Total: 48/50 + 5 bonus points for presenting = 53/50

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1a Suppose $\mu, \lambda \vdash n$, $\mu \leq \lambda$, and $\mu \neq \lambda$. Then $\mu_1 \leq \lambda_1$, $\mu_1 + \mu_2 \leq \lambda_1 + \lambda_2$, etc. Assume $\mu_1 = \lambda_1$ and $\mu_1 + \mu_2 = \lambda_1 + \lambda_2$. Then $\mu_2 = \lambda_2$. Similarly, assume $\mu_1 + \mu_2 + \mu_3 = \lambda_1 + \lambda_2 + \lambda_3$. Then $\mu_3 = \lambda_3$. Suppose we continue making assumptions in this fashion until we reach $\mu_1 + \mu_2 + \dots + \mu_n \leq \lambda_1 + \lambda_2 + \dots + \lambda_n$, where $n = \ell(\mu) = \ell(\lambda)$. We can't have $\mu_1 + \mu_2 + \dots + \mu_n = \lambda_1 + \lambda_2 + \dots + \lambda_n$ because then $\mu_i = \lambda_i$ for $i \in [n]$, which means $\mu = \lambda$, which contradicts the initial supposition. Thus $\mu_1 + \mu_2 + \dots + \mu_n < \lambda_1 + \lambda_2 + \dots + \lambda_n$, so $\mu_n < \lambda_n$, so $\mu \prec \lambda$.

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Suppose $x = \ell(\mu) \neq \ell(\lambda) = y$, and assume that $\mu_i = \lambda_i$ for $i < \min(x, y)$. If $x > y$, then $\mu_y < \lambda_y$, otherwise μ will not be a partition of n since $\mu_{y+1}, \dots, \mu_x$ are unaccounted for. Thus $\mu \prec \lambda$. If $x < y$, then $\mu_y > \lambda_y$ via similar reasoning to the above, so $\mu \geq \lambda$, which is a contradiction. Therefore there must be some $i \leq x$ such that $\mu_i \leq \lambda_i$, so $\mu \prec \lambda$.

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Good. (And as I mentioned during your presentation, one way to avoid considering the length of the partitions is to pad them with parts of 0's so that they have the same length.)

b $\mu = \{4, 4, 2, 2, 2\}$, $\lambda = \{4, 4, 3, 1, 1, 1\}$. $\mu \prec \lambda$, but we have neither $\mu \leq \lambda$ nor $\mu \geq \lambda$. Good.

where $\geq$ is dominance order

2 Claim: $p_\lambda = \alpha^\lambda_\lambda m_\lambda + \sum_{\mu \geq \lambda} \alpha^\lambda_\mu m_\mu$ for coeffs. $\alpha^\lambda_\mu \in \mathbb{C}$. Consider expanding $p_\lambda = (x_1^{\lambda_1} + x_2^{\lambda_1} + \dots)(x_1^{\lambda_2} + x_2^{\lambda_2} + \dots) \dots (x_1^{\lambda_k} + x_2^{\lambda_k} + \dots)$. It is clear that we can obtain a term of $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_k^{\lambda_k}$, so $\alpha^\lambda_\lambda \neq 0$. Additionally, we need only consider $\mu \geq \lambda$ because the exponent vectors of any μ other than λ can be obtained by summing parts of λ, and thus we must have $\mu \geq \lambda$.

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Yes, "summing" or "combining" parts of lambda.

Claim: $e_\lambda = \sum_{\mu \leq \lambda^t} \beta^\lambda_\mu m_\mu$ for coeffs. $\beta^\lambda_\mu \in \mathbb{C}$ w/ $\leq$ being dominance order. Consider expanding $e_\lambda = (x_1 x_2 \dots x_{\lambda_1} + \dots)(x_1 x_2 \dots x_{\lambda_2} + \dots) \dots$. Then the greatest monomial in dominance order we can build is $x_1^{\lambda^t_1} x_2^{\lambda^t_2} \dots \Rightarrow \beta^{\lambda}_{\lambda^t} \neq 0$ & if $\beta^\lambda_\mu \neq 0$, then $\mu \leq \lambda^t$.

Could say a bit more here: the idea is basically the same as with the power sums, but with combining columns instead of rows of the partition (i.e., parts of the conjugate instead of parts of the original partition.) [-1pt]

3 Let $\lambda \vdash n$, and f^λ be the coeff. of $x_1 x_2 \dots x_n$ in the Schur function $S_\lambda(x_1, x_2, \dots)$. Then f^λ represents fillings of a SSYT of $sh = \lambda$ with $x_1, x_2, \dots, x_n$ each appearing once. This means values will be strictly increasing both from left to right and top to bottom. This means filling the rows of λ is identical to filling the columns of λ^t, and the same can be said of filling the columns of λ with respect to the rows of λ^t. Thus $f^\lambda = f^{\lambda^t}$. Good.

Consider $\lambda = \{3\}$, $\lambda^t = \{1,1,1\}$. $S_\lambda = m(3) + m(2,1) + m(1,1,1)$, but $S_{\lambda^t} = m(1,1,1)$.

Good.

4 Suppose $A = (A_{i,j})$ is an $m \times n$ matrix & $B = (B_{i,j})$ is an $n \times m$ matrix.

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Consider the network with source vertices $s_1, \dots, s_m$, internal vertices $k_1, \dots, k_n$, and target vertices $t_1, \dots, t_m$. Give edges $s_i \to k_j$ weight $A_{i,j}$, and edges $k_i \to t_j$ weight $B_{i,j}$. Then each entry $AB_{i,j} = \sum_x^n A_{i,x} B_{j,x}$. Fix some arbitrary $I \subset [n]$. Let P_{SI} be the set of all disjoint paths from all source vertices to only internal vertices w/ indices in I. Similarly, let P_{IT} be the set of all disjoint paths from internal vertices w/ indices in I to any target vertices. Consider $\det(A) \cdot \det(B) =$ $= \sum_{Q \in P_{SI}} \operatorname{sgn}(Q) wt(Q) \cdot \sum_{R \in P_{IT}} \operatorname{sgn}(R) wt(R) = \sum_{P \in P_{SI} \times P_{IT}} \operatorname{sgn}(P) wt(P)$, where $P_{SI} \times P_{IT}$ contains all nonintersecting tuples of paths passing through internal vertices w/ indices in I, effectively restricting to a particular set of columns & rows for A & B. Thusly we obtain $\sum_{I \subseteq [n], \#I = m} \det(A|_{cols = I}) \det(B|_{rows = I})$, which lets us sum over all non-intersecting m-tuples of paths between the source vertices & target vertices. Furthermore, $\sum_{I \subseteq [n], \#I = m} \det(A|_{cols = I}) \det(B|_{rows = I}) = \sum_{I \subseteq [n], \#I = m} \left(\sum_{P \in P_{SI} \times P_{IT}} \operatorname{sgn}(P) wt(P) \right) =$ $= \sum_P \operatorname{sgn}(P) wt(P) = \det(AB)$, so $\det(AB) = \sum_{I \subseteq [n], \#I = m} \det(A|_{cols = I}) \det(B|_{rows = I})$.

Good, but there is one subtle thing to check for this proof: that sgn(Q) * sgn(R) = sgn(P). This is true, but requires at least a little argument. [-1pt]

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5 $m_\lambda(1,1,\ldots,1)$ w/ k 1's represents the number of monomials in k variables w/ exponent sequence equivalent to a permutation of λ. This can be represented by $k!/(k-\ell(\lambda))!$. However, we must account for the arrangements of elements with the same power, so we divide by the factorials of the multiplicities of the nonzero parts of λ.

This gives $m_\lambda(\underbrace{1,1,\ldots,1}_{k\text{ 1's}}) = \dfrac{k!}{(k-\ell(\lambda))! \cdot \prod_{i\geq 1}(m_i(\lambda)!)}$.

Good. (Note this can also be written as a certain mulitnomial coefficient.)