

# Howard Math 273, HW# 1,

Fall 2023; Instructor: Sam Hopkins; Due: Friday, September 29th

1. (Stanley, EC1, #1.66) Let  $p_k(n)$  denote the number of partitions of  $n$  into exactly  $k$  parts. Give a **bijective** proof that

$$p_0(n) + p_1(n) + p_2(n) + \cdots + p_k(n) = p_k(n+k).$$

**Hint:** Think about Young diagrams.

2. (Stanley, EC1, #1.5) Show that

$$\sum_{n_1, \dots, n_k \geq 0} \min(n_1, \dots, n_k) x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} = \frac{x_1 x_2 \cdots x_k}{(1-x_1)(1-x_2) \cdots (1-x_k) \cdot (1-x_1 x_2 \cdots x_k)},$$

where  $\min(n_1, \dots, n_k)$  means the minimum of the integers  $n_1, \dots, n_k$ .

3. (Stanley, EC1, #1.26) Let  $\bar{c}(n, m)$  denote the number of compositions of  $n$  into parts of size at most  $m$ . Show that

$$\sum_{n \geq 0} \bar{c}(n, m) x^n = \frac{1-x}{1-2x+x^{m+1}}.$$

4. Prove that, for any  $n \geq 0$ ,

$$4^n = \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k}.$$

**Hint:** We discussed the generating function  $\sum_{n=0}^{\infty} \binom{2n}{n} x^n$  of the central binomial coefficients. How can you use what we proved about this generating function to deduce this result?

5. Let  $n \geq 1$ , and let  $\text{ODD}(n)$  denote the subset of permutations in the symmetric group  $S_n$  with no cycles of even size. Prove that

$$\sum_{\sigma \in \text{ODD}(n)} 2^{\#\text{cycles}(\sigma)} = 2 \cdot n!.$$

**Hint:** Recall that in class we showed

$$\sum_{n \geq 0} \left( \sum_{\sigma \in S_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots t_n^{c_n(\sigma)} \right) \frac{x^n}{n!} = e^{t_1 \frac{x}{1} + t_2 \frac{x^2}{2} + t_3 \frac{x^3}{3} + t_4 \frac{x^4}{4} + \cdots},$$

where  $c_k(\sigma)$  is the number of cycles of  $\sigma$  of size  $k$ . How can you use this generating function to deduce the desired result?