Math 211 (Modern Algebra II), HW# 2,

Spring 2025; Instructor: Sam Hopkins; Due: Monday, February 10th

(This homework only has 3 problems, but the problems are more involved.)

1. Let K be a field and let $L = K(x)$ be the field of rational functions with coefficients in K. Consider the Galois group $\mathrm{Aut}_K(L)$.
   (a) For $a \in L$ with $a \neq 0$, define $\sigma_a \colon L \to L$ by $\sigma_a(f(x)/g(x)) = f(ax)/g(ax)$. Show that $\sigma_a \in \mathrm{Aut}_K(L)$. Conclude that if K is infinite, then $\mathrm{Aut}_K(L)$ is infinite.
   (b) For $b \in L$, define $\tau_b \colon L \to L$ by $\tau_a(f(x)/g(x)) = f(x+b)/g(x+b)$. Show $\tau_b \in \mathrm{Aut}_K(L)$. Show that if $a \neq 1$ and $b \neq 0$, then $\sigma_a \tau_b \neq \tau_b \sigma_a$. Conclude that $\mathrm{Aut}_K(L)$ is nonabelian.
2. Let $L = \mathbb{R}$, the real numbers, viewed as an extension of $K = \mathbb{Q}$, the rational numbers. Consider the Galois group $\mathrm{Aut}_K(L)$.
   (a) Let $\sigma \in \mathrm{Aut}_K(L)$. Prove that $u \geq 0$ if and only if $\sigma(u) \geq 0$. Conclude that σ preserves the order on $\mathbb{R}$. **Hint:** the nonnegative numbers in $\mathbb{R}$ are exactly those which are squares.
   (b) Use part (a) to show that $\mathrm{Aut}_K(L)$ is trivial. **Hint:** every real number can be "trapped" between two rational numbers that are arbitrarily close to it.
3. Let $L = \mathbb{Q}(\omega, \sqrt[3]{2})$, viewed as an extension of $K = \mathbb{Q}$, where $\omega = e^{2\pi i/3} = \frac{-1+\sqrt{-3}}{2}$ is a primitive cube root of unity. Notice that the roots of $f(x) = x^3 - 2$ are $\sqrt[3]{2}$, $\omega\sqrt[3]{2}$, and $\omega^2\sqrt[3]{2}$, so L is the field we get by adjoining all roots of $f(x)$ to $\mathbb{Q}$ (i.e., L is the *splitting field* of $f(x)$).
   (a) What is the degree $[L : K]$? Find a K-basis of L. **Hint**: looking ahead to the other parts can help you answer this one.
   (b) Prove that L/K is a Galois extension.
   (c) Prove that the Galois group $\mathrm{Aut}_K(L)$ is isomorphic to the symmetric group S_3.
   (d) Draw the subgroup structure of S_3 and the subfield structure of L and show how they match up according to the Fundamental Theorem of Galois Theory.
   (e) Which subgroups of S_3 are normal? Which subfields of L are Galois over K? How do these correspond?