Dynamic Macroeconomics with Numerics: Project II

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Part 1: Blanchard Kahn Approach

First note that we have

$$z_{t+1} = \exp(\epsilon_{t+1}) z_t^{\rho},$$

but we may also use

$$z_{t+1} = \exp(x_{t+1}) = \exp(\rho x_t + \epsilon_{t+1}),$$

However the derivatives evaluated at the steady state remain the same in either notation. Now we summarize the equilibrium conditions in the following four equations:

$$\begin{aligned}
\mathbb{E}c_{t+1} &= \beta c_t \mathbb{E}\Big[\alpha \exp(x_{t+1})^{1-\alpha} U_{t+1}^{\alpha} k_{t+1}^{\alpha-1} + 1 - \delta U_{t+1}^{\phi}\Big] \\
k_{t+1} &= \exp\big([1-\alpha]x_t\big)(k_t U_t)^{\alpha} + (1 - \delta U_t^{\phi})k_t - c_t \\
U_{t+1} &= \left(\frac{\alpha}{\delta\phi}\right)^{\frac{1}{\phi-\alpha}} \exp\left(\frac{1-\alpha}{\phi-\alpha} x_{t+1}\right) k_{t+1}^{\frac{\alpha-1}{\phi-\alpha}} \\
x_{t+1} &= \rho x_t + \epsilon_{t+1}
\end{aligned}$$

For ease of notation we have

$$\forall i \in \{0, 1, 2, \dots\} : \mathbf{s}_{t+i} = \begin{pmatrix} c_{t+i} \\ U_{t+i} \\ k_{t+i} \\ x_{t+i} \end{pmatrix}.$$

Now we write this system of equations as follows:

$$\mathbf{F}(\mathbf{s}_{t+1}, \mathbf{s}_t) = \begin{pmatrix} c_{t+1} - \beta c_t \Big[\alpha \exp(x_{t+1})^{1-\alpha} U_{t+1}^{\alpha} k_{t+1}^{\alpha-1} + 1 - \delta U_{t+1}^{\phi}\Big] \\ U_{t+1} - \left(\frac{\alpha}{\delta\phi}\right)^{\frac{1}{\phi-\alpha}} \exp\left(\frac{1-\alpha}{\phi-\alpha} x_{t+1}\right) k_{t+1}^{\frac{\alpha-1}{\phi-\alpha}} \\ \exp\big([1-\alpha]x_t\big)(k_t U_t)^{\alpha} + (1 - \delta U_t^{\phi})k_t - c_t - k_{t+1} \\ x_{t-1} - \rho x_t \end{pmatrix},$$

where we have $\mathbb{E}_t \mathbf{F}(\mathbf{s}_{t+1}, \mathbf{s}_t) = \mathbf{0}$.

Now we make use of the Jacobian being defined as

$$\mathbf{J} = -[D\mathbf{F}_2(\mathbf{s}^*, \mathbf{s}^*)]^{-1} D\mathbf{F}_1(\mathbf{s}^*, \mathbf{s}^*),$$

so we can look at two separate matrices. We start by taking the derivatives w.r.t. the period t variables of $\mathbf{s}_t$:

$$D\mathbf{F}_1(\mathbf{s}^*, \mathbf{s}^*) = \begin{pmatrix} -\beta[\alpha U_*^{\alpha} k_*^{\alpha-1} + 1 - \delta U_*^{\phi}] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & \delta\phi U_*^{\phi-1} k_* + \alpha k_*^{\alpha} U_*^{\alpha-1} & (1 - \delta U_*^{\phi}) + \alpha k_*^{\alpha-1} U_*^{\alpha} & (1-\alpha)(k_* U_*)^{\alpha} \\ 0 & 0 & 0 & -\rho \end{pmatrix}$$

Next up is the second matrix containing the partial derivatives w.r.t. the period $t+1$ variables of $\mathbf{s}_{t+1}$:

$$D\mathbf{F}_2(\mathbf{s}^*, \mathbf{s}^*) = \begin{pmatrix} 1 & -\beta c_* \Big[\alpha^2 (U_* k_*)^{\alpha-1} + \delta\phi U_*^{\phi-1}\Big] & -\beta c_* \alpha(\alpha-1) U_*^{\alpha} k_*^{\alpha-2} & -\beta c_* \Big[\alpha(1-\alpha) U_*^{\alpha} k_*^{\alpha-1}\Big] \\ 0 & 1 & -\left(\frac{\alpha}{\delta\phi}\right)^{\frac{1}{\phi-\alpha}} \left(\frac{\alpha-1}{\phi-\alpha} k_*^{\frac{\alpha-1}{\phi-\alpha}-1}\right) & -\left(\frac{\alpha}{\delta\phi}\right)^{\frac{1}{\phi-\alpha}} \frac{1-\alpha}{\phi-\alpha} k_*^{\frac{\alpha-1}{\phi-\alpha}} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We use this matrix in MATLAB.