

Math 211 (Modern Algebra II), HW# 4,

Spring 2025; Instructor: Sam Hopkins; Due: Wednesday, March 19th

In this homework, all roots of unity are meant over the complex numbers \mathbb{C} .

1. Let $1 \leq k \leq n$ be integers. Prove that k is a unit in the ring $\mathbb{Z}/n\mathbb{Z}$ if and only if $\gcd(k, n) = 1$. Conclude that the following quantities are all equal to Euler's totient function $\varphi(n)$:

- the order of the group of units $(\mathbb{Z}/n\mathbb{Z})^\times$;
- the number of generators of $(\mathbb{Z}/n\mathbb{Z}, +)$;
- the number of primitive n th roots of unity;
- the degree of the n th cyclotomic polynomial $\Phi_n(x)$;
- $[\mathbb{Q}(\omega) : \mathbb{Q}]$, where ω is a primitive n th root of unity.

2. Let $\Phi_n(x)$ denote the n th cyclotomic polynomial. Prove the following about these $\Phi_n(x)$:

- (a) If $n = p$ is prime, then $\Phi_p(x) = 1 + x + x^2 + \cdots + x^{p-1}$.
- (b) If $n = 2p$ is twice an odd prime p , then $\Phi_{2p}(x) = \Phi_p(-x)$.
- (c) If $n = p^k$ is a power of the prime p , then $\Phi_{p^k}(x) = \Phi_p(x^{p^{k-1}})$.

3. Let $n > 2$, and let ω be a primitive n th root of unity. Prove that $[\mathbb{Q}(\omega + \omega^{-1}) : \mathbb{Q}] = \varphi(n)/2$.
Hint: It suffices to show $[\mathbb{Q}(\omega) : \mathbb{Q}(\omega + \omega^{-1})] = 2$ (why?). To show $[\mathbb{Q}(\omega) : \mathbb{Q}(\omega + \omega^{-1})] \leq 2$, find a degree two polynomial $f(x) \in \mathbb{Q}(\omega + \omega^{-1})[x]$ which has ω as a root. To show that $\mathbb{Q}(\omega + \omega^{-1}) \neq \mathbb{Q}(\omega)$, think about which of these are subfields of \mathbb{R} versus \mathbb{C} .

4. (a) Let $f(x) = ax^3 + bx^2 + cx + d \in \mathbb{Q}[x]$ be a cubic polynomial (so $a \neq 0$). Show that the polynomial $\frac{1}{a} \cdot f(x - \frac{b}{3a})$ has the form $x^3 + px + q$ for $p, q \in \mathbb{Q}$.
- (b) Let $f(x) = x^3 + px + q \in \mathbb{Q}[x]$. Show that one root of $f(x)$ has the form $x = \sqrt[3]{A} + \sqrt[3]{B}$ where

$$A = \frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}, \quad B = \frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

(This solution to the cubic equation is often called *Cardano's formula*.)

Hint: First notice (and explain why!) that with $x = \sqrt[3]{A} + \sqrt[3]{B}$ we get

$$x^3 + px + q = A + B + (3\sqrt[3]{AB} + p)(\sqrt[3]{A} + \sqrt[3]{B}) + q.$$

And then what can you say about the term $(3\sqrt[3]{AB} + p)$?