Howard Math 156: Calculus I   Fall 2023
Instructor: Sam Hopkins   (sam.hopkins@howard.edu)
                call me "Sam"

8/21   Logistics:

Classes: MTWF 2:10-3pm, ASB-B #100
Office hrs: Tue 1-2pm, Annex III - #220
            or by appointment - email me!

Website: samuelfhopkins.com/classes/156.html

Text: Calculus, Early Transcendentals, by Stewart, 7th Ed.

Grading: 35% (in person) quizzes
           45% 3 (in person) midterms
           20% final Exam

There will be 11 in person quizzes taken on Tuesdays
(about 20 mins, we'll go over them for rest of class)
Your lowest 2 scores will be dropped (so 9 will count)

The 3 midterms will happen in class, also on Tuesdays
The final will be during finals week.

Beyond that, I may assign practice problems (not graded)
and I expect you to   SHOW UP TO CLASS
                    + PARTICIPATE :-)

that means... interrupt me by
                    ASKING QUESTIONS!

and please say your name when you ask
a question, so I can start to put names to faces.

What is calculus about?
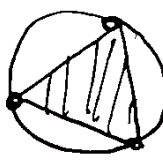
Calculus is different from the math you've seen.
It deals with _change_, with _infinities_ (and _infinitesimals_)
and with _limiting processes_.
It's good to have a preview of this new stuff...

Area of a circle

We all know that the area of a circle of radius
R is πR^2, where $\pi = 3.14159...$ is a special number.
But how would you figure this out if you didn't know?

You could try to _approximate_ the area
by using a simpler shape, like a regular triangle,
whose area you already know how to compute.
But this clearly leaves some of the area out...
So you might consider regular 4-gon, 5-gon, ...

Each inscribed regular n-gon
gives a _better_ and _better_
approximation to area of circle!

And true area can be obtained as _limit_ as $n \to \infty$
(n goes to _infinity_)!

We won't study this exact problem this semester,
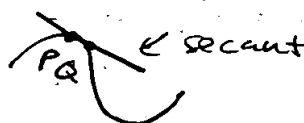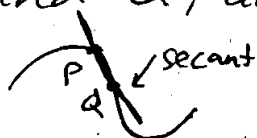but we will consider the _area under a curve_:

this ↙ can also be obtained by a
limit of simpler shapes:
thin _rectangles_ under curve

<u>Tangent</u> to a <u>curve</u>: How would you find the <u>tangent</u> line to a curve at a point?
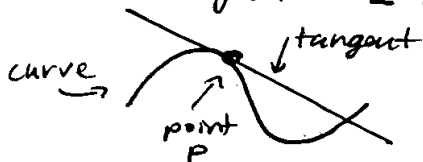
The tangent is the line that "just touches" the curve at that point.

Calling this point P, can draw <u>secant</u> <u>line</u> through P and Q, another nearby point on curve:

As we move this other point Q closer and closer to P, the secant becomes a ~~better~~ and ~~better~~ <u>approximation</u> of the tangent, and in the <u>limit</u>, secant becomes tangent!

Why care about tangents to curves?
They tell us about <u>velocity</u> and <u>acceleration</u> in physics (and rates of <u>change</u> in sciences in general...)

Also, can approx. curve by a series of tangents:
("Newton's method"... used by NASA)
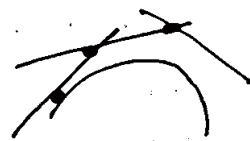
← "linear approx. of curve"

<u>Big</u> i<u>dea</u> of <u>Calculus</u>:
Even though the area problem and the tangent line problem seem pretty different...

They are actually the same <u>problem</u>!

Or more precisely, they are <u>opposite</u> <u>problems</u>!

This semester we'll learn <u>why</u> (+ <u>how</u>)!

Functions (§1.1 of textbook)

Functions are the basic thing we will study in calculus. They are fundamental in all sciences as _models_.

E.g. If we produce x units of some product, our _revenue_ may be given by the function

$$R(x) = p \cdot x \quad \text{where} \quad p = \text{price per unit of product}$$

(Very simple linear model, doesn't take into account costs...) We will see _derivative_ $R'(x)$ (slope of tangent at point x) is what economists would call "_marginal revenue_."

— But what _is_ a function?

Formally, a _function_ f between two sets A and B is a relation between the elements of A and B such that _every element_ of A is related to _exactly one element_ of B.

e.g. $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$
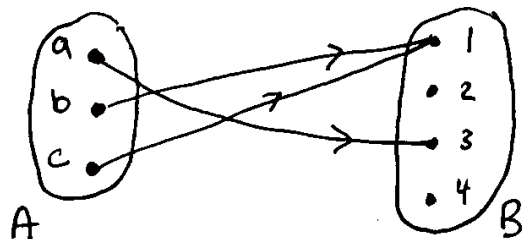
$f(a) = 3$
$f(b) = 1$
$f(c) = 1$

The set A is called the _domain_ of f and the set B is called the _codomain_ of f.

The _range_ of f is the set of all $f(x)$ for $x \in A$.

E.g. Range for f above is $\{1, 3\}$ (actual _values_ of inputs).

The function is called <u>one-to-one</u> if every element in the range is the output of a <u>unique</u> $x \in A$.

E.g: Example f above is <u>not</u> one-to-one since $f(b) = 1$ and also $f(c) = 1$.

That is the formal definition of a function, but we will normally work with functions f whose domain & range are subsets of real numbers $\mathbb{R}$.

Then we'll have several other <u>ways</u> to represent f, beyond an "arrow diagram" or "chart"
(and we'll need other ways since there are ∞-many real #'s!)

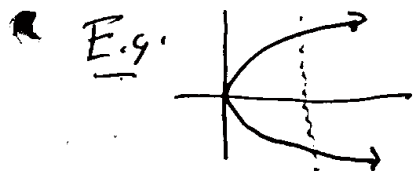You are probably used to functions defined by <u>algebraic formulas</u>, such as
$$f(x) = x^2$$
which we can also represent by a <u>graph</u>:

$y = f(x)$ 'parabola'
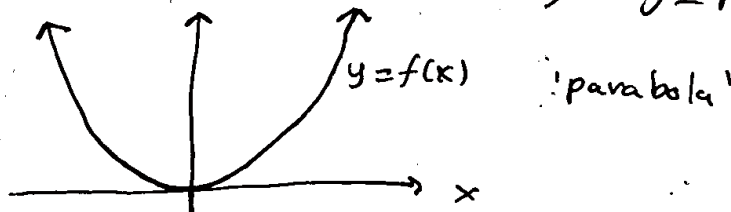
How do we know if a graph represents a function?

"<u>vertical line test</u>": graph represents a function $\Longleftrightarrow$ each vertical line intersects ≤ 1 point on graph
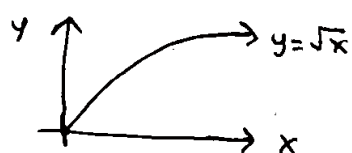
E.g.

graph $x = y^2$ is <u>NOT</u> graph of a function $y = f(x)$, since vertical line $x = 4$ intersects <u>two points</u> on the graph!

The domain of $f(x) = x^2$ is all of the real numbers $\mathbb{R}$, also denoted $(-\infty, \infty)$ in "interval notation."
The range is the nonnegative reals, or $[0, \infty)$.
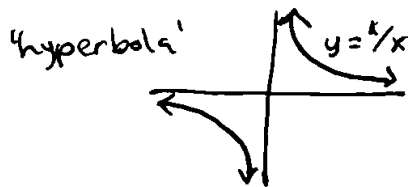
What about $f(x) = \sqrt{x}$?

we mean positive square root when we write this

The domain is $[0, \infty)$, and range is also $[0, \infty)$.

In general, to find the domain of a function $f(x)$ think about what values you're allowed to plug into f.

E.g. Domain of $\sqrt{x-1}$ is $\{x \in \mathbb{R} : x \geq 1\} = [1, \infty)$
  Since can only take square root of a nonneg. #.
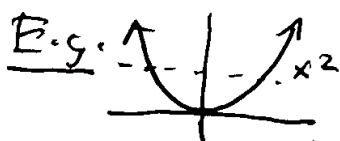
E.g. Function $f(x) = 1/x$ has domain (and also range)
  $\{x \in \mathbb{R} : x \neq 0\} = (-\infty, 0) \cup (0, \infty)$

'hyperbola' $y = 1/x$

Since we are not allowed to divide by zero. (Denominators can never be 0.)

Can also test one-to-one-ness graphically, using:
"horizontal line test": function f is one-to-one $\Longleftrightarrow$ every horizontal line intersects graph $y = f(x)$ in ≤ 1 point.
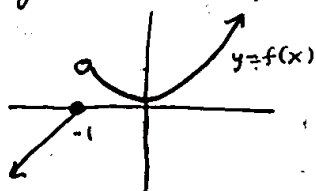
E.g. x^2 $f(x) = x^2$ is Not one-to-one.
  Q: what about $f(x) = x^3$?

Not every function is determined by a single formula.
We can define a piecewise function like

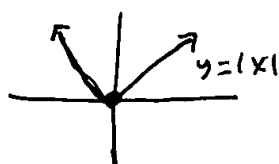$$f(x) = \begin{cases} x+1 & \text{if } x \le -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

The graph of $y = f(x)$ has two parts.

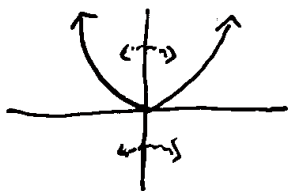(see how we use O (empty circle)
to denote a 'discontinuity')
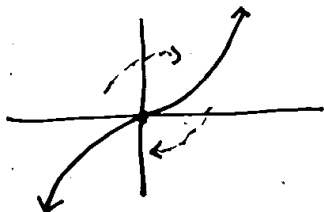
Another important piecewise function is absolute value:

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

← graph of $|x|$ has two parts,
but they 'touch' each other

Symmetries of functions

← The graph of function $f(x) = x^2$
is symmetric about the y-axis:
if I reflect it about the y-axis (vertical axis),
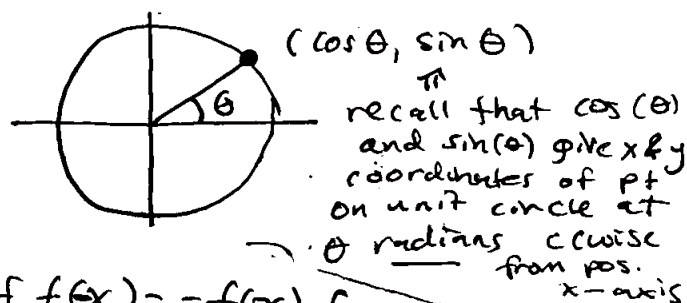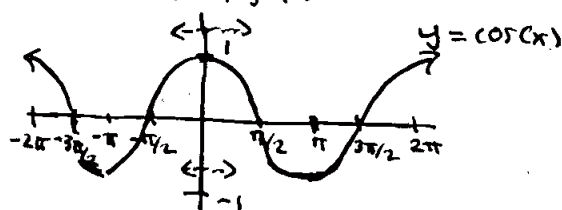I get back the same thing.

← The graph of function $f(x) = x^3$
is symmetric about the origin:
if I rotate it 180° about origin ($(0,0)$ point),
I get back the same thing.

These two kinds of symmetry are called even and odd
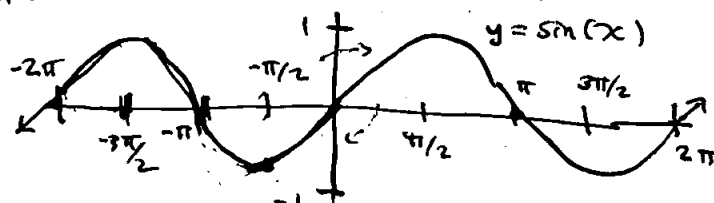for functions $f(x)$.

A function $f(x)$ is called <u>even</u> if $f(x) = f(-x)$ for all x. Same as saying graph is symmetric about y-axis.

Examples of even fn's: x^2, $x^2 + 1$, x^4, $|x|$, $\cos(x)$

$y = \cos(x)$

$(\cos\theta, \sin\theta)$

recall that $\cos(\theta)$ and $\sin(\theta)$ give x & y coordinates of pt on unit circle at θ radians c.cwise from pos. x-axis

A function $f(x)$ is <u>odd</u> if $f(-x) = -f(x)$ for all x. Same as saying graph symmetric about origin.

Examples of odd fn's: x^3, x, $x^5 + x^3$, $\sin(x)$

$y = \sin(x)$

8/28 Can you guess why we use names "<u>even</u>" and "<u>odd</u>"?

§1.3 Transformations of functions!

Given $f(x)$ can make new functions by applying various transformations, like translations!

$y = f(x) + c$ — function whose graph is $f(x)$ translated <u>up</u> by c
$y = f(x) - c$ — graph is $f(x)$ translated <u>down</u> by c
$y = f(x - c)$ — graph is $f(x)$ translated <u>right</u> by c
$y = f(x + c)$ — graph is $f(x)$ translated <u>left</u> by c
  (for $c > 0$)

E.g:

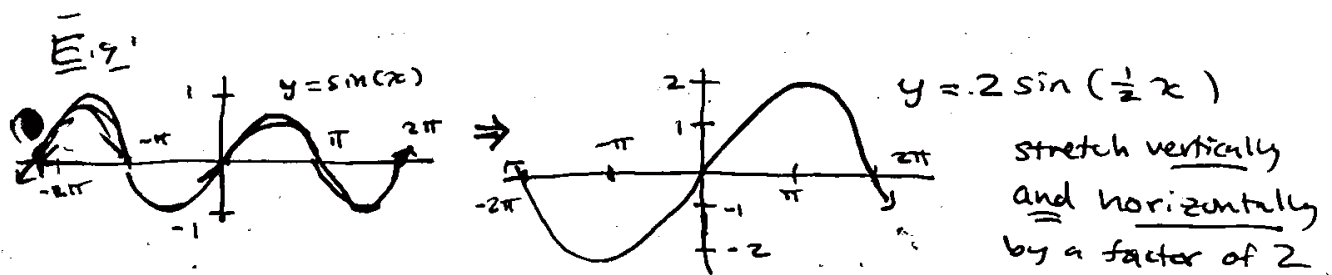$y = x^2$ 　　 $y = x^2 + 1$ 　shift up by 1　　 $y = (x-1)^2$ 　shift right by 1
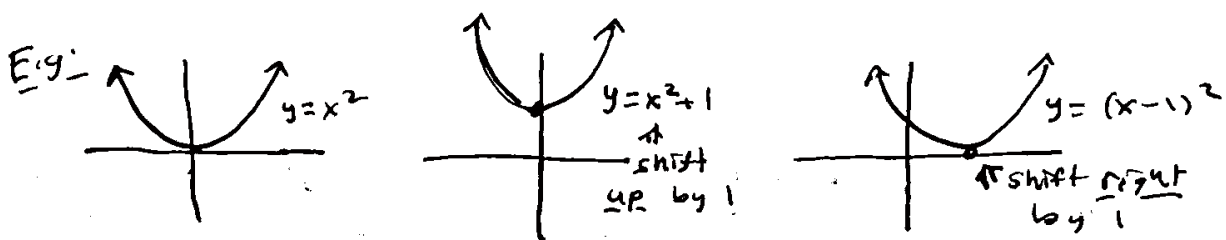
Can also <u>Stretch</u> a function: for $c > 1$,

$y = c \cdot f(x)$ — Stretch graph vertically by factor of c

$y = \frac{1}{c} \cdot f(x)$ — shrink graph vertically by c

$y = f\left(\frac{x}{c}\right)$ — stretch graph horizontally by c

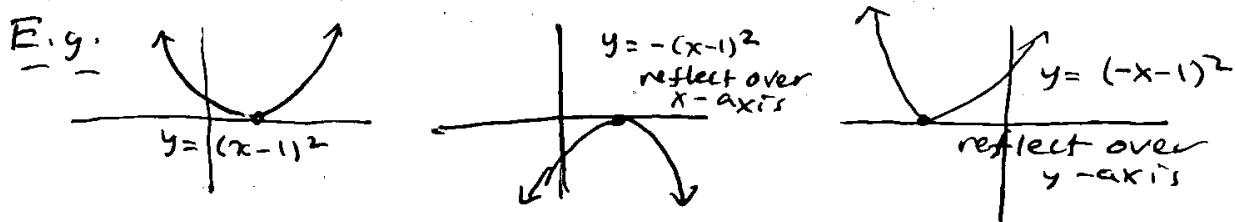$y = f(c \cdot x)$ — shrink graph horizontally by c

E.g:

$y = \sin(x)$ 　　$\Rightarrow$　　 $y = 2\sin\left(\frac{1}{2}x\right)$

stretch vertically and <u>horizontally</u> by a factor of 2

We see in this example how we can <u>combine multiple transformations</u>!
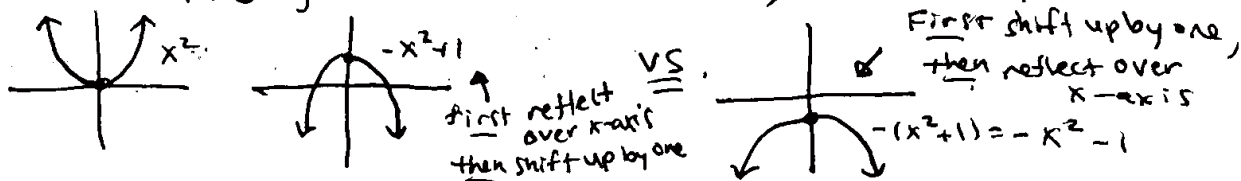
One more geometric transformation: <u>reflection</u>.

$y = -f(x)$ — <u>reflect</u> graph about x-axis

$y = f(-x)$ — <u>reflect</u> graph about y-axis

E.g:

$y = (x-1)^2$ 　　$y = -(x-1)^2$ reflect over x-axis　　 $y = (-x-1)^2$ reflect over y-axis

Q: What happens w/ reflections for <u>even</u> + <u>odd</u> fn's?

§1.3 When applying multiple transformations, order is important!

First reflect over x-axis then shift up by one

VS.

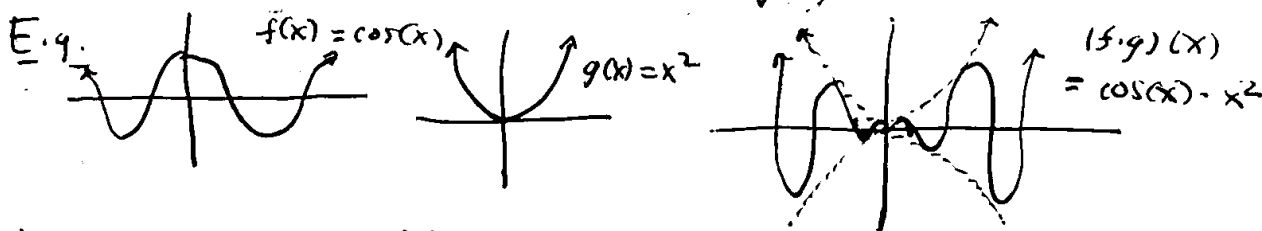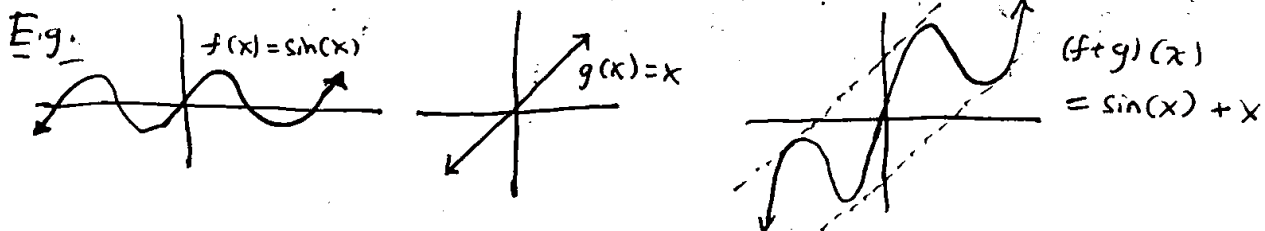First shift up by one, then reflect over x-axis

$-(x^2+1) = -x^2 - 1$

8/30 Another way to get new functions from old ones is by combining functions in various ways.

Def'n If f, g are two fn's, we define their sum, difference, product, and quotient by

$$(f+g)(x) = f(x) + g(x) \qquad (f-g)(x) = f(x) - g(x)$$
$$(f \cdot g)(x) = f(x) \cdot g(x) \qquad (f/g)(x) = f(x)/g(x)$$

E.g.

$f(x) = \sin(x)$

$g(x) = x$

$(f+g)(x) = \sin(x) + x$

E.g.

$f(x) = \cos(x)$

$g(x) = x^2$

$(f \cdot g)(x) = \cos(x) \cdot x^2$

E.g. $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ ← not always easy to graph combinations.

Note: Since $\cos\left(\frac{\pi}{2} + n \cdot \pi\right) = 0$ for all integers $n \in \mathbb{Z}$,

the domain of $\tan(x) = \{x \in \mathbb{R} : x \neq \frac{\pi}{2} + n\pi \text{ for any } n \in \mathbb{Z}\}$
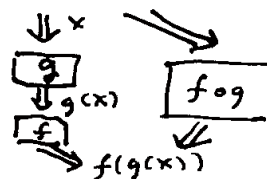
(because we don't want to divide by zero!)

Another very important way to combine functions is composition.

__Def'n__ If f and g are two functions, their __composition__ $f \circ g$ is:
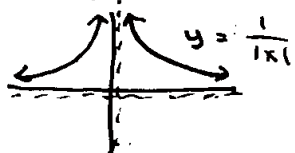$$(f \circ g)(x) = f(g(x))$$

"Do g __first__, then do f to __that__!"

"f __of__ g __of__ x"

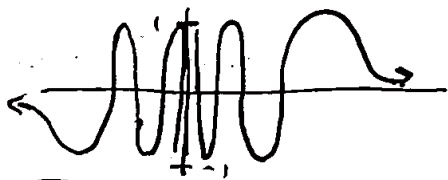__E.g.:__ $f(x) = x^2$, $g(x) = 2x-1$, $(f \circ g)(x) = (2x-1)^2 = 4x^2 - 4x + 1$

__E.g.:__ $f(x) = 1/x$, $g(x) = |x|$, $(f \circ g)(x) = \frac{1}{|x|} = \begin{cases} 1/x & \text{if } x > 0 \\ -1/x & \text{if } x < 0 \end{cases}$

$y = \frac{1}{|x|}$

__Note:__ $\frac{1}{|x|}$ is __even__ since $\frac{1}{|-x|} = \frac{1}{|x|}$ and $x = 0$ is __not__ in the __domain__.

__E.g.:__ $f(x) = \sin(x)$, $g(x) = 1/x$, $(f \circ g)(x) = \sin(1/x)$

What does $\sin(1/x)$ look like? As $|x| \to \infty$, $1/x \approx 0$ barely changes, so $\sin(1/x)$ stops oscillating for big x. But as $|x| \to 0$, $1/x$ changes __a lot__, so $\sin(1/x)$ oscillates a __ton__ near $x = 0$:

$\Leftarrow$ very hard to draw accurate graph of $y = \sin(1/x)$.

Also, notice $x = 0$ __Not__ in domain (since $1/0$ not defined)

If $(f \circ g)(x) = x$, then we say that f is the __inverse function__ of g. f "__undoes__" what g does!

__E.g.:__ $f(x) = \sqrt{x}$, $g(x) = x^2$, $(f \circ g)(x) = \sqrt{x^2} = x$ (for ~~nonnegative~~ nonnegative $x \geq 0$)

The square root function $f(x) = \sqrt{x}$ "__undoes__" the square $g(x) = x^2$ We will be more careful about __domain__ issues for inverses later...
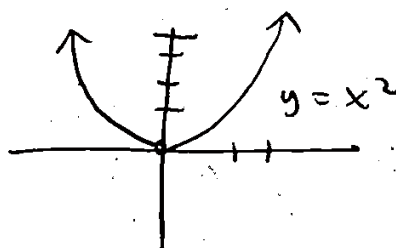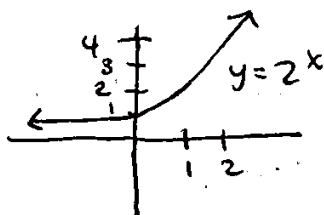
§1.4 Exponential functions

Def'n Fix real number $a > 0$. The _exponential function_
with _base_ a is $f(x) = a^x$.

Do _not_ confuse a^x with _power function_ x^a.

Eg: $f(x) = 2^x$ vs. $g(x) = x^2$

| x	$f(x)$	$g(x)$
0	1	0
1	2	1
2	4	4
3	8	9
4	16	16

At first, x^2 grows more quickly than 2^x, but
this is _misleading_: eventually, 2^x grows much,
much faster than x^2!

In fact, any exponential a^x for $a > 1$ (eventually)
grows much, much faster than any _polynomial_.

(Recall a polynomial is a function $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$
that is a _linear combination_ of power functions.)
We will prove this assertion later (using calculus!).

For $a > 1$, a^x represents _exponential growth_, while
for $0 < a < 1$, a^x represents _exponential decay_:

a^x, $a > 1$

a^x, $0 < a < 1$

a^x, $a = 1$
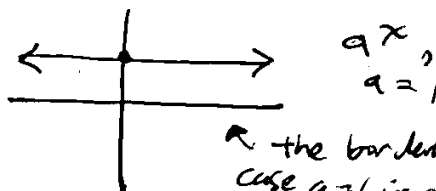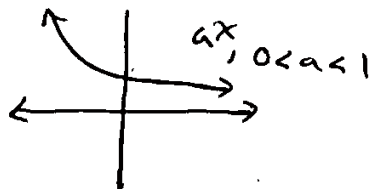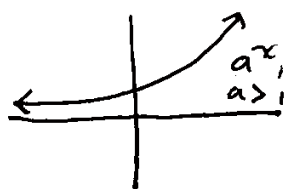
↖ the borderline
case $a = 1$ is _degenerate_

Sometimes we also consider Ca^x for constant C an exponential function.

In sciences, e.g. biology, often see a mix of exponential growth and decay:

e.g. : _then tapers off as resources depleted

_population grows exponentially at first
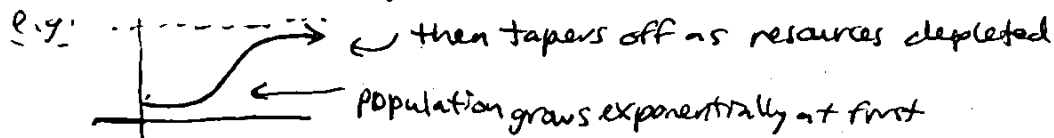
Remember: fixed <u>exponent</u> x^a ⟹ power function
Fixed <u>base</u> ⟹ exponential function
(So something like $f(x) = x^x$ is neither.)

<u>The special number</u> e : There is one base that is "best",
the number $e \approx 2.718\ldots$ e irrational number, like π

How to define e precisely? Can use a <u>limit</u>:

$$e = \lim_{n \to \infty} \left(1 + \tfrac{1}{n}\right)^n$$

There is a way to think of this formula in terms of <u>compound interest</u>:
If you have \$1 invested in an investment with a <u>rate of return</u> of 100% per year that is "<u>continuously compounded</u>",
then at the end of the year you will have \$$e$. $= \$2.718\ldots$
You may remember formula Pe^{rt} for interest.
Principal ↗ rate of return ↑ time ↑

There is also a <u>geometric</u> way to think about e:

e^x — slope of tangent at $x=0$ is $\underline{1}$
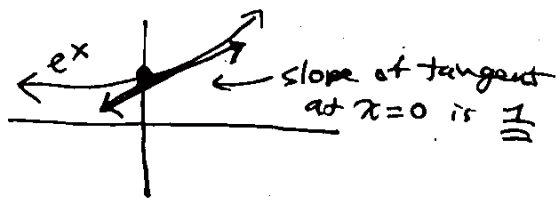
Of all exponential functions $f(x) = a^x$, the unique one that has a <u>tangent line slope</u> of 1 at $x=0$ is for $a = e$.

When we start to talk about <u>derivatives</u>, we will see that this is a desirable property. So $f(x) = e^x$ is by far the <u>most</u> common <u>exponential</u> fn.
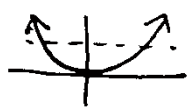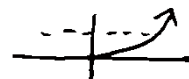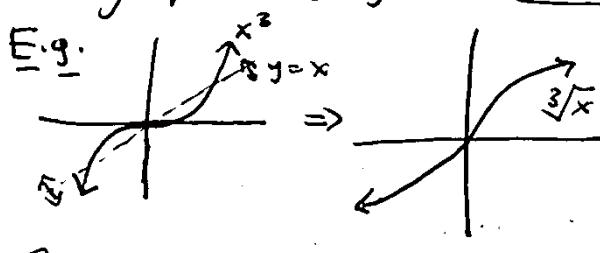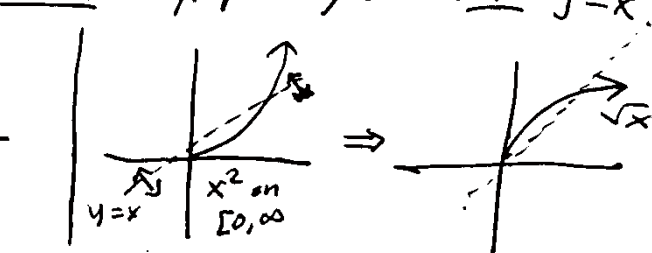
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§1.5 Inverse functions and logarithms

Def'n A function $g(x)$ has an inverse function $f = g^{-1}$ if and only if it is one-to-one. In this case, the inverse function $f = g^{-1}$ is defined by $f(y) = x$ if x is the unique element in domain of g such that $g(x) = y$. (f "undoes" g so that $(f \circ g)(x) = x$).

E.g. Since $g(x) = x^3$, it admits an inverse $f = g^{-1}$ which is $f(x) = \sqrt[3]{x}$.

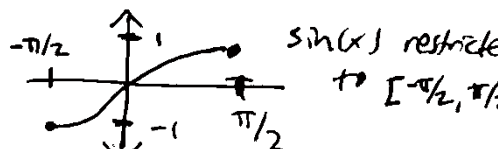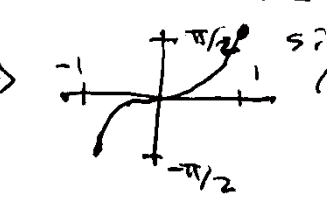**E.g.** Recall $g(x) = x^2$ is not one-to-one: fails horizontal line test!

So it does not have an inverse on all of $\mathbb{R}$.

But if we restrict the domain to $[0, \infty)$, then $f(x) = \sqrt{x}$ is its inverse, like we expect.

There is a geometric way to think about inverses: the graph of $f = g^{-1}$ is reflection of graph of g over line $y = x$.

E.g.

This geometric interpretation also makes clear that domain of f = range of g, and range of f = domain of g for $f = g^{-1}$

E.g. The trig functions are far from 1-to-1, so to define inverse trig functions, we need to restrict domain:

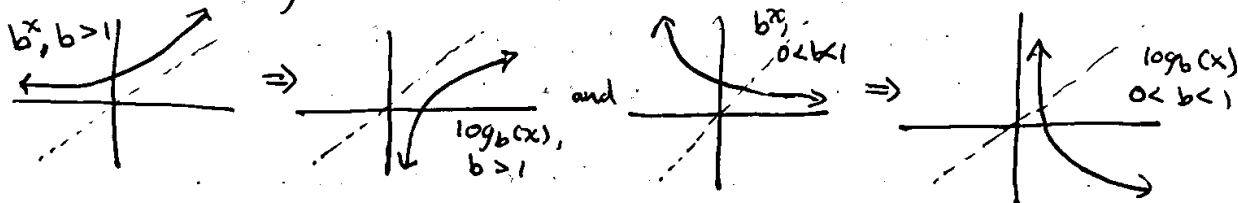

$\sin(x)$ restricted to $[-\pi/2, \pi/2]$ $\Rightarrow$ $\sin^{-1}(x)$ (also called $\arcsin(x)$)

D: $[-1, 1]$
R: $[0, \pi]$

Looking at graph of b^x for any $b>0$, $b \neq 1$, we see
it passes horizontal line test, so it has an inverse:

Def'n $\log_b(x)$, the base b logarithm, is the inverse of
exponential fn. b^x, meaning $\boxed{\log_b(y) = x \iff b^x = y}$

E.g. $\log_{10}(100) = 2$ since $10^2 = 100$.
Graphically, we have:

Note that since range of b^x is $(0, \infty)$ (positive numbers)
domain of $\log_b(x)$ is $(0, \infty)$: can only take log of positive numbers!

Since e^x is the "best" exponential, $\log_e(x)$ is "best" logarithm.
It is also called the natural logarithm, denoted $\ln(x) := \log_e(x)$.
Just like we usually only consider e^x for exponential functions,
we also usually only consider $\ln(x)$ for logarithms.
In fact, these are enough, because of

Thm 1. $b^x = e^{\ln(b) \cdot x}$
2. $\log_b(x) = \dfrac{\ln(x)}{\ln(b)}$.

Pf: For 1.: $e^{\ln(b) \cdot x} = \left(e^{\ln(b)}\right)^x = b^x$. ✓
For 2.: Let $y = \log_b(x)$, so that $b^y = x$.
Taking $\ln$ of both sides $\Rightarrow \ln(b^y) = \ln(x)$
$\Rightarrow y \cdot \ln(b) = \ln(x)$
$\Rightarrow \log_b(x) = y = \dfrac{\ln(x)}{\ln(b)}$. ✓ ▨

In the above proof, we used some important properties of exponentials and logarithms which you hopefully learned in an algebra class:

Prop: 1. $b^{x+y} = b^x b^y$ 2. $b^{x-y} = \dfrac{b^x}{b^y}$

3. $(b^x)^y = b^{xy}$ 4. $(ab)^x = a^x b^x$

Prop: 1. $\log_b(xy) = \log_b(x) + \log_b(y)$

2. $\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$

3. $\log_b(x^r) = r \cdot \log_b(x)$.

Since e^x and $\ln(x)$ are so important, it's also worth remembering these special values:

Prop: 1. $e^0 = 1$ 3. $\ln(1) = 0$

2. $e^1 = e$ 4. $\ln(e) = 1$.

Aside on how to algebraically find inverse function:

To find inverse of $g(x)$, write $y = g(x)$ and "solve for y":

e.g: $g(x) = x^3 - 1 \rightsquigarrow y = x^3 - 1$ ⟍ so inverse $f = g^{-1}$ is

$y + 1 = x^3$

$\sqrt[3]{y+1} = x$ $f(y) = \sqrt[3]{y+1}$

e.g: $g(x) = 5e^x \rightsquigarrow y = 5e^x$ ⟍ so inverse $f = g^{-1}$ is

$\frac{1}{5}y = e^x$

$\ln(\frac{1}{5}y) = x$ $f(y) = \ln(\frac{1}{5}y)$.