

## Sign-reversing involutions + identities involving signs

Some identities w/  $\pm$  signs can be proven like this:

Prop. Given a set  $X$  with a sign function  $\text{sgn}: X \rightarrow \{\pm 1\}$   
a weight function  $\text{wt}: X \rightarrow \mathbb{R}$

and a sign-reversing, weight-preserving, involution  
 $(\text{sgn}(\tau(x)) = -\text{sgn}(x)) \quad (\text{wt}(\tau(x)) = \text{wt}(x)) \quad (\tau^2 = \text{id})$

then  $\sum_{x \in X} \text{sgn}(x) \cdot \text{wt}(x) = \sum_{x \in X^\tau} \text{sgn}(x) \cdot \text{wt}(x)$

Proof:

$$X = \begin{array}{|c|c|} \hline x^+ & x^- \\ \hline \text{cancel} & \\ \hline x & \tau(x) \\ \hline \end{array} \quad X \setminus X^\tau \quad \text{sgn}(x) \cdot \text{wt}(x) + \underbrace{\text{sgn}(\tau(x)) \cdot \text{wt}(\tau(x))}_{-\text{sgn}(x) \cdot \text{wt}(x)} = 0$$

for all  $x \in X \setminus X^\tau$  □

only this left

## Examples

$$(1) \text{(Warm-up)} \quad \sum_{k=0}^n \binom{n}{k} (-1)^k = 0 \text{ for } n \geq 1$$

$$\sum_{\substack{\#S \\ \text{subsets } S \subseteq [n]}} (-1)^{\#S}$$

$$\text{sgn}: X = 2^{[n]} \rightarrow \{\pm 1\}$$

$$S \mapsto (-1)^{\#S}$$

$$\text{wt}: X = 2^{[n]} \rightarrow \mathbb{Z}$$

$$S \mapsto 1$$

$$\tau: X \rightarrow X$$

$$S \mapsto \begin{cases} S - \{1\} & \text{if } 1 \in S \\ S \cup \{1\} & \text{if } 1 \notin S \end{cases}$$

is sign-reversing,  
weight-preserving  
with no fixed points.

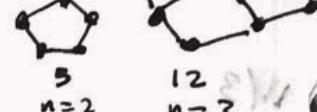
Rmk: This was key identity in pf. of P.I.E. □

Pentagonal numbers:

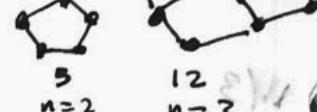
↑



$n=1$



$n=2$



$n=3$

② Recall Jhm (Euler's "Pentagonal Number Theorem")

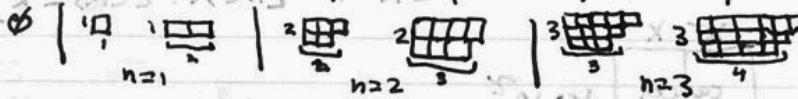
$$\prod_{j \geq 1} (1 - q^j) = 1 + \sum_{n \geq 1} (-1)^n \left( q^{\frac{n(3n-1)}{2}} + q^{\frac{n(3n+1)}{2}} \right)$$

$$= 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \dots$$

Franklin's (1881) proof of Euler's P.N.T.:

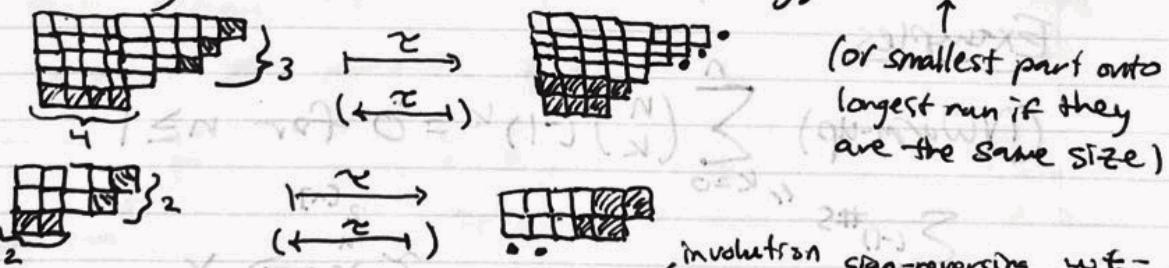
$$\text{LHS} = (1-q)(1-q^2)\dots = \sum_{\lambda} \underbrace{(-1)^{\ell(\lambda)}}_{\substack{\text{partitions} \\ \lambda = \lambda_1, \lambda_2, \dots, \lambda_r > 0 \\ \text{into distinct parts}}} \underbrace{q^{\frac{|\lambda|}{2}}}_{\substack{\text{sgn}(\lambda) \\ \text{wt}(\lambda)}}$$

$$\text{RHS} = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \dots$$

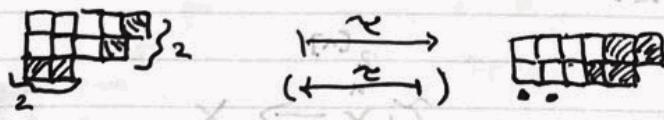


Franklin defined  $\gamma: X := \{\lambda \text{ w/ distinct parts}\} \rightarrow X$  by comparing

- Smallest part and longest initial run  $\lambda_1, \lambda_1-1, \lambda_1-2, \dots$
- and moving the smaller one onto the bigger one:



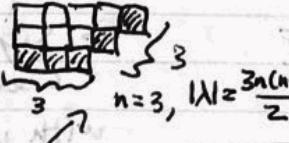
(or smallest part onto  
longest run if they  
are the same size)



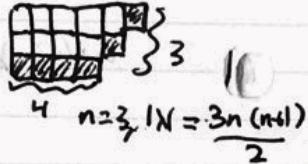
involution sign-reversing weight-preserving

When one can do this, check  $\gamma^2 = \text{id}$ ,  $\ell(\gamma(\lambda)) = \ell(\lambda) \pm 1$ ,  $|\lambda| = |\gamma(\lambda)|$ .  
One cannot do this if:

- smallest part + run have the same size and overlap.



- or run is 1 smaller + they overlap



So  $\gamma$  implies only these shapes contribute to LHS  $\Rightarrow \text{LHS} = \text{RHS}$ .

11/5

(3) Theorem (Kirchoff's Matrix-Tree Theorem)

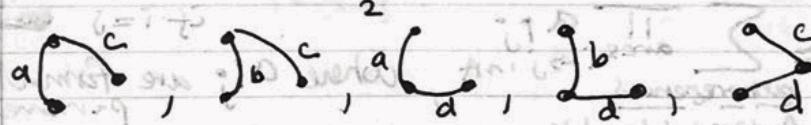
The number of spanning trees in a multigraph  $G = \langle V, E \rangle$   
for any  $i$  is  $\det(\tilde{L}(G)^{i,i})$ , where  $\tilde{A}^{i,i}$  means  $A$  w/ row+column  $i$  removed,  
(multiple/parallel edges allowed)

and  $L(G)$  is the  $n \times n$  Laplacian matrix of  $G$ :

$$L(G)_{v,w} := \begin{cases} \deg(v) & \text{if } v=w \\ -\# \text{edges from } v \text{ to } w & \text{if } v \neq w \end{cases}$$

Note! A spanning tree  $T$  of  $G$  is a subgraph of  $G$  that's a tree and which contains all the vertices  $V$ .

Example  $G = \langle \{a, b, c, d\}, E \rangle$  has 5 spanning trees:



$$\text{And } L(G) = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$$

$$\text{So } \det(L(G)^{1,1}) = \det([3]) = 6 - 1 = 5 \checkmark$$

$$\text{and } \det(L(G)^{3,3}) = \det([-2 3]) = 9 - 4 = 5 \checkmark$$

Example Recall Cayley's formula  $n^{n-2}$  for # of (labeled) trees on  $n$  vertices. These are the ...

Spanning trees of the complete graph  $K_n$  on  $[n]$ .

e.g.  $n=5$    $\overline{L(K_5)}^{5,5} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & n-1 & -1 & -1 & -1 \\ 3 & -1 & n-1 & -1 & -1 \\ 4 & -1 & -1 & n-1 & -1 \\ 5 & -1 & -1 & -1 & n-1 \end{bmatrix} = n \underbrace{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}}_{(n-1) \times (n-1) \text{ identity matrix}} - \underbrace{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}}_{\text{all } 1's \text{ matrix}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

What are eigenvalues of  $\mathbb{1}_{n-1}$ ? It has rank = 1, so  $(n-2)$  eigenvalues = 0  
Also  $\mathbb{1}_{n-1} \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = (n-1) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ , so one eigenvalue =  $(n-1)$ .

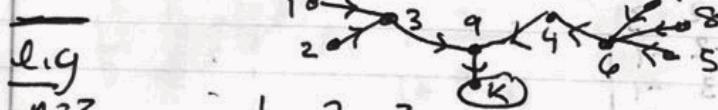
$\mathbb{1}_{n-1}$  has eigenvals  $(0, 0, \dots, 0, n-1) \Rightarrow \overline{L(K)}^{n,n}$  has eigenvals  $(n, n, \dots, n, 1)$

$$\Rightarrow \det(\overline{L(K)}^{n,n}) = n^{n-2} \Rightarrow \text{Cayley's formula.} \checkmark$$

In fact, let's prove a weighted, directed version of Kirchoff!

Thm If  $L = \begin{bmatrix} 1 & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ 2 & -a_{21} & 1 & -a_{23} & \cdots & -a_{2n} \\ 3 & \vdots & \ddots & \ddots & \ddots & \vdots \\ n & -a_{n1} & \cdots & \cdots & \cdots & 1 \end{bmatrix}$  has  $L_{ij} = \begin{cases} -a_{ij} & \text{if } i \neq j \\ a_{i1} + a_{i2} + \cdots + \hat{a_{ij}} + \cdots + a_{in} & \text{if } i = j \end{cases}$

then  $\det(\overline{L}^{K,K}) = \sum_{\substack{\text{arborescences} \\ \text{A directed toward } K}} \prod_{i \rightarrow j} a_{ij}$ , where  $a_{ij}$  are formal parameters.



$$L = \begin{bmatrix} 1 & 2 & 3 \\ 2 & a_{12}+a_{13} & -a_{12} & -a_{13} \\ 3 & -a_{21} & a_{21}+a_{23} & -a_{23} \\ 1 & -a_{31} & -a_{32} & a_{31}+a_{32} \end{bmatrix} \Rightarrow \det(\overline{L}^{3,3}) = \det \begin{bmatrix} a_{12}+a_{13} & -a_{12} & 0 \\ -a_{21} & a_{21}+a_{23} & 0 \\ 0 & 0 & a_{31}+a_{32} \end{bmatrix}$$

$$= (a_{12}+a_{13})(a_{21}+a_{23}) - (-a_{12})(-a_{21})$$

$$= a_{12}a_{23} + a_{12}a_{21} + a_{13}a_{21} + a_{13}a_{23} - a_{12}a_{21}$$

$$= a_{12}a_{23} + a_{13}a_{21} + a_{13}a_{23} \checkmark$$

Note:  $\Rightarrow$  Kirchoff by setting  $a_{ij} = \# \text{edges } i \rightarrow j$  in  $G$

11/8

Proof of Thm:

$$L = \begin{bmatrix} R_1 - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & R_2 - a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & \cdots & R_n - a_{nn} \end{bmatrix}$$

where  $R_i = a_{i1} + a_{i2} + \cdots + a_{ii} + \cdots + a_{in}$

$$= \sum_{j=1}^n a_{ij}$$

$$= (R_i \cdot S_{ij} - a_{ij})_{\substack{i=1, \dots, n \\ j=1, \dots, n}}$$

$\Rightarrow \det(L^{n,n}) = \sum_{w \in S_{n-1}} \text{sgn}(w) \prod_{i=1}^{n-1} L_{i,w(i)}$

Recall:  $\text{sgn}(w) = (-1)^{\text{inv}(w)}$

"Leibniz formula"

$$= \sum_{S \subseteq [n-1]} \prod_{i \in S} (R_i - a_{ii}) \sum_{w \in S_{[n-1] \setminus S}} \text{sgn}(w) \prod_{i \in [n-1] \setminus S} (-a_{i,w(i)})$$

(fixed by w)

$$= \sum_{S \subseteq [n-1]} \sum_{T \subseteq S} \prod_{i \in T} R_i \prod_{i \in S \setminus T} (-a_{ii}) \sum_{w \in S_{[n-1] \setminus S}} \text{sgn}(w) \cdot \prod_{i \in [n-1] \setminus S} (-a_{i,w(i)})$$

derangement

Important property of sign of permutations:  
 $\text{sgn}(v \cdot w) = \text{sgn}(v) \cdot \text{sgn}(w)$

$$= \sum_{T \subseteq [n-1]} \prod_{i \in T} (a_{i1} + a_{i2} + \cdots + a_{in}) \cdot \sum_{w \in S_{[n-1] \setminus T}} \text{sgn}(w) \prod_{i \in [n-1] \setminus T} (-a_{i,w(i)})$$

$$= \sum_{f:T \rightarrow [n]} \prod_{i \in T} a_{i,f(i)}$$

$$= \sum_{\substack{(T,f,w) \\ T \subseteq [n-1] \\ f:T \rightarrow [n] \\ w \in S_{[n-1] \setminus T}}} (-1)^{\#([n-1] \setminus T)} \cdot \frac{\text{sgn}(w)}{\text{sgn}(x)} \cdot \prod_{i \in T} a_{i,f(i)} \prod_{i \in [n-1] \setminus T} a_{i,w(i)}$$

$$\chi := \sum_{\substack{(T,f,w) \\ T \subseteq [n-1] \\ f:T \rightarrow [n] \\ w \in S_{[n-1] \setminus T}}} (-1)^{\#([n-1] \setminus T)} \cdot \text{sgn}(w) \cdot \prod_{i \in T} a_{i,f(i)} \prod_{i \in [n-1] \setminus T} a_{i,w(i)}$$

We will evaluate this signed, weighted sum using a sign-reversing involution ...

Picture of  $(T, f, \omega)$ :



We can define an involution  
 $\chi: X \rightarrow X$   
 that swaps the cycle containing  
 the smallest index  $i \in [n-1]$   
 from  $w$  to  $f$  or back from  $f$  to  $w$ !

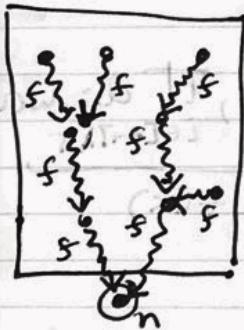
Check that  $\chi$

- is an involution (clear)
- is wt-preserving (preserves arcs)
- is sign-reversing (sgn of a  $k$ -cycle is  $(-1)^{k+1}$ )

What are the fixed points  $X^\chi$ ?

No cycles  $\Rightarrow$   $[n-1] \setminus T$  is empty, i.e.,  $T = [n-1]$   
 in  $w$  or  $f$  and  $f: [n-1] \rightarrow [n]$  has no cycles

$$T = [n-1]$$



(easy) Lemma.

This forces  $f$  to be an arborescence directed toward  $n$  (and conversely, any arborescence is such an  $f$ ).

Hence,  $\det(L^{n,n}) = \sum_{\substack{\text{arborescences} \\ f \text{ of } [n] \text{ directed} \\ \text{toward } n}} \prod_{i \in [n-1]} a_{i, f(i)}.$

11/10

The transfer matrix method (Stanley § 4.7, Ardila § 3.1.2)

Another tool from linear algebra for counting walks in graphs

Thm Let  $G$  be a graph w/ vertex set  $V = [n]$ , and let

$A_G = (a_{ij})_{i,j=1,\dots,n}^{i=1,\dots,n}$  be its adjacency matrix:  $a_{ij} = \# \text{edges from } i \text{ to } j$ .

Then (a) # of walks of length  $\ell$  =  $(A_G^\ell)_{i,j}$  for all  $\ell \geq 0$ .

(b) # of closed walks of length  $\ell$   
 $i = i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_\ell = i$  (for all  $i$ ) =  $\lambda_1^\ell + \lambda_2^\ell + \dots + \lambda_n^\ell$ ,  
where  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the eigenvalues of  $A_G$ .

Pf: (a) is just definition of matrix multiplication:

$$(A_G^\ell)_{i,j} = \sum_{i_1=1}^n \sum_{i_2=1}^n \dots \sum_{i_{\ell-1}=1}^n a_{i,i_1} a_{i_1,i_2} \dots a_{i_{\ell-1},j} = \text{LHS of (a)}.$$

For (b), from (a) it follows that

$$\# \text{closed walks of length } \ell = \sum_{i=1}^n (A_G^\ell)_{i,i} = \text{trace}(A_G^\ell).$$

Since  $A_G$  is real + symmetric, it can be diagonalized,

i.e.,  $\exists P$  s.t.  $P A_G P^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$ . Thus,

$$\begin{aligned} \text{trace}(A_G^\ell) &= \text{trace}((P A_G P^{-1})^\ell) = \text{trace}((P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} P)^\ell) \\ &= \text{trace}(P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}^\ell P) = \text{trace}(\begin{pmatrix} \lambda_1^\ell & 0 \\ 0 & \lambda_2^\ell & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \lambda_n^\ell \end{pmatrix}) = \lambda_1^\ell + \dots + \lambda_n^\ell. \end{aligned}$$

Recall:  
 $\text{tr}(AB) = \text{tr}(BA)$

Example Let  $f(n, k) = \# \text{proper vertex-colorings of } C_n$  cycle graph  
w/  $k$ -colors.  
(no adjacent vertices w/  
Same color)

$$\text{e.g. } n=2 \quad f(2, k) = k(k-1)$$

$\frac{(k-1)(k-2)}{2} = \frac{k(k-1)}{2}$

color 1 first in  $k$  ways color 2 second in  $k-1$  ways

$$\begin{aligned} n=3 \quad & f(3, k) = k(k-1)(k-2) \\ & \text{color 1} \quad \text{color 2} \quad \text{color 3} \end{aligned}$$

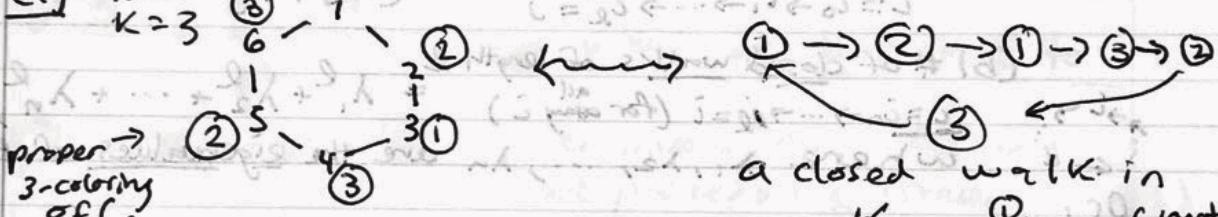
$$\begin{matrix} n=4 \\ \text{color 1} \\ \text{color 2} \\ \text{color 3} \\ \text{color 4} \end{matrix}$$

$$f(4, k) = \underbrace{k(k-1)(k-2)(k-3)}_{\substack{2+4 \text{ have different colors}}} + \underbrace{k(k-1)(k-1)^3}_{\substack{2+4 \text{ have same color}}}$$

color 1 color 2 color 3 color 4 color 1 color 2+4  
 ↘ ↘ ↘ ↘ ↘ ↘  
 same same

Note:  $\{\text{proper } k\text{-colorings of } C_n\} \leftrightarrow \{\text{closed walks of length } n\}$  in complete graph  $K_k$

$$\begin{matrix} n=6 \\ k=3 \end{matrix}$$



proper  $\rightarrow$  3-coloring of  $C_6$

So taking  $A_{K_K} = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}^{K-1} = I_K - I_K$ ,  $K-1$  terms

which has eigenvalues  $(\lambda_1, \dots, \lambda_K) = (k-1, -1, -1, \dots, -1)$   
 (since we saw earlier that  $I_K$  has eigen's  $(k, 0, 0, \dots, 0)$ )

we find that  $f(n, k) = \lambda_1^n + \dots + \lambda_K^n$

$$\begin{aligned} &= (k-1)^n + (-1)^n + \dots + (-1)^n \\ &= (k-1)^n + (k-1)(-1)^n \\ &= (k-1)((k-1)^n + (-1)^n). \end{aligned}$$

e.g.  $f(2, k) = (k-1)(k-1+1) = (k-1)k$

$$f(3, k) = (k-1)((k-1)^2 + 1) = (k-1)(k^2 - 2k + 1)$$

$$f(4, k) = (k-1)((k-1)^3 + 1) = (k-1)(k^3 - 3k^2 + 3k + 1)$$