Total score = 38/50 + 5 bonus points from presentation
= 43 / 50

Homework 2 – Combinatorics 1

1. Fix a positive integer k. We showed the ordinary generating function $F_k(x) := \sum_{n\geq 0} S(n,k)x^n$ of the Stirling numbers of the 2nd kind satisfies $F_k(x) = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$. Find the partial fraction decomposition of $F_k(x)$, i..e., find the coefficients $a_j \in \mathbb{R}$, $j = 1,2,\ldots,k$, for which $F_k(x) = \frac{a_1}{(1-x)} + \frac{a_2}{(1-2x)} + \cdots + \frac{a_k}{(1-kx)}$. Conclude $S(n,k) = \sum_{j=1}^k a_j \cdot j^n$.
Hint: clear denominators, and then plug in $x = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{k}$.
Bonus just to think about, not do: prove $S(n,k) = \sum_{j=1}^k a_j \cdot j^n$ using (i) the *exponential* g.f. $\sum_{n\geq 0} S(n,k)\frac{x^n}{n!} = \frac{1}{k!}(e^x-1)^k$; or (ii) the Principle of Inclusion-Exclusion (P.I.E.).

$$F_k(x) = \sum_{n\geq 0} S(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\ldots(1-kx)} = \frac{a_1}{1-x} + \frac{a_2}{1-2x} + \ldots + \frac{a_k}{1-kx}$$

$$\Longrightarrow x^k = a_1(1-2x)(1-3x)\ldots(1-kx) + a_2(1-x)(1-3x)\ldots(1-kx) +$$
$$a_3(1-x)(1-2x)\ldots(1-kx) + \ldots + a_k(1-x)(1-2x)\ldots(1-(k-1)x).$$

When $x=1$, $1^k = a_1(1-2)(1-3)\ldots(1-k) \Longrightarrow$

$$1^k = a_1(-1)^{k-1}(2-1)(3-1)\ldots(k-1) = -a_1(-1)^k(k-1)! \Longrightarrow$$
$$\frac{1^k}{(-1)^k} = a_1(k-1)! \Longrightarrow a_1 = -\frac{(-1)^k}{(k-1)!} \Longrightarrow a_1 = \frac{(-1)^{k-1}}{(k-1)!}$$

When $x = \frac{1}{2}$, $\left(\frac{1}{2}\right)^k = a_2\left(1-\frac{1}{2}\right)\left(1-\frac{3}{2}\right)\ldots\left(1-\frac{k}{2}\right) \Longrightarrow$

$$\left(\tfrac{1}{2}\right)^k = 2a_2\left(\tfrac{1}{2}\right)^k(2-1)(2-3)\ldots(2-k) \Longrightarrow$$
$$1 = 2a_2(2-3)\ldots(2-k) = 2a_2(-1)^{k-2}(3-2)\ldots(k-2) \Longrightarrow$$
$$1 = 2a_2(-1)^{k-2}(k-2)! \Longrightarrow \frac{1^{k-2}}{(-1)^{k-2}} = 2a_2(k-2)! \Longrightarrow a_2 = \frac{(-1)^{k-2}}{2(k-2)!}$$

When $x = \frac{1}{3}$, $\left(\frac{1}{3}\right)^k = a_3\left(1-\frac{1}{3}\right)\left(1-\frac{2}{3}\right)\left(1-\frac{4}{3}\right)\ldots\left(1-\frac{k}{3}\right) \Longrightarrow$

$$\left(\tfrac{1}{3}\right)^k = 3a_3\left(\tfrac{1}{3}\right)^k(3-1)(3-2)(3-4)\ldots(3-k) \Longrightarrow$$
$$1 = 3\cdot 2a_3(-1)^{k-3}(4-3)\ldots(k-3) \Longrightarrow \frac{1^{k-3}}{(-1)^{k-3}} = 3\cdot 2\, a_3\,(k-3)! \Longrightarrow$$
$$a_3 = \frac{(-1)^{k-3}}{3\cdot 2\,(k-3)!} = \frac{(-1)^{k-3}}{3!\,(k-3)!}$$

Therefore $a_j = \dfrac{(-1)^{k-j}}{j!\,(k-j)!} \Longrightarrow S(n,k) = \displaystyle\sum_{j=1}^{k} \frac{(-1)^{k-j}}{j!\,(k-j)!}\, j^n = \sum_{j=1}^{k} a_j\, j^n \;\square$

Very good! 10/10

2. *(Stanley, EC1, #2.2)* Let A be some finite set of objects, and suppose these objects potentially posses n different *properties* $p_1, p_2, \ldots, p_n$: e.g., $p_1 =$ "is green"; $p_2 =$ "is solid"; et cetera. For $X \subseteq [n]$, let $f_=(X)$ denote the number of elements in A possessing *exactly* the properties p_i for $i \in X$ (and not possessing any of the properties p_j for $j \notin X$); and let $f_\geq(X)$ denote the number of elements in A possessing *at least* the properties p_i for $i \in X$ (but potentially also some properties p_j for $j \notin X$). Give a bijective proof of the P.I.E. identity

$$\sum_{X \subseteq [n]} f_=(X)(1+y)^{\#X} = \sum_{Y \subseteq [n]} f_\geq(Y) y^{\#Y},$$

i.e., give a bijective proof, for each k, that the coefficients of y^k on the L- and RHS are equal.

0/10

Here's the basic idea for this problem. The coefficient of y^k on the RHS counts the number of pairs (a, Y), where a in A is any element and Y is any size k subset of properties it satisfies. Meanwhile, what is the coefficient of y^k on the LHS? By the binomial theorem applied to (1+y)^{#X}, we see that it is the sum over all a in A of (m choose k), where m is the exact number of properties that a satisfies. But this is the same as our interpretation of the RHS, since for any a there are (m choose k) Y's we can pair it with.

3. *(Stanley, EC1, #2.25(a))* Let $f_i(m,n)$ be the number of $m \times n$ matrices of 0's and 1's, with a total of i 1's, and with at least one 1 in each row and column. Use the P.I.E. to show

$$\sum_{i\geq 0} f_i(m,n)t^i = \sum_{k=0}^{n}(-1)^k\binom{n}{k}((1+t)^{n-k}-1)^m.$$

Let $f_i(m,n)$ be the number of $m \times n$ matrices of 0's and 1's with a total of i 1's, and with at least one 1 in each row and column. Let t count the number of 1's, then $((1+t)^n)^m$ is the number of matrices where every row has at least one 1 in it. We only want to count the number of matrices with at least one 1 in each row and column. To count the number of matrices with at least one 1 in each row and column, we have to modify the formula to be $((1+t)^n-1)^m$. Notice that this goes through each row and checks how many combinations there are with at least one 1.

Now consider what happens if a matrix has a column containing only 1's. In this case each row already contains at least one 1, so we can proceed to "ignore" these columns as we check if the other columns in the matrix contain 1's. There are $\binom{n}{k}$ ways to pick those columns, and once those columns are picked, there are $n-k$ spaces to consider in each row so we have to adjust the formula above to only count the "relevant" columns. Replacing n with $n-k$, we get $((1+t)^{n-k}-1)^m$. Now we have $\binom{n}{k}((1+t)^{n-k}-1)^m$ which counts all of the matrices satisfying the criterion. Notice that a column that is not assumed to have all 1's beforehand can still contain all 1's, so some double

As discussed during the presentation, this idea is basically right but instead of focusing on matrices containing all 1's, to properly apply the P.I.E. you want to think about matrices with columns of all 0's.

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counting will occur, we will adjust for this by using an alternating sum, so by P.I.E., we're done.

$\therefore \sum_{i\geq 0} f_i(m,n)t^i = \sum_{k=0}^{n}(-1)^k\binom{n}{k}((1+t)^{n-k}-1)^m$ $\square$

4. *(Stanley, EC1, #2.25(b))* With $f_i(m,n)$ as in the previous problem, show that

$$\sum_{m,n\geq 0}\left(\sum_{i\geq 0} f_i(m,n)t^i\right)\frac{x^m\,y^n}{m!\,n!} = e^{-x-y}\cdot \sum_{m,n\geq 0}(1+t)^{mn}\frac{x^m\,y^n}{m!\,n!}.$$

Hint: use the formula from the previous problem, and do some algebraic manipulations.

$\sum_{m,n\geq 0}\left(\sum_{i\geq 0} f_i(m,n)t^i\right)\frac{x^m y^n}{m!n!} = \sum_{m,n\geq 0}\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}((1+t)^{n-k}-1)^m\right)\frac{x^m y^n}{m!n!} =$

$\sum_{n\geq 0}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\sum_{m\geq 0}\frac{x^m}{m!}((1+t)^{n-k}-1)^m\right)\frac{y^n}{n!} = \sum_{n\geq 0}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(e^{x((1+t)^{n-k}-1)}\right)\frac{y^n}{n!} =$

$e^{-x}\sum_{n\geq 0}\sum_{k=0}^{n}(-1)^k\binom{n}{k}e^{x(1+t)^{n-k}}\frac{y^n}{n!} = e^{-x}\sum_{n\geq 0}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{m\geq 0}\left(\frac{x^m}{m!}((1+t)^{n-k})^m\right)\frac{y^n}{n!} =$

$e^{-x}\sum_{m,n\geq 0}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+t)^{mn-mk}\frac{x^m y^n}{m!n!} = e^{-x}\sum_{m\geq 0}\sum_{n\geq 0}\frac{y^n}{n!}(1+t)^{mn}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+t)^{-mk}\frac{x^m}{m!}$

$= e^{-x}\sum_{m\geq 0}\sum_{n\geq 0}\frac{y^n}{n!}(1+t)^{mn}(1-(1+t)^{-m})^n\frac{x^m}{m!} = e^{-x}\sum_{m\geq 0}\sum_{n\geq 0}\frac{y^n}{n!}((1+t)^m-1)^n\frac{x^m}{m!} =$

$e^{-x}\sum_{m\geq 0}e^{y((1+t)^m-1)}\frac{x^m}{m!} = e^{-x-y}\sum_{m\geq 0}e^{y(1+t)^m}\frac{x^m}{m!} = e^{-x-y}\sum_{m,n\geq 0}(1+t)^{mn}\frac{x^m y^n}{m!n!}.$

$\therefore \sum_{m,n\geq 0}\left(\sum_{i\geq 0} f_i(m,n)t^i\right)\frac{x^m y^n}{m!n!} = e^{-x-y}\sum_{m,n\geq 0}(1+t)^{mn}\frac{x^m y^n}{m!n!}$ $\square$ Very nice! 10/10

5. The q-binomial coefficient satisfies $\left[{n \atop k}\right]_q = \sum_{w\in\mathcal{W}_{n,k}} q^{\mathrm{inv}(w)}$, where $\mathcal{W}_{n,k}$ is the set of words that are rearrangements of $(n-k)$ 0's, and k 1's, and $\mathrm{inv}(w)$ is the number of inversions of w. **Suppose $n=2m$ is even.** Prove that $\left[{n \atop k}\right]_{q:=-1}$ (the evaluation of the q-binomial at $q=-1$) is equal to $\#\mathcal{P}_{n,k}$, where $\mathcal{P}_{n,k}$ is the subset of words $w = w_1w_2\ldots w_n \in \mathcal{W}_{n,k}$ that are *palindromes* (i.e., which satisfy $w_i = w_{n+1-i}$ for all i). Do this by defining a **sign-reversing involution**. That is, define an involution $\tau\colon \mathcal{W}_{n,k}\to\mathcal{W}_{n,k}$ satisfying:

- $\mathrm{inv}(w)$ and $\mathrm{inv}(\tau(w))$ have opposite parity for all $w\in\mathcal{W}_{n,k}$ with $\tau(w)\neq w$;
- $\mathrm{inv}(w)$ is even for all $w\in\mathcal{W}_{n,k}$ with $\tau(w)=w$;
- $\#\{w\in\mathcal{W}_{n,k}\colon \tau(w)=w\} = \#\mathcal{P}_{n,k}$.

Define $\tau: W_{n,k}\to W_{n,k}$ by $\tau(w) = \begin{cases} w' & \text{where the first } w_i \neq w_{n+1-i} \text{ is swapped} \\ w' & \text{if } \forall i,\ w_i = w_{n+1-i}\end{cases}$.

Let $\mathrm{sgn}(W_{n,k}) = (-1)^{\mathrm{inv}(W)}$ and $\mathrm{wgt}(W_{n,k}) = 1$, so that when $W\neq\tau(W)$, $\mathrm{sgn}(W) = -\mathrm{sgn}(\tau(W))$, i.e., $\mathrm{inv}(W)$ and $\mathrm{inv}(\tau(W))$ have opposite parity. This means that $\mathrm{inv}(W)$ is even when $\tau(W)=W$, accounting for the palindromes. And the number of palindromes, $\#P_{n,k} = \#\{w \in W_{n,k} \mid \tau(w) = w\}$

$\Rightarrow \#P_{n,k} = \#\{w \in W_{n,k} \mid \tau(w) = w\} = \sum_{w \in W_{n,k}} (-1)^{inv(w)}$.

$\therefore \left[{n \atop k}\right]_{q:=-1} = \#P_{n,k}$ $\square$

Could do a little more explanation about why the inversions have the right parity, but yes this is a good involution.

10/10