Total score: 50/50 + 5 bonus points from presenting = 55/50

Great job!

# Howard Math 273, HW# 2,

Fall 2021; Instructor: Sam Hopkins; Due: Friday, November 5th

1. Fix a positive integer $k$. We showed the ordinary generating function $F_k(x) := \sum_{n\geq 0} S(n,k)x^n$ of the Stirling numbers of the 2nd kind satisfies $F_k(x) = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$. Find the partial fraction decomposition of $F_k(x)$, i..e., find the coefficients $a_j \in \mathbb{R}$, $j = 1,2,\ldots,k$, for which $F_k(x) = \frac{a_1}{(1-x)} + \frac{a_2}{(1-2x)} + \cdots + \frac{a_k}{(1-kx)}$. Conclude $S(n,k) = \sum_{j=1}^{k} a_j \cdot j^n$.
   **Hint**: clear denominators, and then plug in $x = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{k}$.
   **Bonus just to think about, not do**: prove $S(n,k) = \sum_{j=1}^{k} a_j \cdot j^n$ using (i) the *exponential* g.f. $\sum_{n\geq 0} S(n,k)\frac{x^n}{n!} = \frac{1}{k!}(e^x - 1)^k$; or (ii) the Principle of Inclusion-Exclusion (P.I.E.).

2. *(Stanley, EC1, #2.2)* Let $A$ be some finite set of objects, and suppose these objects potentially posses $n$ different *properties* $p_1, p_2, \ldots, p_n$: e.g., $p_1$ = "is green"; $p_2$ = "is solid"; et cetera. For $X \subseteq [n]$, let $f_=(X)$ denote the number of elements in $A$ possessing *exactly* the properties $p_i$ for $i \in X$ (and not possessing any of the properties $p_j$ for $j \notin X$); and let $f_\geq(X)$ denote the number of elements in $A$ possessing *at least* the properties $p_i$ for $i \in X$ (but potentially also some properties $p_j$ for $j \notin X$). Give a bijective proof of the P.I.E. identity
$$\sum_{X\subseteq[n]} f_=(X)(1+y)^{\#X} = \sum_{Y\subseteq[n]} f_\geq(Y)y^{\#Y},$$
   i.e., give a bijective proof, for each $k$, that the coefficients of $y^k$ on the L- and RHS are equal.

3. *(Stanley, EC1, #2.25(a))* Let $f_i(m,n)$ be the number of $m \times n$ matrices of 0's and 1's, with a total of $i$ 1's, and with at least one 1 in each row and column. Use the P.I.E. to show
$$\sum_{i\geq 0} f_i(m,n)t^i = \sum_{k=0}^{n}(-1)^k\binom{n}{k}((1+t)^{n-k}-1)^m.$$

4. *(Stanley, EC1, #2.25(b))* With $f_i(m,n)$ as in the previous problem, show that
$$\sum_{m,n\geq 0}\left(\sum_{i\geq 0} f_i(m,n)t^i\right)\frac{x^m}{m!}\frac{y^n}{n!} = e^{-x-y}\cdot\sum_{m,n\geq 0}(1+t)^{mn}\frac{x^m}{m!}\frac{y^n}{n!}.$$
   **Hint**: use the formula from the previous problem, and do some algebraic manipulations.

5. The $q$-binomial coefficient satisfies $\left[{n \atop k}\right]_q = \sum_{w\in W_{n,k}} q^{\mathrm{inv}(w)}$, where $W_{n,k}$ is the set of words that are rearrangements of $(n-k)$ 0's, and $k$ 1's, and $\mathrm{inv}(w)$ is the number of inversions of $w$. **Suppose $n = 2m$ is even.** Prove that $\left[{n \atop k}\right]_{q:=-1}$ (the evaluation of the $q$-binomial at $q=-1$) is equal to $\#\mathcal{P}_{n,k}$, where $\mathcal{P}_{n,k}$ is the subset of words $w = w_1w_2\ldots w_n \in W_{n,k}$ that are *palindromes* (i.e., which satisfy $w_i = w_{n+1-i}$ for all $i$). Do this by defining a **sign-reversing involution**. That is, define an involution $\tau\colon W_{n,k} \to W_{n,k}$ satisfying:
   - $\mathrm{inv}(w)$ and $\mathrm{inv}(\tau(w))$ have opposite parity for all $w \in W_{n,k}$ with $\tau(w) \neq w$;
   - $\mathrm{inv}(w)$ is even for all $w \in W_{n,k}$ with $\tau(w) = w$;
   - $\#\{w \in W_{n,k}\colon \tau(w) = w\} = \#\mathcal{P}_{n,k}$.

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Q1 Fix a positive integers $k$. we showed the ordinary generating function $F_k(x) = \sum_{n\geq 0} S(n,k)\, x^n$ of the stirling numbers of the 2nd kind satisfies $F_k(x) = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$.
Find the partial fraction decomposition of $F_k(x)$?

<u>Sol</u>

let $F_k(x) = \sum_{n=0}^{\infty} S(n,k)\, x^n \longrightarrow (1)$

$$F_k(x) = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)} \longrightarrow (2)$$

so, $$F_k(x) = \frac{a_1}{1-x} + \frac{a_2}{1-2x} + \cdots + \frac{a_k}{1-kx} \longrightarrow (3).$$

we want to show $S(n,k) = \sum_{j=1}^{k} a_j\, j^n$.

By plug in $x = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{k}$

$$\Rightarrow a_j = \frac{j^{-k}}{(1-\frac{1}{j})(1-\frac{2}{j})\cdots(1-\frac{j-1}{j})(1-\frac{j+1}{j})\cdots(1-\frac{k}{j})}$$

$$= \frac{j^{k-1}\, j^{-k}}{(j-1)(j-2)\cdots(2)(1)\cdot(-1)^{k-j}\cdot(k-j)(k-j-1)\cdots 2\cdot 1}$$

$$= \frac{(-1)^{k-j}}{(j!)\cdot(k-j)!} = \frac{(-1)^{k-j}\cdot \cancel{j^k}}{k!}\binom{k}{j}$$

typo… this part shouldn't be here

~~As~~ Then $\dfrac{a_j}{1-jx} = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{k-j}}{k!}\cdot\binom{k}{j} j^n \cdot x^n$

$$\Rightarrow F_k(x) = \sum_{j=1}^{k}\left(\frac{a_j}{1-jx}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{j=1}^{k}\frac{(-1)^{k-j}}{k!}\cdot\binom{k}{j}\cdot j^n\right)\cdot x^n$$

$$\Rightarrow F_k(x) = \sum_{n=0}^{\infty} S(n,k)\cdot x^n$$

where $S(n,k) = \displaystyle\sum_{j=1}^{k}\frac{(-1)^{k-j}}{k!}\binom{k}{j} j^n$

$\therefore \quad S(n,k) = \displaystyle\sum_{j=1}^{k} a_j\, j^n$ ⊠

Very good! 10/10

Q2\ Let A be some finite set of objects, and suppose these objects potentially posses n different properties $P_1, P_2, \dots, P_n$ ="is green"; $P_2$ = "is solid" etc.

For $X \subseteq [n]$, let $f_=(X)$ denote the number of elements in A possessing exactly the properties $P_i$ for $i \in X$. and let $f_\geq(X)$ denote the number of elements in A possessing at least the properties $P_i$ for $i \in X$.

Give a bijective proof of P.I.E. identity.

$$\sum_{X \subseteq [n]} f_=(X)(1+y)^{\#X} = \sum_{Y \subseteq [n]} f_\geq(Y)\, y^{\#Y}.$$

proof:

Let

$$\sum_{X \subseteq [n]} f_=(X)(1+y)^{\#X} = \sum_{k=0}^{n} A_k y^k \rightarrow ①$$

$$\sum_{Y \subseteq [n]} f_\geq(Y)\, y^{\#Y} = \sum_{k=0}^{n} B_k y^k \longrightarrow ②$$

Then

$$A_k = \sum_{X \subseteq [n];\, \#X = m \geq k} f_=(X)\binom{m}{k} \longrightarrow ③$$

$$B_k = \sum_{Y \subseteq [n],\, \#Y = k} f_\geq(Y) \longrightarrow ④$$

Now, let us consider

$$f_{=}(X)\binom{m}{k}, \quad \#X = m \geq k \longrightarrow (5)$$

The set $X$ in (5) has $\binom{m}{k}$ different subset $\{Z_i \subseteq X ; \#Z_i = 1, \ldots, \binom{m}{k}\}$, so (5) is the sum of contributions set $X$ makes to the $\binom{m}{k}$ different $f_{\geq}(Z_i)$.

As each set $X \supseteq Z_i$ is present only once in (3), it follows that

$$\sum_{X \subseteq [n], \# X = m \geq k} f_{=}(X)\binom{m}{k} = \sum_{Y \subseteq [n], \#Y = k} f_{\geq}(Y).$$

Very nice! 10/10

Hence approved $\triangle$.

Q3| (a) Let $f_i(m,n)$ be the number of $m \times n$ matrices of 0's and 1's, with a total of $i$ 1's, and with at least one 1 in each row and column. Use P.I.E. to show

$$\sum_{i\geq 0} f_i(m,n)\, t^i = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left((1+t)^{n-k}-1\right)^m .$$

proof:

let us first consider all $m \times n$ matrics which have at least one 1 in each row, regardless of whether they have any 1 in every column, and so use notation $f_i^{(0)}(1,n)$ instead of $f_i(1,n)$. For $m=1$, to have

$$f_i^{(0)}(1,n) = \binom{n}{1} \; ; \rightarrow (1)$$

$$\sum_{i\geq 0} f_i^{(0)}(1,n)\, t^i = \sum_{i\geq 1} f_i^{(0)}(1,n)\, t^i = \sum_{i=0}^{n} \binom{n}{i} t^i - 1 = (1+t)^n - 1$$

For $m>1$, the ~~differen~~ distribution of 1's in each row is independent of the other rows, so

$$\sum_{i\geq 0} f_i^{(0)}(m,n)\, t^i = \left[\sum_{i\geq 0} f_i^{(0)}(1,n)\, t^i\right]^m = \left[(1+t)^n - 1\right]^m \rightarrow (2)$$

Now, we recall that we want 1's in every column and we have to subtract from (2) all such matrices that have any columns made of only 0.

Yes!

(5)

Given a particular column we want to be all 0, this number is the same as (2) with n replaced by n-1, and there are n ways to choose one column out of n.

Therefore, for the metrices with at least one column having no 1's is

$$\sum_{i\geq 0} f_i^{(1)}(m,n)t^i = \binom{n}{1}\sum_{i\geq 0} f_i^{(0)}(1,n)t^i = \binom{n}{1}\left[(1+t)^{n-1}-1\right]^m \quad (3)$$

Next we note that a ~~matrices~~ matrix with two columns without 1's is counted one extra time in (3), so we have to subtract from (3) the number of matrices with at least 2 columns of zeros:

$$\sum_{i\geq 0} f_i^{(2)}(m,n)t^i = \binom{m}{2}\left[(1+t)^{n-2}-1\right]^m \longrightarrow (4)$$

we continue in the same way, following the P.I.E., and end up with

$$\sum_{i\geq 0} f_i(m,n)t^i = \sum_{i\geq 0} f_i^{(0)}(m,n)t^i - \sum_{i\geq 0} f_i^{(1)}(m,n)t^i$$

$$+ \sum_{i\geq 0} f_i^{(2)}(m,n)t^i - \cdots = \sum_{k=0}^{n}(-1)^k\sum_{i\geq 0} f_i^{(k)}(m,n)t^i$$

$$= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\left[(1+t)^{n-k}-1\right]^m \quad \square$$

Excellent! 10/10

Q4(b) with $f_i(m,n)$ as in the previous problem, show that

$$\sum_{m,n\geq 0}\left(\sum_{i\geq 0} f_i(m,n)t^i\right)\frac{x^m y^n}{m!\,n!} = e^{-x-y}$$

$$\cdot \sum_{m,n\geq 0}(1+t)^{mn}\,(x^m y^n)/m!\,n!$$

<u>proof:</u>

we substitute

$$\sum_{m,n\geq 0}\left[\sum_{i\geq 0} f_i(m,n)\,t^i\right]\frac{x^m y^n}{m!\,n!}$$

$$=\sum_{m,n\geq 0}\frac{x^m y^n}{m!\,n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left[(1+t)^{n-k}-1\right]^m$$

$$=\sum_{m,n\geq 0}\frac{y^n}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{m=0}^{\infty}\frac{x^m}{m!}\left[(1+t)^{n-k}-1\right]^m$$

$$=\sum_{m,n\geq 0}\frac{y^n}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}e^{x\left[(1+t)^{n-k}-1\right]}$$

$$=e^{-x}\sum_{m,n\geq 0}\frac{y^n}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}e^{x(1+t)^{n-k}}$$

$$=e^{-x}\sum_{m,n\geq 0}\frac{y^n}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{m=0}^{\infty}\frac{x^m}{m!}(1+t)^{m(n-k)}$$

$$= e^{-x} \sum_{m=0} \frac{x^m}{m!} \sum_{n=0}^{\infty} \frac{y^n}{n!} (1+t)^{mn} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (1+t)^{-mk}$$

$$= e^{-x} \sum_{m=0} \frac{x^m}{m!} \sum_{n=0}^{\infty} \frac{y^n}{n!} (1+t)^{mn} \left[1-(1+t)^{-m}\right]^n$$

$$= e^{-x} \sum_{m=0} \frac{x^m}{m!} \sum_{n=0}^{\infty} \frac{y^n}{n!} \left[(1+t)^m - 1\right]^n$$

$$= e^{-x} \sum_{m=0} \frac{x^m}{m!} e^{y[(1+t)^m - 1]} = e^{-x-y} \sum_{m=0} \frac{x^m}{m!} e^{y(1+t)^m}$$

$$= e^{-x-y} \sum_{m=0} \frac{x^m}{m!} \sum_{n=0}^{\infty} \frac{y^n}{n!} (1+t)^{mn} = e^{-x-y} \sum_{m,n \geq 0} (1+t)^{mn} \frac{x^m y^n}{m!\,n!}$$

$\square$

Hence approved.

Very good! 10/10

Q5| The q-binomial coefficient satisfies $\left[{n \atop k}\right]_q = \sum_{w \in W_{n,k}} q^{inv(w)}$,

where $W_{n,k}$ is the set of words that are rearrangements of $(n-k)$ 0's and $k$ 1's, and $inv(w)$ is the number of inversions of $w$. Suppose $n = 2m$ is even. prove that $\left[{n \atop k}\right]_{q=-1}$ is equal to $\# P_{n,k}$?

proof:

let $q=-1$, the q-binomial coefficient is simply the differente between the number of words whose inversion number is even and the number of words whose inversion number is odd:

$$\left[{n \atop k}\right]_{q=-1} = \#\{w \in W_{n,k} : inv(w) = even\} - \#\{w \in W_{n,k} : inv(w) = odd\} \quad ①$$

we note that pairs of 0's and pairs of 1s don't contribute to the count of inversions, so only pairs of 0s and 1s matter.

Given k 1s, the number of pairs that have some sign, inverted or not, is

$$N_p = k(2m-k) \longrightarrow ②$$

Given $n=2m$, let us first:

consider $k = odd$ and consider involution which reverses the whole word:

$$[\tau(w)]_i = w_{2m+1-i} \quad , \quad i = 1, 2, \dots, 2m \rightarrow ③$$

(example: $\tau(001101) = 101100$) Text

Since $k$ is odd, ② is also odd, therefore involution ③ flips the parity of every word with odd $k$.
As every word with even and odd $k$ so reversed result in a different word, the sum in ① is 0.
As no word with even $n$ and odd $k$ can be a palindrome we get,

$$\begin{bmatrix} 2m \\ k \end{bmatrix}_{q=-1} = \# P_{2m,k} \quad , \quad \forall k = \text{odd} \longrightarrow ④$$

Now, Second let consider $k$ = even

In this case ② is even too, so reversal of the whole word does not change its inversions parity.

Good observation.

Let is now consider the 1st and the last digits if the number of 1's in the word. If they are different, the ~~If they are~~ inner part that includes from 2nd to the 2nd last digit is odd. So we can now define an involution that keeps the 1st and last digit unchanged, but reverses the part of the word inside them.

In the same way as we have seen before the contribution of all such words to ① is 0 and no such words can be palindromes.

If the $1^{st}$ and the last digits are the same, we go deeper and repeat the same with $2^{nd}$ and $2^{nd}$ last digits.

This leads us to the definition of an involution applicable to even k:

Given a word, we start from its outside and compare the pair of digits $w_i$ and $w_{n+1-i}$ for $i=1,2,\dots$ till we find the $1^{st}$ pair of different digits.

Yes! In fact, you could use this for all values of n&k, and not do the reverse the whole word at all.

If we do not find such a pair, this word is a palindrome.

If we do find such a pair, the involution reverses the part of the word inside this pair.

This allows us to see that each word that is not a palindrome has its counterpart of opposite inversion parity and this pair of words cancel each ~~others~~ other's contributions to ①

All that remians in ① are the contributions of palindromes,

So

$$\begin{bmatrix} 2m \\ k \end{bmatrix}_{q=-1} = \# P_{2m,k} \quad , \quad \forall k = 0, \dots, n \Rightarrow$$

Hence approved $\square$. Great! 10/10