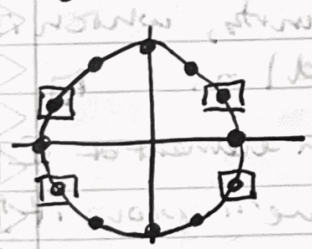
(13) 3/10 Cyclotomic Extensions §5.8

Our goal now is to study finite extensions of $\mathbb{Q}$ of specific forms, leading up to a treatment of the problem which motivated the development of Galois theory: the solvability of polynomials by radicals.

Def'n Recall that a number $u \in \mathbb{C}$ is called an n^{th} root of unity, for some $n \geq 1$, if $u^n = 1$, i.e., if u is a root of $x^n - 1 \in \mathbb{Q}[x]$. If u is an h^{th} root of unity, it is also a $(mn)^{th}$ root of unity for any $m \geq 1$. We say u is a primitive n^{th} root of unity if it is an n^{th} root of unity but not a k^{th} root of unity for any $k < n$.

Prop. The n^{th} roots of unity are $e^{\frac{2\pi i}{n} \cdot j}$ for $j = 0, 1, \ldots, n-1$. The primitive n^{th} roots of unity are those $e^{\frac{2\pi i}{n} \cdot j}$ with $\gcd(j, n) = 1$.

E.g. We've seen before how the n^{th} roots of unity are equally spaced on the unit circle, for instance for $n = 12$ we get



⇐ the primitive 12^{th} roots of unity are circled: they are $e^{\frac{2\pi i}{12} \cdot j}$ for $j = 1, 5, 7, 11$, the integers coprime to 12.

Pf sketch of prop: That the $e^{\frac{2\pi i}{n} \cdot j}$ for $j = 0, 1, 2, \ldots, n-1$ are the n^{th} roots of unity follows from the fact that $e^{\frac{2\pi i}{n} \cdot j} \cdot e^{\frac{2\pi i}{n} \cdot k} = e^{\frac{2\pi i}{n}(j+k \bmod n)}$ (phases of complex #'s add when multiplied).

That the primitive one's are the coprime j's then follows from $e^{\frac{2\pi i}{n} \cdot j}$ is a primitive n^{th} root of unity $\Leftrightarrow$ j is a generator of $(\mathbb{Z}/n\mathbb{Z}, +)$ $\Leftrightarrow$ j is a unit in the ring $\mathbb{Z}/n\mathbb{Z}$ $\Leftrightarrow$ j is coprime to n. You will flesh out this argument on your next HW assignment. ∎

Notice: $\zeta_n = e^{\frac{2\pi i}{n}}$ is always a primitive n^{th} root of unity, and all n^{th} roots of unity are powers of this ζ_n.

Defn Let $n \geq 1$. The n^{th} cyclotomic polynomial $\Phi_n(x) \in \mathbb{C}[x]$ is $\Phi_n(x) = \prod_{\omega \text{ a primitive } n^{th} \text{ root of unity}} (x - \omega)$. (The book uses $g_n(x)$.)

E.g. The primitive 3rd roots of unity are $\omega = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$ and $\omega^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$; so $\Phi_3(x) = (x-\omega)(x-\omega^2) = x^2 + x + 1$.

In fact, the first 6 cyclotomic polynomials are:

$\Phi_1(x) = x - 1$, $\Phi_2(x) = x + 1$, $\Phi_3(x) = x^2 + x + 1$, $\Phi_4(x) = x^2 + 1$

$\Phi_5(x) = x^4 + x^3 + x^2 + x + 1$, $\Phi_6(x) = x^2 - x + 1$.

Thm $x^n - 1 = \prod_{d \mid n} \Phi_d(x)$.

Pf: Every root of $x^n - 1$ is an n^{th} root of unity, which is a primitive d^{th} root of unity for some $d \mid n$. ∎

Note: Even though $\Phi_d(x)$ is a priori defined as an element of $\mathbb{C}[x]$, looks like it belongs to $\mathbb{Q}[x]$. This is true and we'll prove it! In fact the coefficients are integers, which can get arbitrarily big, but take a while ($\Phi_{105}(x)$ is first with a coeff. not in $\{1, -1\}$).

The way we will show cyclotomic polynomials are rational is by studying the extensions of $\mathbb{Q}$ we get by adjoining their roots.

Defn The n^{th} cyclotomic extension of $\mathbb{Q}$ is the splitting field of $x^n - 1$. Equivalently, ...

Thm The n^{th} cyclotomic extension is $\mathbb{Q}(\zeta_n)$, where ζ_n is a primitive n^{th} root of unity.

Pf: Since ζ_n is an n^{th} root of unity, it belongs to splitting field of $x^n - 1$. But on other hand, every root of unity is a power of ζ_n, hence in $\mathbb{Q}(\zeta_n)$. ☐

Thm Let $\psi_k : \mathbb{Q}(\zeta_n) \to \mathbb{Q}(\zeta_n)$ be defined by $\psi_k(\zeta_n) = \zeta_n^k$.
Then $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\zeta_n)) \subseteq \{\psi_k : 1 \le k \le n,\ \gcd(n,k) = 1\}$.

Pf: Any $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\zeta_n))$ is determined by where it sends ζ_n, which must be to some ζ_n^k since these are roots of $x^n - 1$. But it cannot be sent to a non-primitive n^{th} root of unity, since it's not a root of any $x^m - 1$ with $m < n$. ☐

Cor The cyclotomic polynomial $\Phi_n(x) \in \mathbb{Q}[x]$.

Pf: $\mathbb{Q}(\zeta_n)$ is a Galois extension, since it's a splitting field, and every $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\zeta_n))$ fixes $\Phi_n(x)$ since just permutes roots, so in fact coefficients of $\Phi_n(x)$ are rational. ☐

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Thm (Gauss) $\Phi_n(x)$ is irreducible over $\mathbb{Q}$.

Pf: This is non-trivial but I skip it - see the book. ☐

Cor $\Phi_n(x)$ is the minimal polynomial of ζ_n, and every ψ_k for $\gcd(n,k)$ is indeed an element of $G = \mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\zeta_n))$.
Hence $G \cong (\mathbb{Z}/n\mathbb{Z})^\times$, the multiplicative group mod n, via the isomorphism $\psi_k \mapsto k \in (\mathbb{Z}/n\mathbb{Z})^\times$.

Remark: This shows $G \cong (\mathbb{Z}/n\mathbb{Z})^\times$ is an abelian group of order $\varphi(n)$ where $\varphi(n) = \#\{1 \le k \le n : \gcd(n,k) = 1\}$ is Euler's totient function. When $n = p$ is prime we have seen that $(\mathbb{Z}/p\mathbb{Z})^\times$ is in fact cyclic (of order $p-1$), but in general it need not be: e.g. $(\mathbb{Z}/8\mathbb{Z})^\times \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. //

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Cyclic Extensions §5.7

We are almost ready to study the solvability of polynomials by radicals. We just need one more preparatory result...

Def'n An algebraic, Galois extension L/K is called **abelian** if $\mathrm{Aut}_K(L)$ is abelian, it is called **cyclic** if $\mathrm{Aut}_K(L)$ is cyclic, and it is called **cyclic of degree n** if $\mathrm{Aut}_K(L)$ is $\mathbb{Z}/n\mathbb{Z}$.

Remark: We have seen that the cyclotomic extension $\mathbb{Q}(\zeta_n)$ of $\mathbb{Q}$ is always abelian, and sometimes cyclic (e.g. if n is prime) although not always.

In general it is hard to classify cyclic extensions, but there is a nice situation where we can do this.

Def'n For an arbitrary field K, $u \in K$ is called an n^{th} root of unity if $u^n = 1 \in K$, and is called a primitive n^{th} root of unity if $u^0, u^1, \ldots, u^{n-1}$ are all distinct (hence all the n^{th} roots of unity).

For subfields of $\mathbb{C}$, this agrees with our previous definition...

Thm Let K be a field containing a primitive n^{th} root of unity ζ_n for some $n \geq 1$. Then the following are equivalent for L/K:

1) L/K is cyclic of degree d, for some $d \mid n$.
2) L/K is the splitting field of a polynomial of form $f(x) = x^n - a \in K[x]$, in which case $L = K(u)$ for u a m^{th} root of $f(x)$.
3) L/K is a splitting field of irreducible polynomial of form $f(x) = x^d - a$ for some $d \mid n$, in which case $L = K(u)$ for u root of $f(x)$.

E.g. Any degree 2 extension of $\mathbb{Q}$ is the splitting field of a polynomial of the form $x^2 - d$ where d is not a square in $\mathbb{Q}$, and this extension has Galois group $\mathbb{Z}/2\mathbb{Z}$.

E.g. On a previous homework you showed that if $L = \mathbb{Q}(\omega, \sqrt[3]{2})$ (prim. 3rd root of unity) is the splitting field of $x^3 - 2$ over $K = \mathbb{Q}$, then $\mathrm{Aut}_K(L) \cong S_3$, which is not cyclic (not even abelian!). But $\mathbb{Q}$ does not have a prim. 3rd root of unity! If we instead take $K = \mathbb{Q}(\omega)$, then $\mathrm{Aut}_K(L) = \mathbb{Z}/3\mathbb{Z}$.

In the theorem, 2) and 3) are easily seen to be equivalent, just having to do with whether $x^n - a$ is irreducible, equivalently, whether a has a d^{th} root in K for some $d \mid n$. The main point is showing 3) $\Leftrightarrow$ 1). In fact we will mostly care about 3) $\Rightarrow$ 1), which we will prove now; we just need:

Lemma If K is a field with a primitive n^{th} root of unity ζ, then for any $d \mid n$, $\eta = \zeta^{n/d}$ is a primitive d^{th} root of 1. And if L is an extension of K such that $u \in L$ is a root of $x^d - a \in K[x]$, then all the roots of $x^d - a$ are $u, \eta u, \eta^2 u, \ldots, \eta^{d-1} u$ (all distinct). ∎

Pf: Straightforward exercise. ∎

Pf of 3) $\Rightarrow$ 1) in thm: By the lemma, the roots of $x^d - a$ in L are $u, \eta u, \ldots, \eta^{d-1} u$ where u is any root and $\eta = \zeta^{n/d}$ as above. So any $\sigma \in \mathrm{Aut}_K(L)$ is determined by where it sends u (since $\eta \in K$ is fixed by σ). Since $x^d - a$ is irreducible, there must be some σ with $\sigma(u) = \eta u$, and this σ generates all of $\mathrm{Aut}_K(L)$ since $\sigma^k(u) = \eta^k u$, which give all the possible automorphisms in the Galois group by the previous sentence. ∎

We will only sketch the ideas that go into the pf of (1) $\Rightarrow$ (3):

Def'n Let L/K be a finite Galois extension, and suppose that $\mathrm{Aut}_K(L) = \{\sigma_1, \dots, \sigma_n\}$. For any $u \in L$, the norm of u is $N(u) = \sigma_1(u) \cdot \sigma_2(u) \cdot \dots \cdot \sigma_n(u)$.

Eg. Let $K = \mathbb{R}$ and $L = \mathbb{C}$. Recall that $\mathrm{Aut}_{\mathbb{R}}(\mathbb{C}) = \{1, \sigma\}$ where $\sigma : z \mapsto \bar{z}$ is complex conjugation. So the norm of $z = a+bi \in \mathbb{C}$ is $N(z) = z \cdot \bar{z} = a^2 + b^2$, usual complex norm.

Eg. For $K = \mathbb{Q}$ and $L = \mathbb{Q}(i)$, the same is true: the norm of $a+bi$ is $(a+bi)(a-bi) = a^2 + b^2 \in \mathbb{Q}$.

Prop. If L/K is a finite Galois extension, then the norm $N(u)$ of any $u \in L$ is an element of the base field K.

Pf: For any $\sigma \in \mathrm{Aut}_K(L)$, $\sigma(N(u)) = \sigma\sigma_1(u) \cdot \sigma\sigma_2(u) \cdots \sigma\sigma_n(u)$
$= \sigma_{i_1}(u) \cdot \dots \cdot \sigma_{i_n}(u) = N(u)$
(where $i_1, \dots, i_n$ is some permutation of $1, \dots, n$), so because L/K is Galois, $N(u) \in K$ as claimed. ∎

Remark: We can define the norm for non-Galois extensions too, and it remains true that it belongs to the ground field, but it's a little more technical.

Another important property of the norm is multiplicativity:

Prop. We have $N(u) \cdot N(v) = N(uv)$ for $u, v \in L$.

Pf: Straightforward exercise. ∎

The norm is particularly useful for cyclic extensions...

Thm (Hilbert Theorem 90) Let L/K be a finite cyclic (Galois) extension and let $\sigma \in \mathrm{Aut}_K(L)$ be a generator of the Galois group. Then for $u \in L$, $N(u)=1 \iff u = v/\sigma(v)$ for some $v \in L$.

Pf: One direction is easy: if $u = \frac{v}{\sigma(v)}$ then $N(u) = \frac{\sigma_1(v)\cdots\sigma_n(v)}{\sigma_{i_1}(v)\cdots\sigma_{i_n}(v)} = 1$

The other direction is nontrivial - see the book for a proof. ∎

E.g. Consider $L=\mathbb{Q}(i)$ over $K=\mathbb{Q}$. The elements in $\mathbb{Q}(i)$ of norm 1 are $\frac{p}{r}+\frac{q}{r}i$ with $\frac{p^2}{r^2}+\frac{q^2}{r^2}=1$, i.e., $p^2+q^2=r^2$, $p,q,r\in\mathbb{Z}$. These are Pythagorean triples. Hilbert's Thm 90 says they can all be written in form $\frac{a+bi}{a-bi} = \frac{a^2-b^2}{a^2+b^2} + \frac{2ab}{a^2+b^2}i$, $a,b\in\mathbb{Z}$. It is a classic fact going back to Euclid that (primitive) Pythagorean triples can be parameterized in this way.

With Hilbert's thm 90 we can complete the pf of main thm:

Pf of 1) $\Rightarrow$ 3): Let $\sigma \in \mathrm{Aut}_K(L)$ be a generator, and let $\eta = \zeta^{n/d}$ be a primitive dth root of unity. Then $N(\eta) = \eta\cdot\sigma(\eta)\cdots\sigma^{d-1}(\eta) = \eta^d = 1$ (since $\eta\in\mu_n$), so by Hilbert 90 we can write $\eta = v/\sigma(v)$ for some $v\in L$. Notice $\sigma(v^d) = (\sigma(v))^d = (\frac{v}{\eta})^d = \frac{v^d}{\eta^d} = v^d$ (since $\eta^d=1$), so because the extension L/K is Galois, this means that $v^d \in K$. Then v is a root of the polynomial $x^d - v^d \in K[x]$, and it can be shown that this polynomial is in fact irreducible over K and that $L = K(v)$ is the splitting field.

∎

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Radical Extensions & Solving Polynomials §5.9

We come now to one of the major achievements of Galois theory: a precise understanding of when polynomial equations can be solved by expressions using radicals. The famous quadratic formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ says that the roots of any quadratic ax^2+bx+c can be expressed in terms of the coefficients using the basic field operations $(+, -, *, \%)$ together with the square root $\sqrt{}$. Similarly, you saw on HW #4 how for any cubic equation $ax^3+bx^2+cx+d=0$, we can express the solutions in terms of the coefficients using field operations together with square roots and cube roots. In fact, there is also a "quartic formula" expressing the solutions to a degree 4 equation in terms of radicals (i.e., n^{th} roots $\sqrt[n]{}$), but the pattern stops there: as we will see, there is no general "quintic formula."

Def'n Let K be a field. We say a finite (hence, algebraic) extension $L = K(u_1, u_2, \ldots, u_n)$ of K is a radical extension if for each $i = 1, \ldots, n$, there is an $m \geq 1$ such that $u_i^m \in K(u_1, \ldots, u_{i-1})$, i.e. u_i is an "m^{th} root" of an element in $K(u_1, \ldots, u_{i-1})$.

Def'n Let $f(x) \in K[x]$ be a polynomial. We say that $f(x)$ is solvable by radicals if the splitting field of $f(x)$ is a subfield of some radical extension of K.

This captures the notion of the roots of $f(x)$ being expressible from K using the field operations & radicals.

Not only will we show that there is no general formula for equations of degree $n \geq 5$ (using radicals), we will show that, for all degrees $n \geq 5$, there are specific polynomials $f(x) \in \mathbb{Q}[x]$ for which $f(x)$ is not solvable by radicals.

Remark: Notice that we take $K = \mathbb{Q}$ here. If we took, e.g., $K = \mathbb{C}$, then every $f(x) \in \mathbb{C}[x]$ is "solvable by radicals" for the trivial reason that the roots of $f(x)$ belong to the base field $\mathbb{C}$.

The key to showing that some polynomials are not solvable by radicals is to show that the Galois groups of polynomials that are solvable by radicals have a restricted form. So we need to recall some notions from group theory.

Defn Let G be a group. For $x, y \in G$, $[x,y] = xyx^{-1}y^{-1}$ is the commutator of x and y (measures extent to which x and y fail to commute) and for $H_1, H_2 \leq G$ we use $[H_1, H_2] = \langle [x,y] : x \in H_1, y \in H_2 \rangle$. The derived subgroup of G is $G' = [G, G]$, it is $= \{e\}$ exactly when G is abelian. We say that G is solvable if the derived series $G^{(0)} = G$, $G^{(i)} = (G^{(i-1)})'$ eventually reaches the trivial subgroup:

$$\{e\} = G^{(k)} \trianglelefteq G^{(k-1)} \trianglelefteq \cdots \trianglelefteq G^{(1)} \trianglelefteq G^{(0)} = G.$$

(That G' is normal in G is an easy exercise.)

Recall by comparison that G is nilpotent if its lower central series $G^0 = G$, $G^i = [G, G^{i-1}]$ eventually reaches the trivial subgroup:

$$\{e\} = G^k \trianglelefteq G^{k-1} \trianglelefteq \cdots \trianglelefteq G' \trianglelefteq G^0 = G.$$

Every abelian group is nilpotent, and every nilpotent group is solvable (but not conversely).

E.g. The dihedral group D_4 of order 8 is nilpotent but not abelian. The symmetric group S_3 on 3 letters is solvable but not nilpotent. The alternating group A_5 of order 60 is not solvable, since it is simple and non-abelian.

Prop. If G is solvable and $H \subseteq G$ then H is solvable.

Pf: Derived series of H is "smaller" than that of G. ☒

E.g. For any $n \geq 5$, the symmetric group S_n is not solvable, since A_n, a simple nonabelian group, is not solvable.

Thm A group G is solvable if and only if it has a subnormal series $\{e\} = G_k \trianglelefteq G_{k-1} \trianglelefteq \cdots \trianglelefteq G_1 \trianglelefteq G_0 = G$ such that the factor groups G_i/G_{i+1} are all abelian.

Pf: The derived series of a solvable group is such a series, since G/G' is always abelian. We proved the other direction last semester when discussing composition series and the Jordan-Hölder Theorem. ☒

Explaining the name "solvable," we have the following main result:

Thm A polynomial $f(x) \in K[x]$ is solvable by radicals only if its Galois group, i.e. the group $\mathrm{Aut}_K(L)$ where L is its splitting field, is a solvable group.

A "generic" polynomial $f(x) \in \mathbb{Q}[x]$ of degree n has S_n as its Galois group, hence by the previous theorem, it does not have a solution in radicals for $n \geq 5$ (this is the "Abel-Ruffini Theorem").