses as formal power series.

Gao et al. [6, §5] have shown that there are uncountably many different rank generating functions $F(\mathcal{P};x)$ of upho posets $\mathcal{P}$. This prevents us from being able to say much more about the enumerative and structural properties of upho posets in general. However, the situation is different for upho *lattices*.

Let $\mathcal{L}$ be an upho lattice, and let $L := [\hat{0}, s_1 \vee \cdots \vee s_r]$ denote the interval in $\mathcal{L}$ from its minimum to the join of its atoms $s_1, \ldots, s_r$. We refer to the finite graded lattice L as the *core* of the upho lattice $\mathcal{L}$. Rota's cross-cut theorem implies that $\chi^*(\mathcal{L};x) = \chi^*(L;x)$, where $\chi^*(L;x) := \sum_{p \in L} \mu(\hat{0},p)x^{\rho(p)}$ is the (reciprocal) characteristic polynomial of L. Hence,

$$F(\mathcal{L};x) = \chi^*(L;x)^{-1} \tag{1.2}$$

Thus, the rank generating function of an upho lattice is the inverse of a polynomial with integer coefficients. We review (1.1) and (1.2) in Section 2.

Because of (1.2), the core of an upho lattice determines its rank generating function.[1] We caution that the core does not completely determine the upho lattice, i.e., there can be multiple upho lattices with the same core. Nevertheless, any complete understanding of upho lattices would have to start with an answer to the following question.

Question 1. *Which finite graded lattices arise as cores of upho lattices?*

Question 1 can be thought of as a kind of tiling problem: our goal is to tile an infinite, fractal lattice $\mathcal{L}$ using copies of some fixed finite lattice L, or show that no such tiling is possible. In addressing Question 1 here, we provide both positive and negative results.

We start with combinatorial constructions of upho lattices. In Section 3, we construct upho lattices as limits of sequences of finite lattices. We show that any member of a uniform sequence of supersolvable geometric lattices is the core of an upho lattice. Examples of uniform sequences of supersolvable geometric lattices include the Boolean lattices B_n, their q-analogues $B_n(q)$, and the partition lattices Π_n. Hence, these are all cores of upho lattices. Figure 1 depicts an upho lattice produced via this construction.

In addition to combinatorial constructions, we also explore algebraic constructions of upho lattices. In Section 4 we explain how monoids provide one algebraic source of upho lattices. A homogeneously finitely generated monoid M that is left cancellative is

---

[1]In fact, since the flag f-vector of any upho poset is determined by its rank generating function (see [13, §3]), the core of an upho lattice determines its entire flag f-vector.
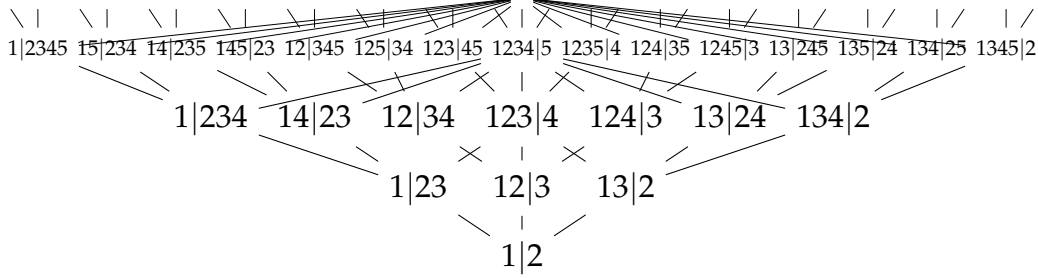
Figure 1: Partitions of sets of the form $\{1, 2, \ldots, n\}$ into 2 blocks, ordered by refinement. This is an upho lattice with core Π_3.
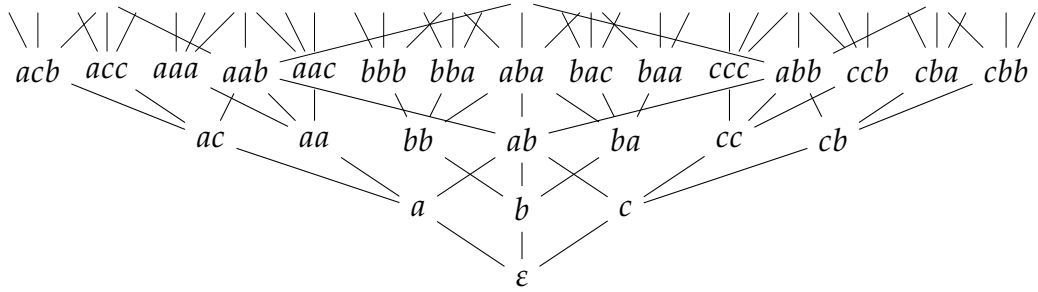

Figure 2: The dual braid monoid $\langle a, b, c \mid ab = bc = ca \rangle$ associated to the symmetric group S_3. This is an upho lattice with core the noncrossing partition lattice of S_3.

an upho poset under left division. Hence, if M is a lattice under left division, it is an upho lattice. An important example of such monoids are the Garside monoids coming from Coxeter theory. Thus, the weak order and noncrossing partition lattice of any finite Coxeter group are cores of upho lattices. Figure 2 depicts an upho lattice of this form.

On the negative side, in Section 5, we show that there are various obstructions which prevent arbitrary finite graded lattices from being realized as cores of upho lattices. There are constraints on the characteristic polynomial of the lattice coming from (1.2). There are also some structural obstructions, requiring the lattice to be partly self-similar. These obstructions allow us to show that the following plausible candidates cannot be realized as cores: the face lattices of the n-dimensional cross polytope and hypercube (for $n \geq 3$); the lattice of flats of the uniform matroid $U(n, k)$ (for $2 < k < n$).

The upshot is that Question 1 is quite subtle: it can be difficult to recognize when a given finite graded lattice is the core of an upho lattice. Many well-behaved finite lattices are cores of upho lattices, but many too are not. In Section 6 we briefly discuss future directions in the study of upho lattices that we intend to pursue.

This is just an extended abstract where we survey our recent results on upho lattices. The full articles containing these results, with proofs, are [7, 8].

Acknowledgements

I thank the following people for useful comments related to this work: Yibo Gao, Joel Lewis, Vic Reiner, David Speyer, Richard Stanley, Benjamin Steinberg, Nathan Williams, and Gjergji Zaimi. SageMath [14] was an important computational aid in this research.

2   Upho poset basics

We use standard notation and terminology for posets, as laid out for instance in [11, §3]. Since we routinely work with both finite and infinite posets, we use the convention that finite posets are denoted by normal script letters (like P and L) and infinite posets are denoted by calligraphic letters (like $\mathcal{P}$ and $\mathcal{L}$).

Recall that a finite poset P is *graded (of rank n)* if we can write $P = \bigsqcup_{i=0}^{n} P_i$ as a disjoint union so that every maximal chain is of the form $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n$ with $x_i \in P_i$. In this case, we define the *rank function* $\rho \colon P \to \mathbb{N}$ by $\rho(x) := i$ if $x \in P_i$. For such a P, we define its *rank generating polynomial* to be $F(P; x) := \sum_{p \in P} x^{\rho(p)}$. If P has a minimum $\hat{0}$, we define its *(reciprocal) characteristic polynomial* to be $\chi^*(P; x) := \sum_{p \in P} \mu(\hat{0}, p) x^{\rho(p)}$, where $\mu(\cdot, \cdot)$ is the Möbius function of P.

We say an infinite poset $\mathcal{P}$ is $\mathbb{N}$-*graded* if we can write $\mathcal{P} = \bigsqcup_{i=0}^{\infty} \mathcal{P}_i$ as a disjoint union so that every maximal chain is of the form $x_0 \lessdot x_1 \lessdot \cdots$ with $x_i \in \mathcal{P}_i$. In this case, we define the *rank function* $\rho \colon \mathcal{P} \to \mathbb{N}$ by $\rho(x) := i$ if $x \in \mathcal{P}_i$. We say that $\mathcal{P}$ is *finite type* $\mathbb{N}$-graded if $\#\mathcal{P}_i < \infty$ for all i. For such a $\mathcal{P}$, we define its *rank generating function* to be $F(\mathcal{P}; x) := \sum_{p \in \mathcal{P}} x^{\rho(p)}$. If $\mathcal{P}$ has a minimum $\hat{0}$, we define its *characteristic generating function* to be $\chi^*(\mathcal{P}; x) := \sum_{p \in \mathcal{P}} \mu(\hat{0}, p) x^{\rho(p)}$.

We say that a poset $\mathcal{P}$ is *upper homogeneous*, or "*upho*," if for every $p \in \mathcal{P}$, we have that $V_p \simeq \mathcal{P}$, where $V_p := \{q \in \mathcal{P} \colon q \geq p\}$ is the principal order filter generated by p. To avoid trivialities, let us assume the upho posets we consider have at least two elements; then they must be infinite. Examples of upho posets include:

- the natural numbers $\mathbb{N} = \{0, 1, \ldots\}$, the nonnegative rational numbers $\mathbb{Q}_{\geq 0}$, and the nonnegative real numbers $\mathbb{R}_{\geq 0}$, all with their usual total orders;

- the poset of finite subsets of X ordered by inclusion, where X is any infinite set.

In order to be able to apply the tools of enumerative and algebraic combinatorics to study upho posets, we need to impose some finiteness conditions. Hence, from now on, **all upho posets are assumed finite type $\mathbb{N}$-graded**. Of the above examples, only $\mathbb{N}$ is finite type $\mathbb{N}$-graded. Here are more examples of (finite type $\mathbb{N}$-graded) upho posets:

Example 1. *Fix $r \geq 1$ and let $\mathcal{P}$ be the "infinite rooted r-ary tree" poset. The case $r = 2$ of this poset is depicted on the left in Figure 3. Note that this $\mathcal{P}$ is the "freest" upho poset with r atoms. It has $F(\mathcal{P}; x) = \frac{1}{1-rx}$ and $\chi^*(\mathcal{P}; x) = 1 - rx$.*
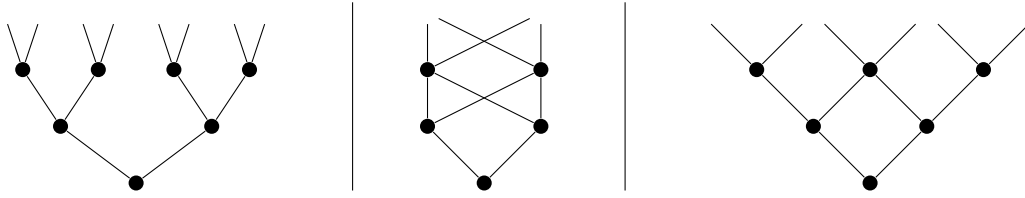
Figure 3: Various upho posets with two atoms.

Example 2. *Fix $r \geq 1$ and let $\mathcal{P} = \bigsqcup_{i=0}^{\infty} \mathcal{P}_i$ be the $\mathbb{N}$-graded poset with $\#\mathcal{P}_0 = 1$ and $\#\mathcal{P}_i = r$ for all $i \geq 1$, and with all cover relations between $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$ for all $i \geq 0$. The case $r = 2$ is depicted in the middle in Figure 3. Note that this $\mathcal{P}$ is the "least free" upho poset with r atoms. It has $F(\mathcal{P}; x) = \frac{1+(r-1)x}{1-x}$ and $\chi^*(\mathcal{P}; x) = \frac{1-x}{1+(r-1)x}$.*

Example 3. *Fix $r \geq 1$ and let $\mathcal{P} = \mathbb{N}^r$, i.e., the (Cartesian) product of r copies of $\mathbb{N}$. The case $r = 2$ is depicted on the right in Figure 3. It has $F(\mathcal{P}; x) = \frac{1}{(1-x)^r}$ and $\chi^*(\mathcal{P}; x) = (1 - x)^r$.*

From the preceding examples, the reader might guess a relationship between the rank and characteristic generating functions of an upho poset. And indeed, the following result is proved by a straightforward application of Möbius inversion (see [11, §3.7]).

Theorem 1. *For an upho poset $\mathcal{P}$, we have $F(\mathcal{P}; x) = \chi^*(\mathcal{P}; x)^{-1}$.*

Lattices have well-behaved Möbius functions, so we can say even more about upho lattices. Let $\mathcal{L}$ be an upho lattice, and let $L := [\hat{0}, s_1 \vee \cdots \vee s_r]$ denote the interval in $\mathcal{L}$ from its minimum to the join of its atoms $s_1, \ldots, s_r$. We call L the *core* of $\mathcal{L}$. Rota's cross-cut theorem (see [11, Corollary 3.9.5]) and Theorem 1 together imply the following.

Corollary 1. *For an upho lattice $\mathcal{L}$ with core L, we have $F(\mathcal{L}; x) = \chi^*(L; x)^{-1}$.*

Of the above examples of (finite type $\mathbb{N}$-graded) upho posets, only $\mathbb{N}^n$ is a lattice. The core of $\mathbb{N}^n$ is B_n, the rank n *Boolean lattice*, i.e., the lattice of subsets of $[n] := \{1, 2, \ldots, n\}$ ordered by inclusion. And indeed, we have $\chi^*(B_n; x) = (1 - x)^n = F(\mathbb{N}^n; x)^{-1}$. In what follows, we focus on Question 1, the question of which finite graded lattices are cores of upho lattices. For example, we just saw that the Boolean lattice B_n is a core for all $n \geq 1$.

3 Upho lattices from sequences of finite lattices

In this section we construct upho lattices as limits sequences of finite lattices that are appropriately embedded in one another. In order to make "appropriately embedded in one another" precise, we need two technical notions from the literature: the notion of a *supersolvable* geometric lattice, as introduced by Stanley in [10]; and the notion of a

uniform sequence of geometric lattices, as introduced by Dowling in [4]. These two notions represent two different ways that a lattice can have a recursive structure.

Let L be a finite lattice. We say L is *atomic* if every element is the join of atoms. We say L is *(upper) semimodular* if it is graded and satisfies $\rho(x) + \rho(y) \geq \rho(x \vee y) + \rho(x \wedge y)$ for all $x, y \in L$. We say L is *geometric* if it is both atomic and semimodular. Geometric lattices are intensely studied because they are exactly the lattices of flats of matroids.

First we review supersolvability. So let L be a geometric lattice. We say $x \in L$ is *modular* if $\rho(x) + \rho(y) = \rho(x \vee y) + \rho(x \wedge y)$ for all $y \in L$. For instance, every element of a modular lattice is modular. The lattice L is called *supersolvable* if it has a maximal chain $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n$ of modular elements. Stanley proved the following remarkable factorization theorem for characteristic polynomials of supersolvable geometric lattices.

Theorem 2 (Stanley [10]). *Let L be a supersolvable geometric lattice with maximal chain of modular elements $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n$. Then $\chi^*(L; x) = (1 - a_1 x)(1 - a_2 x) \cdots (1 - a_n x)$ where $a_i := \#\{\text{atoms } s \in L \colon s \leq x_i, s \not\leq x_{i-1}\}$ for $i = 1, \ldots, n$.*

Next we review uniform sequences. A sequence $L_0, L_1, \ldots$ of geometric lattices is called *uniform* if each L_n is graded of rank n, and $[a, \hat{1}_{L_n}] \simeq L_{n-1}$ for every atom $a \in L_n$.

Now fix a uniform sequence of geometric lattices $L_0, L_1, \ldots$. We define their *Whitney numbers of the second and first kind*, denoted $V(i, j)$ and $v(i, j)$, respectively, to be the coefficients $F(L_i; x) = \sum_{j=0}^{i} V(i, j) x^{i-j}$ and $\chi^*(L_i, x) = \sum_{j=0}^{i} v(i, j) x^{i-j}$. By convention, we set $V(i, j) := 0$ and $v(i, j) := 0$ for $j > i$. Dowling showed that uniform sequences of geometric lattices have the following nice behavior for their Whitney numbers.

Theorem 3 (Dowling [4]). *The matrices $[V(i, j)]_{0 \leq i, j \leq \infty}$ and $[v(i, j)]_{0 \leq i, j \leq \infty}$ are inverses.*

In the case when the geometric lattices in our uniform sequence are supersolvable, we can combine Theorems 2 and 3 to yield a stronger result.

Corollary 2. *Suppose that the geometric lattices L_n in our uniform sequence are all supersolvable. Then their Whitney numbers are*

$$V(i, j) = h_{i-j}(a_1, \ldots, a_{j+1}) \qquad \text{and} \qquad v(i, j) = (-1)^{i-j} e_{i-j}(a_1, \ldots, a_i),$$

where h_k and e_k are the kth complete homogeneous and elementary symmetric polynomials, and $a_n := \#\{\text{atoms } s \in L_n\} - \#\{\text{atoms } s \in L_{n-1}\}$ for all $n \geq 1$.

The key to proving Corollary 2 is to show that each L_{n+1} has a modular coatom t for which $[\hat{0}_{L_{n+1}}, t] \simeq L_n$. In this way, we get rank-preserving embeddings $\iota_n \colon L_n \to L_{n+1}$. By abuse of terminology, we define a *uniform sequence of supersolvable geometric lattices* to be a uniform sequence of geometric lattices $L_0, L_1, \ldots$ for which we have *fixed* such embeddings $\iota_n \colon L_n \to L_{n+1}$, and for which these ι_n are *compatible* with the isomorphisms $[a, \hat{1}_{L_n}] \simeq L_{n-1}$ in the uniformity condition. See [8, §3] for the precise definition.

So now let $L_0, L_1, \ldots$ be a uniform sequence of *supersolvable* geometric lattices, and then define $\mathcal{L}_\infty := \bigcup_{n=0}^\infty L_n$, the direct limit of the L_n with respect to the $\iota_n \colon L_n \to L_{n+1}$. This $\mathcal{L}_\infty$ will almost be an upho lattice, except that it will not be finite type $\mathbb{N}$-graded: it will have infinitely many atoms. We need to "trim" $\mathcal{L}_\infty$ to produce an upho lattice. Hence, for each $k \geq 1$, define $\mathcal{L}_\infty^{(k)} := \{p \in \mathcal{L}_\infty \colon \nu(p) - \rho(p) < k\}$, where for $p \in \mathcal{L}_\infty$ we let $\nu(p) := \min\{n \colon p \in L_n\}$. Then we can prove the following.

Theorem 4. *For each $k \geq 1$, $\mathcal{L}_\infty^{(k)}$ is an upho lattice with core L_k.*

Theorem 4 would not be interesting if there were no interesting examples of uniform sequences of supersolvable geometric lattices. Fortunately, there are many interesting examples, which we now review.

Example 4. *Taking $L_n = B_n$, the rank n Boolean lattice, gives a uniform sequence of supersolvable geometric lattices. For this sequence, $\mathcal{L}_\infty$ is the lattice of all finite subsets of $\{1, 2, \ldots\}$ ordered by inclusion, and $\mathcal{L}_\infty^{(k)} = \{\text{finite } S \subseteq \{1, 2, \ldots\} \colon S \subseteq [\#S + k - 1]\}$. Note that, $\mathcal{L}^{(k)}$ is* not *isomorphic to $\mathbb{N}^k$ (for $k \geq 2$). This is the simplest example a finite graded lattice being the core of two different upho lattices. Of course, we have $F(\mathcal{L}_\infty^{(k)}; x)^{-1} = \chi^*(B_k; x) = (1 - x)^k$.*

Example 5. *Fix a prime power q. Recall that the* subspace lattice *$B_n(q)$ is the lattice of subspaces of $\mathbb{F}_q^n$ ordered by inclusion. Taking $L_n = B_n(q)$ gives a uniform sequence of supersolvable geometric lattices. For this sequence, $\mathcal{L}_\infty$ is the lattice of all finite-dimensional subspaces of $\mathbb{F}_q^\infty$, and $\mathcal{L}_\infty^{(k)} = \{\text{finite-dimensional } U \subseteq \mathbb{F}_q^\infty \colon U \subseteq \mathrm{Span}\{e_1, \ldots, e_{\dim(U)+k-1}\}\}$, with $e_1, e_2, \ldots$ an ordered basis of $\mathbb{F}_q^\infty$. We have $F(\mathcal{L}_\infty^{(k)}; x)^{-1} = \chi^*(B_k(q); x) = (1 - x)(1 - qx) \cdots (1 - q^{k-1}x)$.*

Example 6. *Recall that the* partition lattice *Π_n is the lattice of set partitions of $[n]$ ordered by refinement. Taking $L_n = \Pi_{n+1}$ gives a uniform sequence of supersolvable geometric lattices. For this sequence, $\mathcal{L}_\infty$ is the lattice of all set partitions of $\{1, 2, \ldots\}$ for which all but finitely many blocks are singletons (ordered by refinement). And $\mathcal{L}_\infty^{(k)}$ can be identified with the set partitions of a set of the form $[n]$ into k blocks, still ordered by refinement in the sense that $\pi_1 \leq \pi_2$ if each block of π_1 is a subset of a block of π_2. Figure 1 depicts $\mathcal{L}_\infty^{(2)}$ for this example. We have $F(\mathcal{L}_\infty^{(k)}; x)^{-1} = \chi^*(\Pi_{k+1}; x) = (1 - x)(1 - 2x) \cdots (1 - kx)$.*

Example 7. *Fix a finite group G, say with m elements. In [4], Dowling defined a lattice $Q_n(G)$, now called a* Dowling lattice, *consisting of certain "G-decorated" (partial) set partitions of $[n]$. When G is the trivial group, $Q_n(G) = \Pi_{n+1}$. And when $G = \mathbb{Z}/2\mathbb{Z}$, $Q_n(G)$ is the lattice of flats of the Type B_n Coxeter hyperplane arrangement. Dowling proved that $Q_n(G)$ is a uniform sequence of supersolvable geometric lattices, with $\chi^*(Q_n(G); x) = \prod_{i=1}^n (1 - (1 + m(i - 1))x)$. See [8, §3] for the description of the $\mathcal{L}_\infty$ and $\mathcal{L}_\infty^{(k)}$ for this example.*

4   Upho lattices from monoids

In this section, we explain how monoids give rise to upho lattices. The examples in this section are quite different from those in Section 3. For example, the characteristic polynomials in this section will not necessarily factor over the integers.

Recall that a *monoid* $M = (M, \cdot)$ is a set M with an associative binary product $\cdot$ that has an identity element. The *free monoid* on a set S is the monoid of words over the alphabet S with the product being concatenation and identity the empty word. A *presentation* of a monoid M is a way of writing $M = \langle S \mid R \rangle$ as a quotient of the free monoid over some generating set S by the relations R. A monoid M is *finitely generated* if it has a presentation $M = \langle S \mid R \rangle$ with S finite. Let us say that M is *homogeneously finitely generated* if it has a presentation $M = \langle S \mid R \rangle$ with S finite and R homogeneous. That is, we require that all relations in R are of the form $w_1 = w_2$ with $\ell(w_1) = \ell(w_2)$, where for a word w use $\ell(w)$ to denote its length.

Let M be a monoid. For $a, b \in M$, we say that a is a *left divisor* of b, and b is a *right multiple* of a, if $ax = b$ for some $x \in M$. We use $\leq_L$ for the preorder of left divisibility on M, which is actually a partial order if M is homogeneously finitely generated. The monoid M is called *left cancellative* if whenever $ab = ac$ then $b = c$, for all $a, b, c \in M$.

Lemma 1 (c.f. [6, Lemma 5.1] and [5]). *Let M be a homogeneously finitely generated monoid. If M is left cancellative, then $\mathcal{L} := (M, \leq_L)$ is an upho poset. If moreover every pair of elements in M has a least common right multiple, then $\mathcal{L}$ is an upho lattice.*

The significance of the Möbius function to enumeration in monoids, especially cancellative monoids, was already observed many years ago in the work of Cartier and Foata on free partially commutative monoids [2]. In practice, the left cancellative property of a monoid is not hard to check, but the lattice property is more difficult to establish. Nevertheless, some examples can be produced by hand:

Example 8. *Fix $r \geq 2$ and $n \geq 1$ and let $M := \langle x_1, \ldots, x_r \mid x_i x_1^n = x_1^{n+1}$ for $i = 2, \ldots, r \rangle$. Then M satisfies the conditions of Lemma 1, so $\mathcal{L} := (M, \leq_L)$ is an upho lattice. Its core is $L := \hat{0} \oplus r \cdot [n] \oplus \hat{1}$, i.e., the result of appending a minimum and maximum to the disjoint union of r n-element chains. We have $F(\mathcal{L}; x)^{-1} = \chi^*(L; x) = 1 - rx + (r-1)x^{n+1}$.*

Note in particular that taking $n = 1$ in Example 8 shows how every rank two finite graded lattice (with at least two atoms) can be realized as the core of an upho lattice. We will see in Section 5 that not all rank three lattices can be realized as cores.

To obtain more sophisticated examples of upho lattices from Lemma 1, we need some deeper theory. An important class of monoids satisfying the conditions of Lemma 1 are the (homogeneous) *Garside monoids*. We refer to [3] for a complete account of the theory of Garside monoids. Without giving the full definition, we note that a Garside monoid

is not only left cancellative and a lattice under left divisibility, it is also right cancellative and a lattice under right divisibility.

The most significant examples of Garside monoids come from finite Coxeter groups. We give only a cursory account of the relevant Coxeter theory here; see, e.g., [1] for much more detail. Recall that W is a *finite Coxeter group* if W is a finite group with generating set $S = \{s_1, \ldots, s_r\} \subseteq W$ for which $W = \langle S \mid s_i^2 = 1 \text{ for all } i, (s_i s_j)^{m_{i,j}} = 1 \text{ for } i < j \rangle$ for certain integers $m_{i,j} \geq 2$. We also say (W, S) is a *finite Coxeter system* in this case, and we say S are the *simple reflections* of W. For example, the symmetric group S_n of permutations of $[n]$ is a finite Coxeter group with simple reflections $\{s_1, \ldots, s_{n-1}\}$ the adjacent transpositions $s_i = (i, i+1)$; here we have $m_{i,j} = 2$ if $j - i \geq 2$ and $m_{i,i+1} = 3$.

Now fix a finite Coxeter system $(W, S = \{s_1, \ldots, s_r\})$. The *length* $\ell(w)$ of $w \in W$ is the minimum length of a way of writing $w = s_{i_1} \cdots s_{i_k}$ as a word in the s_i. The *weak order* of W is the partial order on W whose cover relations are $w \lessdot ws$ whenever $\ell(w) < \ell(ws)$, for $w \in W$ and $s \in S$. It is well-known that the weak order is a finite graded lattice, with rank function ℓ. One Garside monoid associated to (W, S) is related to weak order:

Example 9. *Let* $\mathbf{S} = \{\mathbf{s}_1, \ldots, \mathbf{s}_r\}$ *be a collection of letters corresponding to the simple reflections* $S = \{s_1, \ldots, s_r\}$. *For* $\mathbf{s}_i, \mathbf{s}_j \in \mathbf{S}$ *we write* $(\mathbf{s}_i, \mathbf{s}_j)^{[m]} := \mathbf{s}_i \mathbf{s}_j \mathbf{s}_i \mathbf{s}_j \cdots$, *a word with m letters. The* classical braid monoid *associated to* (W, S) *is* $M := \langle \mathbf{S} \mid (\mathbf{s}_i, \mathbf{s}_j)^{[m_{i,j}]} = (\mathbf{s}_j, \mathbf{s}_i)^{[m_{i,j}]} \text{ for } i < j \rangle$. *It is known that M is a Garside monoid (see [3, Chapter IX, §1]), which implies that M satisfies the conditions of Lemma 1, so* $\mathcal{L} := (M, \leq_L)$ *is an upho lattice. Its core is the weak order of W.*

Continue to fix the finite Coxeter system (W, S). There is another Garside monoid associated to (W, S) that is also very interesting. Let $T := \{w^{-1}sw \colon w \in W, s \in S\} \subseteq W$, which is called the set of *reflections* of W. The *absolute length* $\ell_T(w)$ of $w \in W$ is the minimum length of a way of writing $w = t_1 \cdots t_k$ with all $t_i \in T$. The *absolute order* of W is the partial order on W whose cover relations are $w \lessdot wt$ whenever $\ell_T(w) < \ell_T(wt)$, for $w \in W$ and $t \in T$. Absolute order is a graded poset, with rank function ℓ_T, but it is not a lattice since it has multiple maximal elements. However, if $c \in W$ is a *Coxeter element* (a product $c = s_1 \cdots s_r$ of the simple reflections in some order), then the interval $[1, c]$ in absolute order *is* a lattice, whose isomorphism type does not depend on the choice of c. It is called the *noncrossing partition lattice* of W. When $W = S_n$, the noncrossing partition lattice is the restriction of Π_n to those partitions that are noncrossing when the numbers $1, 2, \ldots, n$ are arranged clockwise around a circle, and hence the name. The second Garside monoid attached to (W, S) is related to the noncrossing partition lattice:

Example 10. *Let* $\mathbf{T}$ *be a collection of letters corresponding to the reflections T. For* $\mathbf{s}, \mathbf{t} \in \mathbf{T}$, *we use the notation* $\mathbf{t}^\mathbf{s}$ *for the letter corresponding to the conjugate* $t^s := s^{-1}ts$. *The* dual braid monoid *associated to* (W, S) *is* $M := \langle \mathbf{T} \mid \mathbf{ts} = \mathbf{st}^\mathbf{s} \text{ for all } \mathbf{s} \neq \mathbf{t} \in \mathbf{T} \rangle$. *It is known that M is a Garside monoid (see [3, Chapter IX, §2]), which implies that M satisfies the conditions of Lemma 1, so* $\mathcal{L} := (M, \leq_L)$ *is an upho lattice. Its core is the noncrossing partition lattice of W. For example, Figure 2 depicts this upho lattice in the case when $W = S_3$.*

We note that the noncrossing partition lattice of S_3 happens to be isomorphic to Π_3, so Figures 1 and 2 are examples of non-isomorphic upho lattices with the same core.

5   Obstructions for cores of upho lattices

In this section we explore various methods for showing that a finite graded lattice *cannot* be realized as the core of an upho lattice. We first observe that there are constraints on the characteristic polynomial of a core coming from Corollary 1.

Lemma 2. *Let L be a finite graded lattice which is the core of an upho lattice. Then the formal power series $\chi^*(L;x)^{-1}$ has all positive coefficients.*

Already Lemma 2 can rule out some plausible candidate lattices from actually being realized as cores, as we now explain.

Let P be a convex polytope. The *face lattice* $L(P)$ of P is the poset of faces of P ordered by inclusion. It is always a finite graded lattice. For example, if P is an $(n-1)$-dimensional simplex, then $L(P) = B_n$, which we know is a core. It is reasonable to ask which other face lattices of convex polytopes are cores.

Example 11. *Let P be the octahedron. Then $\chi^*(L(P);x) = 1 - 6x + 12x^2 - 8x^3 + x^4$, and we can compute that $[x^{13}]\chi^*(L(P);x)^{-1} = -123704$, where $[x^n]F(x)$ means the coefficient of x^n in the power series $F(x)$. So by Lemma 2, $L(P)$ is not the core of any upho lattice.*

Let G be a connected, simple graph on vertex set $[n]$. The *bond lattice* $L(G)$ of G is the restriction of Π_n to the those set partitions π for which the induced subgraph of G on each block of π remains connected. It is always finite graded lattice; in fact, it is the lattice of flats of the graphic matroid of G. We have that $\chi^*(L(G);x) = x^n \cdot \chi(G;x^{-1})$ where $\chi(G;x)$ is the chromatic polynomial of G. For example, if $G = K_n$ is the complete graph, then $L(G) = \Pi_n$, which we know is a core. It is reasonable to ask which other bond lattices of graphs are cores.

Example 12. *Consider $G = C_4$, the 4-cycle graph. Then $\chi^*(L(C_4);x) = 1 - 4x + 6x^2 - 3x^3$ and we can compute that $[x^7]\chi^*(L(C_4);x)^{-1} = -80$. So by Lemma 2, $L(C_4)$ is not the core of any upho lattice.*

Beyond characteristic polynomial obstructions, there are also structural obstructions for cores. The following proposition follows trivially from the definition of the core of an upho lattice, but is still worth recording since it rules out many lattices as cores.

Proposition 1. *Let L be a finite graded lattice which is the core of an upho lattice. Then its maximum $\hat{1}$ is the join of its atoms.*

Looking at the join of the elements covering a given element x (but with $x \neq \hat{0}, \hat{1}$) is a good idea, and leads to further, non-trivial obstructions for cores. The following lemma says that a core must already be "partly self-similar" in order to fit into an upho lattice.

Lemma 3. *Let L be a finite graded lattice which is the core of an upho lattice. Let $x \in L \setminus \{\hat{0}, \hat{1}\}$ and let $y_1, \ldots, y_k \in L$ be the elements covering x. Then there are atoms $s_1, \ldots, s_k \in L$ for which we have $[x, y_1 \vee \cdots \vee y_k] \simeq [\hat{0}, s_1 \vee \cdots \vee s_k]$.*

Lemma 3 lets us rule out many further plausible candidate cores, as we now explain.

Example 13. *Let $n \geq 1$ and let P be the n-dimensional cross polytope, i.e., the convex hull of all permutations of the vectors $(\pm 1, 0, \ldots, 0) \in \mathbb{R}^n$. Consider its face lattice $L := L(P)$. Concretely, the elements of L can be represented as length n words in the alphabet $\{0, +, -\}$, where $w \leq w'$ if w' is obtained from w by changing some 0's to $\pm$'s, together with a maximum element $\hat{1}$. Letting x be any atom of L, it can be seen that no isomorphism of the kind required by Lemma 3 exists when $n \geq 3$, so that L is not the core of any upho lattice.*

The 3-dimensional cross polytope is the octahedron, so Example 13 generalizes Example 11. We also remark that it can similarly be shown that the face lattice of the n-dimensional *hypercube* (the dual of the cross polytope) is not a core for $n \geq 3$.

Example 14. *Let $2 \leq k \leq n$ and let L be the lattice of flats of the uniform matroid $U(n, k)$. Concretely, L is obtained from the Boolean lattice B_n by removing all elements at rank k or higher, and then adding a maximum element $\hat{1}$. Letting x be any atom of L, it can be seen that no isomorphism of the kind required by Lemma 3 exists when $2 < k < n$, so that L is not the core of any upho lattice.*

The lattice of flats of the uniform matroid $U(n, n-1)$ is the same as the bond lattice $L(C_n)$ of the n-cycle graph C_n, so Example 14 generalizes Example 12.

6 Future directions

In this section we briefly discuss future directions in the study of upho lattices.

A question naturally suggested by our work here is the following:

Question 2. *For a finite graded lattice L, let $\kappa(L)$ denote the number of different upho lattices with core L. How does $\kappa(L)$ behave?*

In work in progress joint with Joel Lewis [9], we are pursuing Question 2. On the one hand, we can show that $\kappa(L)$ is finite for any lattice L which has no nontrivial automorphisms, suggesting that it may be finite for all L. On the other hand, we can show that $\kappa(L)$ is unbounded even when we restrict to lattices L of rank two.

Finally, if classifying all upho lattices is too difficult, we might instead hope to classify some subvarieties of upho lattices. Two of the most important subvarieties of lattices are the distributive lattices and the modular lattices. In planned future work, we will explore distributive and modular upho lattices. The only distributive upho lattices are $\mathbb{N}^n$, but modular upho lattices are much more interesting (c.f. [6, Conjecture 1.1]).

References

[1]   D. Armstrong. "Generalized noncrossing partitions and combinatorics of Coxeter groups". *Mem. Amer. Math. Soc.* **202**.949 (2009), pp. x+159. DOI.

[2]   P. Cartier and D. Foata. *Problèmes combinatoires de commutation et réarrangements*. Vol. No. 85. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1969, pp. iv+88.

[3]   P. Dehornoy, F. Digne, E. Godelle, D. Krammer, and J. Michel. *Foundations of Garside theory*. Vol. 22. EMS Tracts in Mathematics. European Mathematical Society (EMS), Zürich, 2015, pp. xviii+691. DOI.

[4]   T. A. Dowling. "A class of geometric lattices based on finite groups". *J. Combinatorial Theory Ser. B* **14** (1973), pp. 61–86. DOI.

[5]   Z. Fu, Y. Peng, and Y. Zhang. "The Monoid Representation of Upho Posets and Total Positivity". In this volume of *FPSAC '24 (Bochum)* abstracts. 2024.

[6]   Y. Gao, J. Guo, K. Seetharaman, and I. Seidel. "The rank-generating functions of upho posets". *Discrete Math.* **345**.1 (2022), Paper No. 112629, 14. DOI.

[7]   S. Hopkins. "A note on Möbius functions of upho posets". *Electron. J. Combin.* **29**.2 (2022), Paper No. 2.39, 6. DOI.

[8]   S. Hopkins. "Upho lattices I: examples and non-examples of cores". In preparation, draft available at https://www.samuelfhopkins.com/docs/upho_lattices.pdf. 2024.

[9]   S. Hopkins and J. Lewis. "Upho lattices II: ways of realizing a core". In preparation. 2024.

[10] R. P. Stanley. "Supersolvable lattices". *Algebra Universalis* **2** (1972), pp. 197–217. DOI.

[11] R. P. Stanley. *Enumerative combinatorics. Volume 1*. Second. Vol. 49. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2012, pp. xiv+626.

[12] R. P. Stanley. "From Stern's triangle to upper homogeneous posets". Talk transparencies, available at https://math.mit.edu/~rstan/transparencies/stern-ml.pdf. 2020.

[13] R. P. Stanley. "Theorems and Conjectures on Some Rational Generating Functions". Eprint available at arXiv:2101.02131. Forthcoming, *Eur. J. Combin.* 2021.

[14] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 10.2)*. https://www.sagemath.org. 2023.