Combinatorial reciprocity for non-intersecting paths

Special Session on Enumerative Combinatorics,
AMS Fall Central Sectional Meeting,
Creighton University, Omaha, NE

Sam Hopkins (Howard University)
based on joint work with Gjergji Zaimi

October 7th, 2023

Combinatorial reciprocity

A *combinatorial reciprocity theorem* asserts $f(-n) = \pm g(n)$, where $f(n)$ and $g(n)$ are two related counting functions. It's a "hidden duality."

For example, the most basic combinatorial reciprocity theorem is

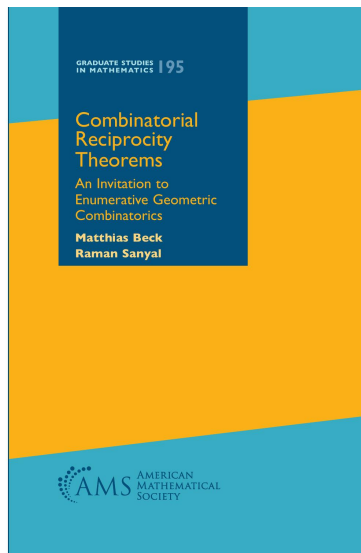$$\binom{-n}{k} = -1^k \left(\!\!\binom{n}{k}\!\!\right)$$

where $\binom{n}{k}$ of course counts the number of k-subsets of $[n] = \{1, 2, \ldots, n\}$, and $\left(\!\!\binom{n}{k}\!\!\right)$ counts the number of k-multisets on $[n]$.

In order to make sense of $\binom{-n}{k}$, we observe that

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-(k-1))}{k!}$$

is a polynomial in n, which can then be evaluated at negative numbers.

Combinatorial reciprocity for polynomials

There are many combinatorial reciprocity theorems for polynomial counting functions, including:

- for the order polynomial $\Omega_P(n)$ of a poset P;
- for the chromatic polynomial $\chi_G(n)$ of a graph G;
- for the Ehrhart polynomial $L_{\mathcal{P}}(n)$ of a lattice polytope $\mathcal{P}$.

Combinatorial reciprocity beyond polynomials

But sometimes we can make sense of $f(-n)$, and prove combinatorial reciprocity theorems, for counting functions $f(n)$ that are not polynomials.

We say that $f: \mathbb{N} \to \mathbb{C}$ satisfies a *linear recurrence* if there are $d \geq 0$ and $\alpha_1, \ldots, \alpha_d \in \mathbb{C}$ (with $\alpha_d \neq 0$) for which

$$f(n+d) + \alpha_1 f(n+d-1) + \alpha_2 f(n+d-2) + \cdots + \alpha_d f(n) = 0$$
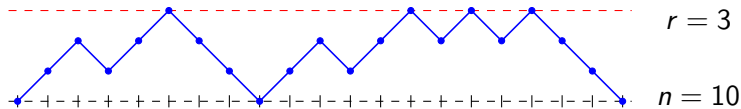
for all $n \geq 0$.

For such an f, we define $f(-n)$ by "running the recurrence backwards." That is, we set

$$f(-n) = \frac{-1}{\alpha_d} \left(f(-n+d) + \alpha_1 f(-n+d-1) + \cdots + \alpha_{d-1} f(-n+1) \right)$$

for all $n \geq 1$.

Bounded Dyck paths

Recall that a *Dyck path* is a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to $(2n, 0)$, whose steps are $(1,1)$ or $(1,-1)$, and which never goes below the x-axis. We say a Dyck path is *r-bounded* if it never goes above the line $y = r$.
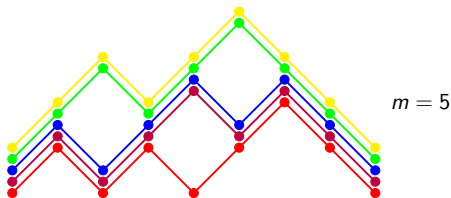

$r = 3$

$n = 10$

Example

Let $f(n)$ be the number of 3-bounded Dyck paths of length $2n$.

Exercise: Show that $f(n) = F_{2n-1}$, where F_n are the *Fibonacci numbers* defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$.

Therefore, $f(n) = \frac{1}{\sqrt{5}}(\varphi^{2n-1} + \varphi^{-2n+1})$, and thus $f(-n) = f(n+1)$.

Reciprocity for fans of bounded Dyck paths

For two Dyck paths D and D', we write $D \leq D'$ if D is weakly below D'. An *m-fan* of Dyck paths is a tuple $D_1 \leq \cdots \leq D_m$ of nested Dyck paths.

$m = 5$

Let $d(m, k; n) = \#$ m-fans of $(2k+1)$-bounded Dyck paths of length $2n$.

Theorem (Cigler–Krattenthaler, 2020)

$d(m, k; n)$ satisfies a linear recurrence, and $d(m, k; -n) = d(k, m; n+1)$.

See also follow up work of Jang–Kim–Kim–Song–Song, 2022 on reciprocity for other kinds of bounded lattice paths (Motzkin, Schröder, et cetera).

Acyclic planar networks

An *acyclic planar network* is an acyclic directed graph $G = (V, E)$ embedded in a disk, with boundary vertices $s_1, \ldots, s_m$ (*sources*) and $t_m, \ldots, t_1$ (*sinks*) in clockwise order, and with *edge weights* $w \colon E \to \mathbb{C}$.

We write $\pi \colon s_i \to t_j$ to mean π is a path in G connecting s_i to t_j, and we write $\Pi = (\pi_1, \ldots, \pi_k) \colon (s_{i_1}, \ldots, s_{i_k}) \to (t_{j_1}, \ldots, t_{j_k})$ to mean Π is a tuple of paths $\pi_\ell \colon s_{i_\ell} \to t_{j_\ell}$. The tuple Π is *non-intersecting* if no two of its paths share any vertices. We set $w(\pi) = \prod_{e \in \pi} w(e)$ and $w(\Pi) = \prod_{\pi \in \Pi} w(\pi)$.
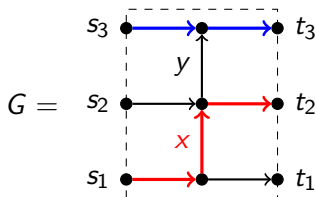
The above non-intersecting tuple $\Pi \colon (s_1, s_3) \to (t_2, t_3)$ has $w(\Pi) = x$, because by convention edges without labels have weight one.

Reciprocity for non-intersecting paths

Let G be an acyclic planar network for which there is a unique, weight one non-intersecting tuple of paths connecting all the sinks to all the sources. Let G^n denote n copies of G glued together like this (red lines = identify):
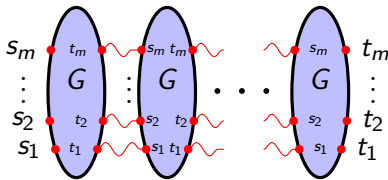
For $I = \{i_1 < \cdots < i_k\}, J = \{j_1 < \cdots < j_k\} \subseteq [m]$ let $f(I, J; n) = \sum w(\Pi)$ a sum over non-intersecting tuples $\Pi \colon (s_{i_1}, \ldots, s_{i_k}) \to (t_{j_1}, \ldots, t_{j_k})$ in G^n.

Theorem (H.–Zaimi, 2023)

$f(I, J; n)$ satisfies a linear recurrence. $f(I, J; -n) = -1^{\sigma(I) + \sigma(J)} f(I^c, J^c; n)$ where for $K \subseteq [m]$ we use $\sigma(K) = \sum_{i \in K} i$ and $K^c = [m] \setminus K$.

Proof ingredients I: LGV lemma

Unsurprisingly, the LGV lemma is a major ingredient in our proof.

For network G, let $P_G = (p_{i,j})$ be *path matrix* of G: $p_{i,j} = \sum_{\pi\colon s_i \to s_j} w(\pi)$.

For an $m \times m$ matrix M and k-subsets $I, J \subseteq [m]$, let $M[I, J]$ denote the $k \times k$ submatrix of M with column indices in I and row indices in J.

Lemma (Lindström–Gessel–Viennot)

For $I = \{i_1 < \cdots < i_k\}, J = \{j_1 < \cdots < j_k\} \subseteq [m]$,

$$\det(P_G[I, J]) = \sum w(\Pi)$$

a sum over non-intersecting tuples $\Pi\colon (s_{i_1}, \ldots, s_{i_k}) \to (t_{j_1}, \ldots, t_{j_k})$ in G.

Proof ingredients II: compound and adjugate matrices

The other ingredient in our proof is a result from elementary linear algebra.

For an $m \times m$ matrix M, let $\mathrm{com}_k(\mathrm{M})$ and $\mathrm{adj}_k(\mathrm{M})$ be the kth *compound* and *adjugate* matrices of M. These are $\binom{m}{k} \times \binom{m}{k}$ matrices whose rows & columns are indexed by k-subsets $I, J \subseteq [m]$. Specifically, the entries are:

$$\mathrm{com}_k(\mathrm{M})_{I,J} = \det(\mathrm{M}[I, J]) \ \mathrm{and} \ \mathrm{adj}_k(\mathrm{M})_{I,J} = -1^{\sigma(I)+\sigma(J)} \det(\mathrm{M}[I^c, J^c])$$

Lemma (Generalized cofactor expansion of determinant)

For any $0 \leq k \leq m$,

$$\mathrm{com}_k(\mathrm{M}) \times \mathrm{adj}_k(\mathrm{M}) = \mathrm{adj}_k(\mathrm{M}) \times \mathrm{com}_k(\mathrm{M}) = \det(\mathrm{M}) \cdot \mathrm{I},$$
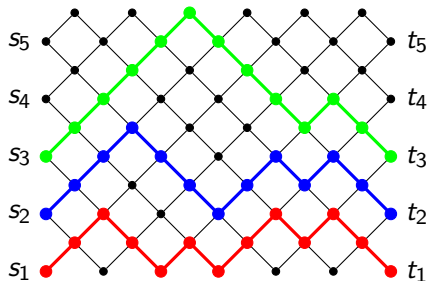
where I *is the* $\binom{m}{k} \times \binom{m}{k}$ *identity matrix.*

Recovering reciprocity for fans of bounded Dyck paths

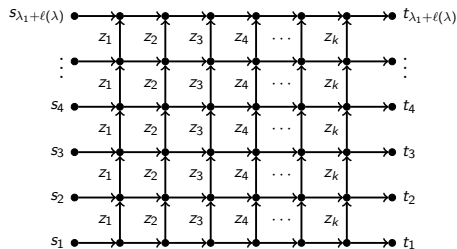To recover the fans of bounded Dyck paths reciprocity from our result, we use this network G:
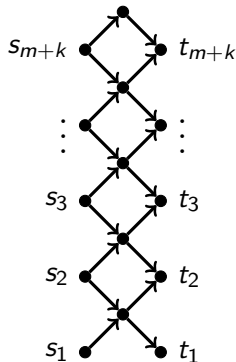
It's easy to see that non-intersecting tuples of paths in G^n correspond to fans of bounded Dyck paths.

Another important network: Schur polynomials

Consider the following network G:

For appropriate I, J depending on λ and μ, non-intersecting tuples in G correspond to SSYT of shape λ/μ.

Thus, the Schur polynomial $s_{\lambda/\mu}(z_1, \ldots, z_n)$ is the generating function of these non-intersecting tuples.
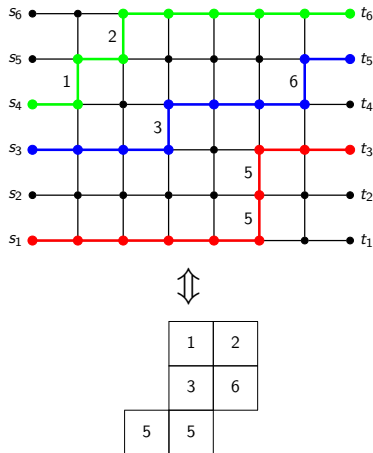
Reciprocity for Schur functions with repeated entries

What does our reciprocity result say for this Schur polynomial network G?

Fix $\mathbf{z} = (z_1, \ldots, z_k) \in \mathbb{C}^k$. Let $\mathbf{z}^n = (z_1, \ldots, z_k, z_1, \ldots, z_k, \ldots, z_1, \ldots, z_k)$, with each value repeated n times. Then, our result yields the following:

Theorem

$s_{\lambda/\mu}(\mathbf{z}^n)$ satisfies a linear recurrence, and $s_{\lambda/\mu}(\mathbf{z}^{-n}) = -1^{|\lambda/\mu|} s_{\lambda^t/\mu^t}(\mathbf{z}^n)$.

More generally, for any homogeneous symmetric function f of degree m, we have that $f(\mathbf{z}^n)$ is a polynomial in n and $f(\mathbf{z}^{-n}) = -1^m \, \omega f(\mathbf{z}^n)$, where $\omega \colon \Lambda \to \Lambda$ is the canonical involution on the ring of symmetric functions Λ. (This will appear as an exercise in the *new edition* of Stanley's EC2).

Extends to quasi-symmetric functions, combinatorial Hopf algebras, etc.

math***overflow***

- `MO:372642`
- `MO:372811`
- `MO:373030`
- `MO:430249`

J. Cigler asked a series of questions on MathOverflow about bounded Dyck paths of "negative length." These attracted comments and answers, including from R. Stanley. Subsequently, J. Cigler and C. Krattenthaler wrote their paper.

I noticed the non-intersecting paths interpretation of Cigler's inquiries, and asked a follow-up MO question. G. Zaimi answered, explaining the argument with compound and adjugate matrices. I later asked an MO question about the symmetric function reciprocity, and again R. Stanley and G. Zaimi provided interesting answers. Then, G. Zaimi and I wrote our joint paper.

Open problems

- Find more **interesting networks** to which we can apply the non-intersecting paths reciprocity theorem. For example, can we recover the Motzkin, Schröder, ... reciprocity of Jang et al. this way?

- In an unpublished manuscript from when he was an undergraduate, D. Speyer proved a combinatorial reciprocity theorem for counting **perfect matchings** in a linearly growing sequence of graphs:

  http://www-personal.umich.edu/~speyer/TransferMatrices.pdf

  Generalizes an earlier reciprocity result of J. Propp for domino tilings. Is there a connection to the non-intersecting paths reciprocity?

- Find a **bijective** proof of the relationship between compound and adjugate matrices, even in the special case of a path matrix P_G.

Thank you!

these slides are on my website:

`https://www.samuelfhopkins.com/docs/reciprocity_talk.pdf`

and the relevant papers are:

- J. Cigler and C. Krattenthaler. "Bounded Dyck paths, bounded alternating sequences, orthogonal polynomials, and reciprocity." Forthcoming, *European J. Combin.*, 2024. `arXiv:2012.03878`
- S. Hopkins, G. Zaimi. "Combinatorial reciprocity for non-intersecting paths." *Enumer. Comb. Appl.* 3, no. 2, 2023. `arXiv:2301.00405`
- J. Jang, D. Kim, J. S. Kim, M. Song, and U.-K. Song. "Negative moments of orthogonal polynomials." *Forum Math. Sigma* 11, 2023. `arXiv:2201.11344`