Total: 47 points + 5 bonus pts for presenting = 52/50

Homework 1 - Combinatorics 2

1. A *k-ary necklace of length n* is a rotation equivalence class of colorings of the vertices of an n-gon with k colors. Use unweighted Pólya counting to show that the number of k-ary necklaces of length n is

$$\frac{1}{n}\sum_{d|n}\varphi(d)k^{\frac{n}{d}}.$$

This formula uses some notation from number theory: $d \mid n$ means "d divides n"; and $\varphi(d)$ is *Euler's totient function*, the number of $1 \leq j \leq d$ with $\gcd(d, j) = 1$.

You need to discuss the cycle structure of elements of the group generated by a single n-cycle.

The point is that the number of cycles of elements of this gp. depends on divisors of n and the totient function. E.g., if we square (1,2,3,4,5,6) we get (1,3,5)(2,4,6) that has 2 cycles. The fact that we get 2 3 cycles is because the gcd of 6 and 2 is 2. Similarly, if we look at the 4th power, it will consist of 2 3 cycles, because the gcd of 6 and 4 is also 2. But if we look at the 3rd power, we get 3 2 cycles, b.c. the gcd of 6 and 3 is 3

Your explanation is kind of getting there, but not quite.

[-2pts]

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Using the unweighted Pólya counting method, we know that there are $\frac{1}{\#G}\sum_{g\in G}(\#Y)^{c(g)}$ k-ary necklaces of length n. In this case, since a k-ary necklace of length n is a rotation equivalence class of colorings of the vertices of an n-gon with k colors, $\#G = n$ and $\#Y = k$.

Now when counting the number of colorings of an n-gon or in this case, of a k-ary necklace of length n, we have to be careful not to overcount when there are repeated coloring patterns. To account for this, we rotate a pattern that's repeated d times, $\frac{n}{d}$ times. These repeated patterns have to be repeated an even number of times, thus we take the sum over $d|n$.

Lastly, the totient function identifies the number of rotations there are of a certain order, giving us $\frac{1}{n}\sum_{d|n}\varphi(d)\,k^{\frac{n}{d}}$ $\square$

2. Continuing the previous problem, now using weighted Pólya counting: how many ways, up to rotation, can the vertices of a hexagon be colored with 2 red, 2 green, and 2 blue vertices?

Using weighted Pólya counting, we have $\frac{1}{6}(t_6 + t_3^2 + t_2^3 + t_3^2 + t_6 + t_1^6)$ $= \frac{1}{6}(t_1^6 + t_2^3 + 2t_3^2 + 2t_6)$. There are three colors, so $\frac{1}{6}((y_1+y_2+y_3)^6 + (y_1^2+y_2^2+y_3^2)^3 + 2(y_1^3+y_2^3+y_3^3)^2 + 2(y_1^6+y_2^6+y_3^6))$.

We're looking to find how many ways, up to rotation, the vertices of a hexagon can be colored with 2 red, 2 green, and 2 blue vertices, so we're looking for the coefficient of $y_1^2 y_2^2 y_3^2$. This means that we can ignore the terms $2(y_1^3 + y_2^3 + y_3^3)^2$ and $2(y_1^6 + y_2^6 + y_3^6)$.

Now we have $\frac{1}{6}((y_1+y_2+y_3)^6 + (y_1^2+y_2^2+y_3^2)^3)$. We can get $y_1^2y_2^2y_3^2$, $\binom{6}{2}\binom{4}{2} = 15 \cdot 6 = 90$ ways using $(y_1+y_2+y_3)^6$; and we can get $y_1^2y_2^2y_3^2$, $3 \cdot 2 = 6$ ways using $(y_1^2+y_2^2+y_3^2)^3$.

$\therefore$ there are a total of $\frac{1}{6}(90+6) = 16$ colorings with 2 red, 2 green, and 2 blue vertices $\square$

Very good. (You could've explained exactly how you got the cycle index polynomial, but that's okay…)

10/10

3. There are 24 orientation-preserving symmetries of a cube– they are all spatial rotations. Use unweighted Pólya counting to give a formula for the number of ways, up to orientation-preserving symmetries, to color the faces of a cube with k colors.

   Hint 1: Your formula should be a polynomial in k.

   Hint 2: This group of symmetries is *abstractly* isomorphic to the symmetric group S_4 (but of course there are *six*, not four, faces of a cube); for more information on this group see for instance the Wikipedia page `https://en.wikipedia.org/wiki/Octahedral_symmetry`.

[I used blog.plover.com/math/polya-burnside.html to help me visualize and identify the rotations of a cube]

To start, note that there are 24 rotations of a cube, so $\#G = 24$.

There's the identity rotation which moves nothing, has six orbits (k^6 fixed).

I was confused at first what you meant by "k^6 fixed" but now I see that you mean that there are k^6 colorings fixed by this element, which is correct. Got it.

There are 8 rotations around an axis that goes through one corner to the opposite corner with two orbits each (k^2 fixed).

There are 6 rotations about an axis that goes through the middle of one edge of the cube and out the middle of the other edge with three orbits each (k^3 fixed).

Lastly, there are 6 $90°$ rotations with three orbits each (k^3 fixed) and 3 $180°$ rotations with four orbits each (k^4 fixed) about an axis that goes through the center of a face and comes out the center of the other face.

Plugging these into the unweighted Pólya counting theorem, we get
$\frac{1}{24}(k^6 + 8k^2 + 6k^3 + 6k^3 + 3k^4) = \frac{1}{24}(k^6 + 3k^4 + 12k^3 + 8k^2)$ $\square$

Very good. 10/10

4. Continuing the previous problem, now using weighted Pólya counting: how many ways, up to orientation-preserving symmetries, can the faces of a cube be colored with 2 red, 2 green, and 2 blue faces?

Using weighted Pólya counting, we get $\frac{1}{24}(t_1^6 + 8t_3^2 + 6t_2^3 + 6t_1^2 t_4 + 3t_1^2 t_2^2)$. Again, there are three colors, so $\frac{1}{24}((y_1+y_2+y_3)^6 + 8(y_1^3+y_2^3+y_3^3)^2 + 6(y_1^2+y_2^2+y_3^2)^3 + 6(y_1+y_2+y_3)^2(y_1^4+y_2^4+y_3^4) + 3(y_1+y_2+y_3)^2(y_1^2+y_2^2+y_3^2)^2)$.

Again, it would be good to explain exactly where you came up with this formula for the cycle index poly.

We're looking for the coefficients of $y_1^2 y_2^2 y_3^2$, so we can ignore the terms $8(y_1^3+y_2^3+y_3^3)^2$ and $6(y_1+y_2+y_3)^2(y_1^4+y_2^4+y_3^4)$ because their powers are already greater than 2. That gives us $\frac{1}{24}((y_1+y_2+y_3)^6 + 6(y_1^2+y_2^2+y_3^2)^3 + 3(y_1+y_2+y_3)^2(y_1^2+y_2^2+y_3^2)^2)$.

There are $\binom{6}{2}\binom{4}{2} = 15 \cdot 6 = 90$ ways to get $y_1^2y_2^2y_3^2$ with $(y_1+y_2+y_3)^6$. There are $6(3\cdot 2) = 36$ ways to get $y_1^2y_2^2y_3^2$ with $6(y_1^2+y_2^2+y_3^2)^3$. And there are $3(3\cdot 2) = 18$ ways to get $y_1^2y_2^2y_3^2$ with $3(y_1+y_2+y_3)^2(y_1^2+y_2^2+y_3^2)^2$. This gives us $\frac{1}{24}(90+36+18) = 144/24 = 6$.

$\therefore$ there are a total of 6 colorings with 2 red, 2 green and 2

Very good. 10/10

blue faces $\square$

5. Let $\mathcal{M}_{n\times m}(k)$ be the set of $n\times m$ matrices with entries from the set $\{1,2,\dots,k\}$. For example,

$$\begin{pmatrix}2&3&4&2\\3&1&2&3\\4&3&5&2\end{pmatrix} \in \mathcal{M}_{3\times 4}(5).$$

The symmetric group S_n acts on $\mathcal{M}_{n\times m}(k)$ by permuting rows: e.g., for $\sigma=(1,2)(3)\in S_3$,

$$\sigma\cdot\begin{pmatrix}2&3&4&2\\3&1&2&3\\4&3&5&2\end{pmatrix} = \begin{pmatrix}3&1&2&3\\2&3&4&2\\4&3&5&2\end{pmatrix}.$$

Let $\widetilde{\mathcal{M}}_{n\times m}(k)$ denote the set of S_n-equivalence classes of $\mathcal{M}_{n\times m}(k)$. Give a formula (in terms of n, m, and k) for $\#\widetilde{\mathcal{M}}_{n\times m}(k)$.

Hint: To simplify your formula you may use the fact, which we proved last semester, that the (unsigned) Stirling numbers of the 1st kind $c(n,j) := \#\{\sigma\in S_n\colon \sigma \text{ has } j \text{ cycles}\}$ have generating function $\sum_{j=1}^{n} c(n,j)t^j = t(t+1)\cdots(t+n-1)$.

From the unweighted Pólya counting, we have $\frac{1}{\#G}\sum_{g\in G}(\#Y)^{c(g)}$.

Immediately we can see that $G=S_n$ and $\sigma=g$, which gives us

$\frac{1}{\#S_n}\sum_{\sigma\in S_n}(\#Y)^{c(\sigma)}$. Recall that $\#S_n=n!$, so $\frac{1}{\#S_n}\sum_{\sigma\in S_n}(\#Y)^{c(\sigma)} = \frac{1}{n!}\sum_{\sigma\in S_n}(\#Y)^{c(\sigma)}$.

Now observe how when $m=1$ we have a necklace problem again. So if

It's not exactly a necklace problem… because we're not modding out just by cyclic symmetries but by ALL ways of permuting the rows.

You need to explain WHY we have #Y = k^m here (that's the main trick behind this problem, although there are other ways to solve it too.)

[-1pt]

we substitute $\#Y$ with k^m, then we have $\frac{1}{n!}\sum_{\sigma\in S_n}(k^m)^{c(\sigma)}$.

Recall that $c(n,j)$ counts the number of permutations in S_n with j cycles. So instead of $\sum_{\sigma\in S_n}(k^m)^{c(\sigma)}$, we can write $\sum_{j=1}^{n} c(n,j)(k^m)^j$. So we have $\frac{1}{n!}\sum_{j=1}^{n}c(n,j)(k^m)^j$.

To simplify, we can use the fact that the generating function for unsigned Stirling numbers of the 1st kind is

$\sum_{j=1}^{n}c(n,j)t^j = t(t+1)\dots(t+n-1) \Rightarrow$

$\frac{1}{n!}\sum_{j=1}^{n}c(n,j)(k^m)^j = \frac{1}{n!}(k^m(k^m+1)\dots(k^m+n-1))$.

$\therefore$ a formula for $\#\widetilde{\mathcal{M}}_{n\times m}(k)$ is $\frac{1}{n!}(k^m(k^m+1)\dots(k^m+n-1))$ $\square$

Okay. 9/10