Total score = 28/50 + 5 bonus points from presentation = 33/50

Caleb DeRose

1 $F_k(x) = \sum_{n\ge 0} S(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\dots(1-kx)}$

$\frac{x^k}{(1-x)(1-2x)\dots(1-kx)} = \frac{\alpha_1}{1-x} + \frac{\alpha_2}{1-2x} + \frac{\alpha_3}{1-3x} + \dots + \frac{\alpha_k}{1-kx}$.

$\therefore\ x^k = \alpha_1(1-2x)(1-3x)\dots(1-kx) + \alpha_2(1-x)(1-3x)\dots(1-kx) + \alpha_3(1-x)(1-2x)\dots(1-kx) + \dots + \alpha_k(1-x)(1-2x)\dots(1-(k-1)x)$

$x=1 \Rightarrow 1^k = 1 = \alpha_1(1-2)(1-3)(\dots)(1-k)$

$1^k = 1 = \alpha_1(-1)^{k-1}(2-1)(3-1)\dots(k-1) = -\alpha_1(-1)^k(k-1)!$

$1^k/(-1)^k = -\alpha_1(k-1)! \Rightarrow \alpha_1 = -\left(\frac{(-1)^k}{(k-1)!}\right) = \alpha_1 = \frac{(-1)^{k-1}}{(k-1)!}$

$x=\frac{1}{2} \Rightarrow \left(\frac{1}{2}\right)^k = \alpha_2\left(1-\frac{1}{2}\right)\left(1-\frac{3}{2}\right)\left(1-\frac{4}{2}\right)\dots\left(1-\frac{k}{2}\right)$

$= 2\alpha_2\left(\frac{1}{2}\right)^k(2-1)(2-3)(2-4)\dots(2-k)$

$1 = 2\alpha_2(2-3)(2-4)\dots(2-k) = 2\alpha_2(-1)^{k-2}(3-2)(4-2)\dots(k-2)$

$1 = 1^{k-2} \qquad = 2\alpha_2(-1)^{k-2}(k-2)!$

$1^{k-2}/(-1)^{k-2} = (-1)^{k-2} \Rightarrow \alpha_2 = \frac{(-1)^{k-2}}{2(k-2)!}$

$x=\frac{1}{3} \Rightarrow \left(\frac{1}{3}\right)^k = \alpha_3\left(1-\frac{1}{3}\right)\left(1-\frac{2}{3}\right)\left(1-\frac{4}{3}\right)\left(1-\frac{5}{3}\right)\dots\left(1-\frac{k}{3}\right)$

$\left(\frac{1}{3}\right)^k = 3\alpha_3\left(\frac{1}{3}\right)^k(3-1)(3-2)(3-4)(3-5)\dots(3-k)$

$1 = (3)(2)\alpha_3(3-4)(3-5)\dots(3-k)$

$1 = (3)(2)\alpha_3(-1)^{k-3}(4-3)(5-3)\dots(k-3) = \alpha_3(-1)^{k-3}3!(k-3)!$

$\Rightarrow \alpha_3 = \frac{(-1)^{k-3}}{3!(k-3)!}$

$\therefore\ \alpha_j = (-1)^{k-j}/j!(k-j)!$

Note that each $\frac{\alpha_i}{1-ix}$ is a geometric sequence $\sum_{n\ge 0}\alpha_i i^n x^n$.

Thus $\sum_{n\ge 0} S(n,k)x^n = \sum_{n\ge 0}\left(\sum_{j=1}^{k}\alpha_j j^n\right)x^n$.

$\therefore S(n,k) = \sum_{j=1}^{k}\alpha_j j^n = \sum_{j=1}^{k}\frac{(-1)^{k-j}}{j!(k-j)!}j^n$

Very good. 10/10

Problem 2 missing? 0/10

3 Consider $m \times n$ matrices of 0's and 1's with at least one 1 in each row & column. Some number of columns will contain all 1's. We can effectively ignore these columns and focus on each other entry in each row. For each entry in a row, it's either 1 or 0.

m rows
n columns

Thus, when constructing a matrix suiting the condition, there are $\binom{n}{k}$ ways to choose columns filled with 1's. For each of the m rows, each of the remaining $(n-k)$ entries is either 1 or 0. Using the variable t to count 1's, we obtain $((1+t)^{n-k})^m$. However, there must be at least one 1 in each row. Taking $((1+t)^{n-k}-1)^m$ ensures each row contributes some power of t.

In this way, $\binom{n}{k}((1+t)^{n-k}-1)^m$ counts the number of matrices that fit the condition after removing k columns. However, we can have matrices with unremoved columns of 1's, so we take the alternating sum to adjust accordingly. (via P.I.E.)

As discussed in the presentation this is almost the right way to think about the problem, but you actually want to focus on matrices containing columns of all 0's (instead of all 1's) to properly apply the P.I.E.

8/10

Thus, $\sum_{i\geq 0} f_i(m,n)t^i = \sum_{k\geq 0}^{n} (-1)^k \binom{n}{k}((1+t)^{n-k}-1)^m$.

- Ignore some number of columns (k). In each row (m), there are $(n-k)$ positions that are either 0 or 1, counted by t, so $(1+t)^{n-k}$. Subtract 1 from that polynom b/c there must be at least one 1 (t^0 is no good).

$\sum_{i=0}^{2} f_i(2,2)t^i = \binom{2}{0}((1+t)^2-1)^2 - \binom{2}{1}((1+t)^1-1)^2 + \binom{2}{2}((1+t)^0-1)^2 = t^4+4t^3+4t^2-2t^2$

$\sum_{i=0}^{2} f_i(1,2)t^i = \binom{2}{0}((1+t)^2-1)^1 - \binom{2}{1}((1+t)^1-1)^1 + \binom{2}{2}((1+t)^0-1)^1 = t^2+2t-2t = t^2$

$\sum_{i=0}^{2} f_i(2,1)t^i = \binom{1}{0}((1+t)^1-1)^2 - \binom{1}{1}((1+t)^0-1)^2 = t^2$

$\sum_{i\geq 0} f_i(1,1)t^i = \binom{1}{0}((1+t)^1-1)^1 - \binom{1}{1}((1+t)^0-1)^1 = t-0$

$\sum_{i\geq 0} f_i(m,n)t^i = \sum_{k\geq 0}^{n} (-1)^k \binom{n}{k}((1+t)^{n-k}-1)^m$

$(t^2+2t)^2$

Problem 4 missing? 0/10

5 & $W_{n,k}$ is the set of words that are rearrangements of $(n-k)$ 0's and n 1's.
& $P_{n,k} \subseteq W_{n,k}$ is the subset of words that are palindromes.
& n is even.

Define $\tau: W_{n,k} \to W_{n,k}$ by $\tau(w) = \begin{cases} w' = w \text{ with the first pair } w_i \neq w_{n+1-i} \text{ swapped} \\ w' = w \text{ if } w_i = w_{n+1-i} \ \forall i \end{cases}$

(e.g. $111000 \leftrightarrow 011001$, and 010010 stays at 010010)

Take $\text{sgn}(W_{n,k}) = (-1)^{inv(w)}$ and $wt(w) = 1$.

$\therefore$ $\text{sgn}(w) = -\text{sgn}(\tau(w))$ for $w \neq \tau(w)$ ($\because$ $inv(w)$ & $inv(\tau(w))$ have opposite parity) Could explain this a little more…

$\therefore$ $inv(w)$ is even for $\tau(w) = w$ ($\because$ palindromes are symmetric, $inv(w) = \frac{n(n-k)}{2}$, but note that both n and $(n-k)$ must be even for w to be a palindrome with an even number of letters, so $\frac{n(n-k)}{2}$ is even)

$\therefore$ $\#\{w \in W_{n,k} : \tau(w) = w\} = \#P_{n,k}$ ($\because$ all non-palindromes will cancel with their counterpart under τ.)

$$\therefore \sum_{w \in W_{n,k}} (-1)^{inv(w)} = \#\{w \in W_{n,k} : \tau(w) = w\} = \#P_{n,k}.$$

Note this is the same as evaluating $\sum_{w \in W_{n,k}} q^{inv(w)}$ with $q = -1$.

$\therefore$ $\begin{bmatrix} n \\ k \end{bmatrix}_q$ with $q = -1$ $= \#P_{n,k}$.

Very good! 10/10