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Parametric Equations § 10.1

The 1st half of the semester for Calc II focused on integration.

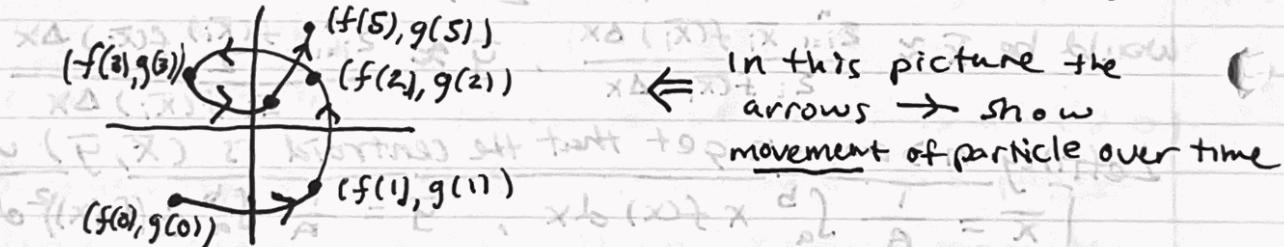
In 2nd half we explore other topics, starting with Chapter 10 on parametric equations & polar coordinates.

Up until now we have considered curves of the form $y = f(x)$ (or more rarely, $f(x, y) = 0$).

A parametrized curve is defined by two equations:

$$x = f(t) \text{ and } y = g(t)$$

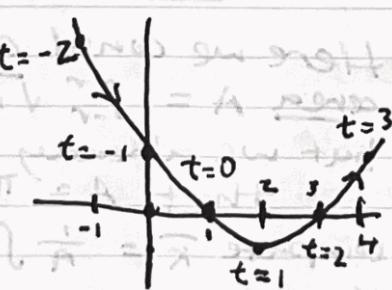
where t is an auxiliary variable. Often we think of t as time, so the curve describes motion of a particle where at time t particle is at position $(f(t), g(t))$:



E.g.: Consider parametrized curve $x = t+1, y = t^2 - 2t$.

We can make a chart with various values of t :

t	x	y
-2	-1	8
-1	0	3
0	1	0
1	2	-1
2	3	0
3	4	3



plot of points
 $\left(f(t), g(t)\right)$ for
 $t = -1, 0, 1, \dots, 4$
looks like a parabola

In this case, we can eliminate the variable t :

$$x = t + 1 \Rightarrow t = x - 1$$

$$y = t^2 - 2t \Rightarrow y = (x-1)^2 - 2(x-1) = x^2 - 4x + 3$$

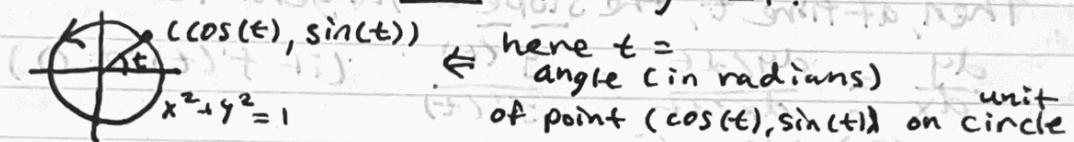
So this parametrized curve is just $y = x^2 - 4x + 3$

E.g.: Consider the parametric curve:

$$x = \cos(t), y = \sin(t) \text{ for } 0 \leq t \leq 2\pi$$

How can we visualize this curve?

Notice that $x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$, so this parametrizes a circle $x^2 + y^2 = 1$.



E.g.: What about $x = \cos(2t)$, $y = \sin(2t)$, $0 \leq t \leq 2\pi$?

Notice we still have $x^2 + y^2 = \cos^2(2t) + \sin^2(2t) = 1$, so the parametrized curve still traces a circle:



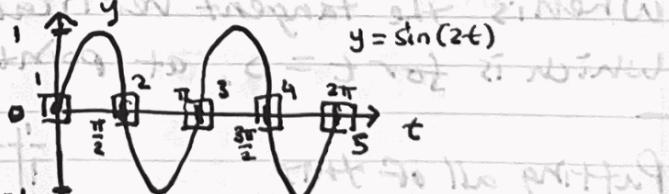
But now the parametrized curve
traces the circle twice:
once for $0 \leq t \leq \pi$
and once for $\pi \leq t \leq 2\pi$

Can think of this particle as moving "faster" than the last one.
We see same curve can be parametrized in different ways!

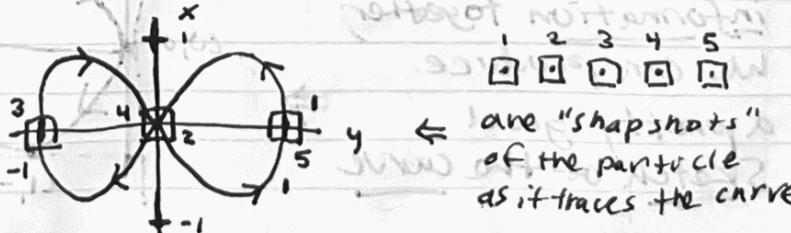
E.g.: Consider the curve $x = \cos(t)$, $y = \sin(2t)$.

It's possible to eliminate t to get $y^2 = 4x^2 - 4x^4$
but that equation is hard to visualize.

Instead, graph $x = f(t)$ and $y = g(t)$ separately:



Then combine
into one picture
showing $(f(t), g(t))$:



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Calculus with parametrized curves §10.2

Much of what we have done with curves of form $y=f(x)$ in calculus can also be done for parametrized curves:

Tangent vectors: Let $(x, y) = (f(t), g(t))$ be a curve.

Then, at time t , the slope of tangent vector is given by:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)} \quad (\text{if } f'(t) \neq 0)$$

chain rule

If $dy/dt = 0$ (and $dx/dt \neq 0$) \Rightarrow horizontal tangent

If $dx/dt = 0$ (and $dy/dt \neq 0$) \Rightarrow vertical tangent

E.g.: Consider curve $x = t^2$, $y = t^3 - 3t$.

First, notice that when $t = \pm\sqrt{3}$ we have

$$x = t^2 = 3 \quad \text{and} \quad y = t^3 - 3t = t(t^2 - 3) = 0,$$

so curve passes thru $(3, 0)$ at two times $t = \sqrt{3}$ and $t = -\sqrt{3}$.

We then compute that:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t} \rightarrow = -6/2\sqrt{3} = -\sqrt{3} \text{ at } t = -\sqrt{3}$$
$$\rightarrow = 6/2\sqrt{3} = \sqrt{3} \text{ at } t = \sqrt{3}$$

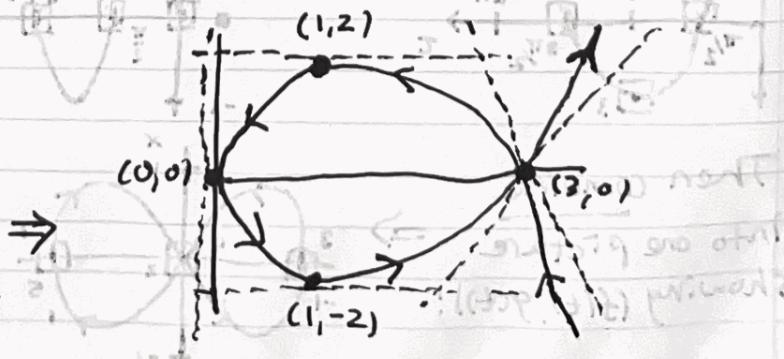
So two tangent lines, of slopes $\pm\sqrt{3}$, for curve at $(3, 0)$.

When is the tangent horizontal? When $dy/dt = 3t^2 - 3 = 0$
which is for $t = \pm 1$, at points $(1, 2)$ and $(1, -2)$.

When is the tangent vertical? When $dx/dt = 2t = 0$,
which is for $t = 0$, at point $(0, 0)$.

Putting all of this
information together,
we can produce

a pretty good
Sketch of the curve



Arc lengths: We saw several times how to find lengths of curves by breaking into line segments:



← recall length of each small segment
 $= \sqrt{(\Delta x)^2 + (\Delta y)^2}$

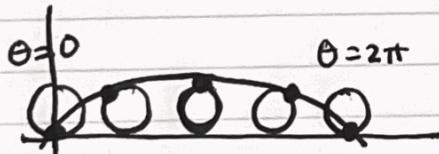
For a parametrized curve $(x, y) = (f(t), g(t))$ with $\alpha \leq t \leq \beta$

we get length of curve $= \int_{\alpha}^{\beta} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt = \boxed{\int_{\alpha}^{\beta} \sqrt{f'(t)^2 + g'(t)^2} dt}$.

Exercise: Using parametrization $x = \cos(t)$, $y = \sin(t)$, $0 \leq t \leq 2\pi$,

Show circumference of unit circle $= 2\pi$ using this formula.

E.g. The cycloid is the path a point on unit circle traces as the circle rolls!



← think of this as an animation of a rolling circle, with point • marked where angle θ = "time"

The cycloid is parametrized by:

$$x = \theta - \sin \theta, y = 1 - \cos \theta \text{ for } 0 \leq \theta \leq 2\pi$$

Q: What is the arclength of the cycloid?

A: We compute $\frac{dx}{d\theta} = 1 - \cos \theta$, $\frac{dy}{d\theta} = \sin \theta$ so that

$$\sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2} = \sqrt{(1 - \cos \theta)^2 + (\sin \theta)^2} = \sqrt{2(1 - \cos \theta)}$$

using trig identity

$$\frac{1}{2}(1 - \cos 2x) = \sin^2 x$$

$$= \sqrt{4 \sin^2(\theta/2)}$$

$$= 2 \sin(\theta/2)$$

$$\Rightarrow \text{length of cycloid} = \int_0^{2\pi} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2} d\theta$$

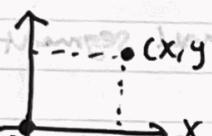
$$= \int_0^{2\pi} 2 \sin(\frac{\theta}{2}) d\theta = \left[-4 \cos(\frac{\theta}{2}) \right]_0^{2\pi}$$

$$\Rightarrow ((-4 \cdot 1) - (-4 \cdot 1)) = \underline{8}$$

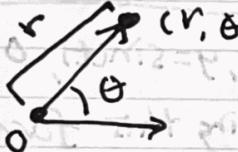
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Polar Coordinates § 10.3

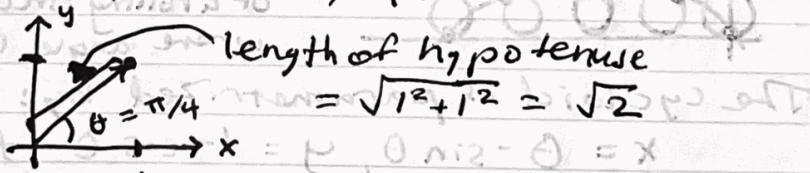
We are used to working with the "Cartesian" coordinate system where a point on the plane is represented by (x, y)

 telling us how far to move along two orthogonal axes to reach that point.

The polar coordinate system is a different way to represent points on the plane by a pair (r, θ) :

 Here we have a fixed axis ray \rightarrow emanating from origin O , and we reach a point (r, θ) by making an angle of θ radians and goint out a distance of r .

E.g. The point $(x, y) = (1, 1)$ in Cartesian coord's is the same as $(r, \theta) = (\sqrt{2}, \frac{\pi}{4})$ in polar coord's.



Notice: There are multiple ways to represent any point in polar coord's because we can add 2π to θ :

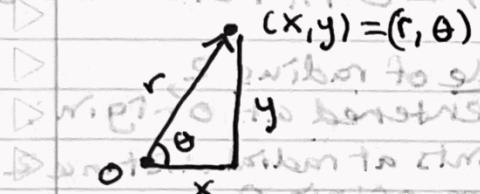
$$\text{→ } (r, \theta) = (\sqrt{2}, \frac{\pi}{4}) \text{ same as } (r, \theta) = (\sqrt{2}, 2\pi + \frac{\pi}{4})$$

$$\leftarrow (r, \theta) = (\sqrt{2}, \frac{\pi}{4}) \text{ same as } (r, \theta) = (-\sqrt{2}, \pi + \frac{\pi}{4})$$

= go backwards that distance along ray.

Question: How to convert between Cartesian & polar coords?

Let's draw a right triangle to help us:



From this picture we see that

$$x = r \cos \theta \text{ and } y = r \sin \theta$$

We also have that:

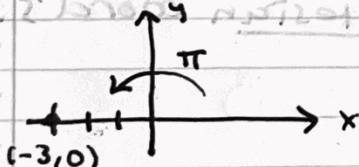
$$r^2 = x^2 + y^2 \text{ and } \tan \theta = \frac{y}{x}$$

which gives us (r, θ) in terms of (x, y) :

$$\text{specifically, } r = \sqrt{x^2 + y^2} \text{ and } \theta = \arctan\left(\frac{y}{x}\right).$$

E.g. Find the polar coordinates of $(x, y) = (-3, 0)$.

To solve this problem, it's easiest to just draw the point.



We see this point is at

angle $\theta = \pi$ and radius $r = 3$.

$$\text{Check: } 3^2 - r^2 = x^2 + y^2 = (-3)^2 + (0)^2$$

$$\text{and } \theta = \tan(\theta) = \frac{y}{x} = \frac{0}{-3} = 0$$

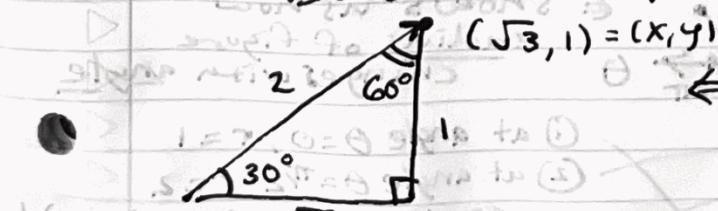
Could have also chosen $(r, \theta) = (-3, 0)$ here...

E.g. Find the Cartesian coordinates of $(r, \theta) = (2, \frac{\pi}{6})$.

$$\text{Here we have } x = r \cos \theta = 2 \cos(\pi/6) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$\text{and } y = r \sin \theta = 2 \sin(\pi/6) = 2 \cdot \frac{1}{2} = 1$$

Can also draw the right triangle to check:



recall that $\theta = \pi/6$ radians

$$= 30^\circ$$

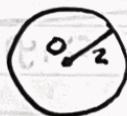
corresponds to a special "30-60-90" triangle

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Polar equations and curves:

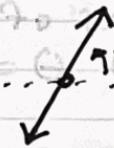
Just like we draw curves $f(x, y) = 0$ in Cartesian coord's, we can draw curves $f(r, \theta) = 0$ in Polar coord's.

E.g.: The equation $r = 2$ gives circle of radius 2, centered at origin:



\Leftarrow circle = all points at radial distance 2 from origin O

E.g.: The equation $\theta = \pi/3$ gives line at angle $\pi/3$ thru origin:



\Leftarrow line thru origin

= all points at given angle

E.g.: What about equation $r = 2 \cos \theta$?

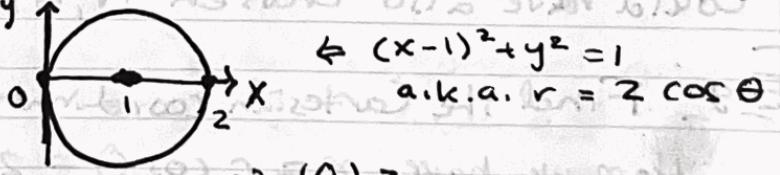
Here it's easiest to switch to Cartesian coord's:

multiplying by r gives $r^2 = 2r \cos \theta$

$$\Leftrightarrow x^2 + y^2 = 2x$$

$$\Leftrightarrow (x-1)^2 + y^2 = 1$$

which is a circle of radius 1 centered at $(x, y) = (1, 0)$:

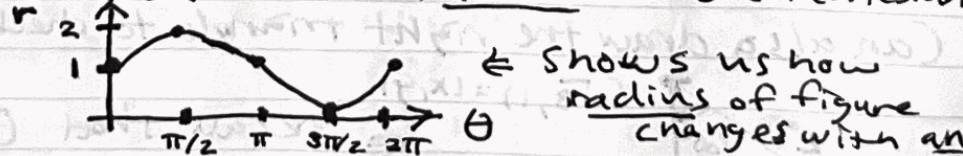


$$\Leftrightarrow (x-1)^2 + y^2 = 1$$

$$\text{a.k.a. } r = 2 \cos \theta$$

E.g.: What about $r = 1 + \sin(\theta)$?

First let's plot r as a function of θ (in Cartesian coord's):



\Leftarrow shows us how radius of figure changes with angle

"cardioid" \Rightarrow

this "heart-shaped" curve is polar curve

$$r = 1 + \sin(\theta)$$

- ① at angle $\theta = 0$, $r = 1$
- ② at angle $\theta = \pi/2$, $r = 2$
so we move out to this point
- ③ at $\theta = \pi$, back to $r = 1$
- ④ at $\theta = 3\pi/2$, radius shinks to $r = 0$

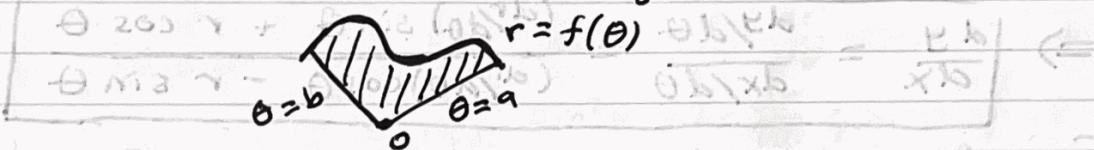
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Calculus in Polar coordinates §10.4

We can do all types of calculus stuff in polar coord's too...

► Areas: How to compute area "inside" polar curve $r = f(\theta)$? where $a \leq \theta \leq b$

► The polar curve looks something like this:



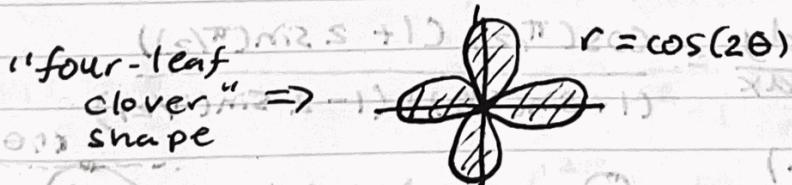
► For a small change $d\theta$ in θ we get roughly a "pie slice":

$$\text{area} = \pi r^2 \cdot \frac{d\theta}{2\pi} \Rightarrow \begin{array}{c} \text{Pie slice} \\ \text{area} = \frac{1}{2} (f(\theta))^2 d\theta \end{array}$$

► As usual, breaking up area into many small pie slices and summing up area gives an integral in limit:

$$\text{area inside polar curve} = \boxed{\int_a^b \frac{1}{2} (f(\theta))^2 d\theta}$$

► E.g. Let's look at the curve $r = \cos(2\theta)$ for $0 \leq \theta \leq 2\pi$:



► What is area inside this curve? Using formula...

$$\text{Area} = \int_0^{2\pi} \frac{1}{2} (f(\theta))^2 d\theta = \int_0^{2\pi} \frac{1}{2} \cos^2 2\theta d\theta$$

► We've seen before that $\int \cos^2 x dx = \frac{1}{2} (x + \sin(x)\cos(x))$ (using int. by parts)

► So w/ a simple u-sub $\int \frac{1}{2} \cos^2 2\theta d\theta = \frac{1}{4} \theta + \frac{1}{8} \sin(2\theta)\cos(2\theta)$.

$$\begin{aligned} \text{Thus, } \text{area} &= \int_0^{2\pi} \frac{1}{2} \cos^2 2\theta d\theta = \left[\frac{1}{4} \theta + \frac{1}{8} \sin(2\theta)\cos(2\theta) \right]_0^{2\pi} \\ &= \left(\left(\frac{1}{4} \cdot 2\pi + \frac{1}{8} \sin(4\pi)\cos(4\pi) \right) - \left(\frac{1}{4} \cdot 0 + \frac{1}{8} \sin(0)\cos(0) \right) \right) = \boxed{\frac{\pi}{2}} \end{aligned}$$

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Tangents: How to find slope of tangent to polar curve $r = f(\theta)$?

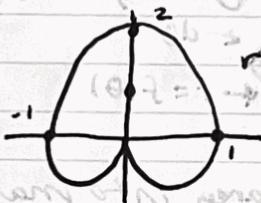
Recall $x = r \cos \theta$ and $y = r \sin \theta$ in Cartesian coord's.

So using the product rule we get:

$$\frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta \text{ and } \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(\frac{dr}{d\theta}) \sin \theta + r \cos \theta}{(\frac{dr}{d\theta}) \cos \theta - r \sin \theta}}$$

E.g.: Consider the cardioid $r = 1 + \sin \theta$:



$$r = 1 + \sin \theta \Rightarrow \frac{dr}{d\theta} = \cos \theta$$

$$\text{Here } \frac{dy}{dx} = \frac{(\frac{dr}{d\theta}) \sin \theta + r \cos \theta}{(\frac{dr}{d\theta}) \cos \theta - r \sin \theta} = \frac{\cos \theta \sin \theta + (1 + \sin \theta) \cos \theta}{\cos \theta \cos \theta - (1 + \sin \theta) \sin \theta}$$

$$= \frac{\cos \theta (1 + 2 \sin \theta)}{1 - 2 \sin^2 \theta - \sin \theta} = \frac{\cos \theta (1 + 2 \sin \theta)}{(1 + \sin \theta)(1 - 2 \sin \theta)}$$

$$\text{So at } \theta = \frac{\pi}{2} \text{ get } \frac{dy}{dx} = \frac{\cos(\pi/2)(1 + 2 \sin(\pi/2))}{(1 + \sin(\pi/2))(1 - 2 \sin(\pi/2))}$$

$$= \frac{0(1+2)}{(1+1)(1-2)} = \frac{0}{-2} \xrightarrow{\text{horizontal tangent at } \theta = \frac{\pi}{2}}$$

$$\text{And at } \theta = \frac{\pi}{3} \text{ get } \frac{dy}{dx} = \frac{\cos(\pi/3)(1 + 2 \sin(\pi/3))}{(1 + \sin(\pi/3))(1 - 2 \sin(\pi/3))}$$

$$= \frac{(\frac{1}{2})(1 + 2 \frac{\sqrt{3}}{2})}{(\frac{1 + \sqrt{3}}{2})(1 - 2 \frac{\sqrt{3}}{2})} = \frac{\frac{1 + \sqrt{3}}{2}}{\frac{(2 + \sqrt{3})(1 - \sqrt{3})}{4}} = \frac{1 + \sqrt{3}}{-1 - \sqrt{3}} = \frac{-1}{-1} = 1$$

\leftarrow tangent slope \leftarrow
 $= 1 \text{ at } \theta = \frac{\pi}{3}$

► Arc length: How to compute length of polar curve $r=f(\theta)$?

► As before, from $x=r\cos\theta$ and $y=r\sin\theta$ we get

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \quad \text{and} \quad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

► so that

$$\begin{aligned} \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r \frac{dr}{d\theta} \cos\theta \sin\theta + r^2 \sin^2\theta \\ &\quad + \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r \frac{dr}{d\theta} \sin\theta \cos\theta + r^2 \cos^2\theta \\ &= \left(\frac{dr}{d\theta}\right)^2 + r^2 \quad (\text{using } \sin^2\theta + \cos^2\theta = 1) \end{aligned}$$

► If we think of (x, y) as parametrized by θ , then

$$\text{length of curve} = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

► which in terms of r and θ is then

$$\text{length} = \boxed{\int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta}$$

► E.g. For a circle $r=m$ centered at origin,

$$\text{this formula gives us length} = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_0^{2\pi} \sqrt{m^2 + 0^2} d\theta$$

$$= \int_0^{2\pi} m d\theta = 2\pi m,$$

which is correct circumference!

► E.g. We saw before that $r=2\cos\theta$, $0 \leq \theta \leq \pi$ gives a circle of radius 1 centered at $(x, y)=(1, 0)$

► Here $\frac{dr}{d\theta} = -2\sin\theta$, so the formula gives...

$$\text{arc length} = \int_0^\pi \sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2} d\theta = \int_0^\pi 2 d\theta = 2\pi.$$
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