

## Math 211 (Modern Algebra II), HW# 5,

Spring 2025; Instructor: Sam Hopkins; Due: Wednesday, April 2nd

1. Let  $K$  be a field and consider  $K(x)$ , the field of rational functions in the variable  $x$ , as a (simple, transcendental) extension of  $K$ . On a previous homework, you found some properties of the Galois group  $\text{Aut}_K(K(x))$ . In this problem, you will fully describe  $\text{Aut}_K(K(x))$ .
  - (a) For a rational function  $0 \neq f/g \in K(x)$  with  $f, g \in K[x]$  relatively prime, define its *degree* to be  $\deg(f/g) := \max(\deg(f), \deg(g))$ . Show that  $[K(x) : K(f/g)] = \deg(f/g)$  if  $\deg(f/g) \geq 1$ . **Hint:**  $x$  is a root of the polynomial  $\varphi(y) = (f/g)g(y) - f(y) \in K(f/g)[y]$ ; you may use without proof the fact that this polynomial is irreducible.
  - (b) Let  $f/g \in K(x)$  with  $\deg(f/g) \geq 1$ . Explain why the assignment  $\sigma: x \mapsto f/g$  induces a homomorphism  $\sigma: K(x) \rightarrow K(x)$ , which is an automorphism if and only if  $\deg(f/g) = 1$ .
  - (c) Conclude that  $\text{Aut}_K(K(x))$  consists exactly of the assignments  $x \mapsto (ax + b)/(cx + d)$  with  $a, b, c, d \in K$  and  $ad - bc \neq 0$ . (These are called *fractional linear transformations*, and can be viewed as invertible  $2 \times 2$  matrices with entries in  $K$ .)
2. Let  $K$  be a field,  $L/K$  an extension, and  $S \subseteq L$  a subset that is algebraically independent over  $K$ . Let  $u, v \in L$  with  $v \in S$  and  $u \notin S$ . Suppose that  $u$  is algebraic over  $K(S)$  but that  $u$  is not algebraic over  $K(S \setminus \{v\})$ . Show that  $v$  is algebraic over  $K((S \setminus \{v\}) \cup \{u\})$ . (This is called the *exchange lemma* for transcendence bases.)
3. In this problem, you will explore  $\text{Aut}_{\mathbb{Q}}(\mathbb{C})$ , the field automorphisms of the complex numbers. We already know that two elements of  $\text{Aut}_{\mathbb{Q}}(\mathbb{C})$  are the identity and complex conjugation  $a + bi \mapsto a - bi$ . You will show that there are many other “wild” elements.
  - (a) Show that a transcendence basis of  $\mathbb{C}$  over  $\mathbb{Q}$  is infinite. **Hint:** First, note  $\mathbb{Q}(x_1, \dots, x_n)$  is countable for any finite  $n \geq 1$  (why?). Then you may use the fact, which we did not prove in class but which is in the book, that if  $K$  is an infinite field, the algebraic closure  $\overline{K}$  of  $K$  has the same cardinality as  $K$ . But  $\mathbb{C}$  is uncountable!
  - (b) Let  $S$  be a transcendence basis of  $\mathbb{C}$  over  $\mathbb{Q}$ . Show that any permutation of  $S$  induces an automorphism in  $\text{Aut}_{\mathbb{Q}}(\mathbb{C})$ . **Hint:** First, observe in general that if the set  $S$  is algebraically independent over the field  $K$ , then any permutation of  $S$  induces an automorphism of  $K(S)$ . Then you may use the fact, which we did not prove in class but which is in the book, that if  $L_1/K$  and  $L_2/K$  are two algebraic closures of a field  $K$ , there is an isomorphism  $\varphi: L_1 \rightarrow L_2$  such that  $\varphi$  is the identity when restricted to  $K$ .
  - (c) Conclude that  $\text{Aut}_{\mathbb{Q}}(\mathbb{C})$  is infinite.

(The only automorphisms in  $\text{Aut}_{\mathbb{Q}}(\mathbb{C})$  that are *continuous* with respect to the standard topology on  $\mathbb{C}$  are the identity and complex conjugation. The other “wild” automorphisms are very wild indeed - their existence depends on the axiom of choice!)