Howard Math 273, HW# 3,

Fall 2021; Instructor: Sam Hopkins; Due: Friday, December 3rd

1. The *complete bipartite graph* $K_{n,m}$ is the graph with vertex set $X \cup Y$ where $X = \{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_m\}$, and with edges $\{x_i, y_j\}$ for all $1 \leq i \leq n, 1 \leq j \leq m$ (but with no edges between the x's, or between the y's). Use the Matrix-Tree Theorem to show that the number of spanning trees of $K_{n,m}$ is $n^{m-1}m^{n-1}$.

   Hint: you can use the fact that for a matrix in *block form* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ we have $\det(M) = \det(A - BD^{-1}C) \cdot \det(D)$ as long as D is invertible (this generalizes $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$).

2. *(Stanley, EC1, #4.69)* Compute the number of closed walks of length ℓ in the complete bipartite graph $K_{n,m}$. Use this computation, together with the Transfer Matrix Method, to find the eigenvalues of the adjacency matrix of $K_{n,m}$.

3. *(Stanley, EC1, #3.34)* Recall that for a poset P, $\mathcal{J}(P)$ denotes the set of *order ideals* of P (i.e., subsets $I \subseteq P$ for which $q \in I$ and $p \leq q \in P$ implies $p \in I$). Find **all** finite posets P for which
$$\sum_{I \in \mathcal{J}(P)} x^{\#I} = (1 + x)(1 + x^2)(1 + x + x^2).$$

   Hint: How many order ideals must such a P have? How many elements must P have? How many *minimal* elements must it have? How many *maximal* elements must it have?

4. Let P be a finite poset. An *antichain* A of P is a subset $A \subseteq P$ of pairwise incomparable elements (i.e., for all $p, q \in A$, we have neither $p \leq q$ nor $q \leq p$). Let $\mathcal{A}(P)$ denote the set of antichains of P. Define a partial order $\preceq$ on $\mathcal{A}(P)$ by $A \preceq A'$ iff for every $p \in A$ there is some $p' \in A'$ with $p \leq p'$. Show that $(\mathcal{A}(P), \preceq)$ is isomorphic to $(\mathcal{J}(P), \subseteq)$, the distributive lattice of order ideals of P under inclusion.

5. *(Stanley, EC1, #3.89)* Let L be a finite lattice, with minimum element $\hat{0}$. Let $f_L(m)$ be the number of m-tuples $(t_1, \ldots, t_m) \in L^m$ such that $t_1 \wedge t_2 \wedge \cdots \wedge t_m = \hat{0}$. Use Möbius inversion to show that
$$f_L(m) = \sum_{t \in L} \mu(\hat{0}, t) \cdot (\#\{s \in L \colon s \geq t\})^m,$$
   where μ is the Möbius function of L.
   Hint: Define $f_L(m, t) := \#\{(t_1, \ldots, t_m) \in L^m \colon t_1 \wedge t_2 \wedge \cdots \wedge t_m = t\}$ for any $t \in L$ (so that $f_L(m) = f_L(m, \hat{0})$), and also define $g_L(m, t) := \#\{(t_1, \ldots, t_m) \in L^m \colon t_1 \wedge t_2 \wedge \cdots \wedge t_m \geq t\}$. How are these f and g related? Can you find a simpler expression for g?