Howard Math 181: Discrete Structures Fall 2022

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(Call me "Sam")

8/22 Logistics:

Classes: MWF 11:10 am - 12pm, ASB-B #0105

Office hrs: Tue 12-1pm, Annex III - #220
or by appointment (email me!)

Website: samuelfhopkins.com/classes/181.html

Text: Discrete Mathematics by Johnsonbaugh, 8e

Grading: 40% (take home) quizzes
40% two (in-person) midterms
20% final exam

There will be 12 takehome quizzes, handed out on mondays + collected on Wednesdays (basically homework)
Your lowest 2 scores will be dropped (so 10/12 count)

The 2 midterms will happen in-class on wednesdays

The final will be during finals week

Beyond that, I may assign additional HW (not graded) and I expect you to SHOW UP TO CLASS + PARTICIPATE! :)

that means... Interrupt me by ASKING QUESTIONS!

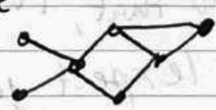
(and please say your names when you ask a question so I learn to put names to faces)

What is "discrete math"?

Discrete	continuous
• • • • •	∿∿∿
finite	infinite
integers	real numbers
$\mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$	$\mathbb{R} = \{\ldots, 0, \frac{1}{3}, \pi, -e, \ldots\}$
-2 -1 0 1 2 3 ...	$-e$ 0 $\frac{1}{3}$ $\pi = 3.1415\ldots$ $= -2.71\ldots$
algebra (ish...)	calculus
computer science	(classical) physics

The main topics we will cover are:

- Basic Mathematical Structures: sets, functions, sequences, relations — Ch's 1 + 3
- Logic and proofs — Ch's 1 + 2
- Basic combinatorics (a.k.a. counting!) — Ch. 5
- And maybe more... Like graph theory — Ch. 8



A major goal of the course is for you to learn how to write proofs, which means convincing mathematical arguments.

A kind of problem you should be able to solve by the end of the semester is...

"If N people are at a party, and each shakes everyone else's hand, how many handshakes happen?"

But... the goal is not just that you know the formula, but you can give a convincing proof why your answer is right!

Sets (§1.1 of textbook):

We will start by reviewing sets, the most basic kind of mathematical object. You probably already saw sets in calculus.

A set is just any collection of objects.

For example, the collection of all the planets in the solar system forms a set.

We use brackets to denote sets; that set is

pluto: :({mercury, venus, earth, mars, Jupiter, Saturn, Uranus, Neptune}

The objects that belong to a set are called its elements.

So mercury is an element of the set of planets.

Often we will work with sets of numbers

for example $A = \{1, 2, 3\}$ is a set of three numbers.

$B = \{2, 5, 9\}$ is another set of three numbers.

We have $2 \in A$, $2 \in B$ where $\in$ = "is an element of".

Some sets of numbers you know about are

the integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
("Zahlen" = "number" in German)

these are infinite sets

the rationals $\mathbb{Q} = \{ \frac{a}{b} : a, b \in \mathbb{Z},\ b \neq 0 \}$
("quotient")

the real numbers $\mathbb{R} = \longleftrightarrow$ $-7/2$ 0 e π

For $\mathbb{Q}$ we used set-builder notation:

Notation $\{x : \text{condition on } x\}$ means the set of x's satisfying this condition

book uses "$\{x \mid \text{condition on } x\}$"

E.g. $\{x : x > 0, x \in \mathbb{Z}\} = \{1, 2, 3, \ldots\}$

Q: What is $\{x : x^2 = 1, x \in \mathbb{R}\}$?

A: $\{-1, 1\}$ since $(-1)^2 = 1$ and $1^2 = 1$
and those are all #'s squaring to one.

8/24 We say set A is a subset of set B if every element of A is an element of B.

E.g. $\{2, 5\}$ is a subset of $\{2, 3, 5, 10\}$

E.g. $\mathbb{Z}$ is a subset of $\mathbb{Q}$, which is a subset of $\mathbb{R}$.

We use $\subseteq$ to denote "is a subset of".
So $\{1, 2\} \subseteq \{1, 2, 3, 4\}$.

There is a special set, called the empty set (or null set) and denoted $\emptyset$ (or $\{\}$) that has no elements: it is a subset of every set.

For any set A, $\emptyset \subseteq A$,

The set of all subsets of a set A is called the powerset of A, denoted $P(A)$.

e.g. If $A = \{a, b, c\}$ then the power set of A is $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$.

Note: A has 3 elements, and powerset of A has $2^3 = 8$ elements.

We use $|A|$ (or $\#A$) to denote the number of elements of a finite set.

In example above, $|A| = 3$ and $|P(A)| = 2^3 = 8$.

Later we will show why $|P(A)| = 2^{|A|}$ always.

Two sets, $\emptyset$ and A itself, are always subsets of a set A. These are called the trivial subsets of A. The nontrivial subsets of A are called the proper subsets of A.

E.g. The proper subsets of $A = \{a, b, c\}$ are $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{a,c\}$, and $\{b,c\}$.

Operations on sets

There are various ways to make new sets from old. Given two sets A and B, their <u>union</u> $A \cup B$ is

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$
(or both!)

and their <u>intersection</u> $A \cap B$ is

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

E.g., if $A = \{1, 3, 5, 6\}$, $B = \{2, 3, 4, 6\}$

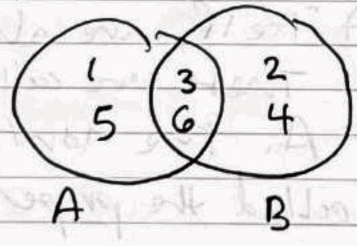
then $A \cup B = \{1, 2, 3, 4, 5, 6\}$ and $A \cap B = \{3, 6\}$

The set <u>difference</u> of B from A (or "A minus B") is

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$$

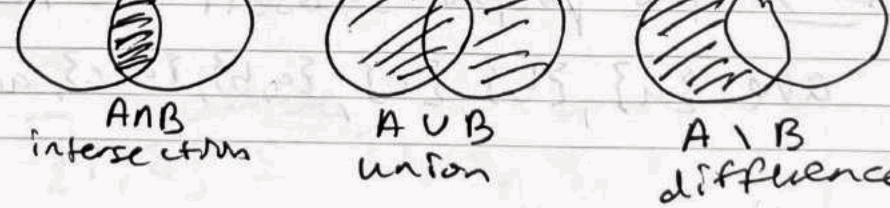
E.g. w/ $A + B$ as above, $A \setminus B = \{1, 5\}$
while $B \setminus A = \{2, 4\}$

It is convenient to use <u>Venn diagrams</u> to represent the relations between sets, unions, intersections:



← Venn diagram puts elements of a set inside circle labeled by that set

Then: we can represent



$A \cap B$ intersection

$A \cup B$ union

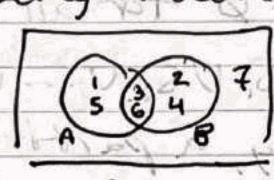
$A \setminus B$ difference

Can also represent subset relation using Venn diagrams:

$B \subseteq A$
subset

Sometimes there is a <u>universal set</u> $\mathcal{U}$ around, with all sets being a subset of this $\mathcal{U}$

We draw that like:



$\mathcal{U} = \{1,2,3,4,5,6,7\}$

The <u>complement</u> of A, denoted A^c, is then $A^c = \mathcal{U} \setminus A$, (things not in A), w/ $\mathcal{U}$ understood from context.

<u>Eg</u>: In this example $A^c = \{2,3,4,6\}$
and $(A \cup B)^c = \{7\}$.

There are many <u>rules</u> that $\cup$, $\cap$, ^{c}, etc. satisfy, some of the most important being:

<u>Thm</u>: (0) Symmetry of $\cup$ and $\cap$: $A \cup B = B \cup A$, $A \cap B = B \cap A$

(1) Associativity of $\cup$ and $\cap$:
$(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$

another important identity is $(A^c)^c = A$

(2) Distributivity of $\cup$ over $\cap$ and $\cap$ over $\cup$:
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

[Think of how $a \times (b + c) = (a \times b) + (a \times c)$]

(3) De Morgan's Laws:
$(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$

<u>Exercise</u>: Think about Venn diagram meaning of these. We may discuss proofs later ... //

Venn diagram of disjoint sets A and B: 

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Let's review a few more discrete structures related to sets.

A *partition* of a set A is a collection $\{S_1, \ldots, S_k\}$ of (nonempty) subsets $S_1, S_2, \ldots, S_k \subseteq A$ such that:

- they are pairwise *disjoint*, meaning $S_i \cap S_j = \emptyset$ for all distinct $i \neq j$,
- their union $S_1 \cup S_2 \cup \cdots \cup S_k = A$ is all of A.

Less formally, a partition is a way of breaking up a set A into (nonempty) subsets $S_1, \ldots, S_k$ so that every element $x \in A$ belongs to a *unique* one of the subsets $S_1, \ldots, S_k$.

E.g. If $A = \{1,2,3,4,5\}$ then one partition of A is $\{\{1,2,4\}, \{3,5\}\}$.

Another is $\{\{1,5\}, \{2,4\}, \{3\}\}$.

Can think of a partition as a way of "grouping together" elements of a set into different parts.

E.g. A partition of {people who live in USA} is {{people in Alabama}, {ppl in Alaska}, ..., {ppl in Wyoming}, {ppl in DC, PR, other territories}}.

Later when we talk about *relations* we will see how set partitions are intimately connected with *equivalence relations*.

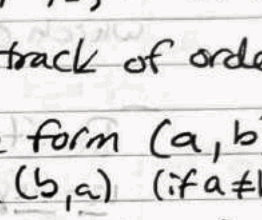
A set is an <u>unordered</u> collection, so
$\{1,2,3\} = \{2,1,3\} = \{3,2,1\} =$ etc...
(and also don't care about "multiplicities", so $\{1,1,2,2,2,3\}$ "=" $\{1,2,3\}$)

But sometimes we do want to keep track of order.

An <u>ordered pair</u> is an object of the form (a,b), which is considered distinct from (b,a) (if $a \neq b$).
For two sets X and Y, the set of all ordered pairs (x,y) with $x \in X$ and $y \in Y$ is denoted $X \times Y$, and called the <u>Cartesian product</u>.

<u>E.g.</u> If $X = \{1,2,3\}$ and $Y = \{a,b\}$ then
$X \times Y = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$
$Y \times X = \{(a,1),(b,1),(a,2),(b,2),(a,3),(b,3)\}$
$Y \times Y = \{(a,a),(a,b),(b,a),(b,b)\}$, etc...

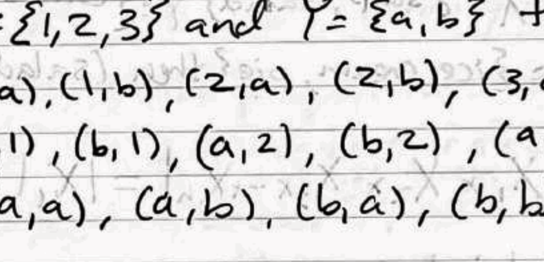
<u>E.g.</u> If $X = \mathbb{R}$ real numbers, then
$X \times X = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\} \rightarrow$



"<u>Cartesian plane</u>" / "<u>Cartesian coordinates</u>"

<u>Thm</u>: If X and Y are finite, then $|X \times Y| = |X| \cdot |Y|$.

<u>Pf</u>: Imagine constructing an ordered pair (x,y) by first choosing $x \in X$ then choosing $y \in Y$:



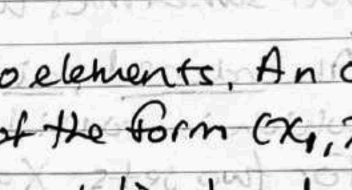
This *decision tree* will have $|X|$ branches at 1st level and each of those branches will break into $|Y|$ branches at 2nd level, giving $|X|\cdot|Y|$ endpoints ("leaves") which correspond to all the elements of $X \times Y$. ☑

Don't have to stop at two elements. An *ordered n-tuple* is something of the form $(x_1, x_2, \ldots, x_n)$ (considered distinct from all permutations) and for sets $X_1, \ldots, X_n$, we let $X_1 \times X_2 \times \cdots \times X_n = \{(x_1, x_2, \ldots, x_n) : x_i \in X_i\}$.

E.g. If $X = \{\text{soup, salad}\}$, $Y = \{\text{chicken, fish, pasta}\}$ and $Z = \{\text{ice cream, pie}\}$ then $(\text{salad, fish, pie}) \in X \times Y \times Z$.

Thm. $|X_1 \times X_2 \times X_3 \times \cdots \times X_n| = |X_1| \cdot |X_2| \cdot \cdots \cdot |X_n|$.

Pf Sketch: Imagine making a decision tree with n different layers:



In Each layer all the branches into $|X_i|$ new branches, so in the end there will be total of $|X_1| \cdot |X_2| \cdots \cdot |X_n|$ leaves. ☑

Exercise: Use decision trees to show why $|P(A)| = 2^{|A|}$ which we mentioned before.

Hint: Think of building a subset of A by *including* or *excluding* each $x \in A$ one-by-one ...

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§1.2 <u>Propositions</u> We've discussed sets for a while. Now we will start a new topic: <u>logic</u>.
The basic things we analyze in logic are propositions.
A <u>proposition</u> is a statement that can be either <u>true</u> or <u>false</u> but not both.

<u>E.g.</u> (a) The boiling point of water at sea level is $100°C$.
(b) August has only 30 days in it.
(c) There is life on Mars.
(d) Take Calculus III next semester!
(e) $x + 4 = 6$
(f) The only positive integers dividing 7 are 1 and 7.

Then (a), (b), (c), (f) are propositions. (a) + (f) are true.
(b) is false. (c) is either true or false but not both (we don't know which!)
(d) is <u>not</u> a proposition because it's not a statement (it's a command). (e) is <u>not</u> a proposition because it could be true or false depending on the value of x. It's true for some x ($x=2$) and false for other x. [It's a <u>formula</u>... we'll discuss it later...]

We use lowercase letters like p and q to denote propositions. We also use notation

$$p: \ 1+1=3$$

to mean p is the proposition "$1+1=3$." (which is false!)

Just like with sets and operations of $\cup$, $\cap$, etc., we have ways of making new propositions from old via different <u>logical operations</u>.

Def'n If p, q are two propositions then we write

$p \wedge q$: p <u>and</u> q ("<u>conjunction</u>")

$p \vee q$: p <u>or</u> q ("<u>disjunction</u>")
(or both!) "inclusive or"

<u>E.g.</u> p: It is raining and q: I have an umbrella

$p \wedge q$: It is raining and I have an umbrella.

p: It is raining, q: I have an umbrella, r: I have a rain jacket.

$p \wedge (q \vee r)$: It is raining and I have an umbrella or a rain jacket (or both)

We can represent compound propositions via <u>truth tables</u>:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

tables show for all the possible truth values of p & q what the truth value of the compound prop. is

Def'n If p is a proposition, then

$\neg p$: <u>not</u> p ("<u>negation</u>")

(also sometimes $!p$)

p	$\neg p$
T	F
F	T

By combining $\wedge$, $\vee$, and $\neg$ can make many more propositions.

<u>Q</u>: How to write the <u>exclusive or</u> of p and q? (XOR)

XOR(p,q): either p or q but not both

: $(p \vee q) \wedge (\neg(p \wedge q))$

p	q	$(p \vee q) \wedge \neg(p \wedge q)$
T	T	F
T	F	T
F	T	T
F	F	F

can check this is right definition of XOR by writing truth table

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§1.3 Conditionals Consider the statement

"If I'm teaching class today, then I'll go to campus."

This is what we call in logic a conditional.

Def'n Given prop.'s p and q, we define the conditional prop.

$p \to q$: if p then q ("p implies q")

In $p \to q$, p is called the hypothesis (or "antecedent") and q is called the conclusion (or "consequent").

When is $p \to q$ true? Let's look at

p = "I'm teaching class today", q = "I'll go to campus".

If I'm teaching class and I go to campus, then $p \to q$ is true.

If I'm teaching and I don't go to campus, then $p \to q$ is false.

But what about if I'm not teaching class?

If I'm not teaching and I don't go to campus, $p \to q$ still true.

On the other hand, if I'm not teaching and I still go to campus (maybe to my office...), $p \to q$ is still true:

$p \to q$ makes no claim about what happens if p is false.

Thus, the truth table of $p \to q$ is:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

$p \to q$ is true if whenever p is true, then q is true (but if p is false, who knows?)

Notice that $q \to p$ is <u>not</u> the same as $p \to q$:
"If I'm teaching, then I'll go to campus" is <u>true</u>
but "If I go to campus, then I'm teaching" is <u>false</u>
(maybe I went to my office to print something, etc.)

The proposition $q \to p$ is called the <u>converse</u> of $p \to q$.
<u>Don't</u> mix up a statement and its converse!

Another way to think about conditionals is in terms of <u>necessary</u> and <u>sufficient</u> <u>conditions</u>.
If q is a <u>necessary condition</u> for p to be true, then $p \to q$.
<u>E.g.</u> Since it is <u>necessary</u> to study hard to get a good grade we can say "If you got a good grade, then you must have studied hard."
On the other hand, if q is a <u>sufficient condition</u> for p to be true, then $q \to p$ (other way around!)
<u>E.g.</u> Since getting a B is <u>sufficient</u> to pass the class, we can say "If you got a B, then you passed the class."

So we see that it's important to treat $p \to q$ and $q \to p$ as different, but sometimes we want to assert <u>both</u>!
<u>Def'n</u> For p, q propositions, their <u>biconditional</u> is
$p \leftrightarrow q$: p <u>if</u> <u>and</u> <u>only</u> <u>if</u> q (same as $p \to q$ <u>and</u> $q \to p$).
Biconditional often used for <u>definitions</u>, and also <u>logical</u> <u>equivalence</u>...

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E.g. for any real number x, the biconditional "$x^3 > 0$ if and only if $x > 0$" is true.

This is because both:

- if $x > 0$, then $x^3 > 0$
- and if $x^3 > 0$, then that must mean $x > 0$.

E.g. Compare to: for any real number x, the conditional "If $x > 0$, then $x^2 > 0$" is true.
But "If $x^2 > 0$, then $x > 0$" is false for $x = -1$, where $(-1)^2 = 1 > 0$ but $-1 < 0$.

The truth table for biconditional is:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

$p \leftrightarrow q$ is true if p and q have exactly the same truth value (both true or both false).

Biconditionals let us define logical equivalence.

Def'n Suppose P and Q are two compound propositions which depend on input propositions $p_1, p_2, \ldots, p_n$.
Then we say that P and Q are logically equivalent, written $P \equiv Q$, if for all truth values of $p_1, p_2, \ldots, p_n$, P and Q have same truth value.
In other words $P \leftrightarrow Q$ for all $p_1, p_2, \ldots, p_n$.
"P and Q are saying the same thing."

E.g. Thm (De Morgan's Laws)

$\neg(p \vee q) \equiv \neg p \wedge \neg q$ and $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

Pf: Let's just verify the 1st DeMorgan's Law.
The way we do this is by writing a truth table:

p	q	$\neg(p\vee q)$	$\neg p \wedge \neg q$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

We see that they have the same truth value no matter what, i.e.
$(\neg(p\vee q)) \leftrightarrow (\neg p \wedge \neg q)$. ▨

E.g. Exercise Show that $p \equiv \neg(\neg p)$
(This is called "double negation.")

E.g. The contrapositive of the conditional $p \to q$
is $\boxed{\neg q \to \neg p}$. For instance, the contrapositive of
"If $x > 0$, then $x^2 > 0$" is
"If not $(x^2 > 0)$, then not $(x > 0)$,"
i.e. "If $x^2 \leq 0$, then $x \leq 0$."
Unlike the converse the contrapositive is logically equivalent to the original conditional.

Thm $p \to q \equiv \neg q \to \neg p$

Pf:

p	q	$p \to q$	$\neg q \to \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

← Check the truth table!

▨