Relations § 3.3

You can think of a __relation__ from one set X to another set Y as a __chart__ that records how elements from X are "related" to elements from Y. For example, we can consider a chart that records for each student in a school the classes they're taking:

Student	Class
Bill	Economics
Bill	English
Alexis	English
Jordan	Chemistry ...

Notice that __unlike__ a __function__, each student can take __multiple classes__. Also, a student may be taking __no classes__ at all (e.g. they're on a leave of absence).

__Def'n__ Formally, a (binary) __relation__ R from set X to set Y is any subset of $X \times Y$, i.e., any set of ordered pairs of form (x, y) with $x \in X$ and $y \in Y$. If $(x, y) \in R$ then we write $x R y$ and we say "x is related to y."

__E.g:__ For the student/class example, the relation is
$$R = \{ (Bill, Econ.), (Bill, Eng.), (Alexis, Eng.), (Jordan, Chem.), ... \}$$
and since Alexis is taking English we could also write Alexis R English.

__Notice:__ A function $f: X \to Y$ is a very special relation from X to Y: one for which each $x \in X$ is related to ex__actly one__ $y \in Y$.
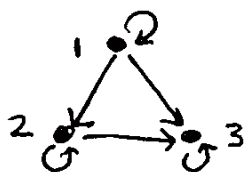
But relations can model things that functions can't...

The most important relations are when $X = Y$:

__Def'n__ If R is a relation from X to X, we say it is a __relation__ __on__ the __set__ X.

__E.g.__ If $X = \{1, 2, 3\}$ then $\leq$ defines a relation on X: we have "a is related to b" if and only if "$a \leq b$".

The set of ordered pairs for this relation is:
$$R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$$

We can represent this same information with a __digraph__:

Here we draw a "__vertex__" (a dot •) for each element of X, and draw an arrow $a \bullet \!\!\longrightarrow\! \bullet b$ whenever $a \, R \, b$.
Notice that if $a \, R \, a$ then we have a __loop__:

__Def'n__ The relation R is called __reflexive__ if $x \, R \, x$ for all $x \in X$.

__E.g.__ The $\leq$ relation on $\{1, 2, 3\}$ is reflexive: means we have a __loop__ at every vertex.
But if we consider the $<$ relation instead:

this is __not__ reflexive (no loops at all here).

Reflexivity captures the difference between $\leq$ (less than or equal to) and $<$ (strictly less than).

Def'n The relation R is called <u>symmetric</u> if whenever $x R y$ then also $y R x$, for all $x, y \in X$.

Eg: The relation $\leq$ on $\{1, 2, 3\}$ is <u>not</u> symmetric, since $1 \leq 2$ but $2 \not\leq 1$.

For a <u>symmetric</u> relation the digraph looks like:

$a \bullet \rightleftharpoons \bullet b$ or $a \bullet \quad _{no \ arrows} \quad \bullet b$ for every a, b.

Eg: An example of a symmetric relation R is
$X = \{$students at Howard$\}$ and
$x R y$ means "x has a class with y."
This is symmetric since if Person x has a class with Person y, then Person y has class with Person x!

Relations $\leq$ are "opposite" from symmetric, so:

Def'n Relation R is called <u>anti-symmetric</u> if whenever $x R y$ and $y R x$ then $x = y$, for all $x, y \in X$.

Eg: The relation $\leq$ (on $X = \{1, 2, 3\}$ or $X = \mathbb{Z}$ or $X = \mathbb{R}, \dots$) is anti-symmetric since if $x \leq y$ and $y \leq x$ then we must have $x = y$.

The relation $<$ is also anti-symmetric: there are no x, y at all with $x < y$ and $y < x$.

For an <u>anti-symmetric</u> relation digraph is:

No $a \bullet \cancel{\rightleftharpoons} \bullet b$ but loops $\overset{\frown}{\underset{a}{\bullet}}$ OK ✓

There is one more important property of $\leq$:

Def'n A relation R on X is called <u>transitive</u>
if for all $x, y, z \in X$, whenever we have
$x R y$ and $y R z$ then we must have $x R z$:

 ⟸ must have the dotted arrow

E.g. The relation $\leq$ (or $<$) is transitive because
if $a \leq b$ and $b \leq c$ then certainly $a \leq c$.

Q: Is relation "has a class with" on students transitive?

A: <u>No</u>! Maybe Bill has English class with Alexis,
and Alexis has Biology class with Cole,
but Bill has no class with Cole.

Def'n A relation R on X that is:
- <u>reflexive</u>,
- <u>anti-symmetric</u>,
- and <u>transitive</u>,

is called a <u>partial order</u> on X.

E.g. $\leq$ is a partial order on $X = \{1, 2, 3\}$
(or on $X =$ any set of numbers).

Partial orders behave like $\leq$: they
let us "<u>compare</u>" things in X.

But... partial orders don't necessarily
let us compare <u>every</u> pair of elements.

E.g. Consider a list of <u>tasks</u> you must do to complete a project. Maybe the project is "make a PB&J sandwich" and the tasks are:
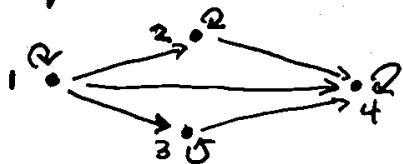
1. Toast two slices of bread.
2. Spread peanut butter on one slice.
3. Spread jelly on the other slice.
4. Put the two slices together.

Some of the tasks must be done <u>before</u> others (1 must be done before 2)

So we can define a relation R on the set of tasks:

$i R j$ if $i = j$ or task i must be done before task j

The digraph for this relation is:

reflexive ✓
anti-symmetric ✓
transitive ✓

There are no arrows between 2 and 3 since these two tasks can be done in either order.

Notice: we get a <u>partial order</u> on the tasks!

If R is a partial order on X and $x, y \in X$, we say x and y are <u>comparable</u> if $x R y$ or $y R x$, and say they are <u>incomparable</u> otherwise.

E.g. In PB&J example, tasks 2 and 3 are incomparable (can be done in either order).

The partial order R on X is called a <u>total order</u> if every pair $x, y \in X$ is comparable.

E.g. Relation $\leq$ on any set of #'s is a total order, but the "do before" relation on tasks is <u>not</u> a total order!

Compositions of relations and inverse relations §3.3

Now we return to discussing relations R from X to Y. Recall that a function $f: X \to Y$ is a special such relation, and we can generalize to relations the important functional notions of *composition* and *inversion*.

Def'n Let R_1 be a relation from X to Y, and R_2 a relation from Y to Z. The *composition* $R_2 \circ R_1$ is a relation from X to Z where for $x \in X$ and $z \in Z$ we have $x \, (R_2 \circ R_1) \, z$ if and only if there is $y \in Y$ with $x \, R_1 \, y$ and $y \, R_2 \, z$.

$$\boxed{X} \underline{\quad R_1 \quad} \boxed{Y} \underline{\quad R_2 \quad} \boxed{Z} \implies \boxed{X} \underline{\quad R_2 \circ R_1 \quad} \boxed{Z}.$$

Def'n Let R be a relation from X to Y. The *inverse* relation R^{-1} is a relation from Y to X where $R^{-1} = \{(y, x) : (x, y) \in R\}$

"reverse" every ordered pair in R

Note: For function $f: X \to Y$, the inverse $f^{-1}: Y \to X$ is defined only when f is a <u>bijection</u>. But the inverse relation R^{-1} is <u>always</u> defined.

If R is a relation on X (i.e. from X to X) then the <u>digraph</u> of R^{-1} is obtained from digraph of R by <u>reversing the direction</u> of all arrows:

$$\implies$$
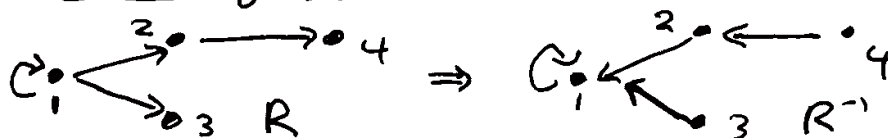
Q: What is inverse of $\leq$? A: $\geq$!

Equivalence Relations §3.4

Let X be a set and recall that a partition of X is a collection S of (nonempty) subsets of X such that every $x \in X$ belongs to exactly one subset in S.

E.g. For $X = \{1,2,3,4,5\}$ one partition is $S = \{ \{1,3,4\}, \{2,5\} \}$

A partition S is a way of "breaking X into groups" and we can use S to define a relation R on X where $x R y$ if and only if x and y are in same subset in S.

E.g. with the above partition, the digraph of R is:
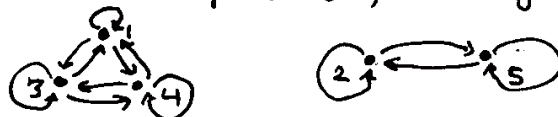
Theorem Relation R defined from a partition S of X is:
- reflexive
- symmetric
- and transitive.

Pf: All properties are easy to check directly.
Reflexive: x is in the same subset of S as itself.
Symmetric: if x is in same subset as y, then vice-versa.
Trans. : if x is same subset as y, and y as z, then same for x and z. ◻

Def'n Any relation R on a set X that is:
- reflexive
- symmetric
- and transitive

← (compare to def. of partial order)

is called an equivalence relation on X.

An equivalence relation on X is a way that elements of X can be "the same".

E.g. Relation R on $\mathbb{R}$ where $x R y$ if $x^2 = y^2$ is an equiv. relation.

E.g. Let n be any positive integer. We define relation R on $\mathbb{Z}$ where $x R y$ if $x - y$ is a multiple of n.

Exercise: This is an equivalence relation on $\mathbb{Z}$.

Partitions give us equivalence relations, and conversely:

Thm Let R be an equiv. relation on X. Let $a \in X$ be any element and define $[a] := \{x \in X : x R a\}$ (things related to element a).

Then $S = \{[a] : a \in X\}$ is a partition of X.

Pf: Need to show every $x \in X$ belongs to exactly one subset in S. By reflexivity of R, have $x \in [x]$. So suppose $x \in [y]$. Want to show then that $[x] = [y]$. So let $z \in [x]$. Then $z R x$, and since $x R y$, have $z R y$ by transitivity, i.e., have $z \in [y]$. By symmetry have $y R x$, so for any $z \in [y]$ have $z \in [x]$ by same argument. Thus, $[x] = [y]$. ▨

Def'n The sets $[a]$ for $a \in X$ from the previous theorem are called the equivalence classes of the equiv. relation R.

E.g. With R being equiv. relation on $\mathbb{R}$ where $x R y$ if $x^2 = y^2$, equivalence classes are $\{a, -a\}$ for $a \in \mathbb{R}$, i.e., each number is grouped with its negative.

E.g. Exercise What are the equivalence classes for the "$x R y$ if $x - y$ is a multiple of n" equivalence relation on the integers $\mathbb{Z}$?

Hint: Consider modular arithmetic mod n.

## Combinatorics: Basic Counting Principles § 6.1

We are starting a new chapter (our last of the semester):
Chapter 6, on combinatorics, a fancy word for "counting."
We'll learn many techniques for counting finite collections.
We start with some basic counting principles:

E.g: Suppose that for a meal you must choose:
• an appetizer: either soup or salad,
• a main course: chicken, fish, or pasta,
• a dessert: either ice cream or pie.

Q: How many different meals is it possible to make?

A: We can represent all the choices in a "decision tree":
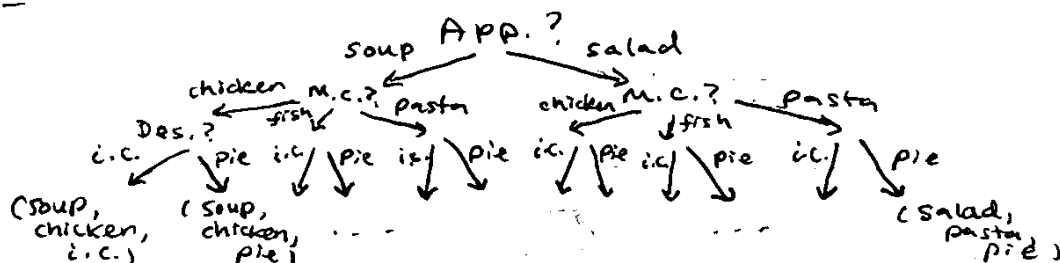
We see that there are $12 = 2 \times 3 \times 2$ total meals.
We multiply the choices at each step to get total!

Thm (Multiplication Principle for Counting)

Suppose we make an object via a series of steps,
where we have k_1 choices for step 1, k_2 choices for step 2,
down to k_m choices for step m. Then the total #
of objects we can make is $k_1 \times k_2 \times \cdots \times k_m$.

Remark: We saw before that for product $X_1 \times X_2 \times \cdots \times X_m$
of sets $X_1, X_2, \ldots, X_m$ which are finite, we have
$$\#(X_1 \times X_2 \times \cdots \times X_m) = \#X_1 \times \#X_2 \times \cdots \times \#X_m.$$
This is basically the same as the multiplication principle.

Let's see some more examples of the multiplication principle:

Eg. A US telephone # has 10 digits, & first digit cannot be 0.
Q: How many telephone #'s are there?
A: We have 9 possibilities for the 1st digit, and 10 for each of the 9 others. So by mult. principle:

$$9 \times \overbrace{10 \times 10 \times \cdots \times 10}^{9 \text{ times}} = 9 \times 10^9 = 9 \text{ billion telephone numbers}.$$

Eg. We saw before that the # of subsets of set $\{1, 2, 3, \ldots, n\}$ is 2^n.
To make a subset, we decide:
- Include 1 or not? (2 choices)
- Include 2 or not? (2 choices)
- ... • Include n or not? (2 choices)

This is n ~~choice~~ steps, with 2 choices at each step, so by mult. principle, # possibilities $= \overbrace{2 \times 2 \times \cdots \times 2}^{n \text{ times}} = 2^n$. ✓

Eg. Q: How many relations on $X = \{1, 2, \ldots, n\}$ are there?
A: For each pair $(x, y) \in X \times X$, we can choose to include (x, y) in our relation R on X, or not. There are $\#X \cdot \#X = n^2$ total pairs of the form (x, y), so we build a relation in n^2 steps, with 2 choices at each step. This gives $\overbrace{2 \times \cdots \times 2}^{n^2 \text{ times}} = \boxed{2^{n^2}}$ possibilities.
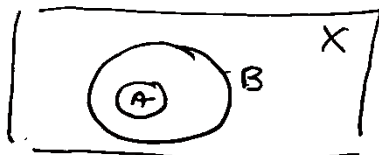
Can also just say that a relation R is <u>any</u> subset of $X \times X$, a set of size n^2, so again get 2^{n^2} such subsets.

Exercise: How many _symmetric_ relations on $\{1, 2, \ldots, n\}$ are there? What about _reflexive_?

Eg: Let $X = \{1, 2, \ldots, n\}$ as before.

$\underline{Q}$: How many ordered pairs (A, B) of subsets of X satisfying $A \subseteq B \subseteq X$ are there?

$\underline{A}$: It is helpful to draw a Venn diagram of our situation:

We see that
$\Leftarrow$ The Venn diagram has 3 _regions_:
• things in A,
• things in $B \setminus A$,
• things in $X \setminus B$.

So to make an ordered pair (A, B) of desired form, we can choose for each $i = 1, 2, \ldots, n$ which of the three regions to place i into:
• Put 1 in A, $B \setminus A$, or $X \setminus B$? (3 choices)
• Put 2 in A, $B \setminus A$, or $X \setminus B$? (3 choices)
⋮
• Put n in A, $B \setminus A$, or $X \setminus B$? (3 choices)

Thus, we have n steps with 3 choices at each step, so total # of possibilities $= 3 \times 3 \times \cdots \times 3 = \boxed{3^n}$

Exercise What about (A, B, C) with
$$A \subseteq B \subseteq C \subseteq \{1, 2, \ldots, n\}?$$
And (A, B, C, D)? And so on... ?

11/7   Addition Principle + Principle of Inclusion-Exclusion   1

Sometimes we are trying to count objects that have multiple "kinds":

Eg. Q: Let $X = \{a, b\}$. How many strings in X^* are there which have length 3 or length 4?

A: The # of strings of length 3 in $X^* = 2 \times 2 \times 2 = 2^3$ by mult. principle
# of strings of length 4 $= 2 \times 2 \times 2 \times 2 = 2^4$
# of strings of length 3 or 4 $= 2^3 + 2^4 = 8 + 16 = 24$.

We see another counting principle in action here:

Theorem (Addition Principle for Counting)
  If $X_1, X_2, \ldots, X_m$ are disjoint sets (meaning $X_i \cap X_j = \varnothing$ for all $i \neq j$, i.e., the sets have no common elements)
  then $\#(X_1 \cup X_2 \cup \cdots \cup X_m) = \#X_1 + \#X_2 + \cdots + \#X_m$.

We see that, as long as the sets are disjoint, we can count any grouping of sets just by adding together:

Eg. Q: # of strings in $\{a, b\}^*$ of length 3 or 4 or 5?
  A: $2^3 + 2^4 + 2^5$, by the addition principle.

Eg: Alexis, Ben, Cole, David and Erica are a 5 person group.
  They have to elect a: President, Vice President, & Treasurer.
Q: a) How many ways are there to do this?
   b) How many ways are there if we require that either Alexis or Ben is the President?

A: a) We can choose any of the 5 people for Prez.
    Then for VP we can choose any of the remaining 4.
    And for treas. we can choose any of remaining 3.

    By the mult. principle this gives: $5 \times 4 \times 3 = 60$
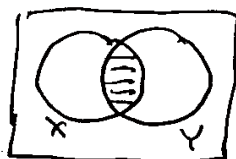   ways...

b) If Alexis is Prez., we have $4 \times 3 = 12$ ways to choose VP + Treas.
If Ben is Prez., also have $4 \times 3 = 12$ ways to choose VP + Treas.
By addition principle, the total # of ways $= 12 + 12 = \underline{24}$.

But what if the sets are __not__ disjoint? Then we use:

__Theorem (Principle of Inclusion - Exclusion)__

$$\#(X \cup Y) = \#X + \#Y - \#(X \cap Y)$$ ← notice that if X and Y are disjoint then this term is $\underline{0}$.

To see why P.I.E. works, look at a Venn diagram:

← when we add #X and #Y we count things in $X \cap Y$ __double__, so have to subtract $-\#(X \cap Y)$ to correct.

E.g. Q: c) How many ways to pick Prez., VP, & treasurer where either Alexis is Prez. or Ben is VP (or both)?

A: c) Let X = elections where Alexis is Prez.
   Then $\#X = 4 \times 3 = 12$, # of choices of VP + treas.
   Let Y = elections where Ben is VP
   Then $\#Y = 4 \times 3 = 12$, # of choices of Prez. + Treas.

We want to compute $\#(X \cup Y)$.
By P.I.E., we also need to know $\#(X \cap Y)$:

$\#(X \cap Y) = 3$, since if Alexis is Prez. & Ben is VP, there are 3 choices left for Treas.

So... $\#(X \cup Y) = \#X + \#Y - \#(X \cap Y) = 12 + 12 - 3$
$= \underline{21}$ ways for Alexis to be Prez. or Ben to be VP.