Models

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Burr state space model

For positive real observations: $y_t \in [0, \infty)$.

$$y_t = \epsilon_t e^{x_t},$$
$$f(\epsilon_t|\eta,\kappa,\lambda) = \eta\kappa\lambda \frac{(\lambda\epsilon_t)^{\eta-1}}{[1+(\lambda\epsilon_t)^{\eta})]^{\kappa+1}}$$

Note that the scale parameter λ is not set to make $E[\epsilon_t] = 1$, as in the exponential, gamma and Weibull state space models, since the Burr mean is not a straightforward function of λ.

Using $\eta = 1.2$, $\kappa = 2.5$, $\lambda = 1.1$ in getting it right example.

Exponential state space model

For positive real observations: $y_t \in [0, \infty)$.

$$y_t = \epsilon_t e^{x_t},$$
$$\epsilon_t \sim \mathrm{Exp}(1)$$
$$f(\epsilon_t) = e^{-\epsilon_t}$$

$E[y_t|x_t] = 1$.

Gamma state space model

For positive real observations: $y_t \in [0, \infty)$.

$$y_t = \epsilon_t e^{x_t},$$
$$\epsilon_t \sim \mathrm{Ga}(\kappa,\kappa)$$
$$f(\epsilon_t|\kappa) = \frac{\Gamma(\kappa)}{\kappa^\kappa}\epsilon_t^{\kappa-1}e^{-\kappa\epsilon_t}$$

$E[y_t|x_t] = 1$.

Using $\kappa = 2$ in the getting it right example.

Gamma Poisson state space model

For non-negative integer observations: $y_t \in \{0, 1, \ldots\}$.

$$y_t \sim \mathrm{NegBin}(r, r/(r + e^{x_t}))$$

We have set the p parameter as $p = r/(r + e^{x_t})$, which makes $E[y_t|x_t] = e^{x_t}$.

Negative binomial interpretation: p is probability of success in independent trials, y_t is number of failures before the r'th success. Interpretation fails for r non-integral, but all values $r > 0$ are possible.

Gamma-Poisson interpretation: draw $\lambda \sim \mathrm{Ga}(r, re^{-x_t})$, draw $y_t \sim \mathrm{Po}(\lambda)$.

Using $r = 10$ in the getting it right example.

Gaussian stochastic volatility model

For real observations: $y_t \in (-\infty, \infty)$.

Given x_t,

$$y_t|x_t \sim \mathrm{N}(0, e^{x_t})$$
$$f(y_t|x_t) = \frac{e^{x_t/2}}{\sqrt{2\pi}} \exp(-\tfrac{1}{2} e^{x_t} y_t^2).$$

Generalized gamma state space model

For positive real observations: $y_t \in [0, \infty)$.

$$f(\epsilon|\lambda, \eta, \kappa) = \frac{\eta}{\Gamma(\kappa)} \lambda^{\eta\kappa} \epsilon^{\eta\kappa - 1} \exp\left[-(\lambda\epsilon)^\eta\right]$$
$$F(\epsilon) = \frac{\gamma(\kappa, (\lambda\epsilon)^\eta)}{\Gamma(\kappa)}$$

where $\gamma(a, b) = \int_0^b t^{a-1} e^{-t} dt$ is the lower incomplete gamma function. with moments given by

$$E[\epsilon^s] = \lambda^{-s} \frac{\Gamma(\kappa + s/\eta)}{\Gamma(\kappa)}.$$

For identification, we fix the mean of the distribution to one by substituting the scale parameter λ by

$$\lambda = \frac{\Gamma(\kappa + 1/\eta)}{\Gamma(\kappa)}$$

in the expression for the density. Hence, the density of the generalized gamma with unit mean is given by

$$p(y_i|x_i; \theta, \vartheta) = \frac{\eta}{\Gamma(\kappa)} \left(\frac{\Gamma(\kappa + 1/\eta)}{\Gamma(\kappa)} \right)^{\eta\kappa} \epsilon^{\eta\kappa - 1} \exp\left[- \left(\epsilon \frac{\Gamma(\kappa + 1/\eta)}{\Gamma(\kappa)} \right)^\eta \right].$$

Mixture of exponentials state space model

For positive real observations: $y_t \in [0, \infty)$.

$$y_t = \epsilon_t e^{x_t},$$

and the distribution $\epsilon_t|x_t$ is a finite mixture of exponentials, not necessarily with a unit mean.

$$f(\epsilon_t|x_t) = \sum_{j=1}^{J} p_j \lambda_j e^{-\lambda_j \epsilon_j}.$$

The p_j are restricted so that $\sum_{j=1}^{J} \pi_j = 1$. $p_j < 0$ is allowed, provided that $f(\epsilon_t|x_t)$ is non-negative for all $\epsilon_t \geq 0$.

Using $p = (0.5, 0.3, 0.2)$ and $\lambda = (1, 2, 4)$ in getting it right example.

Mixture of gammas state space model

For positive real observations: $y_t \in [0, \infty)$.

$$y_t = \epsilon_t e^{x_t},$$

and the distribution $\epsilon_t|x_t$ is a finite mixture of gammas, not necessarily with a unit mean.

$$f(\epsilon_t|x_t) = \sum_{j=1}^{J} p_j \frac{\Gamma(\kappa_j)}{\lambda_j^{\kappa_j}} \epsilon_t^{\kappa_j - 1} e^{-\lambda_j \epsilon_t}$$

The p_j are restricted so that $p_j \geq 0$, $j = 1, \ldots, J$ and $\sum_{j=1}^{J} \pi_j = 1$.

Using $p = (0.5, 0.3, 0.2)$, $\kappa = (1, 2, 4)$ and $\lambda = (2, 3, 4)$ in getting it right example.

Mixture of Gaussians stochastic volatility model

For real observations: $y_t \in (-\infty, \infty)$.

$$y_t = \epsilon_t e^{x_t/2},$$

and the distribution $\epsilon_t|x_t$ is a finite mixture of Gaussians, not necessarily with a unit mean.

$$f(\epsilon_t|x_t) = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{J} p_j \frac{1}{\sigma_j} \exp\left[-\frac{(y_t - \mu_j)^2}{2\sigma_j^2}\right]$$

Using $p = (0.5, 0.3, 0.2)$, $\mu = (0, -1, 1)$ and $\sigma = (1, 2, 2)$ in getting it right example.

Poisson state space model

For non-negative integer observations: $y_t \in \{0, 1, \ldots\}$.

$$y_t|x_t \sim \mathrm{Po}(e^{x_t})$$
$$f(y_t) = \frac{\exp(-e^{x_t}) e^{x_t y_t}}{y_t!}$$

Student's t stochastic volatility model

For real observations: $y_t \in (-\infty, \infty)$.

Given x_t,

$$y_t \exp(-x_t/2) \sim \mathrm{St}(\nu)$$
$$f(y_t|x_t) = \left(\frac{\nu}{\nu + y_t e^{-x_t/2}}\right)^{(\nu+1)/2} e^{-x_t/2}.$$

Using $\nu = 12$ in the getting it right example.

Weibull state space model

For positive real observations: $y_t \in [0, \infty)$.

$$y_t = \epsilon_t e^{x_t},$$
$$\frac{\epsilon_t}{\Gamma(1+\frac{1}{\eta})} \sim \text{Wei}(\eta)$$
$$f(\epsilon_t|x_t) = \frac{\eta}{\Gamma(1+\frac{1}{\eta})}\left(\frac{\epsilon_t}{\Gamma(1+\frac{1}{\eta})}\right)^{\eta-1} \exp(-y_t/\Gamma(1+\tfrac{1}{\eta}))$$

Using $\eta = 2$ in getting it right example.