Notes for Jim's BIRS 2020 DAC Problem

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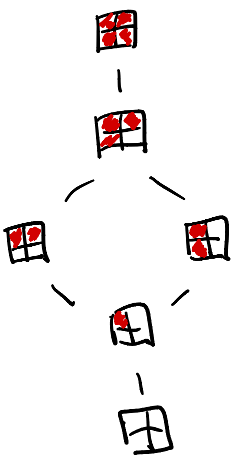
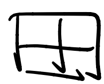
L finite poset (think $L = J(P)$)

uni = uniform distribution on L

maxchain = distribution on L where each $x \in L$ occurs proportional to # of maximal chains containing x

$\mathrm{ddeg}(x)$ = # of things $x \in L$ covers

e.g.

$L =$  $= J(2 \times 2) = J($  $)$

I draw Young diagram posets this way

ddeg	0	1	1	1	2	1
uni	1/6	1/6	1/6	1/6	1/6	1/6
maxchain	2/10	2/10	1/10	1/10	2/10	2/10

for $L = J(P)$ → ddeg = antichain cardinality

$$\mathbb{E}_{\mathrm{uni}}(\mathrm{ddeg}) = \mathbb{E}_{\mathrm{maxchain}}(\mathrm{ddeg}) = \frac{2 \times 2}{2+2} = 1$$

Def. (Reiner-Tenner-Yong) Say L is CDE (= 'coincidental down-degree expectations') if

$$\mathbb{E}_{\text{uni}}(\text{ddeg}) = \mathbb{E}_{\text{maxchain}}(\text{ddeg})$$

Thm (C-LM-P-TiB) $J(a \times b)$ is CDE, w/

$$\mathbb{E}_{\text{uni}}(\text{ddeg}) = \frac{ab}{a+b}.$$

Def. (CHHM) Say distribution μ on $J(P)$ is toggle-symmetric if $\mathbb{E}_\mu(T_p) = 0 \quad \forall\, p \in P$,

where $T_p^+(I)$ = indicator of can I toggle p into I?

$T_p^-(I)$ = indicator of can I toggle p out of I?

$T_p = T_p^+ - T_p^-$ (= 'signed togglability statistic')

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e.g. #1 Uni on $J(P)$ is toggle-symmetric:

toggling p reverses sign of T_p

#2 mchain(m) on $J(P)$ is toggle-symmetric, where

mchain(m) = choose (weakly) order-preserving map $\pi : P \to \{0, 1, \dots, m\}$ uniformly;

Choose $k \in \{0, 1, \ldots, m-1\}$ uniformly;
select ideal $\pi^{-1}(\{0, 1, \ldots, k\})$

Indeed, PL-toggle of π at p demonstrates toggle-symmetry of mchain(m) (check for yourself)

#3 maxchain on $J(P)$ is toggle-symmetric, since $\text{maxchain} = \lim_{m \to \infty} \text{mchain}(m)$

NOTE: uni = mchain(1), so mchain(m) interpolates between uni and maxchain.

Def. (H.) Say $J(P)$ is tCDE (= 'toggle CDE') if

$$\mathbb{E}_\mu(\text{ddeg}) = \mathbb{E}_{\text{uni}}(\text{ddeg}) \ \forall \text{ toggle-symmetric dist. } \mu \text{ on } J(P).$$

NOTE: $J(P)$ tCDE $\Rightarrow$ $J(P)$ CDE

Prop. $J(P)$ is tCDE iff $\exists\, c, c_p \in \mathbb{R}$ s.t.

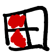
$$\text{ddeg} = c + \sum_{p \in P} c_p T_p.$$

(and here $c = \mathbb{E}_{\text{uni}}(\text{ddeg})$.)

NOTE: $\forall P$, $\text{ddeg} = \sum_{p \in P} T_p^-$

Pf. $(\Leftarrow)$ Take $\mathbb{E}_\mu$. $(\Rightarrow)$ Not too hard... $\square$

E.g. $J(2\times 2) = J\left(\begin{array}{c|c} 11 & 12 \\ \hline 21 & 22 \end{array}\right)$

	T_{11}	T_{12}	T_{21}	T_{22}	$\mathbb{1}$	ddeg
	1	0	0	0	1	0
	-1	1	1	0	1	1
	0	1	-1	0	1	1
	0	-1	1	0	1	1
	0	-1	-1	1	1	2
	0	0	0	-1	1	1

$-\underline{1} + -\tfrac{1}{2} + -\tfrac{1}{2} + 0 + \underline{1} = \text{ddeg}$ ✓

CDE eqn. is asking if certain vector is in column span of certain matrix, so of course can check easily via computer (see Sage code).

Note: Matrix is 'tall' $(\#J(P) \times (\#P+1))$ so it's surprising if a certain vector belongs to its column span.

What's the connection to rowmotion? ...

Prop. (Striker) T_p is 0-mesic for rowmotion acting on $J(P)$, for any $p \in P$, i.e.,

#4 a dist. on $J(P)$ that's constant on rowmotion orbits is toggle symmetric.

pf: Along any Row-orbit $\mathcal{O}$, T_p looks like:

$$\begin{array}{c|l} & \cdots \quad I \quad \mathrm{Row}(I) \quad \cdots \\ \hline T_p & \cdots 1 \;\; -1 \;\; 0 \;\; 0 \cdots 0 \;\; 1 \quad\quad -1 \;\; 0 \;\; 0 \cdots 0 \;\; 1 \;\; -1 \;\; 0 \cdots \end{array}$$

$(T_p^+(I) = 1 \Leftrightarrow T_p^-(\mathrm{Row}(I)) = 1 \;\forall\, I \in J(P))$ $\square$

Cor. $J(P)$ tCDE $\Rightarrow$ ddeg ('antichain card.') is homomesic for Row on $J(P)$.

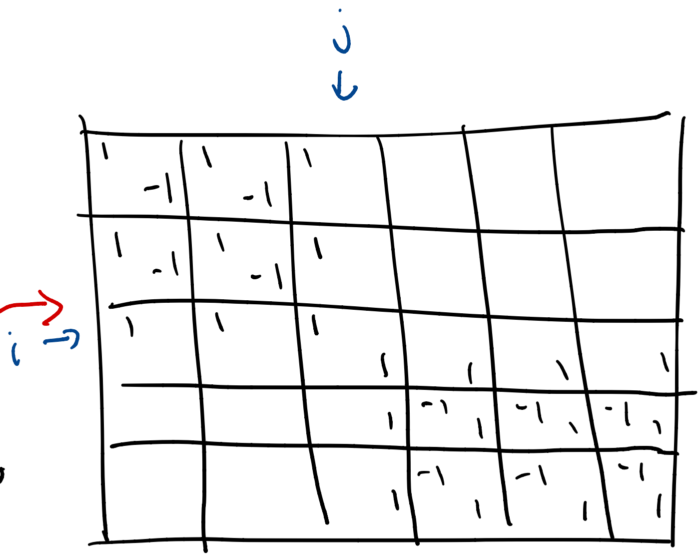
pf: Apply last prop. + unwind definitions... $\square$

So establishing tCDE is a good way to establish antichain card. homomesy for rowmotion.

We can check tCDE easily via computer... but how do you *really* show $J(P)$ is tCDE?

Answer: Rooks!

E.g. $\boxed{P = a \times b}$



Rook R_{ij} is a linear comb. of T_p^+, T_p^- according to this pattern, where for a box (x, y) $\boxed{\begin{matrix} a & \\ & b \end{matrix}}$ means $\rightsquigarrow$ $a T_{xy}^+ + b T_{xy}^-$ (we omit 0's)

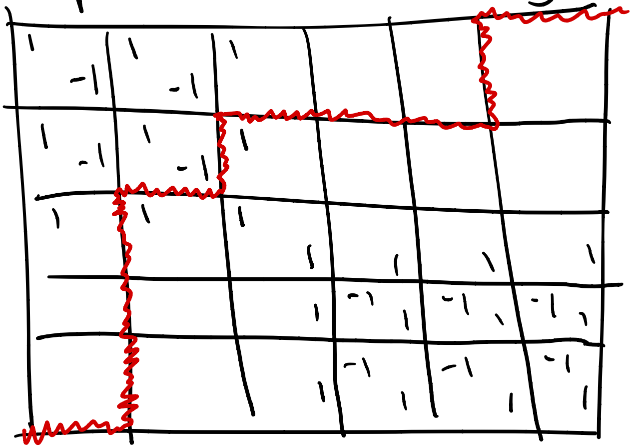
Prop: $$R_{ij} = \sum_{j'} T_{ij'}^- + \sum_{i'} T_{i'j}^- + \sum_{p \in P} c_p T_p$$

Pf: Clear. ☐

So R_{ij} 'attacks' in expectation every box in row i and column j... hence name rook.

Key prop. $R_{ij}(I) = 1 \quad \forall\, I \in J(P)$.

Pf: By example... add #s along lattice path:



$1 + (-1) + 1 = 1$ ✓

$\blacksquare$

So...

place one rook on every square

$$ab = \sum_{p \in P} R_p = \sum_{i,j} \left(\sum_{j'} T_{ij'}^- + \sum_{i'} T_{i'j}^- \right) + \sum_{p \in P} c_p^* T_p$$

every box is attached a+b times

$$= \sum_{p \in P} (a+b) T_p^- + \sum_{p \in P} c_p T_p$$

For any P, $\text{ddeg} = \sum_{p \in P} T_p^-$

$$= (a+b)\,\text{ddeg} + \sum_{p \in P} c_p T_p$$

$\Rightarrow$ **Thm (CHHM)** For $P = a \times b$, $J(P)$ is tCDE w/ $\mathbb{E}_{\text{uni}}(\text{ddeg}) = \dfrac{ab}{a+b}$.

NOTE: CHHM proved Thm for broader family of shapes + skew shapes beyond rectangles ('balanced shapes').
H. extended to shifted shapes. Holds for:
minuscule posets, Type A/B root posets, + more ...

Now for Jim's problem... q-ify all this. ($q>0 \in \mathbb{R}$)

Def. Set $T_p^q := q \cdot T_p^+ - T_p^-$. Say distr. μ on $J(P)$ is q-toggle-symmetric if $\mathbb{E}_\mu(T_p^q) = 0 \;\; \forall p \in P$.

E.g. #1 q-uni = distr. on $J(P)$ where $I \in J(P)$ occurs proportional to $q^{\#I}$.

Then q-uni is q-toggle-symmetric: think abt. toggling.

#2 I think there should be a q-mchain(m) distr. that's q-toggle-symmetric extending q-uni

#3 I dunno about a q-maxchain

#4 I think a distr. that's uniform on a q-rowmotion orbit should be q-toggle symmetric by similar argument:

$$\begin{array}{lcccccccccccc}
 & L & L' & H & H' & H'' & L & L' & H & H' & H'' & \\
T_p \cdots & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & \cdots
\end{array}$$

can toggle out q times as often as in along a q-Row orbit

Def. Say $J(P)$ is q-tCDE if $\mathbb{E}_\mu(\mathrm{ddeg}) = \mathbb{E}_{q\text{-uni}}(\mathrm{ddeg})$ for any q-toggle-symmetric distr. μ.
(Equiv., $\exists\, c^q, c_p^q \in \mathbb{R}$ s.t. $\mathrm{ddeg} = c^q + \sum_{p \in P} c_p^q T_p^q$.)

Conj. For $P = a \times b$, $J(P)$ is q-tCDE w/

$$\mathbb{E}_{q\text{-uni}}(\mathrm{ddeg}) = q \frac{[a]_q [b]_q}{[a+b]_q}.$$

—

Sage code let's you check this for given a, b.

Seems to hold also for shifted staircase and other minuscule posets, but NOT the broader families of tCDE shapes from CHHM... in particular NOT for Type A/B root posets.

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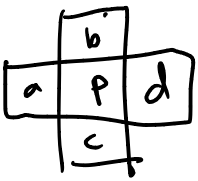
Altogether, the above would imply Jim's q-rowmotion homomesy conjecture (via same argument as in case $q=1$).

Main thing to do would be to try to define q-Rooks...

Pf sketch for key property of rooks:

Use PL lifts of T_p^+, T_p^-



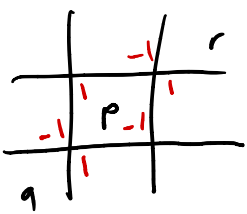
$$T_p^- = \min(c, d) - p$$

$$T_p^+ = p - \max(a, b)$$

p = shorthand for $\pi(p)$

Want to show overall contribution of $p = 0$.

Look at 'generic' case:



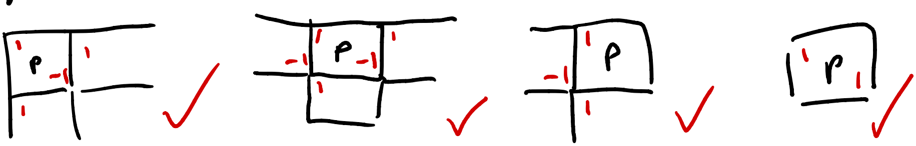
$\min(p,q) + \max(p,q) = p+q$

how much p? :

$\rightsquigarrow p - (-p) - \min(p, q) - \max(p, q)$

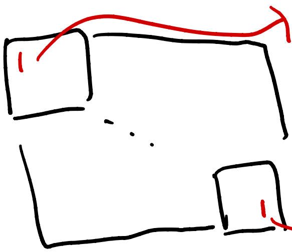
$- \min(p, r) - \max(p, r) = 0 \cdot p$ ✓

Need to check other border cases:



+ symmetric cases ✓

Where does the '1' come from? Corners!



$\max(\phi) = 0$

$\min(\phi) = 1$ ✓

$\boxtimes$

Overview of toggleability statistics technique:

- **Posets**:
  - rectangle
  - shifted staircase (+ other minuscule posets)
  - Type A root poset (+ other root posets)
  - other shapes/shifted shapes ('balanced')
  - beyond distributive lattices: semidistributive
    e.g. weak order intervals
- **Statistics**:
  - antichain cardinality
  - major index
  - order ideal cardinality
  - rank-alternating order ideal cardinality
- **Context**:
  - Span$\{1, T_p\}$
  - PL / birational analogs
  - q-analogy Span$\{1, T_p^q\}$
- **Applications** (toggle-symmetric distributions)
  - uniform distr. / q-uniform distr.
  - rank distr. / q-rank distr.
  - rowmotion orbits / q-rowmotion orbits
  - p-partition distribution / q-analog of
    mchain(m) p-partition distr.
  - linear extension distr. / $\lim_{m\to\infty}$ mchain$_q$(m)
    maxchain = $\lim_{m\to\infty}$ mchain(m)