Total: 49/50

Homework 3 – Combinatorics 2

1. A *plane partition* is an infinite $2D$-array $\pi = (\pi_{i,j})_{i=1,2,\dots}^{j=1,2,\dots}$ of nonnegative integers $\pi_{i,j} \in \mathbb{N}$ such that only finitely many entries are nonzero and the entries are weakly *decreasing* along rows and down columns in the sense that $\pi_{i,j} \geq \pi_{i',j'}$ if $i \leq i'$ and $j \leq j'$. The *size* $|\pi|$ of π is the sum of the entries: $|\pi| := \sum_{i,j\geq 1} \pi_{i,j}$. Prove that

$$\sum_{\pi \text{ a plane partition}} q^{|\pi|} = \prod_{i\geq 1} \frac{1}{(1-q^i)^i} \tag{1}$$

Hint: We proved the following product formula for *reverse* plane partitions of shape λ:

$$\sum_{\pi \in \mathrm{RPP}(\lambda)} q^{|\pi|} = \prod_{u\in\lambda} \frac{1}{1-q^{h(u)}} \tag{2}$$

From the hint, we know that rotating an RPP 180°, gives us the upper left corner of a plane partition.

Now consider the formula for RPPs of shape λ: $\sum_{\pi \in RPP(\lambda)} q^{|\pi|} = \prod_{u\in\lambda} \frac{1}{1-q^{h(u)}}$.

For a square $n\times n$ RPP, there are i # boxes with hook length i, $i \leq n$.

Notice then that for PPs, when you flip an RPP around, you get this $\forall i \Rightarrow$ there are i, $\frac{1}{1-q^i}$ s $\Rightarrow \sum_{\pi \text{ a plane partition}} q^{|\pi|} = \prod_{i\geq 1} \left(\frac{1}{1-q^i}\right)^i = \prod_{i\geq1} \frac{1}{(1-q^i)^i}$ $\square$

Okay. 10/10

2. Recall that a linear extension of a (finite) poset P is a list $p_1, \dots, p_n$ of all its elements (each appearing once) where $p_i \leq p_j$ implies $i \leq j$. $\mathcal{L}(P)$ denotes the set of linear extensions of P.

   (a) Among posets P with n elements, which has the greatest number $\#\mathcal{L}(P)$ of linear extensions? Which has the least?

   (b) The *dual* P^* of a poset P is the poset with the same elements but the reverse order: $p \leq_P q \Leftrightarrow q \leq_{P^*} p$. Prove that $\#\mathcal{L}(P) = \#\mathcal{L}(P^*)$.

   (c) The *(disjoint) union* $P \cup Q$ of two posets P and Q is the poset whose elements are the elements in the union of the two sets, where the order within P and within Q is the same, but all $p \in P$ are incomparable to all $q \in Q$. Give a formula for $\#\mathcal{L}(P \cup Q)$ in terms of $\#\mathcal{L}(P)$, $\#\mathcal{L}(Q)$, and $n = \#P$ and $m = \#Q$.

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a) A linear extension is restricted by the relationships between elements in P, so you have the most linear extensions when there are no relationships between the elements of P. Thus antichains have the greatest number of linear extensions.

Similarly, the least would be when each element can only be in one order for a linear extension. This is the case with chains. Therefore chains have the least number of linear extensions. Good!

b) Define $f: \mathcal{L}(P) \to \mathcal{L}(P^*)$ by $f(E) = f(E_1, E_2, \dots, E_n) = (E_n, \dots, E_2, E_1)$ for some linear extension $E \in \mathcal{L}(P)$. This creates a new list with the same relationships as those in E, but reversed. Thus this new list is a linear extension of P^*, and since f is reversible, $\#\mathcal{L}(P) = \#\mathcal{L}(P^*)$ $\square$ Good!

c) P and Q are disjoint, so to form a linear extension of $P \cup Q$, we can just combine arbitrary extensions of P and Q. So now we have $\#\mathcal{L}(P) \cdot \#\mathcal{L}(Q) \cdot$ the ways to merge the lists together without messing up the order of each list. To find this, let's suppose that we want to put $p \in \mathcal{L}(P)$ into $q \in \mathcal{L}(Q)$. Then we can either put each element in p either before an element of q or at the end. This is the same process that we used for stars and bars, so using the same logic, we get that there are $\binom{n+m}{n}$ ways to merge the lists together while maintaining order. Very good way of putting it ("merging").

$\therefore$ there are $\#\mathcal{L}(P) \cdot \#\mathcal{L}(Q) \cdot \binom{n+m}{n}$ linear extensions of $P \cup Q$ $\square$

3. Recall that f^λ denotes the number of Standard Young Tableaux of shape λ. Give a simple formula for f^λ in the case of a *hook* shaped partition $\lambda = (k, \overbrace{1, 1, \dots, 1}^{n-k})$ for $1 \le k \le n$.

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Standard Young Tableauxs are strictly increasing along both rows and columns. So we only really have the option to choose in the first row, which gives us $k-1$ total choices to define the whole filling.

And there are $n-k+k-1$ values to pick from (you have to subtract 1 because the smallest number must go first), so the final formula is $f^\lambda = \binom{n-1}{k-1}$ $\square$

Nice! You can also use the Hook Length Formula, but your argument is simpler.

4. We used the Robinson-Schensted algorithm to prove that $\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$, the number of permutations in the symmetric group S_n. Prove that $\sum_{\lambda \vdash n} f^\lambda = \#\{\sigma \in S_n : \sigma = \sigma^{-1}\}$, the number of *involutions* in S_n. **Hint**: Use a symmetry property of RS(K) we discussed.

Toggle-based RSK shows the symmetry that transposition of the input
10/10 matrix commutes with RSK $\because$ $RSK(M) = (P,Q) \Longrightarrow RSK(M^t) = (Q,P)$.
We're going to use RSK to go from involutions to Standard Young Tableauxs. So now suppose that M is the permutation matrix of an involution in S_n. Then $M = M^t$, and $RSK(M) = RS(M)$ since M is a permutation matrix. Now since we have that $RSK(M) = (P,Q) \Longleftrightarrow RSK(M^t) = (Q,P)$, we have $P = Q \Longrightarrow RSK(M) = RS(M) = (P,P)$, which means that every involution permutation matrix in S_n is matched with a single Standard Young Tableaux. Yes.

$\therefore \sum_{\lambda \vdash n} f^\lambda = \#\{\sigma \in S_n \mid \sigma = \sigma^{-1}\}$ $\square$

5. For $\sigma \in S_n$, let $\mathrm{lis}(\sigma)$ (resp., $\mathrm{lds}(\sigma)$) denote the length of the longest increasing (resp., decreasing) subsequence in σ. The Erdős-Szekeres theorem says $\max(\mathrm{lis}(\sigma), \mathrm{lds}(\sigma)) \geq \sqrt{n}$ for
9/10 permutations $\sigma \in S_n$. Describe a permutation maximizing $\min(\mathrm{lis}(\sigma), \mathrm{lds}(\sigma))$ among $\sigma \in S_n$.

Alright, so for this one I'm trying to make both $lis(\sigma)$ and $lds(\sigma)$ equal to k or as close to it as possible, because the best result would be if $lis(\sigma) = lds(\sigma) := k$.

Let $\sigma \in S_n$.

If $n = 2k$, then let the permutations contain $1, 2, \ldots, k$ and $2k, 2k-1, \ldots, k+1$ respectively giving us $\min(lis(\sigma), lds(\sigma)) = k$, which is the maximum

This is a good heuristic, but why do you *know* this is the maximum? Answer: because an increasing and a decreasing subsequence can intersect in at most one element. But you should explain that.

And if $n = 2k+1$, then take permutaions containg the sequences $1, 2, \ldots, k+1$ and $2k+1, 2k, \ldots, k+1$. This results in $\min(\mathrm{lis}(\sigma), \mathrm{lds}(\sigma)) = k$, which is max.

And thus this has been described.