Dynamic Macroeconomics with Numerics: Project II

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One-Sector Stochastic Growth Model

Notes on optimality conditions

We have the Euler error given by

$$V_t = c_{t+1} - \beta c_t[f_{k_{t+1}}(k_{t+1}, U_{t+1}, z_{t+1}) + 1 - \delta_{t+1}(k_{t+1})],$$

We have

$$\begin{aligned}
U_t &= \left(\frac{\alpha}{\delta\phi} z_t^{1-\alpha} k_t^{\alpha-1}\right)^{\frac{1}{\phi-\alpha}} \\
U_{t+1} &= \left(\frac{\alpha}{\delta\phi} z_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1}\right)^{\frac{1}{\phi-\alpha}}
\end{aligned}$$

And

$$\begin{aligned}
c_t &= z_t^{1-\alpha} k_t^{\alpha} U_t^{\alpha} - k_{t+1} + k_t - \delta U_t^{\phi} k_t, \\
&= z_t^{1-\alpha} k_t^{\alpha} \left(\frac{\alpha}{\delta\phi} z_t^{1-\alpha} k_t^{\alpha-1}\right)^{\frac{\alpha}{\phi-\alpha}} - k_{t+1} + k_t - \delta \left(\frac{\alpha}{\delta\phi} z_t^{1-\alpha} k_t^{\alpha-1}\right)^{\frac{\phi}{\phi-\alpha}} k_t \\
&= z_t^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} - k_{t+1} + k_t - \delta \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\phi}{\phi-\alpha}} z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_t^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}} \\
c_{t+1} &= z_{t+1}^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} - k_{t+2} + k_{t+1} - \delta \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\phi}{\phi-\alpha}} z_{t+1}^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_{t+1}^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}}
\end{aligned}$$

Then we have

$$\begin{aligned}
f_{k_{t+1}}(\cdot) &= \alpha z_{t+1}^{1-\alpha} U_{t+1}^{\alpha} k_{t+1}^{\alpha-1} \\
&= \alpha z_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} \left(\frac{\alpha}{\delta\phi} z_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1}\right)^{\frac{\alpha}{\phi-\alpha}} \\
&= \alpha z_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} z_{t+1}^{\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\
&= \alpha \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} z_{t+1}^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}}
\end{aligned}$$

And lastly we have

$$\begin{aligned}
\delta_{t+1} = \delta U_{t+1}^{\phi} &= \delta \left(\frac{\alpha}{\delta\phi} z_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1}\right)^{\frac{\phi}{\phi-\alpha}} \\
&= \delta \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\phi}{\phi-\alpha}} z_{t+1}^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}}
\end{aligned}$$

We can put together all terms and have

$$\begin{aligned}
V_t = {} & z_{t+1}^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} - k_{t+2} + k_{t+1} - \delta \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\phi}{\phi-\alpha}} z_{t+1}^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_{t+1}^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}} \\
& - \beta \underbrace{\left\{ z_t^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} - k_{t+1} + k_t - \delta \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\phi}{\phi-\alpha}} z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_t^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}} \right\}}_{=c_t} \alpha \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} z_{t+1}^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\
& - \beta \underbrace{\left\{ z_t^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} - k_{t+1} + k_t - \delta \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\phi}{\phi-\alpha}} z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_t^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}} \right\}}_{=c_t} \\
& + \beta \underbrace{\left\{ z_t^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} - k_{t+1} + k_t - \delta \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\phi}{\phi-\alpha}} z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_t^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}} \right\}}_{=c_t} \delta \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\phi}{\phi-\alpha}} z_{t+1}^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}}
\end{aligned}$$

Now we can take the partial derivatives:

$$\frac{\partial}{\partial k_{t+2}} V_t = -1$$
$$V_3 = -1$$

Next we have

$$\begin{aligned}
\frac{\partial}{\partial k_{t+1}} V_t =& \Big(\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}\Big) z_{t+1}^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta\Big(1 + \frac{(\alpha-1)\alpha}{\phi-\alpha}\Big)\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} z_{t+1}^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\
&- \beta\Big\{ z_t^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} - k_{t+1} + k_t - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_t^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}} \Big\} \\
&\cdot \alpha\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} \Big(\alpha - 1 + \frac{(\alpha-1)\alpha}{\phi-\alpha}\Big) z_{t+1}^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha-2+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\
&- (-1)\alpha\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} z_{t+1}^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\
&- \beta(-1) \\
&+ \beta\Big\{ z_t^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} - k_{t+1} + k_t - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_t^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}} \Big\} \\
&\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} \Big(\frac{(\alpha-1)\alpha}{\phi-\alpha}\Big) z_{t+1}^{\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\alpha}{\phi-\alpha}-1} \\
&- \beta\delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} z_{t+1}^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}}
\end{aligned}$$

and evaluated at the steady state this is

$$\begin{aligned}
V_2 =& \Big(\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}\Big) k_*^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta\Big(1 + \frac{(\alpha-1)\alpha}{\phi-\alpha}\Big)\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} k_*^{\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\
&- \beta\Big\{ k_*^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} k_*^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}} \Big\} \cdot \alpha\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} \Big(\alpha - 1 + \frac{(\alpha-1)\alpha}{\phi-\alpha}\Big) k_*^{\alpha-2+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\
&- (-1)\alpha\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} k_*^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\
&- \beta(-1) \\
&+ \beta\Big\{ k_*^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} k_*^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}} \Big\} \Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} \Big(\frac{(\alpha-1)\alpha}{\phi-\alpha}\Big) k_*^{\frac{(\alpha-1)\alpha}{\phi-\alpha}-1} \\
&- \beta\delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} k_*^{\frac{(\alpha-1)\phi}{\phi-\alpha}}
\end{aligned}$$

The next derivative is given by

$$\begin{aligned}
\frac{\partial}{\partial k_t} V_t =& \beta\Big\{ \Big(\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}\Big) z_t^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta\Big(1 + \frac{(\alpha-1)\phi}{\phi-\alpha}\Big)\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_t^{\frac{(\alpha-1)\phi}{\phi-\alpha}} \Big\} \\
&\alpha\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} z_{t+1}^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\
&- \beta\Big\{ \Big(\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}\Big) z_t^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta\Big(1 + \frac{(\alpha-1)\phi}{\phi-\alpha}\Big)\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_t^{\frac{(\alpha-1)\phi}{\phi-\alpha}} \Big\} \\
&+ \beta\Big\{ \Big(\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}\Big) z_t^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta\Big(1 + \frac{(\alpha-1)\phi}{\phi-\alpha}\Big)\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_t^{\frac{(\alpha-1)\phi}{\phi-\alpha}} \Big\} \\
&\delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} z_{t+1}^{\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\alpha}{\phi-\alpha}}
\end{aligned}$$

and evaluated at the steady state this is

$$
\begin{aligned}
V_1 = \beta\Big\{&\Big(\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}\Big)k_*^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta\Big(1+\frac{(\alpha-1)\phi}{\phi-\alpha}\Big)\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}k_*^{\frac{(\alpha-1)\phi}{\phi-\alpha}}\Big\}\alpha\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}}k_*^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\
- \beta\Big\{&\Big(\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}\Big)k_*^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta\Big(1+\frac{(\alpha-1)\phi}{\phi-\alpha}\Big)\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}k_*^{\frac{(\alpha-1)\phi}{\phi-\alpha}}\Big\} \\
+ \beta\Big\{&\Big(\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}\Big)k_*^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta\Big(1+\frac{(\alpha-1)\phi}{\phi-\alpha}\Big)\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}k_*^{\frac{(\alpha-1)\phi}{\phi-\alpha}}\Big\}\delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}k_*^{\frac{(\alpha-1)\alpha}{\phi-\alpha}}
\end{aligned}
$$

Note: $\frac{\partial}{\partial k_t}c_t = \Big\{\Big(\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}\Big)z_t^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}}k_t^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta\Big(1+\frac{(\alpha-1)\phi}{\phi-\alpha}\Big)\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}}k_t^{\frac{(\alpha-1)\phi}{\phi-\alpha}}\Big\}$
Next we have the derivatives w.r.t. the technology shocks

$$
\begin{aligned}
\frac{\partial}{\partial z_{t+1}}V_t = &\Big(1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}\Big)\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}}z_{t+1}^{\frac{(1-\alpha)\alpha}{\phi-\alpha}-\alpha}k_{t+1}^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}} - \delta\Big(\frac{(1-\alpha)\phi}{\phi-\alpha}\Big)\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}z_{t+1}^{\frac{(1-\alpha)\phi}{\phi-\alpha}-1}k_{t+1}^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}} \\
&- \beta\Big\{z_t^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}}k_t^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} - k_{t+1} + k_t - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}}k_t^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}}\Big\} \\
&\cdot \alpha\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}}\Big(1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}\Big)z_{t+1}^{\frac{(1-\alpha)\alpha}{\phi-\alpha}-\alpha}k_{t+1}^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\
&- \beta\Big\{z_t^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}}k_t^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} - k_{t+1} + k_t - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}}k_t^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}}\Big\}\delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}z_{t+1}^{\frac{(1-\alpha)\phi}{\phi-\alpha}-1}k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}}
\end{aligned}
$$

which, at the steady state is

$$
\begin{aligned}
V_5 = &\Big(1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}\Big)\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}}k_*^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}} - \delta\Big(\frac{(1-\alpha)\phi}{\phi-\alpha}\Big)\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}k_*^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}} \\
&- \beta\Big\{k_*^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}k_*^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}}\Big\}\alpha\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}}\Big(1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}\Big)k_*^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\
&- \beta\Big\{k_*^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}k_*^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}}\Big\}\delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}k_*^{\frac{(\alpha-1)\phi}{\phi-\alpha}}
\end{aligned}
$$

And as the last derivative, we have

$$
\begin{aligned}
\frac{\partial}{\partial z_t}V_t = \beta\Big\{&\Big(1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}\Big)z_t^{\frac{(1-\alpha)\alpha}{\phi-\alpha}-\alpha}k_t^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}\Big(\frac{(1-\alpha)\phi}{\phi-\alpha}\Big)z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}-1}k_t^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}}\Big\} \\
\cdot \alpha&\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}}z_{t+1}^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}}k_{t+1}^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\
- \beta\Big\{&\Big(1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}\Big)z_t^{\frac{(1-\alpha)\alpha}{\phi-\alpha}-\alpha}k_t^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}\Big(\frac{(1-\alpha)\phi}{\phi-\alpha}\Big)z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}-1}k_t^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}}\Big\} \\
+ \beta\Big\{&\Big(1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}\Big)z_t^{\frac{(1-\alpha)\alpha}{\phi-\alpha}-\alpha}k_t^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}\Big(\frac{(1-\alpha)\phi}{\phi-\alpha}\Big)z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}-1}k_t^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}}\Big\} \\
\cdot \delta&\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}z_{t+1}^{\frac{(1-\alpha)\phi}{\phi-\alpha}}k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}},
\end{aligned}
$$

and evaluated at the steady state this is

$$
\begin{aligned}
V_4 = \beta\Big\{&\Big(1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}\Big)k_*^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}\Big(\frac{(1-\alpha)\phi}{\phi-\alpha}\Big)k_*^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}}\Big\} \\
\cdot \alpha&\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}}k_*^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\
- \beta\Big\{&\Big(1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}\Big)k_*^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}\Big(\frac{(1-\alpha)\phi}{\phi-\alpha}\Big)k_*^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}}\Big\} \\
+ \beta\Big\{&\Big(1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}\Big)k_*^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}\Big(\frac{(1-\alpha)\phi}{\phi-\alpha}\Big)k_*^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}}\Big\}
\end{aligned}
$$

$$\cdot\,\delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} k_*^{\frac{(\alpha-1)\phi}{\phi-\alpha}}$$

Note: $\frac{\partial}{\partial z_t} c_t = \left\{ \left(1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}\right) z_t^{\frac{(1-\alpha)\alpha}{\phi-\alpha}-\alpha} k_t^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} - \delta\left(\frac{\alpha}{\delta\phi}\right)^{\frac{\phi}{\phi-\alpha}} \left(\frac{(1-\alpha)\phi}{\phi-\alpha}\right) z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}-1} k_t^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}} \right\}$ Next we have

$$\begin{aligned} z_{t+1} &= \exp(\rho x_t + \epsilon_{t+1}) \\ z_t &= \exp(\rho x_{t-1} + \epsilon_t) \end{aligned}$$

We can rewrite due to $x_t = \rho x_{t-1} + \epsilon_t = \log z_t$to

$$\begin{aligned} z_{t+1} &= \exp(\rho(\rho x_{t-1} + \epsilon_t) + \epsilon_{t+1}) \\ &= \exp(\rho \log z_t + \epsilon_{t+1}) \\ \\ &= \exp(\epsilon_{t+1}) z_t^{\rho} \end{aligned}$$

Now we can take the derivative

$$\frac{\partial}{\partial z_t} z_{t+1} = \rho \exp(\epsilon_{t+1}) z_t^{\rho-1}$$

and evaluated at the steady state this equals

$$\frac{\partial}{\partial z_t} z_* = \rho \exp(0) 1^{\rho-1} = \rho$$

Therefore we have

$$(z_{t+1} - z^*) = \rho(z_t - z^*) + \epsilon_{t+1}$$

where ϵ_{t+1} is the approximation error from the Taylor approximation.

To summarize, we have

$$\begin{aligned} \xi_{t+1} &= V_3(k_{t+2} - k^*) + V_2(k_{t+1} - k^*) + V_1(k_t - k^*) + V_5(z_{t+1} - z^*) + V_4(z_t - z^*) \\ 0 &= (k_{t+1} - k^*) - (k_t - k^*) \\ \epsilon_{t+1} &= (z_{t+1} - z^*) - \rho(z_t - z^*) \end{aligned}$$

with V_i being the partial derivatives defined above. Then we have

$$\begin{pmatrix} V_3 & V_2 & V_5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k_{t+2} - k^* \\ k_{t+1} - k^* \\ z_{t+1} - z^* \end{pmatrix} + \begin{pmatrix} 0 & V_1 & V_4 \\ -1 & 0 & 0 \\ 0 & 0 & -\rho \end{pmatrix} \begin{pmatrix} k_{t+1} - k^* \\ k_t - k^* \\ z_t - z^* \end{pmatrix} = \begin{pmatrix} \xi_{t+1} \\ 0 \\ \epsilon_{t+1} \end{pmatrix}$$

And then

$$\begin{pmatrix} k_{t+2} - k^* \\ k_{t+1} - k^* \\ z_{t+1} - z^* \end{pmatrix} = -\begin{pmatrix} V_3 & V_2 & V_5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & V_1 & V_4 \\ -1 & 0 & 0 \\ 0 & 0 & -\rho \end{pmatrix} \begin{pmatrix} k_{t+1} - k^* \\ k_t - k^* \\ z_t - z^* \end{pmatrix} + \begin{pmatrix} V_3 & V_2 & V_5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \xi_{t+1} \\ 0 \\ \epsilon_{t+1} \end{pmatrix}$$

Note that we can write the V's as follows such that we can split them into nicer parts:

$$
\begin{aligned}
V_1 &= \frac{\partial}{\partial k_t} c_{t+1} - \beta\Big\{c_t \frac{\partial}{\partial k_t}[f_{k_{t+1}}(\cdot) + 1 - \delta_{t+1}] + [f_{k_{t+1}}(\cdot) + 1 - \delta_{t+1}]\frac{\partial}{\partial k_t} c_t\Big\} \\
V_2 &= \frac{\partial}{\partial k_{t+1}} c_{t+1} - \beta\Big\{c_t \frac{\partial}{\partial k_{t+1}}[f_{k_{t+1}}(\cdot) + 1 - \delta_{t+1}] + [f_{k_{t+1}}(\cdot) + 1 - \delta_{t+1}]\frac{\partial}{\partial k_{t+1}} c_t\Big\} \\
V_3 &= \frac{\partial}{\partial k_{t+2}} c_{t+1} - \beta\Big\{c_t \frac{\partial}{\partial k_{t+2}}[f_{k_{t+1}}(\cdot) + 1 - \delta_{t+1}] + [f_{k_{t+1}}(\cdot) + 1 - \delta_{t+1}]\frac{\partial}{\partial k_{t+2}} c_t\Big\} \\
V_4 &= \frac{\partial}{\partial z_t} c_{t+1} - \beta\Big\{c_t \frac{\partial}{\partial z_t}[f_{k_{t+1}}(\cdot) + 1 - \delta_{t+1}] + [f_{k_{t+1}}(\cdot) + 1 - \delta_{t+1}]\frac{\partial}{\partial z_t} c_t\Big\} \\
V_5 &= \frac{\partial}{\partial z_{t+1}} c_{t+1} - \beta\Big\{c_t \frac{\partial}{\partial z_{t+1}}[f_{k_{t+1}}(\cdot) + 1 - \delta_{t+1}] + [f_{k_{t+1}}(\cdot) + 1 - \delta_{t+1}]\frac{\partial}{\partial z_{t+1}} c_t\Big\}
\end{aligned}
$$

where we can use at the equilibrium

$$
\begin{aligned}
c_t|_* &= z_t^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} - k_{t+1} + k_t - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_t^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}} \\
f_{k_{t+1}}|_* &= \alpha\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} z_{t+1}^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\
\delta_{t+1}|_* &= \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} z_{t+1}^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}}
\end{aligned}
$$

We have

$$z_{t+1} = \Big(\exp(\epsilon_{t+1}) z_t^{\rho}\Big)$$

Or alternatively we may use

$$\begin{aligned} z_{t+1} &= \exp(x_{t+1}) = \exp(\rho x_t + \epsilon_{t+1}) \\ z_t &= \exp(x_t) \end{aligned}$$

We have the Euler error given by

$$V_t = c_{t+1} - \beta c_t [f_{k_{t+1}}(k_{t+1}, U_{t+1}, z_{t+1}) + 1 - \delta_{t+1}(k_{t+1})],$$

We have

$$\begin{aligned} U_t &= \Big(\frac{\alpha}{\delta\phi} z_t^{1-\alpha} k_t^{\alpha-1}\Big)^{\frac{1}{\phi-\alpha}} \\ U_{t+1} &= \Big(\frac{\alpha}{\delta\phi} z_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1}\Big)^{\frac{1}{\phi-\alpha}} \end{aligned}$$

or alternatively

$$\begin{aligned} &= \Big(\frac{\alpha}{\delta\phi}\Big(\exp(\epsilon_{t+1}[1-\alpha]) z_t^{\rho 1-\alpha}\Big) k_{t+1}^{\alpha-1}\Big)^{\frac{1}{\phi-\alpha}} \\ &= \Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{1}{\phi-\alpha}} \exp\Big(\epsilon_{t+1}\frac{1-\alpha}{\phi-\alpha}\Big) z_t^{\rho\frac{1-\alpha}{\phi-\rho}} k_{t+1}^{\frac{\alpha-1}{\phi-\alpha}} \end{aligned}$$

And

$$\begin{aligned} c_t &= z_t^{1-\alpha} k_t^{\alpha} U_t^{\alpha} - k_{t+1} + k_t - \delta U_t^{\phi} k_t, \\ &= z_t^{1-\alpha} k_t^{\alpha}\Big(\frac{\alpha}{\delta\phi} z_t^{1-\alpha} k_t^{\alpha-1}\Big)^{\frac{\alpha}{\phi-\alpha}} - k_{t+1} + k_t - \delta\Big(\frac{\alpha}{\delta\phi} z_t^{1-\alpha} k_t^{\alpha-1}\Big)^{\frac{\phi}{\phi-\alpha}} k_t \\ &= z_t^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_t^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} - k_{t+1} + k_t - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} z_t^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_t^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}} \\ c_{t+1} &= z_{t+1}^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} - k_{t+2} + k_{t+1} - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} z_{t+1}^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_{t+1}^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}} \end{aligned}$$

or alternatively:

$$c_{t+1} = \Big(\exp(\epsilon_{t+1}) z_t^{\rho}\Big)^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} - k_{t+2} + k_{t+1} - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} \Big(\exp(\epsilon_{t+1}) z_t^{\rho}\Big)^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_{t+1}^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}}$$

Then we have

$$\begin{aligned} f_{k_{t+1}}(\cdot) &= \alpha z_{t+1}^{1-\alpha} U_{t+1}^{\alpha} k_{t+1}^{\alpha-1} \\ &= \alpha z_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1}\Big(\frac{\alpha}{\delta\phi} z_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1}\Big)^{\frac{\alpha}{\phi-\alpha}} \\ &= \alpha z_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} z_{t+1}^{\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\ &= \alpha\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} z_{t+1}^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \end{aligned}$$

or alternatively

$$= \alpha\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} \Big(\exp(\epsilon_{t+1}) z_t^{\rho}\Big)^{1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}} k_{t+1}^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}}$$

$$= \alpha\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} \exp\Big(\epsilon_{t+1}\Big(1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}\Big)\Big) z_t^{\rho\Big(1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}\Big)} k_{t+1}^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}}$$

And lastly we have

$$\begin{aligned}
\delta_{t+1} = \delta U_{t+1}^{\phi} &= \delta\Big(\frac{\alpha}{\delta\phi} z_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1}\Big)^{\frac{\phi}{\phi-\alpha}} \\
&= \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} z_{t+1}^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}}
\end{aligned}$$

or alternatively

$$\begin{aligned}
&= \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} \Big(\exp(\epsilon_{t+1}) z_t^{\rho}\Big)^{\frac{(1-\alpha)\phi}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}} \\
&= \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} \exp\Big(\epsilon_{t+1}\frac{(1-\alpha)\phi}{\phi-\alpha}\Big) z_t^{\frac{(1-\alpha)\rho\phi}{\phi-\alpha}} k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}}
\end{aligned}$$

USING x_t we alternatively have

Alternatively for the capital utilization rate using x_t we have

$$U_t = \Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{1}{\phi-\alpha}} \exp\Big(\frac{1-\alpha}{\phi-\alpha}x_t\Big)k_t^{\frac{\alpha-1}{\phi-\alpha}}$$
$$U_{t+1}(x_{t+1}) = \Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{1}{\phi-\alpha}} \exp\Big(\frac{1-\alpha}{\phi-\alpha}x_{t+1}\Big)k_{t+1}^{\frac{\alpha-1}{\phi-\alpha}}$$
$$U_{t+1}(x_t) = \Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{1}{\phi-\alpha}} \exp\Big(\frac{1-\alpha}{\phi-\alpha}(\rho x_t + \epsilon_{t+1})\Big)k_{t+1}^{\frac{\alpha-1}{\phi-\alpha}}$$

Now the depreciation rate:

$$\delta_t = \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} \exp\Big(\frac{(1-\alpha)\phi}{\phi-\alpha}x_t\Big)k_t^{\frac{\alpha-1}{\phi-\alpha}}$$
$$\delta_{t+1}(x_{t+1}) = \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} \exp\Big(\frac{(1-\alpha)\phi}{\phi-\alpha}x_{t+1}\Big)k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}}$$
$$\delta_{t+1}(x_t) = \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}} \exp\Big(\frac{(1-\alpha)\phi}{\phi-\alpha}(\rho x_t + \epsilon_{t+1})\Big)k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}}$$

Now we need the partial derivatives of all these terms w.r.t. certain components evaluated at the steady state:

First we have **current consumption**

$$\frac{\partial}{\partial k_{t+2}}c_t = 0$$
$$\frac{\partial}{\partial k_{t+1}}c_t = -1$$
$$\frac{\partial}{\partial k_t}c_t = \Big(\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}\Big)k_t^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta\Big(1 + \frac{(\alpha-1)\phi}{\phi-\alpha}\Big)\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}k_t^{\frac{(\alpha-1)\phi}{\phi-\alpha}}$$
$$\frac{\partial}{\partial z_{t+1}}c_t = 0$$
$$\frac{\partial}{\partial z_t}c_t = \Big(1-\alpha + \frac{(1-\alpha)\alpha}{\phi-\alpha}\Big)k_t^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} - \delta\Big(\frac{(1-\alpha)\phi}{\phi-\alpha}\Big)\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}k_t^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}}$$

Now we have **next-period consumption**

$$\frac{\partial}{\partial k_{t+2}}c_{t+1} = -1$$
$$\frac{\partial}{\partial k_{t+1}}c_{t+1} = \Big(\alpha + \frac{(\alpha-1)\alpha}{\phi-\alpha}\Big)k_{t+1}^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta\Big(1 + \frac{(\alpha-1)\phi}{\phi-\alpha}\Big)\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}}$$
$$\frac{\partial}{\partial k_t}c_{t+1} = 0$$
$$\frac{\partial}{\partial z_{t+1}}c_{t+1} = \Big(1-\alpha + \frac{(1-\alpha)\alpha}{\phi-\alpha}\Big)k_{t+1}^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}\Big(\frac{(1-\alpha)\phi}{\phi-\alpha}\Big)k_{t+1}^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}}$$
$$\frac{\partial}{\partial z_t}c_{t+1} = \Big(\rho - \rho\alpha + \frac{(1-\alpha)\alpha\rho}{\phi-\alpha}\Big)k_{t+1}^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}} - \delta\frac{(1-\alpha)\phi\rho}{\phi-\alpha}k_{t+1}^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}$$

Next we have the **partial derivative of next period output w.r.t. next periods capital**

$$\frac{\partial}{\partial k_{t+2}}f_{k_{t+1}}(\cdot) = 0$$
$$\frac{\partial}{\partial k_{t+1}}f_{k_{t+1}}(\cdot) = \alpha\Big(\alpha - 1 + \frac{(\alpha-1)\alpha}{\phi-\alpha}\Big)\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}}k_{t+1}^{\alpha-2+\frac{(\alpha-1)\alpha}{\phi-\alpha}}$$

$$
\begin{aligned}
\frac{\partial}{\partial k_t} f_{k_{t+1}}(\cdot) &= 0 \\
\frac{\partial}{\partial z_{t+1}} f_{k_{t+1}}(\cdot) &= \alpha\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}}\Big(1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}\Big)k_{t+1}^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}} \\
\frac{\partial}{\partial z_t} f_{k_{t+1}}(\cdot) &= \alpha\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\alpha}{\phi-\alpha}}\Big[\rho\Big(1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}\Big)\Big]k_{t+1}^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}}
\end{aligned}
$$

Lastly we have the **next-period depreciation rate**

$$
\begin{aligned}
\frac{\partial}{\partial k_{t+2}}\delta_{t+1} &= 0 \\
\frac{\partial}{\partial k_{t+1}}\delta_{t+1} &= \frac{(\alpha-1)\phi\delta}{\phi-\alpha}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}-1} \\
\frac{\partial}{\partial k_t}\delta_{t+1} &= 0 \\
\frac{\partial}{\partial z_{t+1}}\delta_{t+1} &= \frac{(1-\alpha)\phi\delta}{\phi-\alpha}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}} \\
\frac{\partial}{\partial z_t}\delta_{t+1} &= \frac{(1-\alpha)\rho\phi\delta}{\phi-\alpha}\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}}
\end{aligned}
$$

We use this in

$$
\begin{aligned}
V_1 &= \frac{\partial}{\partial k_t}c_{t+1} - \beta\Big\{c_t\frac{\partial}{\partial k_t}[f_{k_{t+1}}(\cdot)+1-\delta_{t+1}] + [f_{k_{t+1}}(\cdot)+1-\delta_{t+1}]\frac{\partial}{\partial k_t}c_t\Big\} \\
V_2 &= \frac{\partial}{\partial k_{t+1}}c_{t+1} - \beta\Big\{c_t\frac{\partial}{\partial k_{t+1}}[f_{k_{t+1}}(\cdot)+1-\delta_{t+1}] + [f_{k_{t+1}}(\cdot)+1-\delta_{t+1}]\frac{\partial}{\partial k_{t+1}}c_t\Big\} \\
V_3 &= \frac{\partial}{\partial k_{t+2}}c_{t+1} - \beta\Big\{c_t\frac{\partial}{\partial k_{t+2}}[f_{k_{t+1}}(\cdot)+1-\delta_{t+1}] + [f_{k_{t+1}}(\cdot)+1-\delta_{t+1}]\frac{\partial}{\partial k_{t+2}}c_t\Big\} \\
V_4 &= \frac{\partial}{\partial z_t}c_{t+1} - \beta\Big\{c_t\frac{\partial}{\partial z_t}[f_{k_{t+1}}(\cdot)+1-\delta_{t+1}] + [f_{k_{t+1}}(\cdot)+1-\delta_{t+1}]\frac{\partial}{\partial z_t}c_t\Big\} \\
V_5 &= \frac{\partial}{\partial z_{t+1}}c_{t+1} - \beta\Big\{c_t\frac{\partial}{\partial z_{t+1}}[f_{k_{t+1}}(\cdot)+1-\delta_{t+1}] + [f_{k_{t+1}}(\cdot)+1-\delta_{t+1}]\frac{\partial}{\partial z_{t+1}}c_t\Big\}
\end{aligned}
$$

Dynare: Policy Function using Parameter Set (i)

$$k_t = \underbrace{60.62}_{=G_0} + \underbrace{0.974}_{=G_1}(k_{t-1} - \bar{k}) + \underbrace{2.055}_{=G_2}(x_t - \bar{x})$$

Impulse response: one-time shock

How does capital change from the steady state with a one-time shock to TFP?

$$
\begin{aligned}
(k_t - k^*) &= G_1(k_{t-1} - k^*) + G_2(z_{t-1} - z^*) \\
(k_{t+1} - k^*) &= G_1(k_t - k^*) + G_2(z_t - z^*) \\
(k_{t+1} - k^*) &= G_1(G_1(k_{t-1} - k^*) + G_2(z_{t-1} - z^*)) + G_2(\rho(z_{t-1} - z^*)) \\
&= G_1^2(k_{t-1} - k^*) + G_1G_2(z_{t-1} - z^*) + G_2\rho(z_{t-1} - z^*) \\
&= G_1^2(k_{t-1} - k^*) + (G_1G_2 + G_2\rho)(z_{t-1} - z^*) \\
(k_{t+2} - k^*) &= G_1(G_1^2(k_{t-1} - k^*) + G_1G_2(z_{t-1} - z^*) + G_2\rho(z_{t-1} - z^*)) + G_2(z_{t+1} - z^*) \\
&= G_1(G_1^2(k_{t-1} - k^*) + G_1G_2(z_{t-1} - z^*) + G_2\rho(z_{t-1} - z^*)) + G_2(\rho^2 z_{t-1} - z^*) \\
&= G_1^3(k_{t-1} - k^*) + G_1^2G_2(z_{t-1} - z^*) + G_1G_2(z_{t-1} - z^*) + G_2\rho^2(z_{t-1} - z^*) \\
(k_{t+i} - k^*) &= G_1^i(k_{t-1} - k^*) + G_1^{i-1}G_2(z_{t-1} - z^*) + G_1G_2^{i-1}(z_{t-1} - z^*) + G_2\rho^{i-1}(z_{t-1} - z^*) \\
&= G_1^i(k_{t-1} - k^*) + \left(G_1^{i-1}G_2 + G_1G_2^{i-1} + G_2\rho^{i-1}\right)(z_{t-1} - z^*)
\end{aligned}
$$

So we need $G_1 < 1$ and $G_2 < 1$ in order to have a stable system that always goes back to the steady state.

Furthermore, we have

$$
\begin{aligned}
z_t - z^* &= \rho(z_{t-1} - z^*) \\
z_{t+1} - z^* &= \rho(z_t - z^*) = \rho^2(z_{t-1} - z^*) \\
z_{t+2} - z^* &= \rho(z_{t+1} - z^*) = \rho\rho^2(z_t - z^*) \\
z_{t+i} - z^* &= \rho(z_{t+i-1} - z^*) = \rho^{i+1}(z_{t-1} - z^*)
\end{aligned}
$$

Then we can use this to find the time-path for all other variables based on how evolves:

$$
\begin{aligned}
k_{t+i} &= k^* + G_1^i(k_{t-1} - k^*) + \left(G_1^{i-1}G_2 + G_1G_2^{i-1} + G_2\rho^{i-1}\right)(z_{t-1} - z^*) \\
U_{t+i} &= y_{t+i} & =
\end{aligned}
$$