

Howard Math 273, HW# 1,

Fall 2021; Instructor: Sam Hopkins; Due: Friday, October 1st

1. (Stanley, EC1, #1.66) Let $p_k(n)$ denote the number of partitions of n into k parts. Prove bijectively that

$$p_0(n) + p_1(n) + p_2(n) + \cdots + p_k(n) = p_k(n+k).$$

2. (Stanley, EC1, #1.113) Fix natural numbers k, n . Let $[n]$ denote the set $[n] := \{1, 2, \dots, n\}$. Give a simple formula for the number of ordered k -tuples (T_1, \dots, T_k) of subsets of $[n]$ satisfying

- $T_i \cap T_j = \emptyset$ for all $i \neq j$ (i.e., they are disjoint);
- $\bigcup_{i=1}^k T_i = [n]$ (i.e., their union is the whole set $[n]$).

(This is almost the same as saying that the T_i form a set partition of $[n]$, except that some of these sets may be empty, which we do not usually allow for set partitions.)

3. (Stanley, EC1, #1.5) Show that

$$\sum_{n_1, \dots, n_k \geq 0} \min(n_1, \dots, n_k) x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} = \frac{x_1 x_2 \cdots x_k}{(1-x_1)(1-x_2) \cdots (1-x_k) \cdot (1-x_1 x_2 \cdots x_k)}.$$

4. (Stanley, EC1, #1.26) Let $\bar{c}(n, m)$ denote the number of compositions of n into parts of size at most m . Show that

$$\sum_{n \geq 0} \bar{c}(n, m) x^n = \frac{1-x}{1-2x+x^{m+1}}.$$

5. Prove that, for any $n \geq 0$,

$$4^n = \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k}. \quad (1)$$

Hint: did we discuss the generating function $\sum_{n=0}^{\infty} \binom{2n}{n} x^n$ of the central binomial coefficients?

Hard bonus problem, just to think about, not to do: prove (1) *bijectively* (see Stanley, EC1, #1.3(c)).

6. Let $n \geq 1$, and let $\text{ODD}(n)$ denote the subset of permutations in the symmetric group \mathfrak{S}_n with no cycles of even size. Prove that

$$\sum_{\sigma \in \text{ODD}(n)} 2^{\#\text{cycles}(\sigma)} = 2 \cdot n!. \quad (2)$$

Hint: recall *Touchard's theorem*

$$\sum_{n \geq 0} \frac{1}{n!} \left(\sum_{\sigma \in \mathfrak{S}_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots t_n^{c_n(\sigma)} \right) x^n = e^{t_1 \frac{x}{1} + t_2 \frac{x^2}{2} + t_3 \frac{x^3}{3} + \cdots} = e^{\sum_{j=1}^{\infty} t_j \frac{x^j}{j}},$$

where $c_k(\sigma)$ is the number of cycles of σ of size k .

Hard bonus problem, just to think about, not to do: prove (2) *bijectively*.