Math 211 (Modern Algebra II), HW# 5,

Spring 2025; Instructor: Sam Hopkins; Due: Wednesday, April 2nd

1. Let K be a field and consider $K(x)$, the field of rational functions in the variable x, as a (simple, transcendental) extension of K. On a previous homework, you found some properties of the Galois group $\mathrm{Aut}_K(K(x))$. In this problem, you will fully describe $\mathrm{Aut}_K(K(x))$.

   (a) For a rational function $0 \neq f/g \in K(x)$ with $f, g \in K[x]$ relatively prime, define its *degree* to be $\deg(f/g) := \max(\deg(f), \deg(g))$. Show that $[K(x) : K(f/g)] = \deg(f/g)$ if $\deg(f/g) \geq 1$. **Hint**: x is a root of the polynomial $\varphi(y) = (f/g)g(y) - f(y) \in K(f/g)[y]$; you may use without proof the fact that this polynomial is irreducible.

   (b) Let $f/g \in K(x)$ with $\deg(f/g) \geq 1$. Explain why the assignment $\sigma\colon x \mapsto f/g$ induces a homomorphism $\sigma\colon K(x) \to K(x)$, which is an automorphism if and only if $\deg(f/g) = 1$.

   (c) Conclude that $\mathrm{Aut}_K(K(x))$ consists exactly of the assignments $x \mapsto (ax + b)/(cx + d)$ with $a, b, c, d \in K$ and $ad - bc \neq 0$. (These are called *fractional linear transformations*, and can be viewed as invertible 2×2 matrices with entries in K.)

2. Let K be a field, L/K an extension, and $S \subseteq L$ a subset that is algebraically independent over K. Let $u, v \in L$ with $v \in S$ and $u \notin S$. Suppose that u is algebraic over $K(S)$ but that u is not algebraic over $K(S \setminus \{v\})$. Show that v is algebraic over $K((S \setminus \{v\}) \cup \{u\})$. (This is called the *exchange lemma* for transcendence bases.)

3. In this problem, you will explore $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{C})$, the field automorphisms of the complex numbers. We already know that two elements of $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{C})$ are the identity and complex conjugation $a + bi \mapsto a - bi$. You will show that there are many other "wild" elements.

   (a) Show that a transcendence basis of $\mathbb{C}$ over $\mathbb{Q}$ is infinite. **Hint**: First, note $\mathbb{Q}(x_1, \ldots, x_n)$ is countable for any finite $n \geq 1$ (why?). Then you may use the fact (that we did not prove in class but which is in the book) that if K is an infinite field, the algebraic closure $\overline{K}$ of K has the same cardinality as K. But $\mathbb{C}$ is uncountable!

   (b) Let S be a transcendence basis of $\mathbb{C}$ over $\mathbb{Q}$. Show that any permutation of S induces an automorphism in $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{C})$. **Hint**: First, observe in general that if the set S is algebraically independent over the field K, then any permutation of S induces an automorphism of $K(S)$. Then you may use the fact (that we did not prove in class but which is in the book) that if K_1 and K_2 are fields and L_1/K_1 and L_2/K_1 are algebraic closures, for any isomorphism $\varphi\colon K_1 \to K_2$ there is an isomorphism $\varphi\colon L_1 \to L_2$ extending it.

   (c) Conclude that $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{C})$ is infinite.

   (The only automorphisms in $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{C})$ that are *continuous* with respect to the standard topology on $\mathbb{C}$ are the identity and complex conjugation. The other "wild" automorphisms are very wild indeed - their existence depends on the axiom of choice!)