

Math 211 (Modern Algebra II), HW# 2,

Spring 2026; Instructor: Sam Hopkins; Due: Friday, February 13th

1. Let K be a field and let $L = K(x)$ be the field of rational functions with coefficients in K . Consider the Galois group $\text{Aut}_K(L)$.
 - (a) For $a \in L$ with $a \neq 0$, define $\sigma_a: L \rightarrow L$ by $\sigma_a(f(x)/g(x)) = f(ax)/g(ax)$. Show that $\sigma_a \in \text{Aut}_K(L)$. Conclude that if K is infinite, then $\text{Aut}_K(L)$ is infinite.
 - (b) For $b \in L$, define $\tau_b: L \rightarrow L$ by $\tau_b(f(x)/g(x)) = f(x+b)/g(x+b)$. Show $\tau_b \in \text{Aut}_K(L)$. Show that if $a \neq 1$ and $b \neq 0$, then $\sigma_a\tau_b \neq \tau_b\sigma_a$. Conclude that $\text{Aut}_K(L)$ is nonabelian.
2. Let $L = \mathbb{R}$, the real numbers, viewed as an extension of $K = \mathbb{Q}$, the rational numbers. Consider the Galois group $\text{Aut}_K(L)$.
 - (a) Let $\sigma \in \text{Aut}_K(L)$. Prove that $u \geq 0$ if and only if $\sigma(u) \geq 0$. Conclude that σ preserves the order on \mathbb{R} . **Hint:** the nonnegative numbers in \mathbb{R} are exactly those which are squares.
 - (b) Use part (a) to show that $\text{Aut}_K(L)$ is trivial. **Hint:** every real number can be “trapped” between two rational numbers that are arbitrarily close to it.
3. Let $L = \mathbb{Q}(\omega, \sqrt[3]{2})$, viewed as an extension of $K = \mathbb{Q}$, where $\omega = e^{2\pi i/3} = \frac{-1+\sqrt{-3}}{2}$ is a primitive cube root of unity. Notice that the roots of $f(x) = x^3 - 2$ are $\sqrt[3]{2}$, $\omega\sqrt[3]{2}$, and $\omega^2\sqrt[3]{2}$, so L is the field we get by adjoining all roots of $f(x)$ to \mathbb{Q} (i.e., L is the *splitting field* of $f(x)$).
 - (a) What is the degree $[L : K]$? Find a K -basis of L . **Hint:** looking ahead to the other parts can help you answer this one; also the identity $\omega^2 + \omega + 1 = 0$ may help too.
 - (b) Prove that L/K is a Galois extension.
 - (c) Prove that the Galois group $\text{Aut}_K(L)$ is isomorphic to the symmetric group S_3 .
 - (d) Draw the subgroup structure of S_3 and the subfield structure of L and show how they match up according to the Fundamental Theorem of Galois Theory.
 - (e) Which subgroups of S_3 are normal? Which subfields of L are Galois over K ? How do these correspond?