

Math 210 (Modern Algebra I), HW# 2,

Fall 2024; Instructor: Sam Hopkins; Due: Wednesday, September 11th

1. Prove that the only nontrivial groups with no proper subgroups are $\mathbb{Z}/p\mathbb{Z}$ for p a prime.
2. For a positive integer n , the *multiplicative group* $(\mathbb{Z}/n\mathbb{Z})^\times$ consists of those $a \in \mathbb{Z}/n\mathbb{Z}$ satisfying $\gcd(a, n) = 1$ (i.e., coprime to n), with product given by multiplication modulo n . This group is *not* the same as the additive group $\mathbb{Z}/n\mathbb{Z}$: e.g., the identity element in $(\mathbb{Z}/n\mathbb{Z})^\times$ is 1.
Now let p be a prime. Use Lagrange's Theorem for the group $(\mathbb{Z}/p\mathbb{Z})^\times$ to prove *Fermat's Little Theorem*, which states that $a^p \equiv a \pmod{p}$ for all $a \in \mathbb{Z}$.
3. (a) Let G be a group (not necessarily finite!) and $H \leq G$ a subgroup of G with $[G : H] = 2$. Prove that H is a normal subgroup of G .
(b) Give an example of a group G and a subgroup $H \leq G$ with $[G : H] = 3$ such that H is not a normal subgroup of G .
4. Let D_n denote the dihedral group of symmetries of a regular n -gon. Prove that the map $\varphi: D_n \rightarrow \mathbb{Z}/2\mathbb{Z}$ which sends all reflections to 1 and all other elements to 0 is a homomorphism. Explain why the kernel of φ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.
5. Again letting D_n denote the dihedral group, recall that in class we showed that D_n has a presentation $D_n = \langle r, s : r^n = s^2 = (sr)^2 = 1 \rangle$, where r corresponds to clockwise rotation by $\frac{2\pi}{n}$ radians and s corresponds to one of the reflections. Explain why we also have the presentation $D_n = \langle s, t : s^2 = t^2 = (st)^n = 1 \rangle$.