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Cyclic Sieving for Staircase Plane Partitions via Crystals and Electrical Networks

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Abstract. We prove a cyclic sieving result for the action of promotion on the staircase plane partitions of height two. Our proof has two major algebraic inputs: an interpretation of this promotion action in terms of tensor powers of the spin crystal that was recently studied by Pappe–Pfannerer–Schilling–Simone, and the bush basis of the degree two part of the coordinate ring of the space of electrical networks that was recently introduced by Gao–Lam–Xu. Moreover, we explain how the existence of an electrical canonical basis in all degrees would yield cyclic sieving for promotion of staircase plane partitions of all heights.

Keywords: cyclic sieving, plane partitions, promotion, crystals, electrical networks

1 Introduction

This extended abstract summarizes our work, but for the full paper with proofs, see [9].

Let P be a finite poset. For an integer $m \geq 1$, a *P-partition of height m* is an order-preserving map $P \to \{0, 1, \ldots, m\}$. We use $\mathcal{PP}^m(P)$ to denote the set of P-partitions of height m, and define the *order polynomial* $\Omega_P(m)$ of P to be the polynomial in m for which $\Omega_P(m) = \#\mathcal{PP}^m(P)$.[1]

(Piecewise-linear) rowmotion is an action on the order polytope of a finite poset P which was introduced about ten years ago by Einstein and Propp [4] and has subsequently received significant attention. For any $m \geq 1$, by restricting to the rational points in the order polytope with denominator dividing m, we can consider (piecewise-linear) rowmotion as an invertible operator $\mathrm{Row}\colon \mathcal{PP}^m(P) \to \mathcal{PP}^m(P)$ acting on height m P-partitions. While this rowmotion action is defined for any finite poset, it is only for special families of posets that it exhibits good behavior. In [8], the first author put forward the following meta-conjecture concerning when rowmotion of P-partitions behaves well.

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[1] This differs from the usual definition of order polynomial by a shift by 1, but leads to cleaner formulas.

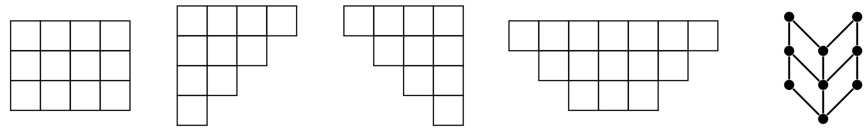


Figure 1: Examples of the families of posets to which Conjecture 1 applies.

Conjecture 1 ([8]). *Let P be a finite graded poset of rank r for which the roots of $\Omega_P(m)$ are all integers or half-integers, and in the latter case possessing an order two automorphism. Let $m \geq 1$ be an integer and define*

$$\Omega_P(m;q) := \prod_{\alpha} \frac{1 - q^{\kappa(m-\alpha)}}{1 - q^{-\kappa\alpha}},$$

where the product is over all the roots α of $\Omega_P(m)$, with multiplicity, and $\kappa := 1$ if the roots are integers or $\kappa := 2$ if they are half-integers. Then $\Omega_P(m;q)$ is a polynomial in q with nonnegative integer coefficients, and $(\mathcal{PP}^m(P), \langle \mathrm{Row} \rangle \simeq \mathbb{Z}/\kappa(r+2)\mathbb{Z}, \Omega_P(m;q))$ exhibits cyclic sieving.

For background on the *cyclic sieving phenomenon* (CSP), consult [18]. Note in particular that when we have a CSP where the sieving polynomial has a product formula as a rational functional, *every* symmetry class under the action is enumerated by a product formula. So Conjecture 1 says that rowmotion of P-partitions for these specials families of posets P is very well behaved indeed.

Families of posets to which Conjecture 1 applies include the *rectangles*, *staircases*, *shifted staircases*, *trapezoids*, and *chains of V's*. Examples of posets in these families are depicted in Figure 1. Notice that rectangles and staircases are Young diagram shapes, shifted staircases and trapezoids are shifted shapes, and chains of V's are neither.

For P a *root poset* (including staircases and maximal trapezoids), the case $m = 1$ of Conjecture 1 was proved by Armstrong, Stump, and Thomas [1]. For P a *minuscule poset* (including rectangles and shifted staircases), the case $m = 1$ of Conjecture 1 was proved by Rush and Shi [17]. For all $m \geq 1$ and P a rectangle, Conjecture 1 was essentially proved by Rhoades [16] (see also [7]). Johnson and Liu [10] established an equivalence of rowmotion for rectangles and trapezoids, which combined with Rhoades's result shows that Conjecture 1 is true also for all $m \geq 1$ when P is trapezoid.

As far as we know, those are all cases of Conjecture 1 which have previously been proved. See [8, §5] for more details and references. Here we prove a new case of Conjecture 1. Let $\delta_n = (n, n-1, \ldots, 1)$ denote a staircase shape.

Theorem 2. *Conjecture 1 is true when $P = \delta_n$ is a staircase and $m = 2$.*

A few terminological remarks concerning Theorem 2 are in order. First of all, when $P = a \times b$ is a rectangle, P-partitions are called *plane partitions*. So for $P = \delta_n$ a staircase, we refer to P-partitions as *staircase plane partitions*. Also, rather than consider rowmotion, which is a composition of *toggles* (local involutions) from "top-to-bottom," in what follows we instead work with *promotion*, which is the composition of these toggles from "left-to-right." Pioneering work of Striker and Williams [19] showed that rowmotion and promotion are always conjugate, so their orbit structures are the same. Hence our main result, Theorem 2, can be restated as a cyclic sieving result for promotion of staircase plane partitions of height two. Let us now briefly explain our approach to understanding promotion of staircase plane partitions.

It is helpful to first review the proof of Rhoades's CSP result for promotion of usual (i.e., rectangular) plane partitions. One way to interpret what Rhoades did in [16] is as follows (see also [7] and especially [12, Thm. 2.2]). Fix $a, b \geq 1$ and let $P = a \times b$ be the $a \times b$ rectangle, so that $\mathcal{PP}^m(a \times b)$ is the set of $a \times b$ plane partitions of height m. The degree m part of the homogeneous *coordinate ring* of the *Grassmannian* $\mathrm{Gr}(a, a+b)$ of a-planes in $\mathbb{C}^{a+b}$ has a basis indexed by the elements of $\mathcal{PP}^m(a \times b)$. In fact, it has many such bases, but a particularly nice one is the Lusztig/Kashiwara *dual canonical basis*. Also, $\mathrm{Gr}(a, a+b)$ caries an action of the general linear group $\mathrm{GL}(a+b)$, and so its coordinate ring does as well. Rhoades showed that a particular lift of the long cycle in the symmetric group S_{a+b} to $\mathrm{GL}(a+b)$ acts as promotion of plane partitions on the dual canonical basis. Then a standard character computation yields Conjecture 1 in this case.

Our proof of Theorem 2 follows a similar structure, where we realize the action of promotion algebraically in terms of a nice basis of the coordinate ring of a homogeneous space. The space in question is the *space of electrical networks* [11, 2, 3]. In [6], the coordinate ring of the space of electrical networks is termed the *grove algebra*. The basis we use to prove our CSP is a basis of the degree two part of the grove algebra called the *bush basis*, which was recently introduced by Gao, Lam, and Xu [6, §4].

In order to relate promotion of height two staircase plane partitions to the bush basis, we use another algebraic incarnation of promotion. Pfannerer, Rubey and Westbury [14] explained how the Henriques–Kamnitzer cactus group action on the *tensor product* of *crystals* gives a promotion operator on these tensor products and their invariants. Pappe, Pfannerer, Schilling, and Simone [13] further explained how, in the particular case of tensor powers of the *spin representation* of a spin group, this crystal promotion operator is promotion of staircase plane partitions. Moreover, because of the exceptional isomorphism of the rank two Lie algebras B_2 and C_2, this crystal interpretation also allows us to show that promotion of height two staircase plane partitions is the same as *rotation* of *3-noncrossing perfect matchings*. These matchings index the bush basis, and rotation corresponds to a natural cyclic action on the basis. Together with a standard character computation for a symplectic group representation, this completes our proof.

Gao–Lam–Xu [6, Conjecture 1.4] conjectured the existence of an *electrical canonical basis* of the grove algebra in all degrees. As we explain below, the existence of this electrical canonical basis would yield a proof of Conjecture 1 for all $m \geq 1$ when $P = \delta_n$.

2 Combinatorial background

2.1 Staircase plane partitions, rowmotion and promotion

Fix positive integers, $n, m \geq 1$. A *staircase plane partition* of size n and height m is a triangular array of nonnegative integers

$$\pi = (\pi_{i,j}) \quad \text{for } 1 \leq i, j \leq n \text{ and } i + j \leq n + 1$$

such that: $\pi_{i,j} \geq \pi_{i+1,j}$ for all (i,j); $\pi_{i,j} \geq \pi_{i,j+1}$ for all (i,j); and $\pi_{1,1} \leq m$. As in the introduction, we denote the set of staircase plane partitions of size n and height m by $\mathcal{PP}^m(\delta_n)$. We view an element of $\mathcal{PP}^m(\delta_n)$ as a triangular array of left-justified boxes that has n boxes in the first row, $n-1$ boxes in the second row, and so on down to one box in the nth row, filled with nonnegative integers that weakly decrease along rows and down columns. We use matrix notation, so position (i,j) means the box in the ith row and jth column from the top left.

The *toggle* at position (i,j) is a piecewise-linear involution $\tau_{i,j} \colon \mathcal{PP}^m(\delta_n) \to \mathcal{PP}^m(\delta_n)$ defined by:

$$\tau_{i,j}(\pi)_{k,l} := \begin{cases} \pi_{k,l}, & \text{if } (k,l) \neq (i,j), \\ \min(\pi_{i,j-1}, \pi_{i-1,j}) + \max(\pi_{i+1,j}, \pi_{i,j+1}) - \pi_{i,j}, & \text{if } (k,l) = (i,j), \end{cases}$$

with the boundary conditions

$$\pi_{0,j} := m, \quad \pi_{i,0} := m, \quad \pi_{i,j} := 0 \text{ if } i + j > n + 1.$$

For $-n+1 \leq k \leq n-1$, let F_k be the composition of toggles along the kth diagonal:

$$F_k := \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n \\ j - i = k}} \tau_{i,j}.$$

For $1 \leq k \leq n$, let R_k be the composition of toggles along the kth antidiagonal:

$$R_k := \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n \\ i + j - 1 = k}} \tau_{i,j}.$$

We define *rowmotion* $\mathrm{Row} \colon \mathcal{PP}^m(\delta_n) \to \mathcal{PP}^m(\delta_n)$ as

$$\mathrm{Row} := R_n \cdot R_{n-1} \cdot \cdots \cdot R_1$$

and *promotion* $\mathrm{Pr} \colon \mathcal{PP}^m(\delta_n) \to \mathcal{PP}^m(\delta_n)$ as

$$\mathrm{Pr} := F_{n-1} \cdot F_{n-2} \cdot \cdots \cdot F_{-n+2} \cdot F_{-n+1}.$$

Theorem 3 (Striker–Williams [19]; see also [7])**.** *The two maps* $\mathrm{Row} \colon \mathcal{PP}^m(\delta_n) \to \mathcal{PP}^m(\delta_n)$ *and* $\mathrm{Pr} \colon \mathcal{PP}^m(\delta_n) \to \mathcal{PP}^m(\delta_n)$ *are conjugate to one another via a sequence of toggles.*

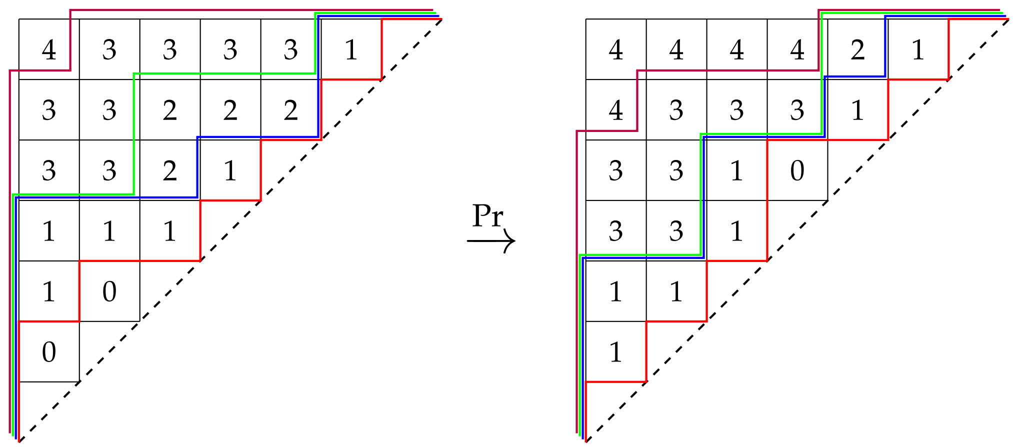


Figure 2: An example of the bijection Φ between staircase plane partitions and fans of Dyck paths. We also depict how promotion behaves on these objects.

2.2 Fans of Dyck paths

We now describe a different way of viewing promotion of staircase plane partitions.

A *Dyck path* of semilength n (and length $2n$) is a lattice walk from $(0,0)$ to (n,n) using only unit steps in north and east direction that stays above the $x = y$ line. We represent a Dyck path of semilength n as a sequence $(0 = d_0, d_1, \ldots, d_{2n} = 0)$ of heights, where d_i is the distance between the path and the $x = y$ line in x-direction after the ith step. We say that a Dyck path $D = (d_0, d_1, \ldots, d_{2n})$ is nested by a Dyck path $D' = (d'_0, d'_1, \ldots, d'_{2n})$ if for all $0 \le i \le 2n$ we have $d_i \le d'_i$ and we write $D \le D'$.

An *m-fan of Dyck paths* is an m-tuple $(D_m, D_{m-1}, \ldots, D_1)$ of Dyck paths of the same semilength such that $D_1 \le D_2 \le \cdots \le D_m$. We denote with $\mathcal{D}_n^m$ the set of m-fans of Dyck paths of semilength n. We represent an m-fan $\mathcal{F}$ of Dyck path with a sequence of vectors of heights $(\emptyset = \mu^0, \mu^1, \ldots, \mu^{2n} = \emptyset)$, where for each $\mu^i = (\mu^i_m, \mu^i_{m-1}, \ldots, \mu^i_1) \in \mathbb{Z}^m_{\ge 0}$ the entries are weakly decreasing and the sequence $(\mu^0_k, \mu^1_k, \ldots, \mu^{2n}_k)$ gives the Dyck path with index k in $\mathcal{F}$. For example, the fan of Dyck paths on the left in Figure 2 is given by

$$(0000, 1111, 2222, 3331, 4442, 5331, 6220, 5311, 6420, 5311, 4200, 3111, 2220, 1111, 0000).$$

Staircase plane partitions of size $n-1$ and height m are naturally in bijection with m-fans of Dyck paths of semilength n, as we now explain. A Dyck path D of length $2n$ is uniquely determined by the set $\Lambda(D)$ of unit squares that lie between the path and lines $x = 0$ and $y = n$. For a plane partition $\pi \in \mathcal{PP}^m(\delta_{n-1})$ let $\pi^{\ge i}$ be the set of all boxes whose entry is at least i. Then for all $1 \le i \le m$ there is a unique Dyck path D_i with $\Lambda(D_i) = \pi^{\ge i}$ and $(D_m, D_{m-1}, \ldots, D_1) \subset \mathcal{D}_n^m$. See Figure 2 for an example. We denote this map by $\Phi \colon \mathcal{PP}^m(\delta_{n-1}) \to \mathcal{D}_n^m$.

We now define promotion on fans of Dyck paths. For a vector $\rho \in \mathbb{Z}^m$ its *dominant representative* $\mathrm{dom}(\rho)$ is the vector obtained from ρ by sorting the absolute values of its entries in weakly decreasing order. Now let $\mathcal{F} = (\varnothing = \mu^0, \dots, \mu^{i-1}, \mu^i, \mu^{i+1}, \dots, \mu^{2n} = \varnothing)$ be an m-fan of Dyck paths of semilength n. Then for each $i = 1, \dots, 2n-1$, we define $\mathrm{BK}_i(\mathcal{F}) := (\varnothing = \mu^0, \dots, \mu^{i-1}, \lambda^i, \mu^{i+1}, \dots, \mu^{2n} = \varnothing)$, where

$$\lambda^i := \mathrm{dom}(\mu^{i-1} + \mu^{i+1} - \mu^i).$$

We call these maps BK_i because they are similar to *Bender–Knuth involutions* acting on tableaux. Indeed, note that each $\mathrm{BK}_i \colon \mathcal{D}_n^m \to \mathcal{D}_n^m$ is a well-defined involution. Finally, we define the *promotion* $\mathrm{Pr}\colon \mathcal{D}_n^m \to \mathcal{D}_n^m$ of m-fans of Dyck paths as the composition

$$\mathrm{Pr} := \mathrm{BK}_{2n-1} \cdot \mathrm{BK}_{2n-2} \cdot \dots \cdot \mathrm{BK}_1.$$

(Note however that BK_1 and BK_{2n-1} always act as the identity, so this is the same as defining $\mathrm{Pr} := \mathrm{BK}_{2n-2} \cdot \mathrm{BK}_{2n-2} \cdot \dots \cdot \mathrm{BK}_2$.)

One way to compute promotion of a fan $\mathcal{F}$ is in terms of the following diagram:

$$\mathcal{F} \qquad \mathrm{BK}_1(\mathcal{F})$$
$$\begin{tikzcd}
\mu^1 \arrow[r] & \mu^2 \arrow[r] & \cdots \arrow[r] & \varnothing \\
\varnothing \arrow[u] \arrow[r] & \lambda^1 \arrow[u] \arrow[r] & \cdots \arrow[u] \arrow[r] & \lambda^{2n-1} \arrow[u]
\end{tikzcd} \tag{2.1}$$
$$\mathrm{BK}_{2n-1} \circ \dots \circ \mathrm{BK}_1(\mathcal{F})$$

We write $\mathcal{F} = (\varnothing = \mu^0, \mu^1, \dots, \mu^{2n} = \varnothing)$ on the corners on the left and top border of the diagram and we apply the local rule $\lambda = \mathrm{dom}(\kappa + \nu - \mu)$ to any square $\begin{tikzcd} \mu \arrow[r] & \nu \\ \kappa \arrow[u] \arrow[r] & \lambda \arrow[u] \end{tikzcd}$.

The promotion $\mathrm{Pr}(\mathcal{F}) = (\varnothing = \lambda^0, \lambda^1, \dots, \lambda^{2n} = \varnothing)$ can be read off at the bottom and right border of the diagram.

A key observation about the map $\Phi\colon \mathcal{PP}^m(\delta_{n-1}) \to \mathcal{D}_n^m$ is that it intertwines toggles and Bender–Knuth involutions and is therefore promotion-equivariant. More formally:

Lemma 4. *For $\pi \in \mathcal{PP}^m(\delta_{n-1})$ we have*

- $\Phi(F_{i-n}(\pi)) = \mathrm{BK}_i(\Phi(\pi))$ *for all* $2 \le i \le 2n-2$, *and hence*
- $\Phi(\mathrm{Pr}(\pi)) = \mathrm{Pr}(\Phi(\pi))$.

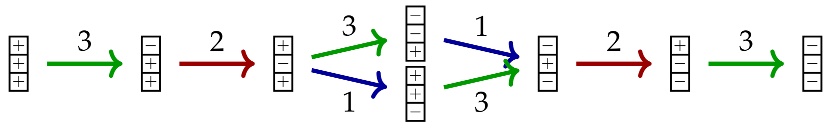


Figure 3: The spin crystal B_3^{spin} for type B_3.

3 Crystal perspective

Crystal bases, introduced by Kashiwara and Lusztig, provide a combinatorial skeleton of representations of quantum groups. A crystal is a set B equipped with:

$$e_i, f_i : B \to B \cup \{\varnothing\}, \quad \varepsilon_i, \varphi_i : B \to \mathbb{Z}, \quad \mathrm{wt} : B \to P,$$

where P is the weight lattice. The operators e_i and f_i are *raising and lowering operators*, and wt gives the weight of an element. Given two crystals B and C, there is a combinatorial construction for the tensor product $B \otimes C$ which corresponds to the crystal of the tensor product of the corresponding representations. The elements in $B \otimes C$ are of the form $b \otimes c$ where $b \in B$ and $c \in C$. The data of a crystal can be represented by an edge-colored directed graph, called *crystal graph*, where we have an edge colored i from a to b if $f_i(a) = b$. The connected components of a crystal graph correspond to irreducible representations and the number of vertices in a connected component is its dimension.

Each connected component has a unique source, which we call the *highest weight element* of its component. A special role are isolated vertices in a crystal graph, which are the *highest weight elements of weight zero*.

Abstractly, promotion is an action on highest weight elements of weight zero in a tensor product of crystals $B^{\otimes n}$. It is defined using the *crystal commutor* introduced by Henriques and Kamnitzer:

$$\mathrm{Pr}(u) := \sigma_{B, B^{\otimes(n-1)}}(u), \tag{3.1}$$

where σ is the commutor built from Lusztig's involution.

3.1 Spin invariants and fans of Dyck paths

For type B_r, the *spin crystal* B_r^{spin} consists of r-tuples $(\pm, \ldots, \pm)$ with crystal operators acting by flipping adjacent signs. See for example Figure 3.

For an r-fan of Dyck paths $\mathcal{F} = (\varnothing = \mu^0, \ldots, \mu^{i-1}, \mu^i, \mu^{i+1}, \ldots, \mu^{2n} = \varnothing)$ we have that the consecutive vectors μ^i and μ^{i-1} differ by a vector of the form $(\pm 1, \pm 1, \ldots, \pm 1)$, which we can naturally identify with an element in B_r^{spin}. Using this identification for each $\mu^{i+1} - \mu^i$ for $1 \le i \le 2n$ we can view a an r-fan of length $2n$ as an element in $(B_r^{\text{spin}})^{\otimes 2n}$.

As observed by Pappe et al. [13], the r-fans of Dyck paths precisely correspond to the highest weight elements of weight zero in this crystal, and equation (2.1) is exactly Lenart's realization of the crystal commutor in terms of local rules. So:

Lemma 5 ([13])**.** *Highest weight elements of weight zero in the crystal $(B_r^{\text{spin}})^{\otimes n}$ can be naturally identified with r-fans of Dyck paths of semilength n. Under this identification, promotion in terms of the crystal commutor in* (3.1) *corresponds to promotion on r-fans of Dyck paths.*

3.2 Symplectic invariants and noncrossing perfect matchings

In order to prove our cyclic sieving result, we need a bijection from 2-fans of Dyck paths to 3-noncrossing perfect matchings which intertwines promotion and rotation. (This is because 3-noncrossing perfect matchings index a basis of the degree two part of the grove algebra.)

Let $n \geq 1$ and $r \geq 1$. A *perfect matching* on $[2n] = \{1, 2, \ldots, 2n\}$ is a partition of $[2n]$ into n disjoint pairs, which we represent as chords in a chord diagram with $2n$ vertices. A perfect matching M is called *$(r+1)$-noncrossing* if it does not contain $r+1$ mutually crossing arcs. That is, there do not exist pairs $(i_1, j_1), (i_2, j_2), \ldots, (i_{r+1}, j_{r+1}) \in M$ such that $i_1 < i_2 < \cdots < i_{r+1} < j_1 < j_2 < \cdots < j_{r+1}$.

There is a remarkable bijection due to Sundaram [20], that maps *r-symplectic oscillating tableaux* of weight zero to $(r+1)$-noncrossing perfect matchings. These r-symplectic oscillating tableau of weight zero of length $2n$ correspond to the highest weight elements of weight zero in the $2n$th tensor power of the crystal of the *standard representation* (i.e., *vector representation*) in type C_r. And Pfannerer–Rubey–Westbury [14] have shown that Sundaram's bijection intertwines promotion and *rotation*. In other words, we have:

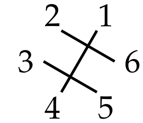
Lemma 6 ([14])**.** *Let C_r^{vec} be the crystal for the standard representation in type C_r. There exists a bijection from highest weight elements of weight zero in $(C_r^{\text{vec}})^{\otimes 2n}$ to $(r+1)$-noncrossing perfect matchings on $[2n]$ that intertwines promotion and rotation.*

3.3 2-fans of Dyck paths and 3-noncrossing perfect matchings

The root systems of type B_2 and C_2 are isomorphic. Under this exceptional isomorphism, the spin representation of type B_2 and the standard representation of type C_2 correspond and their crystals are isomorphic, exchanging the roles of f_1 and f_2.

Thus, combining Lemmas 5 and 6 in this case $r = 2$, we obtain:

Theorem 7. *There exists an explicit bijection from 2-fans of Dyck paths of semilength $2n$ and 3-noncrossing perfect matchings on $2n$ that intertwines promotion and rotation.*

For example, the bijection in Theorem 7 maps $\mathcal{F} = (00, 11, 20, 31, 22, 11, 00)$ to 

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4 Electrical networks and canonical bases

The *space of electrical networks* χ_n [11] is a certain compact algebraic variety whose points correspond to planar electrical networks drawn in a disk with $2n$ boundary vertices. Bychkov, Gorbounov, Kazakov, and Talalaev [2] (see also [3]) showed that the space of electrical networks is abstractly isomorphic to the Lagrangian Grassmannian, but it comes with a distinguished positive part which leads to nice bases of its coordinate ring. Gao, Lam and Xu [6] recently studied the coordinate ring of χ_n, which they called the *grove algebra* and which they showed is defined by certain Plücker-like relations. They conjectured that the grove algebra has an *electrical canonical basis*. Letting $G_{m,n}$ denote the degree m part of the grove algebra, the part of their conjecture relevant to us is:

Conjecture 8 ([6, Conjecture 1.4])**.** *For all $m \geq 1$, $G_{m,n}$ has an electrical canonical basis which, among other properties, satisfies the following:*

- *its elements are indexed by staircase plane partitions of size $n-1$ and height m;*
- *the natural cyclic action of order $2n$ on the space χ_n corresponds to promotion of staircase plane partitions on these basis elements.*

Lemma 9. *If Conjecture 8 is true for some $n, m \geq 1$, then Conjecture 1 is true for that same m and for $P = \delta_{n-1}$ a staircase of size $n-1$.*

Proof. Because of Theorem 3, it is equivalent to prove the cyclic sieving result for promotion instead of rowmotion, which is what we do.

As mentioned, Bychkov et al. [2] showed that χ_n is isomorphic to the Lagrangian Grassmannian $LG(n-1, V)$, where V is the $(2n-2)$-dimensional subspace of $\mathbb{C}^{2n}$ with basis

$$\beta := \{(1,0,1,0,0,\cdots,0),(0,1,0,1,0,\ldots,0),\ldots,(0,\ldots,0,1,0,1)\}$$

and symplectic form given by

$$\Lambda_{2n-2} := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix}$$

when expressed in the β basis. Furthermore, they show that the induced action of $\mathrm{Sp}(2n-2,\mathbb{C})$ on $G_{m,n}$ is an irreducible representation corresponding to $m\omega_{n-1}$, where ω_{n-1} is the long simple root in the type C_{n-1} root system.

The natural cyclic action of order $2n$ on χ^n, corresponding to rotation of the network boundary vertices, can be realized as follows. Let $c\colon \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be the following map:

$$c \cdot (v_1, v_2, \ldots, v_{2n}) := (v_2, \cdots, v_{2n}, (-1)^{n-1} v_1)$$

This c induces an action of $\langle c\rangle \simeq \mathbb{Z}/2n\mathbb{Z}$ on the Lagrangian Grassmannian $LG(n-1,V)$ and thus (via pullback) on the components $G_{m,n}$ of its coordinate ring.

Since the conjectured electrical canonical basis is permuted by this cyclic action according to promotion, the trace of c^k acting on $G_{m,n}$ is equal to the number of basis elements fixed by c^k, i.e., the number of plane partitions in $\mathcal{PP}^m(\delta_{n-1})$ fixed by Pr^k. But the trace of c^k can also be computed through the character theory of the symplectic group. In the β basis, c can be expressed as

$$c = \begin{bmatrix} 0 & 1 & 0 & -1 & \cdots & (-1)^{n-1} \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

It is simple to check that $c\Lambda_{2n-2}c^T = -\Lambda_{2n-2}$. Thus, $ic \in \mathrm{Sp}(2n-2,\mathbb{C})$. Cofactor expansion along the top row of $ic - \lambda I_{2n-2}$ gives a characteristic polynomial of $\frac{\lambda^{2n}-1}{\lambda^2-1}$ and thus ic has eigenvalues

$$\zeta^{\pm 1}, \zeta^{\pm 2}, \ldots, \zeta^{\pm(n-1)}$$

where $\zeta = e^{\frac{2\pi i}{2n}}$. Since the Lagrangian Grassmannian consists of subspaces, c and ic act the same on $G_{m,n}$. So, the trace of c^k acting on $G_{m,n}$ is $\mathrm{Sp}_{2n-2}(m\omega_{n-1}; \zeta^k, \zeta^{2k}, \ldots, \zeta^{(n-1)m})$. By a result of Proctor [15], we have

$$\mathrm{Sp}_{2n-2}(m\omega_{n-1}; q, q^2, q^3, \ldots, q^{n-1}) = \prod_{1\le i\le j\le n-1} \frac{1-q^{i+j+2m}}{1-q^{i+j}}$$

and the cyclic sieving result follows (see also [13, Conjecture 4.23]). ☐

In the case $m=1$, it is not hard to see that the basis of *grove coordinates* (analogues of Plücker coordinates) will satisfy Conjecture 8. For the case $m=2$, Gao–Lam–Xu [6] introduced the *bush basis* of $G_{2,n}$. It is similar to the basis of Temperley–Lieb immanants for the degree two part of the usual Grassmannian. The elements of the bush basis of Gao–Lam–Xu are indexed by 3-noncrossing perfect matchings, and the order $2n$ cyclic action on χ_n corresponds in this basis to rotation of matchings. Hence, by combining Theorem 7 and Lemma 4 we conclude:

Lemma 10. *The bush basis of $G_{2,n}$ satisfies the conditions of Conjecture 8.*

Of course, Lemmas 9 and 10 together imply Theorem 2.

Remark 11. *From the above, it follows that the order of* Row *acting on $\mathcal{PP}^2(\delta_n)$ is $2(n+1)$. In fact, as suggested by Conjecture 1, it should be that, for all $m \ge 1$,* Row^{n+1} *acts on $\mathcal{PP}^m(\delta_n)$ as the order two automorphism, i.e., transposition. But because we work with promotion and not directly with rowmotion, we cannot immediately conclude this, even in this case $m=2$.*

5 Future directions

We conclude with a brief discussion of some future directions. Of course, the most enticing future direction would be to try to define the electrical canonical basis in all degrees. But, since we do not know how to do that, we now discuss a few other things.

The exceptional isomorphism between B_2 and C_2 allowed us to show, using the crystal perspective, that promotion of staircase plane partitions of height two is equivalent to rotation of 3-noncrossing perfect matchings. It is not true that promotion of higher height staircase plane partitions is equivalent to rotation of higher noncrossing perfect matchings. Indeed, the numbers of $(k+1)$-noncrossing perfect matchings and staircase plane partitions of height k are not the same for $k > 2$. Nevertheless, there is still a mysterious "duality" between type B and type C related to staircase plane partitions of higher heights. Namely, these plane partitions arise as indexing sets both for a basis of the space of invariants of a tensor power of spin representations (a type B thing), and for a basis of a component of the coordinate ring of the Lagrangian Grassmannian (a type C thing). Understanding this duality better might help with defining the electrical canonical basis.

Finally, we note that the general paradigm we have followed here, where we realize a cyclic action on a combinatorial set as an appropriate cyclic action on a nice basis of the coordinate ring of a homogeneous variety, could possibly also be followed to resolve other cases of Conjecture 1. Standard Monomial Theory explains how P-partitions often index bases of such coordinate rings. In particular, shifted staircase plane partitions index a basis of the coordinate ring of the (maximal) orthogonal Grassmannian. But, just as we needed here the "right version" of the Lagrangian Grassmannian (namely, the space of electrical networks), it might also be important to choose the right version of the orthogonal Grassmannian. A natural candidate is the Ising model as studied by Galashin and Pylyavskyy [5], which also gives a positive part to the orthogonal Grassmannian. Indeed, we believe this paradigm implicitly relies on the important, but somewhat mysterious, connection between cyclic symmetry and total positivity.

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