Howard Math 157: Calculus II Spring 2024
Instructor: Sam Hopkins (sam.hopkins@howard.edu)
(call me "Sam")

1/8 Logistics:

Classes   MTWF - 2:10 - 3:00pm in Douglass Hall #215

Office HRS: T 1-2pm in Annex III - #220
or by appointment - email me!

website: samuelfhopkins.com/classes/157.html

Text: Calculus, Early Transcendentals by Stewart, 9e

Grading: 35% (in-person) quizzes
45% three (in-person) midterms
20% (in-person) final exam

There will be 11 in-person quizzes taken on Thursdays
(about 20 mins, we will go over answers in class).
Your lowest 2 scores will be dropped (so $9/11$ count).
The 3 midterms will happen in-class, also on Thursdays.
The final will take place during finals week.
This is an in-person class; all assessments must be taken
in - person!

Beyond that, I will assign additional practice
problems from the book.
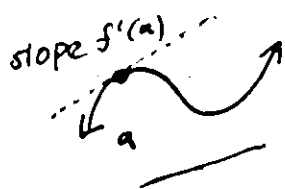and I expect you to SHOW UP TO CLASS
+ PARTICIPATE! ☺

which means ASK QUESTIONS!

Overview of the course:

In Calculus I we learned two important and related operations on functions $f(x): \mathbb{R} \rightarrow \mathbb{R}$:
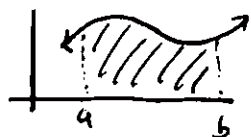- differentiation and • integration

The derivative $f'(a)$ of $f(x)$ at a point $x = a$ is the slope of the tangent to $y = f(x)$ at $(a, f(a))$.

It is also the "instantaneous rate of change" of the function $f(x)$ at $x = a$.

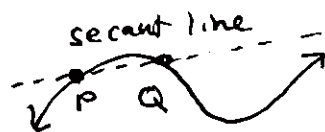The integral $\int_a^b f(x)\, dx$ is the area under the curve $y = f(x)$ from $x = a$ to $x = b$:

$$\text{area}(\text{///}) = \int_a^b f(x)\, dx$$

Both the derivative and integral are formally defined as limits:
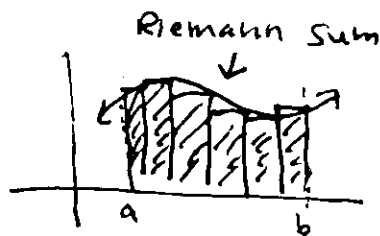
- the derivative is the limit of slopes of secant lines approximating the tangent:
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

secant line

P Q

Riemann sum

- the integral is the limit of Riemann sums (= rectangles) approximating area under curve:
$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

The **Fundamental Theorem of Calculus** says that differentiation and integration are inverse operations:

$$\int_a^b f(x)\, dx = F(b) - F(a),$$
$$\text{where } F'(x) = f(x).$$

In Calculus II we will continue to study derivatives & integrals.
Some of the things we will learn are:

- <u>Applications</u> of <u>integration</u>:
In Calc I we learned many applications of derivatives (minimums & maximums, concavity, etc.)
In Calc II we will learn more things we can compute using integrals (beyond area under curve) like
  - <u>Volumes</u> (3D version of area)
  - <u>lengths</u> (1D version of area)

Also, FTC says that integral represents <u>net change</u>, so we will study some physical applications of integrals like to <u>work</u> (in the sense of <u>force</u>).
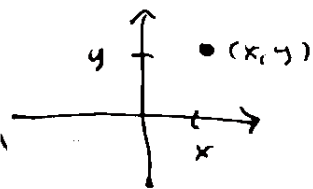
- <u>Techniques for integration</u>:
Using rules for differentiation like <u>product and chain rules</u>, we know how to take the derivative of "any" function,
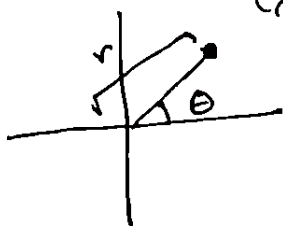 e.g. $d/dx (x \sin(e^{x^2} + 5x - 6))$
But... <u>integrating</u> a "random" function like this can be <u>really hard</u> or <u>not even possible</u>. We will learn more techniques for computing integrals, when possible.
[Recall that we already learned one technique: u-substitution.]

- <u>Polar coordinates</u>: We are used to working with (x,y) a.k.a. "<u>Cartesian coordinates</u>"

vs.

Polar coordinates (r, θ) are a different system where we can also do calculus.

- **Taylor series:**

How do we evaluate a function $f(x)$ at a particular value, e.g. compute $f(1.5)$?

If $f(x)$ is a _polynomial_ like $f(x) = 6x^2 - 2x + 3$ we can use arithmetic: $f(1.5) = 6(1.5)^2 - 2(1.5) + 3 = \ldots$

If it is a _rational_ function like $f(x) = \dfrac{x+1}{x^2 - 2}$

we can use division similarly: $f(1.5) = \dfrac{1.5+1}{(1.5)^2 - 2} = \ldots$

But what about something like $f(x) = \sin(x)$ or $f(x) = e^x$? How to compute $e^{1.5}$? What does your calculator even do?

Even though e^x is _not_ a polynomial, it has a representation as a kind of "infinite" polynomial:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \ldots$$

This is called a _Taylor series_, and it lets us compute things like $e^{1.5}$ (at least approximately).
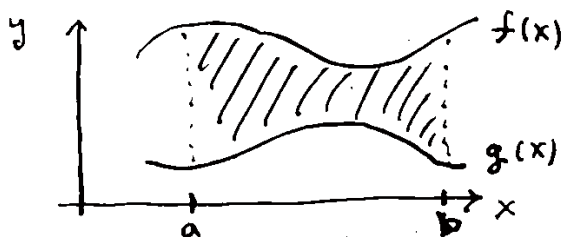
We will learn how to deal with these kind of infinite sums called _series_ (specifically, power series) and related mathematical constructions called _sequences_.

We will also learn Taylor's _theorem_, telling us that the _coefficients_ of the Taylor series can be computed from the _derivative_ of the function (which is where calculus comes in!).
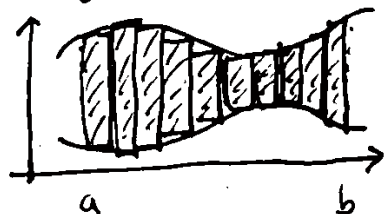
Area between curves (§6.1 of textbook)

The integral computes the area under a curve. What if we have two curves, $y = f(x)$ and $y = g(x)$, and we want to know the <u>area</u> <u>between</u> the curves?

Specifically, suppose that $f(x) \geq g(x)$ for all x in some closed interval from $x = a$ to $x = b$.

Then, as with the integral, we can define the area between the curves on $[a, b]$ by <u>approximating</u> it with a large number of thin <u>rectangles</u>:

Let $\Delta x = \frac{b - a}{n}$ (for some $n \geq 1$) and let $x_i = a + i \cdot \Delta x$ for $i = 0, 1, \ldots, n$ so that $[a, b]$ is divided into n sub-intervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$

For each sub-interval, choose a $x_i^* \in [x_{i-1}, x_i]$, and consider the thin rectangles of <u>width</u> Δx and <u>height</u> $= f(x_i^*) - g(x_i^*)$ ← difference in hts of two curves at $x = x_i^*$
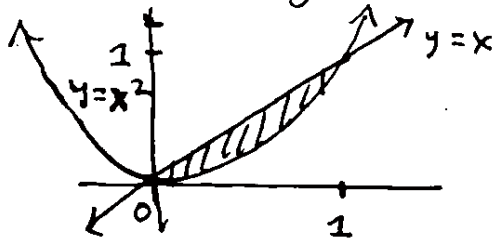
Then area between curves from $x = a$ to $x = b$ $\approx \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*)) \Delta x$

and is exactly $\longrightarrow$ $= \lim_{n \to \infty} \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*)) \Delta x$

$$= \int_a^b f(x) - g(x) \, dx$$

So... area between two curves can be computed as <u>integral</u> of <u>difference</u> function

Note: If we let $g(x) = 0$ be the function corresponding to the x-axis $y = 0$, then we recover the area under the curve as $\int_a^b f(x) \, dx$ from

E.g: Let's compute the area bounded by the curves $y = x$ and $y = x^2$.

Since the problem does not tell us the __bounds__ of integration, let us sketch the curves:

Letting $f(x) = x$, and $g(x) = x^2$, we can find where the curves intersect by setting $f(x) = g(x)$

$$\Rightarrow x = x^2 \Rightarrow x^2 - x = 0$$
$$\Rightarrow x(x-1) = 0$$
$$\Rightarrow x = 0 \text{ or } x = 1$$

Also, choosing $x = \frac{1}{2}$, we see that between $x = 0$ and $x = 1$, $f(x) = \frac{1}{2} \geq g(x) = \frac{1}{4}$, so the curve $y = f(x)$ is __above__ $y = g(x)$ on $[0, 1]$.
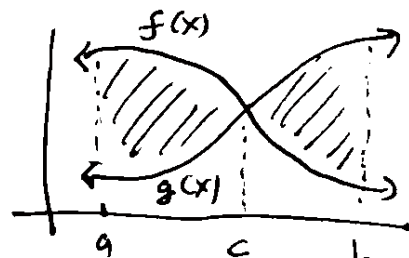
Thus, the area bounded by the curves is
$$\int_a^b f(x) - g(x)\, dx = \int_0^1 x - x^2\, dx = \frac{x^2}{2} - \frac{x^3}{3} \Big]_0^1$$
$$= \left(\frac{1^2}{2} - \frac{1^3}{3}\right) - \left(\frac{0^2}{2} - \frac{0^3}{3}\right) = \frac{1}{2} - \frac{1}{3} = \boxed{\frac{1}{6}}.$$

If on the interval $[a, b]$, sometimes $f(x) > g(x)$ and sometimes $g(x) > f(x)$, then to correctly find area between them, we need to take __absolute value__ of difference:
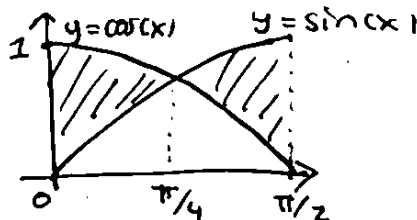$$\text{area between curves} = \int_a^b |f(x) - g(x)|\, dx.$$

In practice, we break up this integral into the parts where $f(x) \geq g(x)$ and where $g(x) \geq f(x)$

$$\Rightarrow \int_a^c f(x) - g(x)\, dx + \int_c^b g(x) - f(x)\, dx$$

E.g: Compute the area between $y = f(x) = \cos(x)$ and $y = g(x) = \sin(x)$ for $x = 0$ to $x = \pi/2$.

Again, good idea to sketch curves to see what's going on:

$\cos(0) = 1 > 0 = \sin(0)$, but $\sin(\pi/2) = 1 > 0 = \cos(\pi/2)$, so which curve is on top changes from $x=0$ to $x = \pi/2$. In fact, have $\cos(\pi/4) = \sin(\pi/4)$ (by symmetry, or isosceles right triangle...)

Thus...

$$\begin{array}{l} \text{area between} \\ y = \cos(x) \text{ and } y = \sin(x) \\ \text{from } x=0 \text{ to } x = \pi/2 \end{array} = \int_0^{\pi/4} \cos(x) - \sin(x)\, dx + \int_{\pi/4}^{\pi/2} \sin(x) - \cos(x)\, dx$$

$$= \sin(x) + \cos(x)\Big]_0^{\pi/4} + -\cos(x) - \sin(x)\Big]_{\pi/4}^{\pi/2}$$

$$= \left(\sin(\pi/4) + \cos(\pi/4) - \sin(0) - \cos(0)\right) + \left(-\cos(\pi/2) - \sin(\pi/2) + \cos(\pi/4) + \sin(\pi/4)\right)$$

$$= \left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}} - 0 - 1\right) + \left(-0 - 1 + \tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}\right) = \boxed{2\sqrt{2} - 2} \; ✓$$

E.g: Sometimes it is easier to __integrate__ w.r.t. y variable. Let's find area between $y = x-1$ and $y^2 = x+1$:

We sketch the curves: they intersect at $y = -1$ and $y = 2$

$x = y^2 - 1 = g(y)$ and $x = y + 1 = f(y)$ set equal $y^2 - 1 = y + 1$ $\Rightarrow$ $y^2 - y - 2 = 0$ $\Rightarrow (y-2)(y+1) = 0$ $\Rightarrow y = 2$ or $y = -1$

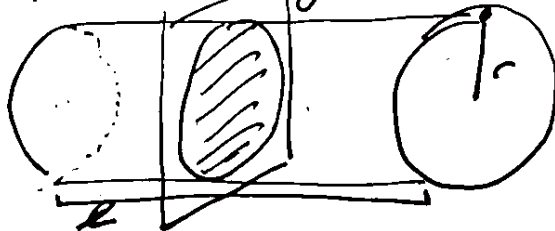Then, since $y = x-1$ is to __right__ of $y^2 = x+1$ for $y = -1$ to $y = 2$:

$$\begin{array}{l} \text{area} \\ \text{between} \\ \text{curves} \end{array} = \int_{-1}^2 f(y) - g(y)\, dy = \int_{-1}^2 (y+1) - (y^2-1)\, dy$$

$$= \int_{-1}^2 -y^2 + y + 2\, dy = \frac{-y^3}{3} + \frac{y^2}{2} + 2y\Big]_{-1}^2 = \left(\frac{-8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right)$$

$$= \boxed{4.5} \; ✓$$

Volumes (§6.2)

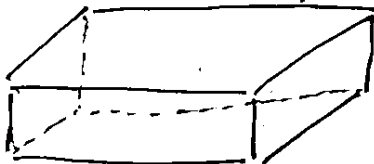Volumes are the 3-dimensional version of areas.
Let's start by considering a circular cylinder:

The cross-section (= intersection w/ y,z-plane) of
this cylinder at any x-coordinate is a circle (of radius r).
We thus define the volume of the cylinder

to be = area of cross-section × length of cylinder

$$= \pi r^2 \cdot \ell.$$

We can also consider cylinders whose cross-sections
are other shapes, e.g., rectangles or triangles:

rectangular prism
(or rectangular cylinder)

triangular cylinder
('Toblerone' bar)

The important thing is that the cylinder has a certain
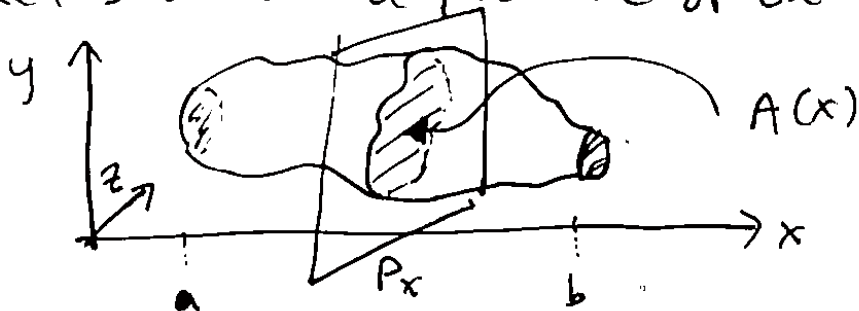length and across the whole length cross-sections are same

Thus, for any cylinder we define

Volume of cylinder = area of cross-section × length.

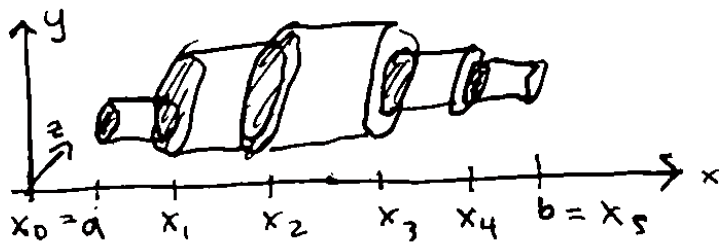E.g.: volume of rectangular prism = width × height × length.

Q: What if the cross-section of our
solid is not constant?

Let's draw a picture of our solid:

$A(x)$

Suppose the solid extends between $x=a$ and $x=b$, and let $A(x)$ for $a \leq x \leq b$ be the <u>area</u> of the cross-<u>section</u> obtained by intersecting with plane P_x perpendicular to x-axis at that point.

We can <u>approximate</u> the volume by dividing the solid into several short cylinders:

← sliced into 5 cylinders

As w/ integral, we break up the interval $[a,b]$ into n sub-intervals $[x_{i-1}, x_i]$, $i=1, \cdots, n$, $x_i = x_{i-1} + \Delta x$

Then the volume of the solid is $\approx \sum\limits_{i=1}^{n} \begin{array}{c} \text{area of cross-section} \\ \text{of } i^{\text{th}} \text{ short cylinder} \end{array} \times \Delta x$

$$= \sum_{i=1}^{n} A(x_i^*) \, \Delta x$$

and is exactly $= \lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} A(x_i^*) \, \Delta x$
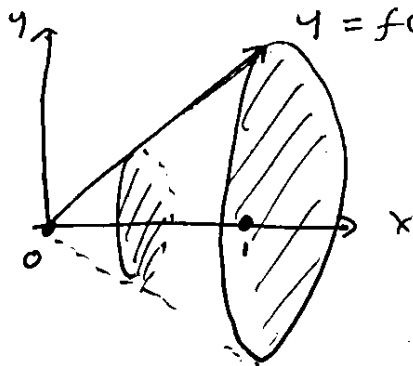
$$\boxed{= \int_a^b A(x) \, dx}$$

This lets us compute volume as an <u>integral</u>!

An important class of solids are the
solids of revolution obtained by
rotating a region in x,y-plane about x-axis;

E.g: Find the volume of the cone obtained by
rotating the area below $y = x$ (and above x-axis)
from $x = 0$ to $x = 1$ about the x-axis.

Sketch:

$y = f(x) = x$

← at any x with $0 \leq x \leq 1$
cross-section of cone
is a circle of
radius $f(x) = x$

Since in this case $A(x) = $ area of circle
of radius $f(x)$

$$= \pi (f(x))^2 = \pi x^2$$

we can use the integral formula for volume to get

$$\text{Volume of cone} = \int_0^1 \pi x^2 dx = \left. \frac{\pi}{3} x^3 \right]_0^1 = \boxed{\frac{\pi}{3}}$$
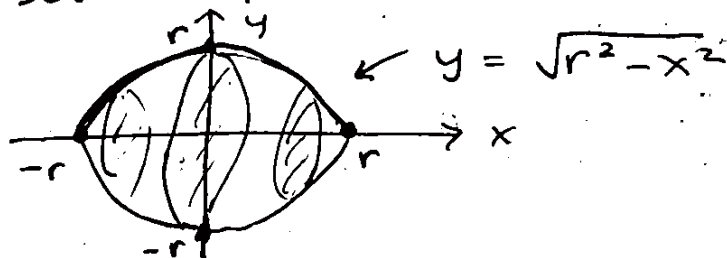
We see that in general the volume of a solid
of revolution obtained by rotating the area
below the curve $y = f(x)$ from $x = a$ to $x = b$
about the x-axis is

$$\boxed{= \int_a^b \pi (f(x))^2 dx}$$

since every cross-section is a circle of radius $= f(x)$

E.g. Find the volume of a _sphere_ of radius r using an integral.

To do this, we have to realize the sphere as a solid of revolution:

$$y = \sqrt{r^2 - x^2}$$

We see that a sphere is obtained by rotating a _semicircle_ of radius r about x-axis, and

$$\text{semicircle of radius } r = \text{area below curve } y = \sqrt{r^2 - x^2} \text{ from } x = -r \text{ to } x = r$$

$$\left(\text{Think:} \qquad \begin{array}{l} y = \sqrt{r^2 - x^2} \\ \text{since } x^2 + y^2 = r^2 \\ \text{by Pythagorean Thm.} \end{array} \right)$$

Thus, according to the formula for volume of a solid of revolution, we have:

$$\begin{aligned}
\text{volume of sphere of radius } r &= \int_{-r}^{r} \pi \left(\sqrt{r^2 - x^2}\right)^2 \, dx \\
&= \pi \int_{-r}^{r} (r^2 - x^2) \, dx \\
&= \pi \left(r^2 x - \frac{x^3}{3} \right]_{-r}^{r} \right) \\
&= \pi \left(\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 - \frac{-r^3}{3}\right) \right) \\
&= \pi \left(2r^3 - \frac{2}{3} r^3 \right) = \boxed{\frac{4}{3} \pi r^3}.
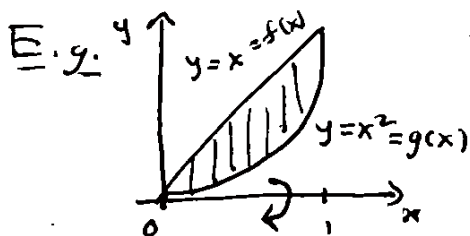\end{aligned}$$

More about volumes §6.2

Solids of revolutions have cross-sections that are circles (or annulus, see below...) but the formula $\int_a^b A(x)\,dx$ for volume works for other shapes too...

E.g.: Let's consider the triangular core (1,0,1) which extends from $x=0$ to $x=1$ and whose cross-section at x is a right isosceles triangle:

← area $= A(x) = \frac{1}{2}$ base × height $= \frac{1}{2}x^2$

Then the volume of this triangular core $= \int_0^1 A(x)\,dx = \int_0^1 \frac{1}{2}x^2\,dx = \frac{1}{2}\cdot\frac{1}{3}x^3\Big]_0^1 = \boxed{\frac{1}{6}}$.
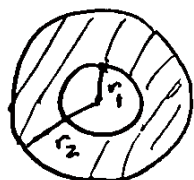
Returning to solids of revolution ... we can also rotate the region between two curves about an axis.

E.g: $y=x=f(x)$, $y=x^2=g(x)$

Let's rotate the region between the curves $y=x$ and $y=x^2$ from $x=0$ to $x=1$ about the x-axis to make a solid.

The cross-section of this solid is an annulus: the region between two circles

"annulus" a.k.a. "washer" shape →

← area of annulus is $\pi(r_2^2 - r_1^2)$

In the case of region between two curves $y=f(x)$ and $y=g(x)$, the area of this cross-section is $A(x) = \pi(f(x)^2 - g(x)^2)$.
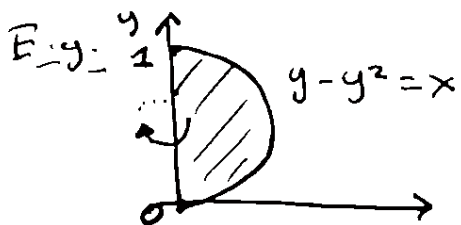
So the volume of the solid is $= \int_a^b \pi(f(x)^2 - g(x)^2)$.

In above example with $f(x) = x$ and $g(x) = x^2$,

we get volume $= \int_0^1 \pi(x^2 - (x^2)^2)\,dx = \int_0^1 \pi(x^2 - x^4)\,dx$

$= \pi\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \boxed{\pi\frac{2}{15}}$
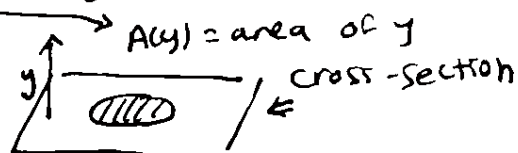
Sometimes we want to rotate across y-axis instead of x-axis.

E.g.

How can we compute the volume of the solid obtained by rotating the region between y-axis and curve $y - y^2 = x$ about the y-axis?

We just do same thing we've been doing, but with respect to y!
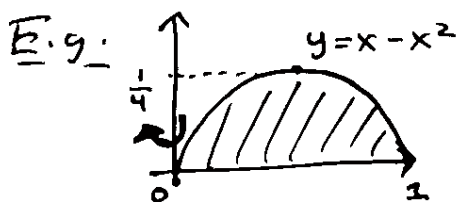
$$\text{volume of solid} = \int_a^b A(y)\, dy$$

$A(y)$ = area of y cross-section

$$= \int_0^1 \pi(y - y^2)^2\, dy$$

since y-cross-section is circle of radius $f(y) = y - y^2$

$$= \int_0^1 \pi(y^2 - 2y^3 + y^4)\, dy$$

$$= \pi\left[\tfrac{1}{3}y^3 - \tfrac{2}{4}y^4 + \tfrac{1}{5}y^5\right]_0^1 = \pi\left(\tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5}\right) = \boxed{\dfrac{\pi}{30}}.$$

What about the following solid of revolution problem?

E.g.

Compute the volume of solid obtained by rotating region below $y = x - x^2$ (and above x-axis) about the y-axis.

To do this following the method above, we would have to realize this region as the region between two curves
- $X = f(y)$ and $X = g(y)$ and integrate w.r.t. y.

(To find $f(y)$ and $g(y)$ we need to "invert" $y = x - x^2$ using the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
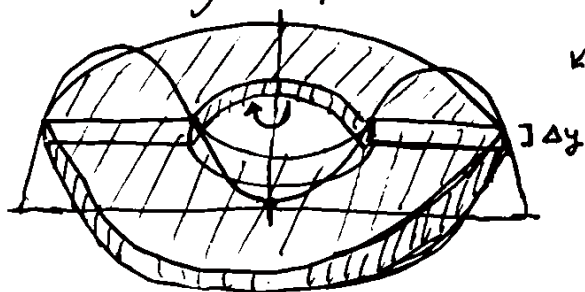
$\Rightarrow f(y) = \dfrac{1 + \sqrt{1 - 4y}}{2}$ and $g(y) = \dfrac{1 - \sqrt{1 - 4y}}{2}$.)

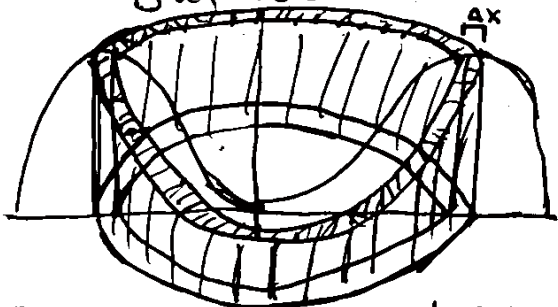But... there is a better approach using integration w.r.t. X

...

The method of cylindrical shells §6.3

To compute the volume of a solid of revolution obtained by rotating the region below $y = f(x)$ about the ~~the~~ _y-axis,_ using the previous method, we break into "thin washers":
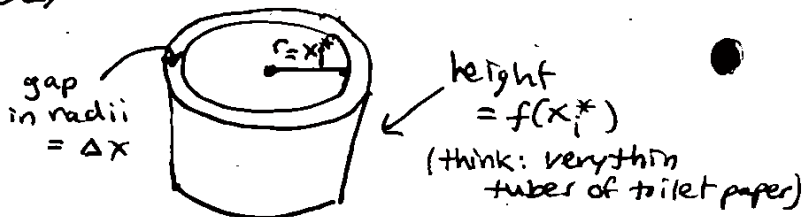
thickness of "washer" = Δy

volume of washer = $\Delta y \times$ area of annulus

$\qquad = \Delta y \cdot \pi(r_2^2 - r_1^2)$

$\Rightarrow \int$ ~~~ dy (integrate w.r.t. y)

But we can also break this solid into hollow_cylindrical_shells:

$\leadsto$ these shells look like:

gap in radii $= \Delta x$

$r = x_i^*$

height $= f(x_i^*)$

(think: very thin tubes of toilet paper)

(please see the text-book for better 3D pictures!)

By breaking the solid into many cylindrical shells in this way we obtain the following formula:

Volume of solid $\approx \sum\limits_{i=1}^{n}$ volume of ith shell

$\qquad = \sum\limits_{i=1}^{n}$ height of ith shell $\cdot$ area of annulus base

$\qquad = \sum\limits_{i=1}^{n} f(x_i^*) \cdot \pi\left((x_i^* + \Delta x)^2 - (x_i^*)^2 \right)$

$\qquad = \sum\limits_{i=1}^{n} f(x_i^*) \cdot 2\pi x_i^* \Delta x + \sum\limits_{i=1}^{n} f(x_i^*) \pi (\Delta x)^2$

and in the limit $n \to \infty$ we get

$\text{Volume} = \lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} f(x_i^*) 2\pi x_i^* \Delta x + \sum\limits_{i=1}^{n} f(x_i^*) \pi (\Delta x)^2$

$\underset{\longrightarrow 0 \text{ as } n \to \infty}{}$

$$\boxed{ = \int_a^b 2\pi f(x) \cdot x \, dx }$$

E.g.

$y = f(x) = x - x^2$

Returning to the example of solid obtained by rotating region below $y = x - x^2$ about y-axis,

its Volume $= \int_a^b 2\pi f(x) x \, dx = \int_0^1 2\pi (x - x^2) x \, dx$

$= 2\pi \int_0^1 x^2 - x^3 dx = 2\pi \left[\frac{1}{3} x^3 - \frac{1}{4} x^4 \right]_0^1 = \boxed{\frac{\pi}{6}}$

Using the "washer" method instead, we would have to compute:

$x = \frac{1 + \sqrt{1-4y}}{2}$

$x = \frac{1 - \sqrt{1-4y}}{2}$

$\text{Volume} = \int_0^{1/4} \pi \left(\left(\frac{1 + \sqrt{1-4y}}{2} \right)^2 - \left(\frac{1 - \sqrt{1-4y}}{2} \right)^2 \right) dy$

$= \cdots = \pi/6$

which involves much harder algebra!

Upshot: Both the "disks/washers" method and the "cylindrical shells" method will work to compute the volume of a solid of revolution, but often one will lead to a much easier integral.

E.g.

$y = f(x)$

for region below curve $y = f(x)$ rotated about x-axis, use "disk/washer" method to get

$$\boxed{\text{volume} = \int_a^b \pi (f(x))^2 \, dx}$$

E.g.

for region below curve $y = f(x)$ rotated about y-axis use "cylindrical shells" method to get

$$\boxed{\text{volume} = \int_a^b 2\pi f(x) \cdot x \, dx}$$

For other kinds of regions... guess or try both methods!

Work §6.4

Intuitively, work is the amount of <u>energy</u> spent accomplishing some task. The formal definition in physics depends on the notion of <u>force</u>.

You can think of <u>force</u> as the push/pull on an object. Its formal definition comes from Newton's 2nd Law:

$$F = ma$$
Force = Mass × Acceleration

E.g. The acceleration <u>due to gravity</u> of an object on Earth is $9.8 \ m/s^2$ (meters per second squared).

So the amount of <u>force</u> that gravity applies to a 10kg object is $10 kg \times 9.8 \ m/s^2 = 98 \ kg \ m/s^2 = 98$ Newtons
this is called <u>weight</u>   SI unit of force $= 98 \ N$

<u>Work</u> is <u>force</u> <u>applied</u> <u>over a</u> <u>distance</u>. Specifically, if an object moves a distance d while experiencing a constant force F (i.e., constant acceleration & mass), we define work done $= F \ d =$ Force × distance.

E.g. What is the work done lifting a 10kg object 100 m in the air? We use the formula:

work = force × distance = $98 \ N \times 100 \ m$

to lift an object we must counteract gravity $= 9800 \ Nm$

SI unit of energy $= 9800$ Joules
$= 9800 \ J$

But what if the object experiences a non-constant force? How do we find work done? We need <u>calculus</u>!

Suppose our object moves from $x = a$ to $x = b$ and at each point x in between experiences force $f(x)$.

As usual, we can <u>approximate</u> the work done breaking the interval $[a, b]$ into sub-intervals $[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$ (of width $\Delta x = \frac{b-a}{n}$) and selecting sample points x_i^* in $[x_{i-1}, x_i]$. The work done moving across the ith sub-interval is

$$W_i \approx \underset{\text{force} \times \text{distance}}{f(x_i^*) \cdot \Delta x}$$
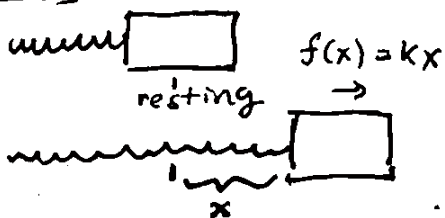
So the total work is then approximately:

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} f(x_i^*) \Delta x.$$

We get an <u>exact</u> value for work as an <u>integral</u>:

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x = \boxed{\int_a^b f(x)\, dx}$$

$$\boxed{\text{Work} = \text{in}\underline{\text{tegr}}\text{al of f}\underline{\text{or}}\text{ce over di}\underline{\text{stan}}\text{ce}}$$

<u>E.g.</u>

$f(x) = kx$

resting

x

Hooke's <u>Law</u> says that the force needed to maintain a <u>spring</u> stretched a distance x from its resting state is given by
$$f(x) = k \cdot x$$
where k is the "<u>spring constant</u>".

Q: Suppose a spring has a spring constant of $k = 10 \frac{N}{m}$. How much work is done stretching this spring $0.5 m$?

A: At a stretch distance of x (meters), we need to apply force
$$f(x) = kx = 10x\ N \text{ by Hooke's Law.}$$

So Work $= \underset{\text{distance}}{\underset{\text{force over}}{\text{integral of}}} = \int_0^{0.5} f(x)\, dx = \int_0^{0.5} 10x\, dx = 10 \tfrac{1}{2} x^2 \Big]_0^{0.5}$

$$= 10 \cdot \tfrac{1}{2} \cdot 0.25 = \boxed{1.25\ J}$$

E.g.: A 100 meter cable hangs off a building. Its weight is 250 Newtons. How much work is done lifting the rope to the top of the building?

Let's show two (related) approaches to this problem:

① Break the cable into n intervals of length $\Delta X = \frac{100}{n}$ m.
Let x_i^* be a point in the ith interval.
All the points in the ith interval must be raised $\approx x_i^*$ meters to bring them to the top.

Since the $\frac{weight}{density}$ of the cable is $\frac{250 \, N}{100 \, m} = 2.5 \, \frac{N}{m}$

the weight of the ith segment is $2.5 \frac{N}{m} \cdot \Delta X \, m$

So total work $W \approx \sum_{i=1}^{n} 2.5 \cdot x_i^* \cdot \Delta X$

and taking limit $n \to \infty$ gives $W = \int_0^{100} 2.5 \, x \, dx$

$$= 2.5 \cdot \frac{1}{2} x^2 \Big]_0^{100} = 2.5 \frac{1}{2} (100)^2$$
$$= \boxed{12500 \, J}$$

② After we have pulled up x meters of the cable, there is $(100-x)$ meters left, and this weighs
$$f(x) = 2.5 \cdot (100-x) \, N.$$
$$\underset{weight \, density}{}$$

Integrating this force over the distance gives:
$$\int_0^{100} 2.5 \, (100-x) \, dx = -\frac{1}{2} 2.5 \, (100-x)^2 \Big]_0^{100}$$

simple u-sub.
to anti-differentiate

$$= \boxed{12500 \, J}$$

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Average value of a function §6.5

To compute the $\underline{average}$ of a finite list $y_1, y_2, \ldots, y_n \in \mathbb{R}$ of real numbers, we add them up and then divide by the number of items in the list:

$$\frac{\text{average}}{\text{value}} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

$\underline{E.g.}$ To compute the average height of a person in this room, we sum the heights of all people and then divide by # of people.

But what about computing: The $\underline{average\ temperature}$ $\underline{during\ a\ day}$. A day has ∞-many times, so we cannot just add all the temperatures and divide.

Instead, we $\underline{approximate}$ by choosing n times to measure temperature at, then let $n \to \infty$.

$\underline{Def'n}$ If $f(x)$ is a continuous function on $[a, b]$, pick some n and let $x_0 = a$, $x_i = x_{i-1} + \Delta x$ for $i = 1, \ldots, n$, where $\Delta x = \frac{b-a}{n}$ as usual. To approximate the average of $f(x)$ on $[a, b]$, we $\underline{sample}$ f at the points $x_1, x_2, \ldots, x_n$ and average these samples:

$$\begin{array}{c}\text{avg. value}\\ \text{of } f(x) \text{ on } [a,b]\end{array} \approx \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$$

And to define average exactly, we let $n \to \infty$:

$$\begin{array}{c}\text{avg. value}\\ \text{of } f(x) \text{ on } [a,b]\end{array} = \lim_{n \to \infty} \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$$

$$= \lim_{n \to \infty} \frac{\sum_{i=1}^{n} f(x_i)}{n} = \frac{1}{b-a} \sum_{i=1}^{n} f(x_i) \Delta x$$

$$\text{since } \Delta x = \frac{b-a}{n}$$

$$\boxed{= \frac{1}{b-a} \int_a^b f(x)\, dx}$$

"average of function on interval $= \dfrac{\text{integral of function on interval}}{\text{length of interval}}$

E.g: Let's compute the average of $f(x) = 1 + x^2$ on $[-1, 2]$.

$$\text{avg.} = \frac{1}{b-a} \int_a^b f(x)\, dx = \frac{1}{2-(-1)} \int_{-1}^2 1 + x^2\, dx$$

$$= \frac{1}{3}\left[x + \frac{1}{3}x^3\right]_{-1}^2 = \frac{1}{3}\left((2 + \frac{8}{3}) - (-1 - \frac{1}{3})\right) = \frac{1}{3} \cdot \frac{18}{3} = \boxed{2}.$$
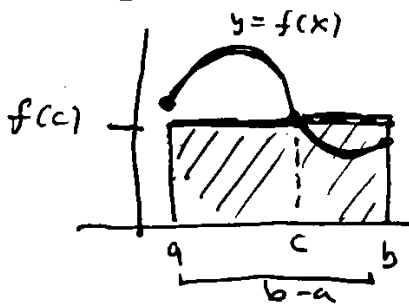
Thm (Mean Value Theorem for Integrals)

If $f(x)$ is a continuous function defined on $[a,b]$, then there exists a point c with $a \leq c \leq b$ s.t.

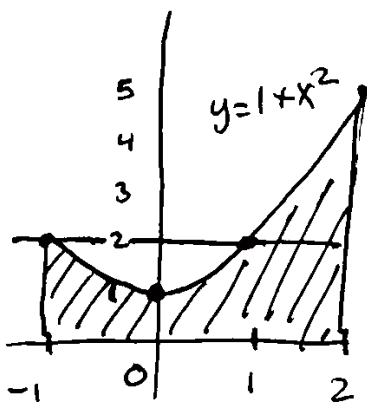$$f(c) = f_{avg} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

MVT for integrals says there is some time during the day when the temperature is exactly the <u>average temperature</u> for that day.

Geometrically:

$\leftarrow$ MVT for integrals says that there is a c in $[a,b]$ s.t. area under curve $y = f(x)$ from $x=a$ to $x=b$ is same as area of <u>rectangle</u> of <u>height</u> $f(c)$ and <u>width</u> $b-a$.

E.g:

Since the average of $f(x) = 1+x^2$ on $[-1,2]$ is $f_{avg} = 2$, MVT for integrals says there is some c in $[-1,2]$ s.t. $f(c) = 2$. Actually, there are <u>two</u> such c's: $c = -1$ and $c = 1$ (since $1 + (-1)^2 = 1 + 1^2 = 2$.) Could solve for c by setting

$$2 = f(c) = 1 + c^2 \implies c = \pm 1.$$