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Longest increasing subsequences

DEFIN Let $\sigma = \sigma_1 \sigma_2 \dots \sigma_n \in S_n$ be a permutation.
A subsequence of σ is $\sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_k}$ for $i_1 < \dots < i_k$
and is increasing if $\sigma_{i_1} < \sigma_{i_2} < \dots < \sigma_{i_k}$.

Let $\text{lis}(\sigma) := \text{length of longest increasing subsequence}$

e.g. For $\sigma = \underline{2} \underline{4} 7 9 \underline{5} 1 3 \underline{6} \underline{8}$ have $\text{lis}(\sigma) = 5$
with longest ~~sub~~ increasing subsequence underlined.

Note: L.I.S. need not be unique: $\underline{1} \quad \underline{2} \quad \underline{4} \quad \underline{3}$

Increasing subsequences are a basic kind of permutation pattern (ask Prof. Burstein for more info...)

Studying LIS's is very natural from point of view of statistical analysis of time series data.

There is a close connection between the Robinson-Schensted Algorithm and longest increasing subsequences:

Thm Suppose $\sigma \xrightarrow{RS} (P, Q)$ w/ $\text{sh}(P) = \lambda = (\lambda_1, \lambda_2, \dots)$.

Then $\lambda_1 = \text{lis}(\sigma)$.

e.g. $\sigma = 5 \underline{2} \underline{3} 6 \underline{4} 1 \underline{7} \xrightarrow{RS} (P = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 7 \\ \hline 2 & 6 & & \\ \hline 5 & & & \\ \hline \end{array}, Q = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 7 \\ \hline 2 & 5 & & \\ \hline 6 & & & \\ \hline \end{array})$

and indeed $\lambda_1 = 4 = \text{lis}(\sigma)$.

But note: 1st row of P (= 1 3 4 7) is not
a LIS of σ (just has same length)

Pf of thm: Suppose $\sigma = p_0, p_1, \dots, p_n = p$ is the sequence of insertion tableaux we build up when inserting $\tau_1, \tau_2, \dots, \tau_n$.

Claim: When inserting τ_k into p_{k-1} , if it enters in the j^{th} column, then the longest increasing subsequence ending at τ_k has length j .

Pf: By induction. The case $k=1$ is fine. So suppose x is entry in p_{k-1} in position $(1, j-1)$ (i.e., left of τ_k). Then by induction there is a subsequence σ' of $\sigma_1, \dots, \sigma_{k-1}$ of length $j-1$ ending at x , and since $x < \tau_k$ (or else we would've bumped it), the concatenation $\sigma' \tau_k$ is a length j increasing subsequence. Similarly, to show there cannot be a longer subsequence, let $y \in \{\sigma_1, \dots, \sigma_{k-1}\}$ be s.t. $y < \tau_k$. By induction, when we inserted y we did so at col. with longest subseq. ending at y , call it j' . Cannot have $j' \geq j$, otherwise we would've inserted τ_k into a later column. So $j' < j$, and so longest inc. subseq. ending at τ_k can have length at most $j' + 1 \leq j$. \checkmark \square

What about the whole shape $\lambda = (\lambda_1, \lambda_2, \dots)$?

Thm (Greene) Suppose $\sigma \mapsto (P, Q)$ w/ $\text{sh}(P) = \lambda$. Then for all k , $\lambda_1 + \lambda_2 + \dots + \lambda_k = \text{length of longest subsequence of } \sigma \text{ that is a union of } k \text{ increasing subsequences.}$

e.g. w/ $\sigma = \underline{2} \underline{4} \underline{7} \underline{9} \underline{5} \underline{1} \underline{3} \underline{6} \underline{8}$ have $P = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 5 & 6 & 8 \\ \hline 2 & 4 & 9 & & \\ \hline 7 & & & & \\ \hline \end{array}$ and $2479 \sqcup 1368$ is a union of 2 increasing subsequences. $k=2$
 $5+3=8 \checkmark$

4/15 Can define decreasing subsequences of perm. σ analogously, and let $lds(\sigma) := \text{length of largest decr. subseq.}$

Thm If $\sigma \mapsto^R (P, Q)$ w/ $sh(P) = \lambda$, then $lds(\sigma) = \ell(\lambda)$
 (length of λ)
 ($= \lambda_1^*$)

In fact, this follows immediately from...

Thm* For $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$ let $\sigma^{rev} = \sigma_n \sigma_{n-1} \dots \sigma_1$. Then if $\sigma \mapsto^R (P, Q)$ have $\sigma^{rev} \mapsto (P', Q')$ where $P' = P^t \leftarrow \text{transpose}$.

To prove this symmetry property of RS, can use column insertion, which works same as (row) insertion, but where we try to put # into 1st column, and bump #'s from i^{th} column to $(i+1)^{\text{th}}$ column, etc.

Key Lemma Row and column insertions commute, i.e., $T \xrightarrow{\text{row}} a \xleftarrow{\text{col}} b = T \xleftarrow{\text{col}} b \xleftarrow{\text{row}} a$.

PS: See Sagan. \square

Pf of thm*: $P' = \sigma_1 \xrightarrow{\text{row}} \dots \sigma_{n-1} \xrightarrow{\text{row}} \sigma_n \xrightarrow{\text{row}} \emptyset$ (1st insertion is same w/ row or col)
 $= \sigma_1 \xrightarrow{\text{row}} \dots \sigma_{n-1} \xrightarrow{\text{row}} \sigma_n \xrightarrow{\text{col}} \emptyset$
 $= \sigma_n \xrightarrow{\text{col}} \sigma_1 \xrightarrow{\text{row}} \dots \sigma_{n-1} \xrightarrow{\text{row}} \emptyset$ (key lemma)
 $= \sigma_n \xrightarrow{\text{col}} \sigma_{n-1} \xrightarrow{\text{col}} \dots \sigma_1 \xrightarrow{\text{col}} \emptyset$ (repeat)
 $= (\sigma_n \xrightarrow{\text{row}} \sigma_{n-1} \xrightarrow{\text{row}} \dots \sigma_1 \xrightarrow{\text{row}} \emptyset)^t$ (transpose of col insert = row insert)
 $= P^t \checkmark$ \square

(or (Erdős-Szekeres Theorem))

For any $\sigma \in S_{(n-1)(m-1)+1}$, have either
 $\text{lis}(\sigma) \geq n$ or $\text{lds}(\sigma) \geq m$.

Pf: Best way to minimize width and length of a partition
is $\lambda =_{\text{maj}} \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$ but we need one more box ✓

Q: What is the expected length of longest incr. subseq.
of a random permutation?

Let $X_n := \text{lis}(\sigma)$ for $\sigma \in S_n$ (uniformly) random.

Ulam's Problem: Compute $\lim_{n \rightarrow \infty} \frac{\mathbb{E} X_n}{\sqrt{n}} = c$.
c. 1960's

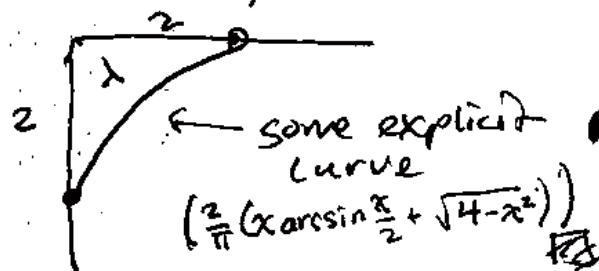
E-S Thm says for any $\sigma \in S_n$, have $\text{lis}(\sigma) \geq \sqrt{n}$ or
 $\text{lis}(\sigma^{\text{rev}}) \geq \sqrt{n}$

so that $c \geq \frac{1}{2}$. In fact...

Thm (Logan-Shepp, Kerov-Vershik, 1977)

Solution to Ulam's Problem is $c = \underline{2}$...

Idea of pf: Same as asking for length of λ when we
insert $\sigma \in S_n$ into RS. In fact, this random
partition λ has
a precise
limit shape
(rescaling by $\frac{1}{\sqrt{n}}$):



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Representation Theory of finite Groups:

In the last couple days, I want to explain why ring of sym. fn.'s is important in algebra.

DEFN Let V be an n -dim'l vector space over \mathbb{C} .
The general linear group $GL(V) = \{ \text{invertible linear maps } V \rightarrow V \}$.
I.e., $GL(V) \cong \{ n \times n \text{ } \mathbb{C}\text{-matrices } M \text{ w/ } \det(M) \neq 0 \}$.

Note: $GL(V)$ is an infinite group.

Let G be a finite group. We want to "represent" G by matrices.

DEFN A representation of G is a group homomorphism
 $\varphi: G \rightarrow GL(V)$ for some v.s. V . In other words,
for each $g \in G$ we have a matrix $\varphi(g)$, and:
• $\varphi(gh) = \varphi(g) \cdot \varphi(h) \quad \forall g, h \in G$,
• $\varphi(e) = I_n$ identity matrix.

A representation of G is very similar to an action,
except it is linear: we act by matrices, not permutations.

e.g. For any V and any G , can set $\varphi(g)(v) = v \quad \forall v \in V$, i.e.,
 $\varphi(g) = I_n$ identity matrix. This is called the trivial
representation and is boring...

e.g. Suppose $G \curvearrowright X$ a finite set. Let $\mathbb{C}[X] := \{ \sum_{x \in X} c_x x : c_x \in \mathbb{C} \}$
be v.s. of formal linear combinations of elements of X .
Then $\mathbb{C}[X]$ is a G representation where $\varphi(g)(x) = g \cdot x$
for all basis vectors $x \in \mathbb{C}[X]$. In other words, each
 $\varphi(g)$ is the permutation matrix of its corresponding permutation.
This is called a permutation representation.

e.g. Let $G = \mathbb{Z}/n\mathbb{Z} = \{0, 1, \dots, n-1\}$. Let $V = \mathbb{C}$.
 We can define a representation $\varphi: G \rightarrow GL(V)$ by
 $\varphi(k) = (e^{2\pi i \cdot k/n}) \times 1 \times 1 \text{ matrix} \quad \forall k = 0, 1, \dots, n-1$.

e.g. Let $G = S_n$ symmetric gp. and let $V = \mathbb{C}$.
 The sign representation $\varphi: S_n \rightarrow GL(\mathbb{C})$ is $\varphi(\sigma) = \begin{pmatrix} \text{sgn}(\sigma) \\ 1 \end{pmatrix}$ ^{1x1 matrix}

e.g. If U, V are G -representations, then direct sum $U \oplus V$
 is another representation; as matrices $\begin{pmatrix} \varphi(g)_U & 0 \\ 0 & \varphi(g)_V \end{pmatrix}$ ^{"block sum"}

DEF'N A repr'n $\varphi: G \rightarrow GL(V)$ is irreducible if we
 cannot find a nontrivial subspace U (i.e., $0 \neq U \neq V$)
 s.t. $gu \in U \quad \forall u \in U, g \in G$ (i.e., invariant under all G).

^{important}
 \hookrightarrow FACT Every representation V of G is a direct sum
 $V = V_1 \oplus V_2 \oplus \dots \oplus V_k$ of irreducible repr's V_i .

e.g. Let $V = \mathbb{C}^n$ w/ standard basis $\{e_1, e_2, \dots, e_n\}$ and $G = S_n$.
 Let $\varphi: S_n \rightarrow GL(V)$ be the standard permutation repr'n,
 i.e. $\varphi(\sigma) e_i = e_{\sigma(i)} \quad \forall \sigma \in S_n, i = 1, \dots, n$. V is reducible,
 since $U_1 = \{c e_1, c_2 e_2, \dots, c_n e_n : c_i \in \mathbb{C}\}$ is a nontrivial invariant subspace.
 With $U_0 = \{(x_1, \dots, x_n) \in V : x_1 + \dots + x_n = 0\}$, we have
 $V = U_1 \oplus U_0$ and U_1, U_0 are irreducible repr's,
^{trivial repr}

The FACT above says that to understand all G -repr's,
 it's enough to understand the irreducible ones...

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Characters of representations

Representations $\rho: G \rightarrow GL(V)$ are matrix-valued functions, hence complicated to understand. It turns out we can "reduce" to studying "ordinary" \mathbb{C} -valued fns $\chi: G \rightarrow \mathbb{C}$.

DEFIN Let ρ be a representation of finite group G .

Its character $\chi_\rho: G \rightarrow \mathbb{C}$ is the function

$$\chi_\rho(g) = \text{Tr}(\rho(g)) \leftarrow \text{trace of matrix} \quad \text{for all } g \in G.$$

e.g. If V is 1-dim \mathbb{C} , then ρ and χ_ρ are the same thing...

e.g. If ρ is the permutation repr'n of an action $G \curvearrowright X$ then $\chi_\rho(g) = \# \text{Fix}(g: X \rightarrow X) \leftarrow \text{why? think abt. perm. matrix.}$

FACT For two G -reps $\rho_1: G \rightarrow GL(V_1)$, $\rho_2: G \rightarrow GL(V_2)$

have $\chi_{\rho_1} = \chi_{\rho_2} \iff \rho_1$ isomorphic to ρ_2

($\rho_1 \cong \rho_2$ means \exists v.s. iso. $V_1 \cong V_2$ that commutes w/ G -action)

Upshot: enough to study characters, in fact, since we have $\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$, enough to study characters of irreducible reps (+ their lin. comb's).

In fact, characters χ are not just any kind of function $G \rightarrow \mathbb{C}$...

DEFIN A conjugacy class of G is set of the form

$$C = \{ghg^{-1} : g \in G\} \text{ for some } h \in G. \text{ A function}$$

$f: G \rightarrow \mathbb{C}$ is called a class function if it is constant on conjugacy classes, i.e. $f(h) = f(ghg^{-1}) \forall g, h \in G$.

Let $\mathcal{C}\ell(G) :=$ v.s. of class functions $f: G \rightarrow \mathbb{C}$.

Prop. Any character χ_ψ is a class function.

PS: $\chi_\psi(g h g^{-1}) = \text{Tr}(g h g^{-1}) = \text{Tr}(g^{-1} \cdot g h) = \text{Tr}(h)$
recall $\text{Tr}(AB) = \text{Tr}(BA)$ for matrices A, B \square

FACT 1. $\{\chi_{\psi_1}, \dots, \chi_{\psi_k}\}$ is a basis of $\text{Cl}(G)$, where ψ_1, \dots, ψ_k are ^{all} the irrep's of G (up to iso.).

2. With the inner product $\langle, \rangle : \text{Cl}(G) \times \text{Cl}(G) \rightarrow \mathbb{C}$
given by $\langle f, f' \rangle := \frac{1}{\#G} \sum_{g \in G} f(g) \overline{f'(g)}$,
the basis $\{\chi_{\psi_1}, \dots, \chi_{\psi_k}\}$ is orthonormal.

3. If $\psi = \bigoplus_m c_m \psi_m$ is decomposition of ψ into irrep's,
then $c_m = \langle \chi_\psi, \chi_{\psi_m} \rangle$.

Note in particular that

$$\begin{aligned} \# \text{irreps (irreducible reps)} &= \dim \text{Cl}(G) \\ &= \# \text{conjugacy classes of } G. \end{aligned}$$

e.g. G acts on itself by multiplication on the left, and
corresponding perm. rep. is called the regular repn. $\mathbb{C}[G]$

How does $\mathbb{C}[G]$ decompose into irrep's?

$$\begin{aligned} \langle \chi_{\mathbb{C}[G]}, \chi_{\psi_m} \rangle &= \frac{1}{\#G} \sum_{g \in G} \chi_{\mathbb{C}[G]}(g) \overline{\chi_{\psi_m}(g)} \\ &= \frac{1}{\#G} \sum_{g \in G} \# \text{Fix}(g: G \rightarrow G) \overline{\chi_{\psi_m}(g)} = \sum_{g \in G} \#G \cdot \delta_{g=e} \overline{\chi_{\psi_m}(g)} \\ &= \frac{1}{\#G} \cdot \#G \cdot \overline{\chi_{\psi_m}(e)} = \dim(\psi_m). \end{aligned}$$

Hence

$$\#G = \dim \mathbb{C}[G] = \dim \left(\bigoplus_m \dim(\psi_m) \cdot \psi_m \right) = \sum_m (\dim \psi_m)^2$$

looks familiar...

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Characters of the Symmetric Group

Finally, by focusing on case $G = S_n$, we see symmetric functions.

Prop. Two permutations $\sigma, \sigma' \in S_n$ belong to same conjugacy class \Leftrightarrow they have the same cycle structure.

Pf: Exercise for you. □

So # conj. classes in $S_n = \#$ cycle structures $= \#$ partitions $\lambda \vdash n$

So # irrep's of $S_n = \# \lambda \vdash n$, and in fact there is a standard way to index irrep's by partitions.

e.g. Let $\text{triv}: S_n \rightarrow GL(\mathbb{C})$ be the trivial rep'n. Then

$$\text{triv} = \psi_{\boxed{1^n}} = \psi_{(1^n)}$$

e.g. For $\text{sgn}: S_n \rightarrow GL(\mathbb{C})$ sign rep'n, $\text{sgn} = \psi_{\boxed{1^n}} = \psi_{(1^n)}$.

e.g. Recall standard perm rep'n $\mathbb{C}^n = U_1 \oplus U_0$
triv ↑ irreducible, dim = n-1

then $U_0 = \psi_{\boxed{n-1, 1}} = \psi_{(n-1, 1)}$

Write $\chi_\lambda = \chi_{\psi_\lambda} =$ character of irrep indexed by $\lambda \vdash n$.

DEF'N The Frobenius characteristic $\text{Fr}: \text{Cl}(S_n) \rightarrow \text{Sym}(n)$

is given by $\text{Fr}(\delta_\lambda) = p_\lambda \leftarrow$ power sum

recall =
sym. fn's
of degree n

where δ_λ is class function $\delta_\lambda(\sigma) = \begin{cases} Z_\lambda & \text{if cycle type}(\sigma) = \lambda \\ 0 & \text{otherwise} \end{cases}$

and $Z_\lambda = \frac{n!}{1^{m_1} 1! \cdot 2^{m_2} 2! \cdot \dots} = \#$ perm's in S_n w/ cycle type $= \lambda = (1^{m_1} 2^{m_2} \dots)$.

Since the δ_λ are a basis of $\text{Cl}(S_n)$ and p_λ are a basis of $\text{Sym}(n)$, this is clearly a v.s. isomorphism.

Thm $Fr(X_\lambda) = S_\lambda \leftarrow$ Schur function.

This is (one reason) why Schur fn's are so important!

Cor $\dim \varphi_\lambda = f^\lambda = \# \text{SYT of sh. } \lambda$

Pf: Via Fr , same as coeff. of $[x_1 x_2 \dots x_n]$ in $S_\lambda = f^\lambda$ ✓ □

More generally...

Cor If $X_\lambda(\mu) = \text{ch. evaluated at a perm. of cycle type } \mu$,

then $S_\lambda = \sum_{\mu} X_\lambda(\mu) \cdot z_\mu^{-1} p_\mu$,

\exists combinatorial rule for these coeff's, called
the Murnaghan-Nakayama rule.

Also note that... by the regular representation, have

$$n! = \# S_n = \sum_{\lambda \vdash n} \dim(\varphi_\lambda)^2 = \sum_{\lambda \vdash n} (f^\lambda)^2, \quad \text{~~expanding~~}$$

which we saw earlier using R.S. algorithm.

Finally, ~~by~~ using something called the induction product
of representations of $S_k \times S_{n-k} \rightarrow S_n$,

we can get ring structure on $\text{Sym} = \bigoplus \text{Sym}(\mathbb{C})$,

structure constants $S_\lambda \cdot S_\mu = \sum_{\nu} c_{\lambda\mu}^\nu S_\nu$ are

called Littlewood-Richardson coefficients,
also very important!

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