Order polynomial product formulas and poset dynamics

GW/Howard Combinatorics & Algebra Seminar

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October 15th, 2025

Plane partitions

A *$a \times b$ plane partition* is an $a \times b$ array of nonnegative integers that are weakly decreasing in rows and columns.

Let $\mathcal{PP}^m(a \times b) := \{a \times b \text{ plane partitions with entries } \leq m\}$:

$$\begin{array}{|c|c|c|c|c|c|} \hline 5 & 3 & 3 & 2 & 2 & 1 \\ \hline 4 & 2 & 1 & 1 & 1 & 0 \\ \hline 4 & 1 & 1 & 1 & 0 & 0 \\ \hline 3 & 1 & 1 & 0 & 0 & 0 \\ \hline \end{array} \in \mathcal{PP}^5(4 \times 6)$$

Theorem (MacMahon's formula (c.1915) for plane partitions in a box)

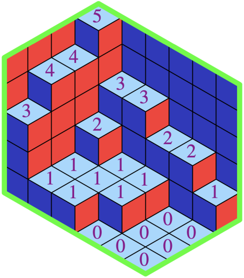
$$\#\mathcal{PP}^m(a \times b) = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{m+i+j-1}{i+j-1}$$

Other guises of plane partitions

Plane partitions have a beautiful $3D$ representation as stacks of boxes:

5	3	3	2	2	1
4	2	1	1	1	0
4	1	1	1	0	0
3	1	1	0	0	0

$\Leftrightarrow$



In this way they correspond to *lozenge tilings* of hexagonal regions of the triangular lattice, a well-studied planar statistical mechanical model.

Plane partitions are also intimately related to the *representation theory of classical groups*, because $\mathcal{PP}^m(a \times b)$ indexes a basis of the irreducible representation V^λ of $\mathfrak{sl}(a+b)$ with highest weight $\lambda = m^a$.

Toggling plane partitions

(Piecewise-linear) toggling of an entry of a plane partition $\pi \in \mathcal{PP}^m(a \times b)$ does the following:

$$\begin{array}{|c|c|c|}\hline & v & \\ \hline w & x & y \\ \hline & z & \\ \hline \end{array} \;\longrightarrow\; \begin{array}{|c|c|c|}\hline & v & \\ \hline w & x' & y \\ \hline & z & \\ \hline \end{array} \qquad \max(y,z) \,\_\, x \,\_\_\_\, x' \,\_\, \min(v,w)$$

with $x' = \max(y,z) + \min(v,w) - x$.[1] Toggling an entry is an involution.

Let $t_{i,j} \colon \mathcal{PP}^m(a \times b) \to \mathcal{PP}^m(a \times b)$ be toggling at entry (i,j).

Example

$$t_{2,2}\left(\begin{array}{|c|c|c|}\hline 5 & 4 & 1 \\ \hline 3 & 1 & 1 \\ \hline \end{array} \right) \in \mathcal{PP}^5(2 \times 3) = \begin{array}{|c|c|c|}\hline 5 & 4 & 1 \\ \hline 3 & 3 & 1 \\ \hline \end{array}$$

[1] If v or w don't exist, treat them as m; if y or z don't exist, treat them as 0.

Rowmotion on rectangular plane partitions

(Piecewise-linear) rowmotion $\mathrm{Row}\colon \mathcal{PP}^m(a \times b) \to \mathcal{PP}^m(a \times b)$ consists of toggling *all* the entries, in sequence, from bottom-right to top-left.

Example

Let's compute rowmotion of a plane partition $\pi \in \mathcal{PP}^5(2 \times 3)$:

$$\pi = \begin{array}{|c|c|c|}\hline 5&4&1\\\hline 3&1&1\\\hline\end{array} \xrightarrow{t_{2,3}} \begin{array}{|c|c|c|}\hline 5&4&1\\\hline 3&1&0\\\hline\end{array} \xrightarrow{t_{2,2}} \begin{array}{|c|c|c|}\hline 5&4&1\\\hline 3&2&0\\\hline\end{array} \xrightarrow{t_{1,3}} \begin{array}{|c|c|c|}\hline 5&4&3\\\hline 3&2&0\\\hline\end{array}$$

$$\xrightarrow{t_{2,1}} \begin{array}{|c|c|c|}\hline 5&4&3\\\hline 4&2&0\\\hline\end{array} \xrightarrow{t_{1,2}} \begin{array}{|c|c|c|}\hline 5&4&3\\\hline 4&2&0\\\hline\end{array} \xrightarrow{t_{1,1}} \begin{array}{|c|c|c|}\hline 4&4&3\\\hline 4&2&0\\\hline\end{array} = \mathrm{Row}(\pi)$$

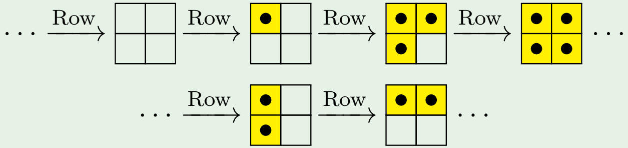
Piecewise-linear rowmotion was introduced by Einstein–Propp, 2013.

Combinatorial rowmotion

Let $\mathcal{J}(a \times b)$ be the set of sub-Young diagrams inside the $a \times b$ rectangle. We have $\mathcal{J}(a \times b) \simeq \mathcal{PP}^1(a \times b)$ via the indicator function.

(Combinatorial) rowmotion $\mathrm{Row}\colon \mathcal{J}(a \times b) \to \mathcal{J}(a \times b)$ sends $\mu \subseteq a \times b$ to the smallest Young diagram containing the minimal elements of $(a \times b) \setminus \mu$.

Example



Combinatorial rowmotion was introduced by Brouwer–Schrijver, 1974. Cameron–Fon-der-Flaass, 1995 showed how to write combinatorial rowmotion as a composition of combinatorial toggles, and Einstein–Propp's contribution was generalizing their description to the piecewise-linear realm.

Periodicity for rowmotion on rectangular plane partitions

Example

One rowmotion orbit in $\mathcal{PP}^5(2 \times 3)$ is:

$$\cdots \xrightarrow{\text{Row}} \begin{array}{|c|c|c|}\hline 5 & 4 & 1 \\ \hline 3 & 1 & 1 \\ \hline\end{array} \xrightarrow{\text{Row}} \begin{array}{|c|c|c|}\hline 4 & 4 & 3 \\ \hline 4 & 2 & 0 \\ \hline\end{array} \xrightarrow{\text{Row}} \begin{array}{|c|c|c|}\hline 5 & 4 & 3 \\ \hline 4 & 4 & 2 \\ \hline\end{array} \xrightarrow{\text{Row}} \begin{array}{|c|c|c|}\hline 3 & 3 & 2 \\ \hline 2 & 1 & 1 \\ \hline\end{array} \xrightarrow{\text{Row}} \begin{array}{|c|c|c|}\hline 4 & 1 & 1 \\ \hline 2 & 1 & 0 \\ \hline\end{array} \cdots$$

Theorem (Grinberg–Roby 2015; conjectured by Einstein–Propp)

The order of $\mathrm{Row}\colon \mathcal{PP}^m(a \times b) \to \mathcal{PP}^m(a \times b)$ *is* $a + b$.

Note: Case $m = 1$ (combinatorial rowmotion) due to Brouwer–Schrijver.

From Kirillov–Berenstein, 1995 and Striker–Williams, 2012 it follows that dynamics are same as *rectangular semistandard Young tableaux promotion*, for which order $a + b$ is known from Schützenberger, Haiman, Rhoades, ...

Rowmotion on plane partitions of other shapes

For a Young diagram λ, a *plane partition of shape λ* is a filling of its boxes with nonnegative integers that are weakly decreasing in rows and columns.

All of the prior constructions make sense for arbitrary shapes λ. But for a "random" λ, rowmotion will not behave well like it does for rectangles.

Example

For $\lambda = (4, 2, 2)$ and for

$$\pi = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 0 & 0 \\ \hline 1 & 1 & \multicolumn{2}{c}{} \\ \cline{1-2} 1 & 0 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \in \mathcal{PP}^1(\lambda)$$

the rowmotion orbit of π has 17 elements. Things get worse from there.

But Grinberg–Roby showed that rowmotion behaves well also for *staircases* and *shifted staircases*, and Johnson–Liu, 2023 showed same for *trapezoids*.

When does rowmotion behave well? The order polynomial...

What distinguishes the shapes with good rowmotion behavior?

For any shape λ, the function $\Omega_\lambda(m) = \#\mathcal{PP}^m(\lambda)$ is a *polynomial* in m, called the *order polynomial* of λ. It was introduced by Richard Stanley.

For example, MacMahon's formula says $\Omega_{a\times b}(m) = \prod_{i=1}^{a}\prod_{j=1}^{b}\frac{m+i+j-1}{i+j-1}$; in particular, all roots of $\Omega_{a\times b}(m)$ are *integers*!

Example

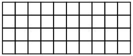
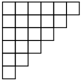
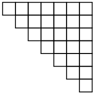
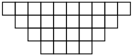
For $\lambda = (4,2,2)$,

$$\Omega_\lambda(m) = \frac{1}{720}(m+1)(m+2)^2(m+3)^2(m+4)(m^2+5m+5),$$

which has an irreducible quadratic factor.

Empirically, shapes λ with good rowmotion behavior are those with *order polynomial product formulas*, i.e., with all roots of $\Omega_\lambda(m)$ in $\mathbb{Z}$ (or $\frac{1}{2}\mathbb{Z}$).

Shapes with order polynomial product formulas

Rectangle 	$\prod_{i=1}^{a}\prod_{j=1}^{b}\frac{m+i+j-1}{i+j-1}$	$\mathfrak{sl}(n)$	MacMahon c. 1915
Staircase 	$\prod_{1\leq i\leq j\leq n}\frac{2m+i+j}{i+j}$	$\mathfrak{sp}(2n)$	Proctor 1988 *"symmetric, self-complementary plane partitions"*
Shifted staircase 	$\prod_{1\leq i\leq j\leq n}\frac{m+i+j-1}{i+j-1}$	$\mathfrak{so}(2n+1)$	Conj. MacMahon 1896, Andrews/Macdonald c. 1977 *"symmetric plane partitions"*
Shifted Trapezoid 	$\prod_{i=1}^{k}\prod_{j=1}^{2n-k+1}\frac{m+i+j-1}{i+j-1}$	$\mathfrak{sp}(2n)$	Proctor 1983 *"transpose-complementary plane partitions"*

More dynamics: promotion of standard Young tableaux

Standard Young Tableaux (SYTs) of a shape λ with n boxes are bijective fillings of the boxes with $1, \ldots, n$, increasing in rows and columns.

Promotion, $\mathrm{Pro}\colon \mathcal{SYT}(\lambda) \to \mathcal{SYT}(\lambda)$, is the following invertible operation on these SYTs:

- Delete the entry 1.
- Slide boxes into the resulting hole.
- Decrement all entries.
- Fill the hole with n.

Example

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline\end{array} \xrightarrow{\text{del. } 1} \begin{array}{|c|c|c|}\hline \bullet & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline\end{array} \xrightarrow{\text{slide}} \begin{array}{|c|c|c|}\hline 2 & 4 & 5 \\ \hline 3 & 6 & \bullet \\ \hline\end{array} \xrightarrow{\text{decr.}} \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & \bullet \\ \hline\end{array} \xrightarrow{\text{fill } 6} \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline\end{array} = \mathrm{Pro}(T)$$

Along with *evacuation*, defined by Schützenberger to study *RSK algorithm*.

When does promotion of SYT behave well? Same shapes!

Promotion behaves chaotically for most shapes, but:

Theorem

- *(Schützenberger 1977) For λ a rectangle, order of Pro is n*
- *(Edelman–Greene 1987) For λ a staircase, order of Pro is $2n$.*
- *(Haiman 1992) For λ a shifted trapezoid or shifted double staircase, order of Pro is n.*
- *(Haiman–Kim 1992) These are the* **only** *four families of shapes with good promotion behavior.*

Remarkably, these are (basically) the same families of shapes that have good plane partition rowmotion behavior!

The main heuristic

To summarize, we have seen that:

shapes with good dynamical properties
= shapes with order polynomial product formulas

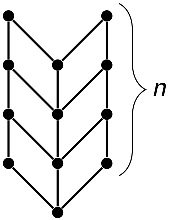
All of the constructions (order polynomial, plane partitions = P-partitions, rowmotion, SYTs = linear extensions, promotion, ...) we discussed make sense for arbitrary *finite posets*. So we put forward the following heuristic:

posets with good dynamical properties
= posets with order polynomial product formulas

What's really cool about this heuristic is that it seems like a powerful tool for mathematical exploration *in both directions!*

Using the heuristic to find good dynamics

Let $V(n)$ be the following poset:



$V(n)$ is *not* a shape, but it has an order polynomial product formula:

Theorem (Kreweras–Niederhausen '81)

$$\Omega_{V(n)}(m) = \frac{\prod_{i=1}^{n}(m+1+i)\prod_{i=1}^{2n}(2m+i+1)}{(n+1)!(2n+1)!}$$

The heuristic lead us to:

Theorem (H.–Rubey 2022)

$\mathrm{Pro}\colon \mathcal{L}(V(n)) \to \mathcal{L}(V(n))$ *has order* $2n$.

Here $\mathcal{L}(P)$ is set of *linear extensions* of a poset P, the analog of SYTs.

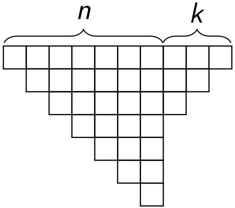
Theorem (Adenbaum 2025)

$\mathrm{Row}\colon \mathcal{PP}^m(V(n)) \to \mathcal{PP}^m(V(n))$ *has order* $2(n+2)$.

Note: for $m = 1$ (combinatorial rowmotion) see Plante–Roby, 2024.

Using the heuristic to find good enumeration

The *shifted double staircase* shape is the following λ:

$\lambda =$ 

Recall that this was one of the families Haiman showed has good behavior of promotion of SYTs.

The heuristic lead us to:

Theorem (H.–Lai 2021, Okada 2021)

$$\Omega_\lambda(m) = \prod_{1 \le i \le j \le n} \frac{m+i+j-1}{i+j-1} \cdot \prod_{1 \le i \le j \le k} \frac{m+i+j}{i+j}$$

Our proof with Lai is based on tilings and *Kuo condensation*.

Okada's proof is algebraic and uses Proctor's *"intermediate" symplectic group* characters.

Aside: counting linear extensions

For P a poset, let $e(P) = \#\mathcal{L}(P)$ be the number of linear extensions of P. Then the leading coefficient of $\Omega_P(m)$ is $e(P)/\#P!$. So whenever there is a product formula for the order polynomial of a poset, there's automatically also a product formula for its number of linear extensions.

But many more posets have a product formula for $e(P)$ than for $\Omega_P(m)$!

Theorem (Hook length formula, Frame–Robinson–Thrall 1953)

For any shape λ with n boxes, the number of SYTs of shape λ is

$$f^\lambda = \#\mathcal{SYT}(\lambda) = \frac{n!}{\prod_{u\in\lambda} h(u)},$$

where $h(u)$ is the hook length of the box u.

We need more refined invariant $\Omega_P(m)$ to identify P with good dynamics.

The cyclic sieving phenomenon

Is there any connection between enumeration and dynamics? Yes, the CSP!

We can ask for even more refined information about a cyclic action than its period, such as its *orbit structure*. A compact way to record orbit structure of a cyclic action is via the *cyclic sieving phenomenon (CSP)*:

Definition (Reiner–Stanton–White 2004)

For $C = \langle c \rangle$ a $\mathbb{Z}/n$-action on a finite set X, and $f(q) \in \mathbb{N}[q]$ a polynomial, we say (X, C, f) *exhibits CSP* if for all k,

$$\#X^{c^k} = f(\zeta^k)$$

with $\zeta := e^{2\pi i/n}$ a primitive nth root of unity.

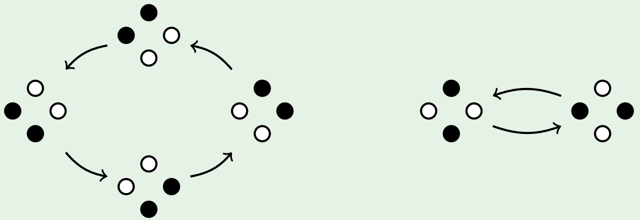
When the sieving polynomial $f(q)$ has a product formula, a CSP result implies that *every* symmetry class has a product formula.

Cyclic sieving example: rotation of subsets

Theorem (Reiner–Stanton–White 2004)

$(\{k\text{-subsets of } \{1,\ldots,n\}\}, \langle i \mapsto i+1 \mod n\rangle \simeq \mathbb{Z}/n, f)$ exhibits CSP, where $f(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^{k} \frac{(1-q^{n+1-i})}{(1-q^i)}$ is the q-binomial coefficient.

Example ($n = 4, k = 2$)



$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4 \Rightarrow \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{q:=1} = 6, \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{q:=\pm i} = 0, \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{q:=-1} = 2$

Cyclic sieving for rectangular rowmotion and promotion

Theorem (Rhoades 2010)

$(\mathcal{PP}^m(a \times b), \langle \mathrm{Row} \rangle \simeq \mathbb{Z}/(a+b), f)$ *exhibits CSP, where*

$$f(q) = \sum_{\pi \in \mathcal{PP}^m(a \times b)} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{(1-q^{i+j+m-1})}{(1-q^{i+j-1})},$$

is MacMahon's size generating function of plane partitions in a box.

Note: case $m = 1$ recovers the subset rotation CSP.

Theorem (Rhoades 2010)

$(\mathcal{SYT}(a \times b), \langle \mathrm{Pro} \rangle \simeq \mathbb{Z}/ab, f)$ *exhibits CSP, where*

$$f(q) = \sum_{T \in \mathcal{SYT}(a \times b)} q^{\mathrm{maj}(T)} = \prod_{i=1}^{ab} (1-q^i) \cdot \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{1}{(1-q^{i+j-1})},$$

is a q-analog of the hook length formula for these SYTs.

General cyclic sieving conjecture from order polynomial

Let P be one of these posets whose order polynomial $\Omega_P(m)$ has a product formula. Define

$$\Omega_P(m;q) = \prod_{\alpha \text{ root of } \Omega_P(m)} \frac{(1-q^{\kappa(m-\alpha)})}{(1-q^{-\kappa\alpha})}, \qquad (\kappa{:=}\min\{k{>}0\colon k\alpha{\in}\mathbb{Z}\forall\alpha\})$$

the natural q-analog of $\Omega_P(m)$. *(Not obviously a polynomial!)*

Conjecture (H. 2020)

$(\mathcal{PP}^m(P), \langle \mathrm{Row} \rangle \simeq \mathbb{Z}/\kappa(\mathrm{rk}(P)+2), \Omega_P(m;q))$ *exhibits CSP (if P graded).*

Define

$$e(P;q) = (1-q^{\kappa})(1-q^{2\kappa})\cdots(1-q^{\#P\cdot\kappa}) \lim_{m\to\infty} \Omega_P(m;q),$$

the natural q-analog of $e(P)$, the number of linear extensions.

Conjecture (H. 2020)

$(\mathcal{L}(P), \langle \mathrm{Pro} \rangle \simeq \mathbb{Z}/\kappa\cdot\#P, e(P;q))$ *exhibits CSP.*

What's behind all the good behavior? Algebra!

Often sophisticated tools from algebra are used to prove these CSP results.

For example, Rhoades used *canonical bases* from Kazhdan–Lusztig theory to prove the rectangular pro/rowmotion CSPs. Subsequent work has connected promotion to *crystals* and tensor invariants, the *monodromy* action on the Wronski map, canonical bases from *cluster algebras*, etc.

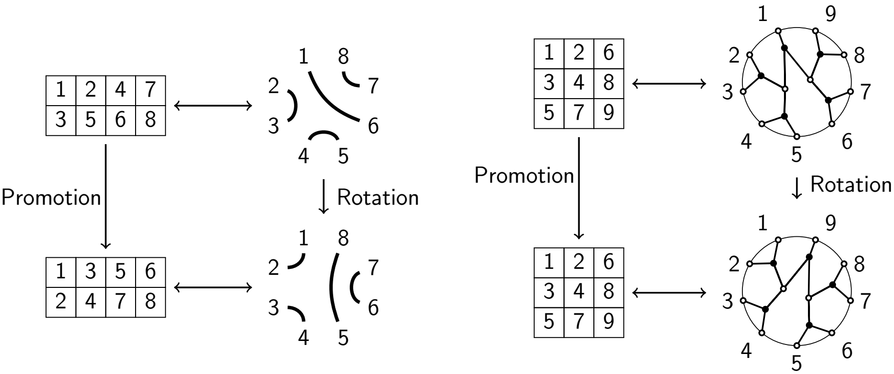
(See my talk at JMM '26 for a new CSP result of this kind, joint with Jesse Kim and Stephan Pfannerer, using the *space of electrical networks*.)

The posets themselves often have direct connection to Lie algebras, being either *root posets* or *minuscule posets*. The Weyl dimension formula often provides the product formula for $\Omega_P(m)$.

Still, we are far from a unified algebraic explanation for all known examples.

Another perspective: pro/rowmotion as rotation

In the best situations, we can find a *diagrammatic model* (like *noncrossing matchings*, *webs*, ...) where pro/rowmotion corresponds to rotation:



Again, we are far from a unified “rotation model” for all known examples.

Further questions

- Can we find a unified algebraic explanation for all the known examples of posets with good behavior? What about a unified rotation model?
- Can we find *direct implications* between the properties in the heuristic (pro/rowmotion dynamics & order polynomial product formula)? This would upgrade the heuristic to an actual theorem!
- Can we find *more examples* of posets satisfying the heuristic?
- How do other aspects of poset dynamics come into play here? For example, the *homomesy* phenomenon, where natural statistics have constant orbit averages. Or, further lifts of the actions to the *birational* and *noncommutative* realms.

Thank you!

- A version of these slides are on my website at:

  `https://www.samuelfhopkins.com/docs/jim_talk.pdf`.
- See my survey `arXiv:2006.01568` for references.