Promotion, webs, and plabic graphs

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BIRS Workshop on Dynamical Algebraic Combinatorics

(sort of) based on joint work w/ Martin Rubey [TU Wien]
arXiv:2005.14031

these slides are on my website:
samuelfhopkins.com
(+ this talk is being recorded)

Promotion is a cyclic action on the set of **Standard Young tableaux (SYTs)** of shape $\lambda \vdash n$:

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} 6 & 7 \\ \cline{1-2}\end{array} \ \overset{\text{delete } 1}{\rightsquigarrow} \ \begin{array}{|c|c|c|}\hline \bullet & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} 6 & 7 \\ \cline{1-2}\end{array} \ \overset{\text{jdt slide into hole}}{\rightsquigarrow} \ \begin{array}{|c|c|c|}\hline 2 & 3 & 5 \\ \hline 4 & 7 \\ \cline{1-2} 6 & \bullet \\ \cline{1-2}\end{array} \ \rightsquigarrow$$

$$\overset{\text{decrement values}}{\rightsquigarrow} \ \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 \\ \cline{1-2} 5 & \bullet \\ \cline{1-2}\end{array} \ \overset{\text{fill hole w/ } n}{\rightsquigarrow} \ \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 \\ \cline{1-2} 5 & 7 \\ \cline{1-2}\end{array} = \mathrm{Pro}(T)$$

Introduced by Schützenberger, together w/ related operation of **evacuation**:

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} 6 & 7 \\ \cline{1-2}\end{array} \ \overset{\text{Pro of } 1,\dots,n}{\rightsquigarrow} \ \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 \\ \cline{1-2} 5 & 7 \\ \cline{1-2}\end{array} \ \overset{\text{Pro of } 1,\dots,n-1}{\rightsquigarrow} \ \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 & 7 \\ \cline{1-2}\end{array} \ \rightsquigarrow$$

← freeze in place

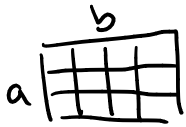
freeze in place

$$\overset{\text{Pro of } 1,\dots,n-2}{\rightsquigarrow} \ \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 4 \\ \cline{1-2} 5 & 7 \\ \cline{1-2}\end{array} \ \rightsquigarrow \ \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 4 \\ \cline{1-2} 5 & 7 \\ \cline{1-2}\end{array} \ \rightsquigarrow \ \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 4 \\ \cline{1-2} 5 & 7 \\ \cline{1-2}\end{array} \ \rightsquigarrow$$

$$\rightsquigarrow \ \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 4 \\ \cline{1-2} 5 & 7 \\ \cline{1-2}\end{array} \ \rightsquigarrow \ \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 4 \\ \cline{1-2} 5 & 7 \\ \cline{1-2}\end{array} = \mathrm{Evac}(T)$$

While $\boxed{\mathrm{Evac}^2(T) = T}$ for any T, the behavior of Pro is in general hard to predict, except for special shapes λ:

Thm (Schützenberger)

For $\lambda =$ 

(the $a \times b$ rectangle),

$\boxed{\mathrm{Pro}^{ab}(T) = T}$ for all T.

Want to study Pro (+Evac) via models:

Thm (Rhoades)

For $\lambda = a \times b \vdash n$, the action of $c = (1, 2, \ldots, n)$ ("long cycle") on $\mathfrak{S}_n$ (sym. gp.) -irrep S^λ corresponds* to Pro in the KL cellular basis.

Thm (Stembridge) For any $\lambda \vdash n$, the action of $w_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}$ ("longest word") on S^λ corresponds* to Evac in KL cell. basis.

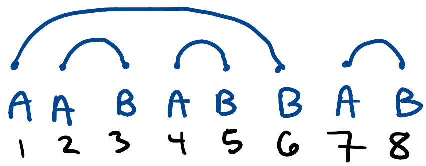
These algebraic models are nice, but what about combinatorial models?

Prop. (D. White)

Promotion of $2 \times n$ SYTs corresponds to rotation of noncrossing matchings of $[2n]$ via standard bijection between these Catalan objects:

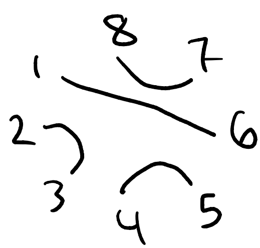
1	2	4	7	A
3	5	6	8	B

$\rightsquigarrow$



1	2	4	7
3	5	6	8

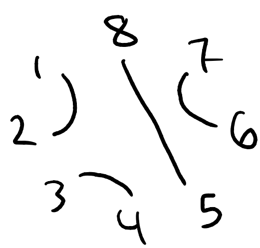
$\longleftrightarrow$



Pro $\downarrow$ $\qquad$ $\downarrow$ Rot

1	3	5	6
2	4	7	8

$\longleftrightarrow$



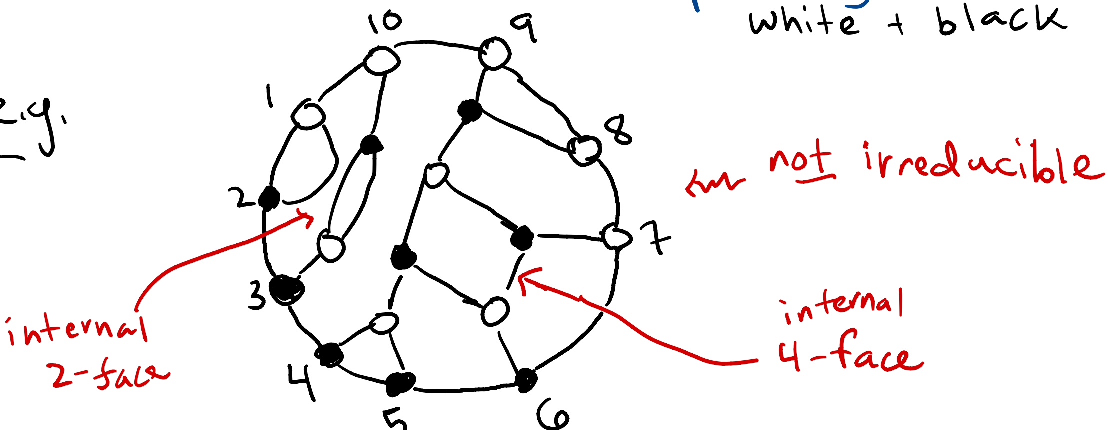
What about more than 2 rows . . . ?

Def'n (Kuperberg)

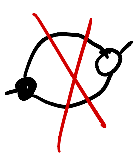
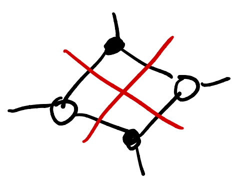
An (sl_3-) web is a planar graph drawn in a disc w/ boundary vertices $1, 2, \ldots, m$, and some internal vertices, such that:

- bdy. vertices have degree one,
- int. vertices are trivalent,
- vertices (bdy. + int.) are properly 2-colored white + black

e.g.



Web is called irreducible (or non-elliptic) if it has no internal (i.e., disjoint from bdy.)

2-faces  and 4-faces 

Thm (Khovanov-Kuperberg, Petersen-Pylyavskyy-Rhoades, Tymoczko)

There is a bijection

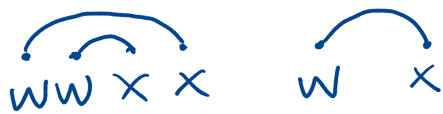
$$\{3\times n \text{ SYTs}\} \xrightarrow{\sim} \left\{\begin{array}{l}\text{irred. webs w/}\\ 3n \text{ white bdy. vert's}\end{array}\right\}$$

such that Pro of T = Rot of W.

How does bij. work? Uses M-diagram:

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 4 & 8 \\ \hline 5 & 7 & 9 \\ \hline \end{array} \begin{array}{c} W \\ X \\ Y \end{array} \rightsquigarrow \begin{array}{ccccccccc} W & W & X & X & Y & W & Y & X & Y \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$

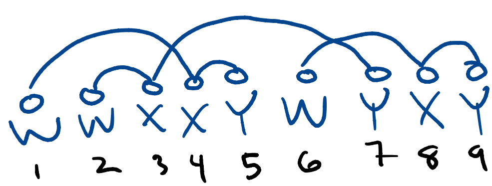
noncross. match of W's + X's:



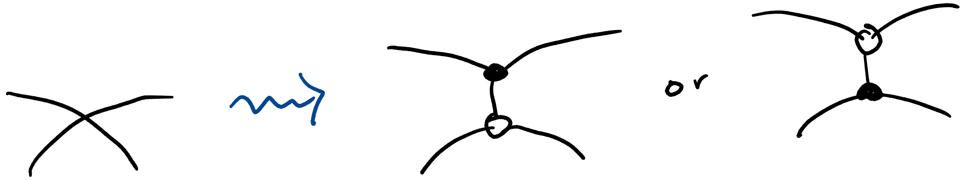
nocross. match of X's + Y's:

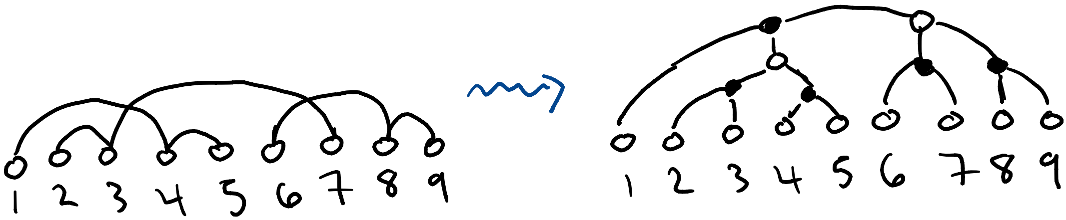
super-impose to get M-diagram:

(compare w/ 'noncrossing tableaux' of Pylyavskyy)



Have to resolve crossings to get web:

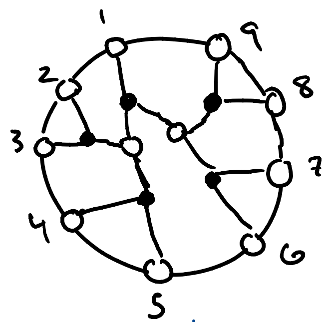




T =

1	2	6
3	4	8
5	7	9

$\longleftrightarrow$



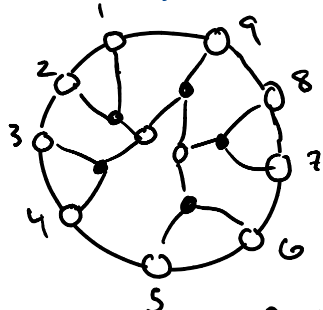
= W

Pro $\downarrow$ $\qquad\qquad$ $\downarrow$ Rot

Pro(T) =

1	3	5
2	6	7
4	8	9

$\longleftrightarrow$



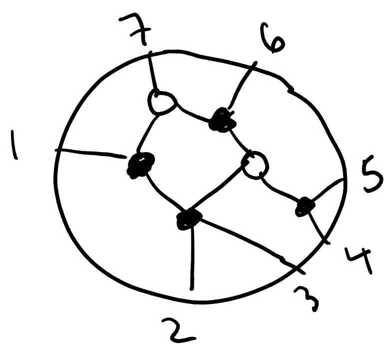
= Rot(W)

Def'n (Postnikov)

A plabic graph is a planar graph, drawn in a disc, w/ deg.=1 bdy. vert's 1, 2, ..., m, and some int. vert's, such that int. vert's are colored black + white.

not necessarily properly!

e.g.,

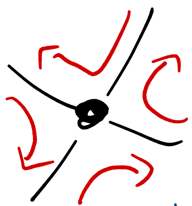


Observation: A web is (basically) a special case of a plabic graph!

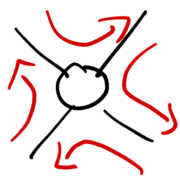
Q: What does this observation buy us?

Def'n (Postnikov)

The trip permutation* of a plabic graph has $\pi(i)$ = bdy. vertex you end at if you start a trip at i and follow the rules of the road:

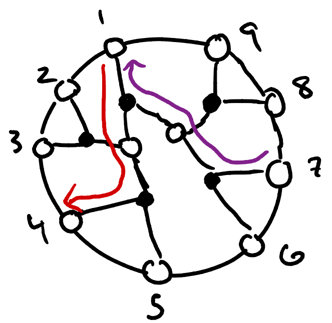


right at black



left at white

e.g.



trip starting at 1 ends at 4

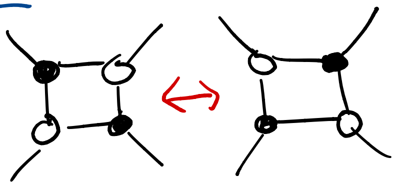
trip starting at 7 ends at 1

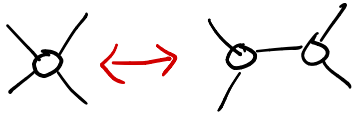
etc...

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 8 & 5 & 2 & 7 & 1 & 9 & 6 \end{pmatrix}$$

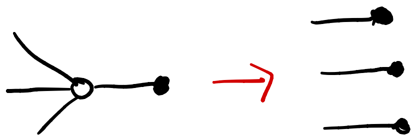
Def'n (Postnikov) Moves and reductions are local transformations of plabic graphs:

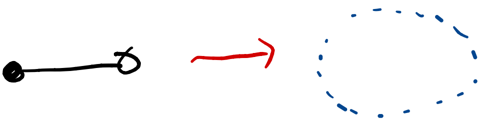
Moves:





Reductions:





Write $G \sim G'$:= G' can be obtained from G via a series of moves.

Say G is reduced if there is no $G' \sim G$ s.t. can apply a reduction to G'.

Thm (Postnikov) G, G' reduced plabic graphs have same trip perm. $\iff$ $G \sim G'$.

Cor Two irreducible webs W, W' have same trip perm. $\overset{*}{\Leftrightarrow}$ $W = W'$.

pf: Only moves/reductions we can do are trivial: $—— \leftrightarrow —\circ—$ ▨

For $3 \times n$ SYT T, let π_{trip} be the trip perm. of associated web W.

So T is determined by π_{trip}.

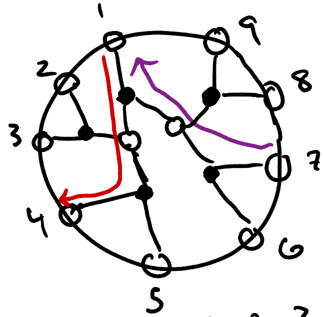
Q: Another way to find π_{trip}?

A: YES! Using promotion:

$\pi_x(i) \equiv$ the # that moves up from 2nd row when applying Pro to $\text{Pro}^{i-1}(T)$ $+ (i-1) \mod 3n$

π_y = same but w/ 3rd row instead of 2nd e.g.

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 6 \\ \hline 3 & 4 & 8 \\ \hline 5 & 7 & 9 \\ \hline \end{array} \longleftrightarrow$$



$= W$

$$\pi_{trip} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 8 & 5 & 2 & 7 & 1 & 9 & 6 \end{pmatrix}$$

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 6 \\ \hline 3 & 4 & 8 \\ \hline 5 & 7 & 9 \\ \hline \end{array} \xrightarrow{\text{Pro.}} \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 6 & 7 \\ \hline 4 & 8 & 9 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array}$$

$4+0=4$ $\quad$ $7+0=7$ $\quad$ $2+1=3$ $\quad$ $4+1=5$ $\quad$ $6+2=8$ $\quad$ $9+2=2$

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & 8 \\ \hline 6 & 7 & 9 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 6 & 7 \\ \hline 5 & 8 & 9 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & 8 \\ \hline 4 & 7 & 9 \\ \hline \end{array}$$

$2+3=5$ $\quad$ $7+3=1$ $\quad$ $7+4=2$ $\quad$ $9+4=4$ $\quad$ $2+5=7$ $\quad$ $4+5=9$

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & 7 \\ \hline 6 & 8 & 9 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 6 & 8 \\ \hline 5 & 7 & 9 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 7 \\ \hline 6 & 8 & 9 \\ \hline \end{array}$$

$4+6=1$ $\quad$ $8+6=5$ $\quad$ $2+7=9$ $\quad$ $5+7=3$ $\quad$ $7+8=6$ $\quad$ $9+8=8$

$\pi_X = 438527196 = \pi_{trip}$ $\qquad$ $\pi_Y = 752149538 = \pi_{trip}^{-1}$

Thm For any $3 \times n$ SYT T,

(a) $\pi_x = \pi_{trip}$,

(b) $\pi_y = \pi_{trip}^{-1}$.

Pf! For (a), first note that π_{trip}, π_x

$w \mapsto cwc^{-1}$

both rotate when we apply Pro to T.

So suffices to prove $\pi_{trip}(1) = \pi_x(1)$.

But it's easy to see both

= what 1 is matched with in M-diagram.

180° rot.

For (b), follows by considering dual SYT. $\blacksquare$

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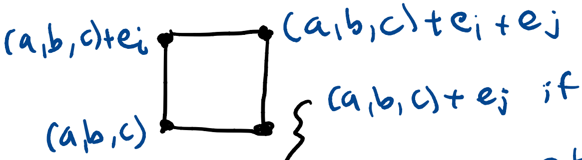
Rmk: Representing T via a chain of

partitions $\lambda^0 - \lambda^1 - \lambda^2 - \cdots - \lambda^{3n-1} - \lambda^{3n}$ in usual way,

can read off π_x, π_y in a beautiful way

from the growth diagram of T.

Rule for growth diagram: (see Fomin, Roby, Stanley...)



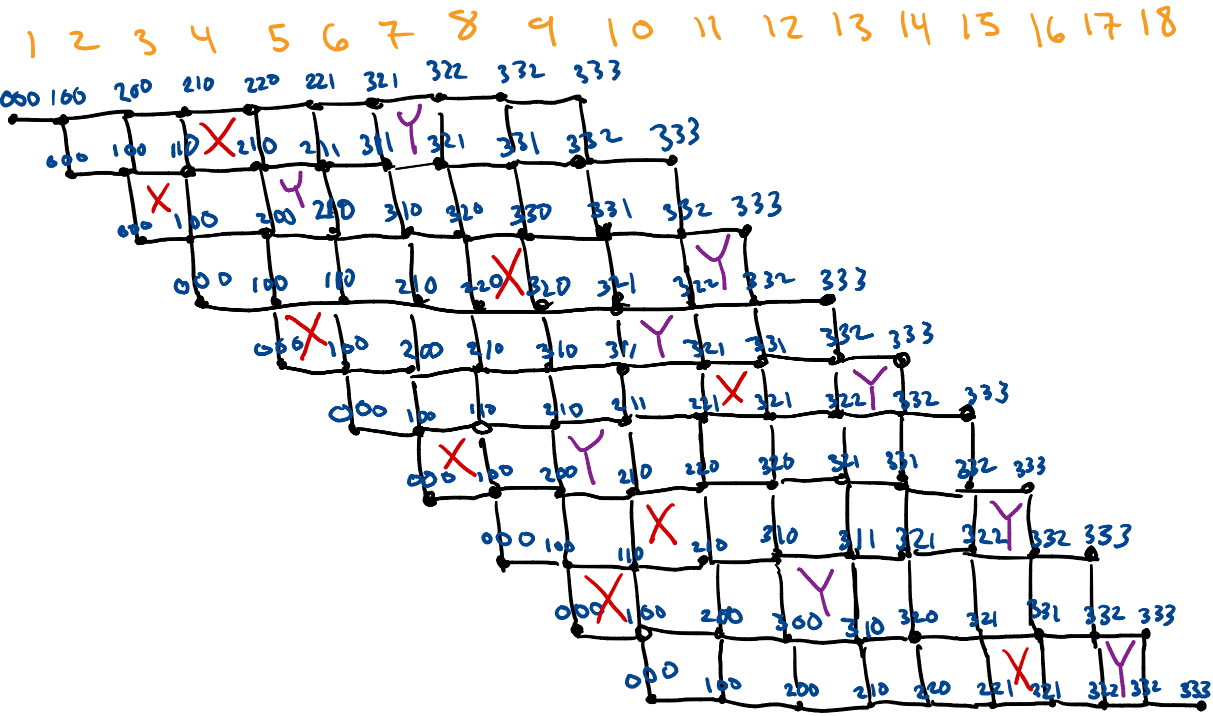
$\begin{cases} (a,b,c)+e_j & \text{if this is a partition} \\ (a,b,c)+e_i & \text{otherwise} \end{cases}$

↳ In this 2^{nd} case mark box w/

X if $j=2$

Y if $j=3$

Growth diagram for full promotion orbit of T:



$\pi_X = 4,3,8,5,11,7,10,9,15 = 4,3,8,5,2,7,1,9,6$

$\pi_Y = 7,5,11,10,13,9,15,12,17 = 7,5,2,1,4,9,6,3,8$

So... what? What can we do w/ this?

Well... I'd love you to tell me!

But here is one thing I know we can do:

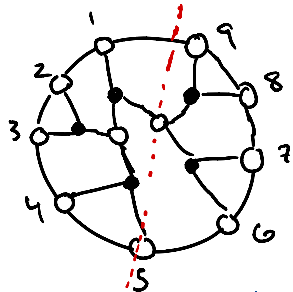
Cor (c.f. Patrias-Pechenik) $\mathrm{Evac}(T) \leftrightarrow \mathrm{Flip}(W)$,

where $\mathrm{Flip}(W)$ = refl. of W across a diam.

e.g.

1	2	6
3	4	8
5	7	9

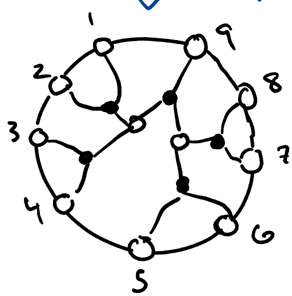
$\longleftrightarrow$



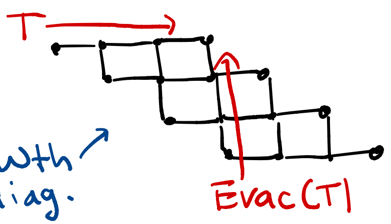
Evac $\downarrow$ $\qquad$ $\downarrow$ Flip

1	3	5
2	6	7
4	8	9

$\longleftrightarrow$



Pf:



hence $\boxed{\pi_X(T)} \longmapsto \boxed{\pi_X(\mathrm{Evac}(T))}$

$w \mapsto w_0 w^{-1} w_0$

inverse rev. comp.

But since trip perm. of $\mathrm{Flip}(W)$ = inv. r.c. of trip perm. of W,

$\mathrm{Flip}(W)$ is only possibility for $\mathrm{Evac}(T)$. $\blacksquare$

But... what else?:

- do this in other, similar contexts:
  e.g., semistandard $3 \times n$ tableaux (Russell)
  or linear extensions of $V \times [n]$ (H.-Rubey)
  or $a \times b$ SYT w/ $a > 3$ (ambitious!)

- can we classify perm's of form π_{trip}, say for $3 \times n$ SYTs?

- well-known that we can either make a (red.) plabic graph trivalent or bipartite via moves. Webs are when we can do both.

  Any significance to that?

- Plabic graphs + Webs both related to the Grassmannian $Gr(k,n)$:
  i) Plabic graphs parameterize cells in the totally nonnegative Grass. $Gr_{\geq 0}(k,n)$,
  ii) Webs yield tensor invariants, which can be viewed as elements of $\mathbb{C}[Gr(k,n)]$.

  Is there a direct connection (c.f. Fraser-Lam-Le)?

  Where does promotion fit in?

Thank you!

(+ hope to see you all in person
sometime soon...
maybe in DC?)