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Longest increasing subsequences:

DEF'N Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n$ be a permutation. A subsequence of σ is $\sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k}$ for $i_1 < \cdots < i_k$ and is increasing if $\sigma_{i_1} < \sigma_{i_2} < \cdots < \sigma_{i_k}$.
Let $\mathrm{lis}(\sigma) :=$ length of longest increasing subsequence

e.g. For $\sigma = \underline{2}\,\underline{4}\,\underline{7}\,9\,\underline{5}\,1\,3\,\underline{6}\,\underline{8}$ have $\mathrm{lis}(\sigma) = 5$ with longest increasing subsequence underlined.

Note: L.I.S. need not be unique: $1\;2\;4\;3$

Increasing subsequences are a basic kind of permutation pattern (ask Prof. Burstein for more info...)
Studying LIS's is very natural from point of view of statistical analysis of time series data.

There is a close connection between the Robinson-Schensted algorithm and longest increasing subsequences:

Thm Suppose $\sigma \xmapsto{RS} (P, Q)$ w/ $\mathrm{sh}(P) = \lambda = (\lambda_1, \lambda_2, \ldots)$.
Then $\lambda_1 = \mathrm{lis}(\sigma)$.

e.g. $\sigma = 5\,\underline{2}\,\underline{3}\,6\,\underline{4}\,1\,\underline{7} \xmapsto{RS} \left(P = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array},\; Q = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array} \right)$

and indeed $\lambda_1 = 4 = \mathrm{lis}(\sigma)$.
But note: 1st row of P $(= 1\,3\,4\,7)$ is not a LIS of σ (just has same length)

Pf of thm: Suppose $\phi = P_0, P_1, \ldots, P_n = P$ is the sequence of insertion tableaux we build up when inserting $\sigma_1, \sigma_2, \ldots, \sigma_n$.

Claim: When inserting σ_k into P_{k-1}, if it enters in the j^{th} column, then the longest increasing subsequence ending at σ_k has length j.

Pf: By induction. The case $k=1$ is fine. So suppose x is entry in P_{k-1} in position $(1, j-1)$ (i.e., left of σ_k). Then by induction there is a subsequence σ' of $\sigma_1, \ldots, \sigma_{k-1}$ of length $j-1$ ending at x, and since $x < \sigma_k$ (or else we would've bumped it), the concatenation $\sigma' \sigma_k$ is a length j increasing subsequence. Similarly, to show there cannot be a longer subsequence, let $y \in \{\sigma_1, \ldots, \sigma_{k-1}\}$ be s.t. $y < \sigma_k$. By induction, when we inserted y we did so at col. with longest subseq. ending at y, call it j'. Cannot have $j' \geq j$, otherwise we would've inserted σ_k into a later column. So $j' < j$, and so longest inc. subseq. ending at σ_k can have length at most $j' + 1 \leq j$. ✓ ▨

What about the whole shape $\lambda = (\lambda_1, \lambda_2, \ldots)$?

Thm (Greene) Suppose $\sigma \overset{RS}{\mapsto} (P, Q)$ w/ $sh(P) = \lambda$. Then for all k, $\lambda_1 + \lambda_2 + \cdots + \lambda_k$ = length of longest subsequence of σ that is a union of k increasing subsequences.

$k=2$

$5+3=8$ ✓

e.g. w/ $\sigma = 247951368$ have $P = \begin{array}{|c|c|c|c|c|}\hline 1&3&5&6&8\\\hline 2&4&9\\\cline{1-3} 7\\\cline{1-1}\end{array}$ and

$2479 \sqcup 1368$ is a union of 2 increasing subsequences.

4/15 Can define decreasing subsequences of perm. σ analogously, and let $\mathrm{lds}(\sigma) :=$ length of longest decr. subseq.

Thm If $\sigma \overset{RS}{\mapsto} (P,Q)$ w/ $sh(P) = \lambda$, then $\mathrm{lds}(\sigma) = \underbrace{\ell(\lambda)}_{\text{length of } \lambda}$. $(= \lambda_1^t)$

In fact, this follows immediately from...

Thm* For $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$ let $\sigma^{rev} = \sigma_n \sigma_{n-1} \cdots \sigma_1$. Then if $\sigma \overset{RS}{\mapsto} (P,Q)$ have $\sigma^{rev} \mapsto (P', Q')$ where
$P' = P^t$ ← transpose.

To prove this symmetry property of RS, can use column insertion, which works same as (row) insertion, but where we try to put # into 1st column, and bump #'s from i^{th} column to $(i+1)^{st}$ column, etc.

Key Lemma Row and column insertions commute, i.e.,
$$T \xleftarrow[row]{} a \xleftarrow[col]{} b = T \xleftarrow[col]{} b \xleftarrow[row]{} a.$$

Pf: See Sagan. ☒

Pf of thm*: $P' = \sigma_1 \xrightarrow[row]{} \cdots \sigma_{n-1} \xrightarrow[row]{} \sigma_n \xrightarrow[row]{} \varnothing$ (1st insertion is same w/ row or col)

$= \sigma_1 \xrightarrow[row]{} \cdots \sigma_{n-1} \xrightarrow[row]{} \sigma_n \xrightarrow[col]{} \varnothing$

$= \sigma_n \xrightarrow[col]{} \sigma_1 \xrightarrow[row]{} \cdots \sigma_{n-1} \xrightarrow[row]{} \varnothing$ (key lemma)

$= \sigma_n \xrightarrow[col]{} \sigma_{n-1} \xrightarrow[col]{} \cdots \sigma_1 \xrightarrow[col]{} \varnothing$ (repeat)

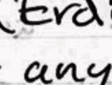
$= (\sigma_n \xrightarrow[row]{} \sigma_{n-1} \xrightarrow[row]{} \cdots \sigma_1 \xrightarrow[row]{} \varnothing)^t$ (transpose of col insert = row insert)

$= P^t$. ✓

☒

Cor (Erdős-Szekeres Theorem)

For any $\sigma \in S_{(n-1)(m-1)+1}$, have either
$\mathrm{lis}(\sigma) \geq n$ or $\mathrm{lds}(\sigma) \geq m$.

Pf: Best way to minimize width and length of a partition is $\lambda =$  but we need one more box ✓ ∎

Q: What is the expected length of longest incr. subseq. of a random permutation?

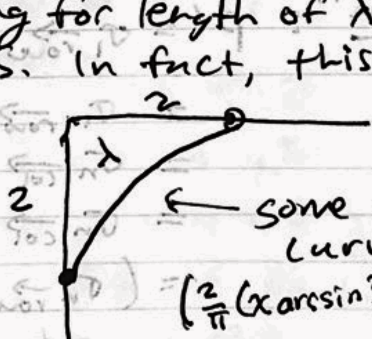
Let $X_n := \mathrm{lis}(\sigma)$ for $\sigma \in S_n$ (uniformly) random.

Ulam's Problem (c. 1960's): Compute $\lim_{n\to\infty} \frac{\mathbb{E}\, X_n}{\sqrt{n}} = c$.

E-S Thm says for any $\sigma \in S_n$, have $\mathrm{lis}(\sigma) \geq \sqrt{n}$ or $\mathrm{lis}(\sigma^{rev}) \geq \sqrt{n}$
so that $c \geq \frac{1}{2}$. In fact...

Thm (Logan-Shepp, Kerov Vershik, 1977)
Solution to Ulam's Problem is $c = 2$.

Idea of pf: Same as asking for length of λ when we insert $\sigma \in S_n$ into RS. In fact, this random partition λ has a precise limit shape (rescaling by $\frac{1}{\sqrt{n}}$):



$\left(\frac{2}{\pi}\left(x \arcsin\frac{x}{2} + \sqrt{4-x^2}\right)\right)$ ∎

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Representation Theory of finite Groups:

In the last couple days, I want to explain why ring of sym. fn.'s is important in algebra...

DEF'N Let V be an n-dim'l vector space over $\mathbb{C}$. The general linear group $GL(V) = \{\text{invertible linear maps } V \to V\}$, I.e., $GL(V) \simeq \{n \times n\ \mathbb{C}\text{-matrices } M \text{ w/ } \det(M) \neq 0\}$.

Note: $GL(V)$ is an infinite group.

Let G be a finite group. We want to "represent" G by matrices.

DEF'N A representation of G is a group homomorphism $\varphi: G \to GL(V)$ for some v.s. V. In other words, for each $g \in G$ we have a matrix $\varphi(g)$, and:

- $\varphi(gh) = \varphi(g) \cdot \varphi(h) \ \forall g, h \in G$,
- $\varphi(e) = I_n$ identity matrix.

A representation of G is very similar to an action, except it is linear: we act by matrices, not permutations.

e.g. For any V and any G, can set $\varphi(g)(v) = v \ \forall v \in V$, i.e., $\varphi(g) = I_n$ identity matrix. This is called the trivial representation and is boring...

e.g. Suppose $G \curvearrowright X$ a finite set. Let $\mathbb{C}[X] := \{\sum_{x \in X} c_x x : c_x \in \mathbb{C}\}$ be v.s. of formal linear combinations of elements of X. Then $\mathbb{C}[X]$ is a G representation where $\varphi(g)(x) = g \cdot x$ for all basis vectors $x \in \mathbb{C}[X]$. In other words, each $\varphi(g)$ is the permutation matrix of its corresponding permutation. This is called a permutation representation.

e.g. Let $G = \mathbb{Z}/n\mathbb{Z} = \{0, 1, \dots, n-1\}$. Let $V = \mathbb{C}$.
We can define a representation $\varphi : G \to GL(V)$ by
$\varphi(k) = (e^{2\pi i \cdot k/n})$ ← 1×1 matrix $\forall\, k = 0, 1, \dots, n-1$.

e.g. Let $G = S_n$ symmetric gp. and let $V = \mathbb{C}$ ← 1×1 matrix.
The sign representation $\varphi : S_n \to GL(\mathbb{C})$ is $\varphi(\sigma) = (\mathrm{sgn}(\sigma))$, $= (-1)^{\mathrm{inv}(\sigma)}$

e.g. if U, V are G-representations, then direct sum $U \oplus V$ is another representation: as matrices $\left(\begin{array}{c|c} \varphi(g)_U & 0 \\ \hline 0 & \varphi(g)_V \end{array}\right)$ ← "block sum".

DEF'N A repr'n $\varphi : G \to GL(V)$ is irreducible if we cannot find a nontrivial subspace U (i.e., $0 \neq U \neq V$) s.t. $gu \in U$ $\forall u \in U, g \in G$ (i.e., invariant under all G).

important → FACT Every representation V of G is a direct sum $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ of irreducible repr'ns V_i.

e.g. Let $V = \mathbb{C}^n$ w/ standard basis $\{e_1, e_2, \dots, e_n\}$ and $G = S_n$. Let $\varphi : S_n \to GL(V)$ be the standard permutation repr'n, i.e. $\varphi(\sigma) e_i = e_{\sigma(i)}$ $\forall \sigma \in S_n, i = 1, \dots, n$. V is reducible, since $U_1 = \{(c, c, \dots, c) \in V : c \in \mathbb{C}\}$ is a nontrivial invariant subspace. With $U_0 = \{(x_1, \dots, x_n) \in V : x_1 + \dots + x_n = 0\}$, we have $V = U_1 \oplus U_0$ and $\underset{\text{trivial repr'n}}{U_1}$, U_0 are irreducible reprns.

The FACT above says that to understand all G-rep's, it's enough to understand the irreducible ones..

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Characters of representations

Representations $\varphi: G \to GL(V)$ are matrix-valued functions, hence complicated to understand. It turns out we can "reduce" to studying "ordinary" $\mathbb{C}$-valued fn's $\chi: G \to \mathbb{C}$.

DEF'N Let φ be a representation of finite group G. Its character $\chi_\varphi: G \to \mathbb{C}$ is the function
$\chi_\varphi(g) = Tr(\varphi(g))$ ← trace of matrix for all $g \in G$.

e.g. If V is 1-dim'l, then φ and χ_φ are the same thing...

e.g. If φ is the permutation rep'n of an action $G \curvearrowright X$ then $\chi_\varphi(g) = \#Fix(g: X \to X)$ ← why? think abt. perm. matrix...

FACT For two G-reps $\varphi_1: G \to GL(V_1)$, $\varphi_2: G \to GL(V_2)$ have $\chi_{\varphi_1} = \chi_{\varphi_2} \Leftrightarrow \varphi_1$ isomorphic to φ_2

($\varphi_1 \simeq \varphi_2$ means $\exists$ v.s. iso. $V_1 \simeq V_2$ that commutes w/ G-action)

Upshot: enough to study characters, in fact, since we have $\chi_{\varphi_1 \oplus \varphi_2} = \chi_{\varphi_1} + \chi_{\varphi_2}$, enough to study characters of irreducible rep'ns (+ their lin. comb.'s).

In fact, characters χ are not just any kind of function $G \to \mathbb{C}$...

DEF'N A conjugacy class of G is set of the form $C = \{ghg^{-1} : g \in G\}$ for some $h \in G$. A function $f: G \to \mathbb{C}$ is called a class function if it is constant on conjugacy classes, i.e. $f(h) = f(ghg^{-1})\ \forall g, h \in G$.
Let $Cl(G) :=$ v.s. of class functions $f: G \to \mathbb{C}$.

Prop. Any character χ_φ is a class function.

Pf: $\chi_\varphi(ghg^{-1}) = Tr(ghg^{-1}) = Tr(g^{-1}\cdot gh) = Tr(h)$

recall $Tr(AB) = Tr(BA)$ for matrices A, B ☒

FACT 1. $\{\chi_{\varphi_1}, \dots, \chi_{\varphi_K}\}$ is a basis of $Cl(G)$, where $\varphi_1, \dots, \varphi_K$ are all the irrep's of G (up to iso.).

2. With the inner product $\langle , \rangle : Cl(G) \times Cl(G) \to \mathbb{C}$ given by $\langle f, f' \rangle := \frac{1}{\#G} \sum_{g \in G} f(g) \overline{f'(g)}$, the basis $\{\chi_{\varphi_1}, \dots \chi_{\varphi_K}\}$ is orthonormal.

3. If $\varphi = \bigoplus_m c_m \varphi_m$ is decomposition of φ into irrep's, then $c_m = \langle \chi_\varphi, \chi_{\varphi_m} \rangle$.

Note in particular that
#irreps (irreducible repr'ns) $= \dim Cl(G)$
$= \#$conjugacy classes of G.

e.g. G acts on itself by multiplication on the left, and corresponding perm. rep. is called the regular rep'n. $\mathbb{C}[G]$
How does $\mathbb{C}[G]$ decompose into irrep's?

$$\langle \chi_{\mathbb{C}[G]}, \chi_{\varphi_m} \rangle = \frac{1}{\#G} \sum_{g \in G} \chi_{\mathbb{C}[G]}(g) \overline{\chi_{\varphi_m}(g)}$$

$$\chi_{\mathbb{C}[G]}(g) = \#Fix(g : G \to G) = \begin{cases} \#G & \text{if } g = e \\ 0 & \text{otherwise} \end{cases}$$

$$= \frac{1}{\#G} \cdot \#G \cdot \overline{\chi_{\varphi_m}(e)} = \dim(\varphi_m).$$

Hence

$$\#G = \dim \mathbb{C}[G] = \dim\left(\bigoplus_m \dim(\varphi_m) \cdot \varphi_m\right) = \sum_m (\dim \varphi_m)^2$$

looks familiar...

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Characters of the Symmetric Group.

Finally, by focusing on case $G = S_n$, we see symmetric functions.

Prop. Two permutations $\sigma, \sigma' \in S_n$ belong to same conjugacy class $\iff$ they have the same cycle structure.

Pf: Exercise for you. $\boxtimes$

So # conj. classes in S_n = # cycle structures = # partitions $\lambda \vdash n$.

So # irrep's of S_n = # $\lambda \vdash n$, and in fact there is a standard way to index irrep's by partitions.

e.g. Let triv: $S_n \to GL(\mathbb{C})$ be the trivial rep'n. Then

$$\text{triv} = \varphi_{\underbrace{\square\square\square\square}_{n \text{ boxes}}} = \varphi_{(n)}$$

e.g. For sgn: $S_n \to GL(\mathbb{C})$ sign rep'n, $\text{sgn} = \varphi_{\text{column}} = \varphi_{(1^n)}$.

e.g. Recall standard perm rep'n $\mathbb{C}^n = \underset{\text{triv}}{U_1} \oplus U_0$ ← irreducible, dim $= n-1$,

then $U_0 = \varphi_{\text{row with one box below}} = \varphi_{(n-1,1)}$.

Write $\chi_\lambda = \chi_{\varphi_\lambda}$ = character of irrep indexed by $\lambda \vdash n$.

DEF'N The Frobenius characteristic $Fr: Cl(S_n) \to Sym(n)$ ← recall = sym. fn's of degree n

is given by $Fr(\delta_\lambda) = p_\lambda$ ← power sum

where δ_λ is class function $\delta_\lambda(\sigma) = \begin{cases} z_\lambda & \text{if cycle type}(\sigma) = \lambda \\ 0 & \text{otherwise} \end{cases}$

and $z_\lambda = 1^{m_1} m_1! \, 2^{m_2} m_2! \cdots = \dfrac{n!}{\#\text{ perm's in } S_n \text{ w/ cycle type} = \lambda = (1^{m_1} 2^{m_2} \cdots)}$.

Since the δ_λ are a basis of $Cl(S_n)$ and p_λ are a basis of $Sym(n)$, this is clearly a v.s. isomorphism.

Thm $Fr(\chi_\lambda) = s_\lambda$ ← Schur function.

This is (one reason) why Schur fn's are so important!

Cor $\dim \varphi_\lambda = f^\lambda = \#SYT$ of sh. λ

Pf: Via Fr, same as coeff. of $[x_1 x_2 \cdots x_n]$ in $s_\lambda = f^\lambda$ ✓ $\boxtimes$

More generally ...

Cor If $\chi_\lambda(\mu)$ = ch. evaluated at a perm. of cycle type μ,

then $s_\lambda = \sum_\mu \chi_\lambda(\mu) \cdot z_\mu^{-1} \, p_\mu$.

∃ combinatorial rule for these coeff's, called the Murnaghan-Nakayama rule.

Also note that ... by the regular representation, have

$$n! = \# S_n = \sum_{\lambda \vdash n} \dim(\varphi_\lambda)^2 = \sum_{\lambda \vdash n} (f^\lambda)^2,$$

which we saw earlier using R.S. algorithm.

Finally, using something called the induction product of representations of $S_k \times S_{n-k} \to S_n$,

we can get ring structure on $Sym = \bigoplus Sym(n)$.

Structure constants $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^{\nu} \, s_\nu$ are called Littlewood-Richardson coefficients, also very important!

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e.g. Character table for S_3:

	$(1)(2)(3)$	$(12)(3)$, $(13)(2)$, $(1)(23)$	(123), (132)
$\chi_{triv} = \chi_{(3)}$	1	1	1
$\chi_{sgn} = \chi_{(1,1,1)}$	1	-1	1
$\chi_{std} = \chi_{(2,1)}$	2	0	1

to compute these, use $S_3 \curvearrowright \mathbb{C}^3 = \underset{u_1}{\rho_{triv}} \oplus \underset{u_0}{\rho_{std}}$

so $\chi_{std}(\sigma) = \#\mathrm{Fix}(\sigma) - 1$ ✓

So e.g. $s_{(2,1)} = \frac{1}{3!}\left(2\cdot 1\cdot p_{(1,1,1)} + 0\cdot 3\cdot p_{(2,1)} + 1\cdot 2\cdot p_{(3)}\right)$ //

Thanks for taking my course!

There are many more things to be said about symmetric functions (+ combinatorics in general) so please don't hesitate to ask me about anything you might be interested in learning more about.

Have a nice summer!