

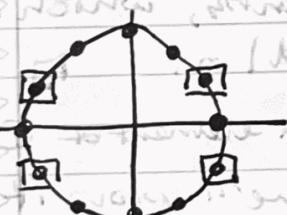
Cyclotomic

3/10 ~~Vieta's~~ Extensions § 5.8

Our goal now is to study finite extensions of \mathbb{Q} of specific forms, leading up to a treatment of the problem which motivated the development of Galois theory: the solvability of polynomials by radicals.

- Def'n Recall that a number $u \in \mathbb{C}$ is called an n^{th} root of unity, for some $n \geq 1$, if $u^n = 1$, i.e., if u is a root of $X^n - 1 \in \mathbb{Q}[X]$.
- If u is an n^{th} root of unity, it is also a $(mn)^{\text{th}}$ root of unity for any $m \geq 1$.
- We say u is a primitive n^{th} root of unity if it is an n^{th} root of unity but not a k^{th} root of unity for any $k < n$.
- Prop. The n^{th} roots of unity are $e^{\frac{2\pi i}{n} j}$ for $j = 0, 1, \dots, n-1$.
- The primitive n^{th} roots of unity are those $e^{\frac{2\pi i}{n} j}$ with $\gcd(j, n) = 1$.

E.g. We've seen before how the n^{th} roots of unity are equally spaced on the unit circle, for instance for $n=12$ we get

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- ↳ the primitive 12^{th} roots of unity are circled: they are $e^{\frac{2\pi i}{12} j}$ for $j = 1, 5, 7, 11$, the integers coprime to 12.

Pf sketch of prop: That the $e^{\frac{2\pi i}{n} j}$ for $j = 0, 1, 2, \dots, n-1$ are the n^{th} roots of unity follows from the fact that $e^{\frac{2\pi i}{n} j} \cdot e^{\frac{2\pi i}{n} k} = e^{\frac{2\pi i}{n} (j+k \bmod n)}$ (phases of complex #'s add when multiplied).

- That the primitive ones are the coprime j 's follows from $e^{\frac{2\pi i}{n} j}$ is a primitive n^{th} root of unity \Leftrightarrow
- j is a generator of $(\mathbb{Z}/n\mathbb{Z}, +)$ \Leftrightarrow
- j is a unit in the ring $\mathbb{Z}/n\mathbb{Z}$ \Leftrightarrow
- j is coprime to n . You will flesh out this argument on your next HW assignment. 

Notice: $\xi_n = e^{\frac{2\pi i}{n}}$ is always a primitive n^{th} root of unity, and all n^{th} roots of unity are powers of this ξ_n .

Def'n Let $n \geq 1$. The n^{th} cyclotomic polynomial $\Phi_n(x) \in \mathbb{C}[x]$

is $\Phi_n(x) = \prod_{w \text{ a primitive } n^{\text{th}} \text{ root of unity}} (x - w)$ (The book uses $\varphi_n(x)$.)

E.g. The primitive 3rd roots of unity are $w = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

and $w^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$; so $\Phi_3(x) = (x - w)(x - w^2) = x^2 + x + 1$.

In fact, the first 6 cyclotomic polynomials are:

$$\Phi_1(x) = x - 1, \quad \Phi_2(x) = x + 1, \quad \Phi_3(x) = x^2 + x + 1, \quad \Phi_4(x) = x^2 + 1$$

$$\Phi_5(x) = x^4 + x^3 + x^2 + x + 1, \quad \Phi_6(x) = x^2 - x + 1.$$

Thm $x^n - 1 = \prod_d \Phi_d(x)$

Pf: Every root of $x^n - 1$ is an n^{th} root of unity, which is a primitive d^{th} root of unity for some $d \mid n$.

Note: Even though $\Phi_d(n)$ is a priori defined as an element of $\mathbb{C}[x]$, books give it belongs to $\mathbb{Q}[x]$. This is true and we'll prove it!

In fact the coefficients are integers, which can get arbitrarily big, but take a while ($\Phi_{105}(x)$ is first with a coeff. not in $\{-1, 1\}$).

The way we will show cyclotomic polynomials are rational is by studying the extensions of \mathbb{Q} we get by adjoining their roots.

Def'n The n^{th} cyclotomic extension of \mathbb{Q} is the splitting field of $x^n - 1$. Equivalently, ...

Thm The n^{th} cyclotomic extension is $\mathbb{Q}(\xi_n)$, where ξ_n is a primitive n^{th} root of unity.

Pf: Since ξ_n is an n^{th} root of unity, it certainly belongs to splitting field of $x^n - 1$.

But on the other hand, every root of unity is a power of ξ_n . \blacksquare

Thm $\text{Aut}_{\mathbb{Q}}(\mathbb{Q}(\xi_n)) = \{\Psi_k : 1 \leq k \leq n, \gcd(n, k) = 1\}$ where

$\Psi_k(\xi_n) = \xi_n^k$. This shows $\text{Aut}_{\mathbb{Q}}(\mathbb{Q}(\xi_n)) \cong (\mathbb{Z}/n\mathbb{Z})^\times$ the group of units of $\mathbb{Z}/n\mathbb{Z}$, with isomorphism $\Psi_k \mapsto k \in \mathbb{Z}/n\mathbb{Z}$.

Pf: Exercise. Point is that primitive n^{th} roots of unity generate $\mathbb{Q}(\xi_n)$, and cannot send ξ_n to a non-primitive n^{th} root of unity because then it would satisfy $x^m - 1$ for some $m < n$. \blacksquare

Cor $\Phi_n(x) \in \mathbb{Q}[x]$. In fact, $\Phi_n(x)$ is min. poly. of ξ_n .

Pf: $\mathbb{Q}(\xi_n)$ is a Galois extension of \mathbb{Q} (since it's a splitting field) and every $\sigma \in \text{Aut}_{\mathbb{Q}}(\mathbb{Q}(\xi_n))$ fixes $\Phi_n(x)$ (since permutes roots), so indeed the coefficients of $\Phi_n(x)$ must be rational. \blacksquare

Remark: Notice that $\text{Aut}_{\mathbb{Q}}(\mathbb{Q}(\xi_n)) \cong (\mathbb{Z}/n\mathbb{Z})^\times$ is always an abelian group, hence every cyclotomic extension is an "abelian extension" (= Galois ext. w/ abelian Galois grp.).

The order of $\text{Aut}_{\mathbb{Q}}(\mathbb{Q}(\xi_n))$ is $\varphi(n) = |\{k \leq n : \gcd(n, k)\}|$, Euler's totient function. If $\varphi(n) = p$ is prime,

then we have seen that $(\mathbb{Z}/p\mathbb{Z})^\times = \mathbb{F}_p^\times = \mathbb{Z}/(p-1)\mathbb{Z}$ is a cyclic group, so $\text{Aut}_{\mathbb{Q}}(\mathbb{Q}(\xi_n))$ is cyclic in this case.

But in general it is not cyclic, just ~~not~~ abelian,

e.g.: $(\mathbb{Z}/8\mathbb{Z})^\times \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$. There is a description of $(\mathbb{Z}/n\mathbb{Z})^\times$ in general, but it is slightly messy (it's an exercise in the textbook...)