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Trigonometric substitution §7.3

It is often possible to compute integrals involving $(a^2 - x^2)$ by writing $x = a\sin(u)$ so that $(a^2-x^2) = (a^2 - a^2\sin^2 u)$
$= a^2(1-\sin^2 u) = a^2\cos^2 u$.

E.g. Let's compute $\int \frac{1}{\sqrt{1-x^2}}\,dx$ this way.

Write $x = \sin(u) \Rightarrow dx = \cos(u)\,du$ so that

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-\sin(u)^2}}\cos(u)\,du = \int \frac{1}{\sqrt{\cos^2(u)}}\cos(u)\,du$$

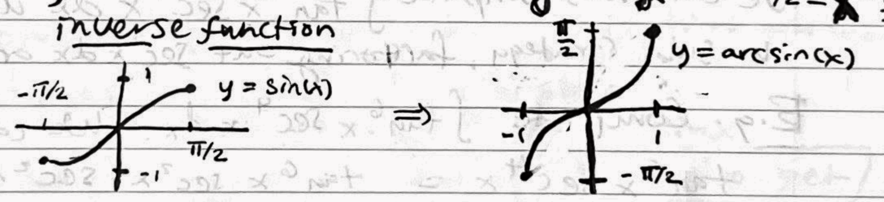
$$= \int \frac{1}{\cos(u)}\cos(u)\,du = \int du = u + C$$

This is the answer in terms of u, but we want the x answer.

Since $x = \sin(u) \Rightarrow u = \arcsin(x)$ (also written $\sin^{-1}(x)$).

Recall: $y = \arcsin(x) \Leftrightarrow \sin(y) = x$ for $-\pi/2 \le y \le \frac{\pi}{2}$

inverse function



e.g. since $\sin(\pi/2) = 1$ have $\arcsin(1) = \pi/2$, etc...

Thus, $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$

Notice: for this problem we used a u-substitution but it was a "reverse" u-substitution where we wrote $x = f(u)$ instead of $u = f(x)$. This is okay as long as you correctly compute the differential $dx = f'(u)\,du$.

Trig substitutions can be very useful when dealing with circles and related shapes...

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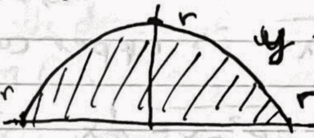
E.g. Let's compute the area of a circle of radius r using an integral.

The equation of a circle is $x^2 + y^2 = r^2$.

If we solve for y we get $y = \sqrt{r^2 - x^2}$,

and the area under this curve $= 1/2$ area of circle:



So area of circle of radius r $= 2 \cdot \int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$. Let's solve this integral by trig. sub.

Since we see $r^2 - x^2$ we set $x = r \cdot \sin(\theta) \Rightarrow dx = r\cos(\theta)\, d\theta$.

$$\Rightarrow \int \sqrt{r^2 - x^2}\, dx = \int \sqrt{r^2 - r^2\sin^2(\theta)}\; r\cos(\theta)\, d\theta$$

$$= \int r\sqrt{1-\sin^2(\theta)}\; r\cos(\theta)\, d\theta = r^2 \int \cos\theta \cos\theta\, d\theta = r^2 \int \cos^2\theta\, d\theta.$$

How to solve $\int \cos^2\theta\, d\theta$? We can do int. by parts:

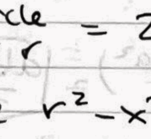
$$\int \underbrace{\cos\theta}_{u}\, \underbrace{\cos\theta\, d\theta}_{dv} = \underbrace{\cos\theta}_{u}\, \underbrace{\sin\theta}_{v} - \int \underbrace{\sin\theta}_{v}\, \underbrace{-\sin\theta\, d\theta}_{du} = \cos\theta\sin\theta + \int \sin^2\theta\, d\theta$$

$$= \cos\theta\sin\theta + \int (1-\cos^2\theta)\, d\theta = \cos\theta\sin\theta + \int d\theta - \int \cos^2\theta\, d\theta$$

$$\Rightarrow 2\int \cos^2\theta\, d\theta = \cos\theta\sin\theta + \theta \Rightarrow \int \cos^2\theta\, d\theta = \tfrac{1}{2}(\cos\theta\sin\theta + \theta)$$

So $\Rightarrow \int \sqrt{r^2 - x^2}\, dx = r^2/2\, (\cos\theta \sin\theta + \theta)$ when $x = r\sin\theta$

Picture of relationship between r & θ:



$\sin\theta = \frac{x}{r}$

$\cos\theta = \frac{\sqrt{r^2-x^2}}{r}$

$\theta = \arcsin\left(\frac{x}{r}\right)$

$$\text{Thus} \Rightarrow \int \sqrt{r^2-x^2}\, dx = r^2/2 \left(\frac{\sqrt{r^2-x^2}}{r}\, \frac{x}{r} + \arcsin\left(\frac{x}{r}\right)\right)$$

$$= \frac{x}{2}\sqrt{r^2-x^2} + r^2/2\, \arcsin\left(\frac{x}{r}\right).$$

$$\Rightarrow 1/2 \text{ area of circle} = \int_{-r}^{r} \sqrt{r^2-x^2}\, dx = \left[\frac{x}{2}\sqrt{r^2-x^2} + r^2/2\, \arcsin\left(\frac{x}{r}\right)\right]_{-r}^{r}$$

$$= \left(0 + \frac{r^2}{2}\arcsin(1)\right) - \left(0 + \frac{r^2}{2}\arcsin(-1)\right) = \frac{r^2}{2}\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) = \frac{\pi r^2}{2}$$ //

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E.g. We can find the area of an ellipse very similarly...

Ellipse equation:

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow y = \frac{b}{a}\sqrt{a^2 - x^2}$ is upper curve of ellipse

$\Rightarrow$ 1/2 area of ellipse $= \int_{-a}^{a} \frac{b}{a}\sqrt{a^2-x^2}\,dx$ take $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$ as before and do same steps

$= \frac{b}{a}\left(\int_{-a}^{a}\sqrt{a^2-x^2}\,dx\right) = \frac{b}{a}\left(\frac{\pi a^2}{2}\right) = \boxed{\frac{ab\pi}{2}}$ //

Sometimes we see expressions of form (a^2+x^2), in that case we take $x = a\tan(u)$ because of identity $\boxed{1+\tan^2\theta = \sec^2\theta}$

E.g. Let's compute $\int \frac{1}{(1+x^2)^2}\,dx$ with a trig. sub.

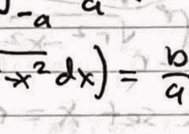
We let $x = \tan(u) \Rightarrow dx = \sec^2(u)\,du$

(recall: $d/dx\,(\tan(u)) = \sec^2(u)$!)

Thus $\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{1}{(1+\tan(u)^2)^2}\sec^2(u)\,du = \int \frac{1}{(\sec^2(u))^2}\sec^2(u)\,du$

$= \int \frac{1}{\sec^2(u)}\,du = \int \cos^2(u)\,du = \underbrace{\frac{\sin(u)\cos(u) + u}{2}}_{\text{we just saw this}} + C$

draw picture of relationship between x and u:



$\tan(u) = x$

$\sin(u) = \frac{x}{\sqrt{1+x^2}}$ $\quad u = \arctan(x)$ (or $\tan^{-1}(x)$)

$\cos(u) = \frac{1}{\sqrt{1+x^2}}$

$\Rightarrow \int \frac{1}{(1+x^2)^2}\,dx = \frac{1}{2}\left(\frac{x}{\sqrt{1+x^2}} \times \frac{1}{\sqrt{1+x^2}} + \arctan(x)\right) + C$

$= \frac{1}{2}\left(\frac{x}{1+x^2} + \tan^{-1}(x)\right) + C$ ✓

Exercise: What if we did $\int \frac{1}{(4+x^2)^2}\,dx$ instead?

or even simpler: $\int \frac{1}{4+x^2}\,dx$. //

Integration of rational functions by partial fractions

Recall that a rational function is $f(x) = \frac{P(x)}{Q(x)}$ where $P(x), Q(x)$ polynomials.

We will now describe procedure for computing $\int \frac{P(x)}{Q(x)}\,dx$.

(0) Recall that the degree of a polynomial $P(x)$ is highest power of x in $P(x)$: e.g. $\deg(P(x)) = 3$ for $P(x) = x^3 + 5x + 4$.

If $\deg(P(x)) \geq \deg(Q(x))$ then we can use long division to write $\frac{P(x)}{Q(x)} = \frac{S(x)}{Q(x)} + R(x)$ where $\deg(S(x)) < \deg(Q(x))$.

E.g. $\frac{2x^3+1}{x^2-1} = 2x + \frac{2x+1}{x^2-1}$

Since it is easy to integrate polynomials, from now on assume $\deg(P(x)) < \deg(Q(x))$.

(1) First suppose the denominator $Q(x)$ factors into distinct linear terms.

E.g. w/ $\frac{P(x)}{Q(x)} = \frac{2x+1}{x^2-1} = \frac{2x+1}{(x+1)(x-1)}$ ← distinct linear factors.

Then we write: $\frac{P(x)}{(x-a)(x-b)\cdots(x-z)} = \frac{A}{x-a} + \frac{B}{x-b} + \cdots + \frac{Z}{x-z}$.

E.g. $\frac{2x+1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$ for some $A, B \in \mathbb{R}$ we need to solve for:

multiply by $Q(x)$ $\Rightarrow$ $2x+1 = A(x-1) + B(x+1)$

$2x+1 = (A+B)x + (-A+B)1$

equate coeff's $\Rightarrow$ $A+B = 2$ and $B - A = 1$

$B = 1 + A$

$A + A + 1 = 2$

$2A = 1 \Rightarrow A = 1/2 \Rightarrow B = 1 + 1/2 = 3/2$

So $\frac{2x+1}{(x+1)(x-1)} = \frac{1/2}{x+1} + \frac{3/2}{x-1}$ ← we can integrate these! using logarithms!

Thus, $\boxed{\int \frac{2x+1}{(x+1)(x-1)}\,dx = \int \frac{1/2}{x+1}\,dx + \int \frac{3/2}{x-1}\,dx = 1/2 \ln(x+1) + 3/2 \ln(x-1) + C}$

NOTE: In general $\int \frac{1}{x+a} = \ln(x+a)$ (easy u-sub).

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② If $Q(x)$ has repeated linear factors, partial fractions is slightly more complicated... let's see an example:

E.g. For $\frac{P(x)}{Q(x)} = \frac{2x+1}{(x-1)^2}$ ← repeated factor, we write:

$$\frac{2x+1}{(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2}$$ ← in general we have powers $(x-a)^i$ up to the multiplicity in $Q(x)$

Then we solve for A & $B \in \mathbb{R}$ as before:

mult. by $Q(x)$

$$2x+1 = A(x-1) + B \xrightarrow{\text{equate coeffs}} A = 2 \quad \& \quad -A+B = 1$$

$$2x+1 = Ax + (-A+B)1 \qquad B = 1+A$$

$$B = 3.$$

Thus $$\boxed{\int \frac{2x+1}{(x-1)^2}\,dx = \int \frac{2}{(x-1)}\,dx + \int \frac{3}{(x-1)^2}\,dx = 2\ln(x-1) - 3(x-1)^{-1} + C}$$

recall to integrate $\frac{1}{(x-a)^r}$ for $r \geq 2$ use simple u-sub. $u = x-a$.

So in general we will get terms like $\ln(x+a)$ and $(x+a)^{-r}$.

③ If $Q(x)$ has irreducible quadratic factors, then partial fractions won't work: instead need trig. sub.

E.g. For $\int \frac{1}{x^2+4}\,dx$ cannot write $(x^2+4) = (x+a)(x+b)$ for real #'s a, b since would need $\sqrt{\ }$ of neg. #

Instead, use $x = 2\tan\theta$

$$\Rightarrow dx = 2\sec^2\theta\, d\theta$$

$$\Rightarrow \int \frac{1}{x^2+4}\,dx = \int \frac{1}{4\tan^2\theta+4} 2\sec^2\theta\, d\theta = \frac{1}{2}\int \frac{1}{\tan^2\theta+1}\sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int \frac{1}{\sec^2\theta}\sec^2\theta\, d\theta = \frac{1}{2}\int d\theta = \frac{1}{2}\theta + C$$

$$= \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$ since $\tan\theta = \frac{x}{2}$.

//

Summary of Strategies for Integration § 7.5

We have now learned many integration techniques. When presented w/ an integral, it can be tricky to decide what to do!

Here are some general guidelines:

① Know and recognize basic integrals such as

$\int x^n dx = \frac{1}{n+1} x^{n+1}$, $\int 1/x = \ln(x)$, $\int e^x dx = e^x$, $\int \sin(x)dx = -\cos(x)$

$\int \cos(x)\,dx = \sin(x)$, $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x)$, $\int \frac{1}{1+x^2}dx = \arctan(x)$, ...

② If you see a function $f(x)$ and its derivative $f'(x)$ in the integrand, try u-substitution.

③ If the integrand is a product of two terms (especially, a polynomial times exponential or trig function...) try integration by parts

④ For things like $\int \sin^n x \cos^m x\,dx$ use the trick we learned of exploiting $\boxed{\sin^2 x + \cos^2 x = 1}$

⑤ If you see $a^2 - x^2$ appear, try trig. sub. $x = a\sin(\theta)$.
If you see $a^2 + x^2$, try trig. sub. $x = a\tan(\theta)$.

⑥ For a rational function $\frac{P(x)}{Q(x)}$, try the technique of partial fraction decomposition.

Sometimes you may need to apply multiple of this steps, and sometimes multiple times.
Even integrals that look similar can require different strategies!

$\int \frac{x}{x^2+1}dx$ — u-sub w/ $u = x^2+1$

$\int \frac{1}{x^2+1}dy$ — trig sub $x = \tan(\theta)$

$\int \frac{1}{x^2-1}dx$ — partial fractions!

//

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Approximate Integration §7.7

Sometimes a definite integral is difficult or impossible to evaluate exactly, and we'd like to get an approximation.

Recall how the definite integral is defined:

- we break $[a,b]$ into n sub intervals $[x_i, x_{i+1}]$ of width $\Delta x = \frac{b-a}{n}$ (so $x_i = a + i\Delta x$ for $i = 0, 1, \dots, n$)
- for each sub interval $[x_{i-1}, x_i]$ we select a point $x_i^* \in [x_{i-1}, x_i]$ (so we get n points $x_1^*, \dots, x_n^*$)
- we let $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$.

We can thus get an approximation for $\int_a^b f(x)\,dx$ by fixing a finite value of n and choosing particular x_i^*.

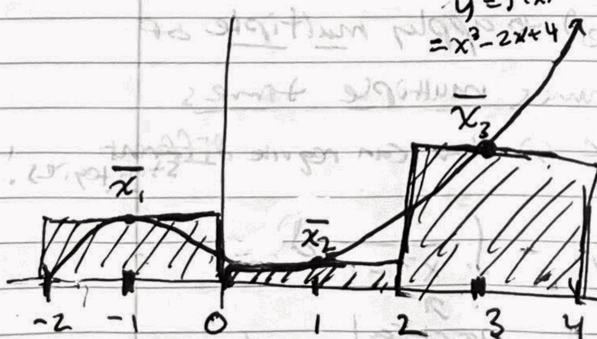
In Calc I we saw the left- and right-endpoint approximations:

$$\int_a^b f(x)\,dx \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x \quad \text{and} \quad \int_a^b f(x)\,dx \approx R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x.$$

A better approximation is to let $x_i^* = \bar{x}_i = \frac{x_{i-1} + x_i}{2}$ be the midpoint of the sub-intervals, giving the midpoint approx.:

$$\int_a^b f(x)\,dx \approx M_n = \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x.$$

Eg: Let's approx. $\int_{-2}^{4} x^3 - 2x + 4\,dx$ using midpoint approx. with $n = 3$ subintervals: $\Delta x = \frac{4-(-2)}{3} = \frac{6}{3} = 2$



The intervals are therefore:

$[-2, 0]$, $[0, 2]$, $[2, 4]$

with midpoints $\bar{x}_1 = -1$, $\bar{x}_2 = 1$, $\bar{x}_3 = 3$

$f(-1) = (-1)^3 - 2(-1) + 4 = 5$

$f(1) = (1)^3 - 2(1) + 4 = 3$

$f(3) = (3)^3 - 2(3) + 4 = 25$

So $M_3 = 5 \cdot 2 + 3 \cdot 2 + 25 \cdot 2$

$= 33 \cdot 2 = \boxed{66}$

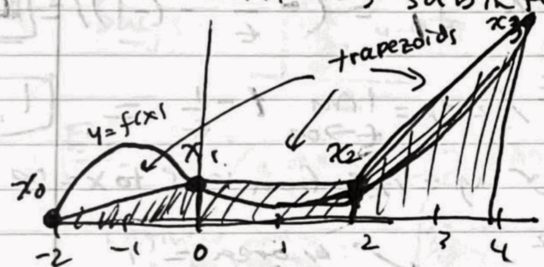
Another good approx. of $\int_a^b f(x)dx$ is the <u>trapezoid approx.</u>:

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right)$$

2's everywhere except x_0 and x_n

It is called "trapezoid" approx. because unlike other approx's using rectangles, it breaks area under curve into trapezoids:

E.g. Let's approx. $\int_{-2}^{4} x^3 - 2x + 4\,dx$ using trapezoid approx. with $n=3$ subintervals: again $\Delta x = \frac{4-(-2)}{3} = 2$



$f(x_0) = f(-2) = (-2)^3 - 2(-2) + 4 = 0$

$f(x_1) = f(0) = (0)^3 - 2(0) + 4 = 4$

$f(x_2) = f(2) = (2)^3 - 2(2) + 4 = 8$

$f(x_3) = f(4) = (4)^3 - 2(4) + 4 = 60$

So $T_3 = \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right)$

$= \frac{2}{2}(0 + 2\cdot 4 + 2\cdot 8 + 60) = \boxed{84}$

The <u>error</u> of an approx. is how much we need to add to get $\int_a^b f(x)dx$:

$$\text{error} = \int_a^b f(x)\,dx - \text{approx.}$$

E.g. We can compute the true value of $\int_{-2}^{4} x^3 - 2x + 4\,dx$ is

$\int_{-2}^{4} x^3 - 2x + 4\,dx = \left[\frac{x^4}{4} - x^2 + 4x\right]_{-2}^{4} = \left(\frac{4^4}{4} - 4^2 + 4(4)\right) - \left(\frac{(-2)^4}{4} - (-2)^2 + 4(-2)\right)$

$= \left(64 - 16 + 16\right) - (4 - 4 - 8) = \boxed{72}$

Thus error of $M_3 = 72 - 66 = \boxed{6}$, error of $T_3 = 72 - 84 = \boxed{-12}$

<u>In general</u>: error of M_n and of T_n <u>have opposite sign</u>,

$|\text{error of } M_n|$ is about $\frac{1}{2}|\text{error of } T_n|$,

and $|\text{error of } M_n|$ and $|\text{error of } T_n| \sim \frac{1}{n^2}$,

meaning if we <u>double</u> n, error gets cut in <u>four</u>.

See book for Simpson's rule which is slightly better error than M_n/T_n but significantly more complicated...

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Improper integrals § 7.8

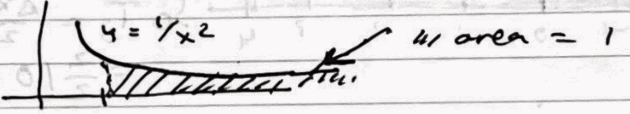
Sometimes we want to find the area under a curve as the curve goes off to infinity. This is called an improper integral:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx.$$

E.g.: $\int_1^t \frac{1}{x^2}\,dx = \left[-x^{-1}\right]_1^t = \left(-\frac{1}{t} - (-1)\right) = \boxed{1 - \frac{1}{t}}$

So $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t 1/x^2\,dx = \lim_{t\to\infty} 1 - \frac{1}{t} = \boxed{1.}$

This means area under $y = 1/x^2$ from $x=1$ to $x=\infty$ is 1:



E.g.: On the other hand, $\int_1^t 1/x\,dx = \left[\ln(x)\right]_1^t = \ln(t) - \ln(1) = \boxed{\ln(t)}$

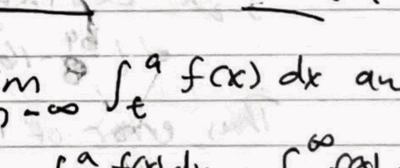
So $\int_1^\infty 1/x\,dx = \lim_{t\to\infty} \int_1^t 1/x\,dx = \lim_{t\to\infty} \ln(t) = \boxed{\text{"}\infty\text{" or D.N.E}}$

We see that $\int_a^\infty f(x)\,dx$ need not exist as a limit!

Similarly, we define $\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty} \int_t^a f(x)\,dx$ and 2-sided improper integral $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$

E.g.: To compute $\int_{-\infty}^\infty \frac{1}{1+x^2}$ write $\int_{-\infty}^\infty \frac{1}{1+x^2} = \int_{-\infty}^0 1/1+x^2\,dx + \int_0^\infty 1/1+x^2\,dx.$

Recall: $\int \frac{1}{1+x^2}\,dx = \arctan(x)$



So $\int_0^\infty \frac{1}{1+x^2} = \lim_{t\to\infty} \left[\arctan(x)\right]_0^t = \lim_{t\to\infty} \arctan(t) - \arctan(0) = \pi/2$

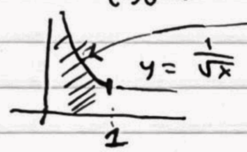
and similarly $\int_{-\infty}^0 \frac{1}{1+x^2}\,dx = \pi/2$, so $\int_{-\infty}^\infty 1/1+x^2\,dx = \pi/2 + \pi/2 = \boxed{\pi}$

Another kind of improper integral is when the integrand is <u>discontinuous</u>.

Suppose $f(x)$ is continuous on $(a,b]$ but discontinuous at $x=a$.

Then we define $\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$.

E.g. $\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\left[2\sqrt{x}\right]_t^1 = \lim_{t\to 0^+} 2-2\sqrt{t} = \boxed{2}$

Says: 

this area = 2
(even though $1/\sqrt{x}$ discontinuous at $x=0$)

E.g. $\int_0^1 1/x\,dx = \lim_{t\to 0^+}\left[\ln(x)\right]_t^1 = \lim_{t\to 0^+} \ln(1)-\ln(t) = \lim_{t\to 0^+} -\ln(t)$

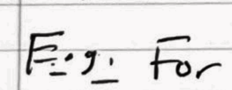
$= \boxed{\text{"}\infty\text{" or D.N.E}}$

Infinite area on region: 

Similarly we define $\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$ for an $f(x)$ that is discontinuous at $x=b$, and if $f(x)$ is continuous on $[a,b]$ except at c then

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ if these are <u>convergent</u>!

E.g. For $\int_{-1}^1 \frac{1}{\sqrt{|x|}}\,dx$, we notice discontinuity at $x=0$;



and write $\int_{-1}^1 \frac{1}{\sqrt{|x|}}\,dx = \int_{-1}^0 \frac{1}{\sqrt{|x|}}\,dx + \int_0^1 \frac{1}{\sqrt{x}}\,dx = 2+2 = \boxed{4}$

by symmetry both are same

E.g. For $\int_{-1}^1 \frac{1}{x^2}\,dx$, notice discontinuity at $x=0$

and write $\int_{-1}^1 1/x^2\,dx = \int_{-1}^0 1/x^2\,dx + \int_0^1 1/x^2\,dx = \lim_{t\to 0^-}\left[-x^{-1}\right]_{-1}^0$

$+ \lim_{t\to 0^+}\left[-x^{-1}\right]_0^1 = \text{"}\infty\text{"} + \text{"}\infty\text{"}$ so $\boxed{\text{D.N.E.}}$

<u>WARNING</u>: If you did $\int_{-1}^1 1/x^2\,dx = \left[-x^{-1}\right]_{-1}^1 = -1-(-1) = 0$

That would give <u>wrong</u> <u>answer</u> because you did not <u>notice the</u> discontinuity!