3/27 Sequences §11.1

We now start a new chapter, Ch. 11, on sequences, series, and power series. This is the final topic of the semester.

Def'n An (infinite) sequence is an infinite list $a_1, a_2, a_3, \ldots, a_n, \ldots$ of real numbers. We also use $\{a_n\}$ and $\{a_n\}_{n=1}^{\infty}$ to denote this sequence.

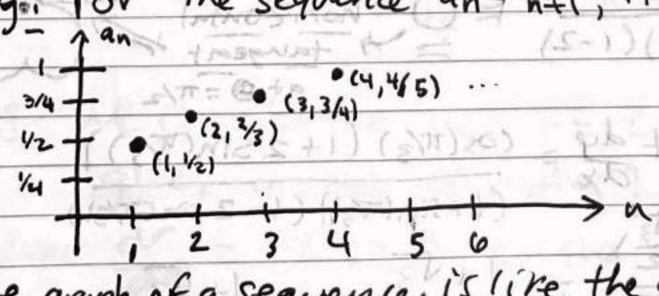
E.g. We can let $a_n = \frac{1}{2^n}$ for $n \geq 1$, which gives the sequence $1/2, 1/4, 1/8, 1/16, 1/32, \ldots$

E.g. $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$ We could also write $\left\{\frac{n}{n+1}\right\}_{n=2}^{\infty} = \left\{\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$ to start at term $n=2$; notice that also $\left\{\frac{n+1}{n+2}\right\}_{n=1}^{\infty} = \left\{\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$.

E.g. Not all sequences have simple formulas for the n^{th} term. For example, with $a_n = n^{th}$ digit of π after the decimal point have $\{a_n\} = \{1, 4, 1, 5, 9, 2, 6, 5, \ldots\}$ but there is no easy way to get the n^{th} term here...

Def'n The graph of sequence $\{a_n\}_{n=1}^{\infty}$ is the collection of points $(1, a_1), (2, a_2), (3, a_3), \ldots$

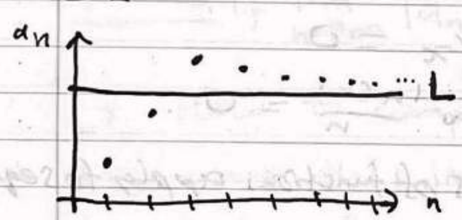
E.g. For the sequence $a_n = \frac{n}{n+1}$, its graph is:

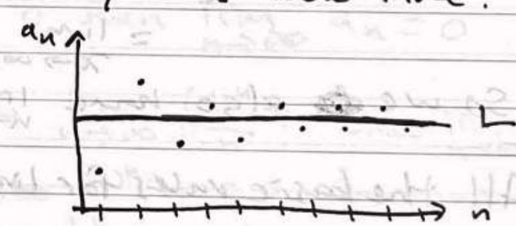


The graph of a sequence is like the graph of a function, but we get discrete points instead of a continuous curve. Notice how for this graph, points approach line $y=1$...

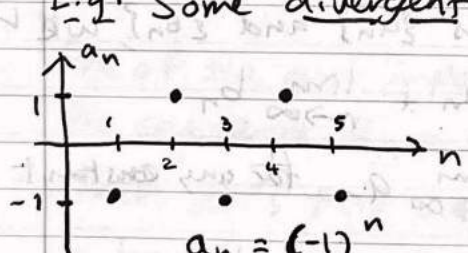
Def'n We say the limit of sequence $\{a_n\}$ is L, written "$\lim_{n\to\infty} a_n = L$" or "$a_n \to L$ as $n \to \infty$," if, intuitively, we can make the terms a_n as close to L as we like by taking n sufficiently large.
(Precise definition uses ε and δ, like in Calc I...)
If $\lim_{n\to\infty} a_n$ exists, we say the sequence converges. Otherwise, we say the sequence diverges.

E.g. The sequence $a_n = \frac{n}{n+1}$ has $\lim_{n\to\infty} a_n = 1$ (we'll prove this below).
E.g. Some other convergent sequences look like:

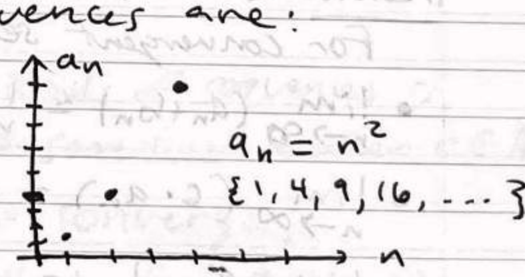




E.g. Some divergent sequences are:



$a_n = (-1)^n$
$\{-1, 1, -1, 1, \dots\}$



Notice how this second example "goes off to ∞"!

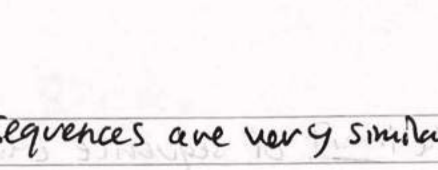
Def'n The notation "$\lim_{n\to\infty} a_n = \infty$" means that for every M there is an N such that $a_n > M$ for all $n > N$.
We define "$\lim_{n\to\infty} a_n = -\infty$" similarly.

E.g. $\lim_{n\to\infty} n^2 = \infty$ and $\lim_{n\to\infty} -n = -\infty$.
Having an infinite limit is one way a sequence can diverge.

Limits of sequences are very similar to limits of functions.

<u>Theorem</u> If $f(x)$ is a function with $f(n) = a_n$ for all integers n, then if $\lim_{x\to\infty} f(x) = L$ also $\lim_{n\to\infty} a_n = L$.

<u>Picture</u>:



<u>E.g.</u> How to find $\lim_{n\to\infty} \frac{\ln(n)}{n}$? Instead, let $f(x) = \frac{\ln(x)}{x}$

then $\lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{1/x}{1}$ (by L'Hôpital's rule)

$= \lim_{x\to\infty} 1/x = 0$

So we also have $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$.

3/29 All the basic rules for limits of functions apply to sequences:

Theorem (Limit Laws for Sequences)

For convergent sequences $\{a_n\}$ and $\{b_n\}$, we have:

- $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} (c \cdot a_n) = c \cdot \lim_{n\to\infty} a_n$ for any constant $c \in \mathbb{R}$
- $\lim_{n\to\infty} a_n \cdot b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = (\lim_{n\to\infty} a_n)/(\lim_{n\to\infty} b_n)$ if $\lim_{n\to\infty} b_n \neq 0$.

<u>E.g.</u> To compute $\lim_{n\to\infty} \frac{n}{n+1}$ we can use these rules:

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} 1/n} = \frac{1}{1+0} = 1.$$

multiply top and bottom by $1/n$

as claimed. ✓

Another very useful Lemma for computing limits of sequences:

Lemma If $\lim_{n\to\infty} a_n = L$ and $f(x)$ is continuous at L then $\lim_{n\to\infty} f(a_n) = f(L)$.

E.g. Q: What is $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right)$?

A: Notice $\lim_{n\to\infty} \pi/n = 0$ and cos is continuous at 0 so that $\lim_{n\to\infty} \cos(\pi/n) = \cos(0) = 1$. ✓

Another useful lemma for limits of sequences with signs:

Lemma If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.

E.g. How to compute $\lim_{n\to\infty} \frac{(-1)^n}{n}$? Since $\lim_{n\to\infty} 1/n = 0$, also have that $\lim_{n\to\infty} (-1)^n/n = 0$. Compare this to $a_n = (-1)^n$, which diverges!

One of the most important kinds of sequences is the sequence $a_n = r^n$ for some fixed number $r \in \mathbb{R}$. When does this sequence converge?

We have seen in Calc I that for $0 < r < 1$,

$$\lim_{x\to\infty} r^x = 0 \Rightarrow \lim_{n\to\infty} r^n = 0.$$

By the absolute value lemma, this also means for $-1 < r < 0$ have $\lim_{n\to\infty} r^n = 0$ for these n too.

Clearly $\lim_{n\to\infty} 1^n = \lim_{n\to\infty} 1 = 1$. Other r's diverge:

Theorem

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \text{does not exist} & \text{for other } r. \end{cases}$$

monotone and bounded sequences § 11.1

Def'n The sequence $\{a_n\}$ is <u>increasing</u> if $a_n < a_{n+1}$ for all $n \geq 1$, and <u>decreasing</u> if $a_n > a_{n+1}$ for all $n \geq 1$. It is called <u>monotone</u> if it is either increasing or decreasing.

<u>E.g.</u> The sequence $a_n = n$ is increasing (hence monotone). The sequence $a_n = (-1)^n$ is neither increasing nor decreasing.

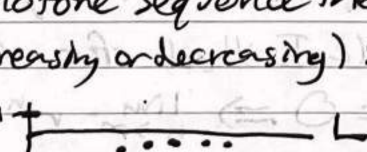
Def'n $\{a_n\}$ is <u>bounded above</u> if there is some M such that $a_n < M$ for all $n \geq 1$; it is <u>bounded below</u> if there is M such that $a_n \geq M$ for all $n \geq 1$; it is <u>bounded</u> if it is <u>both</u> bounded above and below.

<u>E.g.</u> $a_n = (-1)^n$ is bounded (above by 2 and below by -2) but $a_n = n$ is <u>unbounded</u>, since it goes off to ∞.

Clearly a sequence which is unbounded (like $a_n = n$) can not be convergent. Some bounded sequences, like $a_n = (-1)^n$ are also divergent. But, if your sequence is both <u>bounded</u> and <u>monotone</u>, then it must converge:

Thm (Monotone Sequence Theorem) Every bounded, monotone (either increasing or decreasing) sequence converges.

<u>Picture proof</u>:



$\Leftarrow$ increasing sequence bounded by M will converge to an L w/ $L \leq M$.

<u>E.g.</u> $a_n = \frac{1}{n}$ is bounded and monotone (decreasing) so it converges, as we are well aware already...

<u>Exercise</u> Show the sequence $a_1 = 2$, $a_{n+1} = \frac{1}{2}(a_n + 6)$ for $n \geq 1$, is convergent by using the Monotone Sequence Theorem.

//

3/31 **Series §11.2**

A series is basically an "infinite sum."
If we have an (infinite) sequence $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \dots\}$ the corresponding series is

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots$$

An infinite sum like this does not always make sense:

$$\sum_{n=1}^{\infty} n = 1+2+3+4+5+\dots = \text{"}\infty\text{"}$$

But some times we can take a sum of ∞-many terms & get finite number:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = ???$$

Well, $\frac{1}{2} = 0.5$, $\frac{1}{2}+\frac{1}{4} = 0.75$, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8} = 0.875$,
and it seems that if we add up more and more terms, we don't go off to ∞, but instead get closer and closer to 1.

Def'n For series $\sum_{n=1}^{\infty} a_n$, the associated partial sums are $S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \dots + a_n$ for $n \geq 1$.

If $\lim_{n\to\infty} S_n = L$ we write $\sum_{n=1}^{\infty} a_n = L$ and we say the series converges. Otherwise, it diverges.

Key idea: $\boxed{\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} (a_1 + a_2 + \dots + a_n)}$

E.g. Let $a_n = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$. What is $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$?

Well, $S_n = \underbrace{\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right)}_{a_1} + \underbrace{\left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right)}_{a_2} + \cancel{\dots} + \underbrace{\left(\frac{1}{n-1} - \cancel{\frac{1}{n}}\right)}_{a_{n-1}} + \underbrace{\left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)}_{a_n}$

$= 1 - \frac{1}{n+1}$, so that $\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$.

Thus, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$.

One of the most important kind of series are the *geometric series*:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$ for real numbers r and $a \neq 0$.

Notice that $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$

and $r \cdot S_n = \quad ar + ar^2 + \cdots + ar^{n-1} + ar^n$

So $(1-r) S_n = a \qquad - ar^n$

$$\Rightarrow S_n = \frac{a - ar^n}{1-r}$$

Since $\lim_{n\to\infty} r^n = 0$ for $|r|<1$, we have:

$$\sum_{n=1}^{\infty} ar^{n-1} = \lim_{n\to\infty} \frac{a-ar^n}{1-r} = \frac{a}{1-r} \text{ for } |r|<1.$$

Important formula to remember: value of geometric series, when ratio r is $|r|<1$

E.g. $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ is geo. series with $a = 1/2$ and $r = 1/2$ so $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1/2}{1-1/2} = 1$

This is what we expected from before.

For $|r| \geq 1$, geo. series $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

Consider in particular case $a = r = 1$.

Then $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} 1 = 1+1+1+\cdots$, so the partial sums are $S_n = 1+1+\cdots+1 = n$, and $\lim_{n\to\infty} S_n = \infty$.

And similarly for any $a \neq 0$, $\sum_{n=1}^{\infty} a = a+a+a+\cdots$ will diverge. In order to converge, the terms in a series must approach zero:

Theorem (Divergence Test) If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. So if $\lim_{n\to\infty} \neq 0$, $\sum_{n=1}^{\infty} a_n$ diverges.

WARNING: The divergence test says that if terms do not go to zero, series diverges.
But converse does not hold: the a_n can go to 0, while $\sum_{n=1}^{\infty} a_n$ still diverges.

The most important counter example is the harmonic series:
$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$
Of course $\lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ still diverges.
How to see this? Let's ignore the 1 at start and show that
$$\underbrace{\frac{1}{2}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq 2\cdot\frac{1}{4} = \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq 4\cdot\frac{1}{8} = \frac{1}{2}} + \cdots \text{ diverges}$$
The trick, as shown above, is to break the series into pieces consisting of 1, 2, 4, 8, ... terms.
If we add up the terms in each piece, we get a sum bigger than $\frac{1}{2}$. So overall the sum is $\geq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$
But a sum of ∞-many $\frac{1}{2}$'s must diverge.
So the harmonic series must diverge too!

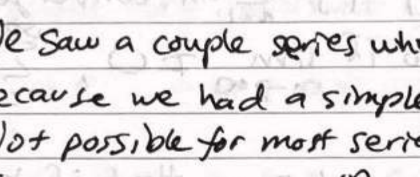
Thm (Laws for Series). Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge.
Then • $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$
and • $\sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n$ for any $c \in \mathbb{R}$.

WARNING: $\sum_{n=1}^{\infty} a_n \cdot b_n \neq \left(\sum_{n=1}^{\infty} a_n\right) \cdot \left(\sum_{n=1}^{\infty} b_n\right)$ //

4/3 <u>Integral</u> <u>test</u> <u>for</u> <u>convergence</u> §11.3

We saw a couple series whose convergence we could establish because we had a simple <u>formula</u> for the partial sums.
<u>Not possible for most series.</u> We need other tools to study convergence.

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. No simple formula for its partial sums.
But let's draw the following picture:



⇐ plot the sequence $a_n = \frac{1}{n^2}$, and use this to make rectangles of width 1 and height $= a_n$

Notice that the area of n^{th} rectangle $= a_n \times 1 = a_n$,
So the sum of areas $= a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$.

Also notice that we plotted the curve $f(x) = \frac{1}{x^2}$.
The area under $y = f(x)$ from $x = 1$ to ∞ is visibly less than $a_2 + a_3 + a_4 + \cdots = \left(\sum_{n=1}^{\infty} a_n\right) - a_1$.

But we can compute this as an <u>improper</u> integral:

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t 1/x^2\,dx = \lim_{t\to\infty} \left[-1/x\right]_1^t = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = \underline{1}.$$

Thus $\sum_{n=1}^{\infty} a_n \leq a_1 + \int_1^{\infty} 1/x^2\,dx = 1 + 1 = 2$, so in particular this series <u>converges</u>: it has a finite value.
(Since all the terms are positive, if it diverged it would go off to ∞, so being bounded means it converges).

This way of comparing a series to an associated integral is called the <u>integral</u> <u>test</u> for <u>convergence</u>, and can be used to show divergence of integrals as well:

Theorem (Integral test for Convergence / Divergence).
Let $f(x)$ be a continuous, positive, (eventually) decreasing function on $[1,\infty)$, and let $a_n = f(n)$ for $n \geq 1$.
1) If $\int_1^\infty f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2) If $\int_1^\infty f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

E.g. We saw before that harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges.
Can also prove this using the integral test:
$$\int_1^\infty 1/x\,dx = \lim_{t\to\infty}\int_1^t 1/x\,dx = \lim_{t\to\infty}[\ln(x)]_1^t = \lim_{t\to\infty}\ln(t) = \infty.$$

Comparing $\sum_{n=1}^{\infty}\frac{1}{n}$ and $\sum_{n=1}^{\infty} 1/n^2$, a natural question is:
for which p does series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converge?
(The book calls these "p-series".)

Theorem The series $\sum_{n=1}^{\infty}\frac{1}{n^p}$:
- diverges for $p \leq 1$
- converges for $p > 1$.

Pf: First notice that if $p \leq 0$ then $\lim_{n\to\infty}\frac{1}{n^p} \neq 0$,
so the series diverges by the Test for Divergence.
So suppose $0 < p < 1$. Then $\int \frac{1}{x^p}\,dx = \frac{1}{1-p}x^{1-p}$
So that $\int_1^\infty 1/x^p\,dx = \lim_{t\to\infty}\left[\frac{1}{1-p}x^{1-p}\right]_1^t = \infty$,
so the series diverges by the integral test.
We have already seen that $\sum_{n=1}^{\infty} 1/n$ diverges,
so finally assume $p > 1$. Then $\int\frac{1}{x^p}\,dx = \frac{-1}{(p-1)\cdot x^{p-1}}$
So that $\int_1^\infty 1/x^p\,dx = \lim_{t\to\infty}\left[\frac{-1}{(p-1)x^{p-1}}\right]_1^t = \frac{1}{p-1}$,
so the series converges by the integral test. $\boxtimes$

4/5

Estimating Remainders with Integrals §11.3

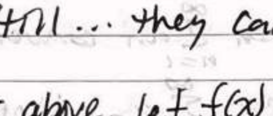
Integrals are useful for proving convergence of series, but don't tell us the exact value of the series. Still... they can be used to estimate the value of the series.

As above, let $f(x)$ be a continuous, positive, decreasing fn. on $[1,\infty)$ and let $a_n = f(n)$ at integers $n \geq 1$. We want to estimate the series $S = \sum_{n=1}^{\infty} a_n$. A simple estimate for any series is just the partial sum $S_n = a_1 + a_2 + \cdots + a_n$ for some finite value of n. How good of an estimate is S_n for the true value S?

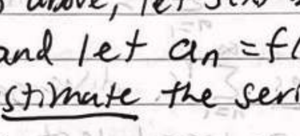
Define the remainder to be $R_n = S - S_n$.

E.g. for $S = \sum_{n=1}^{\infty} \frac{1}{2^n}$, $S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, and we know $S=1$, so $R_n = \frac{1}{4}$.

By looking at the two pictures below:



⇐ ~~Outer~~ over estimate: $R_n = a_{n+1} + a_{n+2} + \cdots \leq \int_n^{\infty} f(x)\,dx$



⇐ under estimate: $R_n = a_{n+1} + a_{n+2} + \cdots \geq \int_{n+1}^{\infty} f(x)\,dx$

Theorem We have $\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)\,dx$.

E.g. For $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$, $S_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \approx 1.42$, and by above

$$\int_5^{\infty} \frac{1}{x^2}\,dx \leq R_4 \leq \int_4^{\infty} \frac{1}{x^2}\,dx$$

$$1/5 \leq R_4 \leq 1/4$$

$$0.2 \leq S - 1.42 \leq 0.25$$

$$1.62 \leq S \leq 1.67$$

pretty good estimate of $\sum_{n=1}^{\infty} \frac{1}{n^2}$!

(In fact, $S = \frac{\pi^2}{6} \approx 1.64\ldots$, but this is a difficult result.)

Comparison Tests for Series §11.4

We know the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges ($|r|<1$).
The series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ seems very similar, but how can we show it converges/diverges? In fact, we can compare the two series:

Theorem (Direct Comparison Test for Series)
Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series whose terms are all positive! Then:

1) If $\sum_{n=1}^{\infty} b_n$ converges and $a_n \le b_n$ for all n then $\sum_{n=1}^{\infty} a_n$ converges too.
2) If $\sum_{n=1}^{\infty} b_n$ diverges and $a_n \ge b_n$ for all n then $\sum_{n=1}^{\infty} a_n$ diverges too.

Note: positive terms here is very important!

E.g. Notice that $\frac{1}{2^n+1} \le \frac{1}{2^n}$ for all $n \ge 1$
(dividing 1 by a bigger number, so smaller)
therefore $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ also converges.

E.g. Easy to show that if $\sum_{n=1}^{\infty} a_n$ diverges/converges, then $\sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n$ also diverges/converges for any nonzero scalar $c \in \mathbb{R} \setminus \{0\}$.
So $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ also diverges, like harmonic series.

And then notice $\frac{1}{2n-1} \ge \frac{1}{2n}$ for all $n \ge 1$,
So therefore $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ also diverges by direct comparison.

The series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ also seems very similar to $\sum_{n=1}^{\infty} \frac{1}{2^n}$, so we expect that it would also converge.
Unfortunately $\frac{1}{2^n-1} > \frac{1}{2^n}$ for all $n \geq 1$, wrong direction of inequality to prove convergence by direct comparison.

—

Instead we can use the following:

Theorem (Limit Comparison Test for Series)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with positive terms.

Suppose $c = \lim_{n\to\infty} \frac{a_n}{b_n}$ exits and $c \neq 0$ and $c \neq \pm\infty$.

Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

—

E.g. Notice $\lim_{n\to\infty} \frac{1/2^n}{1/(2^n-1)} = \lim_{n\to\infty} \frac{2^n-1}{2^n} = \lim_{n\to\infty} \frac{1-\frac{1}{2^n}}{1} = 1$,

So the fact that $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges means $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ does as well.

E.g. Consider a series like $\sum_{n=1}^{\infty} \frac{3n}{5n^2+n-1}$.

How to decide convergence/divergence? Compare to $\sum_{n=1}^{\infty} \frac{1}{n}$:

$\lim_{n\to\infty} \left(\frac{3n}{5n^2+n-1}\right) / \left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{3n^2}{5n^2+n-1} = \frac{3}{5}$, so

By limit comparison $\sum_{n=1}^{\infty} \frac{3n}{5n^2+n-1}$ also diverges.

Exercise: Show $\sum_{n=1}^{\infty} \frac{3n}{5n^3+n-1}$ converges (compare to $\sum_{n=1}^{\infty} \frac{1}{n^2}$).

Key observation: for series whose terms are rational functions, check power of n on top vs. power on bottom!

4/7 **Alternating Series** §11.5

The convergen tests we've seen (integral test, comparison test, etc.) mostly are for series with positive terms only.
Things become more complicated when terms have signs.

—

The most important kind of series with signs are the alternating series, where terms switch positive to negative to positive to negative, etc:

like $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

or $\sum_{n=1}^{\infty} (-1)^n \frac{3n}{4n-1} = -\frac{3}{3} + \frac{6}{7} - \frac{9}{11} + \frac{12}{15} - \cdots$

As we can see, an alternating series has form:

$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ or $\sum_{n=1}^{\infty} (-1)^n b_n$ where b_n is a sequence of positive numbers

(which form it is depends on if it starts positive or negative).

Theorem (Alternating Series Test)

For an alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$ (where $b_n > 0$ are positive), if we have:

- $b_{n+1} \leq b_n$ for all $n \geq 1$ (terms are getting smaller)
- $\lim_{n\to\infty} b_n = 0$ (terms go to zero),

then the series converges.

—

E.g. The alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

Satisfies these conditions: • $\frac{1}{n+1} < \frac{1}{n}$

• and $\lim_{n\to\infty} \frac{1}{n} = 0$,

So $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ converges, unlike usual harmonic series.

Idea: terms cancel each other out, so sum more likely to converge.

Picture Proof of Alternating Series Test:

We start with 0. We add $+b_1$ to get s_1. Then we subtract $-b_2$ to get s_2. Etc. But we never go back further than where we just were, since $b_{n+1} \le b_n$. So we get "trapped" in smaller and smaller space, as the $b_n \to 0$ when $n \to \infty$. Thus the ~~sum~~ series must converge. ∎

In fact, we can use this argument to estimate the series.

Thm Let $s = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ be alternating series satisfying conditions: • $b_{n+1} \le b_n \ \forall n$ and • $\lim_{n\to\infty} b_n = 0$.

Let $s_n = b_1 - b_2 + b_3 - \cdots \pm b_n$ be the n^{th} partial sum and $R_n = s - s_n$ be the remainder (error) of this partial sum.

Then $|R_n| (= |s - s_n|) \le b_{n+1}$.

"Error is bounded by next term."

E.g. Let's compute $s = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ accurately to within 0.1.

We compute $s_9 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{9} = \frac{131}{180} \approx 0.728$

and by thm $|R_n| \le \frac{1}{10}$ (next term), so $s \approx 0.728 \pm 0.1$ //

E.g. Decide if the alternating series

$$\sum_{n=1}^{\infty} (-1)^n \frac{3n}{4n-1} \text{ converges or diverges.}$$

Here: $\lim_{n\to\infty} \frac{3n}{4n-1} = \frac{3}{4} \ne 0$, so we cannot use the alternating series test to establish convergence.

Actually, $\lim_{n\to\infty} (-1)^n \frac{3n}{4n-1}$ Does not exist, so by the limit of terms test, series diverges! //

4/10

Absolute convergence vs. conditional convergence. §11.5

Def'n A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ (series of absolute values) converges.

Thm If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

Pf idea: Adding signs means terms cancel out, so only makes it 'easier' to converge. $\boxtimes$

Def'n Series $\sum_{n=1}^{\infty} a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

E.g. The alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ is conditionally convergent, since it converges, but $\sum_{n=1}^{\infty} |(-1)^{n-1} \frac{1}{n}| = \sum_{n=1}^{\infty} \frac{1}{n}$ (harmonic series) diverges.

Conditionally convergent series are 'fragile' (& weird!).
If you take any finite sum like $1+2+3+4+5=15$ and rearrange the terms $2+5+3+1+4=15$, you of course get the same result.

Thm Any rearrangement of an absolutely convergent series gives the same sum.

However... rearrangements of conditionally convergent sums give different sums (can make sum anything!).
This goes against intuition of how sums should behave... //

The Ratio and Root Tests §11.6

For a geometric series $\sum_{n=1}^{\infty} ar^{n-1}$, convergence/diverge determined by ratio $|a_{n+1}/a_n| = |r|$ of terms.
In fact, this is important for any series:

Theorem (Ratio Test for Absolute convergence)
For series $\sum_{n=1}^{\infty} a_n$, let $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ (limiting ratio of successive terms).

If $L < 1$, then the series converges absolutely and hence, converges.
If $L > 1$ (including $L = \infty$), then the series diverges.
If $L = 1$, the test is inconclusive (could go either way).

E.g. Does the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ converge absolutely?

Here $|a_n| = \frac{n^3}{3^n}$ so

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^3/3^{n+1}}{n^3/3^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} = \lim_{n\to\infty} \frac{1}{3}\left(\frac{n+1}{n}\right)^3$$

$$= \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^3 = \frac{1}{3}(1+0) = \underline{\frac{1}{3}}$$

Since $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{3} < L$, this series converges absolutely.

The ratio test is useful when the series has terms like $2^n, 3^n, e^n$, etc. that are exponential in n.
These terms are "more important" than polynomial terms.

Idea of proof for ratio test:
We compare the series to geometric series $\sum_{n=1}^{\infty} L^{n-1}$ of ratio L, which converges if $L<1$, diverges if $L>1$.

E.g. Let's try applying ratio test to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Here $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = \underline{\underline{1}}$.

So the ratio test fails (= is inconclusive) (even though we know series converges absolutely).

In fact, ratio test fails for any p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

This makes sense, since most of the examples of conditionally convergent series we know are related to p-series, and the ratio test cannot detect conditional convergence.

The following is a variation of the ratio test:

Theorem (Root Test for absolute convergence)

For series $\sum_{n=1}^{\infty} a_n$, let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ (limit of n^{th} root of terms).

If $L < 1$, the series converges absolutely (so converges).

If $L > 1$ (including ∞), then the series diverges.

If $L = 1$ the root test is inconclusive.

The root test is useful when the series has terms like n^n in it ("super exponential" terms).

E.g. Exercise Use the root test to show that

$$\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n \text{ converges.}$$

However, the root test is more obscure than the ratio test, so I will not expect you to memorize the root test ...