

Dynamic Macroeconomics with Numerics: Project II

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One-Sector Stochastic Growth Model

Notes on optimality conditions

We have the Euler error given by

$$V_t = c_{t+1} - \beta c_t [f_{k_{t+1}}(k_{t+1}, U_{t+1}, z_{t+1}) + 1 - \delta_{t+1}(k_{t+1})],$$

We have

$$U_t = \left(\frac{\alpha}{\delta\phi} z_t^{1-\alpha} k_t^{\alpha-1} \right)^{\frac{1}{\phi-\alpha}}$$

$$U_{t+1} = \left(\frac{\alpha}{\delta\phi} z_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} \right)^{\frac{1}{\phi-\alpha}}$$

And

$$\begin{aligned} c_t &= z_t^{1-\alpha} k_t^\alpha U_t^\alpha - k_{t+1} + k_t - \delta U_t^\phi k_t, \\ &= z_t^{1-\alpha} k_t^\alpha \left(\frac{\alpha}{\delta\phi} z_t^{1-\alpha} k_t^{\alpha-1} \right)^{\frac{\alpha}{\phi-\alpha}} - k_{t+1} + k_t - \delta \left(\frac{\alpha}{\delta\phi} z_t^{1-\alpha} k_t^{\alpha-1} \right)^{\frac{\phi}{\phi-\alpha}} k_t \\ &= z_t^{1-\alpha + \frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} - k_{t+1} + k_t - \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_t^{1 + \frac{(\alpha-1)\phi}{\phi-\alpha}} \\ c_{t+1} &= z_{t+1}^{1-\alpha + \frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} - k_{t+2} + k_{t+1} - \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} z_{t+1}^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_{t+1}^{1 + \frac{(\alpha-1)\phi}{\phi-\alpha}} \end{aligned}$$

Then we have

$$\begin{aligned} f_{k_{t+1}}(\cdot) &= \alpha z_{t+1}^{1-\alpha} U_{t+1}^\alpha k_{t+1}^{\alpha-1} \\ &= \alpha z_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} \left(\frac{\alpha}{\delta\phi} z_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} \right)^{\frac{\alpha}{\phi-\alpha}} \\ &= \alpha z_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} z_{t+1}^{\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\ &= \alpha \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} z_{t+1}^{1-\alpha + \frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha-1 + \frac{(\alpha-1)\alpha}{\phi-\alpha}} \end{aligned}$$

And lastly we have

$$\begin{aligned} \delta_{t+1} &= \delta U_{t+1}^\phi = \delta \left(\frac{\alpha}{\delta\phi} z_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} \right)^{\frac{\phi}{\phi-\alpha}} \\ &= \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} z_{t+1}^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}} \end{aligned}$$

We can put together all terms and have

$$\begin{aligned} V_t &= z_{t+1}^{1-\alpha + \frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} - k_{t+2} + k_{t+1} - \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} z_{t+1}^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_{t+1}^{1 + \frac{(\alpha-1)\phi}{\phi-\alpha}} \\ &\quad - \beta \underbrace{\left\{ z_t^{1-\alpha + \frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} - k_{t+1} + k_t - \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_t^{1 + \frac{(\alpha-1)\phi}{\phi-\alpha}} \right\}}_{=c_t} \alpha \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} z_{t+1}^{1-\alpha + \frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha-1 + \frac{(\alpha-1)\alpha}{\phi-\alpha}} \\ &\quad - \beta \underbrace{\left\{ z_t^{1-\alpha + \frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} - k_{t+1} + k_t - \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_t^{1 + \frac{(\alpha-1)\phi}{\phi-\alpha}} \right\}}_{=c_t} \\ &\quad + \beta \underbrace{\left\{ z_t^{1-\alpha + \frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} - k_{t+1} + k_t - \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_t^{1 + \frac{(\alpha-1)\phi}{\phi-\alpha}} \right\}}_{=c_t} \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} z_{t+1}^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}} \end{aligned}$$

Now we can take the partial derivatives:

$$\begin{aligned}\frac{\partial}{\partial k_{t+2}} V_t &= -1 \\ V_3 &= -1\end{aligned}$$

Next we have

$$\begin{aligned}\frac{\partial}{\partial k_{t+1}} V_t &= \left(\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha} \right) z_{t+1}^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta \left(1 + \frac{(\alpha-1)\alpha}{\phi-\alpha} \right) \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} z_{t+1}^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\ &\quad - \beta \left\{ z_t^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} - k_{t+1} + k_t - \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_t^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}} \right\} \\ &\quad \cdot \alpha \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} \left(\alpha - 1 + \frac{(\alpha-1)\alpha}{\phi-\alpha} \right) z_{t+1}^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha-2+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\ &\quad - (-1)\alpha \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} z_{t+1}^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\ &\quad - \beta(-1) \\ &\quad + \beta \left\{ z_t^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} - k_{t+1} + k_t - \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_t^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}} \right\} \\ &\quad \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} \left(\frac{(\alpha-1)\alpha}{\phi-\alpha} \right) z_{t+1}^{\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\alpha}{\phi-\alpha}-1} \\ &\quad - \beta \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} z_{t+1}^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}}\end{aligned}$$

and evaluated at the steady state this is

$$\begin{aligned}V_2 &= \left(\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha} \right) k_*^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta \left(1 + \frac{(\alpha-1)\alpha}{\phi-\alpha} \right) \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} k_*^{\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\ &\quad - \beta \left\{ k_*^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} - \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} k_*^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}} \right\} \cdot \alpha \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} \left(\alpha - 1 + \frac{(\alpha-1)\alpha}{\phi-\alpha} \right) k_*^{\alpha-2+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\ &\quad - (-1)\alpha \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} k_*^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\ &\quad - \beta(-1) \\ &\quad + \beta \left\{ k_*^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} - \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} k_*^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}} \right\} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} \left(\frac{(\alpha-1)\alpha}{\phi-\alpha} \right) k_*^{\frac{(\alpha-1)\alpha}{\phi-\alpha}-1} \\ &\quad - \beta \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} k_*^{\frac{(\alpha-1)\phi}{\phi-\alpha}}\end{aligned}$$

The next derivative is given by

$$\begin{aligned}\frac{\partial}{\partial k_t} V_t &= \beta \left\{ \left(\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha} \right) z_t^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta \left(1 + \frac{(\alpha-1)\alpha}{\phi-\alpha} \right) \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} z_t^{\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\frac{(\alpha-1)\alpha}{\phi-\alpha}} \right\} \\ &\quad \alpha \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} z_{t+1}^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\ &\quad - \beta \left\{ \left(\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha} \right) z_t^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta \left(1 + \frac{(\alpha-1)\alpha}{\phi-\alpha} \right) \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} z_t^{\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\frac{(\alpha-1)\alpha}{\phi-\alpha}} \right\} \\ &\quad + \beta \left\{ \left(\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha} \right) z_t^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta \left(1 + \frac{(\alpha-1)\alpha}{\phi-\alpha} \right) \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} z_t^{\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\frac{(\alpha-1)\alpha}{\phi-\alpha}} \right\} \\ &\quad \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} z_{t+1}^{\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\alpha}{\phi-\alpha}}\end{aligned}$$

$$\begin{aligned} V_1 = & \beta \left\{ \left(\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha} \right) k_*^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta \left(1 + \frac{(\alpha-1)\phi}{\phi-\alpha} \right) \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} k_*^{\frac{(\alpha-1)\phi}{\phi-\alpha}} \right\} \alpha \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} k_*^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\ & - \beta \left\{ \left(\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha} \right) k_*^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta \left(1 + \frac{(\alpha-1)\phi}{\phi-\alpha} \right) \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} k_*^{\frac{(\alpha-1)\phi}{\phi-\alpha}} \right\} \\ & + \beta \left\{ \left(\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha} \right) k_*^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta \left(1 + \frac{(\alpha-1)\phi}{\phi-\alpha} \right) \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} k_*^{\frac{(\alpha-1)\phi}{\phi-\alpha}} \right\} \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} k_*^{\frac{(\alpha-1)\alpha}{\phi-\alpha}} \end{aligned}$$

Next we have the derivatives w.r.t. the technology shocks

which, at the steady state is

$$\begin{aligned} \frac{\partial}{\partial z_t} V_t &= \beta \left\{ \left(1 - \alpha + \frac{(1-\alpha)\alpha}{\phi - \alpha} \right) z_t^{\frac{(1-\alpha)\alpha}{\phi - \alpha} - \alpha} k_t^{\alpha + \frac{(\alpha-1)\alpha}{\phi - \alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi - \alpha}} - \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi - \alpha}} \left(\frac{(1-\alpha)\phi}{\phi - \alpha} \right) z_t^{\frac{(1-\alpha)\phi}{\phi - \alpha} - 1} k_t^{1 + \frac{(\alpha-1)\phi}{\phi - \alpha}} \right\} \\ &\quad \cdot \alpha \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi - \alpha}} z_{t+1}^{1 - \alpha + \frac{(1-\alpha)\alpha}{\phi - \alpha}} k_{t+1}^{\alpha - 1 + \frac{(\alpha-1)\alpha}{\phi - \alpha}} \\ &\quad - \beta \left\{ \left(1 - \alpha + \frac{(1-\alpha)\alpha}{\phi - \alpha} \right) z_t^{\frac{(1-\alpha)\alpha}{\phi - \alpha} - \alpha} k_t^{\alpha + \frac{(\alpha-1)\alpha}{\phi - \alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi - \alpha}} - \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi - \alpha}} \left(\frac{(1-\alpha)\phi}{\phi - \alpha} \right) z_t^{\frac{(1-\alpha)\phi}{\phi - \alpha} - 1} k_t^{1 + \frac{(\alpha-1)\phi}{\phi - \alpha}} \right\} \\ &\quad + \beta \left\{ \left(1 - \alpha + \frac{(1-\alpha)\alpha}{\phi - \alpha} \right) z_t^{\frac{(1-\alpha)\alpha}{\phi - \alpha} - \alpha} k_t^{\alpha + \frac{(\alpha-1)\alpha}{\phi - \alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi - \alpha}} - \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi - \alpha}} \left(\frac{(1-\alpha)\phi}{\phi - \alpha} \right) z_t^{\frac{(1-\alpha)\phi}{\phi - \alpha} - 1} k_t^{1 + \frac{(\alpha-1)\phi}{\phi - \alpha}} \right\} \\ &\quad \cdot \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi - \alpha}} z_{t+1}^{\frac{(1-\alpha)\phi}{\phi - \alpha}} k_{t+1}^{\frac{(\alpha-1)\phi}{\phi - \alpha}}, \end{aligned}$$

$$\begin{aligned}
V_4 = & \beta \left\{ \left(1 - \alpha + \frac{(1-\alpha)\alpha}{\phi - \alpha} \right) k_*^{\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} - \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} \left(\frac{(1-\alpha)\phi}{\phi - \alpha} \right) k_*^{1 + \frac{(\alpha-1)\phi}{\phi-\alpha}} \right\} \\
& \cdot \alpha \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} k_*^{\alpha-1 + \frac{(\alpha-1)\alpha}{\phi-\alpha}} \\
& - \beta \left\{ \left(1 - \alpha + \frac{(1-\alpha)\alpha}{\phi - \alpha} \right) k_*^{\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} - \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} \left(\frac{(1-\alpha)\phi}{\phi - \alpha} \right) k_*^{1 + \frac{(\alpha-1)\phi}{\phi-\alpha}} \right\} \\
& + \beta \left\{ \left(1 - \alpha + \frac{(1-\alpha)\alpha}{\phi - \alpha} \right) k_*^{\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} - \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} \left(\frac{(1-\alpha)\phi}{\phi - \alpha} \right) k_*^{1 + \frac{(\alpha-1)\phi}{\phi-\alpha}} \right\}
\end{aligned}$$

$$\cdot \delta\left(\frac{\alpha}{\delta\phi}\right)^{\frac{\phi}{\phi-\alpha}} k_*^{\frac{(\alpha-1)\phi}{\phi-\alpha}}$$

Note: $\frac{\partial}{\partial z_t} c_t = \left\{ \left(1 - \alpha + \frac{(1-\alpha)\alpha}{\phi-\alpha}\right) z_t^{\frac{(1-\alpha)\alpha}{\phi-\alpha} - \alpha} k_t^{\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} - \delta\left(\frac{\alpha}{\delta\phi}\right)^{\frac{\phi}{\phi-\alpha}} \left(\frac{(1-\alpha)\phi}{\phi-\alpha}\right) z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha} - 1} k_t^{1 + \frac{(\alpha-1)\phi}{\phi-\alpha}} \right\}$ Next we have

$$\begin{aligned} z_{t+1} &= \exp(\rho x_t + \epsilon_{t+1}) \\ z_t &= \exp(\rho x_{t-1} + \epsilon_t) \end{aligned}$$

We can rewrite due to $x_t = \rho x_{t-1} + \epsilon_t = \log z_t$ to

$$\begin{aligned} z_{t+1} &= \exp(\rho(\rho x_{t-1} + \epsilon_t) + \epsilon_{t+1}) \\ &= \exp(\rho \log z_t + \epsilon_{t+1}) \\ &= \exp(\epsilon_{t+1}) z_t^\rho \end{aligned}$$

Now we can take the derivative

$$\frac{\partial}{\partial z_t} z_{t+1} = \rho \exp(\epsilon_{t+1}) z_t^{\rho-1}$$

and evaluated at the steady state this equals

$$\frac{\partial}{\partial z_t} z_* = \rho \exp(0) 1^{\rho-1} = \rho$$

Therefore we have

$$(z_{t+1} - z^*) = \rho(z_t - z^*) + \epsilon_{t+1}$$

where ϵ_{t+1} is the approximation error from the Taylor approximation.

To summarize, we have

$$\begin{aligned} \xi_{t+1} &= V_3(k_{t+2} - k^*) + V_2(k_{t+1} - k^*) + V_1(k_t - k^*) + V_5(z_{t+1} - z^*) + V_4(z_t - z^*) \\ 0 &= (k_{t+1} - k^*) - (k_t - k^*) \\ \epsilon_{t+1} &= (z_{t+1} - z^*) - \rho(z_t - z^*) \end{aligned}$$

with V_i being the partial derivatives defined above. Then we have

$$\begin{pmatrix} V_3 & V_2 & V_5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k_{t+2} - k^* \\ k_{t+1} - k^* \\ z_{t+1} - z^* \end{pmatrix} + \begin{pmatrix} 0 & V_1 & V_4 \\ -1 & 0 & 0 \\ 0 & 0 & -\rho \end{pmatrix} \begin{pmatrix} k_{t+1} - k^* \\ k_t - k^* \\ z_t - z^* \end{pmatrix} = \begin{pmatrix} \xi_{t+1} \\ 0 \\ \epsilon_{t+1} \end{pmatrix}$$

And then

$$\begin{pmatrix} k_{t+2} - k^* \\ k_{t+1} - k^* \\ z_{t+1} - z^* \end{pmatrix} = - \begin{pmatrix} V_3 & V_2 & V_5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & V_1 & V_4 \\ -1 & 0 & 0 \\ 0 & 0 & -\rho \end{pmatrix} \begin{pmatrix} k_{t+1} - k^* \\ k_t - k^* \\ z_t - z^* \end{pmatrix} + \begin{pmatrix} V_3 & V_2 & V_5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \xi_{t+1} \\ 0 \\ \epsilon_{t+1} \end{pmatrix}$$

Note that we can write the V 's as follows such that we can split them into nicer parts:

$$\begin{aligned}
V_1 &= \frac{\partial}{\partial k_t} c_{t+1} - \beta \left\{ c_t \frac{\partial}{\partial k_t} [f_{k_{t+1}}(\cdot) + 1 - \delta_{t+1}] + [f_{k_{t+1}}(\cdot) + 1 - \delta_{t+1}] \frac{\partial}{\partial k_t} c_t \right\} \\
V_2 &= \frac{\partial}{\partial k_{t+1}} c_{t+1} - \beta \left\{ c_t \frac{\partial}{\partial k_{t+1}} [f_{k_{t+1}}(\cdot) + 1 - \delta_{t+1}] + [f_{k_{t+1}}(\cdot) + 1 - \delta_{t+1}] \frac{\partial}{\partial k_{t+1}} c_t \right\} \\
V_3 &= \frac{\partial}{\partial k_{t+2}} c_{t+1} - \beta \left\{ c_t \frac{\partial}{\partial k_{t+2}} [f_{k_{t+1}}(\cdot) + 1 - \delta_{t+1}] + [f_{k_{t+1}}(\cdot) + 1 - \delta_{t+1}] \frac{\partial}{\partial k_{t+2}} c_t \right\} \\
V_4 &= \frac{\partial}{\partial z_t} c_{t+1} - \beta \left\{ c_t \frac{\partial}{\partial z_t} [f_{k_{t+1}}(\cdot) + 1 - \delta_{t+1}] + [f_{k_{t+1}}(\cdot) + 1 - \delta_{t+1}] \frac{\partial}{\partial z_t} c_t \right\} \\
V_5 &= \frac{\partial}{\partial z_{t+1}} c_{t+1} - \beta \left\{ c_t \frac{\partial}{\partial z_{t+1}} [f_{k_{t+1}}(\cdot) + 1 - \delta_{t+1}] + [f_{k_{t+1}}(\cdot) + 1 - \delta_{t+1}] \frac{\partial}{\partial z_{t+1}} c_t \right\}
\end{aligned}$$

where we can use at the equilibrium

$$\begin{aligned}
c_t|_* &= z_t^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} - k_{t+1} + k_t - \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_t^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}} \\
f_{k_{t+1}}|_* &= \alpha \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} z_{t+1}^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\
\delta_{t+1}|_* &= \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} z_{t+1}^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}}
\end{aligned}$$

We have

$$z_{t+1} = \left(\exp(\epsilon_{t+1}) z_t^\rho \right)$$

We have the Euler error given by

$$V_t = c_{t+1} - \beta c_t [f_{k_{t+1}}(k_{t+1}, U_{t+1}, z_{t+1}) + 1 - \delta_{t+1}(k_{t+1})],$$

We have

$$U_t = \left(\frac{\alpha}{\delta\phi} z_t^{1-\alpha} k_t^{\alpha-1} \right)^{\frac{1}{\phi-\alpha}}$$

$$U_{t+1} = \left(\frac{\alpha}{\delta\phi} z_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} \right)^{\frac{1}{\phi-\alpha}}$$

or alternatively

$$= \left(\frac{\alpha}{\delta\phi} \left(\exp(\epsilon_{t+1}) [1 - \alpha] \right) z_t^{\rho^{1-\alpha}} k_{t+1}^{\alpha-1} \right)^{\frac{1}{\phi-\alpha}}$$

$$= \left(\frac{\alpha}{\delta\phi} \right)^{\frac{1}{\phi-\alpha}} \exp \left(\epsilon_{t+1} \frac{1-\alpha}{\phi-\alpha} \right) z_t^{\rho \frac{1-\alpha}{\phi-\alpha}} k_{t+1}^{\frac{\alpha-1}{\phi-\alpha}}$$

And

$$c_t = z_t^{1-\alpha} k_t^\alpha U_t^\alpha - k_{t+1} + k_t - \delta U_t^\phi k_t,$$

$$= z_t^{1-\alpha} k_t^\alpha \left(\frac{\alpha}{\delta\phi} z_t^{1-\alpha} k_t^{\alpha-1} \right)^{\frac{\alpha}{\phi-\alpha}} - k_{t+1} + k_t - \delta \left(\frac{\alpha}{\delta\phi} z_t^{1-\alpha} k_t^{\alpha-1} \right)^{\frac{\phi}{\phi-\alpha}} k_t$$

$$= z_t^{1-\alpha + \frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} - k_{t+1} + k_t - \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_t^{1 + \frac{(\alpha-1)\phi}{\phi-\alpha}}$$

$$c_{t+1} = z_{t+1}^{1-\alpha + \frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} - k_{t+2} + k_{t+1} - \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} z_{t+1}^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_{t+1}^{1 + \frac{(\alpha-1)\phi}{\phi-\alpha}}$$

or alternatively:

$$c_{t+1} = \left(\exp(\epsilon_{t+1}) z_t^\rho \right)^{1-\alpha + \frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} - k_{t+2} + k_{t+1} - \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} \left(\exp(\epsilon_{t+1}) z_t^\rho \right)^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_{t+1}^{1 + \frac{(\alpha-1)\phi}{\phi-\alpha}}$$

Then we have

$$f_{k_{t+1}}(\cdot) = \alpha z_{t+1}^{1-\alpha} U_{t+1}^\alpha k_{t+1}^{\alpha-1}$$

$$= \alpha z_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} \left(\frac{\alpha}{\delta\phi} z_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} \right)^{\frac{\alpha}{\phi-\alpha}}$$

$$= \alpha z_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} z_{t+1}^{\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\alpha}{\phi-\alpha}}$$

$$= \alpha \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} z_{t+1}^{1-\alpha + \frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha-1 + \frac{(\alpha-1)\alpha}{\phi-\alpha}}$$

or alternatively

$$= \alpha \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} \left(\exp(\epsilon_{t+1}) z_t^\rho \right)^{1-\alpha + \frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha-1 + \frac{(\alpha-1)\alpha}{\phi-\alpha}}$$

$$= \alpha \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\alpha}{\phi-\alpha}} \exp \left(\epsilon_{t+1} \left(1 - \alpha + \frac{(1-\alpha)\alpha}{\phi-\alpha} \right) \right) z_t^{\rho \left(1 - \alpha + \frac{(1-\alpha)\alpha}{\phi-\alpha} \right)} k_{t+1}^{\alpha-1 + \frac{(\alpha-1)\alpha}{\phi-\alpha}}$$

And lastly we have

$$\delta_{t+1} = \delta U_{t+1}^\phi = \delta \left(\frac{\alpha}{\delta\phi} z_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} \right)^{\frac{\phi}{\phi-\alpha}}$$

$$= \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} z_{t+1}^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}}$$

or alternatively

$$\begin{aligned} &= \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} \left(\exp(\epsilon_{t+1}) z_t^\rho \right)^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}} \\ &= \delta \left(\frac{\alpha}{\delta\phi} \right)^{\frac{\phi}{\phi-\alpha}} \exp \left(\epsilon_{t+1} \frac{(1-\alpha)\phi}{\phi-\alpha} \right) z_t^{\frac{(1-\alpha)\rho\phi}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}} \end{aligned}$$

Dynare: Policy Function using Parameter Set (i)

$$k_t = \underbrace{60.62}_{=G_0} + \underbrace{0.974}_{=G_1}(k_{t-1} - \bar{k}) + \underbrace{2.055}_{=G_2}(x_t - \bar{x})$$