Midterm #2 Study Guide
Math 157 (Calculus II), Spring 2023

1. Parametrized curves [§10.1, 10.2]
   (a) Curve of form $x = f(t)$ and $y = g(t)$ for some auxiliary variable t ("time") [§10.1]
   (b) Slope of tangent to curve given by chain rule [§10.2]: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{f'(t)}{g'(t)}$
   (c) Arc length [§10.2] is $\int_a^b \sqrt{(\frac{dy}{dt})^2 + (\frac{dx}{dt})^2}\, dt = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt$
2. Polar coordinates and polar curves [§10.3, 10.4]
   (a) Cartesian vs. polar [§10.3]: $(x, y) = (r\cos\theta, r\sin\theta)$ and $(r, \theta) = (\sqrt{x^2 + y^2}, \arctan(\frac{y}{x}))$
   (b) Area inside [§10.4] polar curve $r = f(\theta)$ for $\alpha \le \theta \le \beta$ is $\int_\alpha^\beta \frac{1}{2} r^2\, d\theta = \int_\alpha^\beta \frac{1}{2} f(\theta)^2\, d\theta$
   (c) Arc length [§10.4] of polar curve is $\int_\alpha^\beta \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$
3. Sequences and series [§11.1, 11.2, 11.3, 11.4, 11.5, 11.6, 11.7]
   (a) Sequence $\{a_n\}_{n=1}^\infty = a_1, a_2, \ldots$ is list of numbers, $\lim\limits_{n\to\infty} a_n$ defined like $\lim\limits_{x\to\infty} f(x)$ [§11.1]
   (b) Series $\sum_n^\infty a_n$ is "infinite sum" $a_1 + a_2 + \cdots$ of terms a_n; its value is $s = \lim_{n\to\infty} s_n$ where $s_n = a_1 + a_2 + \cdots + a_n$ is the nth partial sum [§11.2]
   (c) Important series: geometric series [§11.2] $\sum_{n=1}^\infty ar^{n-1}$ converges if and only if $|r| < 1$ (and $= \frac{a}{1-r}$ if it converges); p-series [§11.3] $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if and only if $p > 1$
   (d) Many tests for convergence / divergence of series:
      i. (Divergence test [§11.2]) If $\lim_{n\to\infty} a_n \neq 0$, series $\sum_n^\infty a_n$ diverges.
      ii. (Integral test [§11.3]) If $f(x)$ continuous, decreasing, and positive, with $a_n = f(n)$, then $\sum_n^\infty a_n$ converges if and only if $\int_1^\infty f(x)\, dx$ converges. In this case, have error bounds for remainder $R_n = s - s_n$: $\int_{n+1}^\infty f(x)\, dx \le R_n \le \int_n^\infty f(x)\, dx$.
      iii. (Comparison tests [§11.4]) If $\sum_n^\infty b_n$ converges & $a_n \le b_n$, then $\sum_n^\infty a_n$ converges. If $\sum_n^\infty b_n$ diverges & $a_n \ge b_n$, then $\sum_n^\infty a_n$ diverges. If $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is $\neq 0$, then $\sum_n^\infty a_n$ converges if and only if $\sum_n^\infty b_n$ converges.
      iv. (Alternating series test [§11.5]) Alternating series $\sum_{n=1}^\infty (-1)^{n-1} b_n$ converges as long as $b_{n+1} \le b_n$ and $\lim_{n\to\infty} b_n = 0$. In this case, have error bound: $|R_n| \le b_{n+1}$.
      v. (Ratio test [§11.6]) For series $\sum_{n=1}^\infty a_n$, let $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. If $L < 1$, series converges. If $L > 1$ (including ∞), series diverges. If $L = 1$, test is inconclusive.
4. Power series and Taylor series [§11.8, 11.9, 11.10, 11.11]
   (a) The ratio test tells us that any power series $\sum_{n=0}^\infty c_n (x-a)^n$ has a radius of convergence R such that it converges when $|x - a| < R$ and diverges when $|x - a| > R$ [§11.8]
   (b) Differentiate, integrate, and multiply power series like they are polynomials [§11.9, 11.10]
   (c) Taylor series of $f(x)$ at $x = a$ is $\sum_{n=0}^\infty \frac{f^{(n)}(a)}{n!}(x-a)^n$, where $f^{(n)}$ is nth derivative [§11.10]
   (d) Important Taylor series [§11.10]: $\frac{1}{1-x} = \sum_{n=0}^\infty x^n$ $(R = 1)$; $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$ $(R = \infty)$; $\sin(x) = \sum_{n=0}^\infty \frac{(-1)^{n-1} x^{2n+1}}{(2n+1)!}$ $(R = \infty)$; $\cos(x) = \sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{(2n)!}$ $(R = \infty)$
   (e) Taylor polynomial $T_n(x)$: nth partial sum of series; $f(x) \approx T_n(x)$ if $x \approx a$ [§11.10, 11.11]