Dynamic Macroeconomics with Numerics: Project II

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Part 1: Blanchard Kahn Approach

First note that we have

$$z_{t+1} = \exp(\epsilon_{t+1})z_t^\rho,$$

but we may also use

$$z_{t+1} = \exp(x_{t+1}) = \exp(\rho x_t + \epsilon_{t+1}),$$

Furthermore, we have from the Euler error defined in the last project two different capital utilization functions, differing only in the exponent

$$U_{t+1}^\alpha = \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} \exp\left(\frac{(1-\alpha)\alpha}{\phi-\alpha}x_{t+1}\right)k_{t+1}^{\frac{(\alpha-1)\alpha}{\phi-\alpha}}$$

$$U_{t+1}^\phi = \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\phi}{\phi-\alpha}} \exp\left(\frac{(1-\alpha)\phi}{\phi-\alpha}x_{t+1}\right)k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}}$$

We have a **three-equation system**, having already plugged the values for U_t, U_{t+1} given by

$$\mathbf{F}(\mathbf{s}_{t+1}, \mathbf{s}_t) = \begin{pmatrix} (1) \\ (2) \\ (3) \end{pmatrix},$$

with the elements being given by the equations below:

$$c_{t+1} - \beta c_t\left[\alpha\exp\left(\left[1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}\right]x_{t+1}\right)k_{t+1}^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta\left(\frac{\alpha}{\delta\phi}\right)^{\frac{\phi}{\phi-\alpha}}k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}}\exp\left(x_{t+1}\frac{(1-\alpha)\phi}{\phi-\alpha}\right)\right] \quad (1)$$

$$\exp\left(\left[1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}\right]x_t\right)k_t^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} + k_t - \delta\left(\frac{\alpha}{\delta\phi}\right)^{\frac{\phi}{\phi-\alpha}}\exp\left(\frac{(1-\alpha)\phi}{\phi-\alpha}x_t\right)k_t^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}} - c_t - k_{t+1} \quad (2)$$

$$x_{t+1} - \rho x_t \quad (3)$$

Now we take the partial derivatives w.r.t. $c_t, c_{t+1}, k_t, k_{t+1}, x_t, x_{t+1}$ for all three equations evaluated at the steady state. First the Euler error partial derivatives evaluated at the steady state:

$$\frac{\partial}{c_t}(1) = -\beta\left[\alpha k_*^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta\left(\frac{\alpha}{\delta\phi}\right)^{\frac{\phi}{\phi-\alpha}}k_*^{\frac{(\alpha-1)\phi}{\phi-\alpha}}\right]$$

$$\frac{\partial}{k_t}(1) = 0$$

$$\frac{\partial}{x_t}(1) = 0$$

$$\frac{\partial}{c_{t+1}}(1) = 1$$

$$\frac{\partial}{k_{t+1}}(1) = -\beta c_*\left[\alpha\left(\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}\right)k_*^{\alpha-2+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} - \delta\left(\frac{\alpha}{\delta\phi}\right)^{\frac{\phi}{\phi-\alpha}}k_*^{\frac{(\alpha-1)\phi}{\phi-\alpha}-1}\left(\frac{(\alpha-1)\phi}{\phi-\alpha}\right)\right]$$

$$\frac{\partial}{x_{t+1}}(1) = -\beta c_*\left[\alpha\left[1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}\right]k_*^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} - \delta\left(\frac{\alpha}{\delta\phi}\right)^{\frac{\phi}{\phi-\alpha}}k_*^{\frac{(\alpha-1)\phi}{\phi-\alpha}}\frac{(1-\alpha)\phi}{\phi-\alpha}\right]$$

Now the law of motion for capital's partial derivatives evaluated at the steady state:

$$\frac{\partial}{c_t}(2) = -1$$

$$\frac{\partial}{k_t}(2) = \left(\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}\right)k_*^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta\left(\frac{\alpha}{\delta\phi}\right)^{\frac{\phi}{\phi-\alpha}}\left(1+\frac{(\alpha-1)\phi}{\phi-\alpha}\right)k_*^{\frac{(\alpha-1)\phi}{\phi-\alpha}}$$

$$\frac{\partial}{x_t}(2) = \left[1 - \alpha + \frac{(1-\alpha)\alpha}{\phi - \alpha}\right] k_*^{\alpha + \frac{(\alpha-1)\alpha}{\phi - \alpha}} \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} - \delta\left(\frac{\alpha}{\delta\phi}\right)^{\frac{\phi}{\phi-\alpha}} \frac{(1-\alpha)\phi}{\phi - \alpha} k_*^{1 + \frac{(\alpha-1)\phi}{\phi - \alpha}}$$

$$\frac{\partial}{c_{t+1}}(2) = 0$$

$$\frac{\partial}{k_{t+1}}(2) = -1$$

$$\frac{\partial}{x_{t+1}}(2) = 0$$

And lastly the error term from the law of motion for x's partial derivatives evaluated at the steady state:

$$\frac{\partial}{c_t}(3) = 0$$

$$\frac{\partial}{k_t}(3) = 0$$

$$\frac{\partial}{x_t}(3) = -\rho$$

$$\frac{\partial}{c_{t+1}}(3) = 0$$

$$\frac{\partial}{k_{t+1}}(3) = 0$$

$$\frac{\partial}{x_{t+1}}(3) = 1$$

We have the two matrices

$$D\mathbf{F}_1(\cdot) = \begin{pmatrix} \frac{\partial}{\partial c_t}(1) & \frac{\partial}{\partial k_t}(1) & \frac{\partial}{\partial x_t}(1) \\ \frac{\partial}{\partial c_t}(2) & \frac{\partial}{\partial k_t}(2) & \frac{\partial}{\partial x_t}(2) \\ \frac{\partial}{\partial c_t}(3) & \frac{\partial}{\partial k_t}(3) & \frac{\partial}{\partial x_t}(3) \end{pmatrix}, \quad D\mathbf{F}_2(\cdot) = \begin{pmatrix} \frac{\partial}{\partial c_{t+1}}(1) & \frac{\partial}{\partial k_{t+1}}(1) & \frac{\partial}{\partial x_{t+1}}(1) \\ \frac{\partial}{\partial c_{t+1}}(2) & \frac{\partial}{\partial k_{t+1}}(2) & \frac{\partial}{\partial x_{t+1}}(2) \\ \frac{\partial}{\partial c_{t+1}}(3) & \frac{\partial}{\partial k_{t+1}}(3) & \frac{\partial}{\partial x_{t+1}}(3) \end{pmatrix}$$

Due to the complexity of the following operation we do not provide any detailed analytical description, but it follows that we have the Jacobian of the state space given by

$$\mathbf{J} = -D\mathbf{F}_2(\mathbf{s}_*, \mathbf{s}_*)^{-1} D\mathbf{F}_1(\mathbf{s}_*, \mathbf{s}_*)$$

The result is a 3×3 matrix. We then proceed in the exact same way as we did in the fourth and fifth exercise session, so any details beyond the policy functions are omitted.

Note: our policy function is written to match the dynare output, so we take deviations as the input.
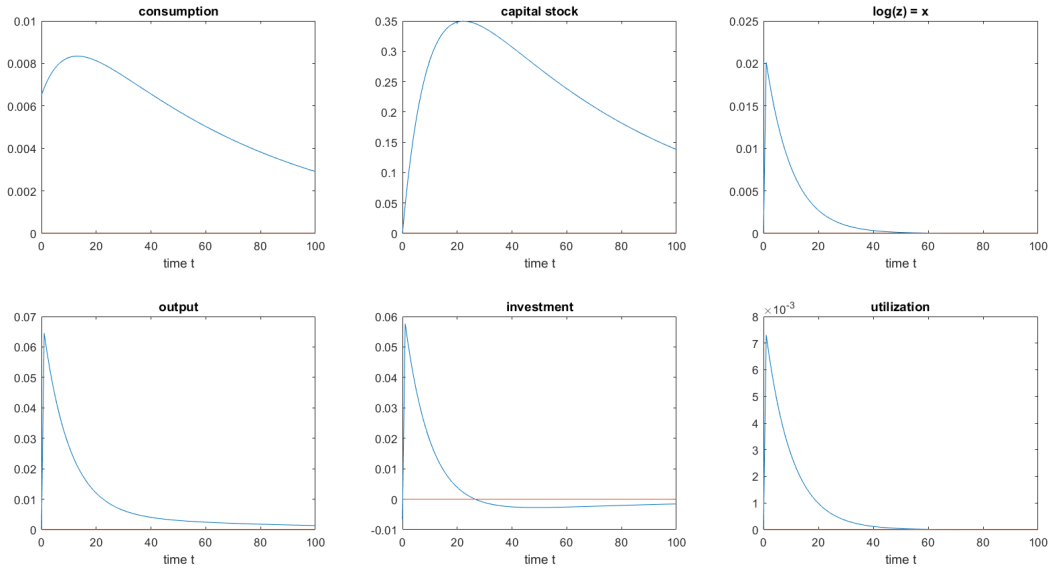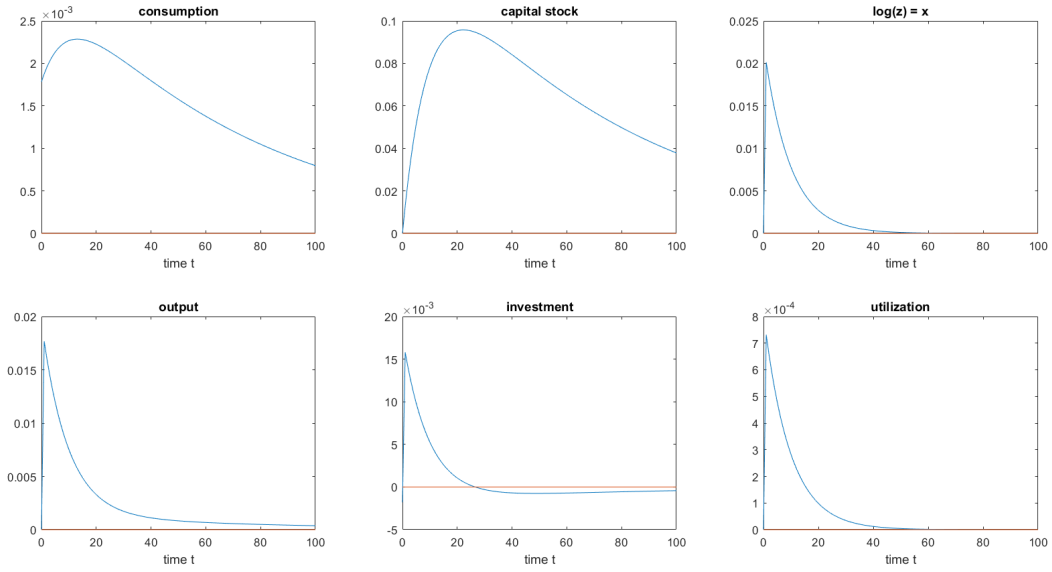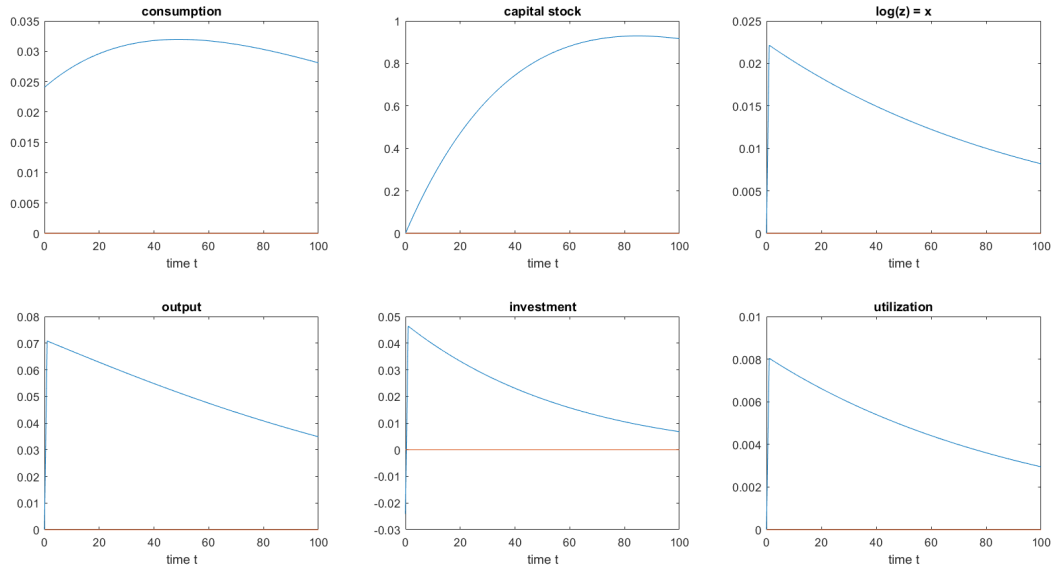
(d) IRFs

Figure 1: (i): $\phi = 1.5,\ \delta = 0.0285,\ \rho = 0.9$

Figure 2: (ii): $\phi = 1.5,\ \delta = 0.9,\ \rho = 0.9$

Figure 3: (iii): $\phi = 1.5,\ \delta = 0.0285,\ \rho = 0.99$

Figure 4: (iv): $\phi = 1.1,\ \delta = 0.0285,\ \rho = 0.9$