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Standard Young Tableaux

The product formula for the g.f. of r.p.p.'s of shape λ is nice, but what about counting a finite combinatorial set?

DEFIN A Standard Young Tableau of Shape λ is a filling of boxes of λ w/ the numbers $1, 2, \dots, n := |\lambda|$, each appearing exactly once, so that numbers are strictly increasing along rows + down columns.

e.g. The 2 SYT's of shape $\lambda = (2, 2)$ are

1	2
3	4

1	3
2	4

Let $f^\lambda := \# \text{SYT}'s$ of shape λ . Note that also

$$f^\lambda = [x_1 x_2 \cdots x_n] S_\lambda(x_1, x_2, \dots) \leftarrow \begin{matrix} \text{"coefficient of} \\ x_1 \cdots x_n \text{ in Schurfn."} \end{matrix}$$

since an SYT is the same as a semi-standard tableau of content $= (1, 1, 1, \dots, 1)$.

Thm ("Hook Length Formula", Frame-Robinson-Thrall, 1954)

$$f^\lambda = n! \cdot \prod_{u \in \lambda} \frac{1}{h(u)} \quad \text{for any partition } \lambda \vdash n$$

e.g. w/ $\lambda = (2, 2)$, hook lengths are:

3	2
2	1

$$\text{so that } f^\lambda = 4! / 3 \cdot 2 \cdot 2 \cdot 1 = 2 \checkmark$$

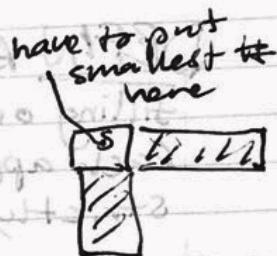
Note that $n! = \#$ ways to fill boxes of λ w/ numbers $1, 2, \dots, n$ (each used once) w/out any requirement on order of #'s

So... HLF has a probabilistic interpretation:

It says that the probability a random filling is an SYT is exactly $\prod_{u \in \lambda} \frac{1}{h(u)}$.

Bogus probabilistic proof of HLF:

- A filling is an SYT iff each entry is smallest among #'s in its hook.
- In a random filling, the probability that box u has entry smallest in its hook is $1/h(u)$
- So prob. random filling is SYT = $\prod_{u \in \lambda} 1/h(u)$



PROBLEM: 1st two bullets are correct, but can only take products for probabilities of independent events, and these events are very much not independent!

There is a valid probabilistic proof of HLF based on construction of a random SYT via "hook walk"

- Choose a random box u in λ to start at,
- Unless we're at a SE border box, move to another random box in the hook of u
- when we hit a SE border box, put the number n there.

Then we repeat w/ where to put $n-1, n-2, \dots$ etc. down to 1.

e.g. In

u_1	u_2	
	u_3	

we might start at u_1 , then go to u_2 , then go to u_3 , and put $n=15$ there

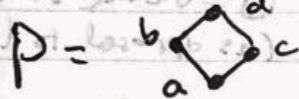
Main thing to show is that this procedure really produces each SYT w/ equal probability ($= \frac{\prod h(u)}{n!}$)

See Sagan G7.3 for proof of this... We'll give different proof...

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Instead, we will deduce HLF for SYTs from g.f. of r.p.p.'s!
Actually, it will be easiest to explain this deduction in the more general setting of (finite) posets!

Recall a poset is a set with a partial order. We draw posets using Hasse diagrams:



$$P = \{a, b, c, d\} \quad a < b, a < c, b < d, c < d$$

(implied: $a < d$)

DEF'N A linear extension of a poset P is a list

(P_1, P_2, \dots, P_n) of all elements s.t. $P_i \leq P_j \Rightarrow i \leq j$.

We let $\mathcal{L}(P) := \{\text{lin. ext.'s of } P\}$.

e.g. w/ P as above, $\mathcal{L}(P) = \{abcd, acbdc\}$

DEF'N We say P is naturally labeled if elts are $P = \{1, 2, \dots, n\}$ and have $i \leq_P j \Rightarrow i \leq j$ (as numbers). In this case, we treat $\mathcal{L}(P) \subseteq S_n$ as a set of permutations.

e.g. $P = \begin{array}{c} 4 \\ 2 \end{array} \diamond \begin{array}{c} 1 \\ 3 \end{array}$ is nat. labeled and $\mathcal{L}(P) = \{1234, 1324\}$

Rmk: Note that the identity $12\dots n$ is always in $\mathcal{L}(P)$.

Recall that a descent of a permutation $\sigma = \sigma_1, \sigma_2, \dots, \sigma_n$ is a position $1 \leq i \leq n-1$ s.t. $\sigma_i > \sigma_{i+1}$. Set $D(\sigma) = \{\text{descents of } \sigma\}$ and recall that the major index of σ is

$$\text{maj}(\sigma) := \sum_{i \in D(\sigma)} i$$

e.g. $D(1234) = \emptyset$ and $D(1324) = \{2\}$ so that

$$\sum_{\sigma \in \mathcal{L}(P)} q^{\text{maj}(\sigma)} = 1 + q^2, \text{ where } P \text{ is nat. labeled poset as above.}$$

DEF'N A P-partition (for poset P) is a function $\pi: P \rightarrow \mathbb{N}$

that is order-reversing: i.e., $p \leq q \Rightarrow \pi(p) \geq \pi(q)$.

We use $|\pi| := \sum_{p \in P} \pi(p)$ (like w/ the r.p.p.'s).

E.g. One P-partition is  $\square = \text{value of } \pi$
 $\square \gg \square \gg \square \gg \square$ (as opposed to label)

Thm (G.F. for P-partitions)

$$\text{For } P \text{ naturally labeled, } \sum_{\substack{\text{all } f: [n] \rightarrow \mathbb{N} \\ \text{P-partition}}} q^{|f|} = \frac{\sum_{\sigma \in \mathcal{L}(P)} q^{\text{maj}(\sigma)}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

E.g. w/ P as before, b/c $1+q^2 = (q)_2 = 1+q+3q^2+4q^3+7q^4\cdots$
g.f. is $= \frac{(1-q)(1-q^2)(1-q^3)(1-q^4)}{0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0}$

Rmk: w/ $P = \{1, 2, \dots, n\}$ an n-element antichain

thm says $\sum_{\substack{\text{all } f: [n] \rightarrow \mathbb{N} \\ \text{P-partition}}} q^{|f|} = \sum_{\sigma \in S_n} q^{\text{maj}(\sigma)}$

$$\frac{(1-q)^n}{(1-q)(1-q^2)\cdots(1-q^n)} \iff \sum_{\sigma \in S_n} q^{\text{maj}(\sigma)} = [n]_q!$$

Something we proved last semester (maj and inv ~~equivalent~~ same distr.).

In fact, proof we give will be same as last semester.

(or for any poset P,

$$\#\mathcal{L}(P) = \lim_{q \rightarrow \infty} \left(\sum_{\text{P-partition}} q^{|\pi|} \right) \cdot (1-q)(1-q^2)\cdots(1-q^n)$$

Pf: mult. both sides by $(1-q)(1-q^2)\cdots(1-q^n)$ in thm above,

and take limit $q \rightarrow 1$ (or just plug in $q=1$). \square

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Before we prove the g.f. for P-partitions them, let's explain how this corollary implies the HLF for SYTs. Basically we just need to match up the various terms.

To any partition $\lambda \vdash n$, associate poset P_λ ($\#P_\lambda = n!$) where elt's are boxes, and $u \geq v \Leftrightarrow u$ northwest of v .

e.g. $\lambda = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \Leftrightarrow P_\lambda = \begin{array}{ccccc} & b & & c & \\ & \swarrow & & \searrow & \\ d & & e & & \end{array}$

With this construction, r.p.p.'s of $sh=\lambda$ = P_λ -partitions and \exists bi.j. between SYTs of $sh=\lambda$ and $\mathcal{L}(P_\lambda)$:

$$T \mapsto \text{box w/ } n, \text{ box w/ } n-1, \dots, \text{ box w/ } 1$$

e.g. $\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \hline\end{array} \mapsto d, e, b, c, a \in \mathcal{L}(P_\lambda)$

So by cor have for any $\lambda \vdash n$ that

$$f_\lambda^k = \# \text{SYTs of } sh=\lambda = \# \mathcal{L}(P_\lambda) = \lim_{q \rightarrow 1} (1-q)(1-q^2)\dots(1-q^n) = \sum_{\pi \text{ r.p.p. of } sh=\lambda} q^{|\pi|}.$$

$$\begin{aligned} &= \lim_{q \rightarrow 1} \frac{(1-q)(1-q^2)\dots(1-q^n)}{\prod_{u \in \lambda} (1-q^{h(u)})} = n! \cdot \prod_{u \in \lambda} \frac{1}{h(u)} \\ &\text{a.k.a. Hillman-Grassl} \quad \text{L'Hopital's rule} \quad \text{Proving HLF!} \end{aligned}$$

Rmk: By choosing a particular natural labeling of P_λ , can also obtain a q-analog of the HLF for SYTs this way.

giving maj-g.f. of tableau

So now to finish everything, need to prove P-partition g.f.:

Pf' (of P-partition g.f. in terms of $\mathcal{L}(P)$):

The idea (which we saw last semester!) is to break $\Sigma P\text{-partitions}$ into pieces correspondingly to lin. ext. Σ

Lemma: Every $f: [n] \rightarrow \mathbb{N}$ has a unique $\sigma \in S_n$ such that f is σ -compatible in sense that

- $f_{\sigma_1} \geq f_{\sigma_2} \geq \dots \geq f_{\sigma_n}$
- if $i \in D(\sigma)$ then $f_{\sigma_i} > f_{\sigma_{i+1}}$

This f is a P-partition $\Leftrightarrow \sigma \in \mathcal{L}(P)$.

Pf: Write $f_{\sigma_1} = f_{\sigma_2} = \dots = f_{\sigma_a} > f_{\sigma_{a+1}} = \dots = f_{\sigma_b} > f_{\sigma_{b+1}} = \dots$

So that $\sigma_1 < \sigma_2 < \dots < \sigma_a$ and $\sigma_{a+1} < \dots < \sigma_b$ and ...

e.g. $f = (2, 0, 5, 0, 3, 3, 2, 0)$ has $f_3 > f_5 = f_6 > f_1 = f_7 > f_2 = f_4 = f_8$

so $\sigma = 3, 5, 6, 1, 7, 2, 4, 8$ is unique perm. f is compatible with. The statement about P-partition \Leftrightarrow lin. ext. is clear. \square

$$\begin{aligned} \text{Thus, } \sum_{\substack{\pi \text{ P-partition} \\ \sigma \in \Sigma(\pi)}} q^{\text{maj}(\sigma)} &= \sum_{\sigma \in \Sigma(P)} \sum_{f \in \mathcal{L}(\sigma)} q^{|\sigma|} \\ &= \sum_{\sigma \in \Sigma(P)} \sum_{\lambda=(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0)} q^{\text{maj}(\sigma) + |\lambda|} \\ &= \sum_{\sigma \in \Sigma(P)} q^{\text{maj}(\sigma)} \cdot \sum_{\lambda: l(\lambda) \leq n} q^{|\lambda|} \\ &= \sum_{\sigma \in \Sigma(P)} q^{\text{maj}(\sigma)} \cdot \frac{1}{(1-q)(1-q^2)\dots(1-q^n)} \end{aligned}$$

subtract off the smallest
 σ -compatible $f_0: [n] \rightarrow \mathbb{N}$
 $(3, 5, 6, 0, 1, 7, 0, 2, 4, 8)$
 $(5, 3, 3, 2, 2, 0, 0, 0) = f$
 $(2, 2, 2, 1, 1, 0, 0, 0) = f_0$
 $(3, 1, 1, 1, 1, 0, 0, 0) = \lambda$

NOTE: $|\sigma| = \text{maj}(\sigma)$

Since $i \in D(\sigma)$
 \Rightarrow have to increase
 by one value
 in 1st i spots to
 get a ~~decrease~~ strict decrease