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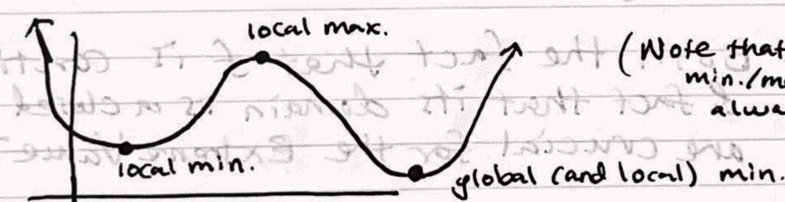
Maximum and minimum values §4.1

One of the most important applications of calculus is to optimization problems: finding "best" option, which ultimately are about locating maxima and minima.

Def'n Let c be in domain of function f. We say $f(c)$ is:
- absolute (or global) maximum if $f(c) \geq f(x)$ $\forall$ x in domain of f,
- absolute (or global) minimum if $f(c) \leq f(x)$ $\forall$ x in domain,
- local maximum if $f(c) \geq f(x)$ for x "near" c,
- local minimum if $f(c) \leq f(x)$ for x "near" c.

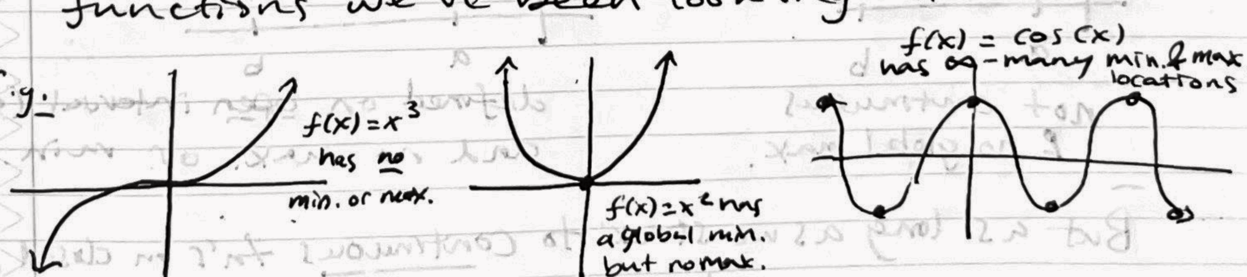
E.g.



(Note that global min./max. are also always local min./max.)

The behavior of min./max. for functions $f: \mathbb{R} \to \mathbb{R}$ can be very complicated, even for the "nice" functions we've been looking at:

E.g.



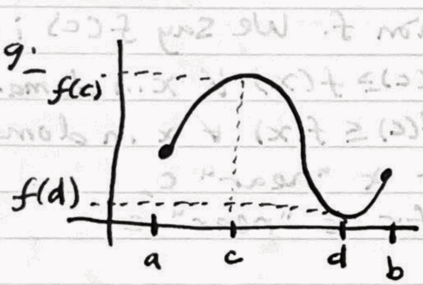
And of course we saw above how local min. & max. do not need to be global min. & max.

Things are much better when we restrict the domain of f to be a closed interval $[a, b]$:

global min./max. are also called "extreme values"

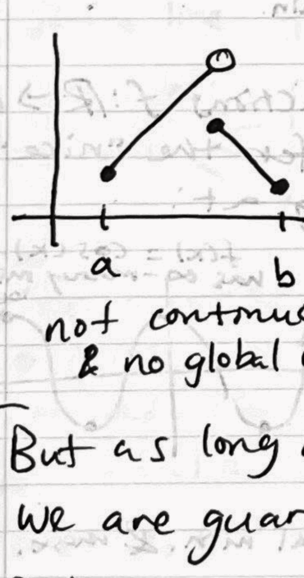
Theorem (Extreme Value Theorem)
Let f be a continuous fn. defined on a closed interval $[a,b]$. Then f attains a global max. value $f(c)$ and a global min. value $f(d)$ at some points $c, d \in [a,b]$.

E.g.

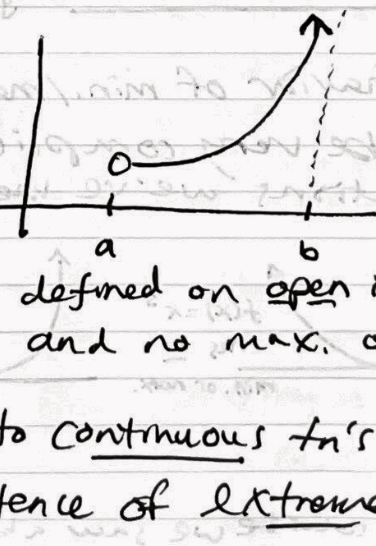


NOTE: can attain max. or min. multiple times.
also can attain max. or min. at endpoints a & b.

WARNING: Both the fact that f is continuous & fact that its domain is a closed interval are crucial for the Extreme Value Thm.



not continuous & no global max.



defined on open interval (a,b) and no max. or min.

But as long as we stick to continuous fn's on closed intervals, we are guaranteed existence of extreme values.

But... how do we find the location of the extreme values that we know must exist?

We use calculus! Specifically: the derivative!

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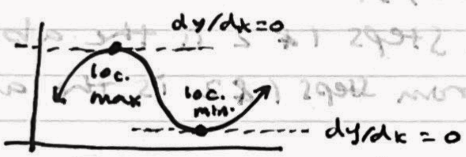
We mentioned before that at (local) min./max., the derivative must be zero.

Thm (Fermat) If f has local min./max. at c, and if $f'(c)$ exists, then $f'(c) = 0$.

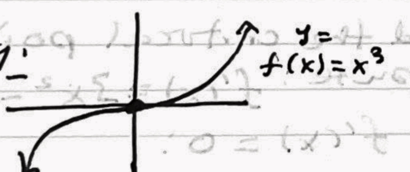
E.g.



← intuitive from tangent line slope definition of derivative.

WARNING: The converse of this theorem is not true, i.e., if $f'(c) = 0$ it does not mean c is location of min./max.

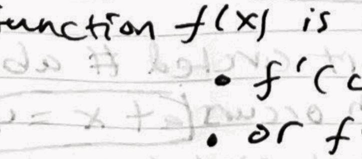
E.g.



For $f(x) = x^3$ we have $f'(0) = 0$ (since $f'(x) = 3x^2$), but 0 is not a local min./max. since there are no local min./max..

WARNING: If $f'(c)$ does not exist, c could be location of a local min./max.!

E.g.



For $f(x) = |x|$ (absolute value) we explained before why $f'(0)$ does not exist, but 0 is a global minimum!

DEF'N A critical point (or critical number) of a function $f(x)$ is a point $x = c$ where either:

- $f'(c) = 0$
- or $f'(c)$ does not exist.

We can use critical points to find extreme values:

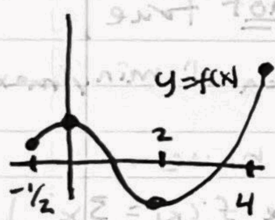
§4.1

The Closed Interval Method

To find the absolute minimum and maximum of a continuous function f defined on a closed interval $[a,b]$:

1. Find the values of f at the critical points of f in (a,b).
2. Find the values of f at the endpoints of interval (i.e., $f(a)$ and $f(b)$).
3. The largest value from steps 1 & 2 is the abs. max. The smallest value from steps 1 & 2 is the abs. min.

E.g. Problem: Find the absolute maximum and minimum of $f(x) = x^3 - 3x^2 + 1$ on interval $-\frac{1}{2} \le x \le 4$



Solution: We use the Closed Interval Method.

1. We need to find the critical points. So we compute: $f'(x) = 3x^2 - 6x$ and solve for $f'(x) = 0$:

$$3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 2.$$

The critical points are $x=0$ and $x=2$. Their f values are:

$\boxed{f(0) = 0^3 - 3\cdot 0^2 + 1 = 1}$ and $\boxed{f(2) = 2^3 - 3\cdot 2^2 + 1 = -3}$

2. We compute the values of f on the endpoints:

$$\boxed{f(-\tfrac{1}{2}) = (-\tfrac{1}{2})^3 - 3\cdot(-\tfrac{1}{2})^2 + 1 = \tfrac{1}{8}}$$

and $\boxed{f(4) = 4^3 - 3\cdot 4^2 + 1 = 17}$

3. The abs. max. is the largest circled # above: i.e., $\boxed{\max = 17}$ which occurs $\boxed{\text{at } x = 4}$

The abs. min. is the smallest circled # above: i.e., $\boxed{\min = -3}$ which occurs $\boxed{\text{at } x = 2}$.

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