Upho posets

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Texas A&M University, College Station, TX

Sam Hopkins (Howard University)

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Section 1

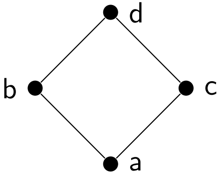
Posets and generating functions

Poset basics

A **poset** $(P, \leq)$ is a set with a partial order satisfying the usual axioms.

We represent posets by their **Hasse diagrams**:

$$P = \{a \leq b, c \leq d\} =$$



The edges of the Hasse diagram are given by the **cover relations** $x \lessdot y$, meaning $x < y$ and there is no $z \in P$ with $x < z < y$.

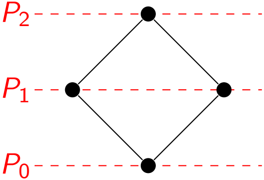
We will consider both finite and infinite posets, but all posets will be at least **locally finite**, meaning **intervals** $[x, y] = \{z : x \leq z \leq y\}$ are finite.

We use normal P for finite posets and caligraphic $\mathcal{P}$ for infinite posets.

Finite graded posets and their generating polynomials

A finite poset P is **graded** (of rank n) if $P = P_0 \sqcup P_1 \sqcup \cdots \sqcup P_n$ where every **maximal chain** in P is of the form $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n$ with $x_i \in P_i$. The **rank** of $p \in P_i$ is $\rho(p) = i$. The **rank generating polynomial** of P is

$$F(P;x) = \sum_{i=1}^{n} \#P_i \, x^i = \sum_{p \in P} x^{\rho(p)}.$$



$$\Rightarrow \quad F(P;x) = 1 + 2x + x^2 = (1+x)^2$$

If P has a **minimum** $\hat{0}$, its (reciprocal) **characteristic polynomial** is

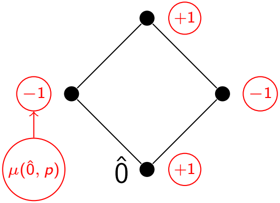
$$\chi(P;x) = \sum_{p \in P} \mu(\hat{0}, p) \, x^{\rho(p)},$$

where $\mu(\cdot,\cdot)$ is the **Möbius function** of P.

Möbius functions

The **Möbius function** $\mu(x, y)$ for $x \leq y \in P$ can be defined recursively by

$$\mu(x, x) = 1 \quad \text{and} \quad \mu(x, y) = -\sum_{x \leq z < y} \mu(x, z) \quad \text{if } x < y.$$



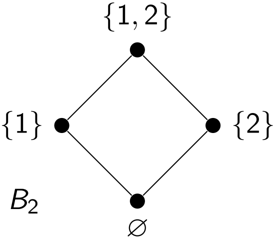
$$\Rightarrow \quad \chi(P; x) = 1 - 2x + x^2 = (1 - x)^2$$

The Möbius function μ is significant for many reasons:

- The **Euler characteristic** of the **order complex** of $[x, y]$ is $\mu(x, y)$.
- In **incidence algebra** of P, $\mu = \zeta^{-1}$ where $\zeta(x, y) = 1$ for all $x \leq y$.
- **Möbius inversion**: $f(x) = \sum_{y \geq x} g(y) \Leftrightarrow g(x) = \sum_{y \geq x} \mu(x, y) f(y)$.

Graded poset examples: Boolean and partition lattices

B_n = **Boolean lattice** of subsets of $[n] = \{1, 2, \ldots, n\}$ under inclusion:

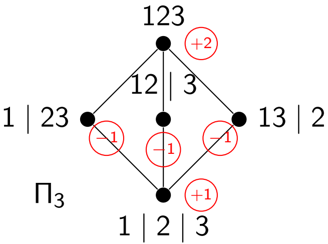


$$F(B_n; x) = \sum_{k=1}^{n} \binom{n}{k} x^k = (1+x)^n,$$

$$\chi(B_n; x) = \sum_{k=1}^{n} (-1)^k \binom{n}{k} x^k = (1-x)^n,$$

with $\binom{n}{k}$ the **binomial coefficients**.

Π_n = **partition lattice** of set partitions of $[n]$ under refinement:



$$F(\Pi_n; x) = \sum_{k=1}^{n} S(n, n-k) x^k,$$

$$\chi(\Pi_n; x) = \sum_{k=1}^{n} s(n, n-k) x^k = \prod_{i=1}^{n-1} (1 - ix),$$

with $S(n,k)$ & $s(n,k)$ the **Stirling numbers** of **2nd & 1st kind**.

Infinite graded posets and their generating functions

We now do something similar for certain infinite posets $\mathcal{P}$ modeled on the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$.

An infinite poset $\mathcal{P}$ is $\mathbb{N}$-**graded** if $\mathcal{P} = \mathcal{P}_0 \sqcup \mathcal{P}_1 \sqcup \mathcal{P}_2 \cdots$ where every maximal chain in $\mathcal{P}$ is of the form $x_0 \lessdot x_1 \lessdot x_2 \lessdot \cdots$ with $x_i \in \mathcal{P}_i$.

We say $\mathcal{P}$ is **finite type** $\mathbb{N}$-graded if $\#\mathcal{P}_i < \infty$ for all i, in which case its **rank generating function** is

$$F(\mathcal{P}; x) = \sum_{i \geq 0} \#\mathcal{P}_i \, x^i = \sum_{p \in \mathcal{P}} x^{\rho(p)},$$

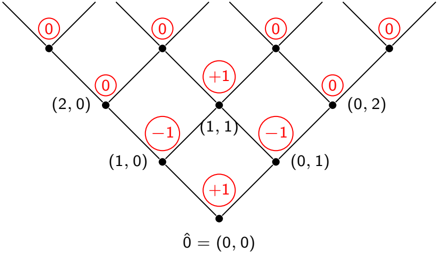
where as before the **rank** of $p \in \mathcal{P}_i$ is $\rho(p) = i$.

If $\mathcal{P}$ has a minimum $\hat{0}$, its **characteristic generating function** is

$$\chi(\mathcal{P}; x) = \sum_{p \in \mathcal{P}} \mu(\hat{0}, p) \, x^{\rho(p)}.$$

Infinite graded poset examples: $\mathbb{N}^2$ and Young's lattice

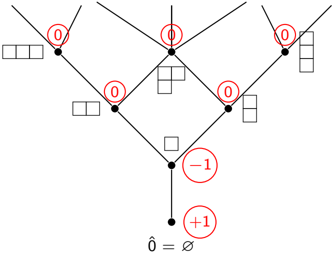
Consider $\mathcal{P} = \mathbb{N}^2$:



$$F(\mathbb{N}^2; x) = \sum_{n \geq 0} (n+1)\, x^n = \frac{1}{(1-x)^2},$$

$$\chi(\mathbb{N}^2; x) = 1 - 2x + x^2 = (1-x)^2.$$

Consider $\mathcal{P} = \mathbb{Y}$, **Young's lattice** of integer partitions:



$$F(\mathbb{Y}; x) = \sum_{n \geq 0} p(n)\, x^n,$$

$$\chi(\mathbb{Y}; x) = 1 - x,$$

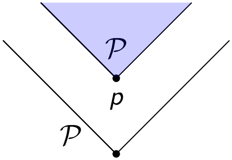
where $p(n) = \#$ partitions $\lambda \vdash n$.

Section 2

Upho posets

Upho posets

A poset $\mathcal{P}$ is **upper homogeneous**, or "**upho**," if for every $p \in \mathcal{P}$ the **principal order filter** $V_p = \{q \colon q \geq p\}$ is isomorphic to whole poset $\mathcal{P}$. Loking up from each $p \in \mathcal{P}$, we see another copy of $\mathcal{P}$:



Examples of upho posets:

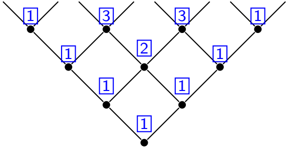
- the natural numbers $\mathbb{N}$, nonnegative rational numbers $\mathbb{Q}_{\geq 0}$, and nonnegative real numbers $\mathbb{R}_{\geq 0}$ (all with their usual total orders);
- the finite subsets of any infinite set X (ordered by inclusion).

Because we want to do combinatorics, from now on all upho posets are assumed **finite type** $\mathbb{N}$**-graded**. Of above, only $\mathbb{N}$ is finite type $\mathbb{N}$-graded.

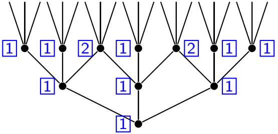
Since $\mathcal{P}$, $\mathcal{Q}$ upho implies $\mathcal{P} \times \mathcal{Q}$ upho, $\mathbb{N}^n$ is upho for any $n \geq 1$.

Aside: planar upho posets and chain counting

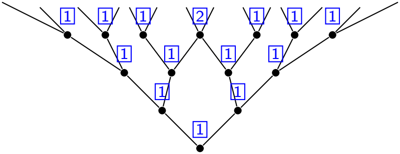
If instead of writing $\mu(\hat{0}, p)$ on each $p \in \mathbb{N}^2$, we write the number of maximal chains from $\hat{0}$ to p, we get **Pascal's triangle**:



Stanley recently introduced upho posets because he was interested in certain analogs of Pascal's triangle coming from other **planar** upho posets:



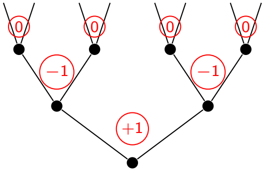
Stern poset



Fibonacci poset

More upho examples: binary tree poset, "necktie" poset

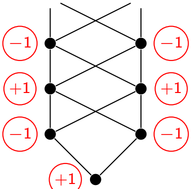
Consider $\mathcal{P}$ = the binary tree poset:



$$F(\mathcal{P};x) = \sum_{n\geq 0} 2^n x^n = \frac{1}{1-2x},$$

$$\chi(\mathcal{P};x) = 1-2x.$$

Consider $\mathcal{P}$ = the "necktie" poset:



$$F(\mathcal{P};x) = 1 + \sum_{n\geq 1} 2\,x^n = \frac{1+x}{1-x},$$

$$\chi(\mathcal{P};x) = 1 + \sum_{n\geq 1} (-1)^n 2\,x^n = \frac{1-x}{1+x}.$$

An **atom** is an element of rank one. These posets have two atoms, like $\mathbb{N}^2$. They have obvious generalizations to any number $r \geq 2$ of atoms, like $\mathbb{N}^r$.

Rank & characteristic generating functions of upho poset

From the examples of rank and characteristic generating functions of upho posets we have seen so far, it is not hard to guess the following:

Theorem (H. 2022)

For any upho poset $\mathcal{P}$, we have $F(\mathcal{P};x) = \chi(\mathcal{P};x)^{-1}$.

Proof.

For $p \in \mathcal{P}$, let $f(p) = \sum_{q \geq p} x^{\rho(q)}$. By Möbius inversion,

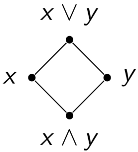
$$1 = x^{\rho(\hat{0})} = \sum_{p \geq \hat{0}} \mu(\hat{0},p) \sum_{q \geq p} x^{\rho(q)} = \sum_{p \in \mathcal{P}} \mu(\hat{0},p)\, x^{\rho(p)}\, F(\mathcal{P};x) = \chi(\mathcal{P};x) \cdot F(\mathcal{P};x),$$

where we used $\sum_{q \geq p} x^{\rho(q)} = x^{\rho(p)}\, F(\mathcal{P};x)$ from the upho-ness of $\mathcal{P}$. $\square$

Unfortunately, Gao–Guo–Seetharaman–Seidel 2022 showed that there are **uncountably many** rank generating functions of upho posets!

Lattices and their Möbius functions

Recall that a poset P is a **lattice** if every pair of elements $x, y \in P$ have a **meet** (greatest lower bound) $x \wedge y$ and a **join** (least upper bound) $x \vee y$:



Lattices have well-behaved Möbius functions:

Theorem (Rota's cross-cut theorem)

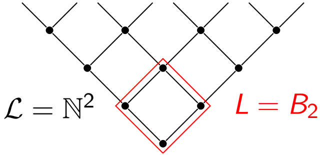
Let L be a finite lattice with minimum $\hat{0}$, maximum $\hat{1}$, and set of atoms S. Then

$$\mu(\hat{0}, \hat{1}) = \sum_{T \subseteq S,\ \bigvee T = \hat{1}} (-1)^{\# T}.$$

In particular, $\mu(\hat{0}, \hat{1}) = 0$ if $\hat{1}$ is not the join of the atoms of L.

Upho lattices and their cores

Let $\mathcal{L}$ be an upho lattice. Let $L = [\hat{0}, s_1 \vee \cdots \vee s_r]$ be the interval from its minimum $\hat{0}$ to the join of its atoms $s_1, \ldots, s_r$, which we call the **core** of $\mathcal{L}$:



Corollary

Let $\mathcal{L}$ be an upho lattice with core L. Then $F(\mathcal{L}; x) = \chi(L; x)^{-1}$.

For example, the core of $\mathbb{N}^n$ is B_n, and $F(\mathbb{N}^n; x) = \frac{1}{(1-x)^n} = \chi(B_n; x)^{-1}$.

Notice that this corollary implies there are only **countably many** rank generating functions of upho lattices, unlike with arbitrary upho poset.

The main question

The core **does not** determine the upho lattice completely. In other words, there are different upho lattices which have the same core.

Nevertheless, a classification of upho lattices must start with an answer to:

Question

Which finite graded lattices L arise as cores of upho lattices?

For example, we saw that the Boolean lattice B_n is a core for any $n \geq 1$.

This can be thought of as a "**tiling**" problem: our goal is to tile an infinite, fractal lattice $\mathcal{L}$ with a given finite lattice L, or show no tiling is possible.

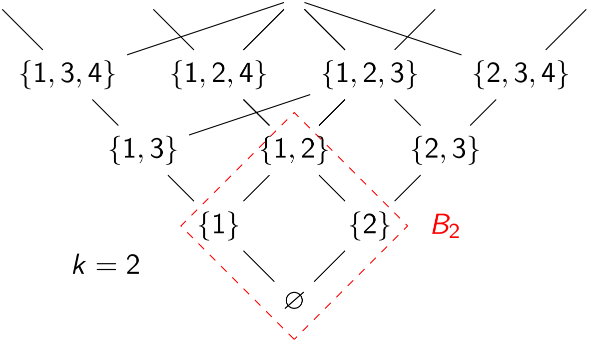
In what remains, we will provide both **positive** and **negative** answers to this question, showing that it is quite subtle.

Section 3

Examples of cores of upho lattices

The Boolean lattice as a core, again

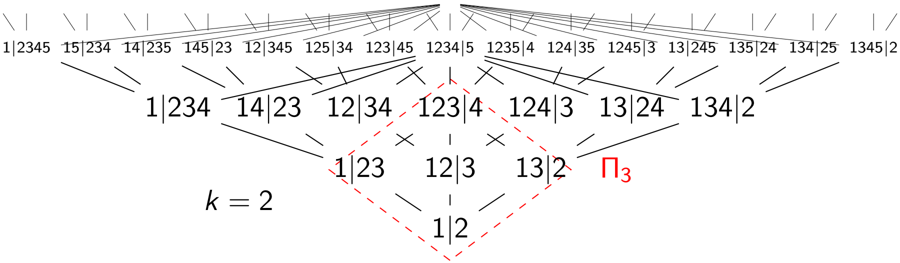
Fix $k \geq 1$ and let $\mathcal{L} = \{\text{finite subsets } A \subseteq \{1, 2, \ldots\} \colon \max(A) < \#A + k\}$ (with $\max(\varnothing) = 0$), ordered by inclusion.



This $\mathcal{L}$ is an upho lattice with core $L = B_k$, but it is **not** isomorphic to $\mathbb{N}^k$. These are the simplest examples of two upho lattices with the same core. Notice that $F(\mathcal{L}; x) = \sum_{n \geq 0} \binom{n+k-1}{n} x^n = \frac{1}{(1-x)^k} = \chi(B_k; x)^{-1}$.

The partition lattice as a core

Fix $k \geq 1$. Let $\mathcal{L}$ be the set partitions of $[n]$ (for any $n \geq k$) into k blocks, ordered by refinement: $\pi \leq \pi'$ if for all $B \in \pi$ there's $B' \in \pi'$ with $B \subseteq B'$.



This $\mathcal{L}$ is an upho lattice with core $L = \Pi_{k+1}$. Notice that

$$F(\mathcal{L}; x) = \sum_{n \geq k} S(n,k) x^{n-k} = \frac{1}{(1-x)(1-2x)\cdots(1-kx)} = \chi(\Pi_{k+1}; x)^{-1}.$$

Uniform sequences of supersolvable geometric lattices

These examples can be generalized. Let $L_0, L_1, \ldots$ be a **uniform sequence** of **supersolvable geometric lattices**, such as $L_n = B_n$ or $L_n = \Pi_{n+1}$. Think: a sequence of graded lattices "nicely embedded in one another."

The **Whitney numbers** of **2nd/1st kind** for these L_n are

$$F(L_i; x) = \sum_{j=0}^{i} V(i,j) x^{i-j} \quad \text{and} \quad \chi(L_i; x) = \sum_{j=0}^{i} v(i,j) x^{i-j}$$

Theorem (Dowling 1973, Stanley 1974)

$$V(i,j) = h_{i-j}(a_1, \ldots, a_{j+1}) \quad \text{and} \quad v(i,j) = (-1)^{i-j} e_{i-j}(a_1, \ldots, a_i),$$

where h_k and e_k denote the complete homogeneous and elementary symmetric polynomials, and $a_i = \#\{\text{atoms } s \in L_i\} - \#\{\text{atoms } s \in L_{i-1}\}$.

In particular, $\chi(L_n; x) = (1 - a_1 x)(1 - a_2 x) \cdots (1 - a_n x)$.

Upho lattices as limits of sequences of finite lattices

We have rank-preserving embeddings $\iota_i \colon L_i \to L_{i+1}$, so let $\mathcal{L}_\infty = \bigcup_{n=1}^{\infty} L_n$, and for any $k \geq 1$, let $\mathcal{L}_\infty^{(k)} = \{p \in \mathcal{L}_\infty \colon \min\{n \colon p \in L_n\} < \rho(p) + k\}$. With $L_n = B_n$ or $L_n = \Pi_{n+1}$, these give the $\mathcal{L}$ we saw before.

Theorem (H. 2024)

For each $k \geq 1$, $\mathcal{L}_\infty^{(k)}$ is an upho lattice with core L_k.

Beyond the Boolean and partition lattices, other examples of uniform sequences of supersolvable geometric lattices are:

- $L_n =$ **lattice of subspaces** of $\mathbb{F}_q^n$, for any prime power q;
- $L_n =$ intersection lattice of **Type** B_n **Coxeter arrangement**;
- $L_n = Q_n(G)$, the **Dowling lattice** associated to any finite group G.

So, these finite graded lattices are all cores.

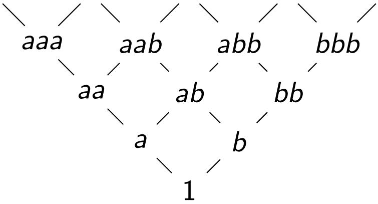
Monoid basics

A **monoid** is a set M with an associative product $\cdot$ and identity element 1.

The **free monoid** on a set S is the collection of words over alphabet S, with product concatenation and identity the empty word.

A **presentation** of a monoid M is a way of writing $M = \langle S \mid R \rangle$ as the quotient of the free monoid on S by the relations in R. We want M having S **finite** and R **homogeneous** (relations equate words of the same length).

For example, consider $M = \langle a, b \mid ab = ba \rangle$:



Here we depict the partial order $\leq_L$ of **left divisibility** on M: $x \leq_L y$ if there is some $z \in M$ such that $xz = y$.

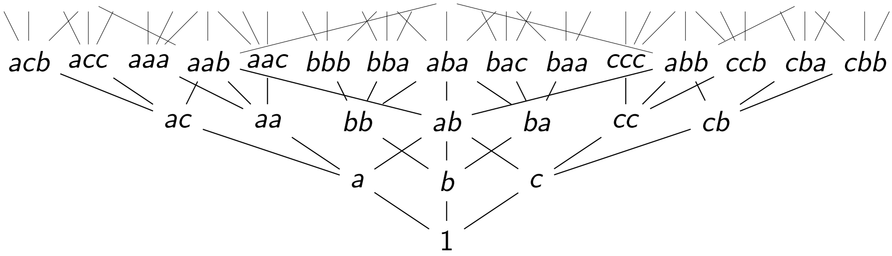
Upho lattices from monoids

M is **left cancellative** if $xy = xz$ implies $y = z$ for every $x, y, z \in M$.

Lemma (c.f. Gao et al. 2022)

Let $M = \langle S \mid R \rangle$ be a homogeneously finitely generated monoid. If M is left cancellative, then $\mathcal{L} = (M, \leq_L)$ is an upho poset. If moreover every $x, y \in M$ have a least common right multiple, then $\mathcal{L}$ is an upho lattice.

For example, this lemma applies to $M = \langle a, b, c \mid ab = bc = ca \rangle$:



Garside monoids and Coxeter groups

Garside monoids are both left and right cancellative, and have both left and right least common multiples for every pair of elements.

The major examples of Garside monoids come from finite Coxeter groups. A **Coxeter group** has presentation $W = \langle s_1, \ldots, s_r \colon s_i^2 = 1 = (s_i s_j)^{m_{ij}} \rangle$, like the **symmetric group** S_n with adjacent transpositions $s_i = (i, i+1)$.

Now fix a finite Coxeter group W.

The **classical braid monoid** $M = \langle s_1, \ldots, s_r \colon \overbrace{s_i s_j s_i \cdots}^{m_{ij}} = \overbrace{s_j s_i s_j \cdots}^{m_{ij}} \rangle$ is a Garside monoid. It gives an upho lattice with core the **weak order** of W.

Let $T = \{s_i^w \colon i = 1, \ldots, r, w \in W\}$, where $g^h = h^{-1}gh$ is conjugation. The **dual braid monoid** $M = \langle T \colon ts = st^s \rangle$ is a Garside monoid as well. It gives an upho lattice with core the **noncrossing partition lattice** of W. The example on the last slide was the dual braid monoid of $W = S_3$.

Section 4

Non-examples of cores of upho lattices

Core obstructions: characteristic polynomial and structural

How can we show that a finite graded lattice L **cannot** arise as a core?

There are restrictions on the characteristic polynomials of cores:

Lemma

If L is the core of an upho lattice, all coefficients of $\chi(L;x)^{-1}$ are positive.

This follows immediately from the fact that $\chi(L;x)^{-1} = F(\mathcal{L};x)$ for a core.

There are also structural obstructions to being a core:

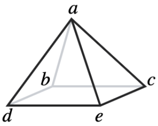
Lemma

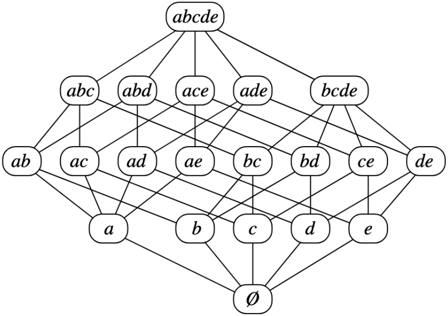
Let L be the core of an upho lattice. Let $x \in L \setminus \{\hat{0},\hat{1}\}$ and let $y_1,\ldots,y_k$ be the elements covering x. Then there are atoms $s_1,\ldots,s_k$ of L for which the interval $[x, y_1 \vee \cdots \vee y_k]$ is isomorphic to $[\hat{0}, s_1 \vee \cdots \vee s_k]$.

This says L must already be "partly self-similar" for it to be a core.

Face lattices of polytopes

Let P be a (convex) **polytope**. The **face lattice** $L(P)$ is the poset of faces of P ordered by containment. It is always a finite graded lattice.

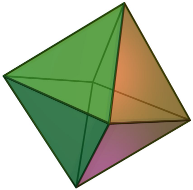


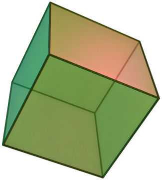


If P is an n-dimensional **simplex**, then $L(P) = B_{n+1}$, which we know is a core. So we can ask: which other face lattices of polytopes are cores?

Face lattices that are not cores

Let P be the **octahedron**. Then $\chi(L(P);x) = 1 - 6x + 12x^2 - 8x^3 + x^4$ and $[x^{13}]\chi(L(P);x)^{-1} = -123704$, where $[x^n]F(x)$ means the coefficient of x^n in the power series $F(x)$. So its face lattice $L(P)$ is not a core.



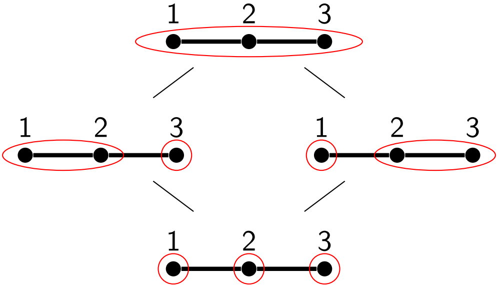


More generally, it can be shown using the structural obstruction that for any $n \geq 3$, $L(P)$ is not a core for P the n-dimensional **cross polytope**, which is the convex hull of all permutations of $(\pm 1, 0, \ldots, 0) \in \mathbb{R}^n$.

Similarly, it can be shown that for any $n \geq 3$, $L(P)$ is not a core when P is the n-dimensional **hypercube**, which is the dual to the cross polytope.

Bond lattices of graphs

Let G be a connected, simple **graph** on vertex set $[n]$. A partition π of $[n]$ is G-**connected** if the restriction of G to each block of π is connected. The **bond lattice** $L(G)$ is the restriction of Π_n to G-connected partitions.

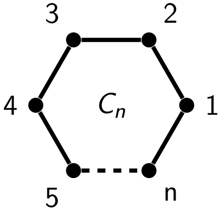


The bond lattice of a graph is always a finite graded lattice, and the **chromatic polynomial** of G is $\chi(G;x) = x^n \cdot \chi(L(G);x^{-1})$.

If $G = K_n$ is the **complete graph**, then $L(G;x) = \Pi_n$, which we know is a core. So we can ask: which other bond lattices of graphs are cores?

Bond lattices that are not cores

Consider a **cycle graph** C_4 on 4 vertices: $\chi(L(C_4); x) = 1 - 4x + 6x^2 - 3x^3$ and $[x^7]\chi(L(C_4); x)^{-1} = -80$. So the bond lattice $L(C_4)$ is not a core.



It can be shown using the structural obstruction that for any $n \geq 4$, the bond lattice $L(C_n; x)$ of the cycle graph C_n on n vertices is not a core.

Even more generally, it can be shown that the **lattice of flats** of the **uniform matroid** $U(n, k)$ is not a core for any $2 < k < n$.

Section 5

Further directions

Number of ways to realize a core

A natural question suggested by our investigation is:

Question

For a finite lattice L, let $\kappa(L)$ be the number of upho lattices with core L. How does $\kappa(L)$ behave?

In work in progress joint with Joel Lewis we are pursuing this question.

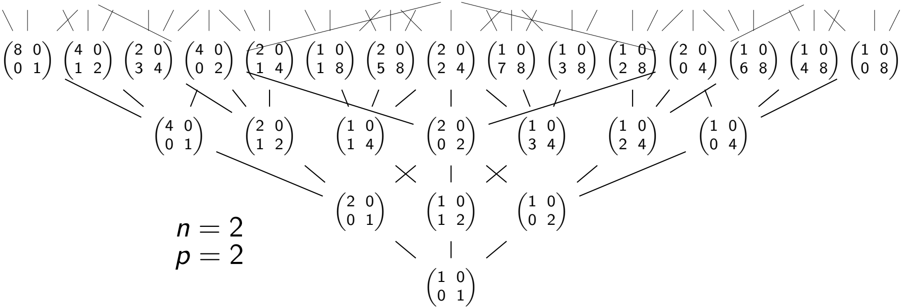
On the one hand, we can show that $\kappa(L)$ is **finite** if L has no nontrivial automorphisms, suggesting it may be finite for all L.

On the other hand, we can show that $\kappa(L)$ is **unbounded** even when restricted to lattices of rank two.

Distributive and modular upho lattices

It is easy to show that the only **distributive** upho lattices are $\mathbb{N}^n$ for $n \geq 1$.

Modular upho lattices are more interesting. Fix $n \geq 1$ and a prime p. Subgroups of $\mathbb{Z}^n$ of index a power of p give a modular upho lattice:



Stanley conjectured that (essentially) all modular upho lattices come from commutative algebra like this example.

Thank you!

these slides are on my website:

`https://www.samuelfhopkins.com/docs/upho_talk.pdf`

and the relevant papers are:

- S. Hopkins. "A note on Möbius functions of upho posets." *Electron. J. Combin.* 29(2), 2022. `arXiv:2202.12103`
- S. Hopkins. "Upho lattices I: examples and non-examples of cores." In preparation, 2024.
- S. Hopkins and J. Lewis. "Upho lattices II: ways of realizing a core." In preparation, 2024.
- S. Hopkins. "Upho lattices III: distributive and modular lattices." In preparation, 2024.