9/7 Arguments and rules of inference §1.4

Consider the following propositions:

- The murderer is Joe or Bob.
- The murderer is right-handed.
- Joe is not right-handed.

If these are all true, it is reasonable to conclude:

- Bob is the murderer.

Drawing a conclusion from a sequence of propositions like this is called deductive reasoning. Let's formalize it:

Def'n A sequence of propositions of the form

$$\boxed{\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}}$$

is called a (deductive) argument. The $p_1, \ldots, p_n$ are called the hypotheses (or premises) and the $q$ is called the conclusion. The "$\therefore$" symbol is read, "therefore".

The argument is valid if: whenever the hypotheses are all true, then the conclusion is true. (If it is not valid, then we say it is invalid.)

NOTE: Saying the argument is valid is not saying it is correct. For example, the hypotheses might be false! When we evaluate the validity of an argument, we look at its form, not its content.

E.g. Thm

$$\begin{array}{c} p \to q \\ p \\ \hline \therefore q \end{array}$$

is a valid argument.

(It is called "modus ponens".)

Pf: One way to prove this is to write a truth table:

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

← We see that whenever $p \to q$ and $p$ are both true, then it must be that $q$ is true too.

Another proof is just to say that by definition of $p \to q$, when $p \to q$ is true and $p$ is true then $q$ is true. $\boxtimes$

We give this argument the special name "modus ponens" because it is a basic rule of inference used often in proofs of validity of other arguments.

Some other common rules of inference are:

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array} \quad \text{"conjunction"}, \qquad \begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array} \quad \text{"disjunctive syllogism"}, \qquad \begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array} \quad \text{"hypothetical syllogism"}$$

See §1.4 of the book for more rules of inference.

Let's prove one more important one:

Thm
$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$
"modus tollens"

Pf: Since the contrapositive $\neg q \to \neg p$ is logically equivalent to $p \to q$, we can "replace" $p \to q$ w/ $\neg q \to \neg p$ to get an equivalent argument (valid if and only if original is valid).

But then $\neg q \to \neg p$, $\neg q$ / $\therefore \neg p$ is an instance of modus ponens $\boxtimes$

We see here the usefulness of logical equivalence for deductive reasoning.

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Now let's consider the 1st argument we saw. Letting

$p$: The murderer is Joe.
$q$: The murderer is Bob.
$r$: The murderer is right-handed,

the argument had the form

$p \vee q$ ("Joe or Bob is murderer")
$r$ ("murderer is right-handed")
$p \rightarrow \neg r$ ("If Joe is the murderer, the murderer is not right-handed")
$\therefore q$ ("Therefore, murderer is Bob")

Pf that this argument is valid: from "double negation" we know that $r$ is logically equivalent to $\neg(\neg r)$.
Then $\neg(\neg r)$ and $p \rightarrow \neg r$ yields $\neg p$ by modus tollens.
Finally $\neg p$ and $p \vee q$ yields $q$ by disjunctive syllogism. ☒

- While it is always theoretically possible to write a truth table to check the validity of arguments, using the common rules of inference is more convenient.

- Now let's look at an invalid argument:

If I get a B on the final, then I will pass the class.
I passed the class
Therefore, I got a B on the final.

This argument has the form

$p \rightarrow q$ where $p$ = "I get a B on the final"
$q$ $\quad$ $q$ = "I passed the class"
$\therefore p$

could check a truth table for this claim...

It is invalid because $p \rightarrow q$ and $q$ can be true while $p$ is false. This kind of invalid argument is so common that it has a name: the fallacy of affirming the consequent.

(Here "fallacy" = "an invalid argument.") //

# Propositional formulas and Quantifiers §1.5

We mentioned a while ago that basic statements in math like

"$n$ is an odd integer"

do not qualify as propositions b.c. they involve a variable (like $n$) and may be true or false depending on the value of $n$. We will now consider these:

**Def'n** A propositional formula $P(x)$ is a statement involving a variable $x$, such that for each $x \in D$, $P(x)$ is a proposition (i.e.) either true or false). Here $D$ is a set called the domain of discourse.

E.g. If the domain of discourse is the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of nonnegative integers, then $P(n) =$ "$n$ is an odd integer" is a propositional formula.

For each $n \in \mathbb{N}$, it determines a proposition:

$P(1) =$ "1 is an odd integer", which is true;

$P(2) =$ "2 is an odd integer," which is false.

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Knowing the domain of discourse $D$ of a prop. formula is very important, but $D$ is often implicit.

E.g. $P(x) =$ "$x^2 \geq 0$" is a propositional formula, where we implicitly assume that the domain of discourse is the set of real numbers $\mathbb{R}$.

Notice something special about this $P(x)$:
for every real number $x \in \mathbb{R}$, the proposition $P(x) =$ "$x^2 \geq 0$" is true.

We will often want to talk about claims like this:

Def'n If $P(x)$ is a prop. formula w/ domain of discourse $D$, the statement

"for every $x \in D$, $P(x)$"

(often abbreviated "for every $x$, $P(x)$")

is called a universally quantified statement.
It is denoted symbolically

$$\forall x \; P(x)$$

Where "$\forall$" is read "for all".

Even though $P(x)$ by itself is not a proposition, $\forall x \, P(x)$ is a proposition: and it is true exactly when for all $x \in D$, $P(x)$ is true.

E.g. The proposition "$\forall x, \; x^2 \geq 0$" (where we assume the domain of discourse is $D = \mathbb{R}$) is true: this expresses a well-known property of real #'s, that for every real number, its square is nonnegative.

E.g. The proposition "$\forall x, \; x^2 > 0$" ($\leftarrow$ strict inequality) (where again $D = \mathbb{R}$) is false: since for $x = 0$, have $x^2 = 0^2 = 0$, which is not $> 0$.

Notice: to show a universally quantified statement is false, just have to find one counterexample

(A <u>counterexample</u> is a $x \in D$ s.t. $P(x)$ is <u>false</u>.)

<u>E.g.</u> The statement "Every planet in the solar system has a moon" is a universally quantified statement;

- the domain of discourse is $D = \{$planets in solar system"
- the prop. formula is $P(x) =$ "planet $x$ has a moon."

It is <u>false</u>, since Mercury has no moons (nor does Venus).

<u>E.g.</u> Consider a different kind of statement:
"There is some planet in the solar system which has a moon."
This proposition is <u>true</u>: Earth has a Moon (as do several other planets). This kind of statement is called an <u>existentially quantified statement</u>.

<u>Def'n</u> For prop. formula $P(x)$ w/ discourse domain $D$, statement
"there is some $x \in D$ such that $P(x)$"
(or "there is $x$ such that $P(x)$") is an <u>existentially quantified statement</u>. Symbolically, written $\exists x\ P(x)$ where $\exists =$ "there exists"

Proposition $\exists x P(x)$ is true when there is <u>at least one</u> $x \in D$ s.t. $P(x)$ is true.

<u>E.g.</u> The statement "$\exists x,\ x^2 = 9$" is <u>true</u> (if we interpret the domain of discourse as $D = \mathbb{R}$) since for $x = 3$, $x^2 = 3^2 = 9$ (and also for $x = -3$) is <u>true</u>.

You might think that the "for all" and "there exists" statements seem "opposite" to each other in the same way that and / or are "opposite". This is true:

Thm (Generalized De Morgan's Laws)

(a) $\neg(\forall x\ P(x)) \equiv \exists x\ \neg P(x)$

(b) $\neg(\exists x\ P(x)) \equiv \forall x\ \neg P(x)$

Pf: We prove (a) since (b) is very similar.
To say $\neg(\forall x\ P(x))$ means exactly that there is some $x$ ($\in D$) for which $P(x)$ is false, i.e., for which $\neg P(x)$ is true.
But this is exactly what $\exists x\ \neg P(x)$ expresses. ∎

The reason this is called "Generalized De Morgan" is because when the domain of discourse is $D = \{x_1, x_2, \ldots\}$
$\neg(\forall x\ P(x))$ is the same as $\neg(P(x_1) \wedge P(x_2) \wedge \cdots)$
which is log. equiv. to $(\neg P(x_1)) \vee (\neg P(x_2)) \vee \cdots$
by repeated use of usual De Morgan's Law.

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E.g. Let $P(x)$ be $\frac{1}{x^2+1} > 1$ (w/ $D = \mathbb{R}$ as usual).
We will show $\exists x\ P(x)$ is false by showing $\forall x\ \neg P(x)$. To see this, recall that

$$\forall x \in \mathbb{R},\quad x^2 \geq 0$$

so $\forall x \in \mathbb{R},\quad x^2 + 1 \geq 1$

dividing both sides by $x^2+1$ (which is $\geq 1$) gives

$$\forall x \in \mathbb{R},\quad 1 \geq \frac{1}{x^2+1}$$

Which is the same as

$$\forall x \in \mathbb{R},\quad \neg\left(\frac{1}{x^2+1} > 1\right).$$

//.

<u>Warning</u>: translating quantified English statements to their symbolic logic equivalents can be even more tricky... have to use <u>common sense</u>!

<u>E.g.</u> Consider the famous idiom

(*) "All that glitters is not gold."

(which just means "looks can sometimes be deceiving.").

If we let $P(x) =$ "x glitters"

and $Q(x) =$ "x is gold"

then a very literal translation of (*) would be

$$\forall x, (P(x) \to \neg Q(x))$$

i.e., "for every thing, if that thing glitters then it is not gold".

But the real meaning of (*) is not that, it is:

$$\neg (\forall x\ P(x) \to Q(x))$$

i.e., "It is Not the case that every thing which glitters is gold."

because (*) is certainly not asserting that gold does not glitter.

<u>Upshot</u>: English is not very consistent about where to put negatives in universally quantified sentences.

<u>Exercises</u>: take some other common idioms like

"Not all those who wander are lost"

"Everything is not as it seems"

"Everyone has their price"

and convert them to symbolic logic statements.