PROBLEMS FROM PROPP'S 64TH BIRTHDAY CONFERENCE

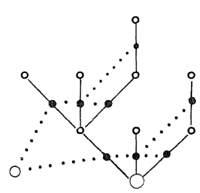
The following $\sqrt{64}$ open problems were presented at "Statistical and Dynamical Combinatorics: a celebration of Jim Propp's $2^{\binom{4}{2}}$th Birthday," which took place at MIT on June 26–29, 2024 (`https://dept.math.lsa.umich.edu/~speyer/JIM/`). These problems were recorded by Sam Hopkins.

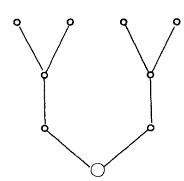
Peter Winkler – "Fencing off the blob". At time zero the blob starts as a unit disc on the plane and, if unimpeded, its radius will grow at a rate of one unit per unit of time. You are capable of placing down a fence anywhere on the plane at a rate of λ total length of fence per unit of time. When the blob encounters this fence, it will be impeded by the fence but will grow around it, as depicted below:

Let λ_{crit} be the critical value of λ below which the blob will grow off to infinity no matter what you do, and above which you can eventually confine the blob to a finite region. The problem is: what is λ_{crit}? For example, $\lambda_{\text{crit}} \leq 2\pi$.

This is a continuous model of firefighting; for a similar discrete model, see [3].

Sam Hopkins – "A cyclic action on plane trees". Consider the set of plane trees with $n+1$ vertices, a set with cardinality the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. We define an operator A acting on this set as follows. In pictures:

 , T.

 , $A(T)$.

In pseudocode:

```
def A(T):
        if T = []:
                return T
        else:
                S = T.pop()
                return A(S) + [A(T)]
```

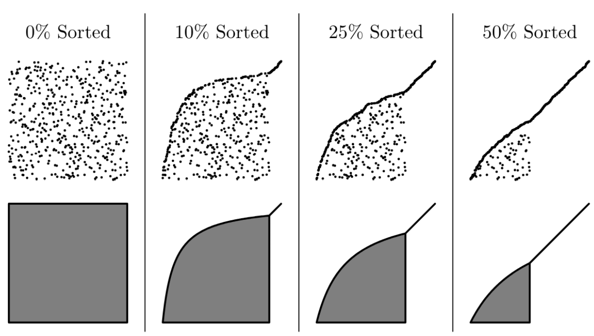
where we encode a plane tree as a list (so the above tree is $T = [[[], [], [[]]], [], [[]]]$), pop removes and returns the first element of a list, and + is concatenation of lists. This operator is invertible and hence defines a bijection on this set of plane trees. It is a variant of the canonical bijection between plane trees and binary trees.

In some regards, the bijection A behaves very chaotically. For any given tree T, it seems hard to determine the minimal m such that $A^m(T) = T$. And for $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots$ the order of A is 1, 2, 6, 6, 30, 120, 720, 15120, 1164240, 15135120, 283931716867999200, …, where this last number $283931716867999200 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 31 \cdot 37 \cdot 47 \cdot 89$ has several large prime factors.

But the bijection A also exhibits surprising regularity. For example, Shapiro [13] showed that it behaves very regularly on a large subset of trees. Namely, consider the subset of plane trees for which all non-root vertices have at most one child; this subset is in bijection with the compositions of n in an straightforward way. Shapiro showed that for the tree T corresponding to a composition α, $A^3(T)$ is the tree corresponding to the conjugate composition α', and hence $A^6(T) = T$. Furthermore, Donaghey [9] showed that the behavior of A on all plane trees can be reduced to its behavior on a certain collection of "primitive" trees.

The problem is: what more can be said about this cyclic action A on plane trees, especially from the modern perspective of dynamical algebraic combinatorics (e.g., homomesy, resonance, etc.)?

Colin Defant – "The c-bubble sort permuton". Let S_n be the symmetric group and let $\tau_i \colon S_n \to S_n$ be the "toggle" which swaps the ith and $(i+1)$st letters of a permutation if they are out of order, or otherwise does not change the permutation. Let $\mathrm{Bubble} = \tau_1 \cdot \tau_2 \cdots \tau_{n-1}$ be the bubble sort operator. It is well-known that this operator really sorts in the sense that there is some minimal $m = o(n)$ such that Bubble^m sends every permutation to the identity. But what happens if we only partially bubble sort? In other words, what is the image of a uniformly random permutation under $\mathrm{Bubble}^{\alpha m}$ for some $0 < \alpha < 1$? The following figure is from DiFranco [8] (see also `https://www.youtube.com/watch?v=Gm8v_MR7TGk`):



As we can see, the resulting random permutation is confined to a region with a curved boundary, and as $n \to \infty$ we obtain an interesting limit shape, which in fancy language could be called a permuton. In his preprint, DiFranco describes and rigorously establishes the existence of this limit shape.

If $i_1, i_2, \dots, i_{n-1}$ is any permutation of the indices $1, 2, \dots, n-1$, we could consider the corresponding "permuted" bubble sort operator $\tau_{i_1} \cdot \tau_{i_2} \cdots \tau_{i_{n-1}}$. In fact, since the toggles satisfy the braid relations, this operator only depends on the choice of a Coxeter element $c = s_{i_1} s_{i_2} \cdots s_{i_{n-1}}$. So let us call this operator the c-bubble sort operator and denote it by Bubble_c. Again, there will be some minimal $m = o(n)$ for which Bubble_c^m sorts all permutations. Extending the work of DiFranco, we are interested in the limit shape of a partially c-bubble sorted permutation, i.e., a uniformly random permutation under $\mathrm{Bubble}_c^{\alpha m}$ for some $0 < \alpha < 1$.

An issue, however, is that we need the make choice of the Coxeter element c uniformly in n. For instance, beyond the standard Coxeter element $c = s_1 s_2 \cdots s_{n-1}$, another sensible choice would be the bipartite Coxeter element $c = s_1 s_3 \cdots s_2 s_4 \cdots$, whose corresponding c-bubble sort operator first sorts odd positions and then even positions. A choice of Coxeter element is the same as an orientation of the Type A Dynkin diagram. Equivalently, a Coxeter element corresponds to a walk with up steps $(1,1)$ and down steps $(1,-1)$, where an oriented edge from i to $i+1$ corresponds to an up step in position i and an oriented edge from $i+1$ to i corresponds to a down step in position i. In this way, the usual bubble sort has a 45° line as its walk, and the bipartite bubble sort has (approximately) a horizontal line as its walk.

Colin suggested that we can enforce uniformity on the Coxeter element by having its corresponding walk approach some limiting curve. So the problem is: if we let $n \to \infty$ and choose a Coxeter element c whose walk (appropriately re-scaled) approaches some limiting curve, does the partially c-bubble sorted permutation yield an interesting permuton whose shape we can describe?

Lionel Levine – "Maximal matchings of the Aztec diamond". In his talk, Kyle Petersen explained that Conway's napkin problem [4] has the following equivalent description in terms of maximal matchings of the cycle graph of length $2n$. Suppose the cycle is bicolored, black and white. Build up a matching by repeatedly doing the following. Choose an unmatched black vertex uniformly at random. If at least one of its neighboring white vertices is unmatched, choose a neighbor uniformly at random and add that edge to our matching. If all its neighboring white vertices have been matched already, do not add an edge; the black vertex will remain unmatched. We terminate with a maximal – but not necessarily perfect! – matching of the cycle. Kyle pointed out that this process makes sense on any bipartite graph, and after the talk David Speyer suggested looking at the process on the Aztec diamond graph.

Using the "copy and paste loop" technique for writing code with AI assistance that he described in his talk, Lionel quickly produced code to simulate this maximal matching process on the Aztec diamond graph, and obtained this image:

[maximal matching of Aztec diamond image here]

This image suggests that there is no frozen region for the random maximal matching constructed via this process, and that about 90% of the vertices end up matched.

The first problem is: can these facts about the maximal matching process on the Aztec diamond be rigorously proved?

Then, Lionel suggested the following extension. It should be possible to upgrade any maximal matching to a perfect matching by greedily selecting augmenting paths. So, we can start with a maximal matching of the Aztec diamond produced by the process described above, and then upgrade it to a perfect matching. We expect that the resulting distribution on perfect matchings is *not* uniform.

The second problem is: analyze the resulting distribution on perfect matchings of the Aztec diamond. Does it have a frozen region?

Mikhail Skopenkov – "Square-tileable polyhedra". The question of which polygons can be tiled by squares is classical. In these tilings, the square tiles can be of varying sizes, and the tiling does not need to be edge-to-edge. For example, a square tiling might look like this:

A famous result of Dehn is that an $a \times b$ rectangle is square-tileable if and only if the ratio a/b is rational. In [11], Kenyon considers square tilings of *surfaces*, and classifies the Euclidean tori which are square-tileable.

The problem is: which polyhedra (i.e., three-dimensional convex polytopes) have boundaries that are square-tileable? In these tilings we allow folding of the square tiles over the edges of the polyhedra, so that the question does not just reduce to whether each face is square-tileable. It turns out that square-tileability questions are related to questions about certain resistor networks, and one motivation for this problem comes from potential theory.

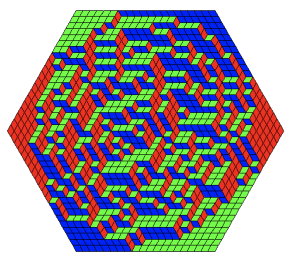
Aaron Abrams – "Longest pattern subsequence". It is a very famous result (of Vershik–Kerov and Logan–Shepp) that the expected length of a longest increasing subsequence of a uniformly random permutation in the symmetric group S_n is asymptotic to $2\sqrt{n}$. See for instance the book by Romik [12].

Stanley [14] showed that the expected length of a longest *alternating* subsequence of a random permutation in S_n is asymptotic to $\frac{2}{3}n$. Notice that this expectation is *linear* in n, unlike the increasing case.

Let U signify an ascent ("up") and D signify a descent ("down"). Then an increasing subsequence can be encoded as the pattern U, and an alternating subsequence can be encoded as the pattern UD. This means that for an increasing subsequence we want an ascent, followed by an ascent, followed by an ascent, and so on, and for an alternating subsequence we want an ascent, followed by a descent, followed by an ascent, followed by a descent, and so on. (So we think of the pattern as "repeating," i.e., $U = UUU\ldots$ and $UD = UDUD\ldots$.) In [1], Abrams and his coauthors showed that for any non-constant pattern of U's and D's, a longest subsequence adhering to this pattern in a random permutation of S_n has expected length linear in n. But they were not able to derive a general formula for the constant a for which this expectation is asymptotic to an.

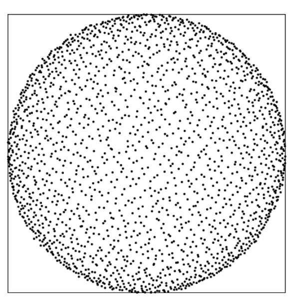
The problem is: for other patterns beyond alternating, determine the constant in this expected longest pattern subsequence length. For example, for the pattern UUD the constant appears empirically to be $0.57744\ldots$, and for the pattern $UUDD$ the constant appears to be $0.561\ldots$.

David Speyer – "Between random plane partitions and sorting networks". We all know that a random lozenge tiling of a large regular hexagon looks like this:



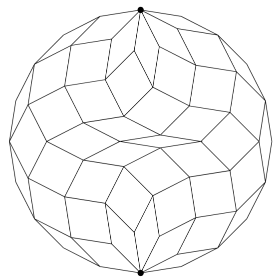
The arctic circle phenomenon for such tilings was established by Cohn–Larsen–Propp [6]. Of course, these random lozenge tilings are nothing other than random boxed plane partitions.

More recently, Angel–Holroyd–Romik–Virag [2] considered random sorting networks. By taking a slice in the middle of a random sorting network, we get a random permutation which also has a distinctive circular pattern:



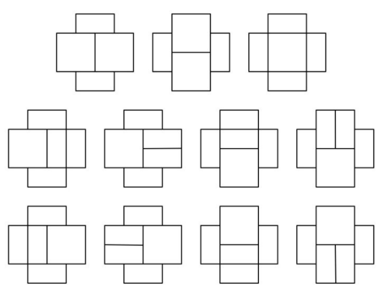
The existence of this limit shape was rigorously established by Dauvergne [7].

Random sorting networks can be thought of as random reduced words for the longest element in the symmetric group S_n. Elnitsky [10] explained how (commutation classes of) these reduced words can be represented by rhombic tilings of the regular $2n$-gon. For example, this figure appears in a paper of Tenner [15]:



The problem is: study random rhombic tilings of a large regular $2m$-gon, where m is greater than 3 but is not growing with the scale parameter. These random structures should be "between" random plane partitions and random sorting networks. David suggested in particular that random rhombic tilings of a large octogon could already be very interesting.

Jim Propp – "2-adic behavior of domino & square Aztec diamond tilings". Consider tilings of the Aztec diamond, where we allow the usual 2×1 and 1×2 domino tiles, but also 2×2 square tiles. For example, here are some of these tilings of the Aztec diamond of order 2:



Using the tiling code that David desJardins described in his talk, Jim found that the number $M(n)$ of such tilings of the Aztec diamond of order n is given by the following table:

n	$M(n)$
0	1
1	3
2	19
3	293
4	10917
5	996599
6	222222039
7	121552500713
8	162860556763865
9	535527565429290907
10	4318205059450240425083
11	85475498697714319842817853
12	4151186175463797888945512144221

This table suggests $M(n)$ does not have any simple product formula.

However, $M(n)$ appears to exhibit interesting behavior modulo powers of 2. It is not difficult to show that $M(n)$ must be odd for all n. And if we look mod 4, we see that $M(n)$ apparently follows the pattern $1, 3, 3, 1,\ 1, 3, 3, 1, \ldots$ which has period 4. Similarly, $M(n) \mod 8$ appears to be periodic with period 8. This suggests that the value of $M(n) \mod 2^i$ might depend only on the value of $n \mod 2^i$, or in other words, that $M(n)$ is a 2-adically continuous function of n.

A result of a similar flavor was proved by Cohn in [5]. Namely, the number of domino tilings of a $2n \times 2n$ square has the form $2^n(f(n))^2$, and Cohn showed that this function $f(n)$ is 2-adically continuous. He did this simply by analzying the famous Kasteleyn/Fisher–Temperley formula for the number of such tilings.

The problem is: prove the 2-adic continuity of the tiling number $M(n)$. Jim suggested that one approach to this problem could be by developing an extension of the Kasteleyn permanent/determinant method which would apply to this setting where we have tiles that are bigger than dimers.

References

[1] Aaron Abrams, Eric Babson, Henry Landau, Zeph Landau, and James Pommersheim. A central limit theorem for repeating patterns. Preprint, available at `https://arxiv.org/abs/1204.2872`, 2012.

[2] Omer Angel, Alexander E. Holroyd, Dan Romik, and Bálint Virág. Random sorting networks. *Adv. Math.*, 215(2):839–868, 2007.

[3] Amir Barghi and Peter Winkler. Firefighting on a random geometric graph. *Random Structures Algorithms*, 46(3):466–477, 2015.

[4] Anders Claesson and T. Kyle Petersen. Conway's napkin problem. *Amer. Math. Monthly*, 114(3):217–231, 2007.

[5] Henry Cohn. 2-adic behavior of numbers of domino tilings. *Electron. J. Combin.*, 6:Research Paper 14, 7, 1999.

[6] Henry Cohn, Michael Larsen, and James Propp. The shape of a typical boxed plane partition. *New York J. Math.*, 4:137–165, 1998.

[7] Duncan Dauvergne. The Archimedean limit of random sorting networks. *J. Amer. Math. Soc.*, 35(4):1215–1267, 2022.

[8] Michael DiFranco. A rigorous derivation of the bubble sort curve. Preprint, available at `https://linesthatconnect.github.io/assets/bubble-sort-derivation/bubble-sort-curve.pdf`, 2024.

[9] Robert Donaghey. Automorphisms on Catalan trees and bracketings. *J. Combin. Theory Ser. B*, 29(1):75–90, 1980.

[10] Serge Elnitsky. Rhombic tilings of polygons and classes of reduced words in Coxeter groups. *J. Combin. Theory Ser. A*, 77(2):193–221, 1997.

[11] Richard Kenyon. Tiling with squares and square-tileable surfaces. Preprint, available at `https://citeseerx.ist.psu.edu/document?doi=eec731c9882dcb80a2fb1dbe2073b85e3ed13753`, 1994.

[12] Dan Romik. *The surprising mathematics of longest increasing subsequences*, volume 4 of *Institute of Mathematical Statistics Textbooks*. Cambridge University Press, New York, 2015.

[13] Louis W. Shapiro. The cycle of six. *Fibonacci Quart.*, 17(3):253–259, 1979.

[14] Richard P. Stanley. Longest alternating subsequences of permutations. *Michigan Math. J.*, 57:675–687, 2008. Special volume in honor of Melvin Hochster.

[15] Bridget Eileen Tenner. Tiling-based models of perimeter and area. *Adv. in Appl. Math.*, 119:102059, 29, 2020.