Involutions on Dyck paths, and piecewise linear & birational lifts

ACPMS special 1-day seminar on Birational Combinatorics

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based on joint work with Michael Joseph (Dalton State College)

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Section 1

Catalan numbers, Dyck paths, Naryana numbers, and the Lalanne–Kreweras involution

Montserrat Mountain, Catalonia, Spain

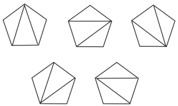
Catalan numbers

The **Catalan numbers** C_n are a famous sequence of numbers

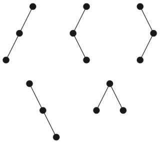
$$1, 2, 5, 14, 42, 132, 429, 1430, \ldots,$$

which count numerous combinatorial collections including:

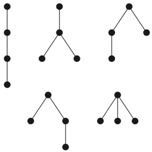
triangulations of an $n+2$-gon



binary trees with n nodes



plane trees with $n+1$ nodes



bracketings of $n+1$ terms

a(b(cd)) a((bc)d)
(ab)(cd) (a(bc))d
((ab)c)d

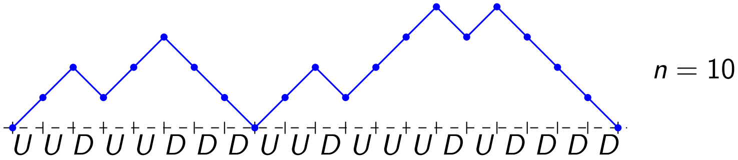
There is a well-known product formula for the Catalan numbers:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Dyck paths

The interpretation of C_n I want to focus on is in terms of Dyck paths.

A **Dyck path** of length $2n$ is a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to $(2n,0)$ consisting of n *up steps* $U=(1,1)$ and n *down steps* $D=(1,-1)$ that never goes below the x-axis:

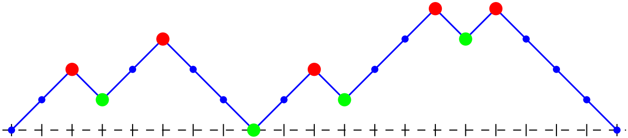


The number of Dyck paths of length $2n$ is C_n:

Peaks and valleys in Dyck paths

Dyck paths look like mountain ranges. So we use some topographic terminology when working with Dyck paths.

A **peak** in a Dyck path is an up step that is immediately followed by a down step; a **valley** is a down step immediately followed by an up step.



Here the peaks are marked by red circles and the valleys by green circles.

It's easy to see that a Dyck path which has k valleys has $k+1$ peaks.

Narayana numbers

The **Narayana number** $N(n,k)$ is the number of Dyck paths of length $2n$ with exactly k valleys.

$n \setminus k$	0	1	2	3
1	1			
2	1	1		
3	1	3	1	
4	1	6	6	1

$\leftarrow$ array of $N(n,k)$

Evidently, the Narayana numbers $N(n,k)$ refine the Catalan number C_n:

$$C_n = \sum_{k=0}^{n-1} N(n,k).$$

They are named after *T.V. Narayana*, who in 1959 showed that

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}.$$

Symmetry of Narayana numbers

From Narayana's formula, it follows immediately that

$$N(n,k) = N(n, n-1-k)$$

for all k. That is, the sequence of Narayana numbers is *symmetric*.

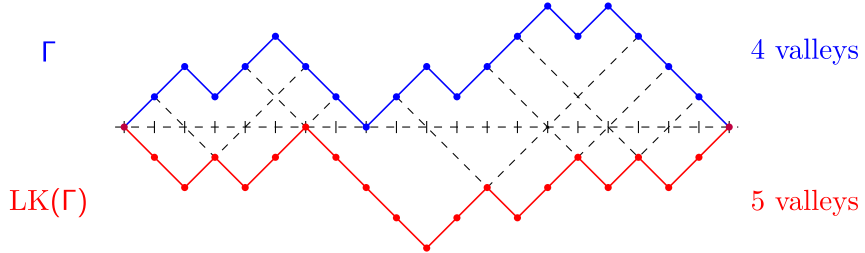
$N(3,0) = 1$ $N(3,1) = 3$ $N(3,2) = 1$

However, it is not combinatorially obvious why the number of Dyck paths with k valleys should be the same as the number with $n-1-k$ valleys.

The Lalanne–Kreweras involution

The **Lalanne–Kreweras involution** is a map on Dyck paths which combinatorially demonstrates the symmetry of the Narayana numbers:

$$\#\text{valleys}(\Gamma) + \#\text{valleys}(\text{LK}(\Gamma)) = n - 1.$$



As depicted above, to compute the LK involution of a Dyck path Γ, we draw dashed lines emanating from the middle of every double up step and every double down step of Γ, at -45° and 45° respectively; these dashed lines intersect at the valleys of (an upside copy of) the Dyck path $\text{LK}(\Gamma)$.

Section 2

Poset description of LK

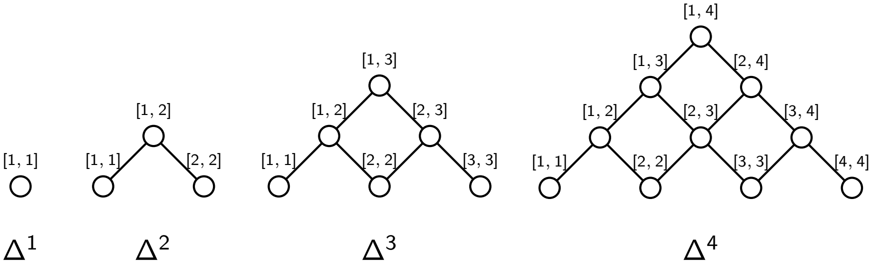
The poset Δ^{n-1}

We can reinterpret the LK involution using a partially ordered set Δ^{n-1}.

Δ^{n-1} is the poset whose elements are **intervals** $[i,j] := \{i, i+1, \ldots, j\}$ with $1 \leq i \leq j \leq n-1$, and with the partial order given by **inclusion**:

$$[i,j] \leq [i',j'] \iff [i,j] \subseteq [i',j'] \iff i' \leq i \leq j \leq j'$$

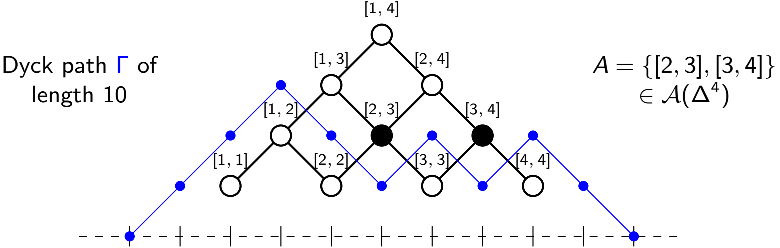
Δ^{n-1} has a "triangular" Hasse diagram:



Dyck paths are antichains in Δ^{n-1}

Recall that an **antichain** $A \subseteq P$ of a poset P is a subset of pairwise incomparable elements. We use $\mathcal{A}(P)$ to denote the set of antichains of P.

The Dyck paths of length $2n$ are in bijection with the antichains of Δ^{n-1}:



The number of valleys of Dyck path Γ is the cardinality of antichain A.

Thus, via this bijection, we can view the LK involution as an involution on antichains $\mathrm{LK}\colon \mathcal{A}(\Delta^{n-1}) \to \mathcal{A}(\Delta^{n-1})$ which satisfies

$$\#A + \#\mathrm{LK}(A) = n - 1.$$

The LK involution on antichains

Panyushev gave a simple description of the LK involution on $\mathcal{A}(\Delta^{n-1})$:

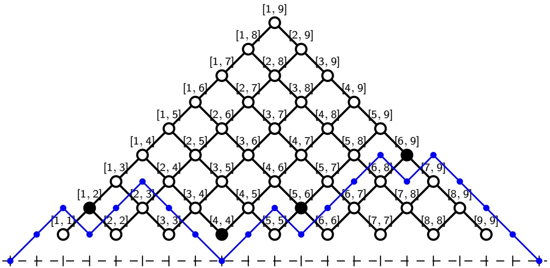
Theorem (Panyushev, 2004)

Let $A = \{[i_1, j_1], [i_2, j_2], \ldots, [i_k, j_k]\} \in \mathcal{A}(\Delta^{n-1})$ *with* $i_1 < i_2 < \cdots < i_k$. *Then* $\mathrm{LK}(A) = \{[i'_1, j'_1], [i'_2, j'_2], \ldots, [i'_{n-1-k}, j'_{n-1-k}]\} \in \mathcal{A}(\Delta^{n-1})$, *where*

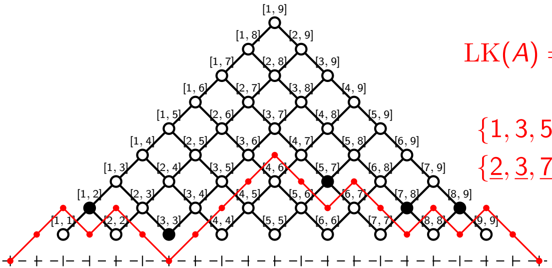
- $\{i'_1 < i'_2 < \cdots < i'_{n-1-k}\} = \{1, 2, \ldots, n-1\} \setminus \{j_1, j_2, \ldots, j_k\}$;
- $\{j'_1 < j'_2 < \cdots < j'_{n-1-k}\} = \{1, 2, \ldots, n-1\} \setminus \{i_1, i_2, \ldots, i_k\}$.

From Panyushev's description, it is immediate that this operation is an involution (i.e., $\mathrm{LK}^2(A) = A$), and that $\#A + \#\mathrm{LK}(A) = n - 1$.

The LK involution on antichains: example



$A = \{[1,\underline{2}],[4,\underline{4}],[5,\underline{6}],[6,\underline{9}]\}$



$\mathrm{LK}(A) = \{[1,\underline{2}],[3,\underline{3}],[5,\underline{7}],[7,\underline{8}],[8,\underline{9}]\}$

$\{1,3,5,7,8\} = \{1,\ldots,9\} \setminus \{\underline{2},\underline{4},\underline{6},\underline{9}\}$

$\{\underline{2},\underline{3},\underline{7},\underline{8},\underline{9}\} = \{1,\ldots,9\} \setminus \{1,4,5,6\}$

Section 3

Toggling

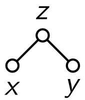
Toggling for antichains

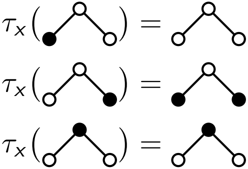
Our first new result gives another expression for the LK involution in terms of certain "local" involutions called **toggles**.

Let P be a poset and $A \in \mathcal{A}(P)$ an antichain. Let $p \in P$ be any element. The **toggle of** p **in** A is the antichain $\tau_p(A) \in \mathcal{A}(P)$, where

$$\tau_p(A) := \begin{cases} A \setminus \{p\} & \text{if } p \in A; \\ A \cup \{p\} & \text{if } p \notin A \text{ and } A \cup \{p\} \text{ remains an antichain;} \\ A & \text{otherwise.} \end{cases}$$

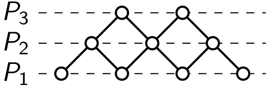
In other words, we "toggle" the status of p in A, if possible:

$P =$ 



Toggling in ranked posets

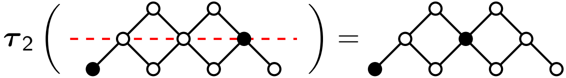
A poset P is **ranked** if we can write $P = P_1 \sqcup P_2 \sqcup \cdots \sqcup P_r$ so that all the edges of the Hasse diagram of P are from P_i (below) to P_{i+1} (above):



Since τ_p and τ_q commute if p and q are incomparable, and all the elements within a rank are incomparable, we can define

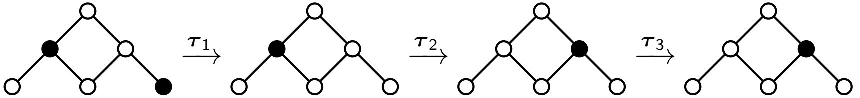
$$\boldsymbol{\tau}_i := \prod_{p \in P_i} \tau_p$$

to be the composition of all toggles at rank i, for $i = 1, \ldots, r$:



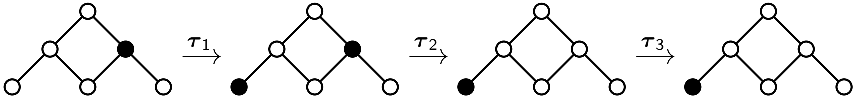
Rowmotion

Rowmotion $\mathrm{Row} := \boldsymbol{\tau}_r \cdots \boldsymbol{\tau}_2 \boldsymbol{\tau}_1 \colon \mathcal{A}(P) \to \mathcal{A}(P)$ is the composition of all rank toggles from bottom to top:



A $\qquad$ $\mathrm{Row}(A)$

Rowmotion has been studied by many authors (Cameron–Fon-Der-Flaass, Striker–Williams, Propp–Roby, etc...) in emerging subfield of **dynamical algebraic combinatorics**. Rowmotion is invertible, but not an involution:



A' $\qquad$ $\mathrm{Row}(A')$

(Actually, $\mathrm{Row} \colon \mathcal{A}(\Delta^{n-1}) \to \mathcal{A}(\Delta^{n-1})$ has order $2n$.)

The LK involution as a composition of toggles

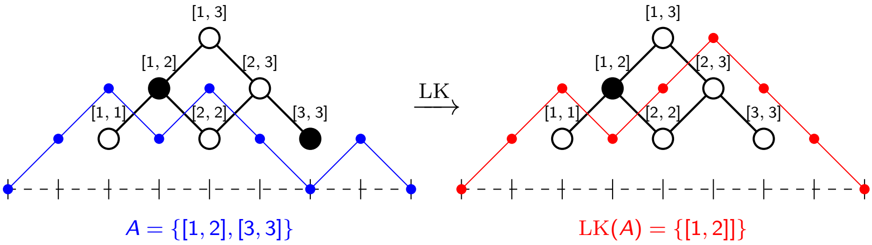
We showed the Lalanne–Kreweras involution can also be written as a composition of rank toggles:

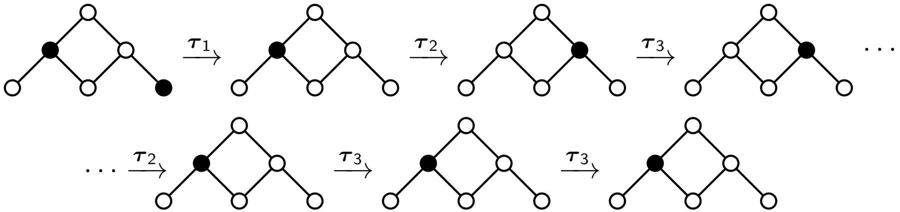
Theorem (H.–Joseph, 2022)

The LK involution $\mathrm{LK}\colon \mathcal{A}(\Delta^{n-1}) \to \mathcal{A}(\Delta^{n-1})$ *can be written as the following composition of toggles:*

$$\mathrm{LK} = (\boldsymbol{\tau}_{n-1})(\boldsymbol{\tau}_{n-1}\boldsymbol{\tau}_{n-2})\cdots(\boldsymbol{\tau}_{n-1}\cdots\boldsymbol{\tau}_3\boldsymbol{\tau}_2)(\boldsymbol{\tau}_{n-1}\cdots\boldsymbol{\tau}_2\boldsymbol{\tau}_1)$$

The LK involution as a composition of toggles: example





Rowvacuation

For any ranked poset P, can define **rowvacuation** $\mathrm{Rvac}\colon \mathcal{A}(P) \to \mathcal{A}(P)$ by same formula: $\mathrm{Rvac} := (\boldsymbol{\tau}_r)(\boldsymbol{\tau}_r\boldsymbol{\tau}_{r-1})\cdots(\boldsymbol{\tau}_r\cdots\boldsymbol{\tau}_3\boldsymbol{\tau}_2)(\boldsymbol{\tau}_r\cdots\boldsymbol{\tau}_2\boldsymbol{\tau}_1)$.

General algebraic properties of the toggles imply:

Proposition

$\langle \mathrm{Row}, \mathrm{Rvac} \rangle$ *gives a dihedral group action on* $\mathcal{A}(P)$*, i.e.,*

- $\mathrm{Rvac} \cdot \mathrm{Row} = \mathrm{Row}^{-1} \cdot \mathrm{Rvac}$*;*
- Rvac *is an involution.*

These names come from Schützenberger's **promotion** and **evacuation** operators acting on the linear extensions of a poset, which can be defined similarly and satisfy analogous properties.

Section 4

Piecewise linear and birational lifts

Lifting combinatorial constructions: overview

Why did we want to write the LK involution as a composition of toggles? In order to **extend** it to the **piecewise linear** and **birational** realms...

A recent trend has been to take some combinatorial construction and realize it as an expression involving $+$ and $-$ and $\min$ and $\max$, and then "de-tropicalize" that PL expression to get a birational transformation.

For example, in 2013, Einstein and Propp introduced **piecewise-linear** and **birational** lifts of **rowmotion**. Remarkably, many theorems lift:

Theorem (Grinberg–Roby, 2015)

The piecewise-linear and birational lifts of $\mathrm{Row}\colon \mathcal{A}(\Delta^{n-1}) \to \mathcal{A}(\Delta^{n-1})$ *still have order* $2n$.

This is surprising, because for other posets these lifts of rowmotion will not even have finite order!

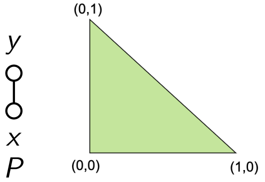
The chain polytope of a poset

In 1986, Richard Stanley associated to any poset P two polytopes in $\mathbb{R}^P$, the **order polytope** $\mathcal{O}(P)$ and the **chain polytope** $\mathcal{C}(P)$.

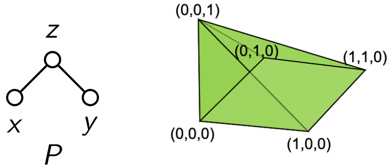
The **chain polytope** $\mathcal{C}(P)$ has **facets**

$$\begin{aligned} 0 \leq x_p & \qquad \forall\, p \in P, \\ \sum_{p \in C} x_p \leq 1 & \qquad \forall \text{ maximal chains } C = \{p_1 < p_2 < \cdots < p_k\} \subseteq P. \end{aligned}$$

Stanley proved that the **vertices** of $\mathcal{C}(P)$ are precisely the indicator functions of **antichains** $A \in \mathcal{A}(P)$:



$\mathcal{C}(P)$



$\mathcal{C}(P)$

Piecewise linear toggling

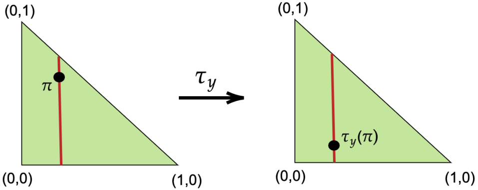
To define the PL extension of rowmotion, Einstein and Propp (c.f. Joseph) introduced a **piecewise linear extension** of the **toggles** τ_p.

For $p \in P$, the **PL toggle** $\tau_p^{\mathrm{PL}} \colon \mathcal{C}(P) \to \mathcal{C}(P)$ is defined by

$$\tau_p^{\mathrm{PL}}(\pi)(q) := \begin{cases} \pi(q) & \text{if } q \neq p; \\ 1 - \max\left\{ \sum_{r \in C} \pi(r) \colon \begin{array}{l} C \subseteq P \text{ a maximal} \\ \text{chain with } p \in C \end{array} \right\} & \text{if } p = q. \end{cases}$$

Restricted to the vertices of the chain polytope $\mathcal{C}(P)$, it is the same as τ_p.

Geometrically, τ_p^{PL} **reflects** π within line segment in $\mathcal{C}(P)$ in direction x_p:



The PL LK involution

As before, for a ranked poset P we use $\boldsymbol{\tau}_i^{\mathrm{PL}} := \prod_{p \in P_i} \tau_p^{\mathrm{PL}}$ to denote the composition of all toggles at rank i.

We define the **PL LK involution** $\mathrm{LK}^{\mathrm{PL}} \colon \mathcal{C}(\Delta^{n-1}) \to \mathcal{C}(\Delta^{n-1})$ to be

$$\mathrm{LK}^{\mathrm{PL}} := (\boldsymbol{\tau}_{n-1}^{\mathrm{PL}})(\boldsymbol{\tau}_{n-1}^{\mathrm{PL}}\boldsymbol{\tau}_{n-2}^{\mathrm{PL}}) \cdots (\boldsymbol{\tau}_{n-1}^{\mathrm{PL}} \cdots \boldsymbol{\tau}_3^{\mathrm{PL}}\boldsymbol{\tau}_2^{\mathrm{PL}})(\boldsymbol{\tau}_{n-1}^{\mathrm{PL}} \cdots \boldsymbol{\tau}_2^{\mathrm{PL}}\boldsymbol{\tau}_1^{\mathrm{PL}})$$

By prior theorem, it's same as LK when restricted to the vertices of $\mathcal{C}(P)$.

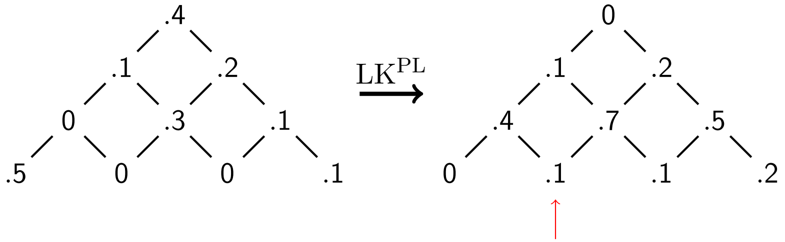
Theorem (H.–Joseph, 2022)

(1) $\mathrm{LK}^{\mathrm{PL}}$ *is an involution.*

(2) *For any* $\pi \in \mathcal{C}(\Delta^{n-1})$, $\sum_{p \in P} \pi(p) + \sum_{p \in P} \mathrm{LK}^{\mathrm{PL}}(\pi)(p) = n - 1$.

Observe that (2) is an extension of the fact that LK combinatorially exhibits the symmetry of the Narayana numbers.

The PL LK involution: example



$.1 = 1 - \max(0+0+.1+.4, 0+.3+.1+.4, 0+.3+.2+.4)$

We can check that

$$(.5+0+0+.1+0+.3+.1+.1+.2+.4)+(0+.1+.1+.2+.4+.7+.5+.1+.2+0) =$$

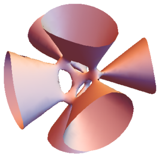
$$1.7 + 2.3 = 4$$

Tropical geometry

Algebraic geometry studies **polynomial** expressions like

$$x^3y + y^3z + z^3x$$

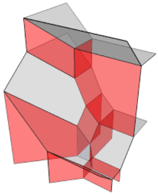
that give "curvy" hypersurfaces



Tropical geometry studies **piecewise linear** expressions like

$$\max(3x + y, 3y + z, 3z + x)$$

that give "flat" polytopal complexes



$$(\times, +) \to (+, \max) = \text{“tropicalization”}$$
$$(+, \max) \to (\times, +) = \text{“de-tropicalization”}$$

Birational toggling

Einstein–Propp (c.f. Joseph–Roby) also introduced a **birational extension** of the **toggles** τ_p (and, using these, rowmotion) via de-tropicalization.

For $p \in P$, the **birational toggle** $\tau_p^{\mathrm{B}} \colon \mathbb{C}^P \dashrightarrow \mathbb{C}^P$ is

$$\tau_p^{\mathrm{B}}(\pi)(q) := \begin{cases} \pi(q) & \text{if } q \neq p; \\ \kappa \cdot \Big(\displaystyle\prod_{\substack{C \subseteq P \\ \text{max. chain,} \\ p \in C}} \sum_{r \in C} \pi(r) \Big)^{-1} & \text{if } p = q, \end{cases}$$

where $\kappa \in \mathbb{C}$ is some fixed constant.

The birational toggle τ_p^{B} tropicalizes to the PL toggle τ_p^{PL}.

The birational LK involution

As before, if P is ranked we set $\boldsymbol{\tau}_i^{\mathrm{B}} := \prod_{p \in P_i} \tau_p^{\mathrm{B}}$.

We define the **birational LK involution** $\mathrm{LK}^{\mathrm{B}} \colon \mathbb{C}^{\Delta^{n-1}} \dashrightarrow \mathbb{C}^{\Delta^{n-1}}$ by

$$\mathrm{LK}^{\mathrm{B}} := (\boldsymbol{\tau}_{n-1}^{\mathrm{B}})(\boldsymbol{\tau}_{n-1}^{\mathrm{B}}\boldsymbol{\tau}_{n-2}^{\mathrm{B}}) \cdots (\boldsymbol{\tau}_{n-1}^{\mathrm{B}} \cdots \boldsymbol{\tau}_3^{\mathrm{B}}\boldsymbol{\tau}_2^{\mathrm{B}})(\boldsymbol{\tau}_{n-1}^{\mathrm{B}} \cdots \boldsymbol{\tau}_2^{\mathrm{B}}\boldsymbol{\tau}_1^{\mathrm{B}})$$

It tropicalizes to $\mathrm{LK}^{\mathrm{PL}}$.

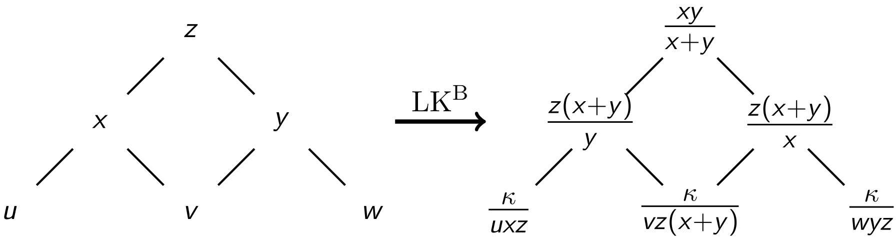
Theorem (H.–Joseph, 2022)

(1) LK^{B} *is an involution.*

(2) *For any* $\pi \in \mathbb{C}^{\Delta^{n-1}}$, $\prod_{p \in P} \pi(p) \cdot \prod_{p \in P} \mathrm{LK}^{\mathrm{B}}(\pi)(p) = \kappa^{n-1}$.

Note that (2) is the birational analog of the fact that LK combinatorially exhibits the symmetry of the Narayana numbers.

The birational LK involution: example

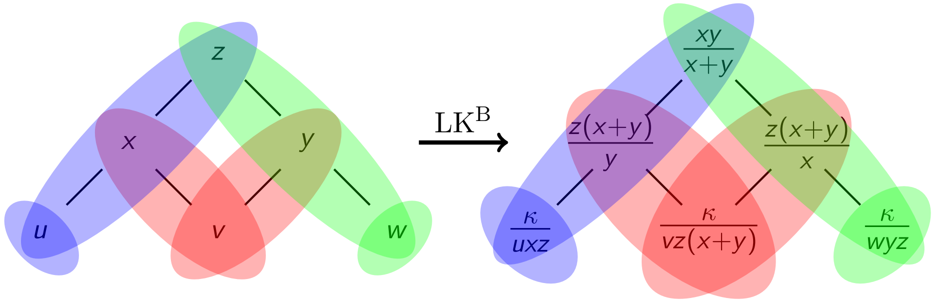


We can check that this operation really is an involution; e.g.,

$$\frac{z'(x'+y')}{y'} = \frac{\frac{xy}{x+y}\cdot\left(\frac{z(x+y)}{y}+\frac{z(x+y)}{x}\right)}{\frac{z(x+y)}{x}} = \frac{zx+zy}{\frac{z(x+y)}{x}} = \frac{z(x+y)}{\frac{z(x+y)}{x}} = x.$$

And if we multiply together all the above values, we get κ^3...

Refined symmetries for birational LK



In fact... if you multiply all the entries shaded by one color (with the apex of the V-shape included twice) you always get κ^2. For red we have:

$$(x \cdot v \cdot v \cdot y) \cdot \left(\frac{z(x+y)}{y} \cdot \frac{\kappa}{vz(x+y)} \cdot \frac{\kappa}{vz(x+y)} \cdot \frac{z(x+y)}{x} \right) = \kappa^2$$

We show that there are always $n-1$ symmetries of LK^{B} like this.

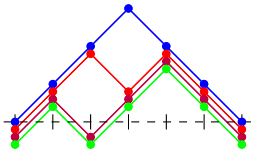
Section 5

Conclusion: so what?

What do the lifts do for us?

(1) **They are more general**
All birational identities tropicalize. But PL identities do *not* always de-tropicalize. So a result proved at the birational level is a strictly stronger result. (Ask Tom and Darij about noncommutative stuff...)

(2) **They imply further combinatorial results**
For any $m \geq 1$, the points in $\frac{1}{m}\mathbb{Z}^{\Delta^{n-1}} \cap \mathcal{C}(\Delta^{n-1})$ correspond to m-tuples of nested Dyck paths:



$n = 3$
$m = 4$
3 valleys

The PL LK involution implies that the generating function over these m-tuples for the (total) number of valleys statistic is still symmetric.

What do the lifts do for us?

(3) **They give new ways of looking at combinatorial constructions**
Writing LK as a composition of toggles leads us to consider this same composition of toggles (i.e., rowvacuation) for other posets.

Δ^{n-1} is the **root poset** of Type A_{n-1}. For any root system Φ, can define Φ-**Narayana numbers** $N(\Phi, k)$ by counting antichains in the root poset Φ^+, and they are again *symmetric*: $N(\Phi, k) = N(\Phi, r - k)$.

Theorem (Defant-H., 2021)

For a root system Φ of classical type A, B, C, or D, rowvacuation is an involution on $\mathcal{A}(\Phi^+)$ which combinatorially exhibits the symmetry of the Φ-Narayana numbers.

Unfortunately, this *fails* for exceptional root systems!

What do the lifts do for us?

(4) **They suggest connections to algebra**
Birational rowmotion has been related to the *Zamolodchikov Periodicity Conjecture*, *Cluster Algebras*, *Geometric Crystals* and *Geometric RSK*, et cetera. So far I don't know of any fancy algebraic connections like this for rowvacuation, but there might be some...

(5) **They give potentially interesting algebro-geometric things**
This is more speculative, but... birational lifts of combinatorial constructions give interesting birational endomorphisms $\mathbb{C}^N \dashrightarrow \mathbb{C}^N$ (of finite order). Could be worth looking at the variety of **fixed points**. See also: our conjectural polynomial **invariants** of birational LK!

Thank you!

these slides are on the conference website
and the paper is at `https://doi.org/10.5802/alco.201`

Exercises 231

6.24. [?] Explain the significance of the following sequence:

un, dos, tres, quatre, cinc, sis, set, vuit, nou, deu, . . .

R. Stanley, *Enumerative Combinatorics*, Vol. 2