

# Order polynomial product formulas and poset dynamics

GW/Howard Combinatorics & Algebra Seminar

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# Plane partitions

A  $a \times b$  *plane partition* is an  $a \times b$  array of nonnegative integers that are weakly decreasing in rows and columns.

Let  $\mathcal{PP}^m(a \times b) := \{a \times b \text{ plane partitions with entries } \leq m\}$ :

5	3	3	2	2	1
4	2	1	1	1	0
4	1	1	1	0	0
3	1	1	0	0	0

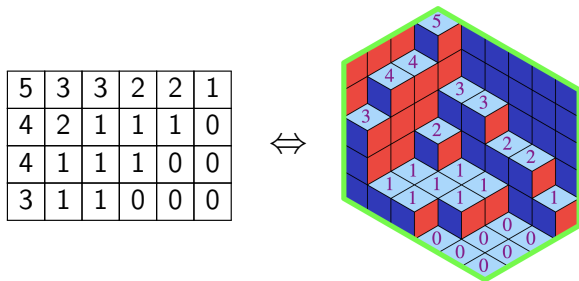
 $\in \mathcal{PP}^5(4 \times 6)$

Theorem (MacMahon's formula (c.1915) for plane partitions in a box)

$$\#\mathcal{PP}^m(a \times b) = \prod_{i=1}^a \prod_{j=1}^b \frac{m+i+j-1}{i+j-1}$$

## Other guises of plane partitions

Plane partitions have a beautiful 3D representation as stacks of boxes:



In this way they correspond to *lozenge tilings* of hexagonal regions of the triangular lattice, a well-studied planar statistical mechanical model.

Plane partitions are also intimately related to the *representation theory of classical groups*, because  $\mathcal{PP}^m(a \times b)$  indexes a basis of the irreducible representation  $V^\lambda$  of  $\mathfrak{sl}(a+b)$  with highest weight  $\lambda = m^a$ .

# Toggling plane partitions

(*Piecewise-linear*) *toggling* of an entry of a plane partition  $\pi \in \mathcal{PP}^m(a \times b)$  does the following:

$$\begin{array}{|c|c|c|} \hline & v & \\ \hline w & x & y \\ \hline & z & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|} \hline & v & \\ \hline w & x' & y \\ \hline & z & \\ \hline \end{array} \quad \max(y, z) - x \rightarrow x' - \min(v, w)$$

with  $x' = \max(y, z) + \min(v, w) - x$ .<sup>1</sup> Toggling an entry is an involution.

Let  $t_{ij}: \mathcal{PP}^m(a \times b) \rightarrow \mathcal{PP}^m(a \times b)$  be toggling at entry  $(i, j)$ .

## Example

$$t_{2,2} \left( \begin{array}{|c|c|c|} \hline 5 & 4 & 1 \\ \hline 3 & 1 & 1 \\ \hline \end{array} \right) \in \mathcal{PP}^5(2 \times 3) = \begin{array}{|c|c|c|} \hline 5 & 4 & 1 \\ \hline 3 & 3 & 1 \\ \hline \end{array}$$

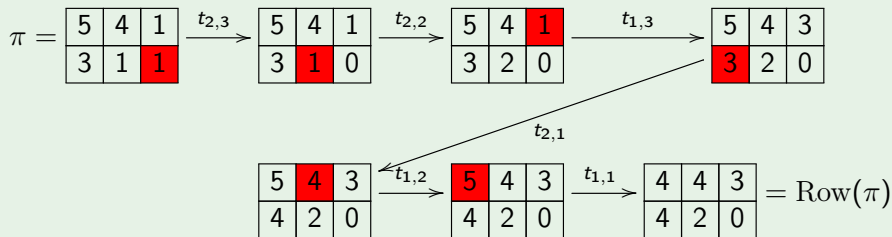
<sup>1</sup>If  $v$  or  $w$  don't exist, treat them as  $m$ ; if  $y$  or  $z$  don't exist, treat them as 0.

# Rowmotion on rectangular plane partitions

(*Piecewise-linear*) rowmotion  $\text{Row}: \mathcal{PP}^m(a \times b) \rightarrow \mathcal{PP}^m(a \times b)$  consists of toggling *all* the entries, in sequence, from bottom-right to top-left.

## Example

Let's compute rowmotion of a plane partition  $\pi \in \mathcal{PP}^5(2 \times 3)$ :



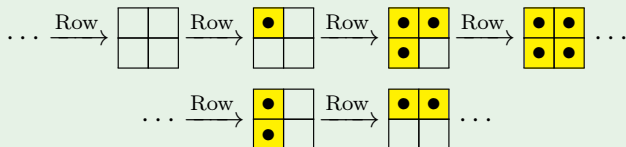
Piecewise-linear rowmotion was introduced by Einstein–Propp, 2013.

# Combinatorial rowmotion

Let  $\mathcal{J}(a \times b)$  be the set of sub-Young diagrams inside the  $a \times b$  rectangle. We have  $\mathcal{J}(a \times b) \simeq \mathcal{PP}^1(a \times b)$  via the indicator function.

*(Combinatorial) rowmotion*  $\text{Row}: \mathcal{J}(a \times b) \rightarrow \mathcal{J}(a \times b)$  sends  $\mu \subseteq a \times b$  to the smallest Young diagram containing the minimal elements of  $(a \times b) \setminus \mu$ .

## Example

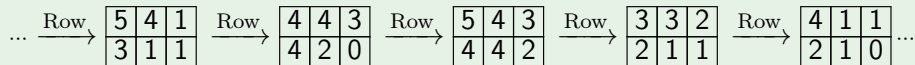


Combinatorial rowmotion was introduced by Brouwer–Schrijver, 1974. Cameron–Fon-der-Flaass, 1995 showed how to write combinatorial rowmotion as a composition of combinatorial toggles, and Einstein–Propp's contribution was generalizing their description to the piecewise-linear realm.

# Periodicity for rowmotion on rectangular plane partitions

## Example

One rowmotion orbit in  $\mathcal{PP}^5(2 \times 3)$  is:



**Theorem** (Grinberg–Roby 2015; conjectured by Einstein–Propp)

*The order of  $\text{Row}: \mathcal{PP}^m(a \times b) \rightarrow \mathcal{PP}^m(a \times b)$  is  $a + b$ .*

**Note:** Case  $m = 1$  (combinatorial rowmotion) due to Brouwer–Schrijver. From Kirillov–Berenstein, 1995 and Striker–Williams, 2012 it follows that dynamics are same as *rectangular semistandard Young tableaux promotion*, for which order  $a + b$  is known from Schützenberger, Haiman, Rhoades, ...

# Rowmotion on plane partitions of other shapes

For a Young diagram  $\lambda$ , a *plane partition of shape  $\lambda$*  is a filling of its boxes with nonnegative integers that are weakly decreasing in rows and columns. All of the prior constructions make sense for arbitrary shapes  $\lambda$ . But for a “random”  $\lambda$ , rowmotion will not behave well like it does for rectangles.

## Example

For  $\lambda = (4, 2, 2)$  and for

$$\pi = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 0 & 0 \\ \hline 1 & 1 & & \\ \hline 1 & 0 & & \\ \hline \end{array} \in \mathcal{PP}^1(\lambda)$$

the rowmotion orbit of  $\pi$  has 17 elements. Things get worse from there.

But Grinberg–Roby showed that rowmotion behaves well also for *staircases* and *shifted staircases*, and Johnson–Liu, 2023 showed same for *trapezoids*.



# When does rowmotion behave well? The order polynomial...

What distinguishes the shapes with good rowmotion behavior?

For any shape  $\lambda$ , the function  $\Omega_\lambda(m) = \#\mathcal{PP}^m(\lambda)$  is a *polynomial* in  $m$ , called the *order polynomial* of  $\lambda$ . It was introduced by Richard Stanley.

For example, MacMahon's formula says  $\Omega_{a \times b}(m) = \prod_{i=1}^a \prod_{j=1}^b \frac{m+i+j-1}{i+j-1}$ ; in particular, all roots of  $\Omega_{a \times b}(m)$  are *integers*!

## Example

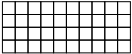



For  $\lambda = (4, 2, 2)$ ,

$$\Omega_\lambda(m) = \frac{1}{720}(m+1)(m+2)^2(m+3)^2(m+4)(m^2+5m+5),$$

which has an irreducible quadratic factor.

Empirically, shapes  $\lambda$  with good rowmotion behavior are those with *order polynomial product formulas*, i.e., with all roots of  $\Omega_\lambda(m)$  in  $\mathbb{Z}$  (or  $\frac{1}{2}\mathbb{Z}$ ).

# Shapes with order polynomial product formulas

<p>Rectangle</p> 	$\prod_{i=1}^a \prod_{j=1}^b \frac{m+i+j-1}{i+j-1}$	$\mathfrak{sl}(n)$	<p>MacMahon c. 1915</p>
<p>Staircase</p> 	$\prod_{1 \leq i \leq j \leq n} \frac{2m+i+j}{i+j}$	$\mathfrak{sp}(2n)$	<p>Proctor 1988  <i>"symmetric, self-complementary plane partitions"</i></p>
<p>Shifted staircase</p> 	$\prod_{1 \leq i \leq j \leq n} \frac{m+i+j-1}{i+j-1}$	$\mathfrak{so}(2n+1)$	<p>Conj. MacMahon 1896,          Andrews/Macdonald c. 1977  <i>"symmetric plane partitions"</i></p>
<p>Shifted Trapezoid</p> 	$\prod_{i=1}^k \prod_{j=1}^{2n-k+1} \frac{m+i+j-1}{i+j-1}$	$\mathfrak{sp}(2n)$	<p>Proctor 1983  <i>"transpose-complementary plane partitions"</i></p>

## More dynamics: promotion of standard Young tableaux

*Standard Young Tableaux (SYTs)* of a shape  $\lambda$  with  $n$  boxes are bijective fillings of the boxes with  $1, \dots, n$ , increasing in rows and columns.

*Promotion*,  $\text{Pro}: \mathcal{SYT}(\lambda) \rightarrow \mathcal{SYT}(\lambda)$ , is the following invertible operation on these SYTs:

- Delete the entry 1.
- Slide boxes into the resulting hole.
- Decrement all entries.
- Fill the hole with  $n$ .

### Example

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} \xrightarrow{\text{del. } 1} \begin{array}{|c|c|c|} \hline \bullet & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} \xrightarrow{\text{slide}} \begin{array}{|c|c|c|} \hline 2 & 4 & 5 \\ \hline 3 & 6 & \bullet \\ \hline \end{array} \xrightarrow{\text{decr.}} \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & \bullet \\ \hline \end{array} \xrightarrow{\text{fill } 6} \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array} = \text{Pro}(T)$$

Along with *evacuation*, defined by Schützenberger to study *RSK algorithm*.

# When does promotion of SYT behave well? Same shapes!

Promotion behaves chaotically for most shapes, but:

## Theorem

- (Schützenberger 1977) For  $\lambda$  a *rectangle*, order of  $\text{Pro}$  is  $n$
- (Edelman–Greene 1987) For  $\lambda$  a *staircase*, order of  $\text{Pro}$  is  $2n$ .
- (Haiman 1992) For  $\lambda$  a *shifted trapezoid* or *shifted double staircase*, order of  $\text{Pro}$  is  $n$ .
- (Haiman–Kim 1992) These are the **only** four families of shapes with good promotion behavior.

Remarkably, these are (basically) the same families of shapes that have good plane partition rowmotion behavior!

# The main heuristic

To summarize, we have seen that:

**shapes with good dynamical properties**  
**= shapes with order polynomial product formulas**

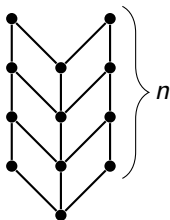
All of the constructions (order polynomial, plane partitions =  $P$ -partitions, rowmotion, SYTs = linear extensions, promotion, ...) we discussed make sense for arbitrary *finite posets*. So we put forward the following heuristic:

**posets with good dynamical properties**  
**= posets with order polynomial product formulas**

What's really cool about this heuristic is that it seems like a powerful tool for mathematical exploration *in both directions!*

# Using the heuristic to find good dynamics

Let  $V(n)$  be the following poset:



$V(n)$  is *not* a shape, but it has an order polynomial product formula:

**Theorem (Kreweras–Niederhausen '81)**

$$\Omega_{V(n)}(m) = \frac{\prod_{i=1}^n (m+1+i) \prod_{i=1}^{2n} (2m+i+1)}{(n+1)!(2n+1)!}$$

The heuristic led us to:

**Theorem (H.–Rubey 2022)**

Pro:  $\mathcal{L}(V(n)) \rightarrow \mathcal{L}(V(n))$  has order  $6n$ .

Here  $\mathcal{L}(P)$  is set of *linear extensions* of a poset  $P$ , the analog of SYTs.

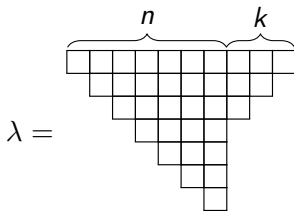
**Theorem (Adenbaum 2025)**

Row:  $\mathcal{PP}^m(V(n)) \rightarrow \mathcal{PP}^m(V(n))$  has order  $2(n+2)$ .

**Note:** for  $m = 1$  (combinatorial rowmotion) see Plante–Roby, 2024.

# Using the heuristic to find good enumeration

The *shifted double staircase* shape is the following  $\lambda$ :



Recall that this was one of the families Haiman showed has good behavior of promotion of SYTs.

The heuristic led us to:

Theorem (H.-Lai 2021, Okada 2021)

$$\Omega_{\lambda}(m) = \prod_{1 \leq i \leq j \leq n} \frac{m + i + j - 1}{i + j - 1} \cdot \prod_{1 \leq i \leq j \leq k} \frac{m + i + j}{i + j}$$

Our proof with Lai is based on tilings and *Kuo condensation*.

Okada's proof is algebraic and uses Proctor's "*intermediate*" symplectic group characters.

## Aside: counting linear extensions

For  $P$  a poset, let  $e(P) = \#\mathcal{L}(P)$  be the number of linear extensions of  $P$ . Then the leading coefficient of  $\Omega_P(m)$  is  $e(P)/\#P!$ . So whenever there is a product formula for the order polynomial of a poset, there's automatically also a product formula for its number of linear extensions.

But many more posets have a product formula for  $e(P)$  than for  $\Omega_P(m)$ !

### Theorem (Hook length formula, Frame–Robinson–Thrall 1953)

*For any shape  $\lambda$  with  $n$  boxes, the number of SYTs of shape  $\lambda$  is*

$$f^\lambda = \#\mathcal{SYT}(\lambda) = \frac{n!}{\prod_{u \in \lambda} h(u)},$$

*where  $h(u)$  is the **hook length** of the box  $u$ .*

We need more refined invariant  $\Omega_P(m)$  to identify  $P$  with good dynamics.



# The cyclic sieving phenomenon

Is there any connection between enumeration and dynamics? Yes, the CSP!

We can ask for even more refined information about a cyclic action than its period, such as its *orbit structure*. A compact way to record orbit structure of a cyclic action is via the *cyclic sieving phenomenon (CSP)*:

**Definition (Reiner–Stanton–White 2004)**

For  $C = \langle c \rangle$  a  $\mathbb{Z}/n$ -action on a finite set  $X$ , and  $f(q) \in \mathbb{N}[q]$  a polynomial, we say  $(X, C, f)$  *exhibits CSP* if for all  $k$ ,

$$\#X^{c^k} = f(\zeta^k)$$

with  $\zeta := e^{2\pi i/n}$  a primitive  $n$ th root of unity.

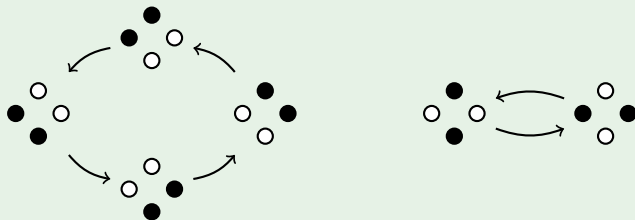
When the sieving polynomial  $f(q)$  has a product formula, a CSP result implies that *every* symmetry class has a product formula.

# Cyclic sieving example: rotation of subsets

## Theorem (Reiner–Stanton–White 2004)

$(\{k\text{-subsets of } \{1, \dots, n\}\}, \langle i \mapsto i+1 \pmod n \rangle \simeq \mathbb{Z}/n, f)$  exhibits CSP, where  $f(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^k \frac{(1-q^{n+1-i})}{(1-q^i)}$  is the  $q$ -binomial coefficient.

## Example ( $n = 4, k = 2$ )



$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4 \Rightarrow \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{q:=1} = 6, \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{q:=\pm i} = 0, \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{q:=-1} = 2$$

# Cyclic sieving for rectangular rowmotion and promotion

## Theorem (Rhoades 2010)

$(\mathcal{PP}^m(a \times b), \langle \text{Row} \rangle \simeq \mathbb{Z}/(a+b), f)$  exhibits CSP, where

$$f(q) = \sum_{\pi \in \mathcal{PP}^m(a \times b)} q^{|\pi|} = \prod_{i=1}^a \prod_{j=1}^b \frac{(1 - q^{i+j+m-1})}{(1 - q^{i+j-1})},$$

is MacMahon's *size generating function* of plane partitions in a box.

**Note:** case  $m = 1$  recovers the subset rotation CSP.

## Theorem (Rhoades 2010)

$(\mathcal{SYT}(a \times b), \langle \text{Pro} \rangle \simeq \mathbb{Z}/ab, f)$  exhibits CSP, where

$$f(q) = \sum_{T \in \mathcal{SYT}(a \times b)} q^{\text{maj}(T)} = \prod_{i=1}^{ab} (1 - q^i) \cdot \prod_{i=1}^a \prod_{j=1}^b \frac{1}{(1 - q^{i+j-1})},$$

is a *q-analog of the hook length formula* for these SYTs.

# General cyclic sieving conjecture from order polynomial

Let  $P$  be one of these posets whose order polynomial  $\Omega_P(m)$  has a product formula. Define

$$\Omega_P(m; q) = \prod_{\alpha \text{ root of } \Omega_P(m)} \frac{(1 - q^{\kappa(m-\alpha)})}{(1 - q^{-\kappa\alpha})}, \quad (\kappa := \min\{k > 0 : k\alpha \in \mathbb{Z}\forall \alpha\})$$

the natural  $q$ -analog of  $\Omega_P(m)$ . (*Not obviously a polynomial!!*)

## Conjecture (H. 2020)

$(\mathcal{PP}^m(P), \langle \text{Row} \rangle \simeq \mathbb{Z}/\kappa(\text{rk}(P) + 2), \Omega_P(m; q))$  exhibits CSP (if  $P$  graded).

Define

$$e(P; q) = (1 - q^\kappa)(1 - q^{2\kappa}) \cdots (1 - q^{\#P \cdot \kappa}) \lim_{m \rightarrow \infty} \Omega_P(m; q),$$

the natural  $q$ -analog of  $e(P)$ , the number of linear extensions.

## Conjecture (H. 2020)

$(\mathcal{L}(P), \langle \text{Pro} \rangle \simeq \mathbb{Z}/\kappa \cdot \#P, e(P; q))$  exhibits CSP.

# What's behind all the good behavior? Algebra!

Often sophisticated tools from algebra are used to prove these CSP results.

For example, Rhoades used *canonical bases* from Kazhdan–Lusztig theory to prove the rectangular pro/rowmotion CSPs. Subsequent work has connected promotion to *crystals* and tensor invariants, the *monodromy* action on the Wronski map, canonical bases from *cluster algebras*, etc.

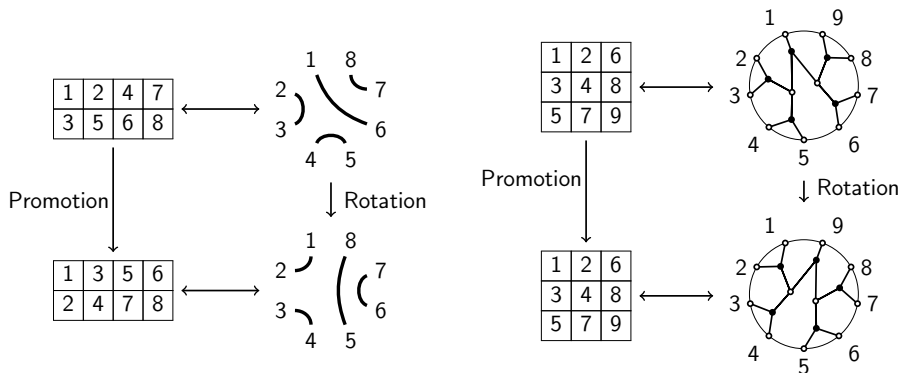
(See my talk at JMM '26 for a new CSP result of this kind, joint with Jesse Kim and Stephan Pfannerer, using the *space of electrical networks*.)

The posets themselves often have direct connection to Lie algebras, being either *root posets* or *minuscule posets*. The Weyl dimension formula often provides the product formula for  $\Omega_P(m)$ .

Still, we are far from a unified algebraic explanation for all known examples.

# Another perspective: pro/rowmotion as rotation

In the best situations, we can find a *diagrammatic model* (like *noncrossing matchings*, *webs*, ...) where pro/rowmotion corresponds to rotation:



Again, we are far from a unified “rotation model” for all known examples.

## Further questions

- Can we find a unified algebraic explanation for all the known examples of posets with good behavior? What about a unified rotation model?
- Can we find *direct implications* between the properties in the heuristic (pro/rowmotion dynamics & order polynomial product formula)?  
This would upgrade the heuristic to an actual theorem!
- Can we find *more examples* of posets satisfying the heuristic?
- How do other aspects of poset dynamics come into play here?  
For example, the *homomesy* phenomenon, where natural statistics have constant orbit averages. Or, further lifts of the actions to the *birational* and *noncommutative* realms.

# Thank you!

- A version of these slides are on my website at:  
[https://www.samuelhopkins.com/docs/jim\\_talk.pdf](https://www.samuelhopkins.com/docs/jim_talk.pdf).
- See my survey arXiv:2006.01568 for references.