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Rules for differentiation § 3.1

Now we will spend a lot of time learning rules for derivatives.
The simplest derivative is for a constant function:

Thm If $f(x) = c$ for some constant $c \in \mathbb{R}$,
then $f'(x) = 0$.

Pf: We could write a limit, but it's easier to just remember the tangent line definition of the derivative.

If $y = f(x)$ is a line, then the tangent line at any point is $y = f(x)$. In this case, the slope = 0 since $f(x) = c$.
 $\longleftrightarrow y = f(x) = c$ (horizontal)

Actually, the same argument works for any linear function $f(x)$.

Thm If $f(x) = mx + b$ is a linear function,
then $f'(x) = m$ (slope of line).

Some other simple rules for derivatives are:

Thm • (sum) $(f+g)'(x) = f'(x) + g'(x)$

• (difference) $(f-g)'(x) = f'(x) - g'(x)$

• (scaling) $(c \cdot f)'(x) = c f'(x)$ for $c \in \mathbb{R}$.

Pf: These all follow from the corresponding limit laws.

E.g., for sum rule have

$$\begin{aligned} (f+g)'(x) &= \lim_{h \rightarrow 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(x) + g'(x). \end{aligned}$$

The first really interesting derivative is for $f(x) = x^n$, a power function. We've seen:

$$\frac{d}{dx}(x^0) = 0, \quad \frac{d}{dx}(x^1) = 1, \quad \frac{d}{dx}(x^2) = 2x$$

Do you see a pattern?

Thm for any nonnegative integer n , if $f(x) = x^n$

$$\text{then } \boxed{f'(x) = n \cdot x^{n-1}}$$

"bring n down"

from the exponent

Pf: We can use an algebra trick:

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a}$$

check that this multiplies correctly!

$$= \lim_{x \rightarrow a} \frac{(x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-2}x + a^{n-1})}{x - a}$$

$$= \lim_{x \rightarrow a} (x^{n-1} + ax^{n-2} + \dots + a^{n-1}) = \underbrace{a^{n-1} + a^{n-1} + \dots + a^{n-1}}_{n \text{ times}}$$

$$= n \cdot a^{n-1} \quad \checkmark \quad \square$$

This is one of the most important formulas in calculus!

Please memorize it.

E.g.: If $f(x) = 3x^4 - 2x^3 + 6x^2 + 5x - 9$ then

$$f'(x) = 12x^3 - 6x^2 + 12x + 5.$$

Can easily take derivative of any polynomial!

E.g.: If $f(x) = x^3$, what is $f''(x)$?

$$\text{Well, } f'(x) = 3x^2, \text{ so } f''(x) = 3 \cdot 2x = 6x.$$

All derivatives of x^n easy to compute this way!

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Derivatives for more kinds of functions § 3.1

Thm For any real number n , if $f(x) = x^n$
 then $\boxed{f'(x) = n \cdot x^{n-1}}$

Exactly same formula as for positive integers n .
 Proof is similar, and we will skip it...

E.g. Q: If $f(x) = \sqrt{x}$, what is $f'(x)$?

A: $f(x) = x^{1/2}$, so $f'(x) = \frac{1}{2} x^{1/2-1} = \frac{1}{2} x^{-1/2}$
 $= \frac{1}{2} \frac{1}{x^{1/2}} = \frac{1}{2\sqrt{x}}$

Q: If $f(x) = \frac{1}{x}$, what is $f'(x)$?

A: $f(x) = x^{-1}$, so $f'(x) = -1 \cdot x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$

The exponential fn. e^x has a surprisingly simple derivative:

Thm If $f(x) = e^x$, then $f'(x) = e^x = (f(x))$.

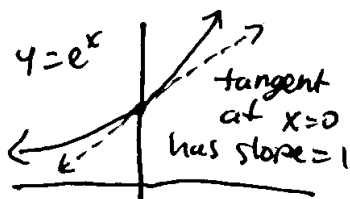
Taking derivative of e^x does not change it!


So also $f''(x) = e^x$, $f'''(x) = e^x$, etc...

Pf: We write

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^x \cdot e^h - e^x \cdot e^0}{h} = e^x \cdot \lim_{h \rightarrow 0} \frac{e^h - e^0}{h} = e^x \cdot f'(0) \end{aligned}$$

So we just need to show $f'(0) = 1$.

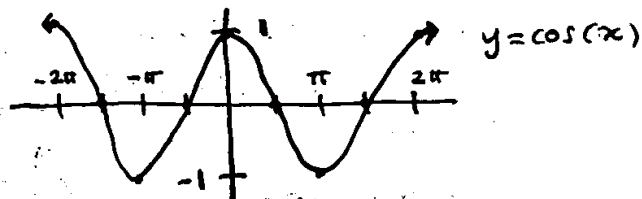
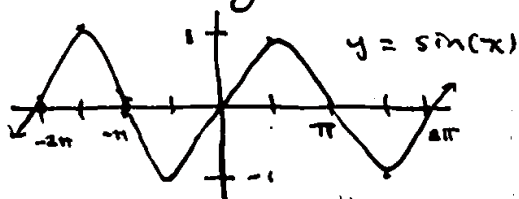


But remember, we defined e as the unique $b > 1$ for which slope of tangent of b^x at $x=0$ is one.
 So $f'(0) = 1$ by definition of e . 

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Derivatives of trigonometric functions § 3.3

Looking at the graphs of $\sin(x)$ and $\cos(x)$:



We notice that:

- | | |
|---|---|
| • $\sin(x)$ is increasing $\Leftrightarrow \cos(x) > 0$ | • $\cos(x)$ is increasing $\Leftrightarrow \sin(x) < 0$ |
| • $\sin(x)$ is decreasing $\Leftrightarrow \cos(x) < 0$ | • $\cos(x)$ is decreasing $\Leftrightarrow \sin(x) > 0$ |
| • $\sin(x)$ has min./max. $\Leftrightarrow \cos(x) = 0$ | • $\cos(x)$ has min./max. $\Leftrightarrow \sin(x) = 0$ |

From these qualitative properties, reasonable to guess:

Thm $\boxed{\begin{aligned} \frac{d}{dx}(\sin(x)) &= \cos(x) \\ \text{and } \frac{d}{dx}(\cos(x)) &= -\sin(x) \end{aligned}}$

E.g. If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$,

so $f''(x) = -\sin(x)$, and $f'''(x) = -\cos(x)$,
and $f^{(4)}(x) = -(-\sin(x)) = \sin(x) = f(x)$.

After 4 derivatives, we get back what we started with!

Can also check that if $f(x) = \cos(x)$, then $f^{(4)}(x) = \cos(x) = f(x)$.

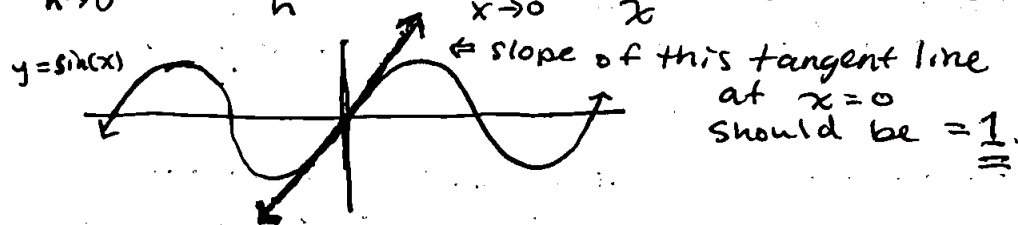
In this way, the trig functions $\sin(x)$ and $\cos(x)$ behave like e^x , where taking enough derivatives gives us back the original function we started with.

Whereas with a polynomial function like $f(x) = 5x^4 - 3x^3 + 6x^2 + 10x - 9$, taking enough derivatives always gives us zero!

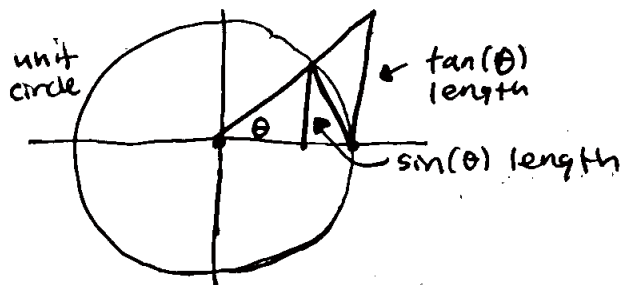
The key step for proving $\frac{d}{dx}(\sin(x)) = \cos(x)$ is this:

Lemma If $f(x) = \sin(x)$, then

$$f'(0) = \lim_{h \rightarrow 0} \frac{\sin(0+h) - \sin(0)}{h} = \lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1 = \cos(0).$$



There is a nice geometric proof of this lemma:



\approx idea of proof is to compare areas of triangles in this drawing.
See book for details!

For our purposes, we will just use the formulas. To summarize, it is worth memorizing the following important derivatives:

$$\frac{d}{dx}(x^n) = n \cdot x^{n-1}$$

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

don't forget this negative sign!
it's important!

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The product and quotient rules § 3.2

Suppose we want to take the derivative of a product $f \cdot g$ of two (differentiable) functions $f(x)$ and $g(x)$.
Might think/hope ^{its} derivative is product of derivatives, but easy to find examples where $(f \cdot g)'(x) \neq f'(x) \cdot g'(x)$.

E.g.: Let $f(x) = x$, $g(x) = x^2$, then $f'(x) \cdot g'(x) = 1 \cdot 2x = 2x$, but $(f \cdot g)(x) = x^3$, so $(f \cdot g)'(x) = 3x^2$.

Instead, we have the product rule:

Thm For two (differentiable) functions $f(x), g(x)$:

$$\boxed{(f \cdot g)'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)}$$

"First times derivative of second plus second times derivative of first."

In differential notation: $\boxed{\frac{d}{dx}(f \cdot g) = f \cdot \frac{dg}{dx} + g \cdot \frac{df}{dx}}$

E.g. With $f(x) = x$ and $g(x) = x^2$, we compute

$$(f \cdot g)'(x) = f(x) g'(x) + g(x) f'(x) = x \cdot 2x + x^2 \cdot 1 = 3x^2 = \frac{d}{dx}(x^3) \quad \checkmark$$

E.g. $\frac{d}{dx}(x e^x) = x \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x) = x e^x + e^x$

E.g. $\frac{d}{dx}(x^2 \sin(x)) = x^2 \frac{d}{dx}(\sin(x)) + \sin(x) \cdot \frac{d}{dx}(x^2) = x^2 \cos(x) + 2x \sin(x)$

Pf. sketch for product rule:

Write $u = f(x)$, $v = g(x)$, $\Delta u = f(x+h) - f(x)$, $\Delta v = g(x+h) - g(x)$.

Then $\Delta(uv) = (u + \Delta u)(v + \Delta v) - uv$

$$= u \Delta v + v \Delta u - \underbrace{\Delta u \Delta v}_{\text{this term goes away in limit!}}$$

See book for details!

The quotient rule is a bit more complicated:

Then for two (differentiable) functions $f(x), g(x)$ (with $g(x) \neq 0$),

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x) \cdot f'(x) - f(x) g'(x)}{g(x)^2}, \text{ or in differential notation:}$$

$$d/dx \left(\frac{f}{g}\right) = (g \frac{df}{dx} - f \frac{dg}{dx}) / g^2.$$

Looks similar in many ways to product rule, but more complicated. When we learn the chain rule, you will see that you don't need to separately memorize the quotient rule.

E.g.: Let $f(x) = x, g(x) = 1-x$, so $\frac{f}{g}(x) = \frac{x}{1-x}$

$$\begin{aligned} \text{Then } \left(\frac{f}{g}\right)'(x) &= \frac{g(x) \cdot f'(x) - f(x) g'(x)}{g(x)^2} = \frac{(1-x) \cdot 1 - x \cdot (-1)}{(1-x)^2} \\ &= \frac{1-x+x}{(1-x)^2} = \frac{1}{(1-x)^2} \end{aligned}$$

Any rational function can be differentiated this way...

E.g.: Recall $\tan(x) = \frac{\sin(x)}{\cos(x)}$

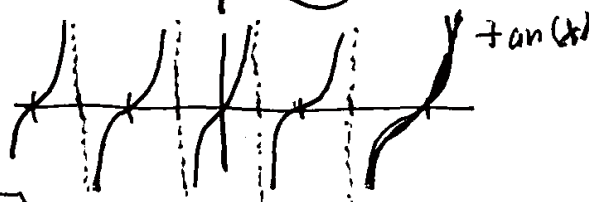
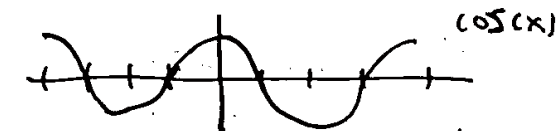
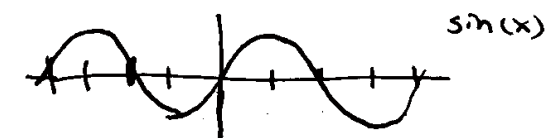
Thus, $\left(\tan(x)\right)' =$

$$\frac{\cos(x) \cdot (\sin)'(x) - \sin(x) (\cos)'(x)}{(\cos(x))^2}$$

$$= \frac{\cos(x) \cdot \cos(x) - \sin(x) (-\sin(x))}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

the one trig identity really worth knowing



using Pythagorean identity:

$$\sin^2(x) + \cos^2(x) = 1$$

for any x

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Chain rule §3.4

Let $f(x) = \sqrt{x^2+1}$. How do we find $f'(x)$?

So far we don't know how... to do this we need the chain rule, which tells us how to take derivatives of compositions of functions:

Theorem For two (differentiable) fn's $f(x)$ and $g(x)$, we have $\boxed{(f \circ g)'(x) = f'(g(x)) \cdot g'(x)}$

In differential notation, this can be written

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

where $y = f(g(x))$ and $u = g(x)$. So roughly speaking the chain rule lets us "cancel" the du 's.

E.g. For $f(x) = \sqrt{x^2+1}$, write $f(x) = h(g(x))$

where $h(x) = \sqrt{x}$ and $g(x) = x^2+1$. Then the chain rule says $f'(x) = h'(g(x)) \cdot g'(x)$

$$= \frac{1}{2}(g(x))^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$$

E.g. Let $f(x) = \sin(x^2)$. Then

$$f'(x) = \underbrace{\cos(x^2)}_{d/dx(\sin(x)), \text{ plug in } x^2} \cdot \underbrace{2x}_{d/dx(x^2)}$$

E.g. Let $f(x) = \sin^2(x)$ (meaning $(\sin(x))^2$).

$$\text{Then } f'(x) = \underbrace{2 \cdot \sin(x)}_{d/dx(x^2), \text{ plug in } \sin(x)} \cdot \underbrace{\cos(x)}_{d/dx(\sin(x))}$$

E.g. with $f(x) = \frac{1}{\sin(x)} = (\sin(x))^{-1}$, we have

$$f'(x) = \underbrace{- (\sin(x))^{-2}}_{d/dx(x^{-1}), \text{ plug in } \sin(x)} \cdot \underbrace{\cos(x)}_{d/dx(\sin(x))} = \frac{-\cos(x)}{\sin^2(x)}$$

With this last example, we see how we don't need the quotient rule. In fact, quotient rule can be deduced from product rule and chain rule:

Let $h(x) = \frac{f(x)}{g(x)} = f(x) \cdot (g(x))^{-1}$

Then $h'(x) = f(x) \cdot \frac{d}{dx}(g(x)^{-1}) + \frac{1}{g(x)} f'(x)$.

But by the chain rule,

$$\begin{aligned} \frac{d}{dx}(g(x)^{-1}) &= -g(x)^{-2} \cdot g'(x) \\ &= \frac{-g'(x)}{g(x)^2} \end{aligned}$$

$$\begin{aligned} \text{So that } h'(x) &= f(x) \cdot \frac{-g'(x)}{g(x)^2} + \frac{f'(x)}{g(x)} \\ &= \frac{-f(x) \cdot g'(x)}{g(x)^2} + \frac{f'(x) g(x)}{g(x)^2} \\ &= \frac{g(x) f'(x) - f(x) g'(x)}{g(x)^2}, \end{aligned}$$

which is exactly the quotient rule we learned. So you never need to separately memorize the quotient rule: the product rule and chain rule are enough.

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§ 3.4, 3.6

Derivatives of exponentials and logarithms

The chain rule allows us to compute derivatives of arbitrary exponential and logarithmic functions.

Let's start with the exponential $f(x) = b^x$ for some base $b > 0$. Recall that

$$b^x = e^{\ln(b) \cdot x} \quad \text{by rules of exponents.}$$

$$\begin{aligned} \text{Thus } d/dx(b^x) &= d/dx(e^{\ln(b) \cdot x}) \\ &= e^{\ln(b) \cdot x} \cdot \ln(b) \quad \text{by chain rule} \\ &= \ln(b) \cdot b^x \end{aligned}$$

So derivative behaves similarly to e^x (but w/ $\ln(b)$ factor)...

What about logarithms? Recall that

$$x = e^{\ln(x)} \quad (\text{because } e^x \text{ and } \ln(x) \text{ are inverses...})$$

Taking d/dx of both sides gives:

$$\begin{aligned} d/dx(x) &= d/dx(e^{\ln(x)}) \\ 1 &= e^{\ln(x)} \cdot d/dx(\ln(x)) \quad \text{by chain rule} \\ 1 &= x \cdot d/dx(\ln(x)) \end{aligned}$$

$$\Rightarrow \boxed{d/dx(\ln(x)) = \frac{1}{x}}$$

How about arbitrary logarithms?

If $f(x) = \log_b(x)$ for some base $b > 0$,

then since $\log_b(x) = \frac{\ln(x)}{\ln(b)}$ by rules of logs

$$\text{We have } f'(x) = \frac{1}{\ln(b)} \cdot \frac{1}{x}.$$

Notice: You might expect that there is some power function $f(x) = a \cdot x^n$ with $f'(x) = 1/x = x^{-1}$. But we would need $n=0$ and $a = "1/0"$ for this to work ($f'(x) = a \cdot n \cdot x^{n-1}$), so there is not such an $f(x)$!

Now that we know:

- Sum, difference, scaling rules:
$$d/dx (c \cdot f(x) + d \cdot g(x)) = c \cdot f'(x) + d \cdot g'(x)$$
- Product rule:
$$d/dx (f(x) \cdot g(x)) = f(x) g'(x) + g(x) f'(x)$$

(and maybe quotient rule...)
- Chain rule:
$$d/dx (f(g(x))) = f'(g(x)) \cdot g'(x).$$
- and derivatives of basic functions:
$$d/dx (x^n) = n \cdot x^{n-1} \text{ for any } n \in \mathbb{R}$$

$$d/dx (e^x) = e^x \text{ and } d/dx (\ln(x)) = 1/x$$

$$d/dx (\sin(x)) = \cos(x) \text{ and } d/dx (\cos(x)) = -\sin(x),$$

We can compute the derivative of essentially any kind of function that we have been studying all semester!

Exercise: find $d/dx (\sin(\ln(x^2)))$.