

IV22

Möbius functions and Möbius inversion (Stanley §3.6, 3.7)

Let's reinterpret inclusion-exclusion as being about the poset $P = B_n = 2^{\binom{[n]}{2}}$ and functions $f = f_z : P \rightarrow R$ & some continuity where we were given a new function

$$g = f_S : P \rightarrow R \text{ such that } g(S) = \sum_{T \subseteq S} f(T) \quad \text{C}$$

i.e., $g(y) = \sum_{x \in P} \varphi(x, y) f(x)$, where $\varphi(x, y) = \begin{cases} 1 & \text{if } x \leq y \text{ in } P, \\ 0 & \text{otherwise} \end{cases}$

and we can invert to get

$$f(S) = f = (S) = \sum_{T \subseteq S} (-1)^{\#S \setminus T} f_{\subseteq}(T),$$

i.e., $f(y) = \sum_{x \in P} \mu(x, y) g(x)$ where $\mu(x, y) = \begin{cases} (-1)^{\#y \setminus x} & \text{if } x \leq y \\ 0 & \text{otherwise.} \end{cases}$

This same set-up works for all (finite) posets P .
Once we find what the $\varphi(x, y), \mu(x, y)$ are and where they live...

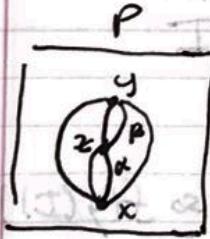
DEFN The incidence algebra $I(P, R)$ of a (finite) poset P (over a comm. ring R) is the ring of all functions

$$f : \text{Int}(P) \longrightarrow R$$

$$\left\{ \begin{array}{l} \text{(non-empty)} \\ \text{intervals} \end{array} \right\} [x, y] := \{z \in P : x \leq z \leq y\} \text{ in } P$$

with pointwise addition $(\alpha + \beta)(x, y) = \alpha(x, y) + \beta(x, y)$

and convolution product $(\alpha * \beta)(x, y) = \sum_{z \in [x, y]} \alpha(x, z) \beta(z, y).$



and

identity element $\delta(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$

We'll want to know that the zeta function

$$\xi(x, y) := \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

is invertible in $I(P, R)$.

Prop. $\alpha \in I(P, R)$ has an inverse $\Leftrightarrow \alpha(x, x) \in R^x \quad \forall x \in P$.

Pf:

$$\alpha * \beta = \xi \Leftrightarrow (\alpha * \beta)(x, y) = \xi(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \quad \forall x, y \in P$$

$$\sum_{z \in [x, y]} \alpha(x, z) \beta(z, y)$$

which forces $\alpha(x, x) \beta(x, x) = 1$ so $\left\{ \begin{array}{l} \alpha(x, x) \in R^x \\ \text{and } \beta(x, x) = \alpha(x, x)^{-1} \end{array} \right\} \quad \forall x \in P$,

and then when $\alpha(x, x) \in R^x$, the values for $\beta(x, y)$ are uniquely determined by induction on $\#([x, y])$ via the formula

$$\alpha(x, x) \beta(x, y) + \sum_{z \in (x, y]} \alpha(x, z) \beta(z, y) = 0 \quad ([x, y] := \{z : x \leq z \leq y\})$$

$$\Rightarrow \beta(x, y) = -\alpha(x, x)^{-1} \cdot \sum_{z \in (x, y]} \alpha(x, z) \beta(z, y) \quad \#([z, y] < \#([x, y]))$$

Note: we can also get a left-inverse $\beta'(\cdot, \cdot)$ for $\alpha(\cdot, \cdot)$ defined recursively by $\beta'(x, y) = -\alpha(y, y)^{-1} \cdot \sum_{z \in (x, y)} \beta'(x, z) \alpha(z, y)$ but then associativity of $*$ forces $\beta' = \beta' * (\alpha * \beta) = (\beta' * \alpha) * \beta = \beta$. \checkmark

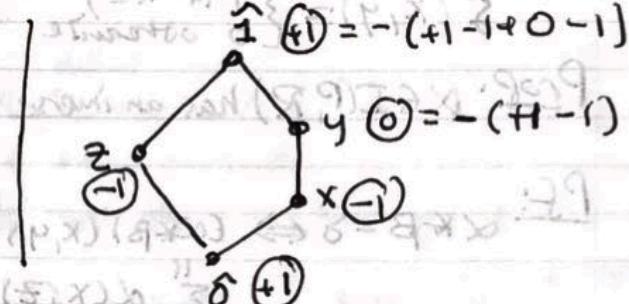
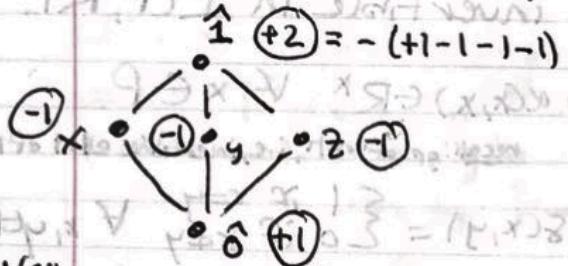
Cor $\xi(\cdot, \cdot) \in I(P, R)$ has an inverse, called the Möbius function $\mu = \xi^{-1}$

defined recursively by $\boxed{\mu(x, x) = 1 \quad \forall x \in P}$

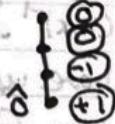
and either $\mu(x, y) = -\sum_{z \in (x, y)} \mu(z, y) \quad \forall x < y$

or $\boxed{\mu(x, y) = -\sum_{z \in [x, y]} \mu(x, z) \quad \forall x < y}$

Examples ① Let's compute $\mu(\overset{\circ}{\sigma}, p)$ & $\mu(\sigma, p)$ (values circled)



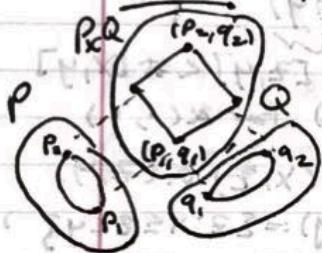
② In a finite chain, $\mu(x, y) = \begin{cases} +1 & \text{if } x=y \\ -1 & \text{if } x < y \\ 0 & \text{otherwise} \end{cases}$



③ Prop. In a product $P \times Q$, $\mu_{P \times Q}((p_1, q_1), (p_2, q_2)) = \mu_p(p_1, p_2) \mu_Q(q_1, q_2)$

Proof: The function $\alpha(\cdot, \cdot) \in I(P \times Q, R)$ defined by the RHS satisfies the correct initial condition

and recurrence: $\alpha((p, q), (p, q)) = \underbrace{\mu_p(p, p)}_{=+1} \underbrace{\mu_Q(q, q)}_{=+1} = +1$



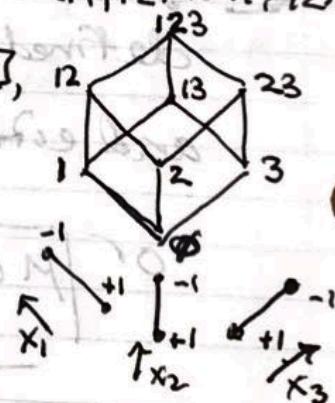
$$\sum_{(p_1, q_1) \in [(p_1, q_1), (p_2, q_2)]} \mu_p(p_1, p_2) \mu_Q(q_1, q_2) = \left(\sum_{p \in [p_1, p_2]} \mu_p(p, p) \right) \left(\sum_{q \in [q_1, q_2]} \mu_Q(q, q) \right)$$

$$= 0 \text{ if } p_1 < p_2 \quad = 0 \text{ if } q_1 < q_2$$

④ Cor In $B_n = 2^{[n]} \cong [2]^n = [2] \times [2] \times \dots \times [2]$, $\mu(T, S) = (-1)^{\#S \setminus T}$ for $T \subseteq S$

$$\mu(T, S) = (-1)^{\#S \setminus T}$$

$$\mu_{2^n} \times V(S \times M) S = (P, X) M$$



PS/1

Theorem (Möbius inversion formula) a commutative ring, e.g. \mathbb{C}

Let P be a poset and $f, g : P \rightarrow R$ related by

$$g(y) = \sum_{x \in P: x \leq y} f(x) \quad \forall y \in P, \text{ then}$$

$$f(y) = \sum_{x \in P: x \leq y} \mu(x, y) g(x). \quad \forall y \in P.$$

(And dually, if we have $g(y) = \sum_{x: x \geq y} f(x)$, then)

$$f(y) = \sum_{x: x \geq y} \mu(y, x) g(x).$$

Proof: Let $R^P := \{\text{all functions } f: P \rightarrow R\}$.

Then $\alpha \in I(P, R)$ acts on such an $f \in R^P$ by

$$(f \cdot \alpha)(y) = \sum_{x \in P} f(x) \alpha(x, y).$$

Check that $(f \cdot \alpha) \cdot \beta = f \cdot (\alpha * \beta)$ since

$$((f \cdot \alpha) \cdot \beta)(y) = \sum_{x \in P} (f \cdot \alpha)(x) \beta(x, y)$$

$$\begin{aligned} &= \sum_{x \in P} \sum_{x' \in P} f(x') \alpha(x', x) \beta(x, y) \\ &= \sum_{x \in P} f(x') \underbrace{\left(\sum_{x \in P} \alpha(x', x) \beta(x, y) \right)}_{(\alpha * \beta)(x, y)} \\ &= (f \cdot (\alpha * \beta))(y) \end{aligned}$$

$$\text{Then } g(y) = \sum_{x \leq y} f(x) = \sum_{x \in P} f(x) \delta(x, y),$$

$$\text{i.e., } g = f \cdot \delta$$

{act on right by $\mu = \delta^{-1}$ }

$$g \cdot \mu = f, \text{ i.e., } \sum_{x \in P} g(x) \mu(x, y) = f(y)$$

$$\sum_{x \leq y} \mu(x, y) g(x) \quad \checkmark$$

Cor with $P = B_n$, get Principle of Inclusion-Exclusion.

11/29

Application of Möbius inversion: Chromatic Polynomials

Defn Let $G = (V, E)$ be a graph. Say that a partition of $[n]$ is G -connected if the restriction of G to each block is connected. Bond lattice Π_G is the sub-poset

Example of the partition lattice Π_n consisting of G -connected partitions,

(so ordered by refinement).

$$G = \begin{array}{c} 1 \\ \circ \end{array} \quad \begin{array}{c} 2 \\ \circ \end{array} \quad \begin{array}{c} 3 \\ \circ \end{array}$$

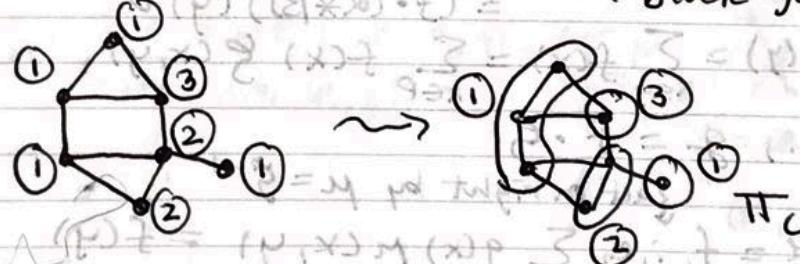


$$\Pi_G = \left(\begin{array}{c} 1 \\ \circ \end{array}, \begin{array}{cc} 2 \\ \circ \end{array}, \begin{array}{c} 3 \\ \circ \end{array}, \begin{array}{ccc} 1 & 2 & 3 \\ \circ & \circ & \circ \end{array} \right) \quad (\text{NOT!}) \quad \left(\begin{array}{ccc} 1 & 2 & 3 \\ \circ & \circ & \circ \end{array} \right)$$

this block is not connected

Let $c: V \rightarrow \{1, 2, 3, \dots\}$ be any coloring of the vertices of G . Associated to c is a G -connected partition $\Pi_c = \max.$ element of Π_G s.t. all vertices in a block get same color.

E.g.



Choose $t \in \mathbb{N}$, the max. # of colors, and let $f, g: \Pi_G \rightarrow \mathbb{C}$ be

$$f(\pi) := \#\{ \text{colorings } c: V \rightarrow \{1, 2, \dots, t\} \text{ s.t. } \Pi_c = \pi \},$$

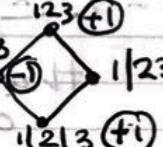
$$g(\pi) := \#\{ \text{colorings } c: V \rightarrow \{1, 2, \dots, t\} \text{ s.t. } \Pi_c \geq \pi \}.$$

Observe $g(\pi) = \sum_{\pi' \geq \pi \in \Pi_G} f(\pi')$, but also
 $g(\pi) = t^{\#\text{blocks}(\pi)}$, since we can get a
 coloring c w/ $\pi'_c \geq \pi$ by coloring each block independently.

Cor For any $\pi \in \Pi_G$, $f(\pi) = \sum_{\pi' \geq \pi \in \Pi_G} \mu(\pi, \pi') t^{\#\text{blocks}(\pi')}$,

and in particular, w/ $\pi = \hat{0} = \{\{1\}, \{2\}, \dots, \{n\}\}$,
 # of proper colorings $c: V \rightarrow \{1, 2, \dots, t\} = \sum_{\pi \in \Pi_G} \mu(\hat{0}, \pi) t^{\#\text{blocks}(\pi)}$
 no two adjacent vertices get same color

Defn This is the chromatic polynomial of G , denoted $\chi_G(t)$.

Example $G = \begin{array}{c} 1 \\ \text{---} \\ 2 \end{array} \quad \begin{array}{c} 1 \\ \text{---} \\ 3 \end{array}$ $\Pi_G = \begin{array}{c} \{1, 2, 3\} \\ \{1, 2\} \\ \{1, 3\} \\ \{2, 3\} \\ \{1\} \\ \{2\} \\ \{3\} \end{array}$  $\mu(\hat{0}, \pi)$ circled

$$\text{So } \chi_G(t) = +1 \cdot t^3 + (-1 - 1) \cdot t^2 + 1 \cdot t = t^3 - 2t^2 + t = t(t-1)^2$$

Cor In the full partition lattice Π_n we have

$$\mu_{\Pi_n}(\hat{0}, \hat{1}) = (-1)^{n-1} (n-1)!$$

Pf. $\Pi_n = \Pi_{K_n}$ for the complete graph K_n .

But choosing colors 1 at a time, we see that

$$\chi_{K_n}(t) = t(t-1)(t-2)\cdots(t-(n-1))$$

$$\text{So } \sum_{\pi \in \Pi_n} \mu(\hat{0}, \pi) t^{\#\text{blocks}(\pi)} = t(t-1)\cdots(t-(n-1))$$

$$\text{Extract coeff. of } t \text{ in } \mu(\hat{0}, \hat{1}) = (-1) \cdot (-2) \cdots (-n+1) = (-1)^{n-1} (n-1)!$$

Rmk: This determines $\mu(\pi, \pi')$ for all $\pi, \pi' \in \Pi_n$ as follows:

If $\pi' = \{S_1, \dots, S_{n'}\}$ and

π refines block S_i into n_i blocks

then $[\pi, \pi'] \cong \Pi_{n_1} \times \Pi_{n_2} \times \dots \times \Pi_{n_r}$

$$\text{so } \mu(\pi, \pi') = (-1)^{n_1}(n_1-1)! \dots (-1)^{n_r}(n_r-1)!$$

12.1 Computing Möbius functions of lattices ($\S 3.8, 3.9$ Stanley)

Def'n: For a lattice L , its Möbius algebra $A(L, \mathbb{C})$, over complex numbers \mathbb{C} , is \mathbb{C}^L with \mathbb{C} -basis $\{f_x\}_{x \in L}$ that multiplies by the rule $f_x \cdot f_y = f_{x \wedge y}$.

Prop. for a finite lattice L , there is a (ring) isomorphism

$$A(L, \mathbb{C}) \xrightarrow{\ell} \mathbb{C}^{|\mathcal{L}|!} := \left\{ \sum_{\substack{x_1, x_2, \dots, x_L \\ \text{1L times}}} c_x e_x \right\}_{c \in \mathbb{C}}$$

$$f_y \mapsto \sum_{x \leq y} e_x \quad \text{that } \mu(x, y) \text{ is } e_x \text{ or } e_x^2 = e_x, \quad e_x e_y = 0 \text{ if } x \neq y,$$

We have $\delta_y := \ell(e_y) = \sum_{x \leq y} \mu(x, y) f_x$, so $f_y = \sum_{x \leq y} \delta_x$.

Hence $\{\delta_x\}_{x \in L}$ are a \mathbb{C} -basis of orthogonal idempotents in $A(L, \mathbb{C})$.

Proof: ℓ is a \mathbb{C} -vector space iso. since its matrix is uppertriangular

$$\ell = f_y \left\{ \begin{bmatrix} * & & \\ & \ddots & \\ & & 1 \end{bmatrix} \right\} \text{ for any linear ordering of } L \text{ that extends } \leq.$$

$$\text{Also can check } \ell(f_y f_z) = \ell(f_{y \wedge z}) = \sum_{x \leq y \wedge z} e_x$$

$$\text{and } (\ell(f_y) \ell(f_z)) = \left(\sum_{x \leq y} e_x \right) \left(\sum_{w \leq z} e_w \right) = \sum_{x \leq y, w \leq z} e_x e_w = \sum_{x \leq y \wedge z} e_x = \ell(f_{y \wedge z})$$

The fact that $\ell^{-1}(e_y) = \sum_{x \leq y} \mu(x, y) f_x$ follows from

$$f_y = \sum_{x \leq y} \ell^{-1}(e_x) \text{ via } \text{Möbius inversion.}$$

Cor (Weisner's Thm)

If $a \neq \vec{1}$ in finite lattice L , then $\sum_{x: ax = \vec{0}} \mu(x, \vec{1}) = 0$.

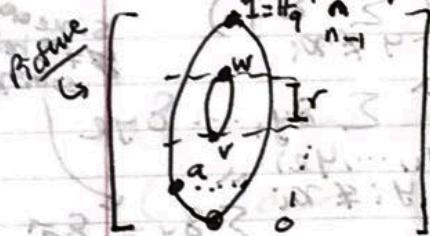
(Dually, if $a \neq \vec{0}$, then $\sum_{x: ax = \vec{1}} \mu(a, x) = 0$.)

Proof: Compute in 2 ways

$$\begin{aligned} (\sum_{b \leq a} \delta_b) \delta_{\vec{1}} &= f_a \delta_{\vec{1}} \quad \text{extract coeffs of } f_a \\ 0, \text{ since } b \leq a \rightarrow b \neq \vec{1} &\quad \sum_{x \in L} \mu(x, \vec{1}) \cdot f_a x \\ 0 &= \sum_{x: ax = \vec{0}} \mu(x, \vec{1}). \end{aligned}$$

Example of use of Weisner's Thm.

Prop: In $\text{Lin}(q)$, $\mu(\vec{0}, \vec{1}) = (-1)^n q^{\binom{n}{2}}$, and hence $\mu(V, W) = (-1)^r q^{\binom{r}{2}}$ if $\dim(W/V) = r$.

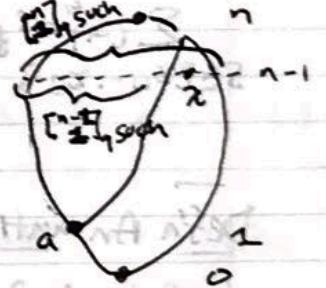


Proof:

Pick a line a , and then

$$0 = \sum_{x: ax = \vec{1}} \mu(\vec{0}, x)$$

$$\begin{aligned} \mu(\vec{0}, \vec{1}) &= - \sum_{x \in \vec{1}, ax = \vec{1}} \mu(\vec{0}, x) \\ \text{count # of } x \text{ of dim } n-1 \text{ s.t. } a \not\in x &\quad \text{forces } x \text{ to have dim } = n-1 \\ &= - \left(\begin{bmatrix} n \\ 1 \end{bmatrix}_q - \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \right) \mu_{L_{n-1}}(\vec{0}, \vec{1}) \end{aligned}$$



$$= - ((1 + q + \dots + q^{n-1}) - (1 + q + \dots + q^{n-2})) \cdot \mu_{L_{n-1}}(\vec{0}, \vec{1}) \leq \dim(x) + 1$$

$$= -q^{n-1} \mu_{L_{n-1}}(\vec{0}, \vec{1}) = (-1)^n q^{(n-1)+(n-2)+\dots+2+1+0} = (-1)^n q^{\binom{n}{2}}$$

induction

□

12/3

To compute μ for distr. lattice $J(P)$, let's use another thm:

Thm (Rota's Crosscut Thm) \Rightarrow elts. $x \leq \hat{1}$

In a finite lattice L , w/ coatoms $\{x_1, \dots, x_e\}$, we have

$$\mu(\hat{0}, \hat{1}) = \sum_{\substack{S \subseteq \{x_1, \dots, x_e\} \\ \wedge S = \hat{0}}} (-1)^{|S|}$$

In particular, $\mu(\hat{0}, \hat{1}) = 0$ if $\hat{0}$ is not a meet of coatoms.

e.g.

$$L = \begin{array}{c} \text{Diagram of } L \text{ with atoms } x_1, x_2, x_3 \text{ and coatoms } \hat{0}, \hat{1}. \\ \mu(\hat{0}, x_1) \end{array}$$

$\sum_{\substack{S \subseteq \{x_1, x_2, x_3\} \\ \wedge S = \hat{0}}} (-1)^{|S|}$

$\begin{aligned} & \{\{x_1\}\} +1 \\ & \{\{x_2\}\} -1 \\ & \{\{x_3\}\} +1 \\ & \{\{x_1, x_2\}\} -1 \\ & \{\{x_1, x_3\}\} +1 \\ & \{\{x_2, x_3\}\} -1 \\ & \{\{x_1, x_2, x_3\}\} +1 \end{aligned}$

$+1 = \mu(\hat{0}, \hat{1}) \quad \checkmark$

Pf. In the Möbius algebra $A(L, C)$, compute in 2 ways:

$$\sum_{\substack{S \subseteq \{x_1, \dots, x_e\} \\ \wedge S = \hat{0}}} (-1)^{|S|} \prod_{i \in S} f_{x_i} = \prod_{i=1}^e (f_{\hat{1}} - f_{x_i}) \Rightarrow \prod_{i=1}^e \left(\sum_{y: y \neq x_i} 8_y \right)$$

extract coeff. of $f_{\hat{1}}$ from $\sum_{y: y \neq x_i} 8_y$

Theorem \checkmark

since $y \neq \hat{1}$ below some x_i

Def'n An antichain $A \subseteq P$ is a subset of pairwise incomparable elts.

(or) In finite distr. lattice $L = J(P)$,

$$\mu(I, I') = \begin{cases} (-1)^{|I' \setminus I|} & \text{if } I' \setminus I \text{ is antichain in } P \\ 0 & \text{otherwise.} \end{cases}$$

Pf:

Check that coatoms of $[I, I']$ are $x_i = I \setminus \{p_i\}$ for maximal $p_i \in I' \setminus I$.

So their meet $x_1, \dots, x_e = I' \setminus \{p_1, \dots, p_e\} = I'$

\hookrightarrow every elt. of $I \setminus I'$ is max, i.e., $I \setminus I'$ is an antichain! \square

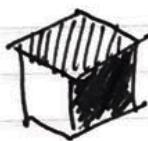
And... that's the end of the material for the course!
Congratulations! and... let me advertise

Math 274 - Combinatorics II - Spring 2022

We will continue the study/enumeration of discrete structures, with a new focus on symmetries!
(a.k.a. algebra!)

Two main topics:

① Enumeration under group action:



↳ How many ways are there to color the faces of a cube w/ 3 colors if we consider colorings the same if we can rotate the cube to get from one coloring to the other?

② Symmetric functions.

Consider polynomial: $P(x) = (x-a)(x-b)(x-c)$
w/ roots a, b, c

$$\text{Expanding} \dots P(x) = x^3 - (a+b+c)x^2 + (ab+bc+ac)x - abc$$

the coefficients of $p(x)$ are themselves poly.'s in a, b, c ,
and invariant under permuting a, b, c : called symmetric polynomials

Symmetric polynomials have rich combinatorial structure!

See samuelhopkins.com/classes/274.html
for more info...