Total: 38 pts
+ 5 bonus pts
for presenting

= 43/50

1. A K-ary necklace of length n is a rotation equivalence class of colorings of the vertices of an n-gon with K colors. Use unweighted Polya counting to show the number of k-ary necklaces of length n is. $1/n \sum_{d|n} \varphi(d) k^{n/d}$

This formula ~~uses~~ uses some notation from number theory: $d|n$ (means d divides n); and $\varphi(d)$ is Euler's totient function, the number of $1 \leq j \leq d$ with $\gcd(d,j) = 1$

An k-ary necklace of length n is a string of n characters, each of k possible types. Rotation is ignored, in the sense that $b_1 b_2, \ldots b_n$ is equivalent to $b_a\ b_{a+1} \ldots b_n\ b_1 b_2 \ldots b_{a-1}$ for any a

A group action is a funct^n $\phi: G \times X \to X$

X = set G = group

$\phi(e, x) = x$

$\phi(g, \phi(h, x))$

$= \phi(gh, x)$

The orbit of $x \in X$ is $\{gx \mid g \in G\}$

In fixed necklaces, reversal of strings is respected, so they represent circular collections of beads in which the necklace may not be picked up out of the plane

The number of fixed necklaces of length n composed of k types of beads $N(n,k)$ is given by

$$N(n,k) = \frac{1}{n} \sum_{i=1}^{\nu(n)} \phi(d_i)\, k^{n/d_i}$$

where d_i are the divisors of n with $d_1 = 1$ $d_2, \ldots, d_{\nu(n)} = n$, $\nu(n)$ is the number of divisors of n and $\varphi(x)$ is the totient function

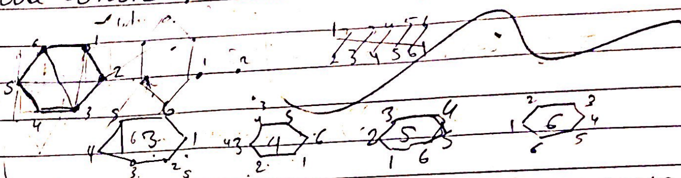
But you need to explain *WHY* this is the correct formula. [-4pts]

The point is to use (unweighted) Polya counting, which will ask you to find the number of cycles in each element of the group generated by an n-cycle like (1,2,…,n). For example, if I square (1,2,3,4,5,6) I get (1,3,5)(2,4,6), which has 2 cycles. The cycle structure of the elements of <(1,2,...,n)> is related to the divisors of n and the totient function.

6/10

2. Continuing the previous problem, now using weighted Pólya Counting: how many ways, up to rotation, can the vertices of a hexagon 6 be colored with 2 red, 2 green and 2 blue vertices?

Since we only consider (rotation).
Since we consider the hexagon then we consider D_6 the symmetry with the



D_6 Symmetry with order 12 $(1)(2)(3)(4)(5)(6)$
6 rotation symmetries, 1^6, $(1\,2\,3\,4\,5\,6)$
6 reflection symmetries

$\frac{1}{6}($

~~$f(n) = \frac{1}{6}(2\cdot n + 2\cdot n^2 + 4\cdot n^3 + 3n^4 + n^6)$~~

when $n :=$ # of collors
$f(n) :=$ # of unique colorings.

$(4)(27) \quad 3\cdot 3^4 + 3^6$

$n=3 \quad \frac{1}{6}(6 + 18 + 32 + 47 + 64)$

$f(2) = \frac{1}{6}(156) = \boxed{26}$

$168 + 243 + 729$

Two problems with your answer: we are considering colorings up to rotation, not all the dihedral group symmetries; and also the problem is about counting the colorings that use exactly 2 red, 2 green, and 2 blue vertices, not any combinations of these three colors. So you need to use *weighted* Polya counting here.

[-4pts]

6/10

3. There are 24 orientation-preserving symmetries of a cube - they are all special rotations.

Use unweighted Pólya counting to give a formula for the number of ways, up to orientation-preserving symmetries, to color the faces of a cube with k ~~colors~~.

Hint 1: Your formula should be a polynomial in k.

Hint 2. This group of symmetries is abstractly isomorphic to the ~~symmetric~~ symmetric group S_4 (but of course ∃ six, not four, faces of a cube). For more information on this group see for instance the wikipedia.

$n = 24$ using polya-Burnside lemma.

I : Identity - no rotation (1 permutation)

Q : 90° face rotation (6 permutations three pairs of opposite faces rotation in either direction)

H: 180° face rotation (3 permutations - three pairs of opposite faces)

D: 120° major diagonal rotations (8 permutations - four pairs of opposite vertices, rotation in either direction)

E: 180° center edge rotation (6 permutations - six pairs of opposite edges)

the number is

(we are considering only orientation preserving symmetries)

$$= \frac{1}{24}(6\Psi(Q) + 3\Psi(H) + 8\Psi(D) + 6\Psi(E))$$

$$= \frac{1}{24}(3k^4 + 12k^3 + 8k^2)$$

Small typo: you're missing the term k^6 coming from the identity. But otherwise all correct.

10/10

4. Continuing the pervious problem, now using weighted Pólya counting: how many ways, up to orientation-preserving symmetries, can the faces of a cube be colored with 2 red, 2 green, and 2 blue faces?

we will assign the numbers 1-6, for cube faces.

we have 24 permutations,

Since its up to orientation-preserving symmetries hence we only consider the even permutations which are the following:

$1: (1)(2)(3)(4)(5)(6)$

$\cancel{6: (12)(34)(56)}$

$3: (1)(2)(34)(56)$

$8: (123)(456)$

So we have, $1+8+3=12$ even permutations

the cycle index polynomial:

$$P = \frac{f_1^6 + 3f_1^2f_2^2 + \cancel{6f_2^3} + 8f_3^2}{12}$$

Since we have 3 colors hence substitute f_1, f_2, and f_3 with 3. we have:

$$P = \frac{3^6 + 3^5 + \cancel{6\cdot 3^3} + 8\cdot 3^2}{12} = \frac{3^2(3^4 + 3^3 + \cancel{2\cdot 3} + 8)}{12} = \boxed{87}$$

So $\boxed{87}$ is the total number of red-green-blue colorings

to find the number of colorings w/ 2 red, 2 green and 2 blue we do the following:

$$P = \frac{(x+y+z)^6 + 3(x+y+z)^2(x^2+y^2+z^2)^2 + 8(x^3+y^3+z^3)^2}{12}$$

Since we have 2 red, 2 blue, and 2 green

we need to Find the terms having the form $x^2y^2z^2$, where x represents the red color, y = blue
z = green

So, the $(x+y+z)^6$, we have the coefficient

$(x+y+z)(x+y+z)(x+y+z)(x+y+z)(x+y+z)(x+y+z)$

$((x^2+xy+xz+yx+y^2+yz+zx+yz+z^2)(\quad '' \quad)(\quad '' \quad)$

$= (x^2+y^2+z^2+2xy+2xz+2yz)(x^2+y^2+z^2+2xy+2xz+2yz)$
$(x^2+y^2+z^2+2xy+2xz+2yz)$

$= \cancel{1x^4} + x^2y^2 + x^2z^2 + \cancel{2x^3y} + \cancel{2x^3z} + 2x^2yz + y^2x^2 + \cancel{y^4} + y^2z^2$
$+ \cancel{2xy^3} + 2xy^2z + \cancel{2y^3z} + z^2x^2 + y^2z^2 + \cancel{z^4} + \cancel{2xz^3} + \cancel{2yz^3}$
$+ \cancel{2x^3y} + \cancel{2xy^3} + 2xyz^2 + 4x^2y^2 + 4x^2zy + \cancel{4xy^3}$
$+ \cancel{2x^3z} + 2xzy^2 + \cancel{2z^3x} + 4x^2zy + 4x^2z^2 + 4y^2z^2$
$+ 2x^2y^2 + \cancel{2y^3z} + \cancel{2yz^3} + 4xy^2z + 4xyz^2$
$+ 4y^2z^2)\ (x^2+y^2+z^2+2xy+2xz+2yz)$

$x^2z^2y^2 + x^2y^2z^2 + 4x^2y^2z^2 + z^2x^2y^2$
$+ x^2y^2z^2 + 4x^2y^2z^2 + z^2x^2y^2$
$8z^2y^2x^2 + 8x^2z^2y^2 + 4x^2z^2y^2 + y^2z^2x^2 + 4x^2y^2z^2 + 4x^2z^2y^2 + 8x^2y^2z^2$
$8x^2y^2z^2 + 8x^2z^2y^2 + 2x^2y^2z^2 + 4x^2y^2z^2 + 4x^2z^2y^2 + 4x^2y^2z^2 + 4x^2y^2$

Thus the factor of $(x^2y^2z^2)$ is $\frac{72}{12} = 6$ which is the nuber of ways to get the cube's faces colored w/ 2 red, 2 blue and 2 green.

6 is the correct answer. And you mostly did the correct steps with applying the weighted Polya counting theorem by plugging in the power sums to the cycle index polynomial.

However, your first step, of restricting to the even permutations only, was not correct: all 24 of the symmetries described above are orientation-preserving (this just means they are rotations, they don't turn the cube "inside out.")

[-1pt]

9/10

5. Let $M_{n\times m}(k)$ be the set of $n\times m$ matrices w/ entries from the set $\{1,2,\dots,k\}$

Let $\bar{M}_{n\times m}(k)$ denote the set of S_n-equivalent classes of $M_{n\times m}(k)$

Give a formula (in terms of n, m and k) for $\# \bar{M}_{n\times m}(k)$

Since the Equivalent class of an element is its orbits so, we will use the formula of finding # orbits but in terms of n, m and k

So $\# \bar{M}_{n\times m}(k) = \frac{1}{n!} \sum_{j=1}^{n} c(n, \cancel{k})$

This is not correct. You should have k^m where you have m here. (Also not good to use k as the variable you're summing over when it already has a meaning in the problem: instead use j or some other letter.)

$\#\text{orbits} = \frac{1}{n!} \sum_{k=1}^{n} c(n,k) \cdot m^k$

where $c(n,k) = \# \{\text{perms } \sigma \text{ in } S_n \text{ w/ } k \text{ total cycles}\}$

The point is, if you're thinking of the rows as the "objects" being permuted by S_n, there are k^m "colors" of the rows, which just correspond to all possible strings of length m with k letters.

$$\Rightarrow \sum_{k=1}^{n} c(n,k)\, m^k = m(m+1)\cdots(m+(n-1))$$

$$\boxed{\begin{aligned}\# \bar{M}_{n\times m}(k) &= \frac{1}{n!}\, m(m+1)\cdots(m+(n-1))\\ &= \binom{n+m-1}{n}\end{aligned}}$$

So the correct answer is (n + k^m - 1 choose n)

[-3pts]

7/10