

9/18

Permutations and cycles (Stanley §1.3)

Recall $S_n = \text{symmetric group on } n \text{ letters}$
 $= \text{permutations of } [n]$

Notations:

- two-line $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 7 & 1 & 3 & 12 & 2 & 10 & 5 & 4 & 11 & 6 & 9 & 8 \end{pmatrix}$
- one-line $\sigma = (7, 1, 3, 12, 2, 10, 5, 4, 11, 6, 9, 8)$

"directed graph" → digraph: $\sigma = \begin{array}{c} 1 \rightarrow 7 \\ \downarrow \\ 2 \leftarrow 5 \end{array} \quad \begin{array}{c} 3 \\ \uparrow \\ 8 \end{array} \quad \begin{array}{c} 4 \rightarrow 12 \\ \downarrow \\ 10 \end{array} \quad \begin{array}{c} 6 \\ \uparrow \\ 11 \end{array} \quad \begin{array}{c} 7 \\ \uparrow \\ 9 \end{array}$

- functional
- cycle notation: $\sigma = (1752) (3) (4128) (610) (911) = (8412) (106) (527) (3) (119) = \dots = (3) (7521) (106) (119) (1284)$

Standard form:

- each cycle has its biggest element first
- cycles appear w/ biggest elements increasing left-to-right.

Q: How many $\sigma \in S_n$ of cycle type $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$

e.g. $n=4$

$$= 1^{c_1} 2^{c_2} 3^{c_3} \dots$$

$$\begin{aligned} \lambda &= 1^4 = \boxed{}(a)(b)(c)(d) & 1 \\ 2^1 1^2 &= \boxed{}(ab)(cd) & \binom{4}{2} = 6 & \text{multiplicity notation:} \\ 2^2 &= \boxed{}(ab)(cd) \circ \binom{4}{2}/2 = 6/2 = 3 & \text{e.g. } \lambda = (5, 5, 5, 3, 2, 2, 2, 1, 1) \\ 3^1 1^1 &= \boxed{}(abc)(d) & 2! \cdot \binom{4}{3} = 2 \cdot 4 = 8 & = 1^2 2^4 3^1 4^0 5^5 \\ 4^1 &= \boxed{}(abcd) & 3! \cdot \binom{4}{4} = 6 & c_1=2, c_2=4, c_3=1, c_4=0, c_5=3 \end{aligned}$$

Prop: There are $\frac{n!}{1^{c_1} c_1! 2^{c_2} c_2! 3^{c_3} c_3! \dots}$ perms in S_n
of cycle type $\lambda = 1^{c_1} 2^{c_2} 3^{c_3} \dots$

Pf of prop! Recall that S_n acts on the set of perms with cycle type λ transitively, by conjugation:

$$\text{e.g. } \underbrace{(1234567)}_{\sigma} (1234) (567) \underbrace{(abcde fg)}_{\sigma^{-1}} = (abcd) (efg)$$

So the # of such perms = size of the orbit

$$\xrightarrow{\text{orbit stabilizer}} = \frac{|S_n|}{|\mathcal{Z}_{S_n}(\sigma_\lambda)|} \quad \begin{array}{l} \text{if } \sigma_\lambda \text{ is a perm} \\ \text{of cycle type } \lambda, \end{array}$$

where $\mathcal{Z}_{S_n}(\sigma) := \{ \tau \in S_n : \tau \sigma \tau^{-1} = \sigma \}$ is the centralizer
i.e., $\tau \sigma \tau^{-1} = \sigma$ of σ in S_n .

Who centralizes $\sigma_\lambda = \underbrace{(ab)(c)}_{c_1 1\text{-cycles}} \underbrace{(cd)(ef)}_{c_2 2\text{-cycles}} \dots$?

- Products of powers of each cycle: there are $1^{c_1} 2^{c_2} 3^{c_3} \dots$ of these

- Perms that swap two cycles of same size, \therefore there are $c_1! c_2! c_3!$ of these preserving cycle order and biggest element, products of these things = $1^{c_1} c_1! 2^{c_2} c_2! \dots$ many

$$\text{e.g. } (1234)(567)(8910) = \sigma_\lambda$$

is centralized by $\tau = \underbrace{(4321)}_{(1234)^3} \underbrace{(59161078)}_{\text{swap}(567) + (8910)}$

Thus $|\text{orbit}| = \frac{n!}{\prod_{j \geq 1} j^{c_j} c_j!}$, as claimed. \square

NOTE: Stanley presents different (but equivalent) proof by considering standard forms of perms σ_λ .

DEFN: For any subgroup G of \mathfrak{S}_n , define its cycle index (indicator) polynomial to be

$$Z_G(t_1, t_2, \dots) := \frac{1}{|G|} \sum_{\sigma \in G} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \dots$$

where $c_i(\sigma) := \# \text{cycles in } \sigma \text{ of size } i$. $\in \mathbb{C}[t_1, t_2, \dots, t_n]$

COR (Touchard): The cycle indicators $Z_{\mathfrak{S}_n}$ have g.f.

$$\sum_{n=0}^{\infty} Z_{\mathfrak{S}_n}(t) x^n = e^{t_1 \frac{x^1}{1} + t_2 \frac{x^2}{2} + t_3 \frac{x^3}{3} + \dots} = e^{\sum_{j \geq 1} t_j \frac{x^j}{j}}$$

Proof (direct but mysterious... we'll see better pf later)

$$\begin{aligned} e^{t_1 \frac{x^1}{1} + t_2 \frac{x^2}{2} + \dots} &= e^{\sum_{i \geq 1} t_i \frac{x^i}{i}} = e^{\sum_{i \geq 1} \frac{(t_i x^i)^{c_i}}{c_i!}} \\ &= \left(\sum_{c_1 \geq 0} \frac{(t_1 x^1)^{c_1}}{c_1!} \right) \left(\sum_{c_2 \geq 0} \frac{(t_2 x^2)^{c_2}}{c_2!} \right) \dots \\ &= \sum_{(c_1, c_2, \dots)} x^{1 \cdot c_1 + 2 \cdot c_2 + \dots} \frac{t_1^{c_1} t_2^{c_2} \dots}{c_1! c_2! \dots} \end{aligned}$$

$$= \sum_{n \geq 0} x^n \frac{1}{n!} \sum_{(c_1, c_2, \dots)} \frac{n!}{c_1! c_2! \dots} t_1^{c_1} t_2^{c_2} \dots$$

$$\sum j c_j = n$$

$$\Rightarrow = \#\{\sigma \in \mathfrak{S}_n : \sigma \text{ has } c_j j\text{-cycles}\}$$

$$= \sum_{n \geq 0} x^n \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \dots$$

$$Z_{\mathfrak{S}_n}(t)$$

Touchard's thm has many consequences...

Let's see a few now...

9/20

$$\textcircled{1} \text{ DEF'N } \sum_{k=1}^n c(n,k) t^k := \sum_{\sigma \in S_n} t^{\#\text{cycles}(\sigma)}$$

(signless) Stirling
number of 1st kind

i.e., $c(n,k) := \#\{ \sigma \in S_n : \sigma \text{ has } k \text{ cycles} \}$.

$$\text{Cor (to Touchard)} \quad \sum_{k=1}^n c(n,k) t^k = t(t+1)(t+2)\dots(t+(n-1))$$

Pf: Set $t_1=t_2=\dots=t$ in

$$\sum_{n \geq 0} \frac{x^n}{n!} \sum_{\sigma \in S_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \dots = e^{t_1 \frac{x^1}{1} + t_2 \frac{x^2}{2} + t_3 \frac{x^3}{3} + \dots}$$

$$\sum_{n \geq 0} \frac{x^n}{n!} \underbrace{\sum_{\sigma \in S_n} t^{\#\text{cycles}(\sigma)}}_{\sum_{k=1}^n c(n,k)t^k} = e^{(x^1 + \frac{x^2}{2} + \frac{x^3}{3} + \dots)}$$

$$= e^{(-\log(1-x))}$$

$$= (1-x)^{-t}$$

$$= \sum_{n \geq 0} \binom{-t}{n} (-x)^n$$

$$= \sum_{n \geq 0} \binom{t+n-1}{n} x^n$$

$$= \frac{t(t+1)(t+2)\dots(t+(n-1))}{n!}$$

Pf follows by comparing coeff's of $\frac{x^n}{n!}$ ◻

Aside on posets: We've mentioned posets, but let's formally introduce them...

A (finite) poset (P, \leq) is a finite set P together with

a partial order \leq on elem's of P :

- (reflexive) $x \leq x \quad \forall x \in P$

- (antisymmetric) $x \leq y \text{ and } y \leq x \Rightarrow x=y \quad \forall x, y \in P$

- (transitive) $x \leq y \text{ and } y \leq z \Rightarrow x \leq z \quad \forall x, y, z \in P$

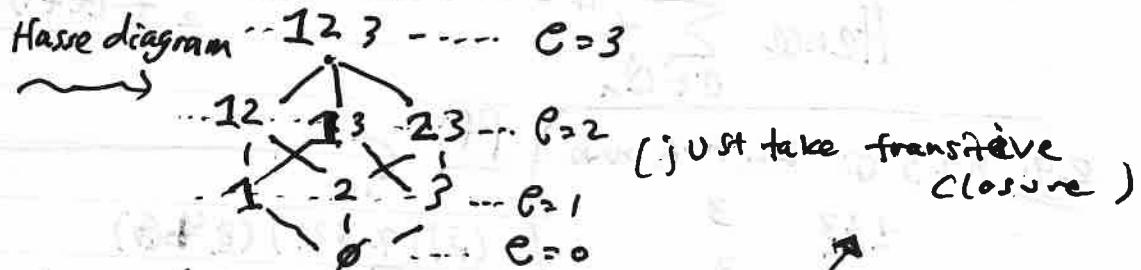
Say y covers x in P , denoted $x < y$, if $x < y$ and there is no $z \in P$ with $x < z < y$.

drawn in the plane!

The Hasse diagram of P is the graph with elements P and with an edge $x \rightarrow y$, and x below y , iff $x \leq y$.

E.g.,

recall $B_3 = \{\text{subsets of } \{1, 2, 3\}, \leq \text{ Boolean lattice}\}$



A finite poset is determined by its Hasse diagram.

DEF'N A chain in P is a totally ordered subset $p_1 < p_2 < \dots < p_m$ of elem's in P . Say P is graded if all maximal chains have the same length. In this case, $\text{length} = m-1$

- \exists unique rank function $c: P \rightarrow \{0, 1, 2, \dots\}$ s.t.
 - $c(x) = 0$ if x is minimal in P
 - $c(y) = c(x) + 1$ if $x \leq y$.

We saw that the binomial coeffs $\binom{n}{k}$ give the rank sizes of a poset: the Boolean lattice B_n . Same is true of Stirling #'s of 1st kind $C(n, k)$:

e.g., $n=3$

$$\begin{aligned} t(t+1)(t+2) &= t^3 + 3t^2 + 2t \\ &= \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} \\ &\quad \text{rank sizes of } B_3 \text{ (graded)} \end{aligned}$$

$c(3, 1) = 2$

$c(3, 2) = 3$

$c(3, 3) = 1$

Partial order on Q_n^V called absolute order (Q_n^V, \leq_{abs})

defined by $\sigma \leq_{abs} \tau$ if $\tau = \sigma \cdot (i, j)$ for some i, j

OTB: Hasse diagram
 $f(Q_n^V, \leq_{abs}) = \text{Cayley graph of } Q_n^V \text{ w/ transpositions as generating set}$

and # cycles $(\tau) = H(\text{cycles}(\sigma)) - 1$
 (as opposed to $H(\text{cycles}(\tau)) = H(\text{cycles}(\sigma)) + 1$, the other possibility for $\tau = \sigma \cdot (i, j)$)

Prop!
her rmk: The map $\Omega_n \rightarrow \Omega_n$ put σ in standard form
 $\sigma \mapsto \hat{\sigma}^2$ and erase parentheses.
 is a bijection, w/ $\# \text{cycles}(\sigma) = \# \text{L-to-R Maxima in } \hat{\sigma}$.
 Hence $\sum_{\sigma \in \Omega_n} t^{\# \text{L-to-R maxima}(\sigma)} = t(t+1) \cdots (t+(n-1))$

g. $n=3 \sigma \# \text{L-to-R max}$

$\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}$
--	---

Pf: $\sigma \xrightarrow{\quad} \hat{\sigma}$

$(3)(\underline{7} \ 5 \ 2 \ 1) (\underline{8} \ 4 \ 6)$	$(3 \ 7 \ 5 \ 2 \ 1 \ 8 \ 4 \ 6)$
--	-----------------------------------

is reversible; just put (before L-to-R maxima, and put) right before the (and at the end.

(2) (or of Touchard) Can compute $E_k(n) = \frac{\text{expected # of } k\text{-cycles}}{\text{in uniformly random } \sigma \in \Omega_n}$

$$E_k(n) = \frac{1}{n!} \sum_{\sigma \in \Omega_n} c_k(\sigma) = \frac{1}{n!} \left[\frac{\partial}{\partial t_k} \sum_{\sigma \in \Omega_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \dots \right]_{t_1=t_2=\dots=1}$$

$$\begin{aligned} \text{So } \sum_{n \geq 0} E_k(n) x^n &= \left[\frac{\partial}{\partial t_k} \sum_{n \geq 0} \frac{x^n}{n!} \sum_{\sigma \in \Omega_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \dots \right]_{t_1=t_2=\dots=1} \\ &= \left[\frac{\partial}{\partial t_k} e^{t_1 \frac{x}{1-x} + t_2 \frac{x^2}{2} + t_3 \frac{x^3}{3} + \dots} \right]_{t_i=1} \\ &= \left[\frac{x^k}{k} e^{t_1 \frac{x}{1-x} + t_2 \frac{x^2}{2} + \dots} \right]_{t_i=1} \\ &= \frac{x^k}{k} e^{\frac{x^1}{1-x} + \frac{x^2}{2} + \dots} = \frac{x^k}{k} e^{-\log(1-x)} = \frac{x^k}{k(1-x)} \\ &= \sum_{n \geq k} \frac{1}{k} x^n \Rightarrow E_k(n) = \begin{cases} 1/k & \text{if } n \geq k, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Note: $E_k(n)$ eventually constant in n . In fact,
 can show $E_k(n)$ converges (as $n \rightarrow \infty$) to a Poisson
random variable w/ expectation $\lambda = 1/k$.

9/23

③ (of Touchard) There are special classes of perms defined by restrictions on their cycle sizes, so all have nice gen.fns.

e.g. no large cycles

$\sigma \in \mathbb{Q}_n$ is an involution ($\sigma^2 = e$)

$\Leftrightarrow \sigma$ has only 1- and 2-cycles ($\sigma = (ab)(cd)\dots(x)(y)(z)$)

$$\text{Hence } \sum_{n \geq 0} \frac{x^n}{n!} \# \{\text{involutions in } \mathbb{Q}_n\} = \left[e^{t_1 \frac{x}{1} + t_2 \frac{x^2}{2} + t_3 \frac{x^3}{3} + \dots} \right] \begin{array}{l} t_1 = t_2 = 1 \\ t_3 = t_4 = \dots = 0 \end{array}$$

$$= e^{x + \frac{x^2}{2}}$$

$$\text{or even } \sum_{n \geq 0} \frac{x^n}{n!} \sum_{\substack{\# 1\text{-cycles } (\sigma) \\ \text{involutions} \\ \sigma \in \mathbb{Q}_n}} = e^{x + \frac{x^2}{2}} \text{ similarly, etc.}$$

What about no small cycles?

DEF'N A derangement $\sigma \in \mathbb{Q}_n$ is a permutation w/ no fixed points, equivalently, w/ $c_1(\sigma) = 0$.

Q: (Derangement / Hat-check problem): 100 people check their hats; the attendant gives people back their hats ^{randomly} ^{completely}; what is the probability that no person gets their own hat back?

i.e., what is $\frac{d_n}{n!}$, where $d_n = \# \{ \sigma \in \mathbb{Q}_n : \sigma \text{ derangement} \}$

$$\sum_{n \geq 0} \frac{x^n}{n!} d_n = \left[e^{t_1 \frac{x}{1} + t_2 \frac{x^2}{2} + \dots} \right] \begin{array}{l} t_1 = 0, t_2 = t_3 = \dots = 1 \end{array}$$

$$= e^{\frac{x^2}{2} + \frac{x^3}{3} + \dots}$$

$$= e^{-\log(1-x)} - \frac{x^1}{1} = \boxed{\frac{e^{-x}}{1-x}}$$

But $\frac{e^{-x}}{1-x} = (1+x+x^2+\dots)(1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\dots)$

 $= \sum_{n \geq 0} x^n \left(\underbrace{\left(-\frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!} \right)}_{\text{consistent w/ } C_1(\mathbb{R}) \rightarrow \text{Poisson w/ mean } \lambda = 1} \right)$
 $\frac{d^n}{n!} e^{-x}$ converges quickly $e^{-1} = \frac{1}{e}$

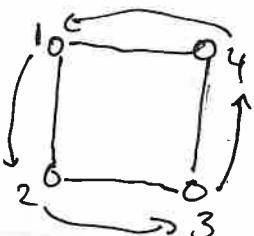
Addition on Polya theory

Recall that we defined the cycle index $Z_G(f) := \left(\sum_{G \in \text{EG}} f_i^{c(G)} \right)_{i \in G}$ for any permutation group $G \subseteq \text{Sym}_n$. Why? Polya theory! = counts G -orbits of colorings of a finite set X (unire $G \setminus X$) with k colors a_1, a_2, \dots, a_k , and more generally studies the pattern inventory

$$\sum_{\substack{\text{G orbits } O \\ \text{of } k\text{-colorings of } X}} q_1^{\# \text{times color } a_1 \text{ is used}} q_2^{\# \text{times color } a_2 \text{ is used}} \dots$$

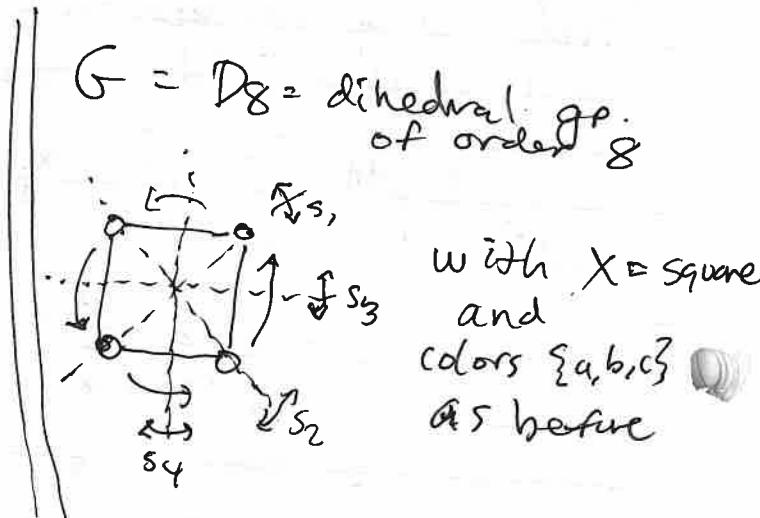
Examples:

$G = C_4 = \text{cyclic group of order 4} = \langle (1234) \rangle$



acting on $X =$ vertices of square
and 3-colorings
via colors $\{a, b, c\}$

$G = D_8 = \text{dihedral gp. of order 8}$



with $X =$ square
and
colors $\{a, b, c\}$
as before

(Note: Colorings with
n-gon up to cyclic symmetry & dptd dihedral symmetries
= "necklaces" = "bracelets")

Pattern inventory for C_4 :

$$\begin{array}{c} \begin{array}{|c|c|c|c|} \hline a-a & a-a & a-a & a-a \\ \hline 1 & 1 & 1 & 1 \\ \hline a-a & a-b & b-b & b-a \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline a-a & a-b & b-c & c-a \\ \hline 1 & 1 & 1 & 1 \\ \hline a-b & b-b & b-a & a-b \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline a-a & a-b & b-c & c-a \\ \hline 1 & 1 & 1 & 1 \\ \hline a-b & b-b & b-a & a-b \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} \\ \begin{array}{l} 4b^4 + c^4 \\ + a^3b \\ + a^3c \\ + a^2b^3 + ac^3 \\ + ab^3 + b^3c \end{array} \quad \begin{array}{l} + 2a^2b^2 + 2a^2c^2 \\ + 2b^2c^2 \\ + 3a^2bc \\ + 3ab^2c \\ + 3abc^2 \end{array} \end{array}$$

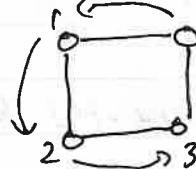
Pattern inventory for D_8 :

$$\begin{array}{c} \begin{array}{|c|c|c|c|} \hline a-a & a-a & a-b & a-b \\ \hline 1 & 1 & 1 & 1 \\ \hline a-a & a-b & b-b & b-a \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline a-a & a-b & b-c & c-a \\ \hline 1 & 1 & 1 & 1 \\ \hline a-b & b-b & b-a & c-b \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} \\ \begin{array}{l} q^4 + b^4 + c^4 + \dots \\ + 2q^2bc \\ + 2ab^2c \\ + 2abc^2 \end{array} \end{array}$$

Same as for C_4

Theorem (Polya) The # of G-orbits of k-colorings of X is $\frac{1}{|G|} \sum_{\sigma \in G} k^{\# \text{cycles}}(\sigma)$
and the pattern inventory is $\left[\frac{1}{|G|} \sum_{\sigma \in G} t_1^{G(\sigma)} t_2^{G(\sigma)} \dots \right]_{T_G(t)}$ $t_j = a_1^j + a_2^j + \dots + a_k^j$

EXAMPLES

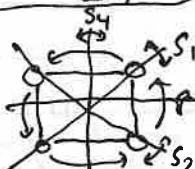
① $G = C_4$ =  $r = \{e, r, r^2, r^3\}$

$$Z_G(t)_{(\text{Polya})} = \frac{1}{4} (t_1^4 + t_1^4 + t_2^4 + t_3^4) = \frac{1}{4} (t_1^4 + t_2^2 + 2t_4)$$

$$t_j = a^j + b^j + c^j$$

Pattern inventory = $\frac{1}{4} ((a+b+c)^4 + (a^2+b^2+c^2)^2 + 2(a^4+b^4+c^4))$ $\# \{ \begin{matrix} a-a \\ b-b \\ c-c \end{matrix} \}$

colorings w/ 2a's = $\frac{1}{4} ((2,1,1) + 0 + 0) = \frac{1}{4} \frac{4!}{2!1!1!} = 3$

② $G = D_8$ =  $r, r^2, r^3, s_1, s_2, s_3, s_4$

$$Z_G(t) = \frac{1}{8} (t_1^4 + t_4 + t_2^2 + t_4 + t_2 + t_1^2 + t_2 + t_1^2 + t_2^2 + t_2^2)$$

$$= \frac{1}{8} (t_1^4 + 3t_2^2 + 2t_4 + 2t_2 + t_1^2)$$

$$t_j = a^j + b^j + c^j$$

Pattern inventory = $\frac{1}{8} ((a+b+c)^4 + 3(a^2+b^2+c^2)^2 + 2(a^4+b^4+c^4) + 2(a^2+b^2+c^2)(a+b+c)^2)$ $\# \{ \begin{matrix} a-a \\ b-b \\ c-c \end{matrix} \}$

colorings w/ 2a's, 1b, 1c = $\frac{1}{8} ((1,2,1) + 0 + 0 + 2 \cdot 2) = \frac{1}{8} \left(\frac{4!}{2!1!1!} + 4 \right) = \frac{1}{8} (12 + 4) = 2$

9/25

Proof of Polya's thm The main result behind it is:

Burnside's Lemma For a group G of permutations of a finite set X , # G -orbits Ω on $X = \frac{1}{|G|} \sum_{g \in G} \#\{x \in X : g(x) = x\}$.

$$\begin{aligned} \text{Pf: } \sum_{g \in G} \#\{x \in X : g(x) = x\} &= \#\{(r, x) \in G \times X : r(x) = x\} \\ &\geq \sum_{x \in X} \#\{g \in G : g(x) = x\} \quad \text{Stabilizer of } x \\ &= \sum_{\substack{\text{G-orbits} \\ \Omega \text{ on } X}} \sum_{x \in \Omega} |G_x| \quad \text{orbit stabilizer lemma} \\ &= |\Omega| \cdot |G| \quad \text{i.e. } |\Omega| = \frac{|G|}{|G_x|} = [G : G_x] \end{aligned}$$

When G permutes X , it also permutes k -colorings of X ,

and $\{f \in G \text{ fixes a } k\text{-coloring} \iff \text{the } k\text{-coloring is constant within cycles of } f\}$

$$\text{e.g. } X = \boxed{\text{graph 1}} \boxed{\text{graph 2}} \boxed{\text{graph 3}} \boxed{\text{graph 4}} \quad \left| \begin{array}{cccc} a & b & c & \\ \begin{matrix} a \\ a \end{matrix} & \begin{matrix} b \\ b \end{matrix} & \begin{matrix} c \\ c \end{matrix} & \\ \begin{matrix} a \\ a \end{matrix} & \begin{matrix} b \\ b \end{matrix} & \begin{matrix} c \\ c \end{matrix} & \\ \begin{matrix} a \\ a \end{matrix} & \begin{matrix} b \\ b \end{matrix} & \begin{matrix} c \\ c \end{matrix} & \\ \begin{matrix} a \\ a \end{matrix} & \begin{matrix} b \\ b \end{matrix} & \begin{matrix} c \\ c \end{matrix} & \end{array} \right| \dots = (a^3 + b^3 + c^3). \\ \text{B}(k\text{-coloring fixed by } f)$$

$$\text{Hence } \sum_{\substack{k\text{-colorings} \\ \text{fixed by } f}} q_1^{\# \text{color 1 used}} q_2^{\# \text{color 2 used}} \dots = \prod_{\substack{\text{cycles} \\ C \text{ of } f}} (q_1^{|C_1|} + q_2^{|C_2|} + \dots + q_k^{|C_k|}) = \left[\frac{q_1^{c_1(f)}}{t_1} + \frac{q_2^{c_2(f)}}{t_2} \dots \right]_{t_j = q_1^{j_1} + q_2^{j_2} + \dots + q_k^{j_k}}$$

$$\begin{aligned} \text{Hence pattern inventory} &= \sum_{\substack{\text{G-orbits} \\ \Omega \text{ of colorings}}} a^{\# \text{colorings in } \Omega} \\ &= \sum_{\underline{c} = (c_1, \dots, c_k)} a^{\underline{c}} \cdot \#\{\text{G-orbits } \Omega\} \quad \text{using } \underline{c} \\ &= \sum_{\underline{c}} a^{\underline{c}} \frac{1}{|G|} \sum_{g \in G} \#\{\text{k-colorings using } \underline{c} \text{ fixed by } g\} \\ &= \left[\frac{1}{|G|} \sum_{g \in G} \left[\frac{q_1^{c_1(g)}}{t_1} + \frac{q_2^{c_2(g)}}{t_2} \dots \right] \right]_{t_j = q_1^{j_1} + q_2^{j_2} + \dots + q_k^{j_k}} \end{aligned}$$

Some theory of ordinary generating functions (Ardila §2.2)

Roughly speaking, if \mathcal{A} is some class of combinatorial structures, w/ $a_n = \#$ (weighted?) \mathcal{A} -structures of wt/size $n \in R$ (a com. ring w/ \mathbb{Z}), then we can form the ordinary generating function $A(x) = \sum_{n \geq 0} a_n x^n \in R[[x]]$.

Prop • If \mathcal{C} structures of size n are a choice of either
an \mathcal{A} - or \mathcal{B} -structure of size n (" $\mathcal{C} = \mathcal{A} + \mathcal{B}$ ")
(i.e., $c_n = a_n + b_n$)
then $C(x) = A(x) + B(x)$.

• If \mathcal{C} -structures of size n are a choice of
- an \mathcal{A} -structure of size i
- a \mathcal{B} -structure of size j
for some $i+j = n$ (" $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ ")

$$\text{(i.e. } c_n = \sum_{\substack{i+j=n \\ i,j \geq 0}} a_i b_j \text{)}$$

then $C(x) = A(x) \cdot B(x)$.

• If \mathcal{C} -structures of size n are a choice of
 \mathcal{B} -structures of sizes i_1, i_2, \dots, i_k for some $i_1 + i_2 + \dots + i_k = n$
 \downarrow
(i.e., $c_n = \sum_{\substack{(i_1, i_2, \dots, i_k), \\ \sum i_j = n, \\ i_j \geq 0}} b_{i_1} b_{i_2} \dots b_{i_k}$) (" $\mathcal{C} = \text{Seq}(\mathcal{B})$ ")

$$\text{then } C(x) = \frac{1}{1 - B(x)},$$

Pf is straightforward. Let's see several examples
of how to apply this proposition --

EXAMPLES (see also Andréa § 2.2.2) (so $\lambda_i \leq k$ vi)

(Partitions w/ bounded part size)

① Let $P_k(n) := \#\{\text{partitions } \lambda \vdash n \text{ w/ } \lambda_i \leq k\}$

$(\lambda_1 \geq \lambda_2 \geq \dots) \downarrow \begin{array}{l} \text{in bijection via} \\ \text{conjugation of partitions} \end{array}$

$= \#\{\text{partitions } \lambda \vdash n \text{ w/ } \ell(\lambda) \leq k\} \quad \lambda \leftrightarrow \lambda^+$

Then $P_k(q) = \sum_{n \geq 0} P_k(n) q^n \left(= \sum_{\substack{\lambda: \lambda_i \leq k \\ \ell(\lambda) \leq k}} q^{|\lambda|} = \sum_{\substack{\lambda: \\ \ell(\lambda) \leq k}} q^{|\lambda|} \right)$

[Flip Young diagram along main diagonal]

example of conjugation:

$\lambda = \begin{matrix} & & 3 \\ & & 2 \\ & & 2 \\ & & 1 \end{matrix} \quad \longleftrightarrow \quad \lambda^+ = \begin{matrix} 4 \\ 3 \\ 2 \\ 1 \end{matrix}$

O.g.f. for λ w/ only parts of size 1

O.g.f. for λ w/ only parts of size 2

w/ only parts of size K

R ↗ we've seen the $K \rightarrow \infty$ limit of this earlier

i.e. $C = \{\lambda \text{ w/ } \lambda_i \leq k\} = \text{Seq(Ones)} \times \text{Seq(Twos)} \times \dots \times \text{Seq(K's)}$

Similarly, $\sum_{\substack{\lambda: \\ \lambda_i \leq k}} q^{|\lambda|} t^{\ell(\lambda)} = \frac{1}{(1-q)(1-q^2)\dots(1-q^k)} \left(= \sum_{\lambda: \ell(\lambda) \leq k} q^{|\lambda|} t^{\lambda_1}\right)$

2 (Compositions)

DEF'N A composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of n , denoted $\alpha \models n$, is a sequence of positive integers $\alpha_i \in \{1, 2, \dots\}$ w/ $\alpha_1 + \alpha_2 + \dots + \alpha_k = n$. (Unlike partitions, no requirement that α_i weakly decrease).

As w/ partitions, α_i are called the parts of α .

Prop # compositions $\alpha \models n$ of n into k parts = $\binom{n-1}{k-1}$.

Df: Say $\alpha = (\alpha_1, \dots, \alpha_k)$ is a weak composition of n if
 α_i are nonnegative integers $\alpha_i \in \{0, 1, 2, \dots\}$ w/ $\alpha_1 + \alpha_2 + \dots + \alpha_k = n$.

weak compositions of n into k parts = $\binom{k}{n}$ (recall
 $= \# \text{size } n \text{ multisets of } \{k\}$)

$$(\alpha_1, \alpha_2, \dots, \alpha_k) \longleftrightarrow \left\{ \underbrace{1, 1, \dots, 1}_{\alpha_1}, \underbrace{2, 2, \dots, 2}_{\alpha_2}, \dots, \underbrace{k, k, \dots, k}_{\alpha_k} \right\}$$

also,
weak compositions of n into k parts = # (usual) compositions
of $(n+k)$ into k parts

$$(\alpha_1, \alpha_2, \dots, \alpha_k) \longleftrightarrow (\alpha_1 + 1, \alpha_2 + 1, \dots, \alpha_k + 1)$$

$$\text{Hence } \# \text{ comp. of } n \text{ into } k \text{ parts} = \# \text{ weak comp. of } (n+k) \text{ into } k \text{ parts} = \binom{k}{n+k} = \binom{k+n-k-1}{n-k} = \binom{n-1}{n-k} = \binom{n-1}{k-1}$$

$$\text{Thus, total } \# \text{ compositions of } n = \sum_{k=1}^n \# \text{ comp. of } n \text{ into } k \text{ parts} = \sum_{k=1}^n \binom{n-1}{k-1} = 2^{n-1} \quad \text{(for } n=0, \text{ get } 1 \text{ instead)}$$

Let's see a different way to get this using O.g.f.'s:

$$\text{Let } a_n := \# \{ \text{comp. } \alpha = (\alpha_1, \dots, \alpha_k) \vdash n \text{ of any length } k \} \quad (\text{convention } \alpha_0 = 1)$$

$$\begin{aligned} \text{Then } A(x) &= \sum_{n \geq 0} a_n x^n = \frac{1}{1 - \underbrace{(x + x^2 + x^3 + \dots)}_{\substack{\text{o.g.f. for} \\ \text{compositions of} \\ n w/ one part}}} = \frac{1}{1 - \frac{x}{1-x}} \\ &\quad \text{"A = Seq(one-part compositions)"} \\ &= \frac{1-x}{1-2x} \\ &\quad \text{since 3 unique} \\ &\quad \text{comp. of } n \\ &\quad \text{into one part} \end{aligned}$$

$$= 1 + \sum_{n \geq 1} 2^{n-1} x^n \quad \checkmark$$

(compositions into odd parts)

③ What about $a_n = \{\text{compositions of } n \text{ into odd parts}\}$?

n	$\{\text{Fn w/ odd parts}\}$	#Exs
0	{}	1
1	{1}	1
2	{1+1}	1
3	{3, 1+1}	2
4	{3+1, 1+3, 1+1+1+1}	3
5	{5, 3+1+1, 1+3+1, 1+1+3, 1+1+1+1+1}	5
6	~	8

Guess $a_n = \begin{cases} 1 & n=0 \\ F_{n+1} & n \geq 1 \end{cases}$ Recall Fibonacci #

and indeed

$$A(x) = \sum_{n \geq 0} a_n x^n = \frac{1}{1 - (x^1 + x^3 + x^5 + \dots)}$$

" $A = \text{Seq}(\text{one odd part compositions})$ " = $\frac{1}{1 - \frac{x}{1-x^2}} = \frac{1-x^2}{1-x^2-x}$

saw this earlier

$$= 1 + \frac{x}{1-x-x^2} = 1 + \sum_{n \geq 1} F_{n+1} x^n$$

④ Stirling #'s of the 2nd kind

DEFN A set partition of $[n]$ is a set $\Pi = \{\Pi_1, \Pi_2, \dots, \Pi_k\}$ of

Subsets $\Pi_i \subseteq [n]$ s.t.

- (nonempty) $\Pi_i \neq \emptyset \forall i$
- (disjoint) $\Pi_i \cap \Pi_j = \emptyset \forall i \neq j$
- (covering) $\bigcup \Pi_i = [n]$.

The Π_i are called the blocks of the set partition Π .

DEFN $S(n, k) := \# \text{ set partitions of } [n] \text{ into exactly } k \text{ blocks}$
 "Stirling #'s of 2nd kind."

$S(n, k) = \text{rank sizes for the poset } (\Pi_n, \leq)$

R(EF ASIDE: Π_n is the lattice of intersections
 the Braided hyperplane arrangement: $\{\{x_i = x_j\}; 1 \leq i, j\}$)

$\{\text{all set partitions of } [n]\}$ refinement,
 i.e. $\Pi_1 \leq \Pi_2 \iff \Pi_1$ refines Π_2

$n=1$	$n=2$	$n=3$	$n=4$
1 $S(1,1)=1$	12 $S(2,1)=1$ 12 $S(2,2)=1$	123 $S(3,1)=1$ 123, 132, 231 $S(3,2)=3$ 1234, 1243, 1324, 1342, 2314, 2341, 2413, 2431, 3124, 3142, 3214, 3241, 3412, 3421, 4123, 4132, 4213, 4231, 4312, 4321 $S(4,2)=6$	1234 $S(4,1)=1$ 1234, 1243, 1324, 1342, 2314, 2341, 2413, 2431, 3124, 3142, 3214, 3241, 3412, 3421, 4123, 4132, 4213, 4231, 4312, 4321, 12345, 12354, 12435, 12453, 12534, 12543, 13245, 13254, 13425, 13452, 13524, 13542, 14235, 14253, 14325, 14352, 14523, 14532, 21345, 21354, 21435, 21453, 21534, 21543, 23145, 23154, 23415, 23451, 23514, 23541, 24135, 24153, 24315, 24351, 24513, 24531, 31245, 31254, 31425, 31452, 31524, 31542, 32145, 32154, 32415, 32451, 32514, 32541, 34125, 34152, 34215, 34251, 34512, 34521, 41235, 41253, 41325, 41352, 41523, 41532, 42135, 42153, 42315, 42351, 42513, 42531, 43125, 43152, 43215, 43251, 43512, 43521, 51234, 51243, 51324, 51342, 51423, 51432, 52134, 52143, 52314, 52341, 52413, 52431, 53124, 53142, 53214, 53241, 53412, 53421, 54123, 54132, 54213, 54231, 54312, 54321, 123456, 123546, 124356, 124536, 125346, 125436, 132456, 132546, 134256, 134526, 135246, 135426, 142356, 142536, 143256, 143526, 145236, 145326, 213456, 213546, 214356, 214536, 215346, 215436, 231456, 231546, 234156, 234516, 235146, 235416, 241356, 241536, 243156, 243516, 245136, 245316, 312456, 312546, 314256, 314526, 315246, 315426, 321456, 321546, 324156, 324516, 325146, 325416, 341256, 341526, 342156, 342516, 345126, 345216, 412356, 412536, 413256, 413526, 415236, 415326, 421356, 421536, 423156, 423516, 425136, 425316, 431256, 431526, 432156, 432516, 435126, 435216, 512346, 512436, 513246, 513426, 514236, 514326, 521346, 521436, 523146, 523416, 524136, 524316, 531246, 531426, 532146, 532416, 534126, 534216, 541236, 541326, 542136, 542316, 543126, 543216, 123457, 123547, 124357, 124537, 125347, 125437, 132457, 132547, 134257, 134527, 135247, 135427, 142357, 142537, 143257, 143527, 145237, 145327, 213457, 213547, 214357, 214537, 215347, 215437, 231457, 231547, 234157, 234517, 235147, 235417, 241357, 241537, 243157, 243517, 245137, 245317, 312457, 312547, 314257, 314527, 315247, 315427, 321457, 321547, 324157, 324517, 325147, 325417, 341257, 341527, 342157, 342517, 345127, 345217, 412357, 412537, 413257, 413527, 415237, 415327, 421357, 421537, 423157, 423517, 425137, 425317, 431257, 431527, 432157, 432517, 435127, 435217, 512347, 512437, 513247, 513427, 514237, 514327, 521347, 521437, 523147, 523417, 524137, 524317, 531247, 531427, 532147, 532417, 534127, 534217, 541237, 541327, 542137, 542317, 543127, 543217, 123467, 1235467, 1243567, 1245367, 1253467, 1254367, 1324567, 1325467, 1342567, 1345267, 1352467, 1354267, 1423567, 1425367, 1432567, 1435267, 1452367, 1453267, 2134567, 2135467, 2143567, 2145367, 2153467, 2154367, 2314567, 2315467, 2341567, 2345167, 2351467, 2354167, 2413567, 2415367, 2431567, 2435167, 2451367, 2453167, 3124567, 3125467, 3142567, 3145267, 3152467, 3154267, 3214567, 3215467, 3241567, 3245167, 3251467, 3254167, 3412567, 3415267, 3421567, 3425167, 3451267, 3452167, 4123567, 4125367, 4132567, 4135267, 4152367, 4153267, 4213567, 4215367, 4231567, 4235167, 4251367, 4253167, 4312567, 4315267, 4321567, 4325167, 4351267, 4352167, 5123467, 5124367, 5132467, 5134267, 5142367, 5143267, 5213467, 5214367, 5231467, 5234167, 5241367, 5243167, 5312467, 5314267, 5321467, 5324167, 5341267, 5342167, 5412367, 5413267, 5421367, 5423167, 5431267, 5432167, 123476, 1235476, 1243576, 1245376, 1253476, 1254376, 1324576, 1325476, 1342576, 1345276, 1352476, 1354276, 1423576, 1425376, 1432576, 1435276, 1452376, 1453276, 2134576, 2135476, 2143576, 2145376, 2153476, 2154376, 2314576, 2315476, 2341576, 2345176, 2351476, 2354176, 2413576, 2415376, 2431576, 2435176, 2451376, 2453176, 3124576, 3125476, 3142576, 3145276, 3152476, 3154276, 3214576, 3215476, 3241576, 3245176, 3251476, 3254176, 3412576, 3415276, 3421576, 3425176, 3451276, 3452176, 4123576, 4125376, 4132576, 4135276, 4152376, 4153276, 4213576, 4215376, 4231576, 4235176, 4251376, 4253176, 4312576, 4315276, 4321576, 4325176, 4351276, 4352176, 5123476, 5124376, 5132476, 5134276, 5142376, 5143276, 5213476, 5214376, 5231476, 5234176, 5241376, 5243176, 5312476, 5314276, 5321476, 5324176, 5341276, 5342176, 5412376, 5413276, 5421376, 5423176, 5431276, 5432176, 123477, 1235477, 1243577, 1245377, 1253477, 1254377, 1324577, 1325477, 1342577, 1345277, 1352477, 1354277, 1423577, 1425377, 1432577, 1435277, 1452377, 1453277, 2134577, 2135477, 2143577, 2145377, 2153477, 2154377, 2314577, 2315477, 2341577, 2345177, 2351477, 2354177, 2413577, 2415377, 2431577, 2435177, 2451377, 2453177, 3124577, 3125477, 3142577, 3145277, 3152477, 3154277, 3214577, 3215477, 3241577, 3245177, 3251477, 3254177, 3412577, 3415277, 3421577, 3425177, 3451277, 3452177, 4123577, 4125377, 4132577, 4135277, 4152377, 4153277, 4213577, 4215377, 4231577, 4235177, 4251377, 4253177, 4312577, 4315277, 4321577, 4325177, 4351277, 4352177, 5123477, 5124377, 5132477, 5134277, 5142377, 5143277, 5213477, 5214377, 5231477, 5234177, 5241377, 5243177, 5312477, 5314277, 5321477, 5324177, 5341277, 5342177, 5412377, 5413277, 5421377, 5423177, 5431277, 5432177, 123478, 1235478, 1243578, 1245378, 1253478, 1254378, 1324578, 1325478, 1342578, 1345278, 1352478, 1354278, 1423578, 1425378, 1432578, 1435278, 1452378, 1453278, 2134578, 2135478, 2143578, 2145378, 2153478, 2154378, 2314578, 2315478, 2341578, 2345178, 2351478, 2354178, 2413578, 2415378, 2431578, 2435178, 2451378, 2453178, 3124578, 3125478, 3142578, 3145278, 3152478, 3154278, 3214578, 3215478, 3241578, 3245178, 3251478, 3254178, 3412578, 3415278, 3421578, 3425178, 3451278, 3452178, 4123578, 4125378, 4132578, 4135278, 4152378, 4153278, 4213578, 4215378, 4231578, 4235178, 4251378, 4253178, 4312578, 4315278, 4321578, 4325178, 4351278, 4352178, 5123478, 5124378, 5132478, 5134278, 5142378, 5143278, 5213478, 5214378, 5231478, 5234178, 5241378, 5243178, 5312478, 5314278, 5321478, 5324178, 5341278, 5342178, 5412378, 5413278, 5421378, 5423178, 5431278, 5432178, 123479, 1235479, 1243579, 1245379, 1253479, 1254379, 1324579, 1325479, 1342579, 1345279, 1352479, 1354279, 1423579, 1425379, 1432579, 1435279, 1452379, 1453279, 2134579, 2135479, 2143579, 2145379, 2153479, 2154379, 2314579, 2315479, 2341579, 2345179, 2351479, 2354179, 2413579, 2415379, 2431579, 2435179, 2451379, 2453179, 3124579, 3125479, 3142579, 3145279, 3152479, 3154279, 3214579, 3215479, 3241579, 3245179, 3251479, 3254179, 3412579, 3415279, 3421579, 3425179, 3451279, 3452179, 4123579, 4125379, 4132579, 4135279, 4152379, 4153279, 4213579, 4215379, 4231579, 4235179, 4251379, 4253179, 4312579, 4315279, 4321579, 4325179, 4351279, 4352179, 5123479, 5124379, 5132479, 5134279, 5142379, 5143279, 5213479, 5214379, 5231479, 5234179, 5241379, 5243179, 5312479, 5314279, 5321479, 5324179, 5341279, 5342179, 5412379, 5413279, 5421379, 5423179, 5431279, 5432179, 1234710, 12354710, 12435710, 12453710, 12534710, 12543710, 13245710, 13254710, 13425710, 13452710, 13524710, 13542710, 14235710, 14253710, 14325710, 14352710, 14523710, 14532710, 21345710, 21354710, 21435710, 21453710, 21534710, 21543710, 23145710, 23154710, 23415710, 23451710, 23514710, 23541710, 24135710, 24153710, 24315710, 24351710, 24513710, 24531710, 31245710, 31254710, 31425710, 31452710, 31524710, 31542710, 32145710, 32154710, 32415710, 32451710, 32514710, 32541710, 34125710, 34152710, 34215710, 34251710, 34512710, 34521710, 41235710, 41253710, 41325710, 41352710, 41523710, 41532710, 42135710, 42153710, 42315710, 42351710, 42513710, 42531710, 43125710, 43152710, 43215710, 43251710, 43512710, 43521710, 51234710, 51243710, 51324710, 51342710, 51423710, 51432710, 52134710, 52143710, 52314710, 52341710, 52413710, 52431710, 53124710, 53142710, 53214710, 53241710, 53412710, 53421710, 54123710, 54132710, 54213710, 54231710, 54312710, 54321710

Table of $S(n, k)$:

$n \setminus k$	1	2	3	4
1	1	0	0	0
2	1	1	0	0
3	1	3	1	0
4	1	7	6	1

(Pascal-like) Recurrence for $S(n, k)$:

$$S(n, k) = S(n-1, k-1) + \underbrace{k S(n-1, k)}_{\substack{n \text{ is a singleton} \\ n \text{ goes into} \\ \text{block}} \underbrace{\text{one of the} \\ k \text{ other blocks}}} \text{ for } k > 1$$

w/ initial conditions $S(n, 1) = 1 \forall n$
and $S(0, 0) = 1$

9/30

Let's study the o.g.f. $F_k(x) = \sum_{n \geq 0} S(n, k) x^n$ in 2 ways.

(a) Solve recurrence: for $k \geq 2$

$$\sum_{n \geq 0} S(n, k) x^n = \sum_{n \geq 0} S(n-1, k-1) x^n + \sum_{n \geq 0} k S(n-1, k) x^n$$

$$F_k(x) = x F_{k-1}(x) + k x F_k(x)$$

$$(1-kx) F_k(x) = x F_{k-1}(x)$$

$$\boxed{F_k(x) = \frac{x}{1-kx} F_{k-1}(x)}$$

(And for $k=1$, $F_1(x) = \sum_{n \geq 0} S(n, 1) x^n = x + x^2 + x^3 + \dots = \frac{x}{1-x}$)

$$\Rightarrow F_k(x) = \frac{x}{1-kx} \cdot \frac{x}{1-(k-1)x} \cdots \frac{x}{1-2x} \cdot \frac{x}{1-x} = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$$

(b) Let A_m := the structure of strings of letters from $[m]$ that start w/ an m , whose size is their length (e.g. $m=3$)

Prop: $\left\{ \begin{array}{l} \text{Set partitions} \\ \text{of } [n] \text{ w/ } k \text{ blocks} \end{array} \right\} \xleftrightarrow{\text{bijection}} \left\{ \begin{array}{l} \text{total size } n \\ \text{structures in } A_1 x A_2 x \cdots x A_k \end{array} \right\}$

$\frac{3}{size 3} \frac{1}{size 1} \frac{2}{size 2}$ or $\frac{3}{size 3} \frac{3}{size 3} \frac{1}{size 1}$

the restricted growth function $[n] \xrightarrow{f} [k]$
associated to π

e.g.,
 $n=6$

$k=4$

$1, 2, 4, 5, 8, 12 | 3, 6, 9, 10 | 11, 16 | 13, 15$

number the blocks of π (1), (2), ..., (k)
according to increasing smaller elements

$f(i) = \begin{cases} 1 & i \in A_1 \\ 2 & i \in A_2 \\ 3 & i \in A_3 \\ 4 & i \in A_4 \end{cases}$

$f(i) := \text{block # of } i$

Pf: Exercise. \square

Cor $F_k(x) = \frac{x}{1-x} \cdot \frac{x}{1-2x} \cdots \frac{x}{1-kx}$

" " " "
 $x + x^2 + x^3 + \dots$
" " " "
 $x + 2x^2 + 4x^3 + \dots$

$\frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$

" " " "
 $x + kx^2 + k^2x^3 + \dots$

Digression on the two kinds of Stirling #'s

How are $S(n, k)$ and $C(n, k) \stackrel{\text{def}}{=} \#\{\sigma \in \mathfrak{S}_n : \# \text{cycles}(\sigma) = k\}$ related?

of 2nd kind (signless) Stirling #'s
of 1st kind

(a) The $C(n, k)$ satisfy a similar recurrence:

$$C(n, k) = \underbrace{C(n-1, k-1)}_{n \text{ goes in } 1\text{-cycle}} + \underbrace{(n-1) C(n-1, k)}_{n \text{ maps to some } i \in [n-1]}$$

$n \backslash k$	1	2	3	4
1	1	0	0	0
2	1	1	0	0
3	2	3	1	0
4	6	11	6	1

(b) They are rank #'s for posets w/ an order+rank-preserving surjection between them:

$(\mathfrak{S}_3, \leq_{abs})$

$$\begin{aligned} C(3, 1) &= 2 & (123) &\xrightarrow{\quad} (132) \\ C(3, 2) &= 3 & (12) &\xrightarrow{\quad} (3) & (13)(2) &\xrightarrow{\quad} (23)(1) \\ C(3, 3) &= 1 & && (11)(21)(3) & \end{aligned}$$

$(\Pi_3, \leq_{refinement})$

$$\begin{array}{ccccccc} & 123 & & & & \Sigma(3,1)=1 & \\ & 123 & 132 & 231 & \xrightarrow{\quad} & \Sigma(3,2)=3 & \\ & 1 & 1 & 1 & & \Sigma(3,3)=1 & \end{array}$$

$\sigma \xrightarrow{\quad} \pi = \text{cycles of } \sigma \text{ (as sets)}$

(a) + (b) are more superficial. The real relation between $S(n, k)$ & $C(n, k)$ is ...

(c) Prop. (i) $x^n = \sum_{k=1}^n S(n, k) (x)_k$ where $(x)_k := x(x-1)(x-2)\cdots(x-(k-1))$

while (ii) $(x)_n = \sum_{k=1}^n (-1)^{n-k} C(n, k) x^k$

$(= (\text{signed}) \text{ Stirling #'s of 1st kind})$

and hence (iii) the infinite (uni)lower triangular matrices

$$(S(n, k))_{\substack{k=0, 1, 2, \dots \\ n=0, 1, 2, \dots}} \text{ and } (C(n, k))_{\substack{k=0, 1, 2, \dots \\ n=0, 1, 2, \dots}}$$

give the inverse change-of-basis matrices between the ordered bases $\{x^n\}_{n=0, 1, 2, \dots}$ and $\{(x)_n\}_{n=0, 1, 2, \dots}$ of $\mathbb{C}[[x]]$.

In particular, (iv) $\sum_{k \geq 1} S(n, k) \delta(k, m) = \delta_{n,m} = \sum_{k \geq 1} S(n, k) S(k, m)$.

Kronecker delta

Pf of prop:

For (i), note that both sides lie in $\mathbb{C}[x]$ (of degree n),] - useful proof
 So it is enough to prove equality holds for $x=1, 2, 3, \dots$ technique,
 (since a polynomial $f(x) \in \mathbb{C}[x]$ that vanishes at $x=1, 2, 3, \dots$ must be $f \equiv 0$)
 LHS-RHS

For $x=1, 2, 3, \dots$

$$x^n = \# \left\{ \begin{array}{l} \text{functions} \\ [n] \xrightarrow{f} [x] \end{array} \right\} = \sum \# \left\{ \begin{array}{l} \text{set of fibers } \{f^{-1}(i)\}_{i \in [x]} \\ \text{set partitions (non-empty)} \\ \pi \end{array} \right\}$$

π of $[n]$

$\xrightarrow{\text{choices of which } i \in [x] \text{ are images of the (non-empty) fibers determined by } \pi}$

$= \sum_{k=1}^n S(n, k) \underbrace{x(x-1)(x-2) \dots (x-(k-1))}_{(x)_k}$

For (ii), recall (1): $x(x+1)(x+2) \dots (x+(n-1)) = \sum_{k=1}^n C(n, k) x^k$

$\downarrow x \mapsto -x$, and multiply by $(-1)^n$

$$x(x-1)(x-2) \dots (x-(n-1)) = \sum_{k=1}^n (-1)^{n-k} C(n, k) x^k$$

Then (iii) + (iv) follow --.



10/2 Back to o.g.s examples ...

(5) Let $a_n := \#\{ \sigma \in G_n : \sigma \text{ is irreducible / indecomposable} \}$, for $n \geq 1$
 i.e. it can't be factored as $\sigma = \sigma_1 \sigma_2 \in \{ \sigma_1, \dots, \sigma_k \} \times \{ \sigma_{k+1}, \dots, \sigma_n \}$

e.g. $\sigma = (135)(2)(4) \in G_5$ is irreducible

$\sigma \in \{ \sigma_1, \dots, \sigma_k \} \times \{ \sigma_{k+1}, \dots, \sigma_n \}$

but $\sigma = \underbrace{(13)}_{\in G_{\{1,2,3\}}} \underbrace{(2)}_{\in G_{\{4,5\}}} \underbrace{(45)}_{\in G_{\{1,2,3,4\}}}$ is not!

Q: How to compute a_n ?

Note: Permutations = Seq (irreducible permutations)

So if we let $A(x) = \sum a_n x^n$

and $B(x) = \sum_{n \geq 0} \frac{n!}{\#\mathfrak{S}_n} x^n$ ($\in \mathbb{C}[x]$ but no radius of convergence!)

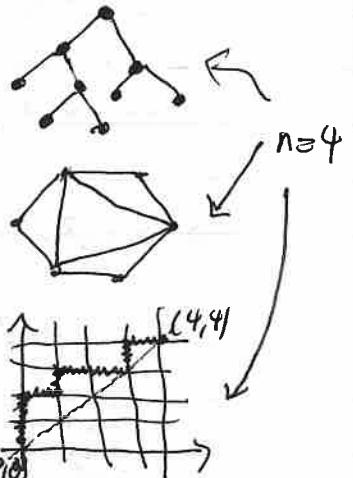
then $B(x) = \frac{1}{1-A(x)}$ and so

$$A(x) = 1 - \frac{1}{B(x)} = 1 - \frac{1}{\sum_{n \geq 0} n! x^n} \stackrel{\text{use some computer algebra software...}}{=} x + x^2 + 3x^3 + 13x^4 + 71x^5 \dots$$

n	σ irreducible in G_n	a_n
1	(1)	1
2	(12) (1345)	1
3	(123), (132), (13)(2)	3
4	...	13

⑥ The Catalan family (See Stanley's other book on this subject!)

$C_n := \text{Catalan number} = \#\left\{ \begin{array}{l} \text{plane binary trees} \\ \text{w/ } n+1 \text{ leaves (or} \\ \text{internal vertices,} \\ \text{each w/ left-right child)} \end{array} \right\}$



$$\begin{aligned} &= \#\{\text{triangulations of } (n+1)\text{-gon}\} \\ &= \#\left\{ \begin{array}{l} \text{lattice paths taking } N, E \\ \text{steps } (0,0) \rightarrow (1,1), \text{ staying} \\ (\text{weakly}) \text{ above diagonal } y=x \end{array} \right\} \end{aligned}$$

Theorem $C_n = \frac{1}{n+1} \binom{2n}{n} \left(= \frac{(2n)!}{(n+1)!n!} = \frac{1}{2n+1} \binom{2n+1}{n} \right)$

n	C_n	Plane binary trees	Triangulations	Lattice paths
0	$1 = \frac{1}{1}(0)$.		
1	$1 = \frac{1}{2}(1)$			
2	$2 = \frac{1}{3}(4)$			
3	$5 = \frac{1}{4}(6)$			
4	$14 = \frac{1}{5}(8)$			

Pf of thm: $C(x) := \sum_{n \geq 0} C_n x^n = 1 + \sum_{n \geq 1} C_n x^n$ fundamental recurrence
 $= 1 + \underbrace{C(x) \cdot x \cdot C(x)}_{\text{if } n \geq 1 \text{ for } i+j=n-1} \quad \text{i.e., } C_n = \sum_{i+j=n-1} C_i C_j$

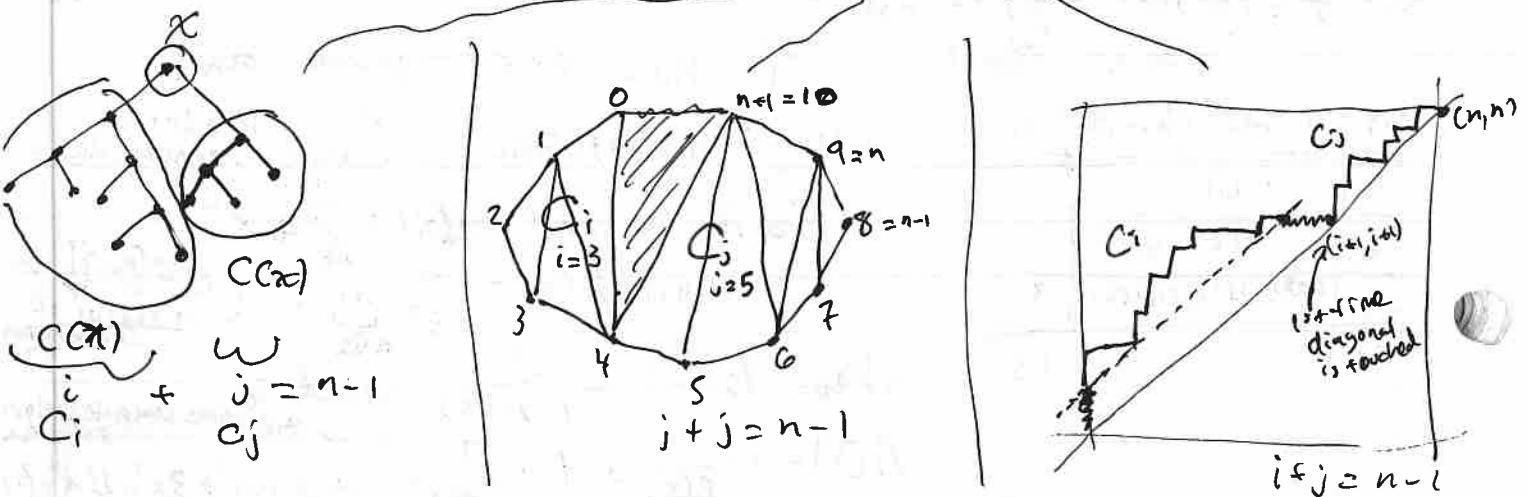


figure out $\pm \sqrt{1-4x}$
by expanding $\sqrt{1-4x}$

$$\text{Consequently, } C(x) = 1 + x(C(x))^2 \Rightarrow 0 = xC(x)^2 - C(x) + 1 \Rightarrow C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

$$\sqrt{1-4x} = (1-4x)^{1/2} = \sum_{n \geq 0} \left(\frac{1}{n}\right) (-4x)^n = \sum_{n \geq 0} \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\cdots\left(\frac{-(2n-3)}{2}\right)}{n!} (-1)^n 4^n x^n$$

$$= 1 - 2 \sum_{n \geq 1} 2^{n-1} \frac{(1)(3)\cdots(2n-3)}{n!} x^n$$

$$= 1 - 2x \sum_{n \geq 1} \frac{(2)(4)\cdots(2n-2)}{(n-1)!} \frac{(1)(3)\cdots(2n-3)}{n!} x^{n-1}$$

$$= 1 - 2x \sum_{n \geq 1} \frac{1}{n} \binom{2n-2}{n-1}$$

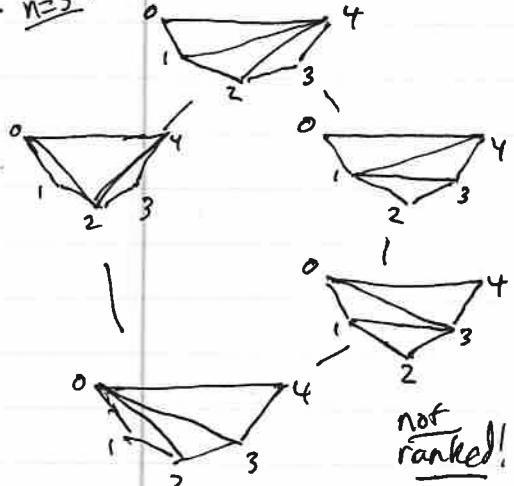
$$\Rightarrow C(x) = \sum_{n \geq 0} C_n x^n = \frac{1 - \sqrt{1-4x}}{2x} = \sum_{n \geq 1} \frac{1}{n} \binom{2n-2}{n-1} x^{n-1} = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} x^n.$$

André:

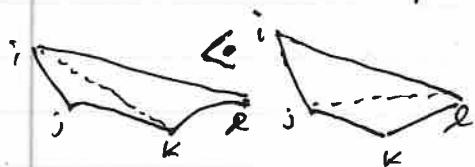
There are (at least) 3 different interesting posets on Catalan objects:

Tamari lattice
on triangulations
of $(n+2)$ -gon

$n=3$:



Cover relations:



Switch ik diagonal to jl diagonal
in quadrilateral

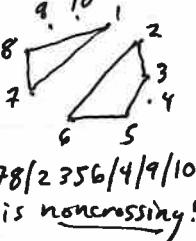
The interval $[0, \frac{1}{n-1}]$ in
Young's lattice of partitions!

$n=3$:



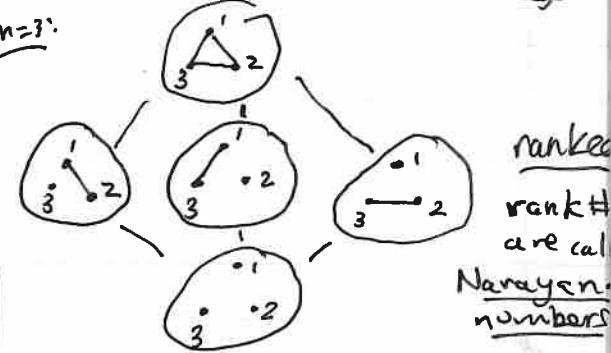
ranked!

$NC(n)$: Noncrossing set partitions of
(w.r.t. $\leq_{\text{refinement}}$)



178/2356/4/9/10
is noncrossing!

$n=3$:



Narayana numbers

NOTE!

$NC(n)$

\cong
in absolute
order on
 $\Pi \mapsto$ orient
cycles of
counter-
clockwise