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## Sequences §11.1

We now start a new chapter, Ch. 11, on sequences, series, and power series. This is the final topic of the semester.

Def'n An (infinite) sequence is an infinite list $a_1, a_2, a_3, \ldots, a_n, \ldots$ of real numbers. We also use $\{a_n\}$ and $\{a_n\}_{n=1}^{\infty}$ to denote this sequence.

E.g. We can let $a_n = \frac{1}{2^n}$ for $n \geq 1$, which gives the sequence $1/2, 1/4, 1/8, 1/16, 1/32, \ldots$

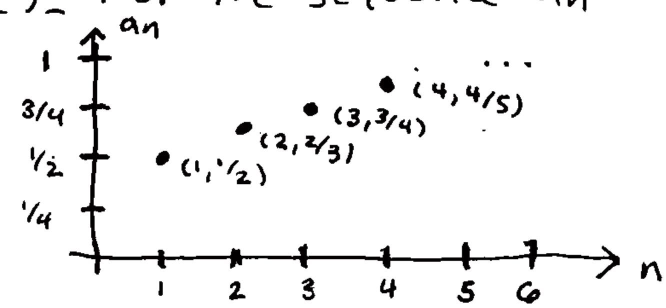
E.g. $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$

Can also write $\left\{\frac{n}{n+1}\right\}_{n=2}^{\infty} = \left\{\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$ to start at $n=2$, or also $\left\{\frac{n+1}{n+2}\right\}_{n=1}^{\infty} = \left\{\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$.

E.g. Not all sequences have simple formulas for the $n^{th}$ term. For example, with $a_n = n^{th}$ digit of $\pi$ after the decimal point, we have $\{a_n\} = \{1, 4, 1, 5, 9, 2, 6, 5, \ldots\}$ but there is no easy way to get the $n^{th}$ term here...

Def'n The graph of sequence $\{a_n\}_{n=1}^{\infty}$ is the collection of points $(1, a_1), (2, a_2), (3, a_3), \ldots$ in the plane.

E.g. For the sequence $a_n = \frac{n}{n+1}$, its graph is




The graph of a sequence is like the graph of a function, but we get discrete points instead of a continuous curve. Notice how for this graph, points approach line $y=1$ ...

Def'n We say the limit of sequence $\{a_n\}$ is $L$, written "$\lim_{n\to\infty} a_n = L$" or "$a_n \to L$ as $n \to \infty$" if, intuitively, we can make the terms $a_n$ as close to $L$ as we'd like by taking $n$ sufficiently large.

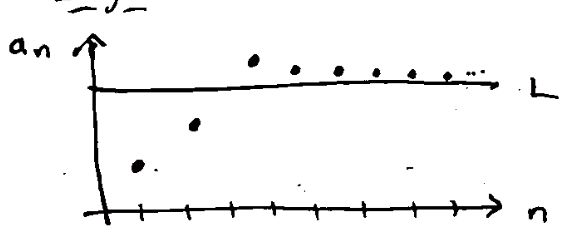
(Precise definition uses $\varepsilon$, like limits in Calc I...)

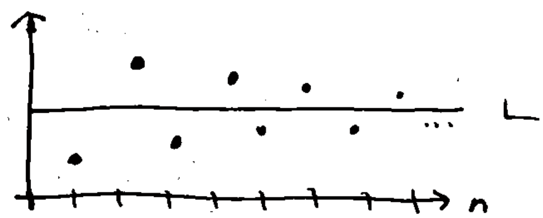
If $\lim_{n\to\infty} a_n$ exists, we say the sequence converges.

Otherwise, we say the sequence diverges.

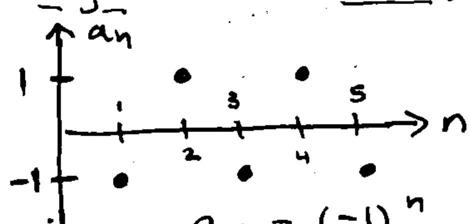
E.g. The sequence $a_n = \frac{n}{n+1}$ has $\lim_{n\to\infty} a_n = 1$ (we'll prove this later...)

E.g. Some other convergent sequences look like:

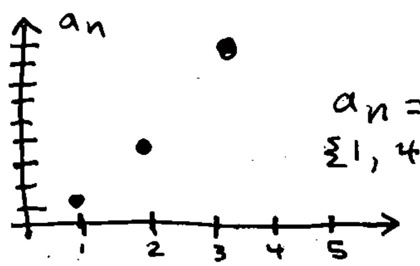



E.g. Some divergent sequences are:




$a_n = (-1)^n$
$\{-1, 1, -1, 1, \dots\}$




$a_n = n^2$
$\{1, 4, 9, 16, \dots\}$

Notice how this $2^{\text{nd}}$ example $a_n = n^2$ "goes off to $\infty$."

Def'n The notation "$\lim_{n\to\infty} a_n = \infty$" means that for every $M$ there is an $N$ such that $a_n > M$ for all $n > N$.

We define "$\lim_{n\to\infty} a_n = -\infty$" similarly.

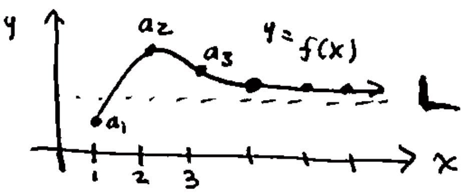
E.g. $\lim_{n\to\infty} n^2 = \infty$ and $\lim_{n\to\infty} -n = -\infty$.

Having an infinite limit is one way a sequence can diverge.

Limits of sequences are very similar to limits of functions:

Theorem If $f(x)$ is a function with $f(n)=a_n$ for all positive integers $n$, then if $\lim_{x\to\infty} f(x) = L$ also $\lim_{n\to\infty} a_n = L$.

Picture:




E.g. How to find $\lim_{n\to\infty} \frac{\ln(n)}{n}$? Instead, let $f(x) = \frac{\ln(x)}{x}$, then $\lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{1/x}{1}$ (by L'Hôpital's Rule)

$$= \lim_{x\to\infty} 1/x = 0$$

So we also have that $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$.

3/29 All the basic rules for limits of functions apply to sequences:

Theorem (Limit Laws for Sequences)

For convergent sequences $\{a_n\}$ and $\{b_n\}$, we have:

- $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.
- $\lim_{n\to\infty} (c \cdot a_n) = c \cdot \lim_{n\to\infty} a_n$ for any constant $c \in \mathbb{R}$.
- $\lim_{n\to\infty} a_n \cdot b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = (\lim_{n\to\infty} a_n)/(\lim_{n\to\infty} b_n)$ if $\lim_{n\to\infty} b_n \neq 0$.

E.g. To compute $\lim_{n\to\infty} \frac{n}{n+1}$, we can use these rules:

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}} = \frac{1}{1+0} = 1$$

multiply top and bottom by $1/n$

as claimed!

Another very useful lemma for computing limits of sequences:

<u>Lemma</u> If $\lim_{n\to\infty} a_n = L$ and $f(x)$ is continuous at $x = L$, then $\lim_{n\to\infty} f(a_n) = f(L)$.

E.g. Q: What is $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right)$?

A: Notice $\lim_{n\to\infty} \frac{\pi}{n} = 0$ and cos is continuous at $0$, so that $\lim_{n\to\infty} \cos(\pi/n) = \cos(0) = 1$.

Another useful lemma for limits of sequences with <u>signs</u>:

<u>Lemma</u> If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.

E.g. How to compute $\lim_{n\to\infty} \frac{(-1)^n}{n}$? Since $\lim_{n\to\infty} \frac{1}{n} = 0$, we also have that $\lim_{n\to\infty} (-1)^n/n = 0$.

Compare this to $a_n = (-1)^n$, which <u>diverges</u>!

One of the most important kind of sequences are the sequences $a_n = r^n$ for some fixed number $r \in \mathbb{R}$.

When does this sequence converge?

We have seen in Calc I that for $0 < r < 1$,

$$\lim_{x\to\infty} (r^x) = 0 \quad (\text{think: } \lim_{x\to\infty} (\tfrac{1}{2})^x = 0)$$

So $\lim_{n\to\infty} r^n = 0$ for $0 < r < 1$ too.

By the absolute value lemma, $\lim_{n\to\infty} r^n = 0$ when $-1 < r < 0$ as well.

Also, clearly $\lim_{n\to\infty} 1^n = \lim_{n\to\infty} 1 = 1$. But other $r$ diverge!

<u>Theorem</u>

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \text{does not exist} & \text{for all other } r. \end{cases}$$

## Monotone and bounded sequences §11.1

**Def'n** The sequence $\{a_n\}$ is increasing if $a_n < a_{n+1}$ for all $n \geq 1$, and decreasing if $a_n > a_{n+1}$ for all $n \geq 1$. It is called monotone if it is either increasing or decreasing.

E.g. The sequence $a_n = n$ is increasing (hence monotone). The sequence $a_n = (-1)^n$ is neither increasing nor decreasing.

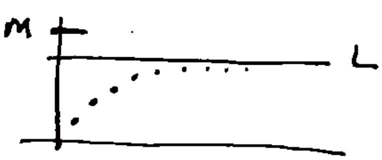
**Def'n** $\{a_n\}$ is bounded above if there is some $M$ such that $a_n < M$ for all $n \geq 1$, it is bounded below if there is $M$ such that $a_n > M$ for all $n \geq 1$, and it is bounded if it is both bounded above and below.

E.g. $a_n = (-1)^n$ is bounded (above by 2 and below by $-2$), but $a_n = n$ is unbounded since it goes off to $\infty$.

Clearly a sequence which is unbounded (like $a_n = n$) cannot be convergent. Some bounded sequences, (like $a_n = (-1)^n$), are also divergent. But if our sequence is both bounded and monotone, then it must converge:

**Thm** (Monotone Sequence Theorem) Every bounded, monotone (either increasing or decreasing) sequence converges.

Picture. proof




$\leftarrow$ an increasing sequence bounded by $M$ will converge to an $L$ with $L \leq M$.

E.g. $a_n = \frac{1}{n}$ is bounded and monotone (decreasing) so it converges, as we were already aware.

**Exercise** Use the Monotone Convergence Theorem to explain why $a_n = \frac{n}{n+1}$ converges (which we also knew...)

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**Series** §11.2

A series is basically an "infinite sum."
If we have an (infinite) sequence $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \dots\}$
the corresponding series is

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots$$

An infinite sum like this does not always make sense:

$$\sum_{n=1}^{\infty} n = 1+2+3+4+5+\dots = \text{"}\infty\text{"}$$

But sometimes we can sum $\infty$-many terms & get a finite number:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = ???$$

Well, $\frac{1}{2} = 0.5$, $\frac{1}{2} + \frac{1}{4} = 0.75$, $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 0.875$,
and it seems that if we add up more and more terms,
we don't go off to $\infty$, but instead get closer and closer to $\underline{\underline{1}}$.

**Def'n** For series $\sum_{n=1}^{\infty} a_n$, the associated partial sums
are $S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \dots + a_n$ for $n \geq 1$.
If $\lim_{n\to\infty} S_n = L$ then we write $\sum_{n=1}^{\infty} a_n = L$ and we
say the series converges. Otherwise, it diverges.

key idea: $\boxed{\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} (a_1 + a_2 + \dots + a_n)}$

E.g.: Let $a_n = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$. What is $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$?

Well, $S_n = \underbrace{\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right)}_{a_1} + \underbrace{\left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right)}_{a_2} + \cancel{\dots} + \underbrace{\left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right)}_{a_{n-1}} + \underbrace{\left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)}_{a_n}$

$= 1 - \frac{1}{n+1}$, so that $\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = \underline{\underline{1}}$.

Thus, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \underline{\underline{1}}$.

One of the most important kind of series are the <u>geometric series</u>:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots, \quad \text{for real numbers } a \text{ and } r \neq 0.$$

Notice that
$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$
and
$$r \cdot S_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$$
$$\Rightarrow \quad (1-r) \cdot S_n = a - ar^n$$
$$\Rightarrow \quad S_n = \frac{a - ar^n}{(1-r)}$$

Since $\lim_{n\to\infty} r^n = 0$ for $|r|<1$, we have:

$$\sum_{n=1}^{\infty} ar^{n-1} = \lim_{n\to\infty} \frac{a-ar^n}{1-r} = \frac{a}{1-r} \quad \text{for } |r|<1.$$

<u>important</u> <u>formula</u> <u>to</u> <u>remember</u>: value of geo. series when $|r|<1$.

E.g.: $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ is geo. series with $a = 1/2$ and $r = 1/2$. So $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1/2}{1-1/2} = 1$.

This is what we expected above!

For $|r| \geq 1$, geo. series $\sum_{n=1}^{\infty} ar^{n-1}$ <u>diverges</u>.

Consider in particular the case $a = r = 1$.

Then $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} 1 = 1+1+1+\cdots$, so the partial sums are $S_n = 1+1+\cdots+1 = n$, and $\lim_{n\to\infty} S_n = \infty$.

In general, in order to converge, the terms in a series must <u>approach zero</u>:

<u>Theorem</u> (Divergence Test) If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. So if $\lim_{n\to\infty} a_n \neq 0$, $\sum_{n=1}^{\infty} a_n$ <u>diverges</u>.

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<u>WARNING</u>: The divergence test says that if terms do not go to zero, the series diverges.

But <u>converse</u> <u>does</u> <u>not</u> hold: the terms $a_n$ can go to 0, while the series $\sum_{n=1}^{\infty} a_n$ still <u>diverges</u>.

The most important (counter)example is the <u>harmonic series</u>:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

Of course, $\lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ <u>still</u> <u>diverges</u>.

How to see this? Ignore the 1 at the start, and consider

$$\underbrace{\frac{1}{2}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq 2\cdot\frac{1}{4} = \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq 4\cdot\frac{1}{8} = \frac{1}{2}} + \cdots \quad \cdots$$

The trick, as shown above, is to break the series into chunks consisting of 1, 2, 4, 8, ... terms.
If we add up the terms in each chunk, we get a sum bigger than $\frac{1}{2}$. So overall sum is $\geq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$
But a sum of $\infty$-many $\frac{1}{2}$'s must <u>diverge</u>!
So the harmonic series, which is bigger than that, <u>diverges</u> too.

<u>Theorem</u> (Laws for series)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge.

Then • $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

and • $\sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n$ for any $c \in \mathbb{R}$.

<u>WARNING</u>: $\sum_{n=1}^{\infty} a_n \cdot b_n \neq \left(\sum a_n\right) \cdot \left(\sum_{n=1} b_n\right)$.

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## Integral test for convergence §11.3

We saw a couple series whose convergence we could establish because we had a simple formula for the partial sums. That's not possible for most series. We need tools to study convergence.

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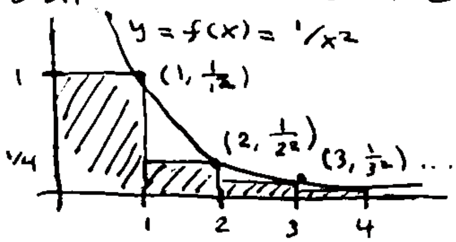
Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. There's no simple formula for its partial sums. But let's draw the following picture:




← plot the sequence $a_n = \frac{1}{n^2}$ and use this to make rectangles of width $= 1$ and height $= a_n$

Notice that the area of the $n^{th}$ rectangle $= a_n \times 1 = a_n$, so the sum of areas $= a_1 + a_2 + a_3 + \dots = \sum_{n=1}^{\infty} a_n$.

Also notice that we plotted the curve $y = f(x) = \frac{1}{x^2}$. The area under $y = f(x)$ from $x = 1$ to $\infty$ is visibly less than $a_2 + a_3 + a_4 + \dots = \left(\sum_{n=1}^{\infty} a_n\right) - a_1$.

But we can compute area under $y = f(x)$ as an improper integral:

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t 1/x^2\,dx = \lim_{t\to\infty} \left[-1/x\right]_1^t = \lim_{t\to\infty} \left(1 - \frac{1}{t}\right) = \underline{\underline{1}}.$$

Thus, $\sum_{n=1}^{\infty} a_n \le a_1 + \int_1^{\infty} 1/x^2\,dx = 1 + 1 = 2$, so in particular this series converges: it has a finite value.

(Since all the terms are positive, if it diverged it would go off to $\infty$. Being bounded means it converges.)

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This way of comparing a series to an associated integral is called the integral test for convergence.

It can be used to establish divergence as well:

<u>Theorem</u> (Integral Test for Convergence / Divergence)
Let $f(x)$ be a continuous, positive, (eventually) decreasing function on $[1,\infty)$, and let $a_n = f(n)$ for all $n \geq 1$.

1) If $\int_1^\infty f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2) If $\int_1^\infty f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

<u>E.g.</u> We saw before that <u>harmonic series</u> $\sum_{n=1}^{\infty} 1/n$ diverges.
We can also prove this using the integral test:

$$\int_1^\infty 1/x \, dx = \lim_{t\to\infty} \int_1^t 1/x \, dx = \lim_{t\to\infty} \left[\ln(x)\right]_1^t = \lim_{t\to\infty} \ln(t) = \underline{\underline{\infty}}.$$

Comparing $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$, a natural question is: for which values of $p$ does the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?
(The book calls these series "p-series".)

<u>Theorem</u> The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$:

- diverges for $p \leq 1$
- converges for $p > 1$.

<u>Pf:</u> First note that if $p \leq 0$ then $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$, so the series diverges by the Test for Divergence.

So suppose $0 < p < 1$. Then $\int \frac{1}{x^p}\,dx = \frac{1}{1-p} x^{1-p}$
So that $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \left[\frac{1}{1-p} x^{1-p}\right]_1^t = \infty$,
So the series diverges by the Integral Test.

We have already seen that $\sum_{n=1}^{\infty} 1/n$ diverges.

So finally assume $p > 1$. Then $\int \frac{1}{x^p}\,dx = \frac{-1}{(p-1)x^{p-1}}$
So that $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \left[\frac{-1}{(p-1)x^{p-1}}\right]_1^t = \frac{1}{p-1}$,
So the series converges by the Integral Test. ∎

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## Estimating Remainders with Integrals §11.3

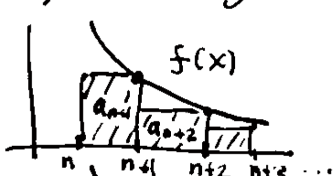
Integrals are useful for establishing convergence of series, but don't tell us the exact value of the series. Still, they can be used to estimate the value of the series.

As above, let $f(x)$ be a continuous, positive, decreasing fn. on $[1,\infty)$ and let $a_n = f(n)$ for all $n \geq 1$. We want to estimate value of series $s = \sum_{n=1}^{\infty} a_n$. A simple estimate for any series is a partial sum $s_n = a_1 + a_2 + \dots + a_n$, for some finite value of $n$. How good of an estimate is $s_n$ for the true value of the series $s$?

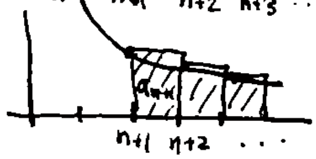
Define the remainder to be $R_n = s - s_n$.

E.g. For $s = \sum_{n=1}^{\infty} \frac{1}{2^n}$, have $s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, and know $s=1$, so $R_2 = \frac{1}{4}$.

By looking at the two pictures below:




⇐ overestimate
$R_n = a_{n+1} + a_{n+2} + \dots \leq \int_n^{\infty} f(x)\,dx$




⇐ underestimate
$R_n = a_{n+1} + a_{n+2} + \dots \geq \int_{n+1}^{\infty} f(x)\,dx$

**Theorem** We have $\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)\,dx$

E.g. For $s = \sum_{n=1}^{\infty} \frac{1}{n^2}$, $s_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \approx 1.42$, and by thm:

$$\int_5^{\infty} \frac{1}{x^2}\,dx \leq R_4 \leq \int_4^{\infty} \frac{1}{x^2}\,dx$$

$$1/5 \leq R_4 \leq 1/4$$

$$0.2 \leq s - 1.42 \leq 0.25$$

$$1.62 \leq s \leq 1.67 \quad \leftarrow \text{good estimate of } \sum_{n=1}^{\infty} \frac{1}{n^2}$$

(In fact, $s = \frac{\pi^2}{6} \approx 1.64\ldots$, but that result is beyond this class...)

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## Comparison Tests for Series §11.4

We know the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges $(|r|<1)$.
The series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ seems very similar, but how can we show it converges or diverges? In fact, we can compare the two series:

**Theorem** (Direct Comparison Test for series)
Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series whose terms are all positive. Then:

1) If $\sum_{n=1}^{\infty} b_n$ converges and $a_n \le b_n$ for all $n$ then $\sum_{n=1}^{\infty} a_n$ converges too.
2) If $\sum_{n=1}^{\infty} b_n$ diverges and $a_n \ge b_n$ for all $n$ then $\sum_{n=1}^{\infty} a_n$ diverges too.

Note: positive terms here is very important!

E.g: Notice that $\frac{1}{2^n+1} \le \frac{1}{2^n}$ for all $n \ge 1$,
(dividing 1 by a bigger number gives something smaller)
So therefore $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ also converges.

E.g: Easy to show that if $\sum_{n=1}^{\infty} a_n$ diverges/converges, then $\sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n$ also diverges/converges for any nonzero scalar $c \in \mathbb{R} \setminus \{0\}$.
So $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ also diverges, like harmonic series.
And then notice $\frac{1}{2n-1} \ge \frac{1}{2n}$ for all $n \ge 1$,
So therefore $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ also diverges by direct comparison.

The series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ also seems very similar to $\sum_{n=1}^{\infty} \frac{1}{2^n}$,
So we expect that it would also converge.
Unfortunately, $\frac{1}{2^n - 1} > \frac{1}{2^n}$ for all $n \geq 1$, wrong direction of inequality to show convergence by direct comparison.
Instead we can use the following:

**Theorem** (Limit Comparison Test for Series)
Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with positive terms.
Suppose $c = \lim_{n \to \infty} \frac{a_n}{b_n}$ exists and $c \neq 0$ and $c \neq \pm\infty$.
Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

E.g. Notice $\lim_{n\to\infty} \frac{1/2^n}{1/(2^n-1)} = \lim_{n\to\infty} \frac{2^n - 1}{2^n} = \lim_{n\to\infty} \frac{1 - \frac{1}{2^n}}{1} = 1$,
So since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, by Limit Comparison $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges too.

E.g. Consider a series like $\sum_{n=1}^{\infty} \frac{3n}{5n^2 + n - 1}$.
How to decide convergence/divergence? Compare to $\sum_{n=1}^{\infty} \frac{1}{n}$:
$\lim_{n\to\infty} \left(\frac{3n}{5n^2+n-1}\right) / \left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{3n^2}{5n^2+n-1} = \frac{3}{5}$, so
by limit comparison, $\sum_{n=1}^{\infty} \frac{3n}{5n^2+n-1}$ also diverges.

Exercise: Show $\sum_{n=1}^{\infty} \frac{3n}{5n^3 + n - 1}$ converges
Hint: Use Limit Comparison to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Key takeaway: For series whose terms are rational functions, check biggest power of n on top vs. on bottom.