

# Final Exam Study Guide

## Math 157 (Calculus II), Spring 2023

### 1. Geometric applications of integrals [§6.1, 6.2, 6.3, 8.1, 8.2]

- (a) Area between curves [§6.1]: area between  $y = f(x)$  and  $y = g(x)$  is  $\int_a^b |f(x) - g(x)| dx$ .
- (b) Volume of general solid [§6.2]: if  $A(x)$  = area of cross-section, then volume is  $\int_a^b A(x) dx$ .
- (c) Volume of solid of revolution [§6.2, 6.3]: “disks/washers” & “cylindrical shells” methods.  
For region below curve  $y = f(x)$  from  $x = a$  to  $x = b$ :
  - i. rotated around  $x$ -axis, “disks method” gives volume  $= \int_a^b \pi f(x)^2 dx$ ;
  - ii. rotated around  $y$ -axis, “shells method” gives volume  $= \int_a^b 2\pi f(x) x dx$ .
- (d) Arc lengths of curves [§8.1]: length of  $y = f(x)$  from  $x = a$  to  $x = b$  is  $\int_a^b \sqrt{1 + (f'(x))^2} dx$ .
- (e) Area of surface of revolution [§8.2]:
  - i. for  $y = f(x)$  from  $x = a$  to  $x = b$  rotated about  $x$ -axis, area is  $\int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx$ ;
  - ii. for  $x = g(y)$  from  $y = c$  to  $y = d$  rotated about  $x$ -axis, area is  $\int_c^d 2\pi y \sqrt{1 + (g'(y))^2} dy$ .

### 2. Other applications of integrals [§6.4, 6.5, 8.3]

- (a) Work [§6.4]: if  $F(x)$  = force as function of distance, then work done is  $W = \int_a^b F(x) dx$ .
- (b) Average of function [§6.5]: the average of  $f(x)$  from  $x = a$  to  $x = b$  is  $\frac{1}{b-a} \int_a^b f(x) dx$ .
- (c) Center of mass [§8.3]: centroid of region below curve  $y = f(x)$  located at  $(\bar{x}, \bar{y})$  where  $\bar{x} = \frac{1}{A} \int_a^b x f(x) dx$ ,  $\bar{y} = \frac{1}{A} \int_a^b \frac{1}{2} (f(x))^2 dx$ , and  $A = \int_a^b f(x) dx$  is area of the region.

### 3. Techniques for computing integrals [§7.1, 7.2, 7.3, 7.4, 7.5]

- (a) Integration by parts [§7.1]:  $\int u dv = uv - \int v du$ ; choose  $u$  using “LIATE” rule
- (b) Trigonometric integrals [§7.2]: for  $\int \sin^n(x) \cos^m(x) dx$ , use the Pythagorean identity  $\sin^2(x) + \cos^2(x) = 1$  to isolate single factor of  $\cos(x) dx$  or  $\sin(x) dx$ , then do a  $u$ -sub.
- (c) Trigonometric substitution [§7.3]:
  - i. for  $a^2 - x^2 \Rightarrow$  sub  $x = a \sin(\theta)$ ,  $dx = a \cos(\theta) d\theta$ , and use  $1 - \sin^2(\theta) = \cos^2(\theta)$ ;
  - ii. for  $a^2 + x^2 \Rightarrow$  sub  $x = a \tan(\theta)$ ,  $dx = a \sec^2(\theta) d\theta$ , and use  $1 + \tan^2(\theta) = \sec^2(\theta)$ .
- (d) Integrating rational functions by partial fractions [§7.4]: find roots of denominator  $Q(x)$  and solve system of equations to write  $P(x)/Q(x) = A/(x-a) + B/(x-b) + \dots + Z/(x-z)$  and use  $\int A/(x-a) dx = A \ln(x-a)$ ; for repeated roots do  $A_1/(x-a) + A_2/(x-a)^2 + \dots$ .

### 4. Other concepts related to integration [§7.7, 7.8]

- (a) Approximating definite integrals [§7.7]: two good approximations of  $\int_a^b f(x) dx$  are
  - i. midpoint approximation  $M_n = \sum_{i=1}^n f(\bar{x}_i) \Delta x$  where  $\bar{x}_i = \frac{x_{i-1} + x_i}{2}$ ;
  - ii. trapezoid approximation  $T_n = \frac{\Delta x}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n))$ .
- (b) Improper integrals [§7.8]:  $\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$ , et cetera.

5. Parametrized curves [§10.1, 10.2]

- (a) Curve of form  $x = f(t)$  and  $y = g(t)$  for some auxiliary variable  $t$  (“time”) [§10.1]
- (b) Slope of tangent to curve given by chain rule [§10.2]:  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$
- (c) Arc length [§10.2] is  $\int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} dt = \int_a^b \sqrt{g'(t)^2 + f'(t)^2} dt$

6. Polar coordinates and polar curves [§10.3, 10.4]

- (a) Cartesian vs. polar [§10.3]:  $(x, y) = (r \cos \theta, r \sin \theta)$  and  $(r, \theta) = (\sqrt{x^2 + y^2}, \arctan(\frac{y}{x}))$
- (b) Area inside [§10.4] polar curve  $r = f(\theta)$  for  $\alpha \leq \theta \leq \beta$  is  $\int_\alpha^\beta \frac{1}{2} r^2 d\theta = \int_\alpha^\beta \frac{1}{2} f(\theta)^2 d\theta$
- (c) Arc length [§10.4] of polar curve is  $\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta$

7. Sequences and series [§11.1, 11.2, 11.3, 11.4, 11.5, 11.6, 11.7]

- (a) Sequence  $\{a_n\}_{n=1}^\infty = a_1, a_2, \dots$  is list of numbers,  $\lim_{n \rightarrow \infty} a_n$  defined like  $\lim_{x \rightarrow \infty} f(x)$  [§11.1]
- (b) Series  $\sum_n a_n$  is “infinite sum”  $a_1 + a_2 + \dots$  of terms  $a_n$ ; its value is  $s = \lim_{n \rightarrow \infty} s_n$  where  $s_n = a_1 + a_2 + \dots + a_n$  is the  $n$ th partial sum [§11.2]
- (c) Important series: geometric series [§11.2]  $\sum_{n=1}^\infty ar^{n-1}$  converges if and only if  $|r| < 1$  (and  $= \frac{a}{1-r}$  if it converges);  $p$ -series [§11.3]  $\sum_{n=1}^\infty \frac{1}{n^p}$  converges if and only if  $p > 1$
- (d) Many tests for convergence / divergence of series:
  - i. (Divergence test [§11.2]) If  $\lim_{n \rightarrow \infty} a_n \neq 0$ , series  $\sum_n a_n$  diverges.
  - ii. (Integral test [§11.3]) If  $f(x)$  continuous, decreasing, and positive, with  $a_n = f(n)$ , then  $\sum_n a_n$  converges if and only if  $\int_1^\infty f(x) dx$  converges. In this case, have error bounds for remainder  $R_n = s - s_n$ :  $\int_{n+1}^\infty f(x) dx \leq R_n \leq \int_n^\infty f(x) dx$ .
  - iii. (Comparison tests [§11.4]) If  $\sum_n b_n$  converges &  $a_n \leq b_n$ , then  $\sum_n a_n$  converges. If  $\sum_n b_n$  diverges &  $a_n \geq b_n$ , then  $\sum_n a_n$  diverges. If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$  exists and is  $\neq 0$ , then  $\sum_n a_n$  converges if and only if  $\sum_n b_n$  converges.
  - iv. (Alternating series test [§11.5]) Alternating series  $\sum_{n=1}^\infty (-1)^{n-1} b_n$  converges as long as  $b_{n+1} \leq b_n$  and  $\lim_{n \rightarrow \infty} b_n = 0$ . In this case, have error bound:  $|R_n| \leq b_{n+1}$ .
  - v. (Ratio test [§11.6]) For series  $\sum_{n=1}^\infty a_n$ , let  $L = \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|}$ . If  $L < 1$ , series converges. If  $L > 1$  (including  $\infty$ ), series diverges. If  $L = 1$ , test is inconclusive.

8. Power series and Taylor series [§11.8, 11.9, 11.10, 11.11]

- (a) The ratio test tells us that any power series  $\sum_{n=0}^\infty c_n(x-a)^n$  has a radius of convergence  $R$  such that it converges when  $|x-a| < R$  and diverges when  $|x-a| > R$  [§11.8]
- (b) Differentiate, integrate, and multiply power series like they are polynomials [§11.9, 11.10]
- (c) Taylor series of  $f(x)$  at  $x = a$  is  $\sum_{n=0}^\infty \frac{f^{(n)}(a)}{n!} (x-a)^n$ , where  $f^{(n)}$  is  $n$ th derivative [§11.10]
- (d) Important Taylor series [§11.10]:  $\frac{1}{1-x} = \sum_{n=0}^\infty x^n$  ( $R = 1$ );  $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$  ( $R = \infty$ );  $\sin(x) = \sum_{n=0}^\infty \frac{(-1)^{n-1} x^{2n+1}}{(2n+1)!}$  ( $R = \infty$ );  $\cos(x) = \sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{(2n)!}$  ( $R = \infty$ )
- (e) Taylor polynomial  $T_n(x)$ :  $n$ th partial sum of series;  $f(x) \approx T_n(x)$  if  $x \approx a$  [§11.10, 11.11]