

"Groups generated by involutions acting on combinatorial objects"

Sam Hopkins [Howard University]

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Disclaimer: This is an **expository talk** based on not my work.
I apologize if I say anything incorrect. All errors = my own!

§1. Tableaux

Semistandard Young tableau = filling of Young diagram λ
w/ pos. integers that is:
• weakly increasing along rows,
• strictly increasing down columns.

e.g. $T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 3 & \\ \hline 3 & & & \\ \hline \end{array}$

$\text{sh}(T) = \text{shape}(T) = \lambda = (\lambda_1, \lambda_2, \dots)$
 $(= (4, 3, 1))$

$\text{con}(T) = \text{content}(T)$
 $= \alpha = (\alpha_1, \alpha_2, \dots)$ w/ $\alpha_i = \# i's$
 $(= (2, 3, 3, 0, \dots))$

$\text{SSYT}(n) :=$ semistandard tableaux w/ entries $\subseteq \{1, 2, \dots, n\}$

$\text{SSYT}(n, \lambda) := T \in \text{SSYT}(n) \text{ w/ } \text{sh}(T) = \lambda$
 $(\neq \emptyset \text{ for } \lambda \text{ w/ #rows} = l(\lambda) \leq n)$

$\text{SSYT}(n, \lambda, \alpha) := T \in \text{SSYT}(n, \lambda) \text{ w/ } \text{con}(T) = \alpha$
 $(\neq \emptyset \text{ for } \alpha = (\alpha_1, \alpha_2, \dots) \text{ w/ } \sum \alpha_i = |\lambda| := \sum \lambda_i)$

$\text{SYT}(\lambda) := \text{SSYT}(n = |\lambda|, \lambda, 1^n = (1, 1, \dots, 1))$
= Standard Young tableaux of sh. λ

For $1 \leq i \leq n-1$, the i^{th} Bender-Knuth involution

$$t_i : \text{SSYT}(n) \rightarrow \text{SSYT}(n)$$

is operation defined on semistandard tableau T as follows:

- first **freeze** in place all i 's above an $i+1$, and all $i+1$'s below an i ,
- then, if a row contains x unfrozen i 's + y unfrozen $i+1$'s, modify it to have y unfrozen i 's + x unfrozen $i+1$'s.

e.g.

$$t_2 \left(\begin{array}{ccccccccc} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ 2 & 2 & 2 & 3 & 3 & 4 & 6 \\ 3 & 4 & 4 & 5 \\ 5 & 5 & 5 & 6 \end{array} \right) = \begin{array}{ccccccccc} 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 & 3 & 4 & 6 \\ 3 & 4 & 4 & 5 \\ 5 & 5 & 5 & 6 \end{array}$$

frozen **unfrozen**

t_i were introduced by Bender & Knuth, 1972.

Easy checks: • $t_i^2 = 1$.

$$\bullet \text{sh}(t_i(T)) = \text{sh}(T),$$

$$\bullet \text{con}(t_i(T)) = \sigma_i(\text{con}(T)), \text{ where}$$

$$\sigma_i(\alpha_1, \alpha_2, \dots, \alpha_i, \alpha_{i+1}, \dots) = (\alpha_1, \alpha_2, \dots, \alpha_{i+1}, \alpha_i, \dots)$$



Basic application: If we define **Schur polynomial**

$$s_\lambda(x_1, \dots, x_n) := \sum_{T \in \text{SSYT}(n, \lambda)} x^{\text{con}(T)}$$

the t_i prove that s_λ is a symmetric polynomial.

In Berenstein & Kirillov, 1995, "Groups generated by involutions..."

they study group $G_n := \langle t_1, \dots, t_{n-1} \rangle \subset \text{SSYT}(n)$.

Let me survey some results from B-K:

(a) Relations in G_n

Have obvious relations:

$$(1) t_i^2 = 1 \quad \forall i, \quad t_i t_j = t_j t_i \quad \forall i, j \text{ w/ } |i-j| \geq 2.$$

Not so obvious relations:

$$(2) (t_1 t_2)^6 = 1$$

$$(3) (t_i q_i)^4 = 1 \quad \forall 3 \leq i \leq n-1$$

where $q_i := \underbrace{t_1}_{\text{ }} \underbrace{t_2 t_1}_{\text{ }} \underbrace{t_3 t_2 t_1}_{\text{ }} \dots \underbrace{t_i t_{i-1} \dots t_1}_{\text{ }}$

Rmk $(t_i t_{i+1})^6 \neq 1$ for $i > 1$.

Rmk Have $q_i^2 = 1 \quad \forall i$ (follows easily from rel's (1)).

Also have $G_n = \langle q_1, \dots, q_{n-1} \rangle$.

Rmk The **cactus group** is $C_n := \langle q_{[i,j]} : 1 \leq i < j \leq n \rangle$

subject to relations:

$$\cdot q_{[i,i]}^2 = 1$$

$$\cdot q_{[i,j]} q_{[i,k]} = q_{[j,k]} q_{[i,j]} \text{ if } j < k$$

$$\cdot q_{[i,j]} q_{[k,l]} q_{[i,j]} = q_{[i+j-2, i+j-k]} \text{ if } i \leq k < l \leq j$$

introduced by Henriques & Kamnitzer, 2006 who were studying tensor products $A_1 \otimes \dots \otimes A_n$ in category of crystals.

("Cactus" name: $C_n = \pi_1(\widetilde{M_{0,n}})$) moduli sp.
of n pts. on \mathbb{P}^1 

Chmutov, Glick, Pylyavskyy, 2020
+ Berenstein & Kirillov, 2016 show \exists homo. $C_n \rightarrow G_n$

$$q_{[i,j]} \mapsto q_{i-1} q_{j-1} q_{i-1}$$

Some of G_n rel's above are implied by cactus gp. homomorphism,
but (2) $(t_1 t_2)^6 = 1$ is not (mysterious...).

Rmk For $1 \leq i \leq n-1$ define $s_i := q_i t_1 q_i$, then

$$s_i^2 = 1, \quad s_i s_j = s_j s_i \text{ if } |i-j| \geq 2, \quad (s_i s_{i+1})^3 = 1,$$

i.e. get **Symmetric gp. action** $G_n = \langle s_i \rangle \subset \text{SSYT}(n)$.

Was previously studied by Lascoux & Schützenberger, 1981
+ has simple interpretation in terms of crystals.

(b) Isomorphism type of G_n as a permutation gp.

Natural question from Prof. Vershik, which prompted this talk:

Q. Acting on set of tableaux, what perm. group is G_n ?

Since $\text{sh}(t_i(T)) = \text{sh}(T)$ and $\text{con}(t_i(T)) = \sigma_i(\text{con}(T))$, to think about this Q., it makes sense to **restrict** to **standard tableaux** of fixed shape.

For $G_n \curvearrowright \text{SYT}(\lambda)$, have additional relations:

on SYT, t_i
swaps i & $i+1$
if non-adjacent

$\rightsquigarrow (t_i t_i t_{i+1})^6 = 1 \text{ for } 2 \leq i \leq n-1,$
 $t_1 = 1$.

Easy to see: Action $G_n \curvearrowright \text{SYT}(\lambda)$ is transitive.

But still not clear what transitive perm. gp. we get...

My (wild) guess: For "generic" shape λ , should get "big" perm. gp., possibly all of $G_{\text{SYT}(\lambda)}$ or (index 2) alternating gp.

But... WARNING: For $\lambda = (n-k, 1^k)$ a hook shape 

have $G_n \curvearrowright \text{SYT}(\lambda) \cong G_{n-1} \curvearrowright \{\text{size } k \text{ subsets of } \{1, 2, \dots, n-1\}\}$

(c) Special elements of G_n

Certain elements of G_n have particular algebraic significance.

Evacuation (a.k.a. "Schützenberger involution") := $q_{n-1} = t_1 t_2 t_3 \dots t_{n-1} \dots t_1$

Promotion := $t_1 t_2 \dots t_{n-1}$

Promotion + evacuation were introduced by Schützenberger, 1972, who showed (1977) connection to RSK algorithm:

$$A \xrightarrow{\text{RSK}} (P, Q) \Leftrightarrow 180^\circ \text{ rot. of } A \xrightarrow{\text{RSK}} (\text{Evac}(P), \text{Evac}(Q)).$$

Thm (Stembridge, 1996; Berenstein & Zelevinsky, 1996) For any λ ,

- action of **longest word** $w_0 = (n^{n-1} \dots 1)$ on G_n -irrep. S^λ corresponds to evacuation in Kazhdan-Lusztig cellular basis ($\cong \text{SYT}(\lambda)$),
- on gl_n -irrep. V^λ , w_0 corresponds to evac. in canonical basis ($\cong \text{SSYT}(n, \lambda)$).

Thm (Rhoades, 2010) For $\lambda = a \times b$ **rectangle** .

- action of **long cycle** $c = (1, 2, \dots, n)$ on G_n -irrep. S^λ corresponds to promotion in KL-basis,
- on gl_n -irrep. V^λ , c corresponds to pr. in canonical basis.

§2. Posets

$\text{SYT}(\lambda)$ = linear extensions of certain poset, so...

Let P be an n -element poset. A linear extension of P is a list (p_1, p_2, \dots, p_n) of all elements of P s.t. $p_i \leq p_j \Rightarrow i < j$.

$$p_i \leq p_j \Rightarrow i < j.$$

Set $\mathcal{L}(P)$:= linear extensions of P .

B-K involutions still make perfect sense: for $1 \leq i \leq n-1$,

$$t_i : \mathcal{L}(P) \rightarrow \mathcal{L}(P)$$

$$t_i(p_1, \dots, p_n) := \begin{cases} (p_1, \dots, p_{i+1}, \overset{\leftarrow}{p_i}, \dots, p_n) & \text{if } p_i, p_{i+1} \text{ are incomparable} \\ (p_1, \dots, p_n) & \text{otherwise.} \end{cases}$$

(a) Relations

As before, have:

- $t_i^2 = 1 \forall i$
- $t_i t_j = t_j t_i$ if $|i - j| \geq 2$,
- $(t_i t_{i+1})^6 = 1$.

(b) Isomorphism type of perm. action

As before, can say that action of t_i on $\mathcal{L}(P)$ is transitive, but not much more than that.

(c) Special elements

As before, define
 evacuation := $t_1 t_2 t_1 \dots t_{n-1} \dots t_1$
 promotion := $t_1 t_2 \dots t_n$

This BK-involution perspective on pro/evac due to Haiman, 1992
 (Schützenberger worked with jeu de taquin...)

See also Stanley, 2009, "Promotion and Evacuation".

A different, but morally very similar, group generated by involutions associated to a poset P was considered in Cameron & Fon-Der-Flaass, 1995, "Orbits of Antichains Revisited"

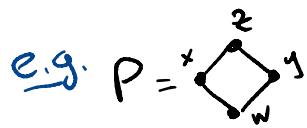
An **order ideal** of P is a downwards-closed subset $I \subseteq P$:

$$y \in I \text{ and } x \leq y \in P \Rightarrow x \in I.$$

Let $\mathcal{J}(P) :=$ order ideals of P .

For $p \in P$, define **toggle** $\gamma_p : \mathcal{J}(P) \rightarrow \mathcal{J}(P)$ by

$$\gamma_p(I) := \begin{cases} I \cup \{p\} & \text{if } p \notin I \text{ and } I \cup \{p\} \in \mathcal{J}(P), \\ I \setminus \{p\} & \text{if } p \in I \text{ and } I \setminus \{p\} \in \mathcal{J}(P), \\ I & \text{otherwise.} \end{cases}$$



$$\gamma_x(\text{---}) = \text{---}$$

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$$\gamma_x(\text{---}) = \text{---}$$

(a) Relations

$$\gamma_p^2 = 1$$

Easy to see: $\gamma_p \gamma_q = \gamma_q \gamma_p$ if neither p covers q , nor vice versa

Slightly less obvious: $(\gamma_p \gamma_q)^6 = 1$ if p covers q or vice versa.

(b) Iso-type of perm. gp.

Thm (Cameron - Fon-Der-Flaass, 1995)

For **connected** P , $\langle \gamma_p \rangle \subseteq \mathcal{J}(P)$ is either whole sym. gp.

$\mathbb{G}_{\mathcal{J}(P)}$, or (index 2) alternating gp.

Rmk Proof is a simple induction argument, deleting maximal element, based on fact that alternating gp. An is simple for $n \geq 5$.

Rmk Toggles τ_p make sense acting on any set system \mathcal{F} . Striker, 2018 considered various \mathcal{F} where one can show that $\langle \tau_p \rangle$ is at least alt. gp. (e.g., independent sets of a graph).

(c) Special elements

The operator **Rowmotion** := $\tau_{p_1} \tau_{p_2} \dots \tau_{p_n}$,
(where (p_1, p_2, \dots, p_n) is any linear extension)

has been studied by many people. It is known to have good behavior for certain P (e.g., rectangles).

It is also known to be closely related to promotion

(see, e.g., Striker & Williams, 2012, "Promotion and Rowmotion").

We will explain one precise relation between t_i 's + τ_p 's via ...

§3. Piecewise-linear maps

Let $T \in \text{SSYT}(n)$. Its **Gelfand-Tsetlin pattern** is the triangular array of nonnegative integers:

$$\begin{matrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{22} & x_{23} & \dots & x_{2n} \\ \vdots & \ddots & \vdots & & \\ x_{n-1,n-1} & x_{n-1,n} \\ x_{nn} \end{matrix}$$

where k^{th} diagonal $(X_{1,k+1}, X_{2,k+2}, \dots, X_{n-k,n})$
 $= \text{shape}(T_{\leq n-k})$
 with $T_{\leq i} := T$ restricted to entries $\{1, 2, \dots, i\}$

e.g. $T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 3 & 4 \\ \hline 3 & 3 & 4 & \\ \hline 4 & & & \\ \hline \end{array}$

$$\text{sh}(T_{\leq 4}) = (4, 4, 3, 1)$$

$$\text{sh}(T_{\leq 3}) = (4, 3, 2)$$

$$\text{sh}(T_{\leq 2}) = (3, 2)$$

$$\text{sh}(T_{\leq 1}) = (2)$$

$$\begin{matrix} T_{\leq 4} & T_{\leq 3} & T_{\leq 2} & T_{\leq 1} \\ \nearrow GT & & & \\ 4 & 4 & 3 & 2 \\ & 4 & 3 & 2 \\ & & 3 & 2 \\ & & & 1 \end{matrix}$$

Rmk GT patterns satisfy inequalities

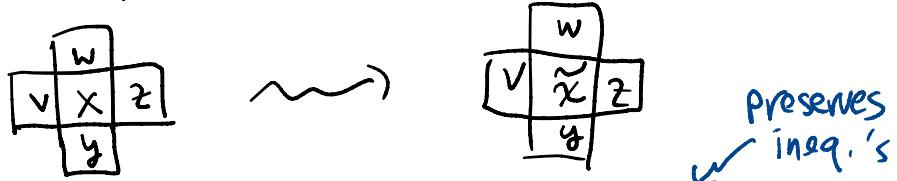
$$X_{i,j} \geq X_{i,j+1}$$

\swarrow

$$X_{i+1,j+1} \geq 0$$

and hence = integer pt.s of a polyhedral cone.

Berenstein & Kirillov, 1995 showed how BK involutions behave on GT patterns. Consider local involution



where $\tilde{x} := \min(v, w) + \max(y, z) - x$

Thm (Berenstein & Kirillov, 1995)

At the level of GT patterns, BK involution t_i corresponds to $x \rightarrow \tilde{x}$ for all entries x on $(n-i)$ th diagonal.

e.g.

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 3 & 4 \\ \hline 3 & 3 & 4 & \\ \hline 4 & & & \\ \hline \end{array}$$

GT
↔

$$\begin{matrix} 4 & 4 & 3 & 2 \\ 4 & 3 & 2 & \\ 3 & 2 & & \\ & & & \\ & & & \downarrow \\ & & & 1 \end{matrix}$$

$x \mapsto \tilde{x}$

$\{t_2\}$

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 3 & 4 \\ \hline 3 & 3 & 4 & \\ \hline 4 & & & \\ \hline \end{array}$$

GT
↔

$$\begin{matrix} 4 & 4 & 4 & 2 \\ 4 & 3 & 2 & \\ 3 & 2 & & \\ & & & \downarrow \\ & & & 1 \end{matrix}$$

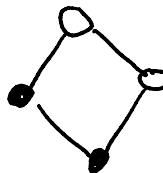
Inspired by B&K, Einstein & Propp, 2021 introduced

a PL extension of poset toggles for poset P .

Let $\gamma^m(P) :=$ order-preserving maps $\pi: P \rightarrow \{0, 1, \dots, m\}$.

Note $\gamma(P) \cong \gamma^m(P)$ via indicator fn:

I



=

$$\begin{matrix} & 1 & & \\ & / & \backslash & \\ 0 & & 0 & \\ & \backslash & / & \\ & 0 & & \end{matrix} \quad \pi$$

↑
of complement

For $p \in P$, define PL toggle $\tilde{\pi}_p : J^m(P) \rightarrow J^m(P)$
by

$$\tilde{\pi}_p(\pi)(q) := \begin{cases} \pi(q) & \text{if } q \neq p \\ \min \{\pi(r) : r \leq p\} + \max \{\pi(r) : r < p\} & \text{if } q = p \end{cases}$$

*same as
 $x \mapsto \tilde{x}$*

(w/ $\min \emptyset = m$ and $\max \emptyset = 0$)

These are same as Σ_P for $J(P)$ when $m=1$.

Prop. For $\lambda = a \times b$ a rectangle, \exists bijection

$$\text{SSYT}(n, \alpha \times b) \xrightarrow{\sim} \mathcal{T}^b(\alpha \times (n-\alpha))$$

s.t. t_i = composition of γ_p 's along $(n-i)^{th}$ diag.

e.g.
 $\text{GT} = \text{pattern}$

$\begin{matrix} b & b & b & b \\ b & b & b & b \\ b & b & b & b \\ b & b & b & b \end{matrix}$ $\begin{matrix} * & * & * \\ * & * & * \\ * & * & * \end{matrix}$ ≥ 0

this is $\pi \in \mathcal{Y}^b(a \times (n-a))$

Rmk Since $t_i = \text{product}$ of Σ_p 's, the group

$\mathbb{G}_n = \langle t_i \rangle$ will be a subgroup of $\langle x_p \rangle$.

But cannot say much more than that..

§4. Conclusion

My own interest in this: finding P that have good behavior of promotion + rowmotion.

In preprint "Order poly. product formulas + poset dynamics"
I propose heuristic that these should be P w/ product formulas for $\# \mathcal{J}^m(P)$, e.g.:

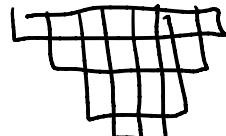
Thm (MacMahon)

$$\# \mathcal{J}^m(a \times b) = \prod_{i=1}^a \prod_{j=1}^b \frac{i+j+m-1}{i+j-1}$$

Beyond rectangles, other examples are:



staircases



trapezoids



CPIACN60!

(slides available upon request)

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