4/13

Longest increasing subsequences

DEF'N Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n$ be a permutation. A subsequence of σ is $\sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k}$ for $i_1 < \cdots < i_k$ and is increasing if $\sigma_{i_1} < \sigma_{i_2} < \cdots < \sigma_{i_k}$.

Let $\mathrm{lis}(\sigma) :=$ length of longest increasing subsequence

e.g. For $\sigma = \underline{2}\,\underline{4}\,7\,9\,\underline{5}\,1\,3\,\underline{6}\,\underline{8}$ have $\mathrm{lis}(\sigma) = 5$ with longest increasing subsequence underlined.

Note: L.I.S. need not be unique: $\underline{1}\,\underline{2}\,\underline{4}\,\underline{3}$

Increasing subsequences are a basic kind of permutation pattern (ask Prof. Burstein for more info...)
Studying LIS's is very natural from point of view of statistical analysis of time series data.

There is a close connection between the Robinson-Schensted algorithm and longest increasing subsequences:

Thm Suppose $\sigma \overset{RS}{\longmapsto} (P,Q)$ w/ $\mathrm{sh}(P) = \lambda = (\lambda_1, \lambda_2, \ldots)$.
Then $\lambda_1 = \mathrm{lis}(\sigma)$.

e.g. $\sigma = 5\,\underline{2}\,\underline{3}\,6\,\underline{4}\,1\,\underline{7} \overset{RS}{\longmapsto} \left(P = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}, Q = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array} \right)$

and indeed $\lambda_1 = 4 = \mathrm{lis}(\sigma)$.

But note: 1st row of P $(= 1\,3\,4\,7)$ is not a LIS of σ (just has same length)

Pf of thm: Suppose $\emptyset = P_0, P_1, \ldots, P_n = P$ is the sequence of insertion tableaux we build up when inserting $\sigma_1, \sigma_2, \ldots, \sigma_n$.

Claim: When inserting σ_k into P_{k-1}, if it enters in the j^{th} column, then the longest increasing subsequence ending at σ_k has length j.

Pf: By induction. The case $k=1$ is fine. So suppose x is entry in P_{k-1} in position $(1, j-1)$ (i.e., left of σ_k). Then by induction there is a subsequence σ' of $\sigma_1, \ldots, \sigma_{k-1}$ of length $j-1$ ending at x, and since $x < \sigma_k$ (or else we would've bumped it), the concatenation $\sigma' \sigma_k$ is a length j increasing subsequence. Similarly, to show there cannot be a longer subsequence, let $y \in \{\sigma_1, \ldots, \sigma_{k-1}\}$ be s.t. $y < \sigma_k$. By induction, when we inserted y we did so at col. with longest subseq. ending at y, call it j'. Cannot have $j' \geq j$, otherwise we would've inserted σ_k into a later column. So $j' < j$, and so longest inc. subseq. ending at σ_k can have length at most $j' + 1 \leq j$. ✓ ▨

What about the whole shape $\lambda = (\lambda_1, \lambda_2, \ldots)$?

Thm (Greene) Suppose $\sigma \overset{RS}{\mapsto} (P, Q)$ w/ $sh(P) = \lambda$. Then for all k, $\lambda_1 + \lambda_2 + \cdots + \lambda_k$ = length of longest subsequence of σ that is a union of k increasing subsequences.

k=2

$5+3=8$ ✓

e.g. w/ $\sigma = \underline{2}\,\underline{4}\,\underline{7}\,\underline{9}\,5\,\underline{1}\,\underline{3}\,\underline{6}\,\underline{8}$ have $P = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 5 & 6 & 8 \\ \hline 2 & 4 & 9 \\ \cline{1-3} 7 \\ \cline{1-1}\end{array}$ and

$2479 \sqcup 1368$ is a union of 2 increasing subsequences.

4/15 Can define decreasing subsequences of perm. σ analogously, and let $\mathrm{lds}(\sigma)$:= length of longest decr. subseq.

Thm If $\sigma \overset{RS}{\mapsto} (P,Q)$ w/ $sh(P) = \lambda$, then $\mathrm{lds}(\sigma) = \ell(\lambda)$ length of λ $(= \lambda^t_1)$

In fact, this follows immediately from ...

Thm* For $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ let $\sigma^{rev} = \sigma_n \sigma_{n-1} \cdots \sigma_1$. Then if $\sigma \overset{RS}{\mapsto} (P,Q)$ have $\sigma^{rev} \mapsto (P', Q')$ where

$P' = P^t$ ← transpose.

To prove this symmetry property of RS, can use column insertion, which works same as (row) insertion, but where we try to put # into 1st column, and bump #'s from i^{th} column to $(i+1)^{st}$ column, etc.

Key Lemma Row and column insertions commute, i.e.,

$$T \xleftarrow[\text{row}]{} a \xleftarrow[\text{col}]{} b = T \xleftarrow[\text{col}]{} b \xleftarrow[\text{row}]{} a.$$

Pf: See Sagan. ▨

Pf of thm*:

$$\begin{aligned} P' &= \sigma_1 \xrightarrow[\text{row}]{} \cdots \sigma_{n-1} \xrightarrow[\text{row}]{} \sigma_n \xrightarrow[\text{row}]{} \emptyset \quad \text{(1st insertion is same w/ row or col)} \\ &= \sigma_1 \xrightarrow[\text{row}]{} \cdots \sigma_{n-1} \xrightarrow[\text{row}]{} \sigma_n \xrightarrow[\text{col}]{} \emptyset \\ &= \sigma_n \xrightarrow[\text{col}]{} \sigma_1 \xrightarrow[\text{row}]{} \cdots \sigma_{n-1} \xrightarrow[\text{row}]{} \emptyset \quad \text{(key lemma)} \\ &= \sigma_n \xrightarrow[\text{col}]{} \sigma_{n-1} \xrightarrow[\text{col}]{} \cdots \sigma_1 \xrightarrow[\text{col}]{} \emptyset \quad \text{(repeat)} \\ &= \left(\sigma_n \xrightarrow[\text{row}]{} \sigma_{n-1} \xrightarrow[\text{row}]{} \cdots \sigma_1 \xrightarrow[\text{row}]{} \emptyset \right)^t \quad \text{(transpose of col insert = row insert)} \\ &= P^t. \ ✓ \end{aligned}$$

▨

Cor (Erdős-Szekeres Theorem)
For any $\sigma \in S_{(n-1)(m-1)+1}$, have either
$\text{lis}(\sigma) \geq n$ or $\text{lds}(\sigma) \geq m$.

Pf: Best way to minimize width and length of a partition
is $\lambda =$ 

but we need one more box ✓ ☒

Q: What is the expected length of longest incr. subseq. of a random permutation?

Let $X_n := \text{lis}(\sigma)$ for $\sigma \in S_n$ (uniformly) random.

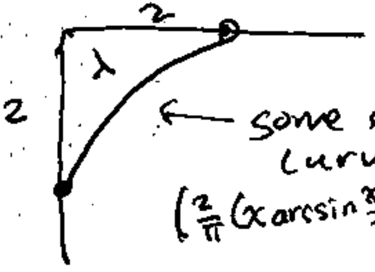
Ulam's Problem: Compute $\lim_{n \to \infty} \frac{\mathbb{E}\, X_n}{\sqrt{n}} = c$.
c. 1960's

E-S Thm says for any $\sigma \in S_n$, have $\text{lis}(\sigma) \geq \sqrt{n}$ or $\text{lis}(\sigma^{\text{rev}}) \geq \sqrt{n}$

so that $c \geq \frac{1}{2}$. In fact...

Thm (Logan-Shepp, Kerov Vershik, 1977)
Solution to Ulam's Problem is $c = 2$.

Idea of pf: Same as asking for length of λ when one insert $\sigma \in S_n$ into RS. In fact, this random partition λ has a precise limit shape (rescaling by $\frac{1}{\sqrt{n}}$):



← some explicit curve
$\left(\frac{2}{\pi}\left(x \arcsin\frac{x}{2} + \sqrt{4-x^2}\right)\right)$ ☒

//

4/18

Representation Theory of finite Groups:

In the last couple days, I want to explain why ring of sym. fn.'s is important in algebra...

DEF'N Let V be an n-dim'l vector space over $\mathbb{C}$.
The general linear group $GL(V) = \{\text{invertible linear maps } V \to V\}$.
I.e., $GL(V) \simeq \{n \times n\ \mathbb{C}\text{-matrices } M \text{ w/ } \det(M) \neq 0\}$.

Note: $GL(V)$ is an infinite group.

Let G be a finite group. We want to "represent" G by matrices.

DEF'N A representation of G is a group homomorphism $\varphi : G \to GL(V)$ for some v.s. V. In other words, for each $g \in G$ we have a matrix $\varphi(g)$, and:

- $\varphi(gh) = \varphi(g) \cdot \varphi(h)\ \forall g, h \in G$,
- $\varphi(e) = I_n$ identity matrix.

A representation of G is very similar to an action, except it is linear: we act by matrices, not permutations.

e.g. For any V and any G, can set $\varphi(g)(v) = v\ \forall v \in V$, i.e., $\varphi(g) = I_n$ identity matrix. This is called the trivial representation and is boring...

e.g. Suppose $G \curvearrowright X$ a finite set. Let $\mathbb{C}[X] := \{\sum_{x \in X} c_x x : c_x \in \mathbb{C}\}$ be v.s. of formal linear combinations of elements of X. Then $\mathbb{C}[X]$ is a G representation where $\varphi(g)(x) = g \cdot x$ for all basis vectors $x \in \mathbb{C}[X]$. In other words, each $\varphi(g)$ is the permutation matrix of its corresponding permutation. This is called a permutation representation.

e.g. Let $G = \mathbb{Z}/n\mathbb{Z} = \{0, 1, \dots, n-1\}$. Let $V = \mathbb{C}$.
We can define a representation $\varphi : G \to GL(V)$ by
$\varphi(k) = (e^{2\pi i \cdot k/n})$ ← 1×1 matrix $\forall\ k = 0, 1, \dots, n-1$.

e.g. Let $G = S_n$ symmetric gp. and let $V = \mathbb{C}$ ← 1×1 matrix
The sign representation $\varphi : S_n \to GL(\mathbb{C})$ is $\varphi(\sigma) = (\operatorname{sgn}(\sigma))$.

e.g. If U, V are G-representations, then direct sum $U \oplus V$ is another representation; as matrices: $\begin{pmatrix} \varphi(g)_U & 0 \\ 0 & \varphi(g)_V \end{pmatrix}$ ← "block sum".

DEF'N A repr'n $\varphi : G \to GL(V)$ is *irreducible* if we cannot find a *nontrivial* subspace U (i.e., $0 \neq U \neq V$) s.t. $gu \in U\ \forall u \in U, g \in G$ (i.e., *invariant* under all G).

important ↳ **FACT** Every representation V of G is a direct sum $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ of *irreducible* repr'ns V_i.

e.g. Let $V = \mathbb{C}^n$ w/ standard basis $\{e_1, e_2, \dots, e_n\}$ and $G = S_n$, Let $\varphi : S_n \to GL(V)$ be the standard permutation repr'n, i.e. $\varphi(\sigma) e_i = e_{\sigma(i)}\ \forall \sigma \in S_n, i = 1, \dots, n$. V is *reducible*, since $U_1 = \{(c, c, \dots, c) \in V : c \in \mathbb{C}\}$ is a nontrivial invariant subspace. With $U_0 = \{(x_1, \dots, x_n) \in V : x_1 + \cdots + x_n = 0\}$, we have $V = U_1 \oplus U_0$ and $\underset{\substack{\uparrow \\ \text{trivial repr'n}}}{U_1}$, U_0 are *irreducible* repr'ns.

The FACT above says that to understand all G-reps, it's enough to understand the irreducible ones...

4/20

Characters of representations

Representations $\varphi: G \to GL(V)$ are matrix-valued functions, hence complicated to understand. It turns out we can "reduce" to studying "ordinary" $\mathbb{C}$-valued fn's $\chi: G \to \mathbb{C}$.

DEF'N Let φ be a representation of finite group G. Its character $\chi_\varphi: G \to \mathbb{C}$ is the function $\chi_\varphi(g) = \mathrm{Tr}(\varphi(g))$ ← trace of matrix for all $g \in G$.

e.g. If V is 1-dim'l, then φ and χ_φ are the same thing...

e.g. If φ is the permutation rep'n of an action $G \circlearrowright X$ then $\chi_\varphi(g) = \#\mathrm{Fix}(g: X \to X)$ ← why? think abt. perm. matrix..

FACT For two G-rep's $\varphi_1: G \to GL(V_1)$, $\varphi_2: G \to GL(V_2)$ have $\chi_{\varphi_1} = \chi_{\varphi_2} \Leftrightarrow \varphi_1$ isomorphic to φ_2

($\varphi_1 \simeq \varphi_2$ means $\exists$ v.s. iso. $V_1 \simeq V_2$ that commutes w/ G-action)

Upshot: enough to study characters, in fact, since we have $\chi_{\varphi_1 \oplus \varphi_2} = \chi_{\varphi_1} + \chi_{\varphi_2}$, enough to study characters of irreducible rep'ns (+ their lin. comb.'s).

In fact, Characters χ are not just any kind of function $G \to \mathbb{C}$..

DEF'N A conjugacy class of G is set of the form $C = \{ghg^{-1} : g \in G\}$ for some $h \in G$. A function $f: G \to \mathbb{C}$ is called a class function if it is constant on conjugacy classes, i.e. $f(h) = f(ghg^{-1}) \; \forall g, h \in G$.

Let $Cl(G) :=$ v.s. of class functions $f: G \to \mathbb{C}$.

Prop. Any character χ_φ is a class function.

Pf. $\chi_\varphi(ghg^{-1}) = \mathrm{Tr}(ghg^{-1}) \underset{\uparrow}{=} \mathrm{Tr}(g^{-1}\cdot gh) = \mathrm{Tr}(h)$

recall $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for matrices A, B ▨

FACT 1. $\{\chi_{\varphi_1}, \dots, \chi_{\varphi_m}\}$ is a basis of $C\ell(G)$, where $\varphi_1, \dots, \varphi_m$ are all the irrep's of G (up to iso.).

2. With the inner product $\langle\ ,\ \rangle : C\ell(G) \times C\ell(G) \to \mathbb{C}$ given by $\langle f, f' \rangle := \frac{1}{\#G} \sum_{g \in G} f(g)\, \overline{f'(g)}$, the basis $\{\chi_{\varphi_1}, \dots \chi_{\varphi_m}\}$ is orthonormal.

3. If $\varphi = \bigoplus c_m \varphi_m$ is decomposition of φ into irrep's, then $c_m = \langle \chi_\varphi, \chi_{\varphi_m} \rangle$.

Note in particular that
#irreps (irreducible rep'ns) = $\dim C\ell(G)$
= #conjugacy classes of G.

e.g. G acts on itself by multiplication on the left, and corresponding perm. rep. is called the regular rep'n. $\mathbb{C}[G]$
How does $\mathbb{C}[G]$ decompose into irrep's?

$$\langle \chi_{\mathbb{C}[G]}, \chi_{\varphi_m} \rangle = \frac{1}{\#G} \sum_{g \in G} \underset{\#\mathrm{Fix}(g : G \to G) = \begin{cases} \#G & \text{if } g = e \\ 0 & \text{otherwise} \end{cases}}{\underset{\parallel}{\chi_{\mathbb{C}[G]}(g)}}\ \overline{\chi_{\varphi_m}(g)}$$

$$= \frac{1}{\#G} \cdot \#G \cdot \overline{\chi_{\varphi_m}(e)} = \dim(\varphi_m).$$

Hence

$$\#G = \dim \mathbb{C}[G] = \dim\left(\bigoplus \dim(\varphi_m) \cdot \varphi_m\right) = \underset{\uparrow}{\sum_m} (\dim \varphi_m)^2$$

looks familiar...

4/22

Characters of the Symmetric Group.

Finally, by focusing on case $G = S_n$, we see symmetric functions.

Prop. Two permutations $\sigma, \sigma' \in S_n$ belong to same conjugacy class $\iff$ they have the same cycle structure.

Pf: Exercise for you. ☒

So # conj. classes in S_n = # cycle structures = # partitions $\lambda \vdash n$

So # irrep's of S_n = # $\lambda \vdash n$, and in fact there is a standard way to index irrep's by partitions.

e.g. Let triv : $S_n \to GL(\mathbb{C})$ be the trivial rep'n. Then

$$\text{triv} = \varphi_{\square\square\square\square\square}{}^{\leftarrow n \text{ boxes}} = \varphi_{(n)}$$

e.g. For sgn : $S_n \to GL(\mathbb{C})$ sign rep'n, $\text{sgn} = \varphi_{\begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}} = \varphi_{(1^n)}$.

e.g. Recall standard perm rep'n $\mathbb{C}^n = \underset{\text{triv}}{U_1} \oplus U_0$ ← irreducible, dim = n-1,

then $U_0 = \varphi_{\begin{smallmatrix}\square\square\square\square\\\square\end{smallmatrix}} = \varphi_{(n-1,1)}$

Write $\chi_\lambda = \chi_{\varphi_\lambda}$ = character of irrep indexed by $\lambda \vdash n$.

DEF'N The Frobenius characteristic $Fr : Cl(S_n) \to Sym(n)$ ← recall = sym. fn's of degree n

is given by $Fr(\delta_\lambda) = p_\lambda$ ← power sum

where δ_λ is class function $\delta_\lambda(\sigma) = \begin{cases} z_\lambda & \text{if cycle type}(\sigma) = \lambda \\ 0 & \text{otherwise} \end{cases}$

and $z_\lambda = 1^{m_1} m_1! \, 2^{m_2} m_2! \cdots = \frac{n!}{\#}$ perm's in S_n w/ cycle type $= \lambda = (1^{m_1} 2^{m_2} \cdots)$.

Since the δ_λ are a basis of $Cl(S_n)$ and p_λ are a basis of $Sym(n)$, this is clearly a v.s. isomorphism.

Thm $Fr(\chi_\lambda) = s_\lambda$ ← Schur function.

This is (one reason) why Schur fn's are so important!

Cor $\dim \varphi_\lambda = f^\lambda = \#\text{SYT of sh. } \lambda$

Pf: Via Fr, same as coeff. of $[x_1 x_2 \cdots x_n]$ in $s_\lambda = f^\lambda$ ✓ ☒

More generally ...

Cor If $\chi_\lambda(\mu)$ = ch. evaluated at a perm. of cycle type μ,

then $s_\lambda = \sum_\mu \chi_\lambda(\mu) \cdot z_\mu^{-1} p_\mu$.

∃ combinatorial rule for these coeff's, called the Murnaghan-Nakayama rule.

Also note that ... by the regular representation, have

$$n! = \# S_n = \sum_{\lambda \vdash n} \dim(\varphi_\lambda)^2 = \sum_{\lambda \vdash n} (f^\lambda)^2,$$

which we saw earlier using R.S. algorithm.

Finally, using something called the induction product of representations of $S_k \times S_{n-k} \longrightarrow S_n$,

we can get ring structure on $Sym = \bigoplus Sym(n)$.

Structure constants $s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$ are called Littlewood-Richardson coefficients, also very important!

//