Total: 48/50 + 5 bonus points from presentation = 53/50

Homework 3 – Combinatorics 1

1. The *complete bipartite graph* $K_{n,m}$ is the graph with vertex set $X \cup Y$ where $X = \{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_m\}$, and with edges $\{x_i, y_j\}$ for all $1 \le i \le n, 1 \le j \le m$ (but with no edges between the x's, or between the y's). Use the Matrix-Tree Theorem to show that the number of spanning trees of $K_{n,m}$ is $n^{m-1}m^{n-1}$.

   Hint: you can use the fact that for a matrix in *block form* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ we have $\det(M) = \det(A - BD^{-1}C) \cdot \det(D)$ as long as D is invertible (this generalizes $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$).

The Matrix-Tree Theorem tells us that the number of spanning trees of $K_{n,m}$ is $\det(L(K_{n,m})^{i,i})$.

$L(K_{n,m})$ is in block form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A = mI_{n,n}$, $B = -J_{n,m}$, $C = -J_{m,n}$ and $D = nI_{m,m}$.

Now WLOG let's consider the matrix $L(K_{m,n})^{n+1,n+1}$, which is also in block form with $A = mI_{n,n}$, $B = -J_{n,m-1}$, $C = -J_{m-1,n}$ and $D = nI_{m-1,m-1}$.

We know that $\det(L(K_{m,n})^{n+1,n+1}) = \det(A - BD^{-1}C) \cdot \det(D)$.

$\det(D) = n^{m-1}$, and $BD^{-1}C = (-J_{n,m-1} \cdot \frac{1}{n} I_{m-1,m-1}) \cdot -J_{n,m-1} = \frac{m-1}{n} J_{n,n}$

$\implies A - BD^{-1}C = mI_{n,n} - \frac{m-1}{n} J_{n,n} \implies \det(A - BD^{-1}C) = m^{n-1}$. yes but could explain this slighty more

$\therefore$ By the Matrix-Tree Theorem, the number of spanning trees of $K_{n,m}$ is $n^{m-1}m^{n-1}$ $\square$ Good. 10/10

2. *(Stanley, EC1, #4.69)* Compute the number of closed walks of length ℓ in the complete bipartite graph $K_{n,m}$. Use this computation, together with the Transfer Matrix Method, to find the eigenvalues of the adjacency matrix of $K_{n,m}$.

Consider the complete bipartite graph $K_{n,m}$.

Now if we start with a point in n, there are m options for the first move. Likewise, if we start with a point in m, there are n options for the first move. So in all closed walks of $K_{n,m}$, there are $m+n$ ways to make the first move.

Also note that since these are closed walks, the last move needs to return us to the starting point, so there is only one option for the final move.

By nature of bipartite graphs, any closed walks on bipartite graphs must have an even length, otherwise we will end on the side opposite to the one we started on. Thus there are $\frac{\ell}{2}$ moves to either side, but we've already accounted for the first and the last moves, so there are $n^{\frac{\ell}{2}-1} m^{\frac{\ell}{2}-1} = (nm)^{\frac{\ell}{2}-1}$ intermediary choices.

$\therefore$ there are $(n+m)(nm)^{\frac{\ell}{2}-1}$ closed walks of length ℓ on $K_{n,m}$.

Not quite. Should be (mn)^{ell/2} for both starting on left and right = 2(mn)^{ell/2} total

$A_{K_{n,m}}$, the adjacency matrix, is in block form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A = O_{n,n}$, $B = J_{n,m}$, $C = J_{m,n}$ and $D = O_{m,m}$. Note that the roots of the characteristic polynomial of $A_{K_{n,m}}$ are the eigenvalues of $A_{K_{n,m}}$. The chacteristic polynomial of $A_{K_{n,m}}$ is $\det(tI - A_{K_{n,m}})$, and since $A_{K_{n,m}}$ is in block form, $tI - A_{K_{n,m}}$ is also in block form, where $A = tI_{n,n}$, $B = -J_{n,m}$, $C = -J_{m,n}$, $D = tI_{m,m}$. From the hint in #1, we know that $\det(tI - A_{K_{n,m}}) = \det(A - BD^{-1}C) \cdot \det(D)$. $\det(D) = t^m$, and $BD^{-1}C = (-J_{n,m} \cdot \frac{1}{t} I) \cdot -J_{m,n} = \frac{m-1}{t} J_{n,n} \Longrightarrow A - BD^{-1}C =$ $tI_{n,n} - \frac{m-1}{t} J_{n,n} \Longrightarrow \det(A - BD^{-1}C) = t^n - nmt^{n-2}$. Therefore, $\det(tI - A_{K_{n,m}}) =$ $\det(A - BD^{-1}C) \cdot \det(D) = (t^n - nmt^{n-2})(t^m) = t^{n+m} - nmt^{n+m-2} =$ $t^{n+m-2}(t^2 - mn)$.

$\therefore$ The eigenvalues of $A_{K_{n,m}}$ are 0 and $\pm\sqrt{mn}$ $\square$

Many thanks to Wolframalpha (on both this one and the last one)!

Ah, okay… you just directly computed eigenvalues of matrix. But problem wanted you to use the walk computation + transfer matrix method to get eigenvalues instead.

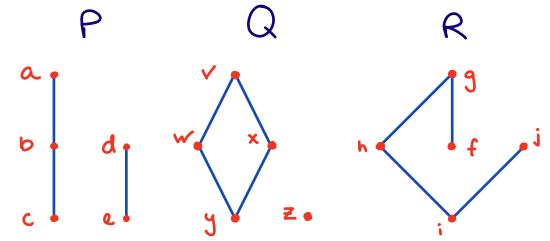
9/10

3. *(Stanley, EC1, #3.34)* Recall that for a poset P, $\mathcal{J}(P)$ denotes the set of *order ideals* of P (i.e., subsets $I \subseteq P$ for which $q \in I$ and $p \leq q \in P$ implies $p \in I$). Find **all** finite posets P for which

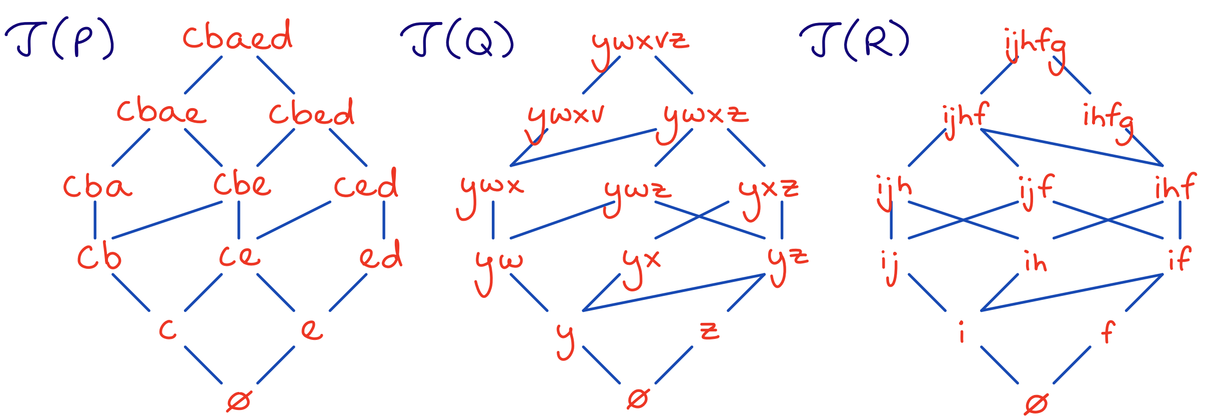
$$\sum_{I \in \mathcal{J}(P)} x^{\#I} = (1+x)(1+x^2)(1+x+x^2).$$

Hint: How many order ideals must such a P have? How many elements must P have? How many *minimal* elements must it have? How many *maximal* elements must it have?

The finite posets P for which $\sum_{I \in \mathcal{J}(P)} x^{\#I} = (1+x)(1+x^2)(1+x+x^2) = 1 + 2x + 3x^2 + 3x^3 + 2x^4 + x^5$ are



To check that these satisfy $\sum_{I \in \mathcal{J}(P)} x^{\#I} = 1 + 2x + 3x^2 + 3x^3 + 2x^4 + x^5$, we can write out the distributed lattice of ordered ideals for each poset:



Very nice pictures, and correct (but could explain why no others). 10/10

4. Let P be a finite poset. An *antichain* A of P is a subset $A \subseteq P$ of pairwise incomparable elements (i.e., for all $p, q \in A$, we have neither $p \leq q$ nor $q \leq p$). Let $\mathcal{A}(P)$ denote the set of antichains of P. Define a partial order $\preceq$ on $\mathcal{A}(P)$ by $A \preceq A'$ iff for every $p \in A$ there is some $p' \in A'$ with $p \leq p'$. Show that $(\mathcal{A}(P), \preceq)$ is isomorphic to $(\mathcal{J}(P), \subseteq)$, the distributive lattice of order ideals of P ordered by containment.

Whenever $I \subseteq I'$, all maximums of I have a relation that's less than or equal to it with a max of I' $\Rightarrow$ g and f are

I understand implicitly you are associating to I the antichain of maximal elements of I (this is the correct bijection), but you could write a little more. 9/10

isomorphic by construction $\square$

5. *(Stanley, EC1, #3.89)* Let L be a finite lattice, with minimum element $\hat{0}$. Let $f_L(m)$ be the number of m-tuples $(t_1, \ldots, t_m) \in L^m$ such that $t_1 \wedge t_2 \wedge \cdots \wedge t_m = \hat{0}$. Use Möbius inversion to show that

$$f_L(m) = \sum_{t \in L} \mu(\hat{0}, t) \cdot (\#\{s \in L \colon s \geq t\})^m,$$

where μ is the Möbius function of L.
Hint: Define $f_L(m,t) := \#\{(t_1, \ldots, t_m) \in L^m \colon t_1 \wedge t_2 \wedge \cdots \wedge t_m = t\}$ for any $t \in L$ (so that $f_L(m) = f_L(m, \hat{0})$), and also define $g_L(m,t) := \#\{(t_1, \ldots, t_m) \in L^m \colon t_1 \wedge t_2 \wedge \cdots \wedge t_m \geq t\}$. How are these f and g related? Can you find a simpler expression for g?

Define $f_L(m,t) := \#\{(t_1, \ldots, t_m) \in L^m : t_1 \wedge t_2 \wedge \ldots \wedge t_m = t\}$ and $g_L(m,t) := \#\{(t_1, \ldots, t_m) \in L^m : t_1 \wedge t_2 \wedge \ldots \wedge t_m \geq t\}$, $\forall t \in L$.
Writing g_L in terms of f_L, we have that $\forall y \in P$,
$g_L(m,y) = \sum_{x \in P | x \geq y} f_L(m,x)$. Inverting this gives us
$f_L(m,y) = \sum_{x \in P | x \geq y} \mu(x,y) g(m,x)$. yes
$g_L(m,t) = \#\{$subsets of L^m with size m that are above $t\} =$ true
$(\#\{s \in L \mid s \geq t\})^m \Longrightarrow f_L(m) = \sum_{t \in L} \mu(\hat{0}, t) \cdot (\#\{s \in L \mid s \geq t\})^m$ $\square$

Good. 10/10

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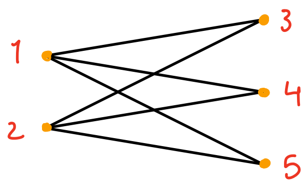
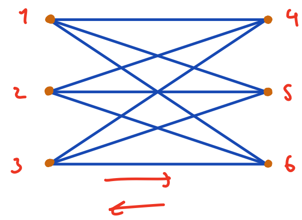


$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

adjacency matrix: $a_{K_{n,m}}$



$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Walks of length 2 $= \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}^2$

$$\begin{pmatrix} 3 & 3 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 2 \end{pmatrix}$$

closed walks

Eigenvalues

Closed walk

~~Can't go anywhere more than once~~

must end at starting point