ON THE PADÉ APPROXIMANTS ASSOCIATED WITH THE CONTINUED FRACTION AND SERIES OF STIELTJES*

BY

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INTRODUCTION

The well known series of Stieltjes is a power series with real coefficients:

$$(1) \qquad \frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} - \cdots$$

such that the persymmetric determinants

$$A_n \equiv \begin{vmatrix} c_0 & , & c_1, & \cdots & , & c_{n-1} \\ c_1 & , & c_2, & \cdots & , & c_n \\ \cdot & & \cdot & & \cdot & \\ c_{n-1}, & & c_n, & \cdots & , & c_{2n-2} \end{vmatrix}, \qquad B_n \equiv \begin{vmatrix} c_1, & c_2 & , & \cdots & , & c_n \\ c_2, & c_3 & , & \cdots & , & c_{n+1} \\ \cdot & & & & \cdot & \\ c_n, & c_{n+1}, & & \cdots & , & c_{2n-1} \end{vmatrix}$$

($n = 1, 2, 3, \cdots, A_0 \equiv 1, B_0 \equiv 1$) are all positive. Stieltjes† connected the series (1) with a continued fraction

$$(2) \qquad \frac{1}{a_1 z} + \frac{1}{a_2} + \frac{1}{a_3 z} + \cdots$$

in which all the a_i are real and positive, and are uniquely determined by the c_i in accordance with the relations

$$(3) \qquad a_{2n} = A_n^2/(B_n B_{n-1}), \qquad a_{2n+1} = B_n^2/(A_n A_{n+1}).$$

Conversely, the positive real numbers a_i ($i = 1, 2, 3, \cdots$) uniquely determine coefficients c_i of a Stieltjes power series. Thus to every Stieltjes continued fraction (2) with positive a_i there corresponds a Stieltjes series (1) and vice versa.

In the work of Stieltjes the convergents of (2) separate into two sets, the odd and the even convergents. When the series‡ $\sum a_i$ is divergent, these

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† *Recherches sur les fractions continues*, Annales de Toulouse, vol. 8, J, pp. 1–122, and vol. 9, A, pp. 1–47, 1894–95. Published also in vol. 32 of the *Mémoires présentés à l'Académie des Sciences de l'Institut National de France.*

‡ Here and henceforth we write $\sum$ in place of $\sum_{i=1}^{\infty}$.

two sets of alternate convergents have one and the same limit which is a function analytic throughout the complex plane, excepting the whole or a part of the negative real axis. When, on the other hand, the series $\sum a_i$ converges, these two sets have distinct limits p_1/q_1 and p/q respectively, which are meromorphic functions having only simple poles all of which are real and non-positive. The numerators and denominators p, p_1, q, q_1 are analytic functions between which there is the relation

$$(4) \qquad\qquad p_1 q - p q_1 = +1.$$

In the case of a convergent Stieltjes series (1) the former alternative is always realized, but for a divergent series either alternative may be realized. Thus for some divergent series the continued fraction of Stieltjes picks out two functional equivalents, while for others only a single equivalent is yielded.

In this article it will be convenient to replace $1/z$ by z in (1) and (2) and then drop the unessential factor z. We then have the Stieltjes series

$$(5) \qquad\qquad c_0 - c_1 z + c_2 z^2 - c_3 z^3 + \cdots$$

with the same inequalities:

$$(6) \qquad\qquad A_n > 0, \quad B_n > 0,$$

while the continued fraction becomes

$$(7) \qquad\qquad \frac{1}{a_1} + \frac{z}{a_2} + \frac{z}{a_3} + \frac{z\cdot}{a_4} + \cdots .$$

When $\sum a_i$ is convergent, the two sets of alternate convergents have limits which are distinct and meromorphic over the entire plane *except at the origin*, while if $\sum a_i$ is divergent both sets have the same limit which is a function analytic over the entire plane except the whole or a part of the negative real axis.

With any power series

$$(8) \qquad\qquad k_0 + k_1 z + k_2 z^2 + \cdots \qquad\qquad (k_0 \neq 0),$$

Padé connects a table of approximants[*]

	$n = 0$	$n = 1$	
$m = 0$	$\dfrac{N_{0,0}}{D_{0,0}} = k_0,$	$\dfrac{N_{0,1}}{D_{0,1}} = k_0 + k_1 z,$	$\cdots$
$m = 1$	$\dfrac{N_{1,0}}{D_{1,0}},$	$\dfrac{N_{1,1}}{D_{1,1}},$	$\cdots$
	$\cdots$	$\cdots$	$\cdots$

[*] Padé, Thesis, published in the Annales de l'Ecole Normale Supérieure, (3). vol 9. supplement, pp. 1–93, 1892.

where $N_{m,n}(z)/D_{m,n}(z)$ is the rational fraction, taken in its *lowest terms*, in which the degrees of numerator and denominator do not exceed n and m respectively, and whose expansion into a power series, $P(z)$, agrees with (8) for a greater number of terms than any other such rational fraction.

If a particular approximant does not appear more than once in the table, that approximant is said to be normal, and when every approximant is normal, the table is called *normal*. The power series $P(z)$ for a normal approximant, in which necessarily the degrees of numerator and denominator are exactly n and m respectively, agrees with the given series term by term up to and including the term of degree $m+n$ and no further.

Professor Van Vleck[*] showed that the series of Stieltjes (5) has a normal Padé table. The odd convergents of the continued fraction (7) fill the principal diagonal of the table, while the even convergents fill the parallel diagonal file immediately below it. Consequently these two diagonal files of approximants will either converge to a common limit analytic throughout the entire plane except over the whole or a part of the negative real axis, or to two distinct functions, meromorphic everywhere except at the origin, according as $\sum a_i$ diverges or converges.

The first main question which I have considered in this article is the following:

1. *What holds regarding the convergence of the other diagonal files parallel to the principal diagonal in the Padé table for the series of Stieltjes?*

I find that the sequence of convergents in any file parallel to the principal diagonal converges to a limit. Three cases arise.

Case I. If the series $\sum a_i$ is convergent, no two files to the right of and parallel to the principal diagonal have the same limit, and no two parallel files below the principal diagonal have the same limit. In this case all the limit functions are meromorphic over the entire plane except at the origin. The poles of the limits of the files to the right of the principal diagonal are all simple and lie upon the negative half of the real axis.

When $\sum a_i$ diverges there are two possibilities which exist, namely:

Case II. The file of approximants in the principal diagonal and the successive parallel files to the right of it up to the nth file inclusive converge to one and the same limit, while subsequent to the nth they converge to distinct limits. Likewise the successive parallel files below the principal diagonal, down to an mth file inclusive, may converge to the same limit, while subsequent to the mth file they converge to distinct limits.

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[*] *On an extension of the* 1894 *memoir of Stieltjes,* these Transactions, vol. 4 (1903), pp. 297–332.

All the limit functions in this case are meromorphic over the entire plane except at the origin.

Case III. Every diagonal file parallel to the principal diagonal converges to one and the same function, which is analytic over the entire plane excepting the whole or a part of the negative half of the real axis.

In Case I, the series (5) diverges, while in Cases II and III it may either diverge or converge. Whereas Stieltjes obtained two distinct functional equivalents of the divergent series (5) in Case I, we now have an infinite set; and whereas Stieltjes obtained a single functional equivalent when $\sum a_i$ diverges, I obtain an infinite number in Case II, and but one in Case III.

Generalizing (1), I shall call any series

$$(9) \qquad\qquad \pm (c_0 z^n - c_1 z^{n+1} + c_2 z^{n+2} - \cdots),$$

in which n is a positive or negative integer or zero, a Stieltjes series if the coefficients c_i satisfy the Stieltjes inequalities (6).

If the first n terms of (5) be removed, the remainder is still a Stieltjes series. Instead of removing terms I have concerned myself with the following question:

2. *Can the series* (5) *be so continued to the left by successive addition of new terms that the extended series shall be Stieltjes series?*

I find that the necessary and sufficient condition for a first extension is that the series $\sum a_{2i}$, with even subscripts, shall converge.

Two cases arise:

Case A. If the series $\sum a_i$ converges, the Stieltjes series (5) may be extended to the left by the addition of an arbitrary number of new terms, forming thereby a series whose coefficients satisfy the Stieltjes inequalities (6). The coefficients of the prefixed terms are not unique.

Case B. If the series $\sum a_i$ diverges and $\sum a_{2i}$ converges, the series (5) may be extended by the addition of a term, $-c_{-1} z^{-1}$, in which c_{-1} may be taken equal to or greater than $\sum a_{2i}$. If $c_{-1} > \sum a_{2i}$, further extension is never possible; but if $c_{-1} = \sum a_{2i}$, the like series $\sum a'_{2i}$ in the new Stieltjes continued fraction belonging to the extended series may or may not converge. If it does converge, a second extension is possible. The coefficient of the new term, $c_{-2} z^{-2}$, prefixed may be taken equal to or greater than $\sum a'_{2i}$. Only if $c_{-2} = \sum a'_{2i}$, can a third term be prefixed, and so on.

When n terms can be prefixed, all the coefficients in the added terms are unique with the exception of c_{-n}. For an infinite extension all the coefficients are unique. There exist series which can be extended to just n terms and others which can be infinitely extended.

Chapter I. The Padé table of approximants

To each pair of non-negative integers (m, n) there corresponds one rational function of the complex variable $z = x + iy$: $N_{m,n}(z)/D_{m,n}(z)$ taken in its *lowest* terms, in which the degrees of numerator and denominator do not exceed n and m respectively, and whose expansion into a series in ascending powers of z agrees term by term with a given series

$$(8) \qquad\qquad k_0 + k_1 z + k_2 z^2 + \cdots \qquad\qquad (k_0 \neq 0)$$

to a higher degree than all other such rational functions. In fact, if we let $N_{m,n}(z) = s_0 + s_1 z + \cdots + s_n z^n$ and $D_{m,n}(z) = t_0 + t_1 z + \cdots + t_m z^m$, the conditions that the series

$$(10) \quad (k_0 + k_1 z + \cdots)(t_0 + t_1 z + \cdots + t_m z^m) - (s_0 + s_1 z + \cdots + s_n z^n)$$

shall begin with the $(m+n)$th power of z are the following:

$$(11) \qquad\qquad k_i t_0 + k_{i-1} t_1 + \cdots + k_{i-m} t_m = s_i$$

$$(i = 0, 1, \cdots, n ; \quad k_j = 0 \quad \text{if} \quad j < 0),$$

$$(12) \qquad\qquad k_{n+i} t_0 + k_{n+i-1} t_1 + \cdots + k_{n+i-m} t_m = 0$$

$$(i = 1, 2, \cdots, m ; \quad k_j = 0 \quad \text{if} \quad j < 0).$$

If we take t_0 equal to the determinant

$$\Delta_{m-1,n} \equiv \begin{vmatrix} k_{n-m+1} & \cdots & k_n \\ \cdot & \cdot & \cdot \\ k_n & \cdots & k_{n+m-1} \end{vmatrix},$$

and assume $\Delta_{m-1,n} \neq 0$, then all the other t_i and also the s_i are determined uniquely, and $N_{m,n}$ and $D_{m,n}$ may be written in the form of determinants:

$$(13) \quad N_{m,n}(z) = \begin{vmatrix} k_{n-m+1} & \cdots & k_n & k_{n-m}z^n + k_{n-m-1}z^{n-1} + \cdots + k_0 z^m \\ \cdot & \cdot & \cdot & \cdot \\ k_{n+1} & \cdots & k_{n+m} & k_n z^n + k_{n-1}z^{n-1} + \cdots + k_0 \end{vmatrix},$$

$$(14) \quad D_{m,n}(z) = \begin{vmatrix} k_{n-m+1} & \cdots & k_n & z^m \\ \cdot & \cdot & \cdot & \cdot \\ k_{n+1} & \cdots & k_{n+m} & 1 \end{vmatrix}.$$

The table of approximants is normal when and only when all the determinants

$$\Delta_{m,n} \equiv \begin{vmatrix} k_{n-m} & \cdots & k_n \\ \cdot & \cdot & \cdot \\ k_n & \cdots & k_{n+m} \end{vmatrix} \qquad \begin{pmatrix} m, n = 0, 1, 2, \cdots \\ k_i = 0, \text{ for } i < 0 \end{pmatrix}$$

are different from zero.* We will henceforth assume the normality of the table.

In a normal table Padé distinguished three types of regular continued fractions. We shall need to use here the *stairlike* type. This is obtained by starting from a border element of the table of approximants, and passing hence alternately one step to the right and one downward or vice versa. If, for example, we start from the nth element of the upper border of the table and proceed hence one step to the right and one down alternately, we get the sequence of approximants

$$(15) \qquad \frac{N_{0,n-1}(z)}{D_{0,n-1}(z)}, \quad \frac{N_{0,n}(z)}{D_{0,n}(z)}, \quad \frac{N_{1,n}(z)}{D_{1,n}(z)}, \quad \frac{N_{1,n+1}(z)}{D_{1,n+1}(z)}, \quad \cdots$$

whose terms are the successive convergents of the continued fraction

$$(16) \qquad k_0 + k_1 z + k_2 z^2 + \cdots + k_{n-1} z^{n-1} + z^n \left[\frac{1}{h_1^{(n)}} + \frac{z}{h_2^{(n)}} + \frac{z}{h_3^{(n)}} + \cdots \right],$$

in which

$$(17) \qquad h_{2i}^{(n)} = - \frac{\Delta_{i-1,i+n-1}^2}{\Delta_{i-1,i+n}\Delta_{i-2,i+n-1}}, \qquad h_{2i+1}^{(n)} = \frac{\Delta_{i-1,i+n}^2}{\Delta_{i-1,i+n-1}\Delta_{i,i+n}}.$$

If we start with the corner element of the table and proceed hence one step downward and one to the right alternately, we get the sequence of approximants

$$(18) \qquad \frac{N_{0,0}(z)}{D_{0,0}(z)}, \quad \frac{N_{1,0}(z)}{D_{1,0}(z)}, \quad \frac{N_{1,1}(z)}{D_{1,1}(z)}, \quad \frac{N_{2,1}(z)}{D_{2,1}(z)}, \quad \cdots,$$

whose terms are the successive convergents of a continued fraction

$$(19) \qquad \frac{1}{a_1} + \frac{z}{a_2} + \frac{z}{a_3} + \frac{z}{a_4} + \cdots,$$

in which

$$a_{2i} = - \frac{\Delta_{i-1,i-1}^2}{\Delta_{i-1,i}\Delta_{i-2,i-1}}, \qquad a_{2i+1} = \frac{\Delta_{i-1,i}^2}{\Delta_{i-1,i-1}\Delta_{i,i}}.$$

It is to be understood that $\Delta_{-1,0}$ is to be taken here and hereafter equal to unity.

If now the continued fraction (16) converges, the two adjacent diagonal files of the table from which (15) is obtained by selecting approximants from each alternately, converge to one and the same limit, namely the limit of (16). On the other hand, if these two diagonal files converge to separate

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* For details, see Padé, Thesis, loc. cit; or O. Perron, *Die Lehre von den Kettenbrüchen*, 1913, Chap. X, pp. 418–465.

limits, the sequences of alternate convergents of (16) must have separate limits.

It is easily seen that not only does the series (8) uniquely determine the continued fraction (19) in the case of a normal table but conversely (19) determines the series. The study of the convergence of the files of approximants parallel to the principal diagonal of the table can therefore be made to depend upon the sequence of numbers a_i in (19). In the next chapter such a study is made for the diagonal files to the right of the principal diagonal of the table in the case of the continued fraction of Stieltjes.

To investigate the convergence of the diagonal files below the principal diagonal which start on the left border of the table, I find it desirable to turn to the Padé table of the reciprocal series

$$(20) \qquad\qquad \delta_0 + \delta_1 z + \delta_2 z^2 + \delta_3 z^3 + \cdots,$$

which is determined by the identity

$$(21) \qquad\qquad (k_0 + k_1 z + \cdots)(\delta_0 + \delta_1 z + \cdots) = 1.$$

The approximant corresponding to (m, n) for this series is the reciprocal of the approximant corresponding to (n, m) for the original series (8); and when, as supposed, the Padé table of (8) is normal, the table of the reciprocal series is also normal, inasmuch as the first $m+n+1$ terms of the latter series are determined by the first $m+n+1$ terms of the former and conversely. Thus the question of the convergence of the diagonal files to the left of the principal diagonal in a normal Padé table may be replaced by the consideration of the convergence of the diagonal files to the right of the principal diagonal in the table of the reciprocal series.

Let now $N'_{n,m}(z)/D'_{n,m}(z)$, $m > n$, be an approximant of the series (20). Then

$$(22) \qquad\qquad D'_{n,m} = w(m, n - m)\, N_{m,n},$$

$$(23) \qquad\qquad N'_{n,m} = w(m, n - m) D_{m,n},$$

where $w(m, n - m)$ is a numerical factor independent of z. By (13), (14) the relation (22) may be written

$$(24) \quad \begin{vmatrix} \delta_{n-m+1} \cdots \delta_n & , & z^n \\ \cdot & \cdot & \cdot & \cdot \\ \delta_{n+1} & \cdots \delta_{n+m}, & z^0 \equiv 1 \end{vmatrix}$$

$$= w(m, n - m) \begin{vmatrix} k_{m-n+1} \cdots k_m, & k_{m-n} z^m + k_{m-n-1} z^{m-1} + \cdots \\ \cdot & \cdot & \cdot & \cdot \\ k_{m+1} & \cdots k_{m+n}, & k_m z^m + k_{m-1} z^{m-1} + \cdots \end{vmatrix},$$

with a similar equation resulting from (23).

On setting $z = 0$, and replacing n by $n+m$ in (24) the latter becomes

$$(25) \qquad \Delta'_{m-1,n+m} = k_0 w(m,n) \cdot \Delta_{m+n-m},$$

where $\Delta'_{m,n}$ denotes the determinant $\Delta_{m,n}$ expressed in terms of the δ_i instead of the k_i.

We will now determine the constant $w(m,n)$.

From the identity (21) we have

$$\delta_0 k_0 = 1,$$

$$\delta_1 k_0 + \delta_0 k_1 = 0,$$

$$(26)$$

$$\qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$\delta_n k_0 + \delta_{n-1} k_1 + \cdots + \delta_0 k_n = 0,$$

which gives

$$(27) \qquad \delta_n = (-1)^{n(n+1)/2} \cdot \frac{1}{k_0^{n+1}} \cdot \Delta_{n-1,1}.$$

Also, setting $m = 1$ and replacing n by $n-1$ in (25) we have

$$(28) \qquad \delta_n = k_0 w(1, n-1)\Delta_{n-1,1},$$

which with (27) gives

$$(29) \qquad w(1,n) = (-1)^{(n+1)(n+2)/2} \cdot \frac{1}{k_0^{n+3}}.$$

I shall now prove by mathematical induction that[*]

$$(30) \qquad w(m,n) = (-1)^{(n+1)(n+2)/2+m-1} \cdot \frac{1}{k_0^{2m+n+1}}.$$

We see from (29) that the value of $w(m, n)$ given by (30) is correct for all values of n if $m = 1$.

I shall assume the correctness of formula (30) for all values of n when $m = 1, 2, 3, \cdots, p-1$, and prove, on that hypothesis, that it is correct for $m = p$, and hence for all m.

By a well known rule for multiplying a determinant by one of its minors,

$$(31) \qquad \Delta'_{p-1,n+p} = \frac{\Delta'_{p-2,n+p-1}\Delta'_{p-2,n+p+1} - (\Delta'_{p-2,n+p})^2}{\Delta'_{p-3,n+p}}.$$

Then by (25) and what we have assumed, (31) may be written

$$(32) \quad \Delta'_{p-1,n+p} = \frac{(-1)^{(n+1)(n+2)/2+p-1}\left[-\dfrac{\Delta_{n+p,p-1}}{\Delta_{n+p-1,p}} - \dfrac{\Delta^2_{n+p-1,p-1}}{\Delta_{n+p-1,p}\Delta_{n+p-2,p-1}} \right]}{-k_0^{2p+n+2}(\Delta_{n+p-1,p-2}/\Delta_{n+p-1,p}\Delta_{n+p-2,p-1})}.$$

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[*] Cf. Hadamard, Liouville's Journal, (4), vol. 8 (1892), §24, p. 138. Hadamard obtains (25) with $w(m, n)$ as in (30) but by a different method.

But by the above mentioned rule for multiplying a determinant by one of its minors, the quantity within the bracket is equal to $-\Delta_{n+p-1,p-2}/\Delta_{n+p-2,p-1}$. Consequently, (32) may be written

$$(33) \qquad \Delta'_{p-1,n+p} = \frac{(-1)^{(n+1)(n+2)/2+p-1}\Delta_{n+p-1,p}}{k_0^{2p+n+2}}.$$

Comparison of (25) and (33) shows that the value of $w(m, n)$ as given by (30) is correct for $m = p$, as was to be proved.

As particular instances of (25) we have

$$(34) \qquad \Delta'_{m-1,m} = (-1)^m\Delta_{m-1,m}/k_0^{2m},$$

$$(35) \qquad \Delta'_{m-1,m-1} = (-1)^{m-1}\Delta_{m-2,m}/k_0^{2m-1}.$$

Due to symmetry of the identity (21), the δ_i and the k_i may be interchanged in (35), giving the relation

$$(36) \qquad \Delta'_{m-2,m} = (-1)^{m-1}\Delta_{m-1,m-1}/k_0^{2m-1}.$$

We will now consider the continued fraction of the form (19) belonging to the normal* series

$$(37) \qquad \delta_1 + \delta_2 z + \delta_3 z^2 + \cdots.$$

If we denote by a_i' the a_i obtained from (37), then by (34), (35), and (36) we obtain

$$a'_{2i} = -(\Delta'_{i-1,i})^2/[\Delta'_{i-1,i+1}\Delta'_{i-2,i}] = \Delta^2_{i-1,i}/[\Delta_{i,i}\Delta_{i-1,i-1}] = a_{2i+1},$$

$$a'_{2i+1} = (\Delta'_{i-1,i+1})^2/[\Delta'_{i-1,i}\Delta'_{i,i+1}] = -\Delta^2_{i,i}/[\Delta_{i-1,i}\Delta_{i,i+1}] = a_{2i+2}.$$

We have thus proved the following theorem:

THEOREM 1. *If $k_0 + k_1 z + k_2 z^2 + \cdots$ is any normal power series giving rise to the continued fraction*

$$(38) \qquad \frac{1}{a_1} + \frac{z}{a_2} + \frac{z}{a_3} + \cdots$$

and $\delta_0 + \delta_1 z + \delta_2 z^2 + \cdots$ its reciprocal, then the continued fraction of the form (19) associated with the series $\delta_1 + \delta_2 z + \delta_3 z^2 + \cdots$ is

$$(39) \qquad \frac{1}{a_2} + \frac{z}{a_3} + \frac{z}{a_4} + \cdots.$$

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* A normal series is one giving rise to a normal Padé table.

CHAPTER II. THE CONVERGENCE OF THE DIAGONAL FILES FOR THE SERIES
OF STIELTJES

I pass now to the series of Stieltjes

(5) $$c_0 - c_1 z + c_2 z^2 - \cdots .$$

The work of the preceding chapter is applicable here if we replace through-out k_{2i} by c_{2i} and k_{2i-1} by $-c_{2i-1}$. In particular the continued fraction (16) becomes[*]

(40) $$G_n(z) + (-z)^n \left[\frac{1}{a_1{}^n +} \frac{z}{a_2{}^n +} \frac{z}{a_3{}^n +} \cdots \right],$$

where $G_n(z) = c_0 - c_1 z + \cdots + (-1)^{n-1} c_{n-1} z^{n-1}$ and the $a_i{}^n$ are obtained by replacing every k_{i+n} by $(-1)^i c_{i+n}$ in (17). It is easily seen that now, in place of (17), we have

(41)
$$a_{2i}^{n} = \Delta_{i-1,i+n-1}^2 / [\Delta_{i-1,i+n}\Delta_{i-2,i+n-1}],$$
$$a_{2i+1}^{n} = \Delta_{i-1,i+n}^2 / [\Delta_{i-1,i+n-1}\Delta_{i,i+n}],$$

in which it is to be understood that the $\Delta_{m,n}$ are the same as in Chapter I with every k_i replaced by c_i.

In particular,

(42)
$$a_{2i} = A_i{}^2/(B_i B_{i-1}) = \Delta_{i-1,i-1}^2/(\Delta_{i-1,i}\Delta_{i-2,i-1}),$$
$$a_{2i+1} = B_i{}^2/(A_i A_{i+1}) = \Delta_{i-1,i}^2/(\Delta_{i-1,i-1}\Delta_{i,i}).$$

Thus each $a_i{}^n$ is obtained from the corresponding a_i by means of the substitution of c_{i+n} for c_i throughout.

Since, by hypothesis, A_n and B_n, for $n = 1, 2, 3, \cdots$, are positive, all the determinants $\Delta_{m,n}$ formed from them by removing the first r rows and r columns are positive.[†] Therefore, in (40), the bracketed part,

(43) $$\frac{1}{a_1{}^n +} \frac{z}{a_2{}^n +} \frac{z}{a_3{}^n +} \cdots ,$$

is a continued fraction of Stieltjes with positive $a_i{}^n$. We may therefore apply his convergence results for the continued fraction (2), bearing in mind that, as noted in the introduction, z has been replaced by $1/z$ in (2) and then the factor z has been dropped. It is readily seen that if $P_i(z)$, $Q_i(z)$ denote the numerator and denominator of the convergents of (2) (after a_i has been re-

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[*] Henceforth for convenience I write a_i^n in place of $a_i^{(n)}$.

[†] Sylvester, Philosophical Magazine, (4), vol. 4 (1852), pp. 140–141.

placed by a_i^*), and if $P_i'(z)$, $Q_i'(z)$ denote the corresponding numerators and denominators of (43), then*

$$P_{2i}'(z) = z^{i-1}P_{2i}\left(\frac{1}{z}\right), \qquad Q_{2i}'(z) = z^i Q_{2i}\left(\frac{1}{z}\right),$$

$$P_{2i+1}'(z) = z^i P_{2i+1}\left(\frac{1}{z}\right), \qquad Q_{2i+1}'(z) = z^{i+1}Q_{2i+1}\left(\frac{1}{z}\right).$$

The sequence of convergents accordingly separates into two sets, the even and the odd convergents. Stieltjes showed that when $\sum a_i^n$ is convergent, the numerators and denominators of the even convergents converge to limits $p(z)$, $q(z)$ respectively; likewise those of the odd convergents to limits $p_1(z)$, $q_1(z)$, and these limits are *entire* functions whose zeros all lie upon the negative half of the real axis. These functions are connected by the relation

$$(44) \qquad\qquad p_1(z)q(z) - p(z)q_1(z) = 1.$$

On the other hand, when $\sum a_i^n$ is divergent, the even and the odd convergents converge to a common function $F(z)$ which is analytic over the entire plane excepting the whole or a part of the negative half of the real axis.

If now we denote the ith convergent of (40) by $U_i(z)/V_i(z)$, we have

$$(45) \qquad \frac{U_i(z)}{V_i(z)} = G_n(z) + (-z)^n\frac{P_i'(z)}{Q_i'(z)} = G_n(z) - (-z)^{n-1}\frac{P_i\left(\dfrac{1}{z}\right)}{Q_i\left(\dfrac{1}{z}\right)}.$$

The odd and even convergents of (40) fill respectively the nth and $(n+1)$st diagonal files of the Padé table. When $\sum a_i^n$ is convergent, they have separate limits in consequence of the foregoing results of Stieltjes, and these limits are meromorphic over the entire plane except at $z=0$, with poles lying only upon the negative half of the real axis. If now we put

$$u(z) = G_n(z)q\left(\frac{1}{z}\right) - (-z)^{n-1}p\left(\frac{1}{z}\right),$$

$$v(z) = q\left(\frac{1}{z}\right),$$

the limit of the even convergents $U_{2i}(z)/V_{2i}(z)$, for $i = \infty$, is $u(z)/v(z)$. Similarly the limit of the odd convergents will be $u_1(z)/v_1(z)$ where u_1 and v_1

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* Cf. the formulas for $P_i(z)$, $Q_i(z)$ given by Stieltjes, loc. cit., §2.

are the same as u and v except that p, q are replaced by p_1, q_1. In consequence of (44) we have the relation

(46) $u(z)v_1(z) - u_1(z)v(z) = (-z)^{n-1}.$

When $\sum a_i{}^n$ is divergent, the application of the results of Stieltjes to (45) shows immediately that the two sets of convergents have a common limit which is analytic over the entire plane except the whole or a part of the negative real axis. We conclude therefore that whether $\sum a_i{}^n$ converges or diverges every diagonal file of approximants to the right of the principal diagonal converges.

For a complete discussion of the relationship between the limits of these files we shall need to connect our successive series $\sum a_i^n$ with the primary series $\sum a_i$. For that purpose I shall now obtain expressions for the $a_i{}^n$ in terms of the a_i^{n-1}.

Let the numerator and denominator of the ith convergent of (40) be denoted by $U_i^n(z)$ and $V_i^n(z)$. Except for a common numerical factor, these may be obtained from (13) and (14). Thus, in particular, on setting $m=k$ and replacing n by $n+k$ in (14) and introducing this factor, we have*

(47) $V_{2k+1}^n(-z) = (-1)^k \begin{vmatrix} c_{n+1} & , \cdots , & c_{n+k}, & z^k \\ \cdot & \cdot & \cdot & \cdot \\ c_{n+k+1}, & \cdots , & c_{n+2k}, & 1 \end{vmatrix} : \Delta_{k,n+k}.$

From (47) we see that the coefficient of z^k in $V_{2k+1}(z)$ is $\Delta_{k-1,n+k+1}/\Delta_{k,n+k}$. Now $V_{2k+1}^n(z)$ is the denominator of the $(2k+1)$st convergent of the Stieltjes continued fraction (43), and it is easily proved by mathematical induction that the coefficient of z^k is also $a_1^n + a_3^n + \cdots + a_{2k+1}^n$. Therefore

(48) $a_1{}^n + a_3{}^n + \cdots + a_{2k+1}^n = \Delta_{k-1,n+k+1}/\Delta_{k,n+k}.$

If now we form by (41) the ratios a_{2i}^n/a_{2i+1}^{n-1} and $a_{2i+1}^n/a_{2i+2}^{n-1}$ and then employ (48), we obtain the very fundamental relations†

(49) $a_{2i}^n = \dfrac{a_{2i+1}^{n-1}}{\displaystyle\sum_{i=0}^{i-1} a_{2i+1}^{n-1} \sum_{i=0}^{i} a_{2i+1}^{n-1}}$ $(n = 1,2,3,\cdots),$

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* Cf. Perron, loc. cit., p. 428. Note that Perron's $\Delta_{\mu\nu}$ differs by a factor $(-1)^{\mu(\mu+1)/2}$ from that used by me.

† After this article was finished I found these relations (unused) in Stieltjes, loc. cit., §78.

and

(50) $$a_{2i+1}^{n} = a_{2i+2}^{n-1}\left(\sum_{i=0}^{i} a_{2i+1}^{n-1}\right)^{2}.$$

For $n = 0$ we are to understand that $a_i^0 \equiv a_i$ $(i = 1, 2, 3, \cdots)$.

Equation (49) may evidently be written in the form

(49a) $$a_{2i}^{n} = \frac{1}{\displaystyle\sum_{i=0}^{i-1} a_{2i+1}^{n-1}} - \frac{1}{\displaystyle\sum_{i=0}^{i} a_{2i+1}^{n-1}}.$$

It now follows from (49) that when $\sum a_{2i-1}^{n-1}$ is convergent and has the limit L, then from and after some index i,

$$a_{2i}^{n} < a_{2i+1}^{n-1}/(L - e)^{2},$$

where e is an arbitrary fixed positive number. Therefore $\sum a_{2i}^{n}$ is convergent when $\sum a_{2i-1}^{n}$ is convergent. Furthermore, $\sum a_{2i}^{n}$ is convergent even when $\sum a_{i-1}^{n}$ is divergent. For by summing (49a) for $i = 1, 2, 3, \cdots, k$, we get

$$\sum_{i=1}^{k} a_{2i}^{n} = \frac{1}{a_1^{n-1}} - \frac{1}{\displaystyle\sum_{i=0}^{k} a_{2i+1}^{n-1}},$$

and this has the limit $1/a_1^{n-1}$ when k becomes infinite. We thus have proved the theorem:

THEOREM 2. *The successive series* $\sum a_{2i}^{n}$, $n = 1, 2, 3, \cdots$, *are all convergent.*

We now turn to the series with odd subscripts, $\sum a_{2i-1}^{n}$. We see at once by (50) that, when $\sum a_i^{n-1}$ is convergent,

$$a_{2i+1}^{n} < a_{2i+2}^{n-1}(L')^{2} \qquad (i = 0, 1, 2, \cdots),$$

where $L' = \sum a_{2i-1}^{n-1}$, and therefore $\sum a_i^{n}$ is convergent. Taking $n = 1, 2, 3, \cdots$ in succession, we obtain at once the following theorem:

THEOREM 3. *If in the Stieltjes continued fraction* (7) *the series* $\sum a_i$ *is convergent, then all the associated series* $\sum a_i^{n}$, $n = 1, 2, 3, \cdots$, *are convergent.*

Consider next the case in which $\sum a_i$ diverges. Then, while $\sum a_{2i}^{1}$ will necessarily converge, $\sum a_{2i-1}^{1}$ may either converge or diverge. On the former hypothesis all subsequent series $\sum a_i^{n}$, $n = 2, 3, 4, \cdots$, must converge, by application of Theorem 3. If $\sum a_{2i-1}^{1}$ diverges, then as before $\sum a_{2i}^{2}$ must converge but $\sum a_{2i-1}^{2}$ may either converge or diverge, etc. Proceeding in this manner we perceive that in addition to the possibility of the convergence of

all $\sum a_i^n$, $n=0$, 1, 2, $\cdots$, Case I, there are just two other possibilities: either, Case II, all of the series $\sum a_i^n$, $n=0$, 1, 2, $\cdots$, will diverge; or else, Case III, the first r will diverge while all subsequent to the rth will converge.

It remains to see that both the last possibilities can be realized. For this purpose the following lemma will be established.

LEMMA 1. *Let*

$$b_1 + b_2 + b_3 + \cdots$$

be any divergent series in which there is a limit to the ratio b_i/b_{i-1}: $\lim b_i/b_{i-1} = s$. Then if $s \geq 1$ we have

$$\lim_{i=\infty} \frac{b_1 + b_2 + \cdots + b_{i+k}}{b_1 + b_2 + \cdots + b_i} = s^k.$$

According to the hypothesis, we may take a fixed j sufficiently large to insure that b_{j+k+i}/b_{j+i} $(i \geq 0)$ shall lie between the limits $s^k - e'/2$ and $s^k + e'/2$ where e' is an arbitrarily small prescribed positive quantity. Then

$$\frac{b_{j+k} + b_{j+k+1} + \cdots + b_{j+k+i}}{b_j + b_{j+1} + \cdots + b_{j+i}}$$

$$= \frac{1 - \dfrac{b_1 + b_2 + \cdots + b_{j+k-1}}{b_1 + b_2 + \cdots + b_{j+k+i}}}{1 - \dfrac{b_1 + b_2 + \cdots + b_{j-1}}{b_1 + b_2 + \cdots + b_{j+i}}} \cdot \frac{b_1 + b_2 + \cdots + b_{j+k+i}}{b_1 + b_2 + \cdots + b_{j+i}}$$

will lie between the same limits inasmuch as the ratio of each term of the numerator of the left member to the corresponding term of the denominator lies between these limits. But since $\sum b_i = +\infty$, the first factor on the right by taking i sufficiently large, say for $i > N$, becomes $1/(1+e'')$, where $|e''| < e'$. Thus

$$\left| \frac{b_1 + b_2 + \cdots + b_{j+k+i}}{b_1 + b_2 + \cdots + b_{j+i}} - s^k \right| \leq e'(s^k + 2) < \epsilon \quad \text{if} \quad e' < \frac{\epsilon}{s^k + 2}, \; i > N.$$

The lemma accordingly results.

Construct now a Stieltjes continued fraction for which $\lim (a_{2i+1}/a_{2i-1}) = s$, $\lim(a_{2i}/a_{2i-2}) = r$. Let $0 < r < 1$, $s > 1$. Then by the lemma and (49), (50) for $n=1$ we obtain

$$\lim_{i=\infty} (a_{2i+1}^1/a_{2i-1}^1) = rs^2, \quad \lim_{i=\infty} (a_{2i}^1/a_{2i-2}^1) = 1/s.$$

If we so choose s that $rs^2 < 1$, $1/s < 1$, then $\sum a_i$ will diverge while $\sum a_i^1$ will converge. But if $rs^2 > 1$, $1/s < 1$, $\sum a_i^1$ as well as $\sum a_i$ will diverge. In

the latter event, we may again apply the lemma and (49), (50) for $n=2$ and obtain

$$\lim_{i=\infty} (a_{2i+1}^2/a_{2i-1}^2) = r^2s^3, \quad \lim_{i=\infty} (a_{2i}^2/a_{2i-2}^2) = 1/(rs^2).$$

If we so choose s that $rs^2>1$, $r^2s^3<1$, the series $\sum a_i$, $\sum a_i^1$ will diverge, while $\sum a_i^2$ will converge. But if $r^2s^3>1$, $1/(rs^2)<1$, $\sum a_i$, $\sum a_i^1$, and $\sum a_i^2$ will all diverge, and we may again apply the lemma, etc.

Continuing in this manner we find that if $0<r<1$, $1/r^{(k-1)/k}<s<1/r^{k/(k+1)}$, all of the series $\sum a_i$, $\sum a_i^1$, $\cdots$, $\sum a_i^{k-1}$ will diverge, while the series $\sum a_i^k$, $\sum a_i^{k+1}$, $\cdots$, will converge. When $s\geq 1/r$, all of the series $\sum a_i^n$, $n=0, 1, 2, 3, \cdots$, are divergent. Thus the two possibilities indicated above when $\sum a_i$ is divergent can be realized.

Whenever the Stieltjes series (5) is convergent, the series $c_n-c_{n+1}z+c_{n+2}z^2-\cdots$, $n=1, 2, 3, \cdots$, are convergent, and therefore the numbers $1/(a_i^n a_{i+1})$, $i=1, 2, 3, \cdots$, must be bounded for each $n=1, 2, 3, \cdots$.* Hence none of the associated series $\sum a_i^n$, $n=1, 2, 3, \cdots$, can be convergent.

These associated series may also diverge even when the Stieltjes series (5) is *divergent*. For consider the Stieltjes continued fraction for which $a_{2i-1}=1/(ir^i)$, $a_{2i}=r^i$, $0<r<1$. In this the numbers $1/(a_i a_{i+1})$ increase without limit, and $\lim(a_{2i+1}/a_{2i-1})=1/r$, $\lim(a_{2i+2}/a_{2i})=r$. Accordingly the series (5) and the associated series diverge.

We have seen that there are *three cases* which arise. Of these we will now consider the first.

Case I. The series $\sum a_i$ is convergent.

Then by Theorem 3 all the series $\sum a_i^n$, $n=1, 2, 3, \cdots$, converge. Hence by the discussion at the beginning of the chapter all the diagonal files of approximants which lie to the right of the principal diagonal have limits which are meromorphic over the entire plane excepting $z=0$; with poles lying only on the negative half of the real axis; and the limits for no two successive files are identical.

To show that *no* two of these diagonal files have the same limit, I will show that for two of the denominators v_n of these limits the roots of $v_n(1/z)$ (different from 0) nearest the origin are distinct.

Stieltjes found that the roots of $Q_{2k}(z)$ and $Q_{2k+1}(z)$ for (2) are all real, non-positive, distinct, and separate each other. The former is of degree k and has no root equal to zero, while the latter is of degree $k+1$ and has a zero root. The same is true of $Q_{2k}^n(z)$, and $Q_{2k+1}^n(z)$, obtained from these by

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* Cf. Stieltjes, loc. cit., §10.

replacing a_i by a_i^n. If then we denote by $r_k^{(n)}$ and $r_k^{(n+1)}$ the non-zero root nearest the origin of $Q_{2k}^n(z)$ and $Q_{2k+1}^n(z)$, we have

$$r_k^{(n)} > r_k^{(n+1)}. \qquad (n = 1, 2, 3, \cdots ; \ k = 1, 2, 3, \cdots).$$

But $r_k^{(n+1)}$ is also the non-zero root nearest the origin of $Q_{2k}^{n+1}(z)$. If then we let $k \to \infty$, $r_k^{(n)}$ will approach a limit $R_n \neq 0$, and this limit is the non-zero root nearest the origin of the limit function $v_n(1/z)$.* We have

$$(51) \qquad\qquad R_1 \geqq R_2 \geqq R_3 \geqq R_4 \geqq \cdots .$$

The equality sign can nowhere hold in (51), since, by (46), $v_n(1/z)$ and $v_{n+1}(1/z)$ cannot vanish for the same value of z not zero, and therefore $R_n \neq R_{n+1}$ for all values of n. The distinctness of the functions $v_n(1/z)$ therefore follows, and hence no two diagonal files can converge to the same limit.

Case II. All the series $\sum a_i^n$, $n = 1, 2, 3, \cdots$, diverge.

Any two consecutive diagonal files to the right of and parallel to the principal diagonal converge to the same limit, and hence all these files have a common limit. The limit is analytic throughout the complex plane excepting the whole or a part of the negative half of the real axis.

Case III. The series $\sum a_i^n$, $n = 1, 2, 3, \cdots, r$, diverge while $\sum a_i^{r+j}$, $j = 1, 2, 3, 4, \cdots$, converge.

By the first part of the hypothesis the principal diagonal and all parallel files on its right up to the $(r+1)$th diagonal file inclusive converge to one and the same limit, while by virtue of the second part of the hypothesis all the files beginning with the $(r+1)$th converge to limits which are meromorphic over the entire plane except at $z = 0$. Thus the limits of all these files are of the same meromorphic character, but the distinctness of the limits begins only on passing the $(r+1)$th file.

We will now turn our attention to the diagonal files below the principal diagonal which start with an approximant on the vertical border of the Padé table. Instead of considering directly the convergence of these diagonal files of approximants, we may consider instead the convergence of their reciprocals which are files of approximants to the right of the principal diagonal of the Padé table for the reciprocal series of (5). Let this be

$$(52) \qquad\qquad d_0 + d_1 z + d_2 z^2 + \cdots .$$

By (34) and (35) the determinants A_n and B_n with every c_i replaced by $(-1)^i d_i$ are not all positive, but to the series

$$(53) \qquad\qquad d_1 + d_2 z + d_3 z^2 + \cdots$$

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* Stieltjes, loc. cit., §19.

belongs (Theorem 1, Chapter I) a Stieltjes continued fraction

$$(54) \qquad \frac{1}{a_2} + \frac{z}{a_3} + \frac{z}{a_4} + \cdots .$$

Consequently (53) is a Stieltjes series and we may apply the results previously obtained for (7) and (43) using (53) as our primary series.

Three cases may be distinguished.

Case I. If the series $\sum a_i$ converges, the diagonal files of approximants for (53) to the right of the principal diagonal converge each to a different function, which is meromorphic except at $z = 0$ with poles only on the negative half of the real axis. Consequently the diagonal files of approximants for (5) below the principal diagonal converge, each to a different function, which is meromorphic except at $z = 0$ with *zeros* only on the negative half of the real axis.

If $\sum a_i$ is divergent, two cases may be distinguished corresponding to II and III above.

Case II. If all the diagonal files of approximants for (53) to the right of the principal diagonal have a common limit $F_1(z)$, then those below the principal diagonal for (5) have the limit $F(z) = 1/(d_0 + zF_1)$, which is the limit of the continued fraction (7), and is analytic except over all or a part of the negative real axis.

Case III. If the first r files of approximants of (53) to the right of the principal diagonal have the same limit, while after passing the rth file the limits are distinct, then in the table of approximants of (5) the files below the principal diagonal will have the same limit until after passing the $(r+1)$th file, whereupon they become distinct from one another. The limit functions are all meromorphic except at $z = 0$ with *zeros* only on the negative half of the real axis.

One may ask the question whether, if the rth and $(r+1)$th diagonal files to the right of the principal diagonal have a common [different] limit [limits], a corresponding pair of adjacent files below the principal diagonal must have a common [different] limit [limits]. In Case I the answer is obviously in the affirmative, for then all the files to the right of and also those below the principal diagonal have distinct limits. In Case II when (5) converges, since its reciprocal also converges, both the series $\sum a_i^n$, $n = 0$, 1, 2, 3, $\cdots$, and the series similarly associated with (54) (which we shall denote for future reference by $\sum \alpha_i^n$, $n = 0$, 1, 2, $\cdots$, $\alpha_i^0 \equiv a_{i+1}$) must all diverge. Hence *all* the files have the same limit. Also for the divergent Stieltjes series for which $a_{2i} = r^i$, $a_{2i-1} = 1/(ir^i)$, $0 < r < 1$ (Chapter II) it is easily verified that $\sum a_i^n, \sum \alpha_i^n, n = 0, 1, 2, 3, \cdots$, all diverge. Finally, when

$\lim(a_{2i}/a_{2i-2}) = r$, $\lim(a_{2i+1}/a_{2i-1}) = s$, $0 < r < 1$, $s > 1$, it may be easily seen that a_{2i+1}/a_{2i-1}^{n} and $\alpha_{2i+1}^{n+1}/\alpha_{2i-1}^{n+1}$ both have the limit $r^{n}s^{n+1}$ for $i = \infty$. Consequently (except possibly when $r^{n}s^{n+1} = 1$) $\sum a_i^n$ and $\sum \alpha_i^{n+1}$ converge or diverge together. In all these examples pairs of adjacent files equally distant from the *first diagonal file below* the principal diagonal behave alike (i.e. the two files in each pair have a common limit or else they have distinct limits). In the following example pairs of adjacent files equidistant from the *principal diagonal* behave alike. Choose $a_{2i+3} = b_i/(a_2 + a_4 + \cdots + a_{2i+2})^2$, where $\sum b_i$ is any convergent positive-term series. Then $\sum \alpha_{2i+1}^1 = \sum b_i$ converges and hence by Theorem 3 all $\sum \alpha_i^n$ converge. But if $\sum a_{2i}$ is divergent, $\sum a_i$, $\sum a_i^1$ are divergent, but $\sum a_i^2$ and all subsequent series are convergent.

Now we shall examine the zeros and poles of the approximants $N_{m,n}(z)/D_{m,n}(z)$ of the Stieltjes series (5). If $n \geq m$, these by (45) are equal to $G_{2p+1}(z) - z^{2p}P_j(1/z)/Q_j(1/z)$, where $j = 2m$ or $2m+1$ according as $n - m = 2p$ or $2p+1$; and if $n < m$ their reciprocals, by the discussion centering about (52), are equal to $d_0 + d_1 z + \cdots + d_{2p+1}z^{2p+1} + z^{2p+1}P_j''(1/z)/Q_j''(1/z)$, where P_j''/Q_j'' is of the same form as P_j/Q_j and $j = 2n$ or $2n+1$ according as $m - n = 2p+1$ or $2p+2$. If we replace z by $1/z$ and set $G_n^1(z) = d_0 + d_1 z + \cdots + d_{n-1}z^{n-1}$, these become

$$G_{2p+1}\left(\frac{1}{z}\right) - \frac{1}{z^{2p}}\frac{P_j(z)}{Q_j(z)} \quad \text{and} \quad G_{2p}'\left(\frac{1}{z}\right) + \frac{1}{z^{2p+1}}\frac{P_j''(z)}{Q_j''(z)},$$

where $P_j(z)/Q_j(z)$ and $P_j''(z)/Q_j''(z)$ are the jth convergents of Stieltjes continued fractions of the form (2). Now it follows from the work of Stieltjes that when $n \geq m - 1$ the roots of the polynomials $D_{m,n}(z)$ are all distinct and lie on the negative half of the real axis. If $n \leq m$ the same is true of the roots of $N_{m,n}(z)$. If $n = m$ or $m-1$, all the roots of $N_{m,n}(z)$ and $D_{m,n}(z)$ are real and alternate along the negative half of the real axis. Consider now the roots of the numerator and denominator of an arbitrary approximant $N_{m,n}(z)/D_{m,n}(z)$. In the above representation for it, $P_j(z)/Q_j(z)$ and $P_j''(z)/Q_j''(z)$ are monotone functions of z which decrease from $+\infty$ to $-\infty$ as z increases between two adjacent roots of the denominators.* But the polynomials $G_n(1/z)$ and $G_n'(1/z)$ are continuous and bounded in any interval not containing the origin. Hence $G_{2p+1}(1/z)$ becomes equal to $P_j(z)/z^{2p}Q_j(z)$ at least once between every two adjacent non-zero roots of $Q_j(z)$. Therefore *if $n > m$, $N_{m,n}(z)$ vanishes at least once between every two adjacent roots of $D_{m,n}(z)$; similarly, if $n < m-1$, $D_{m,n}(z)$ vanishes at least once between every two adjacent roots of $N_{m,n}(z)$.*

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* Cf. Stieltjes, loc. cit., §3.

Chapter III. Extended Stieltjes series

I pass now to the question whether, given the Stieltjes series (5), it is possible to find a number c_{-1} such that the series

$$(55) \qquad\qquad -c_{-1}z^{-1} + c_0 - c_1 z + c_2 z^2 - \cdots$$

shall be a Stieltjes series, i.e. such that the determinants A_n and B_n in terms of the coefficients of (55) shall all be positive. The B_n for (55), being the A_n for the original series (5), are positive. Consequently it is both necessary and sufficient for the extension that a number $t=c_{-1}>0$ shall exist such that

$$f_i(t) = \begin{vmatrix} t & c_0 & \cdots & c_i \\ c_0 & c_1 & \cdots & c_{i+1} \\ & \cdot & \cdot & \cdot \\ c_i & & \cdots & c_{2i+1} \end{vmatrix}$$

shall be positive for all values of $i = 0, 1, 2, 3, \cdots$.

Now

$$(56) \qquad\qquad f_i(t) \equiv \Delta_{i,i+1} \cdot t + f_i(0).$$

Since the coefficient of t in (56) is positive, c_{-1} must be taken greater than the root of $f_i(t)$, i.e. greater than

$$(57) \qquad\qquad t_i = -f_i(0)/\Delta_{i,i+1},$$

for all values of i. It follows that the extension is possible when and only when the numbers t_i, $i = 0, 1, 2, \cdots$, have an upper bound.

Now[*]

$$(58) \qquad\qquad t_i = a_2 + a_4 + \cdots + a_{2i+2}.$$

In fact, by the rule for multiplying a determinant by one of its minors

$$(59) \qquad\qquad f_i(t) \cdot \Delta_{i-1,i} = f_{i-1}(t)\Delta_{i,i+1} - \Delta_{i,i}^2.$$

If we divide both members of this equation by $\Delta_{i-1,i}\Delta_{i,i+1}$ and then replace i by $i-1, i-2, \cdots, 3, 2, 1, 0$, we have with the aid of (42)

$$a_{2i+2} = f_{i-1}(t)/\Delta_{i-1,i} - f_i(t)/\Delta_{i,i+1},$$

$$a_{2i} = f_{i-2}(t)/\Delta_{i-2,i-1} - f_{i-1}(t)/\Delta_{i-1,i},$$

$$\cdot \qquad\qquad \cdot \qquad\qquad \cdot$$

$$a_4 = f_0(t)/\Delta_{0,1} - f_1(t)/\Delta_{1,2}.$$

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[*] Stieltjes, loc. cit., §35, eq. 11, obtained this in a different manner.

If to these equations we adjoin the identity

$$a_2 = t - f_0(t)/\Delta_{0,1},$$

and then add them together and set $t = t_i$, we obtain (58).

It follows from (58) that the numbers t_i, $i = 0, 1, 2, \cdots$, will have an upper bound when and only when the series $\sum a_{2i}$ is convergent. When this condition is satisfied, we may take $c_{-1} = \sum a_{2i} + \sigma$ where σ is zero or an arbitrary positive number. We state this result in the following theorem.

THEOREM 4. *Given a Stieltjes series*

(5) $$c_0 - c_1 z + c_2 z^2 - \cdots ,$$

there exists a Stieltjes series

(55) $$- c_{-1} z^{-1} + c_0 - c_1 z + c_2 z^2 - \cdots$$

when and only when the series $\sum a_{2i}$ in the Stieltjes continued fraction belonging to (5) is convergent.

When the condition is fulfilled the number c_{-1} may be taken equal to $\sum a_{2i}$ or any greater number.

Let us now suppose that $\sum a_i$ in (7) is convergent. Denote by*

(60) $$\cfrac{1}{a_1^{-1} + \cfrac{z}{a_2^{-1} + \cfrac{z}{a_3^{-1} + \cdots}}}$$

the Stieltjes continued fraction belonging to (55), in which the a_n^{-1} are necessarily positive. We must now express the a_n^{-1} in terms of the a_n, which is obviously the same problem as that of expressing the a_i^{n-1} in terms of the a_i^n. To do this we will need to develop some necessary formulas.

If we equate the two values of t_{i-2} obtained from (57) and (58), and then advance the subscripts of every c_i by n, we get the equation

(61) $$a_2^n + a_4^n + \cdots + a_{2i-2}^n = - f_{i-2}^n(0)/\Delta_{i-2,n+i-1},$$

where

$$f_{i-2}^n(0) = \begin{vmatrix} 0 & c_n & \cdots & c_{n+i-2} \\ c_n & c_{n+1} & \cdots & c_{n+i-1} \\ \cdot & & \cdot & \cdot \\ c_{n+i-2} & & \cdots & c_{n+2i-3} \end{vmatrix} .$$

Now

(62) $$\Delta_{i-1,n+i-2} = f_{i-2}^n(0) + c_{n-1}\Delta_{i-2,n+i-1},$$

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* The superscripts as before are written for conven ence without parentheses.

and therefore (61) may be written in the form

$$(63) \qquad \Delta_{i-1,n+i-2} = f_{i-2}^n(0)' \left(c_{n-1} - \sum_{i=1}^{i-1} a_{2i}^n \right).$$

By (41),

$$(64) \qquad \frac{a_{2i}^{n-1}}{\Delta_{i-1,n+i-2}^2} = \frac{1}{\Delta_{i-1,i+n-1}\Delta_{i-2,i+n-2}}.$$

Squaring the members of (63) and multiplying them into the corresponding members of (64), we have

$$(65) \qquad a_{2i}^{n-1} = a_{2i-1}^n \left(c_{n-1} - \sum_{i=1}^{i-1} a_{2i}^n \right)^2 \qquad (i = 2,3,4,\cdots) ;$$

while for $i=1$ we have

$$(66) \qquad a_2^{n-1} = a_1^n c_{n-1}^2.$$

Also by (41), (63),

$$(67) \qquad a_{2i+1}^{n-1} = \frac{a_{2i}^n}{\left(c_{n-1} - \sum_{i=1}^{i-1} a_{2i}^n \right)\left(c_{n-1} - \sum_{i=1}^{i} a_{2i}^n \right)} \qquad (i = 2,3,4,\cdots),$$

while for $i=0, 1$ we have

$$(68) \qquad a_1^{n-1} = 1/c_{n-1} ; \quad a_3^{n-1} = a_2^n / [c_{n-1}(c_{n-1} - a_2^n)].$$

Then we have, setting $n=0$, $c_{-1}=\sum a_{2i}+\sigma$, $\sigma>0$, in (65), (67),

$$a_{2i}^{-1} = a_{2i-1}\left(\sum a_{2i} + \sigma - \sum_1^{i-1} a_{2i} \right)^2 < a_{2i-1}(\sum a_{2i} + \sigma)^2,$$

and

$$a_{2i+1}^{-1} = a_{2i}\Big/\left(\sum a_{2i} + \sigma - \sum_1^{i-1} a_{2i} \right)\left(\sum a_{2i} + \sigma - \sum_1^{i} a_{2i} \right) < a_{2i}/\sigma^2.$$

Hence since $\sum a_i$ is convergent, it follows that $\sum a_i^{-1}$ is also convergent.

We may now take $c_{-2}=\sum a_{2i}^{-1}+\sigma'$, where $\sigma'>0$, and extend (55) to form the Stieltjes series

$$(69) \qquad c_{-2}z^{-2} - c_{-1}z^{-1} + c_0 - c_1 z + c_2 z^2 - \cdots ,$$

etc.

Continuing in this manner, we see that the series (5) can be extended indefinitely to the left. We have proved the following theorem:

THEOREM 5. *If in the Stieltjes continued fraction belonging to the Stieltjes series*

(5) $$c_0 - c_1 z + c_2 z^2 - \cdots$$

the series $\sum a_i$ converges, then the series (5) can be extended to the left by prefixing any finite or infinite number of terms: $-c_{-1}z^{-1}$, $+c_{-2}z^{-2}$, $-c_{-3}z^{-3}$, $\cdots$, in such wise that the extended series shall be Stieltjes series.

The coefficients in the prefixed terms are not unique.

Let us next suppose the series $\sum a_i$ to diverge. By Theorem 4, $\sum a_{2i}$ must converge in order that extension shall be possible, and consequently $\sum a_{2i-1}$ is divergent. We then take $c_{-1} = \sum a_{2i} + \sigma$, where σ is *zero* or a positive number. I shall prove that unless $\sigma = 0$, the series $\sum a_{2i}^{-1}$ will be divergent, and therefore further extension will be impossible. In fact, by (65), if $\sigma \neq 0$, $a_{2i}^{-1} > \sigma^2 a_{2i-1}$, $i = 1, 2, 3, \cdots$, and therefore each term of $\sum a_{2i}^{-1}$ is greater than the corresponding term of the divergent series $\sigma^2 \sum a_{2i-1}$ so that extension is impossible.

When $\sigma \neq 0$, it has been shown that $\sum a_{2i}^{-1}$ is necessarily divergent. I shall give examples presently to show that when $\sigma = 0$, $\sum a_{2i}^{-1}$ may either converge or diverge. On the former hypothesis $\sum a_{2i-1}^{-1}$ must be divergent, and therefore also $\sum a_i^{-1}$. For suppose the contrary. Then by Theorem 3, since $\sum a_i^{-1}$ converges, $\sum a_i$, $\sum a_i^1$, $\cdots$ converge. But by hypothesis $\sum a_i$ is divergent. We thus have a contradiction. Hence:

THEOREM 6. *If in the Stieltjes continued fraction*

$$\cfrac{1}{a_1 +} \cfrac{z}{a_2 +} \cfrac{z}{a_3 +} \cdots$$

belonging to the Stieltjes series

(5) $$c_0 - c_1 z + c_2 z^2 - \cdots$$

the series $\sum a_i$ diverges, and if (5) admits the extension

$$\pm c_{-n} z^{-n} \mp c_{-n+1} z^{-n+1} \pm \cdots + c_0 - c_1 z + c_2 z^2 - \cdots ,$$

then the coefficients $c_{-1}, c_{-2}, \cdots, c_{-n}$ must be chosen successively in accordance with the relations

(70)
$$c_{-i} = \sum a_{2k}^{-i+1} \qquad (i = 1, 2, \cdots, n-1),$$
$$c_{-n} \geqq \sum a_{2k}^{-n+1}.$$

For examples illustrating the extension of Stieltjes series to the left the following lemma will be needed.

LEMMA 2, *Let $b_1+b_2+b_3+\cdots$ be any convergent series in which $\lim_{i=\infty} b_i/b_{i-1}=r$; then if $0<r<1$,*

$$\lim_{i=\infty} \frac{b_{i+1}+b_{i+2}+\cdots}{b_{i+k+1}+b_{i+k+2}+\cdots}=\frac{1}{r^k}.$$

According to the hypothesis we may take a fixed integer m sufficiently large to insure that for all $n\geq m$ the quotient b_n/b_{n+k} will lie between $(1/r^k)-\epsilon/4$ and $(1/r^k)+\epsilon/4$, where ϵ is an arbitrarily small assigned positive number. Then

$$\frac{b_m+b_{m+1}+\cdots+b_{m+p}}{b_{m+k}+b_{m+k+1}+\cdots+b_{m+k+p}}$$

will lie between these same limits inasmuch as the terms are all of the same sign and the ratio of each term of the numerator to the corresponding term of the denominator lies between these limits. But since $\sum b_i$ is convergent there will be a limit to this fraction for $p\to\infty$ which lies between these limits.

As ϵ may be taken as small as desired by taking m sufficiently large, the lemma accordingly results.

Construct now a Stieltjes continued fraction in which $a_{2i}=r^i$, $\lim_{i=\infty} a_{2i+1}/a_{2i-1}=s$, $0<r<1$, $s>1$. As $\sum a_i$ diverges, c_{-1} must be taken in accordance with (70). Then by (65) and (67), with $n=0$,

$$\lim_{i=\infty}\frac{a_{2i}^{-1}}{a_{2i-2}^{-1}}=\lim_{i=\infty}\frac{a_{2i-1}}{a_{2i-3}}\frac{(a_{2i}+a_{2i+2}+\cdots)^2}{(a_{2i-2}+a_{2i}+\cdots)^2}=r^2 s,$$

$$\lim_{i=\infty}\frac{a_{2i+1}^{-1}}{a_{2i-1}^{-1}}=\lim_{i=\infty}\frac{a_{2i}}{a_{2i-2}}\frac{(a_{2i-2}+a_{2i}+\cdots)}{(a_{2i+2}+a_{2i+4}+\cdots)}=1/r.$$

If we now choose r so that $r^2 s<1$, then $\sum a_{2i}^{-1}$ will be convergent and both a first and second extension of (5) is possible. This requires that we have $r<1/s^{1/2}<1$. Similarly, if $r^2 s<1$, we have

$$\lim_{i=\infty}\frac{a_{2i}^{-2}}{a_{2i-2}^{-2}}=\lim_{i=\infty}\frac{a_{2i-1}^{-1}}{a_{2i-3}^{-1}}\frac{(a_{2i}^{-1}+\cdots)^2}{(a_{2i-2}^{-1}+\cdots)^2}=r^3 s^2,$$

$$\lim_{i=\infty}\frac{a_{2i+1}^{-2}}{a_{2i-1}^{-2}}=\lim_{i=\infty}\frac{a_{2i}^{-1}}{a_{2i-2}^{-1}}\frac{(a_{2i-2}^{-1}+\cdots)}{(a_{2i+2}^{-1}+\cdots)}=1/(r^2 s).$$

Then if $r^3 s^2<1$, that is, if $r<1/s^{2/3}$, a third extension is possible and $\lim a_{2i}^{-3}/a_{2i-2}^{-3}=r^4 s^2$, etc. Thus, if

$$\frac{1}{s^{k/(k+1)}} < r < \frac{1}{s^{(k-1)/k}},$$

the series $\sum a_{2i}, \sum a_{2i}^{-1}, \cdots, \sum a_{2i}^{-k+1}$ will converge, while $\sum a_{2i}^{-k}$ is divergent. Hence extension of (5) to the left for k terms is then possible, but a further extension is impossible. If $r = 1/s$, an extension of infinite order is possible.

The following theorem follows immediately from (49), (50) of the preceding chapter.*

THEOREM 7. *Given any series of positive numbers $\sum a_i$, there is uniquely determined a power series of Stieltjes*

$$c_0 - c_1 z + c_2 z^2 - \cdots$$

belonging to the Stieltjes continued fraction

$$\frac{1}{a_1} + \frac{z}{a_2} + \frac{z}{a_3} + \cdots$$

The c_i are given in terms of the a_i by the following formulas:

$$c_i = 1/a_1{}^i \qquad\qquad (i = 0,1,2,\cdots),$$

(71)
$$\overset{i}{a_{2k}} = \overset{i-1}{a_{2k+1}} / \sum_{k=1}^{k} \overset{i-1}{a_{2k-1}} \sum_{k=1}^{k+1} \overset{i-1}{a_{2k-1}},$$

$$\overset{i}{a_{2k-1}} = \overset{i-1}{a_{2k}} \left(\sum_{k=1}^{k} \overset{i-1}{a_{2k-1}} \right)^2.$$

CHAPTER IV. A PARTICULAR EXAMPLE

The continued fraction†

$$\frac{1}{1} + \frac{z}{1} + \frac{z}{1} + \frac{z}{1} + \cdots .$$

I will determine the c_i and the q_i^n (which are the same as the α_i^n in this case) for the above continued fraction. I will show that the radius of convergence of the corresponding Stieltjes series is $1/4$. The associated series $\sum a_i^n$, $n = 1, 2, 3, \cdots$, are divergent, but the $\sum a_{2i}^n$ converge, and consequently

(72)
$$c_n = \sum a_{2i}^{n+1} \qquad\qquad (n = 0,1,2,3,\cdots).$$

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* Stieltjes, Annales de Toulouse, vol. 3, H, pp. 1–17, 1889, shows the one-to-one correspondence between his series and continued fractions. He obtains the c_i in terms of the a_i by a different method.

† Laurent, *Note sur les fractions continues*, Nouvelles Annales de Mathématiques, (2), vol. 5 (1866), pp. 540–552.

The c_n being known, (72) serve to give the sums of certain positive-term series.

To determine the c_i, write

$$F_1(z) = 1 + \frac{z}{1} + \frac{z}{1} + \cdots = 1 + \frac{z}{F_1(z)} \cdot$$

Hence,

$$F_1(z) = \frac{1 + (1 + 4z)^{1/2}}{2},$$

and

$$F(z) = \frac{1}{1} + \frac{z}{1} + \frac{z}{1} + \cdots$$

$$= (F_1 - 1)/z$$

$$= \frac{-1 + (1 + 4z)^{1/2}}{2z} \cdot$$

If we develop the numerator of the last expression in a Taylor's power series about $z = 0$, we may then divide by $2z$ and obtain

(73) $$F(z) = c_0 - c_1 z + c_2 z^2 - \cdots,$$

where

(74)
$$c_{2k} = \frac{(2k + 3)(2k + 5) \cdots (4k - 1)2^k}{k!},$$

$$c_{2k+1} = \frac{(2k + 3)(2k + 5) \cdots (4k + 1)2^k}{k + 1!},$$

and therefore satisfy the recurrence relations

$$c_{2k+1} = [(4k + 1)/(k + 1)]c_{2k},$$

(75) $$c_{2k} = 2[(4k - 1)/(2k + 1)]c_{2k-1},$$

$$c_0 = 1.$$

Since $\lim(c_i/c_{i-1}) = 4$ it follows that the radius of convergence of (73) is equal to $1/4$.

We will next get the elements in the associated continued fractions. Using (49), (50):

$$\overset{1}{a}_{2k} = 1/[k(k + 1)],$$

$$\overset{1}{a}_{2k+1} = (k + 1)^2.$$

Then since $1^2+2^2+ \cdots +(k+1)^2=(k+1)(k+2)(2k+3)/2\cdot 3$, we have

$$a_{2k}^{2} = (2\cdot 3)^2/[k(k + 2)(2k + 1)(2k + 3)],$$

$$a_{2k+1}^{2} = (k + 1)(k + 2)(2k + 3)^2/(2\cdot 3)^2.$$

The next step will involve the formula $1\cdot 2\cdot 3^2+2\cdot 3\cdot 5^2+ \cdots +(k+1)$ $(k+2)(2k+3)^2 \equiv (k+1)(k+2)(k+3)(2k+3)(2k+5)/5$. We have

$$a_{2k}^{3} = (2\cdot 3)^2(5)^2/[k(k + 1)(k + 2)(k + 3)(2k + 1)(2k + 5)],$$

$$a_{2k+1}^{3} = (k + 1)(k + 2)^2(k + 3)(2k + 3)(2k + 5)/[(2\cdot 3)^2(5)^2].$$

In general:

$$a_{2k+1}^{2i+1} = \frac{(k + 1) \cdots (k + i)(k + i + 1)^2(k + i + 2) \cdots (k + 2i + 1)(2k + 3) \cdots (2k + 4i + 1)}{(2\cdot 3)^2(5)^2(2\cdot 7)^2 \cdots [2(4i - 1)]^2(4i + 1)^2},$$

$$a_{2k}^{2i+1} = \frac{(2\cdot 3)^2(5)^2(2\cdot 7)^2 \cdots [2(4i - 1)]^2(4i + 1)^2}{k \cdots (k + 2i + 1)(2k + 1) \cdots (2k + 2i - 1)(2k + 2i + 3) \cdots (2k + 4i + 1)},$$

$$a_{2k+1}^{2i} = \frac{(k + 1) \cdots (k + 2i)(2k + 3) \cdots (2k + 2i - 1)(2k + 2i + 1)^2(2k + 2i + 3) \cdots (2k + 4i - 1)}{(2\cdot 3)^2(5)^2(2\cdot 7)^2 \cdots (4i - 3)^2[2(4i - 1)]^2},$$

$$a_{2k}^{2i} = \frac{(2\cdot 3)^2(5)^2(2\cdot 7)^2 \cdots (4i - 3)^2[2(4i - 1)]^2}{k \cdots (k + i - 1)(k + i + 1) \cdots (k + 2i)(2k + 1) \cdots (2k + 4i - 1)}.$$

These formulas are seen to hold for all k when $i=1$. Assuming them true for all k and for all $i \leq I-1$, they may, with the aid of the following formulas, be proved* for $i=I$. Set

$$Y_k^i = (k + 1) \cdots (k + i - 1)(k + i)^2(k + i + 1) \cdots (k + 2i - 1)(2k + 3)$$
$$\cdots (2k + 4i - 3),$$

$$Z_k^i = (k + 1) \cdots (k+2i)(2k+3) \cdots (2k+2i-1)(2k+2i+1)^2(2k+2i+3)$$
$$\cdots (4k + 4i - 1).$$

Then,

$$\sum_{k=0}^{k} Y_k^i = (k + 1) \cdots (k+2i)(2k+3) \cdots (2k+2i-1)(2k+2i+1)^2(2k+2i+3)$$
$$\cdots (2k + 4i - 1)/(2(4i - 1)),$$

$$\sum_{k=0}^{k} Z_k^i = (k + 1) \cdots (k + 2i + 1)(2k + 3) \cdots (2k + 4i + 1)/(4i + 1).$$

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*The completion of the proof, which offers no difficulty, will be omitted.

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