"Groups generated by involutions acting on combinatorial objects"

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Disclaimer: This is an expository talk based on not my work. I apologize if I say anything incorrect. All errors = my own!

§1. Tableaux

Semistandard Young tableau = filling of Young diagram λ w/ pos. integers that is:

- weakly increasing along rows,
- strictly increasing down columns.

e.g. $T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array}$

$sh(T)$ = shape(T) $= \lambda = (\lambda_1, \lambda_2, \ldots)$
$(= (4,3,1))$

$con(T)$ = content(T)
$= \alpha = (\alpha_1, \alpha_2, \ldots)$ w/ $\alpha_i = \#\, i$'s
$(= (2,3,3,0,\ldots))$

$SSYT(n)$:= semistandard tableaux w/ entries $\leq \{1,2,\ldots,n\}$

$SSYT(n,\lambda)$:= $T \in SSYT(n)$ w/ $sh(T) = \lambda$
($\neq \emptyset$ for λ w/ # rows $= \ell(\lambda) \leq n$)

$SSYT(n,\lambda,\alpha)$:= $T \in SSYT(n,\lambda)$ w/ $con(T) = \alpha$
($\neq \emptyset$ for $\alpha = (\alpha_1, \alpha_2, \ldots)$ w/ $\Sigma \alpha_i = |\lambda| := \Sigma \lambda_i$)

$SYT(\lambda)$:= $SSYT(n = |\lambda|, \lambda, 1^n = (1,1,\ldots,1))$
= Standard Young tableaux of sh. λ

For $1 \le i \le n-1$, the i^{th} Bender-Knuth involution

$$t_i : SSYT(n) \longrightarrow SSYT(n)$$

is operation defined on semistandard tableau T as follows:

- first freeze in place all i's above an $i+1$, and all $i+1$'s below an i,
- then, if a row contains x unfrozen i's + y unfrozen $i+1$'s, modify it to have y unfrozen i's + x unfrozen $i+1$'s.

e.g.

$$t_2\left(\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&2&3&3\\\hline 2&2&2&3&3&4&6\\\cline{1-7} 3&4&4&5\\\cline{1-4} 5&5&5&6\\\cline{1-4}\end{array}\right) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&2&2&3\\\hline 2&3&3&3&3&4&6\\\cline{1-7} 3&4&4&5\\\cline{1-4} 5&5&5&6\\\cline{1-4}\end{array}$$

frozen unfrozen

t_i were introduced by Bender & Knuth, 1972.

Easy checks:
- $t_i^2 = 1$,
- $sh(t_i(T)) = sh(T)$,
- $con(t_i(T)) = \sigma_i(con(T))$, where

$$\sigma_i(\alpha_1, \alpha_2, \ldots, \alpha_i, \alpha_{i+1}, \ldots) = (\alpha_1, \alpha_2, \ldots, \alpha_{i+1}, \alpha_i, \ldots)$$

Basic application: If we define Schur polynomial

$$s_\lambda(x_1, \ldots, x_n) := \sum_{T \in SSYT(n,\lambda)} x^{con(T)},$$

the t_i prove that s_λ is a symmetric polynomial.

In Berenstein & Kirillov, 1995, "Groups generated by involutions ..."

they study group $G_n := \langle t_1, \dots, t_{n-1} \rangle \circlearrowright SSYT(n)$.

Let me survey some results from B-K:

(a) Relations in G_n

Have obvious relations:

(1) $t_i^2 = 1 \ \forall i$, $\quad t_i t_j = t_j t_i \ \forall i,j$ w/ $|i-j| \geq 2$.

Not so obvious relations:

(2) $(t_1 t_2)^6 = 1$

(3) $(t_1 q_i)^4 = 1 \quad \forall \ 3 \leq i \leq n-1$

where $q_i := \underbrace{t_1}\ \underbrace{t_2 t_1}\ \underbrace{t_3 t_2 t_1} \cdots \underbrace{t_i t_{i-1} \cdots t_1}$

Rmk $(t_i t_{i+1})^6 \neq 1$ for $i > 1$.

Rmk Have $q_i^2 = 1 \ \forall i$ (follows easily from rel's (1)).

Also have $G_n = \langle q_1, \dots, q_{n-1} \rangle$.

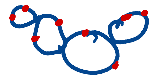
Rmk The cactus group is $C_n := \langle q_{[i,j]} := 1 \leq i < j \leq n \rangle$

subject to relations:

- $q_{[i,j]}{}^2 = 1$
- $q_{[i,j]} q_{[j,k]} = q_{[j,k]} q_{[i,j]}$ if $j < k$
- $q_{[i,j]} q_{[k,l]} q_{[i,j]} = q_{[i+j-l,\, i+j-k]}$ if $i \leq k < l \leq j$

Introduced by Henriques & Kamnitzer, 2006 who were

studying tensor products $A_1 \otimes \cdots \otimes A_n$ in category of crystals.

("Cactus" name: $C_n = \pi_1(\overline{M_{0,n}})$ ← moduli sp. of n pts. on $\mathbb{P}^1$ )

Chmutov, Glick, Pylyavskyy, 2020
+ Berenstein & Kirillov, 2016

show $\exists$ homo. $C_n \to G_n$

$$q_{[i,j]} \mapsto q_{i-1} q_{j-1} q_{i-1}$$

Some of G_n rel's above are implied by cactus gp. homomorphism, but (2) $(t_1 t_2)^6 = 1$ is not (mysterious...).

Rmk For $1 \leq i \leq n-1$ define $s_i := q_i t_1 q_i$, then

$$s_i^2 = 1, \quad s_i s_j = s_j s_i \text{ if } |i-j| \geq 2, \quad (s_i s_{i+1})^3 = 1,$$

i.e. get symmetric gp. action $\mathfrak{S}_n = \langle s_i \rangle \circlearrowright SSYT(n)$.

Was previously studied by Lascoux & Schützenberger, 1981 + has simple interpretation in terms of crystals.

(b) Isomorphism type of G_n as a permutation gp.

Natural question from Prof. Vershik, which prompted this talk:

Q. Acting on set of tableaux, what perm. group is G_n?

Since $sh(t_i(T)) = sh(T)$ and $con(t_i(T)) = \sigma_i(con(T))$, to think about this Q., it makes sense to restrict to standard tableaux of fixed shape.

For $G_n \circlearrowright SYT(\lambda)$, have additional relations:

$\boxed{\text{on SYT, } t_i \text{ swaps } i \text{ \& } i+1 \text{ if non-adjacent}}$ ⇝

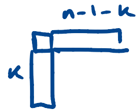
$$(t_i t_{i+1})^6 = 1 \quad \text{for} \quad 2 \leq i \leq n-1,$$

$$t_1 = 1.$$

Easy to see: Action $\mathfrak{S}_n \curvearrowright SYT(\lambda)$ is **transitive**.

But still not clear what transitive perm. gp. we get...

My (wild) guess: For "generic" shape λ, should get "big" perm. gp., possibly **all** of $\mathfrak{S}_{SYT(\lambda)}$ or (index 2) **alternating** gp.

But... **WARNING**: For $\lambda = (n-k, 1^k)$ a **hook** shape 

have $\mathfrak{S}_n \curvearrowright SYT(\lambda) \simeq \mathfrak{S}_{n-1} \curvearrowright \{\text{size } k \text{ subsets of } \{1,2,\ldots,n-1\}\}$

(c) Special elements of $\mathfrak{S}_n$

Certain elements of $\mathfrak{S}_n$ have particular algebraic significance.

Evacuation (a.k.a. "Schützenberger involution"): $= q_{n-1} = \underbrace{t_1}\,\underbrace{t_2 t_1} \cdots \underbrace{t_{n-1} \cdots t_1}$

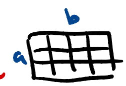
Promotion $:= t_1 t_2 \cdots t_{n-1}$

Promotion + evacuation were introduced by Schützenberger, 1972, who showed (1977) connection to RSK algorithm:

$$A \xmapsto{RSK} (P, Q) \Leftrightarrow 180^\circ \text{ rot. of } A \xmapsto{RSK} (Evac(P), Evac(Q)).$$

Thm (Stembridge, 1996; Berenstein & Zelevinsky, 1996) For any λ,

- action of **longest word** $w_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}$ on $\mathfrak{S}_n$-irrep. S^λ corresponds to evacuation in Kazhdan-Lusztig cellular basis ($\simeq SYT(\lambda)$),
- on gl_n-irrep. V^λ, w_0 corresponds to evac. in canonical basis ($\simeq SSYT(n,\lambda)$).

Thm (Rhoades, 2010) For $\lambda = a \times b$ **rectangle** 

- action of **long cycle** $c = (1, 2, \ldots, n)$ on $\mathfrak{S}_n$-irrep. S^λ corresponds to promotion in KL-basis,
- on gl_n-irrep. V^λ, c corresponds to pro. in canonical basis.

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§2. Posets

$SYT(\lambda)$ = linear extensions of certain poset, so...

Let P be an *finite!* n-element poset. A **linear extension** of P is a list $(p_1, p_2, \ldots, p_n)$ of all elements of P s.t.

$$p_i \leq p_j \Rightarrow i < j.$$

Set $\mathcal{L}(P)$:= linear extensions of P.

B-K involutions still make perfect sense: for $1 \leq i \leq n-1$,

$$t_i : \mathcal{L}(P) \to \mathcal{L}(P)$$

$$t_i(p_1, \ldots, p_n) := \begin{cases} (p_1, \ldots, p_{i+1}, p_i, \ldots, p_n) & \text{if } p_i, p_{i+1} \text{ are } \underline{\text{incomparable}} \\ (p_1, \ldots, p_n) & \text{otherwise.} \end{cases}$$

(a) Relations

As before, have:
- $t_i^2 = 1 \quad \forall i$
- $t_i t_j = t_j t_i$ if $|i-j| \geq 2$,
- $(t_i t_{i+1})^6 = 1$.

(b) Isomorphism type of perm. action

As before, can say that action of t_i on $\mathcal{L}(P)$ is <u>transitive</u>, but not much more than that.

(c) Special elements

As before, define **evacuation** $:= t_1 \, t_2 t_1 \, \cdots \, t_{n-1} \cdots t_1$

promotion $:= t_1 t_2 \cdots t_n$

This BK-involution perspective on pro./evac. due to Haiman, 1992

(Schützenberger worked with *jeu de taquin*...)

See also Stanley, 2009, "Promotion and Evacuation".

A different, but morally very similar, group generated by involutions associated to a poset P was considered in Cameron & Fon-Der-Flaass, 1995, "Orbits of Antichains Revisited"

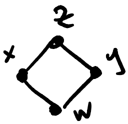
An **order ideal** of P is a downwards-closed subset $I \subseteq P$:

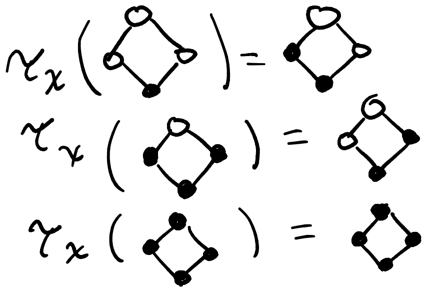
$$y \in I \text{ and } x \le y \in P \Rightarrow x \in P.$$

Let $\mathcal{J}(P) :=$ order ideals of P.

For $p \in P$, define **toggle** $\tau_p : \mathcal{J}(P) \to \mathcal{J}(P)$ by

$$\tau_p(I) := \begin{cases} I \cup \{p\} & \text{if } p \notin I \text{ and } I \cup \{p\} \in \mathcal{J}(P), \\ I \setminus \{p\} & \text{if } p \in I \text{ and } I \setminus \{p\} \in \mathcal{J}(P), \\ I & \text{otherwise.} \end{cases}$$

e.g. $P =$ 



(a) Relations

$\tau_p^2 = 1$

Easy to see: $\tau_p \tau_q = \tau_q \tau_p$ if neither p covers q, nor vice versa

Slightly less obvious: $(\tau_p \tau_q)^6 = 1$ if p covers q or vice versa.

(b) Iso. type of perm. gp.

Thm (Cameron - Fon-Der-Flaass, 1995)

For any P, $\langle \tau_p \rangle \curvearrowright \mathcal{J}(P)$ is either whole sym. gp. $\mathfrak{S}_{\mathcal{J}(P)}$, or (index 2) alternating gp.

Rmk Proof is a simple induction argument, deleting maximal element, based on fact that alternating gp. A_n is simple for $n \geq 5$.

Rmk Toggles τ_p make sense acting on any set system $\mathcal{F}$. Striker, 2018 considered various $\mathcal{F}$ where one can show that $\langle \tau_p \rangle$ is at least alt. gp. (e.g., independent sets of a graph).

(c) Special elements

The operator Rowmotion $:= \tau_{p_1} \tau_{p_2} \cdots \tau_{p_n}$,
(where $(p_1, p_2, \ldots, p_n)$ is any linear extension)

has been studied by many people. It is known to have good behavior for certain P (e.g., rectangles).

It is also known to be closely related to promotion (see, e.g., Striker & Williams, 2012, "Promotion and Rowmotion").

We will explain one precise relation between t_i's + τ_p's via...

§3. Piecewise-linear maps

Let $T \in SSYT(n)$. Its Gelfand-Tsetlin pattern is the triangular array of nonnegative integers:

$$\begin{array}{ccccccc}
x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\
& x_{22} & x_{23} & \cdots & x_{2n} \\
& & \ddots & \ddots & \vdots \\
& & & x_{n-1,n-1} & x_{n-1,n} \\
& & & & x_{nn}
\end{array}$$

where k^{th} diagonal $(X_{1,k+1}, X_{2,k+2}, \dots, X_{n-k,n})$

$= \text{shape}(T_{\leq n-k})$

with $T_{\leq i} := T$ restricted to entries $\{1, 2, \dots, i\}$

e.g. $T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 3 & 4 \\ \hline 3 & 3 & 4 \\ \cline{1-3} 4 \\ \cline{1-1} \end{array}$

$sh(T_{\leq 4}) = (4,4,3,1)$

$sh(T_{\leq 3}) = (4,3,2)$

$sh(T_{\leq 2}) = (3,2)$

$sh(T_{\leq 1}) = (2)$

GT $\rightsquigarrow$ $\begin{array}{cccc} T_{\leq 4} & T_{\leq 3} & T_{\leq 2} & T_{\leq 1} \\ 4 & 4 & 3 & 2 \\ & 4 & 3 & 2 \\ & & 3 & 2 \\ & & & 1 \end{array}$

Rmk GT patterns satisfy inequalities

$$\begin{array}{ccc} X_{ij} & \geq & X_{i\,j+1} \\ & & \vee\!| \\ & & X_{i+1\,j+1} \geq 0 \end{array}$$

and hence = integer pt.s of a polyhedral cone.

Berenstein & Kirillov, 1995 showed how BK involutions behave on GT patterns. Consider local involution

$$\begin{array}{|c|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & w & \multicolumn{1}{c}{} \\ \hline v & x & z \\ \hline \multicolumn{1}{c|}{} & y & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array} \rightsquigarrow \begin{array}{|c|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & w & \multicolumn{1}{c}{} \\ \hline v & \tilde{x} & z \\ \hline \multicolumn{1}{c|}{} & y & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array}$$

preserves ineq.'s

where $\tilde{x} := \min(v,w) + \max(y,z) - x$

Thm (Berenstein & Kirillov, 1995)

At the level of GT patterns, BK involution t_i corresponds to $x \to \tilde{x}$ for all entries x on $(n-i)$th diagonal.

e.g.

$$T = \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&2&3&4\\\hline 3&3&4\\\cline{1-3} 4\\\cline{1-1}\end{array} \quad \overset{GT}{\rightsquigarrow} \quad \begin{array}{ccccccc} 4&&4&&3&&2\\ &4&&3&&2&\\ &&3&&2&&\\ &&&1&&& \end{array} \quad x \mapsto \tilde{x}$$

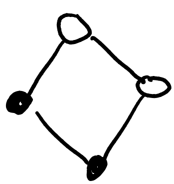
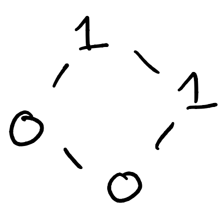
$$\Big\downarrow t_2 \qquad\qquad\qquad\qquad \Big\downarrow$$

$$\begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 2&2&3&4\\\hline 3&3&4\\\cline{1-3} 4\\\cline{1-1}\end{array} \quad \overset{GT}{\rightsquigarrow} \quad \begin{array}{ccccccc} 4&&4&&4&&2\\ &4&&3&&2&\\ &&3&&2&&\\ &&&1&&& \end{array}$$

Inspired by B&K, Einstein & Propp, 2021 introduced a PL extension of poset toggles for poset P.

Let $\mathcal{J}^m(P) :=$ order-preserving maps $\pi : P \to \{0,1,\dots,m\}$.

Note $\mathcal{J}(P) \simeq \mathcal{J}^m(P)$ via indicator fn. of complement

I  $=$  π

For $p \in P$, define PL toggle $\tau_p : \mathcal{J}^m(P) \to \mathcal{J}^m(P)$ by

$$\tau_p(\pi)(q) := \begin{cases} \pi(q) & \text{if } q \neq p \\ \min\{\pi(r) : p \lessdot r\} + \max\{\pi(r) : r \lessdot p\} - \pi(p) & \text{if } q = p \end{cases}$$

same as $x \mapsto \tilde{x}$

covers

(w/ $\min \emptyset = m$ and $\max \emptyset = 0$)

These are same as τ_p for $J(P)$ when $m = 1$.

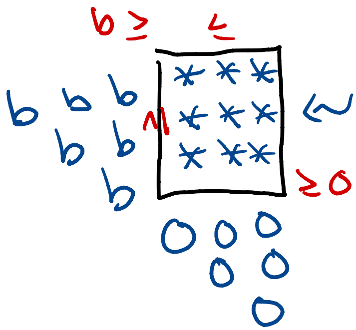
Prop. For $\lambda = a \times b$ a rectangle, $\exists$ bijection

$$SSYT(n, a \times b) \xrightarrow{\sim} \mathcal{J}^b(a \times (n-a))$$

s.t. t_i = composition of τ_p's along $(n-i)$th diag.

e.g.

GT = pattern



$\leftsquigarrow$ this is $\pi \in \mathcal{J}^b(a \times (n-a))$

Rmk Since t_i = product of τ_p's, the group $\langle t_i \rangle$ will be a subgroup of $\langle \tau_p \rangle$. But cannot say much more than that...

§4. Conclusion

My own interest in this: finding P that have good behavior of promotion + rowmotion.

In preprint "Order poly. product formulas + poset dynamics" I propose heuristic that these should be P w/ product formulas for $\#\mathcal{J}^m(P)$, e.g.:

Thm (MacMahon)

$$\#\mathcal{J}^m(a\times b) = \prod_{i=1}^{a}\prod_{j=1}^{b}\frac{i+j+m-1}{i+j-1}$$

Beyond rectangles, other examples are:

Спасибо!

(slides available upon request)

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