

(Piecewise linear & birational) involutions on Dyck paths

Howard Mathematics Colloquium

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Section 1

Catalan numbers, Dyck paths, Naryana numbers, and the Lalanne–Kreweras involution



Montserrat Mountain, Catalonia, Spain

Catalan numbers

The **Catalan numbers** C_n are a famous sequence of numbers

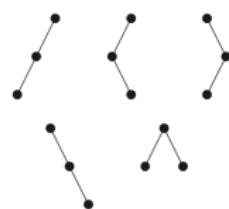
$$1, 2, 5, 14, 42, 132, 429, 1430, \dots,$$

which count numerous combinatorial collections including:

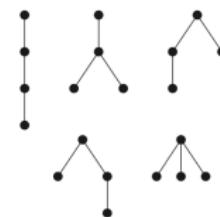
triangulations
of an $n + 2$ -gon



binary trees
with n nodes



plane trees with
 $n + 1$ nodes



bracketings of
 $n + 1$ terms

$$\begin{aligned} &a(b(cd)) \quad a((bc)d) \\ &(ab)(cd) \quad (a(bc))d \\ &((ab)c)d \end{aligned}$$

There is a well-known product formula for the Catalan numbers:

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$$

History of Catalan numbers

The Catalan numbers are named after Belgian mathematician *Eugène Catalan* (1814 – 1894), who studied them in conjunction with bracketings.

But they were studied combinatorially much earlier by *Leonhard Euler* (1707 – 1783), who showed they count triangulations of convex polygons.

In fact, even earlier, Mongolian mathematician/scientist *Minggatu* (c.1692 – c.1763) used Catalan numbers in certain trigonometric identities.



E. Catalan



L. Euler



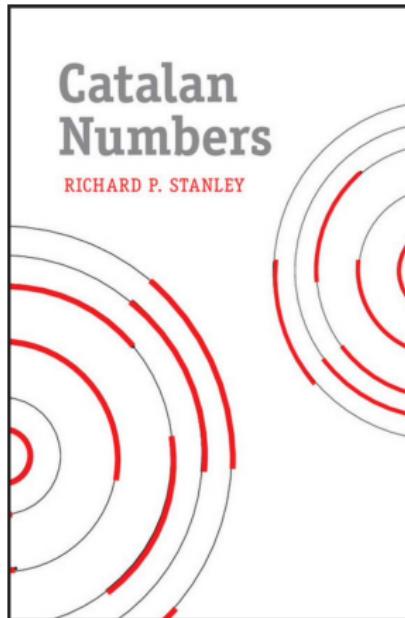
Minggatu

It's a good thing the C_n are not named after Euler, since there are already

- *Euler numbers & Eulerian numbers*, counting certain permutations;
- *Euler's number* $e \approx 2.71$ & the *Euler–Mascheroni constant* $\gamma \approx 0.57$.

Catalan numbers: the book

Richard Stanley has a whole book devoted to the Catalan numbers.

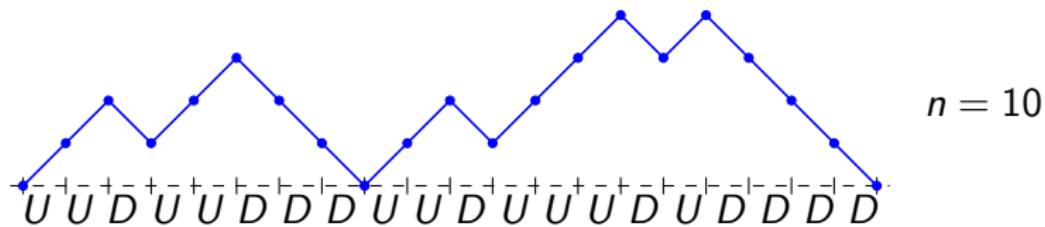


In it, he gives an astounding 214 different interpretations of C_n .

Dyck paths

The interpretation of C_n I want to focus on is in terms of Dyck paths.

A **Dyck path** of length $2n$ is a lattice path in \mathbb{Z}^2 from $(0, 0)$ to $(2n, 0)$ consisting of n up steps $U = (1, 1)$ and n down steps $D = (1, -1)$ that never goes below the x -axis:



The number of Dyck paths of length $2n$ is C_n :

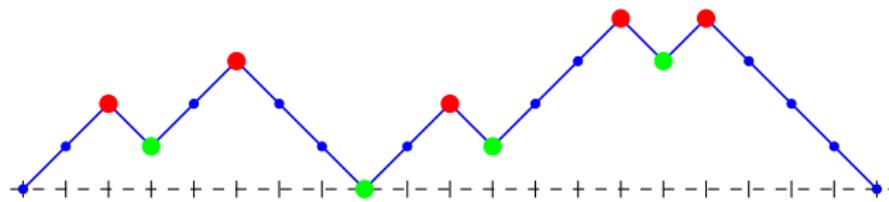


They are named after German algebraist *Walther von Dyck* (1856 – 1934).

Peaks and valleys in Dyck paths

Dyck paths look like mountain ranges. So we use some topographic terminology when working with Dyck paths.

A **peak** in a Dyck path is an up step that is immediately followed by a down step; a **valley** is a down step immediately followed by an up step.



Here the peaks are marked by red circles and the valleys by green circles.
It's easy to see that a Dyck path which has k valleys has $k + 1$ peaks.

Narayana numbers

The **Narayana number** $N(n, k)$ is the number of Dyck paths of length $2n$ with exactly k valleys.

$n \setminus k$	0	1	2	3
1	1			
2	1	1		
3	1	3	1	
4	1	6	6	1

\leftarrow array of $N(n, k)$

Evidently, the Narayana numbers $N(n, k)$ refine the Catalan number C_n :

$$C_n = \sum_{k=0}^{n-1} N(n, k).$$

They are named after Canadian mathematician/statistician *Tadepalli Venkata Narayana* (1930 – 1987), who in 1959 showed that

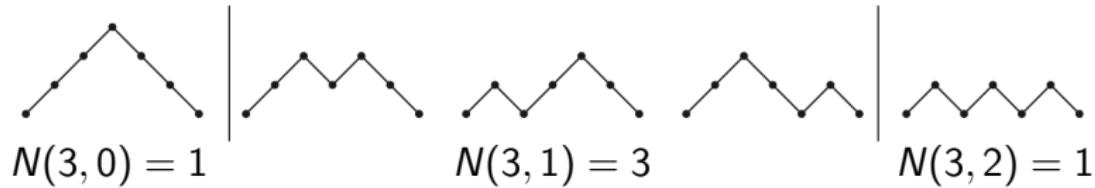
$$N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k+1}.$$

Symmetry of Narayana numbers

From Narayana's formula, it follows immediately that

$$N(n, k) = N(n, n - 1 - k)$$

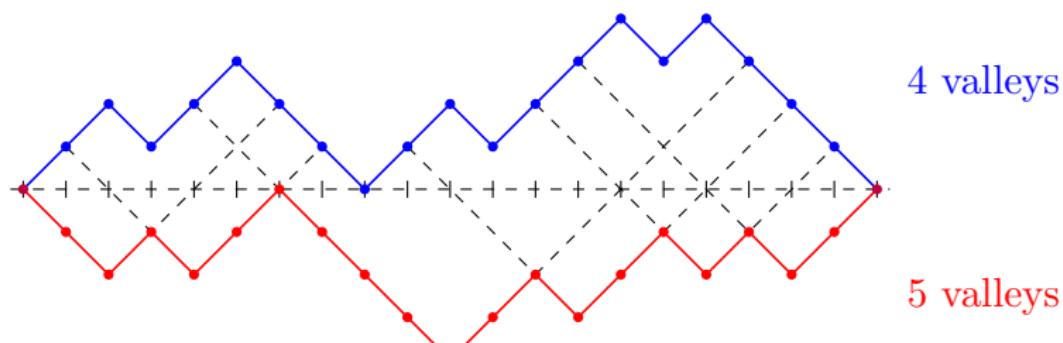
for all k . That is, the sequence of Narayana numbers is *symmetric*.



However, it is not combinatorially obvious why the number of Dyck paths with k valleys should be the same as the number with $n - 1 - k$ valleys.

The Lalanne–Kreweras involution

The **Lalanne–Kreweras involution** (after G. Kreweras and J.-C. Lalanne) is a map on Dyck paths which combinatorially demonstrates the symmetry of the Narayana numbers: $\#\text{valleys}(\Gamma) + \#\text{valleys}(\text{LK}(\Gamma)) = n - 1$.



As depicted above, to compute the LK involution of a Dyck path Γ , we draw dashed lines emanating from the middle of every double up step and every double down step of Γ , at -45° and 45° respectively; these dashed lines intersect at the valleys of (an upside copy of) the Dyck path $\text{LK}(\Gamma)$. That LK is an involution means $\text{LK}^2(\Gamma) = \Gamma$ for all Dyck paths Γ .

Section 2

Posets

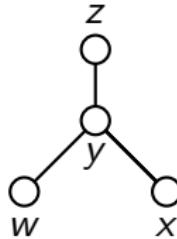
Posets

We will now reinterpret the LK involution using the theory of finite posets.

A (finite) **poset**, or *partially ordered set*, is a (finite) set P together with a relation \leq satisfying the usual axioms of a partial order:

- *transitivity* ($x \leq y, y \leq z \Rightarrow x \leq z$);
- *anti-symmetry* ($x \leq y, y \leq x \Rightarrow x = y$);
- *reflexivity* ($x \leq x$).

We represent posets via their **Hasse diagrams**:

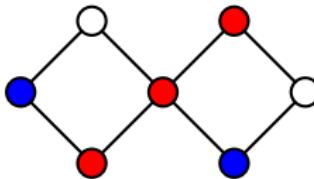


Here an edge from x (below) to y (above) represents the **cover relation** $x \lessdot y$ in P , which means $x < y$ and there is no $p \in P$ with $x < p < y$.

Chains and antichains

Two elements x, y in a poset P are **comparable** if either $x \leq y$ or $y \leq x$. A **chain** $C \subseteq P$ of P is a subset of pairwise comparable elements (i.e., a chain is a *totally ordered* subset $C = \{x_1 < x_2 < \dots < x_k\}$). A chain C is **maximal** if it is not strictly contained in another chain.

Two elements $x, y \in P$ are **incomparable** if they are not comparable. An **antichain** $A \subseteq P$ of P is a subset of pairwise incomparable elements. We use $\mathcal{A}(P)$ to denote the set of antichains of P .



Here the red elements form a maximal chain C , and the blue elements form an antichain $A \in \mathcal{A}(P)$.

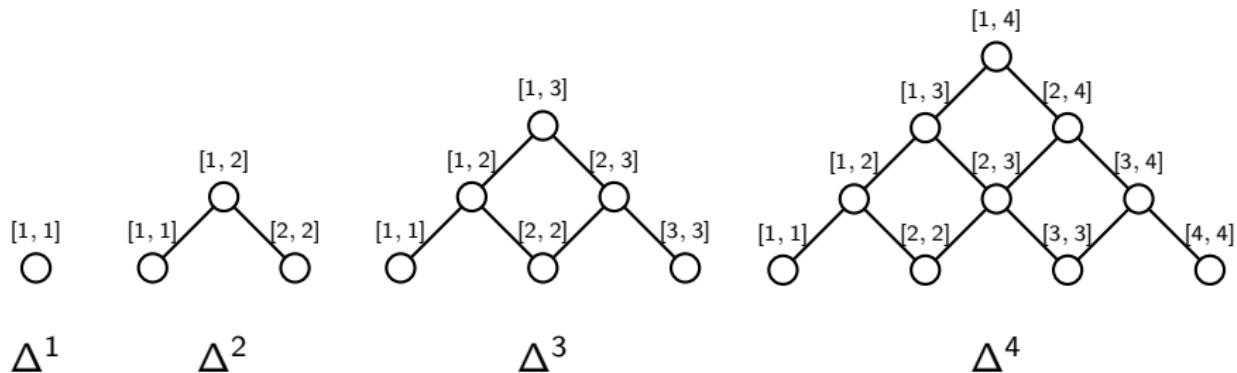
The poset Δ^{n-1}

One particular family of posets Δ^{n-1} is relevant to the LK involution.

Δ^{n-1} is the poset whose elements are **intervals** $[i, j] := \{i, i + 1, \dots, j\}$ with $1 \leq i \leq j \leq n - 1$, and with the partial order given by **inclusion**:

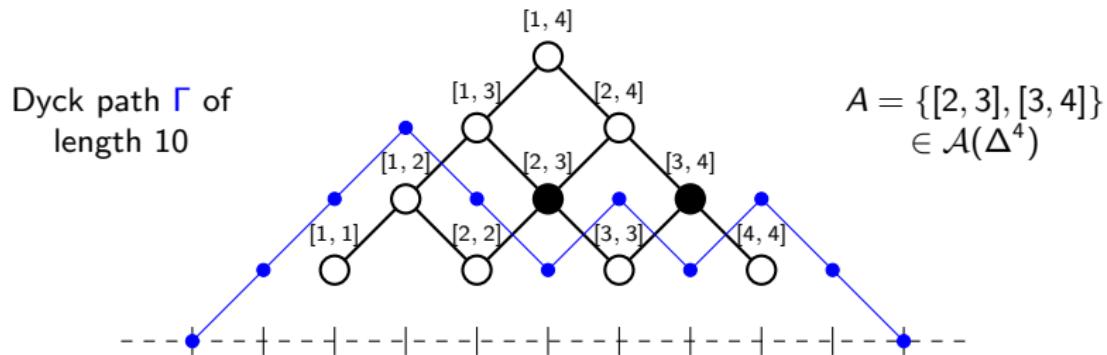
$$[i, j] \leq [i', j'] \iff [i, j] \subseteq [i', j'] \iff i \leq i' \leq j' \leq j$$

Δ^{n-1} has a “triangular” Hasse diagram:



Dyck paths are antichains in Δ^{n-1}

There is a natural, pictorial bijection between the Dyck paths of length $2n$ and the antichains of Δ^{n-1} :



Observe how, under this bijection, the number of valleys of a Dyck path Γ becomes the number of elements of an antichain A .

Via this bijection, we can view the LK involution as an involution on antichains $\text{LK}: \mathcal{A}(\Delta^{n-1}) \rightarrow \mathcal{A}(\Delta^{n-1})$ which satisfies

$$\#A + \#\text{LK}(A) = n - 1.$$

The LK involution on antichains

D. Panyushev gave a simple description of the LK involution on $\mathcal{A}(\Delta^{n-1})$:

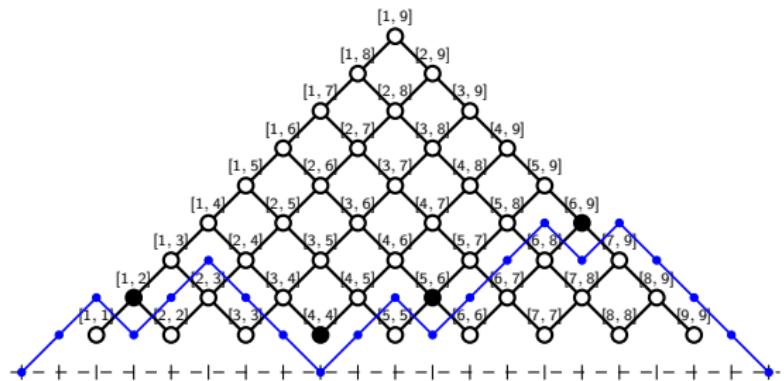
Theorem (Panyushev, 2004)

Let $A = \{[i_1, j_1], [i_2, j_2], \dots, [i_k, j_k]\} \in \mathcal{A}(\Delta^{n-1})$ with $i_1 < i_2 < \dots < i_k$.
 Then $\text{LK}(A) = \{[i'_1, j'_1], [i'_2, j'_2], \dots, [i'_{n-1-k}, j'_{n-1-k}]\} \in \mathcal{A}(\Delta^{n-1})$, where

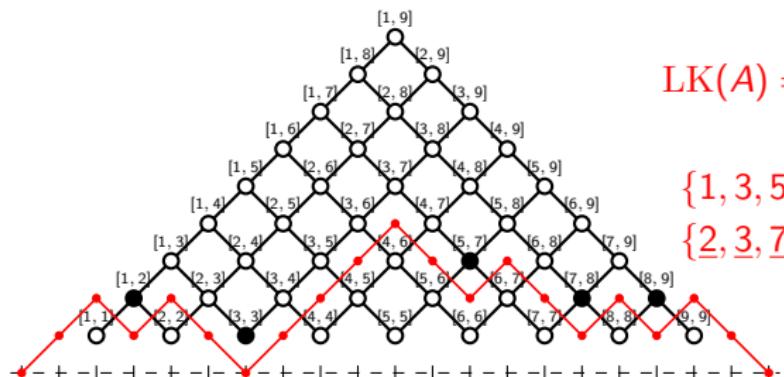
- $\{i'_1 < i'_2 < \dots < i'_{n-1-k}\} = \{1, 2, \dots, n-1\} \setminus \{j_1, j_2, \dots, j_k\}$;
- $\{j'_1 < j'_2 < \dots < j'_{n-1-k}\} = \{1, 2, \dots, n-1\} \setminus \{i_1, i_2, \dots, i_k\}$.

From Panyushev's description, it is immediate that this operation is an involution (i.e., $\text{LK}^2(A) = A$), and that $\#A + \#\text{LK}(A) = n - 1$.

The LK involution on antichains: example



$$A = \{[1, \underline{2}], [4, \underline{4}], [5, \underline{6}], [6, \underline{9}]\}$$



$$\text{LK}(A) = \{[1, \underline{2}], [\underline{3}, \underline{3}], [\underline{5}, \underline{7}], [\underline{7}, \underline{8}], [\underline{8}, \underline{9}]\}$$

$$\begin{aligned} \{1, 3, 5, 7, 8\} &= \{1, \dots, 9\} \setminus \{\underline{2}, \underline{4}, \underline{6}, \underline{9}\} \\ \{\underline{2}, \underline{3}, \underline{7}, \underline{8}, \underline{9}\} &= \{1, \dots, 9\} \setminus \{1, 4, 5, 6\} \end{aligned}$$

Section 3

Toggling

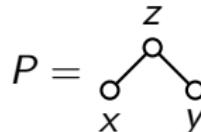
Toggling for antichains

Our first new result gives another expression for the LK involution in terms of certain “local” involutions called **toggles**.

Let P be a poset and $A \in \mathcal{A}(P)$ an antichain. Let $p \in P$ be any element. The **toggle of p in A** is the antichain $\tau_p(A) \in \mathcal{A}(P)$, where

$$\tau_p(A) := \begin{cases} A \setminus \{p\} & \text{if } p \in A; \\ A \cup \{p\} & \text{if } p \notin A \text{ and } A \cup \{p\} \text{ remains an antichain;} \\ A & \text{otherwise.} \end{cases}$$

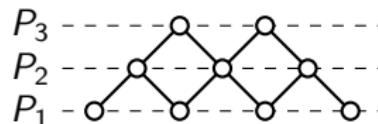
In other words, we “toggle” the status of p in A , if possible:



$$\begin{aligned} \tau_x(\bullet \diagup \circ \diagdown \circ) &= \circ \diagup \circ \diagdown \circ \\ \tau_x(\circ \diagup \bullet \diagdown \bullet) &= \bullet \diagup \circ \diagdown \bullet \\ \tau_x(\circ \diagup \bullet \diagdown \circ) &= \circ \diagup \bullet \diagdown \circ \end{aligned}$$

Toggling in ranked posets

A poset P is **ranked** if we can write $P = P_1 \sqcup P_2 \sqcup \cdots \sqcup P_r$ so that all the edges of the Hasse diagram of P are from P_i (below) to P_{i+1} (above):



Since τ_p and τ_q commute if p and q are incomparable, and all the elements within a rank are incomparable, we can define

$$\tau_i := \prod_{p \in P_i} \tau_p$$

to be the composition of all toggles at rank i , for $i = 1, \dots, r$:



The LK involution as a composition of toggles

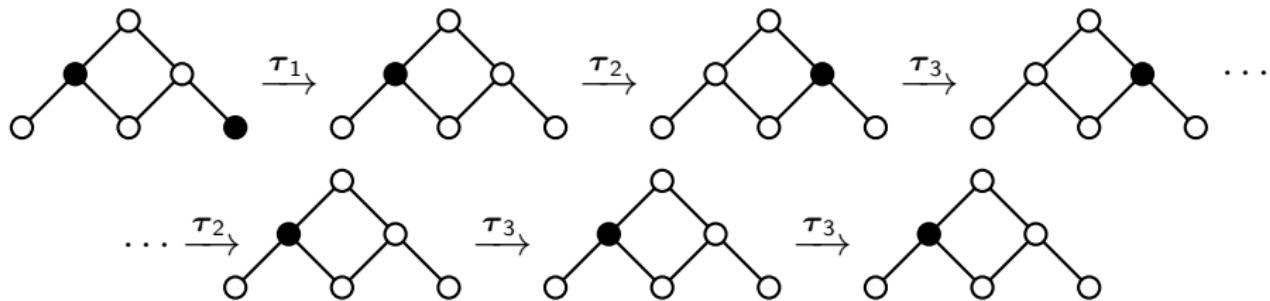
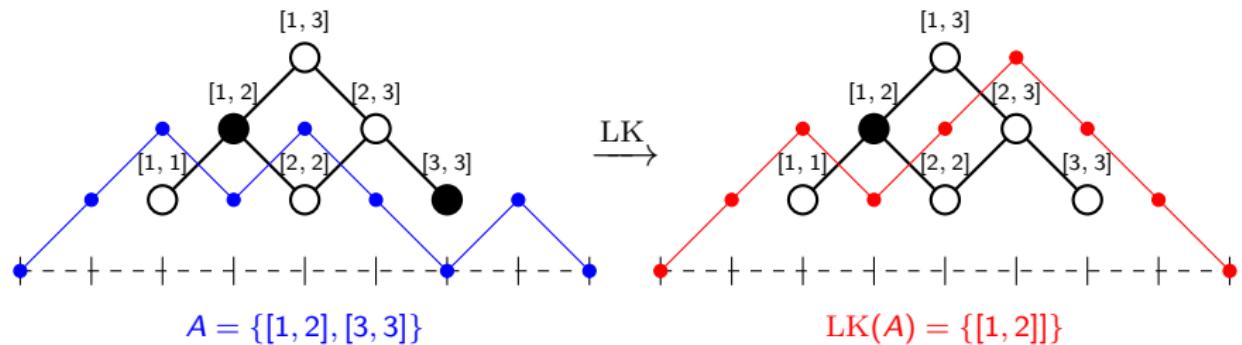
Theorem (H.-Joseph, 2021)

The LK involution $\text{LK}: \mathcal{A}(\Delta^{n-1}) \rightarrow \mathcal{A}(\Delta^{n-1})$ can be written as the following composition of toggles:

$$\text{LK} = (\tau_{n-1})(\tau_{n-1}\tau_{n-2}) \cdots (\tau_{n-1} \cdots \tau_3\tau_2)(\tau_{n-1} \cdots \tau_2\tau_1)$$

Remark: for a ranked poset P , the composition of toggles $\tau_r \cdots \tau_2\tau_1$ “from bottom to top” is called **rowmotion** and has been studied by many authors (Cameron–Fon-Der-Flaass, Striker–Williams, Propp–Roby, Joseph, etc...) in the emerging subfield of **dynamical algebraic combinatorics**.

The LK involution as a composition of toggles: example



Section 4

Piecewise linear and birational lifts

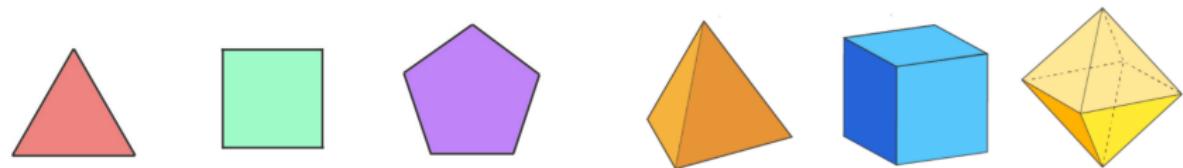
Convex polytopes

Why did we want to write the LK involution as a composition of toggles?
In order to **extend** it to the **piecewise linear** realm...

A **convex polytope** in \mathbb{R}^n can be defined either as

- a convex hull of finitely many points (**vertices**);
- a bounded intersection of finitely many linear inequalities (**facets**).

In dimensions 2 and 3, these are familiar shapes:



There is a rich interplay between combinatorics and convex geometry,
because combinatorial objects can often be “realized” polytopally: e.g.,
the subsets of $\{1, 2, \dots, n\}$ correspond to the vertices of the n -hypercube.

The chain polytope of a poset

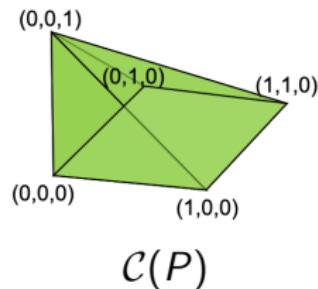
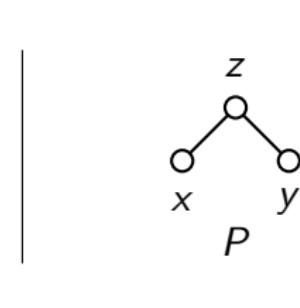
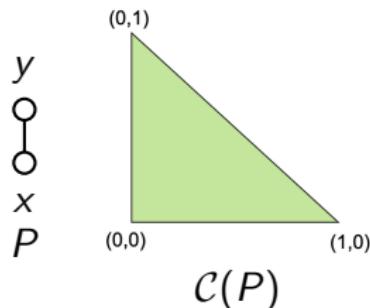
In 1986, Richard Stanley associated to any poset P two polytopes in \mathbb{R}^P , the **order polytope** $\mathcal{O}(P)$ and the **chain polytope** $\mathcal{C}(P)$.

The **chain polytope** $\mathcal{C}(P)$ has facets

$$0 \leq x_p, \quad \forall p \in P$$

$$\sum_{p \in C} x_p \leq 1, \quad \forall C \subseteq P \text{ a maximal chain.}$$

Stanley proved that the **vertices** of $\mathcal{C}(P)$ are precisely the **indicator functions of antichains** $A \in \mathcal{A}(P)$:



Piecewise linear toggling

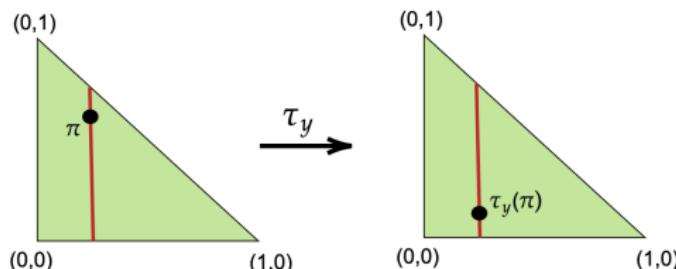
In 2013, D. Einstein and J. Propp (c.f. Joseph) introduced a (continuous) **piecewise linear extension** of the toggles τ_p .

For $p \in P$, the **PL toggle** $\tau_p^{\text{PL}}: \mathcal{C}(P) \rightarrow \mathcal{C}(P)$ is defined by

$$\tau_p^{\text{PL}}(\pi)(q) := \begin{cases} \pi(q) & \text{if } q \neq p; \\ 1 - \max \left\{ \sum_{r \in C} \pi(r) : \begin{array}{l} C \subseteq P \text{ a maximal} \\ \text{chain with } p \in C \end{array} \right\} & \text{if } p = q. \end{cases}$$

Restricted to the vertices of the chain polytope $\mathcal{C}(P)$, it is the same as τ_p .

Geometrically, τ_p **reflects** π within line segment in $\mathcal{C}(P)$ in direction x_p :



The PL LK involution

As before, for a ranked poset P we use $\tau_i^{\text{PL}} := \prod_{p \in P_i} \tau_p^{\text{PL}}$ to denote the composition of all toggles at rank i .

We define the **PL LK involution** $\text{LK}^{\text{PL}} : \mathcal{C}(\Delta^{n-1}) \rightarrow \mathcal{C}(\Delta^{n-1})$ to be

$$\text{LK}^{\text{PL}} := (\tau_{n-1}^{\text{PL}})(\tau_{n-1}^{\text{PL}} \tau_{n-2}^{\text{PL}}) \cdots (\tau_{n-1}^{\text{PL}} \cdots \tau_3^{\text{PL}} \tau_2^{\text{PL}})(\tau_{n-1}^{\text{PL}} \cdots \tau_2^{\text{PL}} \tau_1^{\text{PL}})$$

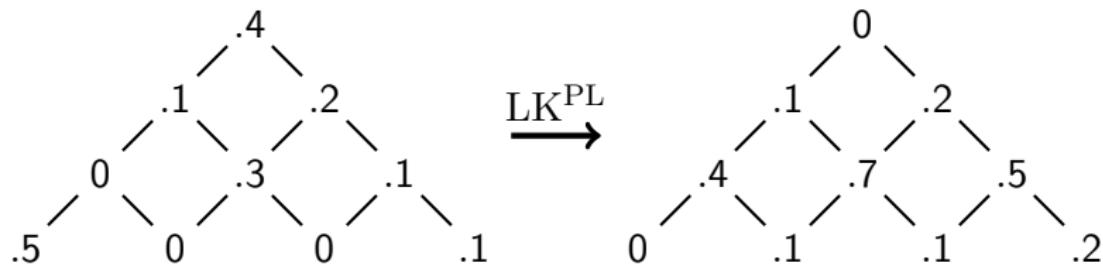
By prior theorem, it's same as LK when restricted to the vertices of $\mathcal{C}(P)$.

Theorem (H.-Joseph, 2021)

- (1) LK^{PL} is an involution.
- (2) For any $\pi \in \mathcal{C}(\Delta^{n-1})$, $\sum_{p \in P} \pi(p) + \sum_{p \in P} \text{LK}^{\text{PL}}(\pi)(p) = n - 1$.

Observe that (2) is an extension of the fact that LK combinatorially exhibits the symmetry of the Narayana numbers.

The PL LK involution: example



We can check that

$$(0.5 + 0 + 0 + 0.1 + 0 + 0.3 + 0.1 + 0.1 + 0.2 + 0.4) + (0 + 0.1 + 0.1 + 0.2 + 0.4 + 0.7 + 0.5 + 0.1 + 0.2 + 0) =$$

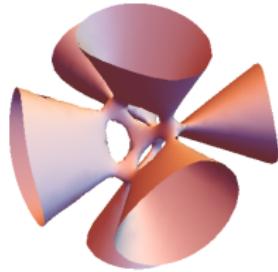
$$1.7 + 2.3 = 4$$

Tropical geometry

Algebraic geometry studies
polynomial expressions like

$$x^3y + y^3z + z^3x$$

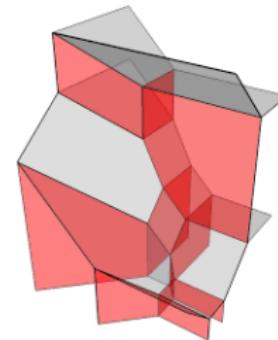
which lead to “curvy” hypersurfaces
like



Tropical geometry studies
piecewise linear expressions like

$$\max(3x + y, 3y + z, 3z + x)$$

which lead to “flat” polytopal
complexes like



“De-tropicalization”

The process of replacing $(\times, +)$ with $(+, \max)$ in a polynomial expression is called **tropicalization**:

$$x^3y + y^3z + z^3x \mapsto \max(3x + y, 3y + z, 3z + x)$$

It lead to important interactions between algebraic & convex geometry.

(Adjective “tropical” comes from fact that computer scientist & pioneer of tropical geometry Imre Simon worked at University of São Paulo,  

The process of replacing $(+, \max)$ with $(\times, +)$ in a piecewise linear expression is called **de-tropicalization***:

$$\max(3x + y, 3y + z, 3z + x) \mapsto x^3y + y^3z + z^3x$$

It is often interesting to try to de-tropicalize PL maps, like those coming from classical combinatorial constructions.

Birational toggling

Einstein–Propp (c.f. Joseph–Roby) also introduced a **birational extension** of the toggles τ_p , via de-tropicalization.

For $p \in P$, the **birational toggle** $\tau_p^B : \mathbb{C}^P \dashrightarrow \mathbb{C}^P$ is

$$\tau_p^B(\pi)(q) := \begin{cases} \pi(q) & \text{if } q \neq p; \\ \kappa \cdot \left(\prod_{\substack{C \subseteq P \\ \text{max. chain,} \\ p \in C}} \sum_{r \in C} \pi(r) \right)^{-1} & \text{if } p = q, \end{cases}$$

where $\kappa \in \mathbb{C}$ is some fixed constant.

The birational toggle τ_p^B tropicalizes to the PL toggle τ_p^{PL} .

The birational LK involution

As before, if P is ranked we set $\tau_i^B := \prod_{p \in P_i} \tau_p^B$.

We define the birational LK involution $\text{LK}^B : \mathbb{C}^{\Delta^{n-1}} \dashrightarrow \mathbb{C}^{\Delta^{n-1}}$ by

$$\text{LK}^B := (\tau_{n-1}^B)(\tau_{n-1}^B \tau_{n-2}^B) \cdots (\tau_{n-1}^B \cdots \tau_3^B \tau_2^B)(\tau_{n-1}^B \cdots \tau_2^B \tau_1^B)$$

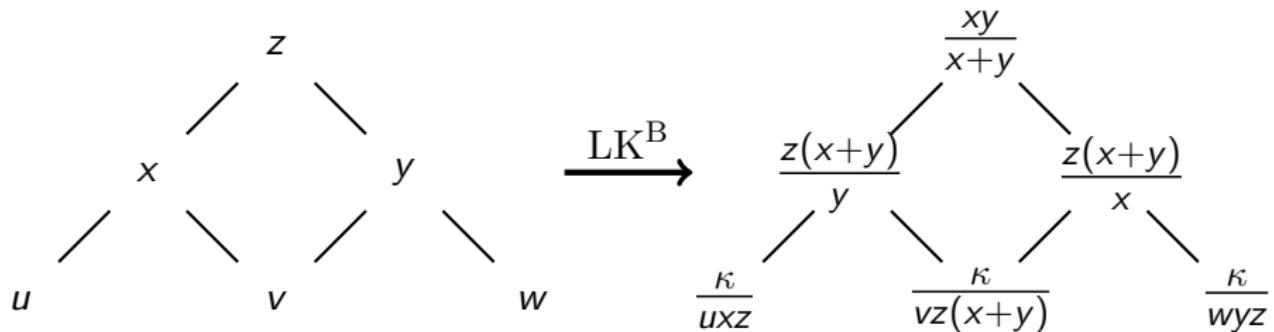
It tropicalizes to LK^{PL} .

Theorem (H.-Joseph, 2021)

- (1) LK^B is an involution.
- (2) For any $\pi \in \mathbb{C}^{\Delta^{n-1}}$, $\prod_{p \in P} \pi(p) \cdot \prod_{p \in P} \text{LK}^B(\pi)(p) = \kappa^{n-1}$.

(2) is the birational analog of the fact that LK combinatorially exhibits the symmetry of the Narayana numbers.

The birational LK involution: example



We can check that this operation really is an involution; e.g.,

$$\frac{z'(x' + y')}{y'} = \frac{\frac{xy}{x+y} \cdot \left(\frac{z(x+y)}{y} + \frac{z(x+y)}{x} \right)}{\frac{z(x+y)}{x}} = \frac{zx + zy}{\frac{z(x+y)}{x}} = \frac{z(x+y)}{\frac{z(x+y)}{x}} = x.$$

And if we multiply together all the above values, we get κ^3 .

Thank you!

these slides are available on my website
and the paper on the arXiv: arXiv:2012.15795

Exercises

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- 6.24.** [?] Explain the significance of the following sequence:

un, dos, tres, quatre, cinc, sis, set, vuit, nou, deu, . . .

R. Stanley, *Enumerative Combinatorics*, Vol. 2