Dynamic Macroeconomics with Numerics

Lecture and Reading Notes

SS2021

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Why do economists rely on such unrealistic assumptions? It is because a well-chosen simplification can remove the dust and smoke that obscures our view of the workings of economic forces. Although we celebrate the resulting theorems for the insights they deliver, we can apply them successfully only by being vigilant, working hard to understand not just the insights that simplified analyses provide but also how the designs and rule choices they inspire must be adapted to withstand the dust and smoke and also the much larger disturbances of the particular worlds in which the mechanisms will operate.

- Milgrom, Paul (*AER* 2021)

1 Introduction: Lucas' Critique

Why do we work with dynamic stochastic models in modern macroeconomics at all?

These models help us illustrate the quantitative implications of economic theories, thereby empirically testing those theories. Experimental situations may not be easily exploited in macro, thus we use such models and program them, simulating how things change.

According to the Lucas' Critique (1976), meaningful analysis of policy and economic forecasts should have an analytical framework with individuals with two viewed characteristics

- optimal behavior (homo oeconomicus)
- action under uncertainty, i.e. forming expectations about uncertain future events

Prior to this, most macroeconomic models were so-called *large-scale macroeconometric models*. Typical characteristics include *ad hoc* behavioral equations (e.g. your typical Keynesian consumption function, consisting of income and some constant: $C(Y^d) = a + bY^d$, where $Y^d = Y - T$, where T is taxes and transfers; investment as a function of the real interest rate (downward sloping)) aiming at adequate depiction of individual behavior, and pursuing policy analysis or economic forecasting. Behavior equations were considered stable and independent of policy parameters. Their size was large, a typical macroeconomic model consisted of lots of *ad hoc* behavioral equations which were estimated using econometric methods. Prominent representatives include Lawrence Klein (U Pennsylvania), and Wilhelm Krelle (Bonn University).

The Lucas' Critique can be summarized as follows:

> Economic models which ignore individuals' reactions to changes in policy parameters deliver questionable and typically wrong policy recommendations and forecasts.

Consider the following motivation based on a later result: this stability assumption (here: Phillips curve) may not hold when public policy changes. In our example, when agents realize that the government intentionally exploits this relationship in order to reduce unemployment, then the agents learn and the relationship between the output and inflation does not hold anymore. Therefore, we have no parameter stability.

Lucas required "meaningful economic models need to be built on stable characteristics, i.e. on first principles which are invariant to policy changes". For example, production technology, preferences (*this may be questionable*), or market structure (for the sake of the analysis, e.g. BC analysis or growth analysis, the time-span at which this was developed gives plausibility for the assumption). In the long run, all three of these examples will change.

Lucas' critique was decisive in two ways:

- the *microfounded approach* to modern macroeconomics which has become equally popular among neoclassical and Keynesian economists
- development and application of the concept of *rational expectations* in macroeconomics (agents are forwards looking)

Lucas 1987, ch. 2:

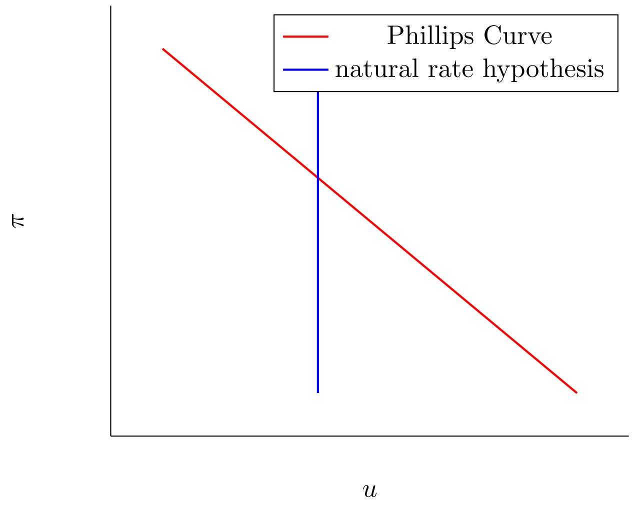
> The main purpose of applied economic analysis is the evaluate the welfare consequences of economic policy.

To actually do this, we need quantifiable models, which serve as a laboratory to simulate the implications that policy changes have on an economy. Such models are called *computable model economies*. The dilemma is as follows, large-scale econometric models contain optimal decision rules of economic agents, those rules vary systematically with exogenous policy changes. Thus, policy changes alter the structure of econometric models, and assumed stability of parameters and functional forms is fundamentally wrong, yielding meaningless forecasts.

Example 1.1 (Lucas' Island Model (AER 1973))**.** The issue to discuss is the question whether or not a stable inflation-unemployment trade-off exists, i.e. a menu from which politicians can choose their preferred combination? We have two dissenting viewpoints here. Economic theory suggests no, according to the natural-rate hypothesis (Friedman 1968), while econometric analysis suggests that it does exist, see the empirical evidence in Menil, Enzler (1972).

The origin of this apparent contradiction comes from what precisely?

Figure 1.1: The Phillips Curve and the Natural Rate Hypothesis



We now build the model. We assume the following:

- rational agents
- market clearing
- N distinct markets i: $i = 1, 2, \ldots, N$
- incomplete information (about the other islands)
- one homogeneous good produced and traded

The supply of the good, logarithmized, in market i in period t is given by

$$y_{it} = y_{it}^p + y_{it}^c, \tag{1.1}$$

where we have an additive supply, consisting of a variable and a permanent (normal), unresponsive component, which is unresponsive to price changes y_{it}^p. Note that this normal, or secular, component y_{it}^p is a reflection of capital accumulation and population growth, i.e. the effect of a fully employed capital stock and full employment of the labor force. Furthermore, it is common to all markets. It is assumed to be a linear trend, around which the cycle, or transitory, component fluctuates. This additivity of the log implies a multiplicative property for the non-logarithmized values.

In the end we are interested in the slope of the aggregate supply, therefore - while not mentioned as much - the trend component of supply plays a crucial role.

The transitory supply varies with the expected relative price change:

$$y_{it}^c = \beta(p_{it} - p_t^e), \ \beta > 0, \tag{1.2}$$

and the observed good price (by the supplier in that market) in market i in period t is

$$p_{it} = p_t + z_{it}, \tag{1.3}$$

where we have (which is known by the suppliers)

$$\begin{aligned} p_t &\sim \mathcal{N}(\bar{p}_t, \sigma^2), \\ z_{it} &\sim \mathcal{N}(0, \tau^2), \\ \mathbb{E} z_{it} z_{js} &= 0 \text{ if } i \neq j, s \neq t, \\ \mathbb{E} p_t z_{it} &= 0. \end{aligned}$$

This implies that the shocks are limited to one period (covariance), and the same holds across markets. Thus, shocks may influence all markets in the period they start, but do not have a *trickle* effect on markets other than the one they occurred in, in the periods following the shock. The last term implies that the price in period t is independent of the shock, i.e. the shock does not influence the *base price*, around which the realized price of market i varies.

The actual general price level in period t is defined as

$$N^{-1} \sum_{i=1}^{N} p_{it} = p_t + N^{-1} \sum_{i=1}^{N} z_{it} = p_t, \tag{1.4}$$

and the supply decision in market i depends on p_t^e in the following way (as information always takes some time to reach a specific island, think of newspaper delivery times in the 60's):

$$\mathbb{E}[p_t \mid p_{it}, I_{t-1}] = (1-\theta) p_{it} + \theta \bar{p}_t, \tag{1.5}$$

where $\theta = \tau^2 / (\sigma^2 + \tau^2)$ is the noise ratio (the fraction of the total price variation - local and aggregate - due to the local disturbances), and I_{t-1} is the information set, i.e. a weighted mean of the past prices. This is equivalent to a normal equation in linear regression. The more volatile local demand shocks are, the less weight I would want to put on the local shocks. Inserting (5) into (2) yields the transitory supply in market i:

$$y_{it}^c = \beta[p_{it} - (1-\theta) p_{it} - \theta \bar{p}_t] = \beta\theta(p_{it} - \bar{p}_t). \tag{1.6}$$

As β is constant, we know that any changes must come from the noise ratio. Averaging over all N markets yields the aggregate transitory supply

$$y_t^c = \beta\theta(p_t - \bar{p}_t). \tag{1.7}$$

Thus, the total aggregate supply in the economy is

$$y_t = \beta\theta(p_t - \bar{p}_t) + y_t^p. \tag{1.8}$$

Aggregate supply is increasing for a positive deviation from the average price, based on past prices. The slope is increasing when the local disturbance is large in relation to the total disturbances. If θ is small (country-specific variations are small compared to the global variations in prices), then we know a country has relatively stable prices, thus giving larger weight to the trend component.

So, the question arises whether or not parameter stability is likely to be fulfilled. This may not be the case as β is ad-hoc and we have variation in economies (and their demand shock variances). Assume that the aggregate price level follows a random walk, where the error normal with the mean being equal to inflation.

$$p_t = p_{t-1} + \epsilon_t, \ \epsilon_t \sim \mathcal{N}(\pi, \sigma^2).$$

Then, $\bar{p}_t = p_{t-1} + \pi$, i.e. the evaluation of the expectation, and (8) can be rewritten as

$$y_t = \beta\theta(p_t - p_{t-1}) - \beta\theta\pi + y_t^p. \tag{1.9}$$

This has two implications:

- Short-run: stable inflation-output trade-off
- Long-run: $y_t = y_t^p$, i.e. no such trade-off

This raises some key questions, namely whether equation (1.9) is appropriate for forecasting when the inflation rate π rises. The answer here would be that it depends. If the politicians have exploited this trade-off too often in the past, the agents probability of being being fooled does not work as well, or at all. Thus, it may not be very wise to use this as a forecast.

Some International Evidence on Output-Inflation Tradeoffs (Lucas, 1973)

Note 1.1 (Reading Notes)**.** The paper considers the empirical plausibility of the hypothesis that "average real output levels are invariant under changes in the time pattern of the rate of inflation". This takes into consideration how the natural rate theory performs empirically, what can be tested, and whether or not these tested restrictions work together.

Nominal output is determined by the aggregate demand, while the function giving the real output, i.e. the division by prices, is dependent on the behavior of suppliers of labor and goods. We assume partial rigidities due to lack of information on some prices. Lastly we assume optimal behavior of the agents, depending on the stochastic character of the economy. Such theories do not imply testable restrictions on the estimated coefficients (Phillips curve, single equation expressions of this trade-off). We have

$$\text{supply parameters} \overset{\text{theory links}}{\leftrightarrow} \text{parameters governing stochastic nature of demand shifts}$$

The natural rate theory suggests such a dependence, thus we may test it.

Aggregate prices are viewed as intersection points of aggregate demand and supply schedule. Thus we assume cleared money markets (represents output-price level relationship, IS-LM diagram). Demand shift due to monetary, fiscal, and export (demand) policy. Supply schedule assumes a cleared labor market.

Agents form their demand based on relative prices only, thus they are unable to distinguish relative from general price movements. We assume a large number of competitive markets. Demand is not distributed evenly over the markets, thus we have relative and general price movements. We define an multiplicative-logarithmized supply (trend + cycle component):

$$y_t(z) = y_{nt} + y_{ct}(z)$$

Note that we (at least at this point) only have notational differences to the earlier model from the lecture. The first component reflects capital accumulation and population change, following a trend

$$y_{nt} = \alpha + \beta t.$$

The cycle component varies with perceived relative prices and the own lagged value:

$$y_{ct}(z) = \gamma\big[P_t(z) - \mathbb{E}[P_t \mid I_t(z)]\big] + \lambda y_{c,t-1}(z)$$

Note that $|\lambda| < 1$ as we only view a deviation from a trend. Thus we exclude an *explosion* in response to shocks. We assume the information set on which we condition allows for the determination of the prior for the expected price level $P_t|I_t(z) \sim \mathcal{N}(\bar{P}_t, \sigma^2)$. Markets are indexed by their price deviation from average, $z \perp P_t$ and $z \sim \mathcal{N}(0, \tau^2)$. $P_t(z)$ (in logs) is

$$P_t(z) = P_t + z.$$

The information considered is the observed price $P_t(z)$ and the history summarized in $\bar{P}_t$. The distribution of P_t conditional on this information is normal with the first moment being

$$\begin{aligned}\mathbb{E}[P_t \mid I_t(z)] &= \mathbb{E}[P_t \mid P_t(z), \bar{P}_t] \\ &= (1-\theta)P_t(z) + \theta\bar{P}_t,\end{aligned}$$

where $\theta = \tau^2/(\sigma^2 + \tau^2)$, and variance $\theta\sigma^2$. We get the supply function for market z

$$y_t(z) = y_{nt} + \theta\gamma[P_t(z) - \bar{P}_t] + \lambda y_{c,t-1}(z).$$

Integrating w.r.t. the distribution of z, we have the aggregate supply function

$$y_t = y_{nt} + \theta\gamma(P_t - \bar{P}_t) + \lambda[y_{t-1} - y_{n,t-1}]$$

Thus, the slope of the supply depends on the fraction θ of the total individual price variance, $\sigma^2 + \tau^2$, which is due to relative price variation. τ^2 small implies individual price changes reflecting the general price changes almost with certainty. The supply curve tends towards being vertical with $\tau^2 \downarrow$, as we have it being independent of the price, i.e. the middle term gets *removed*, so to speak. For $\tau^2 \to \infty$ we have the slope converging towards γ. In other words, if the individual price variations are the dominant part of the total variation, then we observe a stable output-price relation.

Consider now the demand function

$$x_t = y_t + P_t,$$

where x_t is some exogenous shift variable, equal to the observable log of nominal GNP. $\{\Delta x_t\}_{t=1}^T$ is i.i.d. with $\Delta x_t \sim \mathcal{N}(\delta, \sigma_x^2)$. Due to linearity we can reasonably assume

$$P_t = \pi_0 + \pi_1 x_t + \pi_2 x_{t-1} + \cdots + \eta_1 y_{t-1} + \eta_2 y_{t-2} + \cdots + \xi_0 y_{nt}.$$

Then, conditional on all information, when plugging in $x_t = x_{t-1} + \delta$, we have the mean price

$$\bar{P}_t = \bar{P}_0 + \pi_1(x_{t-1} + \delta) + \pi_2 x_{t-1} + \cdots + \eta_1 y_{t-1} + \eta_2 y_{t-2} + \cdots + \xi_0 y_{nt}.$$

From the demand function and the aggregate supply, we can eliminate the y's, such that we can estimate the parameters $\{\pi_i, \eta_j, \xi_0\}$. This can the be inserted into the last two equations. Thus, we have

$$\begin{aligned} P_t &= \frac{\theta\gamma\delta}{1+\theta\gamma} - \lambda\beta + \frac{1}{1+\theta\gamma} x_t + \frac{\theta\gamma}{1+\theta\gamma} x_{t-1} + \lambda y_{t-1} - (1-\lambda) y_{nt} \\ y_t &= -\frac{\theta\gamma\delta}{1+\theta\gamma} + \lambda\beta + \frac{\theta\gamma}{1+\theta\gamma} \Delta x_t + \lambda y_{t-1} + (1-\lambda) y_{nt}. \end{aligned}$$

Then, we get

$$\begin{aligned} y_{ct} &= -\pi\delta + \pi\Delta x_t + \lambda y_{c,t-1}, \\ \Delta P_t &= -\beta + (1-\pi)\Delta x_t + \pi \Delta x_{t-1} - \lambda \Delta y_{c,t-1}, \end{aligned}$$

where $\pi = \theta\gamma/(1+\theta\gamma)$. P_t is normal around $\bar{P}_t$ and has constant variance $1/(1+\theta\gamma)^2\sigma_x^2$, so the assumption of suppliers' knowledge of prices is true in this economy. The last two equations give equilibrium real output and inflation rate. Thus, we have the intersection points, aggregate demand shifted by changes in x_t and supply shifted by variables which determine expectation (our information set, i.e. past values of prices that influence our prior). We cannot evaluate this as a cross-sectional model, as it depends on the past values and therefore we have an equilibrium path. Change in Δx_t has an immediate effect which decays geometrically. Periods of inflation and below-average real output may occur in this setup, arising from the observation of lagged demand changes.

The model implies the existence of a natural rate of output, the average rate of demand expansion is δ, coefficient equal in size but opposite sign to the coefficient of the current rate. Changes in the average rate of nominal income growth will thus not have an effect on the average real output.

Unanticipated demand shifts have output effects with magnitude π. This is *fooling* the suppliers, thus π will be larger the smaller the variance of demand shifts (more unexpected shock, e.g. unusually large shock $\Rightarrow$ larger reaction)

$$\pi = \frac{\tau^2\gamma}{(1-\pi)^2\sigma_x^2 + \tau^2(1+\gamma)}.$$

Fixing τ^2 and γ, we have the limiting values of $\gamma(1+\gamma)$ for $\sigma_x^2 = 0$ and monotonic decrease towards zero for $\sigma_x^2 \to \infty$. Due to the assumptions on residuals in an OLS context (mean zero), we base tests of the natural rate hypothesis on the last equation; a relationship between observable variance and a slope parameter.

Looking at the national level, the second and third to last equations should perform decently. Under the assumption that the main source in price level variation are demand fluctuations, the fit of the model should be good. We have five slope slope parameters, but only two theoretical ones, thus the estimated π and λ should work well for explaining variation in ΔP_t. The main aim is to investigate the empirical plausibility of the natural rate theory, i.e. to look at how this theory fares across countries. We use the last equation for this, and further assume that τ^2 and γ are relatively stable across countries. We predict that $\hat{\pi}$ should be decreasing in the sampling variance of Δx_t.

The data shows two different types of countries, highly volatile and expansive policies such as in Argentina and Paraguay, and smooth with moderately expansive policies in the remaining countries. The issue seems to be that the types are separated very strongly, demand variance is ten times as volatile in high inflation countries compared to the rest.

The estimated π and λ are all $\in (0, 1)$ except for Argentina ($\lambda = -0.126$), and Puerto Rico ($\lambda = 1.029$). From the R^2 we can assume output determinants being omitted, Argentina, Honduras, and the UK have values below 0.5, Argentina at 0.018. It appears that this equation performs worse for highly volatile countries. For the inflation rate equation, the R^2s are very high for some countries, such as Argentina. However, they generally appear slightly lower, which makes sense as we would expect them to behave more erratically. The last column gives the R^2 when imposing the estimated coefficients from the third-to-last equation.

The $\hat{\pi}$ appears to be conform to the natural rate hypothesis. Stable price level countries have $\hat{\pi} \in [0.28, 0.91]$, while the volatile countries have estimates smaller by an order of ten. Comparing the USA to Argentina, we get

$$\begin{aligned}
y_{ct,USA} &= -0.049 + 0.910\Delta x_t + 0.887 y_{c,t-1} \\
y_{ct,ARG} &= -0.006 + 0.011\Delta x_t - 0.126 y_{c,t-1} \\
\Delta P_{t,USA} &= -0.028 + 0.119\Delta x_t + 0.758\Delta x_{t-1} - 0.637\Delta y_{c,t-1} \\
\Delta P_{t,ARG} &= -0.047 + 1.140\Delta x_t - 0.083\Delta x_{t-1} + 0.102\Delta y_{c,t-1}
\end{aligned}$$

In a stable county like the USA, nominal income increasing policy has a large initial effect on the real output, and a small positive effect on the inflation rate. Short term trade-offs are favorable here as long as it remains unused. Looking at Argentina, we can see a large increase in the price level change (1.140), but no real discernible effect on the real output (0.011). These results are inconsistent with a stable Phillips curve, but follow from the view that inflation stimulates real output iff it succeeds in fooling suppliers of goods and labor into thinking that relative prices are moving in their favor. The estimation rate of Argentina and Paraguay are inconsistent with the traditional view of a downward sloping Phillips curve, those two would have a vertical supply curve, whereas the United States would have a downward slope (this was the conventional view: the larger inflation, the lower unemployment - higher output).

The conventional Phillips curve is the observed co-movement of inflation and employment in the same direction, relative to the trend. This trade-off arises from stable features of the economy, thus being independent from the actual nature of the aggregate demand policy. Alternatively, the trade-off might be due to the suppliers misinterpreting the price changes as relative price changes, even if they are general price movements. Thus they face a decreased real wage, increasing the labor demand and thus increased output. From this explanation we have that changes in average inflation rates not increasing average output, as well as a higher variance in average prices, the less favorable will the observed trade-off. This, however, is empirically hard to measure, as it requires some measure of average inflation rates, average output - relative to the norm, or full employment. This would imply defining an obviously positive value to be the average unemployment rate. Furthermore, measuring *normal* output is hard to do satisfactory. Thus, sample averages are not ideal. However, we may use the variance, as a stable relationship would also apply to them.

When looking at the variances, it appears that there is no clear co-movement. This leads to the conclusion that trade-offs tend to fade away the more often it is used (and abused). Therefore the effect may only be observable in the short run, as seen in the model description before this paper. The simplifications are still detailed enough to capture the main phenomenon, namely the one predicted by the natural rate theory, the higher the variance of demand, the less favorable the terms of the Phillips trade-off.

2 Equilibrium with Complete Markets (LS, 4^{th} edition)

We want to highlight aspects that are particularly important. Arrow (1964) and Debreu (1959) contribute two classical pieces, important for the development of general equilibrium theory. While prices are not part of the fundamentals in an economy, they can influence resource allocation through relative valuation of commodities. Modern macroeconomics emphasizes quantitative issues in addition to qualitative relationships. It is expected to explain observed magnitudes.

The Keynesian spending multiplier is a good example for the differences between modern and classic macroeconomics. It tells us, given the agents propensity to consume, how much government spending is *inflated* when circling through the economy. This multiplier is given by

$$\text{Multiplier} = \frac{1}{1 - MPC} = \frac{1}{MPS - MPT - MPM},$$

where we have the marginal propensity to consume, to save, to tax, and to import. Modern macroeconomics uses microfoundation, we can link the multiplier to the individual decisions (the microfoundation). In classical macroeconomics, we would simply specify a basic functional relationship, it would be derived from ad-hoc equations. These aggregate relationships are numerous, like the Phillips curve, and microfoundation may allow us to investigate stability of such relationships.

2.1 Time 0 versus sequential trading

We describe two systems of markets; an Arrow Debreu structure (convex preferences, perfect competition, and demand independence) with complete markets in dated contingent claims (traded at time 0), and a sequential-trading structure with complete one-period Arrow securities (a security that pays one unit of numeraire if a particular state of the world is reached and zero otherwise). The timing of trades may differ, but the consumption allocations are identical. The reason agents trade is that they want to insure each other against variations due to states of the world (for example, the weather can be good or bad).

2.2 Preferences and Endowments

We have some realization of a stochastic event $s_t \in S$ in each period, and denote the history up to t by $s^t = [s_0, s_1, \ldots, s_t]$. The unconditional probability of observing a particular sequence s^t is given by $\pi_t(s^t)$. For $t > \tau$ we can write the conditional probability (conditional on the history realized up to τ) as $\pi(s^t \mid s^\tau)$. We set $\pi_0(s_0) = 1 \forall s_0$, as trading occurs after observing the initial state s_0.

We have I consumers ($i = 1, 2, \ldots, I$), and each consumer owns a stochastic endowment $y_t^i(s^t)$ which depends on the history s^t, which is publicly observable (*these allocations drop from the sky*). Each consumer purchases a history-dependent consumption plan $c^i = \{c_t^i(s^t)\}_{t=0}^\infty$ and orders these consumption streams by

$$U_i(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u_i[c_t^i(s^t)] \pi_t(s^t), \tag{2.1}$$

with $\beta \in (0, 1)$ being a discount factor. Thus we have the utility of consumption for any state, multiplied by the history conditioned probability of reaching this state. This implies an expectation, so we have $\mathbb{E}_0 \sum_{t=0}^\infty \beta^t u_i(c_t^i)$, i.e. the expectation captures the sum across all histories and the associated probabilities, all conditional on the realization s_0 of course. We assume $u_i(c)$ to be increasing, twice continuously differentiable, and strictly concave function of consumption $c \geq 0$ of one good, thus being *risk averse* (concavity is key here). Furthermore, to ensure interior solutions, the utility function satisfies the Inada condition

$$\lim_{c \downarrow 0} u_i'(c) = +\infty.$$

Another assumption is that $\pi_t(s^t)$ is common across all consumers for all t and s^t.

A feasible allocation (*note: this is not a budget constraint as we do not have prices*) of consumption satisfies

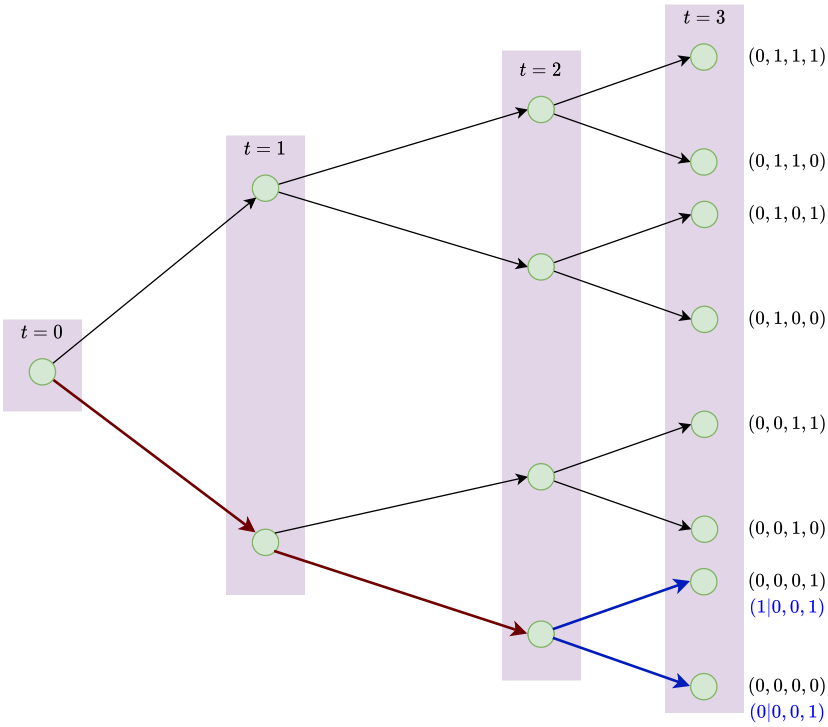
$$\sum_{i \in I} c_t^i(s^t) \leq \sum_{i \in I} y_t^i(s^t), \tag{2.2}$$

for all t and s^t. Consumption of all agents thus must be weakly smaller than the stochastic endowment of all agents.

2.3 Alternative trading arrangements

Consider a two-event stochastic process $s_t \in S = \{0, 1\}$. Now, for $t = 3$ and some initial $s_0 = 0$, we have $2^{|t|} = 8$ different possible outcomes which can be reached. Figure 2.1 generally illustrates all possible histories at time 3. However, the red line would indicate a history in which the time 2 is already reached, allowing for only two - conditional on $s^2 = (0, 0, 1)$ - possible histories, marked in blue.

Figure 2.1: The commodity space with Arrow securities



We consider two separate trading agreements, both of which support the same equilibrium allocations:

- Time-0-trading (Arrow-Debreu structure): at time 0 all markets meet and make trades for all possible nodes (up to t, s^t). After that, no trading occurs but deliveries at the specified times take place. This corresponds to the overall structure of the below graph.
- Sequential trading: contracts are only made one period ahead, thus depending on the history up to that point. Trades for history s^{t+1}, i.e. with contingency date $t+1$ occur only at the date t history s^t, and no further ahead.

The reached allocations share the property that they are functions only of the aggregate endowment realization $\sum_{i \in I} y_t^i(s^t)$ and time-invariant parameters describing the initial distribution of wealth.

2.3.1 History dependence

We may measure the luck of any random consumer i by $\{y_0^i(s_0), y_2^i(s_1), \dots\}$, which clearly depends on s^t. However, it is not quite as clear whether or not this is still history dependent after trading took place. Using the complete markets model in this chapter, the consumption allocation at time t depends only on some time-invariant parameters which describe the initial distribution of wealth at $t = 0$, and not the particular history which led to that aggregate endowment. Market incompleteness of course complicates this.

2.4 Pareto Problem

A benchmark for market economies is the efficient allocation. Efficient is used in the sense of Pareto optimality, meaning that any deviation that makes one agent strictly better off, makes at least one other agent worse off. To find such an allocation, we construct the social planner problem. This planner gives *Pareto weights* λ_i to the utility of consumers, i.e. it attaches a weight to each consumer, and then chooses allocations c^i to maximize

$$W = \sum_{i=1}^{I} \lambda_i U_i(c^i), \tag{2.3}$$

subject to the feasible consumption allocations given by equation (2.2). We call the allocation *efficient* if it solves this problem for some set of $\lambda_i \geq 0$. Let $\theta_t(s^t)$ be some non-negative Lagrange multipliers on the feasibility constraint (2.2) for time t and history s^t, then we form the Lagrangian as

$$\mathfrak{L} = \sum_{t=0}^{\infty} \sum_{s^t} \left\{ \sum_{i=1}^{I} \lambda_i \beta^t u_i\big(c_t^i(s^t)\big)\pi_t(s^t) + \theta_t(s^t) \sum_{i=1}^{I} [y_t^i(s^t) - c_t^i(s^t)] \right\}.$$

Maximizing this w.r.t. $c_t^i(s^t)$ yields the FOC*

$$\beta^t u_i'(c_t^i(s^t))\pi_t(s^t) = \lambda_i^{-1}\theta_t(s^t), \tag{2.4}$$

for each i, t, s^t. Furthermore, we have a second FOC:

$$\frac{\partial}{\partial \theta_t(s^t)} \mathfrak{L} = \sum_{i=1}^{I} [y_t^i(s^t) - c_t^i(s^t)] = 0 \forall t, s^t.$$

Total endowment varies across time and histories, therefore the Lagrangian multiplier varies over histories . As the time discount factor, the probabilities for each state of the world, and the Lagrange multipliers are independent of i, we can take the ratio of (2.4) for consumers i and 1:

$$\frac{u_i'(c_t^i(s^t))}{u_1'(c_t^1(s^t))} = \frac{\lambda_1}{\lambda_i}, \tag{2.5}$$

which implies, when reformulating and taking the inverse of the utility function of consumer i

$$c_t^i(s^t) = u_i'^{-1}(\lambda_i^{-1}\lambda_1 u_1'(c_t^1(s^t))). \tag{2.6}$$

We can now substitute (2.6) into the feasibility condition, and using equality we have

$$\sum_i u_i'^{-1}(\lambda_i^{-1}\lambda_1 u_1'(c_t^1(s^t))) = \sum_i y_t^i(s^t). \tag{2.7}$$

This equation contains only one unknown $c_t^1(s^t)$, and the right side is the realized aggregate endowment. Thus, the LHS is a function only of the aggregate endowment. Therefore, given $\{\lambda_i\}_{i=1}^I$, $c_t^1(s^t)$ depends only on the current realization of the aggregate endowment, and not separately either on the date t or on the specific history s^t leading up to this endowment, or the cross-section distribution of individual endowments realized at t. Equation (2.6) then implies that $\forall i$, $c_t^i(s^t)$ depends only on the aggregate endowment realization.

> **Proposition 2.1.** An efficient allocation is a function of the realized aggregate endowment and does not depend separately on either the specific history s^t leading up to that aggregate endowment, or on the cross-section distribution of individual endowments realized at t : $c_t^i(s^t) = c_\tau^i(\tilde{s}^\tau)$ for s^t and $\tilde{s}^\tau$ such that $\sum_j y_t^j(s^t) = \sum_j y_\tau^j(\tilde{s}^\tau)$

To compute the optimal allocation, we can solve (2.7) for $c_t^1(s^t)$, and then solve (2.6) for $c_t^i(s^t)$. We may further normalize the Pareto weights to 1, as only the ratio matters.

*Is it wrong to say that each consumer faces a collection of history-dependent sequences of FOCs?

2.4.1 Time invariance of Pareto weights

The allocation $c_t^i(s^t)$ assigned to the consumer through (2.7) and (2.6) depends in a time-invariant way on the aggregate endowment $\sum_j y_t^j(s^t)$. The consumer's share varied directly with their assigned Pareto weight λ_i.

2.5 Time 0 trading: Arrow-Debreu securities

Now we look at a competitive equilibrium allocation obtained with Arrow-Debreu timing. Consumers trade a complete set of dated, history-contingent claims to consumption. Trades occur at $t = 0$, after realization of s_0. Consumers can exchange claims to consumption in time t, contingent on history s^t at price $q_t^0(s^t)$. The superscript denotes the period in which the trade was made, so this only changes when we allow other timings. The subscript t denotes the contracted time when it is delivered. Now, the budget constraint is given by

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)c_t^i(s^t) \leq \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)y_t^i(s^t). \tag{2.8}$$

The consumer's problem is to choose c^i to maximize equation (2.1) subject to this inequality in (2.8).

We only have one budget constraint as we assume some underlying clearing operation that keeps track of the net claims when doing multilateral trades. All trades occur at time zero, afterwards no trades occur but the agreed upon trades are executed.

It is straighforward to set up the Lagrangian, using equality on the budget constraint, and assuming a Lagrangian multiplier μ_i that is independent of time, such that

$$\mathfrak{L} = \sum_{t=0}^{\infty}\sum_{s^t} \beta^t u_i\left[c_t^i(s^t)\right]\pi_t(s^t) + \mu_i\left[q_t^0(s^t)y_t^i(s^t) - q_t^0(s^t)c_t^i(s^t)\right].$$

This implies, following from the FOC of the Lagrangian

$$\beta^t u_i'\left[c_t^i(s^t)\right]\pi_t(s^t) = \mu_i q_t^0(s^t). \tag{2.9}$$

Definition 2.1. A *price system* is a sequence of functions $\{q_t^0(s^t)\}_{t=0}^{\infty}$. An *allocation* is a list of sequences of functions $c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$, one for each i.

Definition 2.2. A *competitive equilibrium* is a feasible allocation and a price system such that, given the price system, the allocation solves each consumer's problem.

For any pair (i, j) we can define the ratio of equations (2.9), again noting the time individual-invariant terms such as the time-discount factor, the probability of any given history, or the price sequence, such that

$$\frac{u_i'[c_t^i(s^t)]}{u_j'[c_t^i(s^t)]} = \frac{\mu_i}{\mu_j}. \tag{2.10}$$

This implies that the ratio of individuals' marginal utilities is constant across all histories and dates.

An equilibrium allocation solves (2.2), (2.8), and (2.10). Furthermore, (2.10) implies the solution for the consumption sequence as

$$c_t^i(s^t) = u_i'^{-1}\left\{u_1'\left[c_t^1\right]\frac{\mu_i}{\mu_1}\right\}, \tag{2.11}$$

which can be substituted into (2.2) such that we have

$$\sum_i u_i'^{-1}\left\{u_1'\left[c_t^1\right]\frac{\mu_i}{\mu_1}\right\} = \sum_i y_t^i(s^t). \tag{2.12}$$

As the right side constitutes the current realization of the aggregate endowment, the left side - and thus $c_t^1(s^t)$ - must also depend only on the current aggregate endowment, as well as on the sequence of ratios $\left\{\frac{\mu_i}{\mu_1}\right\}_{i=2}^{I}$. From (2.11) something similar follows, the equilibrium allocation $c_t^i(s^t)$ for each i depends only on the economy's aggregate endowment as well as on $\left\{\frac{\mu_j}{\mu_1}\right\}_{j=2}^{I}$. The first one implies one specific allocation for only one individual, the second implication is a more general statement on the equilibrium allocations.

Proposition 2.2. The competitive equilibrium allocation is a function of the realized aggregate endowment and does not depend on time t or the specific history, or on the cross section distribution of endowments: $c_t^i(s^t) = c_\tau^i(\tilde{s}^\tau)$ for all histories s^t and $\tilde{s}^\tau$ such that $\sum_j y_t^j(s^t) = \sum_j y_\tau^j(\tilde{s}^\tau)$.

Thus, the competitive outcome is similar to the social planner one, differing only in the ratios of Lagrange multipliers. Section 2.5.2 further elaborates on the optimality.

2.5.1 Equilibrium pricing function

Suppose c^i, $i = 1, 2 \ldots, I$ is an equilibrium allocation. Then the marginal condition

$$\frac{\partial U_i(c^i)}{\partial c_t^i(s^t)} = \mu_i q_t^0(s^t), \tag{2.13}$$

as well as (2.9) may be regarded as determining the price system $q_t^0(s^t)$ as a function of the equilibrium allocation assigned to consumer i, for any i. However, this is computationally difficult as it requires us to calculate the prices and the allocation simultaneously. To exploit the above mentioned fact, we need to compute them separately.

Units in the price system are chosen arbitrarily, thus we can normalize them to any positive value. We set $q_0^0(s_0) = 1$, thus putting the price system in units of time 0 goods. This choice implies from (2.9), as $\beta^0 = 1$ and $\pi_0(s^0) = 1$, that we have $\mu_i = u_i'[c_0^i(s_0)]$.

2.5.2 Optimality of equilibrium allocation

A competitive equilibrium allocation is a particular Pareto optimal allocation, one that sets the Pareto weights $\lambda_i = \mu_i^{-1}$. These weights are unique up to multiplication by a positive scalar. At a competitive equilibrium allocation, we need the prices $q_t^0(s^t)$ to equal the implied *shadow prices* $\theta_t(s^t)$ (see equation (2.4) for comparison) of the associated planer's problem. That these allocations are identical reflects the two fundamental theorems of welfare economics. The first states that a competitive equilibrium allocation is efficient, the second that there exists a price system and an initial distribution of wealth that can support an efficient allocation as a competitive equilibrium allocation. Note that the welfare theorems require us to be in world with no externalities, and where all agents are price takers.

2.5.3 Interpretation of trading arrangement

In the competitive equilibrium with Arrow-Debreu timing, we require a vast clearing or credit system operating at $t = 0$, ensuring that equation (2.8) holds for each consumer i. A symptom of the once-and-for-all and net-clearing trading agreement is that each consumer faces one budget constraint that restricts trades across all dates and histories.

2.5.4 Equilibrium computation

To compute the equilibrium, we need to determine the ratios of the Lagrange multipliers, μ_i/μ_1, $i = 1, \ldots, I$, that appears in (2.11) and (2.12). The following describes a *Negishi algorithm* that accomplishes this goal.

1. Fix a positive value for one μ_i, e.g. μ_1, throughout the algorithm. Guess positive values for the remaining μ_i's, then solve (2.11) and (2.12) for a candidate consumption allocation c^i, $i = 1, \ldots, I$.

2. Use (2.9) for any consumer i to solve for the price system $q_t^0(s^t)$.

3. For $i = 1, \dots, I$, check the budget constraint (2.8). For those i's for which the cost of consumption exceeds the value of their endowment, raise μ_i, while for those i's for which the reverse inequality holds, lower μ_i.

4. Iterate to convergence on steps 1.3.

Multiplying the μ_i's by a positive value simply changes the units of the price system, with no real consequences, thus we may simply normalize one value to 1, as we did in step 1.

Generally, the equilibrium price system and the distribution of wealth are mutually determined. Along with the equilibrium allocation, they solve a lot of simultaneous equations. The above presented algorithm is hard to implement in practice. Thus, we often formally state preferences such that that we eliminate the dependence of equilibrium prices on the distribution of wealth.

2.6 Simpler computational algorithm

Using a specific preference to avoid iterating on Pareto weights can be achieved by the following motivating examples.

2.6.1 Example 1: risk sharing

Suppose all agents have a one-period utility function with constant relative risk-aversion, i.e.

$$u(c) = (1-\gamma)^{-1} c^{1-\gamma}, \ \gamma > 0,$$

which implies the derivative w.r.t. consumption to be

$$\frac{\partial}{\partial c} u(c) = c^{-\gamma}.$$

Now consider equation (2.10). Plugging in our result from above, we have

$$\left[c_t^i(s^t)\right]^{-\gamma} = \left[c_t^j(s^t)\right]^{-\gamma} \frac{\mu_i}{\mu_j}.$$

or, in simpler terms,

$$c_t^i(s^t) = c_t^j(s^t) \left(\frac{\mu_i}{\mu_j}\right)^{-\frac{1}{\gamma}}. \tag{2.14}$$

This equation states that the elements of consumption allocations to distinct agents are constant fractions of one another, at time t. Power utility functions* say that individual consumption is perfectly correlated with the aggregate endowment or aggregate consumption. This would mean that the individual consumption is independent, conditional on the history s^t, from the individual endowment at the same point in time.†

These fractions assigned to each individual are also independent of the realization of s^t. This implies cross-history and cross-time consumption sharing‡. This constant-fractions-of-consumption comes from two aspects: the assumptions of complete markets, and a homothetic one-period utility function. Preferences are intertemporally homothetic if, across time periods, rich and poor decision makers (independence of individual endowments) are equally averse to proportional fluctuations in consumption. For within-period power utility functions, their assumptions imply that the elasticity of intertemporal substitution, and its inverse, the coefficient of (risk) aversion, are constant (as mentioned above).

*the one we used is one such power utility function, alternatively we could use $u(c) = (1-\gamma)^{-1}(c^{1-\gamma} - 1)$ as an example of one such utility function.

†As the footnotes in the book says, this appears to not hold. So what assumption results in this conditional independence of individual consumption from the individual endowment? Timing of trades, or simply the assumed utility function?

‡Of individuals, or across individuals?

2.6.2 Implications for equilibrium computation

Equation (2.14) and the pricing formula (2.9) imply that an equilibrium price vector satisfies

$$q_t^0(s^t) = \mu_i^{-1}\alpha_i^{-\gamma}\beta^t\left(\bar{y}_t(s^t)\right)^{-\gamma}\pi_t(s^t), \tag{2.15}$$

where we have

$$c_t^i(s^t) = \alpha_i \bar{y}_t(s^t),\ \bar{y}_t = \sum_i y_t^i(s^t),$$

and α_i is consumer i's fixed consumption share of the aggregate endowment. We may normalize the price system by setting $\mu_i^{-1}\alpha_i^{-\gamma}$ to some arbitrary positive number.

The assumption of homothetic CRRA preferences leading to equation (2.15) allows us to compute an equilibrium in two steps:

1. Use (2.15) to compute an equilibrium price system.
2. Use this price system and consumer i's budget constraint to compute

$$\alpha_i = \frac{\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t) y_t^i(s^t)}{\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)\bar{y}_t(s^t)}$$

Thus, consumer i's fixed consumption share α_i equals its share of aggregate wealth evaluated at the competitive equilibrium price vector.

2.6.3 Example 2: no aggregate uncertainty

For this example we assume a sufficiently simple endowment structure, such that we do not need such strong assumptions on the utility functions. Denote a stochastic event s^t, taking on values on the closed unit interval. We only consider two consumers, where $y_t^1(s^t) = s_t$, and $y_t^2(s^t) = 1 - s_t$. On aggregate we face no uncertainty, as $\sum_{i=1}^2 y_t^i(s^t) = 1$. Equation (2.12) then implies that $c_t^1(s^t)$ is constant over time and across histories. Furthermore, equation (2.11) then implies the same for $c_t^2(s^t)$. Since it is constant across time, it has to equal the mean endowment over time, i.e. the equilibrium allocation satisfies $c_t^i(s^t) = \bar{c}^i$ for all t and s^t, as well as $i = 1, 2$. From equation (2.9), we now have, plugging in the previous result:

$$q_t^0(s^t) = \beta^t \pi_t(s^t)\frac{u_i'(\bar{c}^i)}{\mu_i}. \tag{2.16}$$

$\forall t, s^t$, and $i = 1, 2$. This allows us to rewrite the budget constraint of each individual consumer, see equation (2.9), such that

$$\frac{u_i'(\bar{c}^i)}{\mu_i}\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t(s^t)\left[\bar{c}^i - y_t^i(s^t)\right] = 0.$$

Solving this equation for $\bar{c}^i$ gives us

$$\bar{c}^i = (1-\beta)\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t(s^t)y_t^i(s^t). \tag{2.17}$$

Summing this equation verifies $\bar{c}^1 + \bar{c}^2 = 1$.

2.6.4 Example 3: periodic endowment process

This builds upon the previous example, but now we assume s_t to be deterministic and alternate between 1 and 0, starting at $s_0 = 1$. This implies $s_1 = 0$, $s_2 = 1$, and so on. Endowment processes are perfectly predictable, the first agent faces $(1, 0, 1, \dots)$, and the second agent faces $(0, 1, 0, \dots)$. Now let $\tilde{s}^t$ be the history of $(1, 0, 1, \dots)$, up to t. Per construction we have $\pi_t(\tilde{s}^t) = 1$, and thus the probability for all other histories up to t is zero. Then, we have the equilibrium price system

$$q_t^0(s^t) = \begin{cases} \beta^t, & \text{if } s^t = \tilde{s}^t, \\ 0, & \text{otherwise;} \end{cases}$$

when using the time 0 good as numeraire (i.e. normalizing with respect to it), $q_0^0(\tilde{s}_0) = 1$. From (2.17), we have

$$\bar{c}^1 = (1-\beta)\sum_{j=0}^{\infty}\beta^{2j} = \frac{1}{1+\beta}, \tag{2.18a}$$

$$\bar{c}^2 = (1-\beta)\beta\sum_{j=0}^{\infty}\beta^{2j} = \frac{\beta}{1+\beta}. \tag{2.18b}$$

To see why this result holds, consider the following intermediate result:

$$(1-\beta)\sum_{t=0}^{\infty}\beta^{2t} = \frac{1-\beta}{1-\beta^2} = \frac{1}{1+\beta}.$$

For the second equality, note the following steps:

$$\frac{1-\beta}{1-\beta^2} = \frac{1}{1-\beta^2} - \frac{\beta}{1-\beta^2} = \frac{1}{1-\beta^2} + \frac{\beta}{(\beta-1)(\beta+1)}$$

then, using partial fraction expansion, we have

$$= \frac{1}{1-\beta^2} + \frac{1}{2(\beta+1)} + \frac{1}{2(\beta-1)},$$

Now, doing the same for the first fraction (note that only the signs differ), we have

$$\begin{aligned} &= \frac{1}{2(\beta+1)} - \frac{1}{2(\beta-1)} + \frac{1}{2(\beta+1)} + \frac{1}{2(\beta-1)} \\ &= 2\cdot\frac{1}{2(\beta+1)} = \frac{1}{1+\beta}. \end{aligned}$$

Going back to the final result, we can see that consumer 1 has a higher consumption every period because they are richer, simply due to receiving earlier endowment than consumer 2.

2.6.5 Example 4

Again, we assume the one-period CRRA utility function. There are again two consumers, i.e. $i = 1, 2$. We assume endowments of $y_t^1 = y_t^2 = 0.5$ for $t = 0, 1$, and $y_t^1 = s_t$ and $y_t^2 = 1 - s^t$ for $t \geq 2$. The event space $S = \{0, 1\}$ is governed by a Markov chain with initial state probabilities $\pi(s_0 = 1) = 1$ (this may be expressed as an initial-state column vector of $(0, 1)'$). Now, for the transition probabilities from $t = 0$ to $t = 1$, we have the probability $\pi_1(s_1 = 1 \mid s_0 = 1) = 1$, i.e. the first two entries of the history are always 1's. From period 1 to period 2, we have $\pi_2(s_2 = 0 \mid s_1 = 1) = \pi_2(s_2 = 1 \mid s_1 = 1) = 0.5$. For $t > 2$, we have $\pi_t(s_t = 1 \mid s_{t-1} = 1) = 1$, and $\pi_t(s_t = 0 \mid s_{t-1} = 0) = 1$, i.e. depending on the state in period 2, we do not change the state any further. This implies the transition matrix

$$\Pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where we know that the row sum must always equal 1, which is trivially satisfied. This specification implies that $\pi_t(1, 1, \ldots, 1, 1, 1) = 0.5$, and $\pi_t(0, 0, \ldots, 0, 1, 1) = 0.5$ for all $t > 2$.

We may use the method of section 2.6.2 to compute an equilibrium. The aggregate endowment is $\bar{y}_t(s^t) = 1$ for all t and all s^t (as for the first two periods both individuals have 0.5, and from then on only one of them has 1 and the other 0). Therefore, in the equilibrium, we have $q_1^0(1,1) = \beta$, $q_2^0(0,1,1) = q_2^0(1,1,1) = 0.5\beta^2$, and $q_t^0(1,1,\ldots,1,1,1) = q_t^0(0,0,\ldots,0,1,1) = 0.5\beta^t$ for $t > 2$. Using these prices we compute the value of agent i's endowment

$$\sum_t \sum_{s^t} q_t^0(s^t) y_t^i(s^t) = \sum_t \beta^t 0.5[0.5 + 0.5 + 0 + 0 + \cdots + 0] + \sum_t \beta^t 0.5[0.5 + 0.5 + 1 + 1 + \cdots + 1]$$
$$= 2\sum_t \beta^t 0.5[0.5 + 0.5 + 0.5 + 0.5 + \cdots + 0.5] = 0.5\sum_t \beta^t = \frac{0.5}{1-\beta}.$$

Clearly, the budget constraint of consumer i is satisfied if they consumer a constant consumption value of 0.5 in each period in each state: $c_t^i(s^t) = 0.5$ for all t and s^t.

2.7 Sequential Trading

2.7.1 Arrow securities

At each date $t \geq 0$, only at the history s^t which is actually realized, trades occur in a complete set of claims to one-period-ahead state-contingent consumption. A competitive equilibrium of this trading structure attains the same allocation as the time 0 trading structure described earlier.

2.7.2 Financial wealth as an endogenous state variable

We need to identify a variable to serve as the state in a value function for the consumer at date t and history s^t. Finding this state is done by taking an equilibrium allocation and price system from the time 0 trading structure, and then using the guess-and-verify method. To start this, what is the implied continuation of wealth of consumer i at time t for the realized history s^t? This is answered by summing up the value of the consumer's holding of claims to current and future consumption at time t and history s^t. We can discard any claims contingent on time t history $\tilde{s}^t \neq s^t$, as we already know that s^t is realized. The implies wealth is determined by the trades undertaken by consumer i at the time 0 trading equilibrium, i.e. where you can view this as having sold the entire endowment stream on the right side of the budget constraint in order to acquire the contingent consumption claims on the left side of the budget constraint in equation (2.8).

This trading structrue then implies two main differences. Firstly, the consumer retains their endowment flow through time, and secondly the one-period securities are traded on a period-by-period basis. The wealth of the consumer i at any point in time can be decomposed into financial, and non-financial wealth. Financial wealth at time t after history s^t is beginning-of-period holdings of Arrow securities that are contingent on the current state s_t being realized* (but are not yet realized), while non-financial wealth is the present value of the consumer's current and future endowments. From the insight that the two trading structures yield identical equilibrium, a consumer's financial wealth in a sequential trading should equal the continuation wealth in a time 0 trading equilibrium minus the continuation value of its current and future endowment, evaluated in terms of prices for a time-0-trading competitive equilibrium. This implies the financial wealth of consumer i at time t after history s^t, expressed in terms of the date t, history s^t consumption good is

$$\Upsilon_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau)[c_\tau^i(s^\tau) - y_\tau^i(s^\tau)]. \tag{2.19}$$

Note that the budget constraint (2.8) at equality implies that each consumer starts with zero financial wealth at time 0, $\Upsilon_0^i(s^0) = 0$ for all i. At $t > 0$, financial wealth $\Upsilon_t^i(s^t)$ typically differs from zero for each individual consumer, but it sums to zero in aggregate

$$\sum_{i=1}^{I} \Upsilon_t^i(s^t) = 0 \forall t, s^t. \tag{2.20}$$

which follows from the feasibility constraint (2.2) at equality. The securities making up the financial wealth are in zero net supply, i.e. any positive holdings directly imply indebtedness of another consumer who issued the security.

*The traded securities at time $t-1$, contingent on the realization of s_t?

2.7.3 Reopening markets

The formula

$$q_t^\tau(s^t) \equiv \frac{q_t^0(s^t)}{q_\tau^0(s^\tau)} = \frac{\beta^t u_i'[c_t^i(s^t)]\pi_t(s^t)}{\beta^\tau u_i'[c_\tau^i(s^\tau)]\pi_\tau(s^\tau)} = \beta^{t-\tau}\frac{u_i'[c_t^i(s^t)]}{u_i'[c_\tau^i(s^\tau)]}\pi_t(s^t \mid s^\tau) \tag{2.21}$$

takes the form of a pricing function for a complete markets economy with date- and history-contingent commodities whose markets can be regarded as having reopened at date τ, history s^τ, starting from a distribution of wealth implied by the tails of each consumer's endowment and consumption streams for a complete markets economy that originally convened at $t = 0$.

> **Proposition 2.3** (Relating to time-zero trading)**.** Start from the distribution of time t, history s^t financial wealth that is implicit in a time 0 Arrow-Debreu equilibrium. If markets are reopened at date t after history s^t, no trades occur. That is, given the price system (2.21), all consumers choose to continue the tails of their original consumption plans.

At the beginning of every period $t > 0$ we do not know our state of the world, i.e. we do not know if we got lucky or not. Then we trade for the next period before we realize the current state of the world. This is why we would not want to trade in the reopened markets, as no re-optimization is needed.

2.7.4 Debt limits

Moving from the Arrow-Debreu economy to one with sequential trading, we propose to match the time t, history s^t wealth off the consumer in the sequential economy with the equilibrium tail wealth $\Upsilon_t^i(s^t)$ from the Arrow-Debreu economy computed in equation (2.19). However, there is a need to discuss debt limits at this point, something only implicit in the time 0 budget constraint (2.8). Asset trades are restricted to rule out Ponzi schemes, at the weakest possible restriction. To formulate these restrictions, the starting point is the equilibrium allocation from the time 0 markets and then find some state-by-state debt limits that support the equilibrium allocation that emerged from the Arrow-Debreu economy under sequential trading. These *natural debt limits* come from the common sense requirement that it has to be feasible for a consumer to repay their state contingent debt in every possible state. Together with the assumption of non-negative consumption $c_t^i(s^t)^*$, that feasible requirement leads to natural debt limits.

$q_\tau^t(s^\tau)$ denotes the Arrow-Debreu price, denominated in units of the date t, history s^t consumption good. The value of the tail of agent i's endowment sequence at time t in history s^t is

$$A_t^i(s^t) = \sum_{\tau=t}^{\infty}\sum_{s^\tau \mid s^t} q_\tau^t(s^\tau) y_\tau^i(s^\tau). \tag{2.22}$$

This term is called the natural debt limit at time t and history s^t. This is the maximal value that agent i can repay, starting from that period, assuming zero consumption going forward. Sequential trading now applies that this is viewed one-period previously. Consumer i at time $t-1$ and history s^{t-1} cannot promise to pay more than $A_t^i(s^t)$ conditional on the realization of s_t tomorrow. Consumer i at time $t-1$ faces one such borrowing constraint for each possible realization of s_t.

2.7.5 Sequential trading

There is a sequence of markets in one-period-ahead state-contingent claims. At each date $t \geq 0$, consumers trade claims to $t+1$ consumption, the payment being contingent on the realization s_{t+1}. Let $\tilde{a}_t^i(s^t)$ denote the claims to time t consumption, other than its time t endowment $y_t^i(s^t)$ that consumer i brings into time t and history s^t. Suppose $\tilde{Q}_t(s_{t+1} \mid s^t)$ is a *pricing kernel* with the following interpretation: $\tilde{Q}_t(s_{t+1} \mid s^t)$ is the price of one unit of time $t+1$ consumption, contingent on the realization s_{t+1} at $t+1$, when the history at time t is

*Inada condition on the utility function.

s^t. The consumer faces a sequence of budget constraints for $t \geq 0$, where at time t, history s^t budget constraint is given by

$$\tilde{c}_t^i + \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1} \mid s^t) \leq y_t^i(s^t) + \tilde{a}_t^i(s^t). \tag{2.23}$$

At time t, a consumer chooses $\tilde{c}_t^i(s^t)$ and $\{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}$, a vector of claims on $t+1$ consumption, there being one element of the vector for each value of $t+1$ realizations of s_{t+1}. Ruling out Ponzi schemes, we impose the state-by-state borrowing constraints

$$-\tilde{a}_{t+1}^i(s^{t+1}) \leq A_{t+1}^i(s^{t+1}), \tag{2.24}$$

where the RHS is computed in equation (2.22).

Now let $\eta_t^i(s^t)$ and $\upsilon_t^i(s^t; s_{t+1})$ be nonnegative Lagrange multipliers on the budget constraint (2.23), and the borrowing constraint (2.24) respectively, for time t and history s^t. Now, we have the Lagrangian

$$\begin{aligned}\mathfrak{L}^i = \sum_{t=0}^{\infty}\sum_{s^t} \Bigg\{ &\beta^t u_i(\tilde{c}_t^i(s^t))\pi_t(s^t) \\ &+ \eta_t^i(s^t)\Bigg[y_t^i(s^t) + \tilde{a}_t^i(s^t) - \tilde{c}_t^i - \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1} \mid s^t)\Bigg] \\ &+ \sum_{s_{t+1}} \upsilon_t^i(s^t; s_{t+1})[A_{t+1}^i(s^{t+1}) + \tilde{a}_{t+1}^i(s^{t+1})]\Bigg\},\end{aligned}$$

for a given initial wealth $\tilde{a}_0^i(s_0)$. As we maximize w.r.t. $\tilde{c}_t^i(s^t)$ and $\{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}_{s_{t+1}}$, the FOCs are

$$\beta^t u_i'(\tilde{c}_t^i(s^t))\pi_t(s^t) - \eta_t^i(s^t) = 0, \tag{2.25a}$$

$$-\eta_t^i(s^t)\tilde{Q}_t(s_{t+1} \mid s^t) + \upsilon_t^i(s^t; s_{t+1}) + \eta_{t+1}^i(s_{t+1}, s^t) = 0, \tag{2.25b}$$

for all s_{t+1}, t, s^t. The natural debt limits (2.24) do not bind and the Lagrange multipliers $\upsilon_t^i(s^t; s_{t+1})$ all equal zero. This is the case because if there were any history s^{t+1} leading to a binding natural debt limit, the consumer from then on have to set consumption equal to zero in order to honor the debt. Now, since the utility function satisfies the Inada condition, $\lim_{c\downarrow 0} u_i'(c) = +\infty$, i.e. all future marginal utilities would be infinite, implying that there must be alternative affordable allocations yielding higher expected utility by simply postponing earlier consumption to periods after such a binding constraint.

Setting this multiplier to zero in equation (2.25b), and then combining the two FOCs implies the following restriction on the optimal consumption allocation

$$\tilde{Q}_t(s_{t+1} \mid s^t) = \beta \frac{u_i'(\tilde{c}_{t+1}^i(s^{t+1}))}{u_i'(\tilde{c}_t^i(s^t))}\pi_t(s^{t+1} \mid s^t), \tag{2.26}$$

for all s_{t+1}, s^t, t.

Definition 2.3. A *distribution of wealth* is a vector $\vec{\tilde{a}}_t(s^t) = \{\tilde{a}_t^i(s^t)\}_{i=1}^I$ satisfying $\sum_i \tilde{a}_t^i(s^t) = 0$.

Definition 2.4. A *competitive equilibrium with sequential trading of one-period Arrow securities* is an initial distribution of wealth $\vec{\tilde{a}}_0(s_0)$, a collection of borrowing limits $\{A_t^i(s^t)\}$ satisfying (2.22) for all i, for all t, and for all s^t a feasible allocation $\{\tilde{c}_t^i\}_{i=1}^I$, and pricing kernels $\tilde{Q}_t(s_{t+1} \mid s^t)$ such that

(a) given the pricing kernels and $\tilde{a}_0^i(s_0)$ and the borrowing limits $\{A_t^i(s^t)$ for all i, the consumption allocation $\tilde{c}^i$ and portfolio $\{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}\}$ solves the consumer's problem for all i; and

(b) for all realizations of $\{s^t\}_{t=0}^{\infty}$ allocations and portfolios $\{\tilde{c}_t^i(s^t), \{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}\}_i$ satisfy $\sum_i \tilde{c}_t^i(s^t) = \sum_i y_t^i(s^t)$ and $\sum_i \tilde{a}_{t+1}^i(s_{t+1}, s^t) = 0$.

This definition does leave open the initial wealth distribution, which is discussed at some other point.

2.7.6 Equivalence of allocations

We are able to show, assuming an appropriate guess on the form of pricing kernels, that the competitive allocation of complete markets with time 0 trading is also the an allocation for the competitive allocation with sequential trading with Arrow securities, with a specific initial wealth distribution. Taking $q_t^0(s^t)$ as given from the Arrow-Debreu equilibrium and suppose that the pricing kernel makes the following recursion true:

$$q_{t+1}^0(s^{t+1}) = \tilde{Q}_t(s_{t+1} \mid s^t) q_t^0(s^t),$$

or alternatively

$$\tilde{Q}_t(s_{t+1} \mid s^t) = q_{t+1}^t(s^{t+1}), \tag{2.27}$$

where we recall that the ratio of the price in period $t+1$, expressed in terms of period 0 good, to the price of period t good in terms of period 0 good, is then simply the price of period $t+1$ goods expressed in terms of period t goods:

$$q_{t+1}^t(s^{t+1}) = \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)}.$$

Thus we might say that the pricing kernel captures the price changes exhibited by a realization of state s_{t+1} conditional on the history s^t, i.e. the pricing kernel gives us a one-period conditional growth rate for prices of goods in terms of their previous-period counterparts.

Let $\{c_t^i(s^t)\}$ be a competitive equilibrium allocation in the Arrow-Debreu economy. If equation (2.27) is satisfied, that allocation is also a sequential trading competitive equilibrium allocation. To see why, take the consumer's FOC (2.9) for the Arrow-Debreu economy from two successive periods, and divide one by the other to get

$$\frac{\beta^{t+1} u_i'\left[c_{t+1}^i(s^{t+1})\right] \pi_{t+1}(s^{t+1})}{\beta^t u_i'\left[c_t^i(s^t)\right] \pi_t(s^t)} = \frac{\mu_i q_{t+1}^0(s^{t+1})}{\mu_i q_t^0(s^t)},$$

or, when simplifying

$$\beta \frac{u_i'[c_{t+1}^i(s^{t+1})]}{u_i'[c_t^i(s^t)]} \pi(s^{t+1} \mid s^t) = \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)} = \tilde{Q}_t(s_{t+1} \mid s^t). \tag{2.28}$$

If this pricing kernel satisfies (2.27), this equation is equivalent with the FOC (2.26) for the sequential trading competitive equilibrium. Now, we need to choose the initial wealth such that the sequential trading competitive equilibrium duplicates the Arrow-Debreu competitive equilibrium allocation.

The conjecture here is that the initial wealth vector $\vec{a}_0(s_0)$ of the sequential trading economy is the zero vector. This is a natural choice as it implies that each individual must finance their own consumption stream via their own endowment stream. This yields the same implication as in the time 0 trading arrangement economy where each individual is constrained to finance their history-contingent purchases for the infinite future at time 0. To prove this, it needs to be shown that zero initial wealth enables consumer i to finance $\{c_t^i(s^t)\}$ and leaves no room to increase consumption in any period after any history*.

The proof proceeds by guessing that, at time $t \geq 0$ and history s^t, a consumer i chooses a portfolio given by $\tilde{a}_{t+1}^i(s_{t+1}, s^t) = \Upsilon_{t+1}^i(s^{t+1})$ for all s_{t+1} (the claims to period $t+1$ consumption contingent on s^t being equal to the financial wealth in $t+1$). The value of this portfolio expressed in terms of date t, history s^t consumption good is

$$\begin{aligned} \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t) \tilde{Q}_t(s_{t+1} \mid s^t) &= \sum_{s^{t+1} \mid s^t} \Upsilon_{t+1}^i(s^{t+1}) q_{t+1}^t(s^{t+1}) \\ &= \sum_{\tau=t+1}^{\infty} \sum_{s^\tau \mid s^t} q_\tau^t(s^\tau)[c_\tau^i(s^\tau) - y_\tau^i(s^\tau)], \end{aligned} \tag{2.29}$$

*Budget constraint binding with equality?

where we have used expressions (2.19) and (2.27). Also note the following:

$$\sum_{s^{t+1}|s^t} \left\{ \sum_{\tau=t+1}^{\infty} \sum_{s^\tau|s^{t+1}} q_\tau^{t+1}(s^{t+1})\left[c_\tau^i(s^\tau) - y_\tau^i(s^\tau)\right] \right\} q_{t+1}^t(s^{t+1})$$

and the identity

$$q_\tau^{t+1}(s^\tau)q_{t+1}^t(s^{t+1}) = \frac{q_\tau^0(s^\tau)}{q_{t+1}^0(s^{t+1})} \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)} = q_\tau^t(s^\tau)\forall\tau > t.$$

To demonstrate that consumer i can afford this portfolio strategy, we can use the budget constraint (2.23) to compute the implied consumption plan $\{\tilde{c}_\tau^i(s^\tau)\}$. In period 0 with $\tilde{a}_0^i(s_0) = 0$, substitution of equation (2.29) into the budget constraint (2.23) at equality

$$\tilde{c}_0^i(s_0) + \sum_{t=1}^{\infty} \sum_{s^t} q_t^0(s^t)[c_t^i(s^t) - y_t^i(s^t)] = y_t^i(s_0) + 0.$$

This, together with the budget constraint (2.8) at equality implies $\tilde{c}_0^i(s_0) = c_0^i(s_0)$.* The proposed portfolio is affordable in period 0 and the associated consumption plan is the same as in the competitive equilibrium of the Arrow-Debreu economy. In all consecutive future periods $t > 0$ and histories s^t, we replace $\tilde{a}_t^i(s^t)$ in constraint (2.23) by $\Upsilon_t^i(s^t)$, and after noticing that the present value of the asset portfolio in equation (2.29) can be written as

$$\sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1} \mid s^t) = \Upsilon_t^i(s^t) - [c_t^i(s^t) - y_t^i(s^t)], \tag{2.30}$$

it follows immediately from equation (2.23) that $\tilde{c}_t^i(s^t) = c_t^i(s^t)$ for all periods and histories. To see this, we start by reformulating (2.23) in the way described to obtain this result:

$$\tilde{c}_t^i + \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1} \mid s^t) \leq y_t^i(s^t) + \tilde{a}_t^i(s^t),$$

plugging in $\tilde{a}_t^i(s^t) = \Upsilon_t^i(s^t)$ yields

$$\tilde{c}_t^i + \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1} \mid s^t) \leq y_t^i(s^t) + \Upsilon_t^i(s^t).$$

Now, using the above statement on the present value of of the asset portfolio in (2.29), we have

$$\tilde{c}_t^i + \Upsilon_t^i(s^t) - \left[c_t^i(s^t) - y_t^i(s^t)\right] \leq y_t^i(s^t) + \Upsilon_t^i(s^t),$$

which, at equality, gives the implication

$$\tilde{c}_t^i = c_t^i(s^t).$$

We have shown that the proposition attains the same consumption plan as in the competitive equilibrium of the Arrow-Debreu economy. But what induces the consumer to not increase further present consumption by reducing some component of the asset portfolio? The answer stems from the debt limit restrictions. If the consumer wants to ensure that the consumption plan $\{c_\tau^i(s^\tau)\}$ can be attained starting next period in all possible future states, the consumer should subtract the value of this commitment to future consumption from the natural debt limit in (2.22). Thus, the consumer is facing a state-by-state borrowing constraint that is more restrictive than (2.24): for any s^{t+1},

$$-\tilde{a}_{t+1}^i(s^{t+1}) \leq A_{t+1}^i(s^{t+1}) - \underbrace{\sum_{\tau=t+1}^{\infty} \sum_{s^\tau|s^{t+1}} q_\tau^{t+1}(s^\tau)c_\tau^i(s^\tau)}_{\text{the value of the commitment}}.$$

*Wouldn't it only imply $\tilde{c}_0^i(s_0) = y_t^i(s_0)$?

When plugging in A^i_{t+1}, we have

$$\begin{aligned}
&\leq \sum_{\tau=t+1}^{\infty} \sum_{s^\tau|s^{t+1}} q^{t+1}_\tau(s^\tau) y^i_\tau(s^\tau) - \sum_{\tau=t+1}^{\infty} \sum_{s^\tau|s^{t+1}} q^{t+1}_\tau(s^\tau) c^i_\tau(s^\tau), \\
&\leq \sum_{\tau=t+1}^{\infty} \sum_{s^\tau|s^{t+1}} q^{t+1}_\tau(s^\tau) \left[y^i_\tau(s^\tau) - c^i_\tau(s^\tau) \right]
\end{aligned}$$

which is simply

$$-\tilde{a}^i_{t+1}(s^{t+1}) \leq -\Upsilon^i_{t+1}(s^{t+1}),$$

or

$$\tilde{a}^i_{t+1}(s^{t+1}) \geq \Upsilon^i_{t+1}(s^{t+1}).$$

Consumer i does not want to increase consumption at time t by reducing next period's wealth below $\Upsilon^i_{t+1}(s^{t+1})$ because that would imply a consumption plan violating the satisfaction of the FOC in equation (2.26) for all future periods and histories.

Some remarks from the lecture

Note the difference between uncertainty and risk:

- uncertainty implies uncertainty about the probability distribution of some event, so we cannot insure against this,
- risk implies knowledge about the probability distribution of some event, thus we can insure against the risk.

In the Arrow-Debreu setting this corresponds to the following. We cannot insure against an aggregate bad endowment, but since we know $s \in S$ and the probability $\mathbb{P}\mathrm{r}(s)$, we can insure against individual bad endowments.

The following section will elaborate on models with endogenous and exogenous variables. The latter one for example is the endowment in the AD setting, while the former is the consumption decisions of the individuals.

3 Recursive Competitive Equilibrium II (LS, 4^{th} edition)

Non-Stochastic Environment (Lecture)

Deterministic One-Sector Growth Model - Application of the Arrow-Debreu Setup

We have a dynamic general equilibrium model with a production sector. This model captures standard resource allocation problems in economics such as the key trade-offs between leisure and consumption, as well as between consumption and investment.

We assume a single production good which can either be consumed or invested, denoted c and i respectively. We study movement of aggregate entities, abstract from distributional issues. As we assume an infinite number of individuals and firms, uniformly distributed over their respective interval, we may simply study the interaction of one representative firm and one representative agent, having equality of per-capita and aggregate numbers.

We assume discrete time, i.e. $t = 1, 2, \ldots$, agents are then infinitely lived. Households have a fixed time endowment which is normalized to 1, so we have $1 = \ell_t + h_t$. Furthermore, we assume some initial capital stock of k_0. A single consumer has preferences over the sequences $\{c_t\}_{t=0}^{\infty}$ and $\{1 - h_t\}_{t=0}^{\infty}$, given by the discounted lifetime utility

$$\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t).$$

We assume $u : \mathbb{R}_+ \times [0,1] \mapsto \mathbb{R}$, where u is strictly increasing in both arguments, is strictly concave, as well as twice continuously differentiable. Lastly, the discount factor is given by $\beta \in (0,1)$.

The output production follows the production function

$$y_t = F(k_t, h_t),$$

where $F : \mathbb{R}_+ \times [0,1] \mapsto \mathbb{R}_+$, i.e. positive inputs are required to produce a positive amount of output. We assume constant returns to scale, as well as that if capital is zero, then for all labor inputs, the production is zero. It is strictly increasing in both arguments as well as twice continuously differentiable. Additionally, the Inada conditions are assumed to be fulfilled for any $h > 0$:

$$\lim_{k\to 0} F_k(k,h) = \infty, \ \lim_{k\to\infty} F_k(k,h) = 0.$$

Overall, the economy faces a resource constraint

$$y_t \leq c_t + i_t \forall t.$$

Capital accumulates as follows: physical capital stock k increases through gross investment i and decreases due to capital depreciation at rate δ. This implies the following law of motion for the capital accumulation:

$$k_{t+1} = (1-\delta)k_t + i_t, \ \delta \in (0,1), \ k_t \geq 0.$$

In every period, the household faces the decision of consumption demand or savings supply equaling investment, as well as between supplying labor or consuming leisure. The firm on the other hand decides on the output supply, the labor demand, as well as the capital demand in every period.

An *allocation* for this economy is a list of sequences for households and firms respectively:

$$\left\{c_t^d, h_t^s\right\}_{t=0}^{\infty}, \ \left\{y_t^s, h_t^d, k_t^d\right\}_{t=0}^{\infty},$$

such that $c_t^d \geq 0 \forall t$, $h_t^d \geq 0 \forall t$, $k_t^d \geq 0 \forall t$, and $h_t^s \in [0,1] \forall t$, $y_t^s = F(k_t^d, h_t^d) \forall t$. As we need to ensure that this allocation sequence is feasible, we need to ensure that the sequences also satisfy

$$k_{t+1}^d = (1-\delta)k_t^d + i_t^s \forall t,$$

as well as goods market clearing

$$y_t^s = c_t^d + i_t^d \forall t,$$

and lastly labor market clearing

$$h_t^s = h_t^d \equiv h_t \forall t.$$

k_0 equals the initial endowment. Note that if all markets clear, we can drop the superscripts, as we always have demand being equal to the supply. The capital market is assumed to clear by the Walras' Law. We face three markets, if two of them clear then we have the implication that the third one must do so as well. Walras' law asserts that an examined market must be in equilibrium if all other markets are in equilibrium, and also that excess supply in one market needs to be met with excess demand in another market, balancing each other out.

It is easy to write this social planner's problem, given $h_t = 1$, as a Lagrangian

$$\mathfrak{L}(c_t, k_{t+1}, \lambda_t) = \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t, 0) + \lambda \big[f(k_t) + (1-\delta)k_t - k_{t+1} - c_t \big] \right\},$$

which yields the FO(N)Cs

$$\begin{aligned}
\frac{\partial}{\partial c_t}\mathfrak{L}(\cdot) &= u'(c_t) - \lambda_t = 0 \forall t, \\
\frac{\partial}{\partial k_{t+1}}\mathfrak{L}(\cdot) &= -\lambda_t \beta^t + \lambda_{t+1}\beta^{t+1}[f'(k_{t+1}) - (1-\delta)] = 0 \forall t,
\end{aligned}$$

$$\frac{\partial}{\partial \lambda_t}\mathfrak{L}(\cdot) = c_t \leq f(k_t) + (1-\delta)k_t - k_{t+1}.$$

and k_0 is given. We can rewrite the first two FO(N)Cs such that for all t

$$u'(c_t) = \beta u'(c_{t+1})[f'(k_{t+1}) - (1-\delta)],$$
$$c_t \leq f(k_t) + (1-\delta)k_t - k_{t+1}.$$

Further consolidating the FOCs yields

$$u'\big(f(k_t) + (1-\delta)k_t - k_{t+1}\big) = \beta u'\big(f(k_{t+1}) + (1-\delta)k_{t+1} - k_{t+2}\big)\big[f'(k_{t+1}) - (1-\delta)\big].$$

This is a second-order non-linear difference equation in k, and thus not trivial to solve. However, under certain circumstances it may be easy to solve. Consider Brock and Mirman (1972), where under $\delta = 1$, $u(\cdot) = \log(\cdot)$ and $f(k) = k^\alpha$ collapses to a first-order linear difference equation with the following analytical solution:

$$k_{t+1} = \alpha\beta k_t^\alpha,$$

or, logarithmized, we have

$$\log k_{t+1} = \log \alpha\beta + \alpha \log k_t.$$

Noteworthy is that capital acts as a state variable. It is a solutions because it gives us some *law of motion* for capital, only missing a starting value. Thus, for specifying a full analytical solution we always assume k_0 as given. However, in the general case we do not have an analytical solution.

Solow's version of the model makes two assumptions:

1. $h_t = 1 \forall t$,
2. $i_t = \tilde{s} \cdot y_t \forall t$, where $\tilde{s} \in [0, 1]$ is a time-constant savings rate, i.e. we have an ad-hoc behavioral equation here.

This has two implications:

$$c_t = (1-\tilde{s})y_t \forall t,$$
$$f(k_t) \equiv F(k_t, 1) \forall t.$$

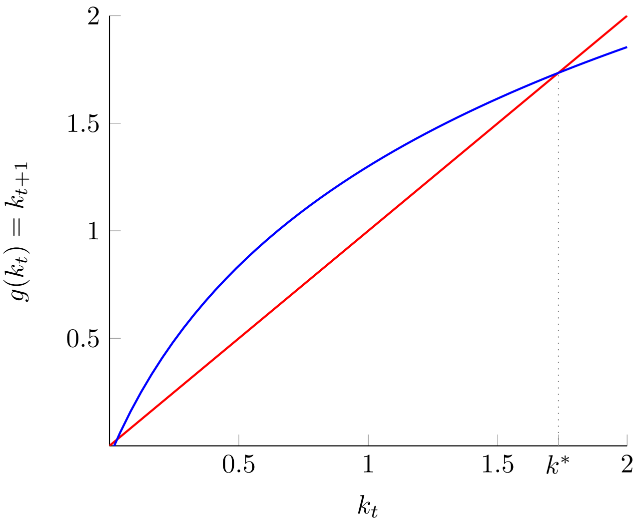
This implies that the consumer chooses the savings rate $\tilde{s}$. We can now characterize the allocations for all period using an iterative procedure:

$$\left.\begin{array}{c} k_0 \text{ given} \\ h_0 = 1 \\ y_0 = f(k_0) \\ i_0 = \tilde{s}f(k_0) \\ c_0 = (1-\tilde{s})f(k_0) \end{array}\right\} \quad \left.\begin{array}{c} k_1 = (1-\delta)k_0 + i_0 \\ y_1 = f(k_1) \\ i_1 = \tilde{s}f(k_1) \\ c_1 = y_1 - i_1 \end{array}\right\} \text{and so on} \ldots$$

We can iterate to solve for the sequences $\{k_t, h_t, y_t, i_t, c_t\}_{t=0}^\infty$. Clearly capital is the central variable, all other variables depend on it in every period. $\{k_t\}_{t=0}^\infty$ is given by $k_{t+1} = (1-\delta)k_t + \tilde{s}f(k_t)\forall t$, i.e. capital follows a first-order difference equation. We now define a function $g(k)$ by $g(k) = (1-\delta)k + \tilde{s} \cdot f(k)$. Then we have the sequence $\{k_t\}_{t=0}^\infty$ being described by $k_{t+1} = g(k_t)\forall t$ and k_0 given. Assuming $\tilde{s} > 0$, we observe the following properties on the function $g(\cdot)$:

- $g(0) = 0$,
- g is strictly increasing in k,
- g is strictly concave in k,
- Inada conditions hold, i.e. $\lim_{k\to 0} g'(k) = \infty > 1$ and $\lim_{k\to\infty} g'(k) = 1 - \delta < 1$.

Figure 3.1: Graphical representation of $g(\cdot)$ and a 45° line



To conclude, note that there is no population growth, and also no aggregate growth. Growth to the steady state is captured, but in the long run there is no permanent growth as the model does not feature technology progress, furthermore, population growth cannot give permanent growth, only technological progress. Also note that k denotes the aggregate per capita capital, which is the same as individual or firm specific capital due to the agents being atomic, i.e. each agent is of measure zero.

There exists a unique $k^* > 0$ such that $g(k^*) = k^*$ (another fix point would be $k^* = 0$). We have two cases:

- if $k \in (0, k^*)$, then $g(k) > k$, i.e. above the 45° line,
- if $k \in (k^*, \infty)$ and $k > 0$, then $g(k) < k$, i.e. below the 45° line.

In fact, if $0 < k < k^*$, then $g(k^*) = k^* > g(k) > k$, and if $k > k^* > 0$, then $k^* = g(k^*) < g(k) < k$. Furthermore, how many steady states an economy has is independent of initial conditions. These initial conditions are only need for the time-path.

Deterministic One-Sector Growth Model - Competitive Equilibrium Version

Definition 3.1. A competitive equilibrium for the economy is the list of sequences $\{c_t^*\}_{t=0}^{\infty}$, $\{p_t^*\}_{t=0}^{\infty}$, $\{k_t^*\}_{t=0}^{\infty}$, $\{h_t^*\}_{t=0}^{\infty}$, $\{w_t^*\}_{t=0}^{\infty}$, and $\{r_t^*\}_{t=0}^{\infty}$ such that

- the consumer takes the prices $\{p_t^*\}_{t=0}^{\infty}$, $\{w_t^*\}_{t=0}^{\infty}$, $\{r_t^*\}_{t=0}^{\infty}$ as given and then chooses $\{c_t^*\}_{t=0}^{\infty}$, $\{k_t^*\}_{t=0}^{\infty}$, and $\{h_t^*\}_{t=0}^{\infty}$ to solve

$$\max_{\{c_t, k_t, h_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to, using $i_t + (1-\delta)k_t = k_{t+1}$,

$$\sum_{t=0}^{\infty} p_t^* \big(c_t + k_{t+1} - (1-\delta)k_t\big) \le \sum_{t=0}^{\infty} \big(r_t^* k_t + w_t^* h_t\big),$$
$$c_t \ge 0, h_t \in [0,1] \forall t, \;\; k_0 \text{ given.}$$

This means that any consumer either borrows or lends in any given period, i.e. they save and dis-save over time. This feature requires complete markets.

- The firm takes the prices $\{p_t^*\}_{t=0}^{\infty}$, $\{w_t^*\}_{t=0}^{\infty}$, and $\{r_t^*\}_{t=0}^{\infty}$ as given, then chooses $\{k_t^*\}_{t=0}^{\infty}$ and $\{h_t^*\}_{t=0}^{\infty}$ to solve the maximization problem

$$\max_{\{k_t,h_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[p_t^* F(k_t,h_t) - w_t^* h_t - r_t^* k_t\right],$$

subject to

$$k_t \geq 0, h_t \geq 0 \forall t.$$

- All three markets then clear, i.e.

$$c_t^* + k_{t+1}^* - (1-\delta)k_t^* = F(k_t^*, h_t^*) \forall t$$

To simplify this problem, we can substitute as many variables as possible, thus reducing the number of FO(N)Cs. In equilibrium we have a fully employed labor force, i.e. $h_t^* = 1 \forall t$. But plugging this into the equations removes the labor market, eliminating the possibility of determining the equilibrium wage rate w_t^*.

To actually characterize the competitive equilibrium, we follow a certain strategy here:

1. Assume $\{c_t^*\}_{t=0}^{\infty}$, $\{p_t^*\}_{t=0}^{\infty}$, $\{k_t^*\}_{t=0}^{\infty}$, $\{h_t^*\}_{t=0}^{\infty}$, $\{w_t^*\}_{t=0}^{\infty}$, and $\{r_t^*\}_{t=0}^{\infty}$ is a competitive equilibrium and then derive the necessary conditions that these sequences must satisfy.

2. Show, using the derived optimality conditions that the market clearing is implied by the sequences.

Now we focus on the consumer's problem where we assumed $h_t^* = 1 \forall t$. In principle there may be negative amounts of good, e.g. via short sales, but in equilibrium we assume strictly positive values for capital and consumption in all periods, i.e. $c_t^*, k_t^* > 0 \forall t$, since there is only one consumer (per construction). Furthermore, we assume the Lagrange multiplier $\lambda > 0$ on the life-time budget constraint, i.e. we assume strictly positive shadow prices of the budget constraint, i.e. it expresses the quantity of utils that could be obtained with the next dollar of consumption. Note that this only holds for the equilibrium values. Essentially, the shadow price gives the *utility value* of relaxing the budget constraint by one unit. Now, we write the Lagrangian for the consumer's decision problem:

$$\mathfrak{L}(c_t, k_t, \lambda) = \sum_{t=0}^{\infty} \left\{\beta^t u(c_t) + \lambda\big[r_t^* k_t + w_t^* h_t - p_t^*(c_t + k_{t+1} - (1-\delta)k_t)\big]\right\},$$

which yields the following FO(N)Cs

$$\begin{aligned}
\frac{\partial}{\partial c_t}\mathfrak{L}(\cdot) &= \beta^t u'(c_t) - \lambda p_t^* = 0 \forall t,\\
\frac{\partial}{\partial k_t}\mathfrak{L}(\cdot) &= -\lambda p_{t-1}^* + \lambda p_t^*(1-\delta) + \lambda r_t^* = 0 \forall t,\\
\frac{\partial}{\partial \lambda}\mathfrak{L}(\cdot) &= \sum_{t=0}^{\infty}(r_t^* k_t + w_t^* h_t) - \sum_{t=0}^{\infty} p_t^*(c_t + k_{t+1} - (1-\delta)k_t) = 0,\\
& k_0 \text{ given.}
\end{aligned}$$

Now we can divide the FO(N)C for c_t by the one for c_{t+1} such that we have something that looks very familiar:

$$\frac{\beta^t u'(c_t)}{\beta^{t+1} u'(c_{t+1})} = \frac{\lambda p_t^*}{\lambda p_{t+1}^*} = \frac{p_t^*}{p_{t+1}^*} \forall t.$$

Furthermore, we have some terminal condition, i.e. a transversality condition

$$\lim_{t\to\infty} \beta^{t-1} u'(c_{t-1}) k_t = 0.$$

Think of a model with only a finite number of periods, would it make sense to save in the last period? No, any positive left-over capital must have an economic value of zero.. A similar logic applies to infinitely lives agents, at some point (this point is in the infinite future) there has to be ensured that there is no saving anymore and the consumer uses up all their savings. But as there is no final period, we need to take the limit as this terminal condition. As $t \to \infty$, for the solution to be actually optimal, it must be the case that if there is capital left over at the end, this must imply that the agents do not assign any economic value to this capital. Otherwise it cannot be optimal, and we need to impose this condition (it is **not** derived from the optimization). Furthermore, the transversality condition only ever applies to endogenous state variables.

To summarize, for an interior solution to the optimization problem we need three components. First, we need the initial conditions for all state variables. Second, we need the transversality condition for all endogenous state variables. Lastly, from the FONCs we need the Euler equations satisfied.

Now, we take a look at the FO(N)Cs of the firm's decision problem, where we denote π_t as the profit as a function of capital and labor:

$$\frac{\partial}{\partial k_t}\pi_t(k_t, h_t) = p_t^* F_1(k_t, h_t) - r_t^* = 0,$$
$$\frac{\partial}{\partial h_t}\pi_t(k_t, h_t) = p_t^* F_2(k_t, h_t) - w_t^* = 0.$$

Note that there is no transversality condition for the firm's problem, this is due to this problem being a static problem. It is easy to see why this is the case. In every period the output decision only depends on variables that have the same subscript.

Summary 3.1 (Conditions that Characterize a CE)**.** The following collections of conditions fully characterize a CE in the given setting:

1. $\frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} = \frac{p_t^*}{p_{t+1}^*}$,
2. $h_t^* = 1$,
3. $p_t^* = (1-\delta)p_{t+1}^* + r_{t+1}^*$,
4. $\sum_{t=0}^{\infty}(r_t^* k_t^* + w_t^* h_t^*) = \sum_{t=0}^{\infty} p_t^*(c_t^* + k_{t+1}^* - (1-\delta)k_t^*)$,
5. $\lim_{t\to\infty} \beta^{t-1} u'(c_{t-1}^*) k_t^* = 0$
6. k_0 given
7. $p_t^* F_1(k_t^*, h_t^*) = r_t^*$
8. $p_t^* F_2(k_t^*, h_t^*) = w_t^*$,
9. $c_t^* + k_{t+1}^* - (1-\delta)k_t^* = F(k_t^*, h_t^*)$ (due to Euler's theorem on first-order linearly homogeneous functions),
10. $\frac{p_t^*}{p_{t+1}^*} = (1-\delta) + \frac{r_{t+1}^*}{p_{t+1}^*}$ (follows from the third condition),
11. $\frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} = (1-\delta) + \frac{r_{t+1}^*}{r_t^*}$,
12. $\frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} = (1-\delta) + F_1(k_{t+1}^*, h_{t+1}^*)$
13. $\frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} = (1-\delta) + f'(k_{t+1}^*)$

Remember Euler's theorem, that we can decompose a first-order linearly homogeneous function into the inputs and their partial derivatives:

$$F(k, h) = kF_k(k, h) + hF_h(k, h).$$

Adding the subscript and substituting this into the RHS of the 9^{th} condition listed above, we have

$$\begin{aligned} c_t^* + k_{t+1}^* - (1-\delta)k_t^* &= k_t^* F_1(k_t^*, h_t^*) + h_t^* F_2(k_t^*, h_t^*), \\ c_t^* + k_{t+1}^* - (1-\delta)k_t^* &= k_t^* \frac{r_t^*}{p_t^*} + h_t^* \frac{w_t^*}{p_t^*}, \\ p_t^*[c_t^* + k_{t+1}^* - (1-\delta)k_t^*] &= k_t^* r_t^* + h_t^* w_t^* \forall t. \end{aligned}$$

Note that the budget constraint always holds with equality, this is implied from the optimality of the solution. This is also why we do not explicitly consider it in the social planner version of the problem.

This implies that we are able to further summarize the conditions characterizing the competitive equilibrium:

- $\frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} = (1-\delta) + f'(k_{t+1}^*)$ (IMRS = MRT),
- $\lim_{t\to\infty} \beta^{t-1} u'(c_{t-1}^*) k_t^* = 0$,
- k_0 given,
- $c_t^* + k_{t+1}^* - (1-\delta)k_t^* = F(k_t^*, h_t^*)$ (CRS, homogeneous of degree one; goods, labor, and capital market clear),
- $h_t^* = 1$.

These are the same conditions as derived for the solution to the social planner problem. We argued that these conditions uniquely characterize the solution sequences to the social planner problem. In other words, an equilibrium exists and it is unique. In conclusion, given that the optimality conditions associated in the social planner problem and those associated with the decentralized market economy are identical, the equilibrium allocation for $\{c_t^*\}_{t=0}^\infty$,$\{k_t^*\}_{t=0}^\infty$, and $\{h_t^*\}_{t=0}^\infty$ are identical as well.

This yields the question of how the equilibrium price sequences are determined. In order to guarantee unique sequences of prices, we normalize $p_0^* = 1$ and then iterate on the FONCs.

1. given p_0^*, we can use conditions 7 and 8 to infer the factor prices r_0^*, and w_0^* as functions of the factors k_0^* and h_0^*,
2. going to condition 1, given $\{p_0^*, c_0^*, c_1^*\}$, this condition delivers p_1^*, then we keep iterating again

Generally, once we know equilibrium quantities, we can solve for the equilibrium prices, because we have one (representative) consumer and one (representative) firm. With more consumers, we will have a whole family of allocations.

Numerical Estimation of the Equilibrium and Transition Path in the Deterministic Neoclassical Growth Model (Exercise Session)

Numerical estimation sections are based on the exercise session from Alexander Hansak.

Remark 3.1 (Estimation errors)**.** We face two kinds of errors, where $\hat{x}$ differs from x. The first one is the *truncation error*, iterative numeric algorithms theoretically arrive at the true solution after infinitely many step, but we can obviously only go through finitely many iterations. The second one is the *rounding error*, a result of the computer only knowing finitely many numbers.

Of course these errors are an issue, but we can approximate the result reasonably well by setting the tolerance for our solutions to an appropriate number. Concerning the rounding error, it might be helpful to see how Matlab represents numbers. The default is double-precision (64 bits). Bit 63 is used for the sign (0 is positive, 1 is negative), bits 62-52 are used for an exponent e of 2^e, and bits 51-0 are used as a fraction f of $1.f$. Thus, Matlab describes any number as

$$(-1)^{\text{sign}} \cdot 2^{e-1023} \cdot 1.\text{fraction}.$$

In more mathematical terms, we have

$$x = \pm 2^{e-1023}\left[1 + \sum_{k=1}^{52} a_k 2^{-k}\right],$$

where $e \in \{1, 2, \ldots, 2046\}$, and $a_k \in \{0, 1\}$. Note that rounding errors might be negligible, but sometimes lead to unstable solutions, for example:

$$\begin{pmatrix} 1 & 2 \\ 2 & 3.999 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 7.999 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Introducing minor distortions on either side can give very different results:

$$\begin{pmatrix} 1 & 2 \\ 2 & 3.999 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4.001 \\ 7.998 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3.9999 \\ 4 \end{pmatrix},$$
$$\begin{pmatrix} 1.001 & 2.001 \\ 1.999 & 3.998 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 7.999 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1.5 \end{pmatrix}.$$

Why is this the case? The coefficient matrix is almost singular, thus the inversion process with rounding errors yields very change-sensitive results.

We consider a first-order difference equation of the general form

$$x_{t+1} = f(x_t), \tag{**}$$

where $x_t \in X \subseteq \mathbb{R}^n$ for all t, and $f : \mathbb{R}^n \mapsto \mathbb{R}^n$. A point $x^* \in \mathbb{R}^n$ is called a *steady state* if $x^* = f(x^*)$. Close to this steady state we can approximate the behavior of this system by a linear system

$$x_{t+1} - x^* = \mathfrak{J}(x^*)(x_t - x^*),$$

assuming the Jacobian has no eigenvalue with absolute value 1. Note that the Jacobian $\mathfrak{J}(x^*) \in \mathbb{R}^{n\times n}$ denotes the Jacobian of f evaluated at x^*:

$$\mathfrak{J}(x^*) = Df(x^*) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x^*) & \cdots & \frac{\partial f_1}{\partial x_n}(x^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(x^*) & \cdots & \frac{\partial f_n}{\partial x_n}(x^*) \end{pmatrix}.$$

Consider now the neoclassical growth model, where the equilibrium system for $t \geq 0$ is given by

$$\begin{aligned} k_{t+1} &= f(k_t) + (1-\delta) - c_t \\ u'(c_t) &= \beta[1 - \delta + f'(k_{t+1})]u'(c_{t+1}). \end{aligned} \tag{1}$$

Utility maximizing w.r.t. the choices of capital and consumption for $t \in \mathbb{N}_0$ also requires some initial capital stock k_0 and a transversality condition $\lim_{t\to\infty} \beta^t u'(c_t)k_{t+1} = 0$. The transversality condition pins down a unique c_0. From the definition of a steady state, we know that $c_t = c^*,\ k_t = k^* \forall t \in \mathbb{N}_0$, and therefore

$$\begin{aligned} &1 = \beta[1 - \delta + f'(k^*)] \Leftrightarrow k^* = (f')^{-1}(\delta + \beta^{-1} - 1), \\ &c^* = f(k^*) - \delta k^*. \end{aligned}$$

We are now concerned with describing the behavior around the steady state. For describing this, we can start by finding our Jacobian. This can be done either directly, or via the implicit function theorem.

Calculating the Jacobian directly: rewrite (1) in the form $x_{t+1} = f(x_t)$, such that we have

$$\mathfrak{J}(x^*) = Df(x^*) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x^*) & \cdots & \frac{\partial f_1}{\partial x_n}(x^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(x^*) & \cdots & \frac{\partial f_n}{\partial x_n}(x^*) \end{pmatrix},$$

which - in our example - uses

$$x_{t+1} = \begin{pmatrix} k_{t+1} \\ c_{t+1} \end{pmatrix} = \begin{pmatrix} f(k_t) + (1-\delta)k_t - c_t \\ (u')^{-1}\left\{\frac{u'(c_t)}{\beta[1-\delta+f'(k_{t++1})]}\right\} \end{pmatrix}.$$

Using the Implicit Function Theorem: rewrite (1) in the form $F(x_t, x_{t+1})$, and then calculate

$$\mathfrak{J}_F(x^*, x^*) = DF(x^*, x^*) = \left(\begin{array}{cc|cc} \frac{\partial F_1}{\partial k_t}(x^*) & \frac{\partial F_1}{\partial c_t}(x^*) & \frac{\partial F_1}{\partial k_{t+1}}(x^*) & \frac{\partial F_1}{\partial c_{t+1}}(x^*) \\ \frac{\partial F_2}{\partial k_t}(x^*) & \frac{\partial F_2}{\partial c_t}(x^*) & \frac{\partial F_2}{\partial k_{t+1}}(x^*) & \frac{\partial F_2}{\partial c_{t+1}}(x^*). \end{array} \right)$$

Note that the LHS of the matrix can be denoted by $DF_1(x^*, x^*)$ and the RHS is $DF_1(x^*, x^*)$. We can then obtain the Jacobian as

$$\mathfrak{J}(x^*) = -[DF_2(x^*, x^*)]^{-1} DF_1(x^*, x^*).$$

Regardless of the method, we arrive at the same matrix:

$$\mathfrak{J}(x^*) = \begin{pmatrix} \beta^{-1} & -1 \\ -A\beta^{-1} & 1 + A \end{pmatrix},$$

where $A = \beta f''(k^*) \frac{u'(c^*)}{u''(c^*)} > 0$. It can be shown that the eigenvalues of $\mathfrak{J}$ satisfy $0 < \lambda_1 < 1 < \lambda_2$. x^* is a saddle-point of the linearized system because of the eigenvalues satisfy $0 < \lambda_1 < 1 < \lambda_2$. In other words, in a neighborhood around x^* we have existence of a saddle path such that $\forall x_t$ on this saddle path it holds that $x_{t+1} = f(x_t)$. Locally, the saddle path (*stable path*) corresponds to the eigenvector v_1 to λ_1, and the unstable path (larger than zero eigenvalue) to the eigenvector v_2 to λ_2.

Remark 3.2 (Computation of the transition path to the steady state for a given k_0)**.** In the models used in this course, the *stable* neighborhood around the steady state is usually quite large, so we have a large neighborhood in which there exists a saddle path. We can therefore try to pick a $c_0 \in \mathbb{R}$ on the saddle path. The properties of the saddle path then guarantee $x_t \to x^*$.

There are multiple ways to find these transition paths, one method is called *forward shooting.*

Definition 3.2 (Forward Shooting Algorithm)**.**

1. guess a $c_0 \in \mathbb{R}$
2. starting from (k_0, c_0) iterate forward on (1) to obtain $(k_t, c_t)_{t=0}^T$, where T is sufficiently large to ensure convergence
3. compute the residual $k_T - k^*$
4. repeat 1-3 until $k_T \approx k^*$

Note that this requires is to already have calculated the steady state. Furthermore, this solution method is *not numerically stable* for two reasons. Firstly, a small change in the initial value may lead to a large change in $k_T - k^*$. Secondly, even if we start from the theoretically correct c_0, carrying-over of rounding errors across iterations may result in the algorithm not converging. The reason for this numerical instability is quite simple, once we deviate a little from the stable manifold, the dynamics of the system push us away.

One takeaway here is that the choice of c_0 is critical, but also that T should not be chosen too large. This raises the question on the size of T. When implementing forward shooting, modifying the procedure may help alleviate some of the issues. Instead of solving for a root of $k_T - k^*$, we may consider choosing c_0 to minimize the following quadratic loss function

$$\Delta = \left(\frac{k_T - k^*}{k^*} \right)^2 + \left(\frac{c_T - c^*}{c^*} \right)^2.$$

If T is too small, then from a given k_0 there is no path that reaches the stable point (k^*, c^*) in T periods. But we can obtain a $\tilde{c}_0$ that leads into the direction of the steady state. This $\tilde{c}_0$ can then be used as an initial guess, using a larger T. We can then increase T and refine the starting guesses until $\Delta \approx 0$.

In Matlab, the function *lsqnonlin* can be used here, but requires a modification of the tolerances.

Listing 1: Forward shooting - modified procedure

```
myoptions = optimset('TolX',10e-12,'TolFun',10e-12);
c0 = lsqnonlin(@(x) transitionpath(T,x), cguess,[],[],myoptions);
```

Example 3.1 (Deterministic Growth Model with Fiscal Policy)**.** We assume a representative household maximizes their discounted lifetime utility

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

and we have a feasibility constraint on the economy

$$g + c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t.$$

We assume a Cobb-Douglas production function

$$f(k) = k^\alpha.$$

The government faces a budget constraint

$$\sum_{t=0}^{\infty} q_t[\tau_t^c c_t + \tau_t^h] = \sum_{t=0}^{\infty} q_t g.$$

In equilibrium, we have

$$\begin{aligned} k^* &= (f')^{-1}\left[\beta^{-1} + \delta - 1\right], \\ c^* &= f(k^*) - \delta k^* - g. \end{aligned}$$

Furthermore, we have the time-path towards the equilibrium for capital and consumption:

$$\begin{aligned} k_{t+1} &= f(k_t) + (1-\delta)k_t - c_t - g \\ c_{t+1} &= (u')^{-1}\left\{\frac{u'(c_t)}{\beta[1-\delta+f'(k_{t+1})]}\right\}. \end{aligned}$$

We have the equilibrium prices following the time path

$$q_{t+1} = \frac{q_t}{1-\delta+f'(k_{t+1})}.$$

Lastly, the equilibrium tax (either one of the taxes is assumed to be zero when the other one is non-zero) is given by

$$\tau^h = \sum_{t=0}^{\infty} q_t g \left\{\sum_{t=0}^{\infty} q_t\right\}^{-1}$$

or

$$\tau^c = \sum_{t=0}^{\infty} q_t g \left\{\sum_{t=0}^{\infty} q_t c_t\right\}^{-1}$$

One-Sector Stochastic Growth Model - Alternative Representations

Remark 3.3 (Indirect Utility Function)**.** First some remarks on the indirect utility function. Consider the static utility maximization

$$\max_x u(x) \text{ s.t. } px \leq y.$$

From the FOC of the Lagrangian we get the Marshallian demand of the form

$$x_M^* = x(p, y).$$

Now we insert this demand in the utility function, this then describes the indirect utility function

$$u(x_M^*) = V(p, y) = u\big(x(p, y)\big).$$

Note that the indirect utility is weakly decreasing in the prices and weakly increasing in the income.

Remark 3.4 (Value Function)**.** The value function in the context of this section describes something similar to an indirect utility function. The input consists of the good prices and the income, i.e. it takes the result of the maximization problem as an input. However, the value function is not the exact same thing as the indirect utility function but rather the dynamic counterpart to the static indirect utility function.

In this section we consider a modified version of the previously described models. The modification describes an output in every period being stochastic. The output is subject to some i.i.d. disturbance ϵ_t, where $\log \epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$. This disturbance is the Solow residual, we model TFP shocks by an AR(1) process.

The planner's dynamic optimization problem can now be written as

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $\beta \in (0, 1)$, and we face the resource constraint

$$k_{t+1} = \epsilon_t k_t^\alpha + (1 - \delta) k_t - c_t,$$

with $\alpha \in (0, 1)$ and the initial conditions $k_0 \geq 0$, $\epsilon_0 = 1$.

If we want to interpret this, at the beginning of every period t the agents observe the TFP shock ϵ_t, produce output given the current available input. Afterwards they decide on consumption and investment.

There are multiple alternative ways to solve the optimization problem from the resource constrained maximization problem, we usually focus on two of these. Firstly the familiar Lagrangian setup, and secondly a dynamic programming approach which uses a value function as a functional. Functional means that the function itself is unknown and we have to determine this function.

Definition 3.3. A value function V is the maximized value of the objective function subject to the constraints. It depends on the state vector of the model, e.g. on (k, ϵ).

The value function is the outcome of a maximization problem. It obeys the Bellman principle of optimality.

Definition 3.4 (Bellman Principle of Optimality)**.** In order for the total sum of discounted payoffs over the total time interval of our $[0, \infty)$ time horizon to be maximized, the subtotal of the sum of payoffs over $[0, 1]$ and $[1, \infty)$ must be maximized. To put it in the original wording according to Bellman (1957), "[a]n optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

Example 3.2. Let $V(k_0, \epsilon_0)$ denote the maximized value of the maximization problem subject to the resource constraint, given that at $t = 0$ capital stock is k_0 and the shock is ϵ_0. Then the following holds true:

$$V(k_0, \epsilon_0) = \max_{c_0, k_1} u(c_0) + \beta \mathbb{E}_0 \left\{ V(k_1, \epsilon_1) \mid \epsilon_0 \right\},$$

subject to

$$c_0 + k_1 = \epsilon_0 k_0^\alpha + (1 - \delta) k_0.$$

This indicates that the maximization problem can be alternatively expressed as

$$V(k_0, \epsilon_0) = \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to

$$k_{t+1} = \epsilon_t k_t^{\alpha} + (1-\alpha)k_t - c_t,$$

given $k_0 \geq 0$ and $\epsilon_0 = 1$.

More generally, we can put this into a recursive formulation:

$$V(k_t, \epsilon_t) = \max_{c_t, k_{t+1}} u(c_t) + \beta \mathbb{E}_t \{V(k_{t+1}, \epsilon_{t+1}) \mid \epsilon_t\},$$

subject to

$$k_{t+1} = \epsilon_t k_t^{\alpha} + (1-\delta)k_t - c_t,$$

for $\alpha, \beta \in (0,1)$. Thus, $V(k_{t+1}, \epsilon_{t+1})$ represents the maximized value of the objective from $t+1$ onward. If $\{c, k\}$ is maximizing, it must maximize over the two sub-intervals $[t, t+1]$ and $[t+1, \infty)$. Deleting the subscripts and letting x' denote next period's value for some variable x, we can use the commonly used notation from a *Bellman equation*:

$$V(k, \epsilon) = \max_{c, k'} u(c) + \beta \mathbb{E} \{V(k', \epsilon') \mid \epsilon\}$$

subject to

$$c \leq \epsilon k^{\alpha} + (1-\delta)k - k',$$

and the initial conditions of weakly positive values for k and ϵ.

The solution to this dynamic programming problem can be represented by a time-invariant policy function $g : K \mapsto K$, determining which value of k' to choose for a given value of the state variable k. The micro-foundation of this model acts as an *anchor* shielding against the Lucas critique. The time-invariant policy drops out of the model via the iteration process. This faces multiple possible complications. For one, a value of $c^*(g(k), k)$ may not exist. Secondly, the solution to the functional equation may not involve a policy function but rather a policy correspondence, because there may be more than one maximizer for a given k. Assume we have a correspondence as the output, this would imply a set of choices rather than a unique result. This is an issue when doing policy recommendation, as you would need to tell the politician that they may choose from a range of options, which is not ideal*. But, if the value function exists and is well-behaved, i.e. if it satisfies certain requirements, the optimal policy follows directly from the solution to the value function.

Remark 3.5 (Characteristics of the Bellman equation)**.** The sequence problem and the Bellman equation yield the same solution. To see this, consider

$$\begin{aligned}
V(x_0) &= \sup_{x_{t+1} \in \Gamma(x)} \sum_{t=0}^{\infty} \beta^t u(x_t, x_{t+1}) \\
&= \sup_{x_{t+1} \in \Gamma(x)} \left\{ u(x_0, x_1) + \sum_{t=1}^{\infty} \beta^t u(x_t, x_{t+1}) \right\} \\
&= \sup_{x_{t+1} \in \Gamma(x)} \left\{ u(x_0, x_1) + \beta \sum_{t=1}^{\infty} \beta^{t-1} u(x_t, x_{t+1}) \right\} \\
&= \sup_{x_1 \in \Gamma(x_0)} \left\{ u(x_0, x_1) + \beta \sup_{x_{t+1} \in \Gamma(x)} \sum_{t=0}^{\infty} \beta^t u(x_{t+1}, x_{t+2}) \right\}
\end{aligned}$$

*It's not like they would listen properly anyway.

$$= \sup_{x_1 \in \Gamma(x_0)} u(x_0, x_1) + \beta V(x_1).$$

We would need to show the implication in the other direction at this point, but I will omit this here. The sufficient condition is $\lim_{t\to\infty} \beta^t u(x_t) = 0$ for all feasible sequences of x.

The Bellman equation breaks the large life-cycle problem into smaller static problems. The key here is that we have a memory-less process, dependence is only on the value of state variables at the time of the decision.

The unknown value function can be pinned down by backwards induction.

Remark 3.6 (What exactly is the result of solving a Bellman equation?)**.** Once we know the functional equation, we can solve for k' as a function g of the current capital stock k. This function $g(k) = k'$ is known as the policy function.

Remark 3.7 (More on Dynamic Programming)**.** Assume the functional equation representation of the deterministic growth model

$$V(k) = \max_{k' \in G(k)} \Big\{ u(k, k') + \beta V(k') \Big\} \tag{4.1}$$

for all $k \in K$, where $K \subseteq \mathbb{R}^n$ is compact. G is non-empty, compact, and continuous. $u(\cdot)$ is concave, continuously differentiable on the interior of its domain, and for each $k' \in J$, $u(\cdot, k')$ is strictly increasing in k.

The goal is to solve for the pair of function V and $g(k)$. Now let $g(k)$ denote the unique and time-invariant *optimal solution* to (4.1), then $g(k)$ is characterized by the Euler equation (from the FONC of the interior solution):

$$\frac{\partial}{\partial k'} u\big(k, g(k)\big) + \beta V'\big(g(k)\big).$$

So, we have a comparison of today's marginal utility of consumption (or capital) tomorrow, to the discounted future marginal utility.

Once V is determined, the policy function $g(k)$ satisfies

$$V(k) = u\big(k, g(k)\big) + \beta V\big(g(k)\big) \forall k \in K. \tag{4.2}$$

That is, the RHS of the Bellman's equation is maximal when valuated at the optimal policy function, i.e. when you plug in the optimal policy function and the functional, then the overall term is maximized. Furthermore, we have

$$V'(k) = u_1\big(k, g(k)\big) + u_2\big(k, g(k)\big) \frac{\partial}{\partial k} \Big[g(k) \Big] + \beta V'\big(g(k)\big) \frac{\partial}{\partial k} \Big[g(k) \Big].$$

Since $g(k)$ is optimal, this expression reduces to

$$V'(k) = u_1\big(k, g(k)\big). \tag{4.2}$$

This last formula is commonly referred to as *Benveniste-Scheinkman formula* which is very useful for solving dynamic problems. It is used evaluate welfare changes from changes in capital.

Remark 3.8 (Some Remarks on the Previous Derivations)**.** Say we have the optimum policy function $k' = g(k)$, this implies

$$V(k) = u\big(k, g(k)\big) + \beta V\big(g(k)\big)$$

by definition. Similarly, $g(k)$ satisfies the FONC:

$$u_2(k, k') + \beta V'(k') = 0,$$

assuming an interior solution. Evaluated at the optimum (the policy function), we have

$$u_2\big(k, g(k)\big) + \beta V'\big(g(k)\big) = 0.$$

While we cannot give a general $V(\cdot)$, we can find the derivative. Using the first equation from this remark, we can differentiate both sides w.r.t. k, since the equation holds for all k. We obtain

$$V'(k) = u_1\big(k, g(k)\big) + \underbrace{g'(k)\big\{u_2\big(k, g(k)\big) + \beta V'\big(g(k)\big)\big\}}_{\text{indirect effect through the choice of } k'}.$$

From the FONC, we know that the bracket reduces to zero in optimum. This leaves us with

$$V'(k) = u_1\big(k, g(k)\big).$$

We can update this term, such that $V'\big(g(k)\big) = u_1\big(g(k), g(g(k))\big)$. We then rewrite the FONC as follows:

$$u_2\big(k, g(k)\big) + \beta u_1\big(g(k), g(g(k))\big) = 0 \forall k.$$

This is the Euler equation stated as a functional equation. It does not contain the unknowns k_t, k_{t+1}, k_{t+2}, where we now have the unknown being $g(\cdot)$.

Definition 3.5 (Kuhn-Tucker Conditions)**.** Let f and g_j for $j = 1, \ldots, m$ be *differentiable functions of* n *variables* and let $c_j \in \mathbb{R}^j$ for $j = 1, \ldots, m$. Define the function $\mathfrak{L}$ of n variables by

$$\mathfrak{L}(\mathbf{x}) = f(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j \{g_j(\mathbf{x}) - c_j\} \forall \mathbf{x} \in \mathbf{X} \subseteq \mathbb{R}^n.$$

The *Kuhn-Tucker conditions* for the problem

$$\max_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x}) \text{ subject to } g_j(\mathbf{x}) \leq c_j \text{ for } j = 1, \ldots, m$$

are given by

$$\frac{\partial}{\partial x_i} \mathfrak{L}(\mathbf{x}) = 0$$

for $i = 1, \ldots, n$, and $\lambda_j \geq 0$, $g_j(\mathbf{x}) \leq c_j$, and $\lambda_j[g_j(\mathbf{x}) - c_j] = 0$ for $j = 1, \ldots, m$. These conditions constitute the FONCs, given some satisfied regularity conditions.

For the existence of a solution we would require the column rank of the Jacobian (of the constraints) to be of full rank (non-singular). First we define the constraints to be of the form $h_i(\cdot) = 0$ for $i = 1, \ldots, m$. This is

$$\begin{aligned} g_1(\mathbf{x}) &= c_1 \\ g_2(\mathbf{x}) &= c_2 \\ g_3(\mathbf{x}) &= c_3 \\ g_4(\mathbf{x}) &= c_4 \\ &\vdots\vdots \\ g_j(\mathbf{x}) &= c_j \end{aligned}$$

The Jacobian is defined as the matrix of partial derivatives

$$\mathfrak{J} = \begin{pmatrix} g_{1,x_1}(\mathbf{x}) & g_{1,x_2}(\mathbf{x}) & g_{1,x_3}(\mathbf{x}) & g_{1,x_4}(\mathbf{x}) & \ldots & g_{1,x_n}(\mathbf{x}) \\ g_{2,x_2}(\mathbf{x}) & g_{2,x_2}(\mathbf{x}) & g_{2,x_3}(\mathbf{x}) & g_{2,x_4}(\mathbf{x}) & \ldots & g_{2,x_n}(\mathbf{x}) \\ \vdots & & & & & \vdots \\ g_{m,x_1}(\mathbf{x}) & g_{m,x_2}(\mathbf{x}) & g_{m,x_3}(\mathbf{x}) & g_{m,x_4}(\mathbf{x}) & \ldots & g_{m,x_n}(\mathbf{x}) \end{pmatrix}.$$

This matrix needs to be of full rank, otherwise the Lagrangian does not lead to a solution to the maximization problem. This is called the regularity condition, or the constraint qualification in this context.

Numerical Estimation of the Equilibrium and Transition Path in the Stochastic Neoclassical Growth Model (Exercise Session)

We consider the planner's problem:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to

$$\begin{aligned} k_{t+1} &= z_t f(k_t) + (1-\delta)k_t - c_t, \\ z_{t+1} &= z(1-\rho) + \rho z_t + \epsilon_{t+1}, \end{aligned}$$

where $\beta \in (0,1)$, $|\rho| < 1$, and ϵ_t is white noise. Note that $\mathbb{E} z_t = z$.

We have the following optimality condition. First, the stochastic Euler equation

$$u'(c_t) = \mathbb{E}_t\big[\beta(1-\delta + z_{t+1} f'(k_{t+1}))u'(c_{t+1})\big],$$

the aggregate resource constraint

$$k_{t+1} = z_t f(k_t) + (1-\delta)k_t - c_t,$$

and the law of motion for TFP

$$z_{t+1} = z(1-\rho) + \rho z_t + \epsilon_t.$$

We can rewrite this to the form $F(c_t, k_t, z_t, c_{t+1}, k_{t+1}, z_{t+1}) = \xi_{t+1}$, where we have

$$F(\cdot) = \begin{pmatrix} u'(c_t) - \beta[1-\delta + z_{t+1} f'(k_{t+1})]u'(c_{t+1}) \\ z_t f(k_t) + (1-\delta)k_t - c_t - k_{t+1} \\ \rho(z_t - z) - (z_{t+1} - z) \end{pmatrix},$$

and the mean zero error vector

$$\xi_{t+1} = \begin{pmatrix} \eta_{t+1} \\ 0 \\ \epsilon_{t+1} \end{pmatrix}.$$

In a non-stochastic steady state, we have $c^* = c_t = c_{t+1},\ k^* = k_t = k_{t+1},\ z^* = z_t = z_{t+1} \forall t$, and $\xi_{t+1} = 0$. Thus (c^*, k^*, z^*) satisfy $F(c^*, k^*, z^*, c^*, k^*, z^*) = \mathbf{0}$, which leads to

$$\begin{aligned} z^* &= z, \\ k^* &= (f')^{-1}\left(\frac{\beta^{-1} + \delta - 1}{z}\right), \\ c^* &= z f(k^*) - \delta k^*. \end{aligned}$$

Evaluated at the steady state, the Jacobian is given by

$$\begin{aligned} \mathfrak{J}(c^*, k^*, z^*) &= \begin{pmatrix} \frac{\partial c_{t+1}}{\partial c_t} & \frac{\partial k_{t+1}}{\partial c_t} & \frac{\partial z_{t+1}}{\partial c_t} \\ \frac{\partial c_{t+1}}{\partial k_t} & \frac{\partial k_{t+1}}{\partial k_t} & \frac{\partial z_{t+1}}{\partial k_t} \\ \frac{\partial c_{t+1}}{\partial z_t} & \frac{\partial k_{t+1}}{\partial z_t} & \frac{\partial z_{t+1}}{\partial z_t} \end{pmatrix} \\ &= \begin{pmatrix} 1 + A & -A\beta^{-1} & -Af(k^*) - \rho f'(k^*)\frac{u'(c^*)}{u''(c^*)} \\ -1 & \beta^{-1} & f(k^*) \\ 0 & 0 & \rho \end{pmatrix}, \end{aligned}$$

where $A := \beta z^* f''(k^*) \frac{u'(c^*)}{u''(c^*)} > 0$. We linearize around the steady state, this allows us to express the dynamics in the following way:

$$\begin{pmatrix} \Delta c_{t+1} \\ \Delta k_{t+1} \\ \Delta z_{t+1} \end{pmatrix} = \begin{pmatrix} 1+A & -A\beta^{-1} & -Af(k^*) - \rho f'(k^*)\frac{u'(c^*)}{u''(c^*)} \\ -1 & \beta^{-1} & f(k^*) \\ 0 & 0 & \rho \end{pmatrix} \begin{pmatrix} \Delta c_t \\ \Delta k_t \\ \Delta z_t \end{pmatrix} + \begin{pmatrix} \eta_{t+1} \\ 0 \\ \epsilon_{t+1} \end{pmatrix},$$

where $\Delta x_t := x_t - x^*$.

Remark 3.9 (Stability Analysis)**.** The eigenvalues of the Jacobian $\mathfrak{J}(c^*, k^*, z^*)$ are the roots of the characteristic polynomial $p(\lambda) = |\mathfrak{J}(c^*, k^*, z^*) - \lambda \mathbb{I}_3| = 0$. It can be shown that

$$|\mathfrak{J}(c^*, k^*, z^*) - \lambda \mathbb{I}_3| = (\rho - \lambda) \cdot |\mathfrak{J}(c^*, k^*) - \lambda \mathbb{I}_2|.$$

Why is this interesting for us? $\mathfrak{J}(c^*, k^*)$, which is the upper left 2×2 submatrix of $\mathfrak{J}$, is the Jacobian from the deterministic growth model, hence we know that the roots of this term satisfy $0 < \lambda_1 < 1 < \lambda_2$. The 3^{rd} eigenvalue of the Jacobian is $\rho \in (-1, 1)$. This implies that (c^*, k^*, z^*) is a saddle point with two stable eigenvalues (the second one is not stable). We have two predetermined variables k_0 and z_0. Blanchard-Kahn (1990) have shown that the linearized system possesses a unique stable solution satisfying

$$\mathbb{E}_0[\Delta c_t, \Delta k_t, \Delta z_t] \to \mathbf{0}.$$

The following result holds for deterministic systems of first-order difference equations:

Theorem (Deterministic Dynamic Equilibrium)**.** Assume that $\mathfrak{J}(x^*)$ has n distinct eigenvalues λ_i with corresponding eigenvectors v_i, then x_t is a solution of $\Delta x_{t+1} = \mathfrak{J}(x^*)\Delta x_t$ if and only if it has the form

$$\Delta x_t = \sum_{i=1}^{n} \alpha_i \lambda_i^t v_i.$$

It converges to x^* if and only if $\alpha_i = 0$ whenever $|\lambda_i| \geq 1$.

The previous result can be generalized to stochastic dynamic equilibria.

Theorem (Stochastic Dynamic Equilibrium)**.** Assume that $\mathfrak{J}(x^*)$ has n distinct eigenvalues λ_i with corresponding eigenvectors v_i, then x_t is a solution of $\Delta x_{t+1} = \mathfrak{J}(x^*)\Delta x_t + \xi_{t+1}$ if and only if it has the form

$$\Delta x_t = \sum_{i=1}^{n} \alpha_i \lambda_i^t v_i + \sum_{s=1}^{t} \mathfrak{J}(x^*)^{t-s} \xi_s.$$

Taking expectations and considering that $\mathbb{E}_0 \xi_s = \mathbb{E}_0 \mathbb{E}_{s-1} \xi_s = \mathbf{0}$ yields

$$\mathbb{E}_0 \Delta x_t = \sum_{i=1}^{n} \alpha_i \lambda_i^t v_i.$$

Therefore, Δx_t converges *in expectation* to x^* if and only if $\alpha_i = 0$ whenever $|\lambda_i| \geq 1$.

Remark 3.10 (Application to the stochastic growth model)**.** We have a general solution at $t = 0$:

$$\begin{pmatrix} \Delta c_0 \\ \Delta k_0 \\ \Delta z_0 \end{pmatrix} = \sum_{i=1}^{3} \alpha_i v_i = \alpha_1 \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \end{pmatrix} + \alpha_2 \begin{pmatrix} v_{21} \\ v_{22} \\ v_{23} \end{pmatrix} + \alpha_3 \begin{pmatrix} v_{31} \\ v_{32} \\ v_{33} \end{pmatrix}.$$

From the previous remarks we know that for stability we need $\alpha_2 = 0$. But we do have a predetermined k_0 and z_0, thus also Δk_0 and Δz_0. Then we have three linear equations in three unknowns $\alpha_1,\ \alpha_3,\ c_0$. This implies a unique stable solution $(c_t, k_t, z_t)_{t=0}^{\infty}$ for given (k_0, z_0).

The solution of the system of linear equations is given by

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} v_{12} & v_{32} \\ v_{13} & v_{33} \end{pmatrix}^{-1} \begin{pmatrix} \Delta k_0 \\ \Delta z_0 \end{pmatrix},$$

and the initial consumption level on the stable path is then given by

$$\Delta c_0 = \begin{pmatrix} v_{11} & v_{31} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_3 \end{pmatrix}.$$

The next remark is of particular importance, as it describes the policy functions we use very often.

Remark 3.11 (Policy functions)**.** A policy function expresses c_t and k_{t+1} as functions of the state vector $(\Delta k_t, \Delta z_t)$. From the remarks on linearized dynamics we have the linearized dynamics for k_{t+1} in terms of the second row of the Jacobian, and the state vector:

$$\Delta k_{t+1} = \mathfrak{J}_{(2,1)} \Delta c_t + \mathfrak{J}_{(2,2)} \Delta k_t + \mathfrak{J}_{(2,3)} \Delta z_t. \tag{1}$$

Calculating the policy function of consumption at time zero is trivial given the above remark, namely

$$\Delta c_0 = \begin{pmatrix} v_{11} & v_{31} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} v_{11} & v_{31} \end{pmatrix} \begin{pmatrix} v_{12} & v_{32} \\ v_{13} & v_{33} \end{pmatrix}^{-1} \begin{pmatrix} \Delta k_0 \\ \Delta z_0 \end{pmatrix}.$$

As time itself is of no importance in the model (period-to-period behavior is analyzed), this relationship holds for all $t \geq 0$. Therefore, we start by calculating the first term

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} v_{11} & v_{31} \end{pmatrix} \begin{pmatrix} v_{12} & v_{32} \\ v_{13} & v_{33} \end{pmatrix}^{-1}.$$

Then the policy function for Δc_t is given by

$$\Delta c_t = \gamma_1 \Delta k_t + \gamma_2 \Delta z_t.$$

This can be substituted into (1) in order to eliminate Δc_t and express Δk_{t+1} as a linear function of only $(\Delta k_t, \Delta z_t)$:

$$\begin{aligned} \Delta k_{t+1} &= \mathfrak{J}_{(2,1)} \left[\gamma_1 \Delta k_t + \gamma_2 \Delta z_t \right] + \mathfrak{J}_{(2,2)} \Delta k_t + \mathfrak{J}_{(2,3)} \Delta z_t, \\ &= \Big[\gamma_1 \mathfrak{J}_{(2,1)} + \mathfrak{J}_{(2,2)} \Big] \Delta k_t + \Big[\gamma_2 \mathfrak{J}_{(2,1)} + \mathfrak{J}_{(2,3)} \Big] \Delta z_t \end{aligned}$$

3.1 Endogenous aggregate state variable

In order to transfer an equilibrium with time-0 trading into ne with sequential trading, it was accounted for how individuals' wealth evolves as time passes in a time-0 trading economy. While no agent makes any trades after time 0, the present value of their portfolio evolves as uncertainty is resolved. So at time t and history s^t, we used the Arrow-Debreu prices to compute the value of an individual's purchased claims to current and future goods, net of their outstanding liabilities. It was then shown that these wealth levels, and their associated consumption choices, could also be obtained in a sequential trading economy where markets only deal in one-period Arrow securities that reopen each period.

In the pure exchange economies this is fairly easily done, as the relevant state variable wealth is a function solely of the current realization of the exogenous Markov state variable. This is more subtle in an economy in which part of the aggregate state is endogenous in the sense that it emerges from the history of equilibrium interactions of agents' decisions. In this section we use the basic stochastic growth model, the RBC model, to illustrate the implications of moving from a time-0 trading scheme to sequential trading.

3.2 The stochastic growth model

In each period $t \geq 0$, there is a realization of a stochastic event $s_t \in S$. We again denote the history of realized stochastic events by $s^t = [s_t, s_{t-1}, \ldots, s_0]$. The unconditional probability of observing any particular history is given by $\pi_t(s^t)$, while the conditional probability is given by $\pi_\tau(s^\tau \mid s^t)$. Furthermore, we assume $\pi_0(s_0) = 1$ for a particular $s_0 \in S$. s^t is used as a commodity space in which goods are differentiated by histories.

A representative household has preferences over nonnegative streams of consumption and leisure $\{c_t(s^t), \ell_t(s^t)\}$ that are ordered by

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[c_t(s^t), \ell_t(s^t)] \pi_t(s^t), \tag{3.1}$$

where $\beta \in (0,1)$ and $u : \mathbb{R}_+ \times [0,1] \mapsto \mathbb{R}$ is strictly increasing in its two arguments, twice continuously differentiable, strictly concave, and satisfies the Inada conditions

$$\lim_{c \to 0} u_c(c, \ell) = \lim_{\ell \to 0} u_\ell(c, \ell) = \infty.$$

In each period, we assume a household to be endowed with 1 unit time, where the agent faces the labor-leisure constraint:

$$1 = \ell_t(s^t) + n_t(s^t). \tag{3.2}$$

The only other endowment is a capital stock k_0 at the beginning of period 0.

The technology is

$$c_t(s^t) + x_t(s^t) \leq A_t(s^t) F\left[k_t(s^{t-1}), n_t(s^t)\right], \tag{3.3a}$$

$$k_{t+1}(s^t) = (1-\delta) k_t(s^{t-1}) + x_t(s^t), \tag{3.3b}$$

where F is twice continuously differentiable and exhibits constant returns to scale w.r.t. the inputs capital $k_t(s^{t-1})$ and labor $n_t(s^t)$. $A_t(s^t)$ is a stochastic process of Harrod-neutral technology shocks, i.e. stochastic shocks leading to technology progress faster than labor force size growth (labor saving). We have two output goods, the consumption good $c_t(s^t)$, and the investment good $x_t(s^t)$. In (3.3.b), the investment good augments a rate-δ depreciating capital stock. We allow for $x_t(s^t)$ to be negative, i.e. we can recover consumption goods from the investment good. Furthermore, we assume certain properties about the production function. It satisfies the assumption of positive but diminishing marginal products,

$$F_i(k, n) > 0, \;\; F_{ii}(k, n) < 0 \text{ for } i = k, n,$$

and the Inada conditions

$$\lim_{i \to 0} F_i(k, n) = \infty \text{ for } i = k, n$$

$$\lim_{i \to \infty} F_i(k, n) = 0 \text{ for } i = k, n.$$

Since we assume the production function to have constant returns to scale, we can redefine the function as

$$F(k, n) = n f(\hat{k}), \;\; \text{where } \hat{k} = \frac{k}{n}. \tag{3.4}$$

Another property of linearly homogeneous functions is that its first derivatives are homogeneous of degree 0, thus the first derivatives are functions only of the ratio $\hat{k}$, in particular

$$F_k(k, n) = \frac{\partial}{\partial k} n f(k/n) = f'(\hat{k}), \tag{3.5a}$$

$$F_n(k, n) = \frac{\partial}{\partial n} n f(k/n) = f(\hat{k}) - f'(\hat{k})\hat{k} \tag{3.5b}$$

3.3 Lagrangian formulation of the planning problem

The planner chooses an allocation $\{c_t(s^t), \ell_t(s^t), x_t(s^t), n_t(s^t), k_{t+1}(s^t)\}_{t=0}^{\infty}$ to maximize (3.1), subject to (3.2), (3.3), the initial capital stock k_0 and the stochastic process governing the technology level $A_t(s^t)$. To solve this planning problem, we form the Lagrangian

$$\mathfrak{L} = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) \Big\{ u(c_t(s^t), 1 - n_t(s^t)) + \mu_t(s^t) \big[A_t(s^t) F(k_t(s^{t-1}), n_t(s^t)) + (1-\delta) k_t(s^{t-1}) - c_t(s^t) - k_{t+1}(s^t) \big] \Big\},$$

where $\mu_t(s^t)$ is a process of Lagrange multipliers on the technology constraint. We now have the FOCs w.r.t. $c_t(s^t)$, $n_t(s^t)$, and $k_{t+1}(s^t)$ respectively:

$$u_c(s^t) = \mu_t(s^t) \tag{3.6a}$$

$$u_\ell(s^t) = u_c(s^t) A_t(s^t) F_n(s^t) \tag{3.6b}$$

$$u_c(s^t)\pi_t(s^t) = \beta \sum_{s^{t+1}|s^t} u_c(s^{t+1}) \pi_{t+1}(s^{t+1}) [A_{t+1}(s^{t+1}) F_k(s^{t+1}) + (1-\delta)], \tag{3.6c}$$

where the summation over $s^{t+1}|s^t$ means that we sum over all possible histories $\tilde{s}^{t+1}$ such that $\tilde{s}^t = s^t$.

3.4 Time 0 trading: Arrow-Debreu securities

Like before, we can support the allocation that solves the planning problem by a competitive equilibrium with time 0 trading of a complete set of date- and history-contingent securities. Trades occur among a representative household and two types of representative firms. We let $[q^0, w^0, r^0, p_{k0}]$ be a price system where p_{k0} is the price of a unit of the initial capital stock, and each of the other three is a stochastic process of prices for output, renting labor and capital, and the time t component of each is indexed by the history s^t. The household sells labor services to a type I firm and in turn purchases the consumption goods from this firm. Firm I operates the production technology (3.3a). The household owns the initial capital stock k_0 and at date 0 sells it to a type II firm, which operates the capital storage technology (3.3b), which purchases new investment goods x_t from firm I, and rents stocks of capital back to firm I.

3.4.1 Household

The household maximizes

$$\sum_t \sum_{s^t} \beta^t u[c_t(s^t), 1 - n_t(s^t)] \pi_t(s^t), \tag{3.7}$$

subject to

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t(s^t) \le \sum_{t=0}^{\infty} \sum_{s^t} w_t^0(s^t) n_t(s^t) + p_{k0} k_0. \tag{3.8}$$

This implies the FOCs w.r.t. the labor and consumption choices:

$$\beta^t u_c(s^t) \pi_t(s^t) = \eta q_t^0(s^t) \tag{3.9a}$$

$$\beta^t u_\ell(s^t) \pi_t(s^t) = \eta w_t^0(s^t), \tag{3.9b}$$

where $\eta > 0$ is a multiplier on the budget constraint.

3.4.2 Firm of type I

Firm I operates the production technology (3.3a) with capital and labor that it rents at market prices. For each period t and history s^t the firm enters into state-contingent contracts at time 0 to rent capital $k_t^I(s^t)$ and labor services $n_t(s^t)$. The type I firm maximizes

$$\sum_{t=0}^{\infty} \sum_{s^t} \{ q_t^0(s^t)[c_t(s^t) + x_t(s^t)] - r_t^0(s^t) k_t^I(s^t) - w_t^0(s^t) n_t(s^t) \} \tag{3.10}$$

subject to

$$c_t(s^t) + x_t(s^t) \leq A_t(s^t)F(k_t^I(s^t) - w_t^0(s^t)). \tag{3.11}$$

We can substitute the latter into the former equation, then using (3.4) to obtain the objective function without any constraints:

$$\sum_{t=0}^{\infty}\sum_{s^t} n_t(s^t)\left\{q_t^0(s^t)A_t(s^t)f\big(\hat{k}_t^I(s^t)\big) - r_t^{=}(s^t)\hat{k}_t^I(s^t) - w_t^0(s^t)\right\} \tag{3.12}$$

and the maximization problem can then be decomposed into two parts. First, conditional on operating the production technology in period t and history s^t, the firm solves for the profit-maximizing capital-labor ratio, denoted $k_t^{I*}(s^t)$. Then, given the capital-labor ratio, the firm determines the profit-maximizing level of its operation by solving for the optimal employment level $n_t^*(s^t)$.

The firm finds the profit-maximizing capital-labor ratio by maximizing the term in curly brackets in the previous equation. The FOC w.r.t. $\hat{k}_t^I(s^t)$ is

$$q_t^0(s^t)A_t(s^t)f'\big(\hat{k}_t^I(s^t)\big) - r_t^0(s^t) = 0 \tag{3.13}$$

At the optimal capital labor ratio $\hat{k}_t^{I*}(s^t)$ that satisfies (3.13), the firm evaluates the expression in brackets in (3.12) in order to determine the optimal level of employment $n_t(s^t)$. This level is optimally set to zero or infinity if the expression in the curly brackets is strictly negative or positive, respectively. If the expression in curly brackets is zero in some period t and history s^t, the firm would be indifferent to the level of $n_t(s^t)$, since profits are then equal to zero for all levels of operation in the period and state. In these cases, we summarize the optimal employment decision by using (3.13) to eliminate $r_t^0(s^t)$ in the expression in curly brackets in (3.12);

$$n_t^*(s^t) = \begin{cases} 0, & \text{if } q_t(s^t)A_t(s^t)\left[f(\hat{k}_t^{I*}(s^t)) - f'(\hat{k}_t^{I*}(s^t))\hat{k}_t^{I*}(s^t)\right] - w_t^0(s^t) < 0 \\ \text{indeterminate}, & \text{if } q_t(s^t)A_t(s^t)\left[f(\hat{k}_t^{I*}(s^t)) - f'(\hat{k}_t^{I*}(s^t))\hat{k}_t^{I*}(s^t)\right] - w_t^0(s^t) = 0 \\ \infty, & \text{if } q_t(s^t)A_t(s^t)\left[f(\hat{k}_t^{I*}(s^t)) - f'(\hat{k}_t^{I*}(s^t))\hat{k}_t^{I*}(s^t)\right] - w_t^0(s^t) > 0. \end{cases} \tag{3.14}$$

In an equilibrium, we have $k_t^I(s^t)$ and $n_t(s^t)$ being strictly positive and finite, so the expressions (3.14) and (3.13) imply the following equilibrium prices:

$$q_t^0(s^t)A_t(s^t)F_k(s^t) = r_t^0(s^t) \tag{3.15a}$$
$$q_t^0(s^t)A_t(s^t)F_n(s^t) = w_t^0(s^t), \tag{3.15b}$$

where we used (3.5).

3.4.3 Firm of type II

The firm of type II operates technology (3.3b) to transform output into capital. The firm purchases capital at time 0 from the household sector and from then on invests in new capital, the revenue coming from renting this capital to the type I firm. It maximizes

$$-p_{k0}k_0^{II} + \sum_{t=0}^{\infty}\sum_{s^t}\{r_t^0(s^t)k_t^{II}(s^{t-1}) - q_t^0(s^t)x_t(s^t)\}, \tag{3.16}$$

subject to

$$k_{t+1}^{II}(s^t) = (1-\delta)k_t^{II}(s^{t-1}) + x_t(s^t). \tag{3.17}$$

Note that the capital stock in period 0, k_0^{II}, is bought without any uncertainty about the rental price in that period, while all future values $k_t^{II}(s^{t-1})$ are conditioned on the realized history s^{t-1}. This implies that firm II can be viewed as a risk-manager. Type II firm manages the risk associated with technology constraint (3.3b) that states that capital must be assembled one period prior to its intended use as an input for production. Firm II however, can choose how much capital $k_t^I(s^t)$ to rent in period t, conditioned on history s^t.

Substituting the constraint into the objective function, then rearranging, lets us write the objective function as

$$k_0^{II}\Big\{-p_{k0}+r_0^0(s^0)+q_0^0(s_0)(1-\delta)\Big\}+\sum_{t=0}^{\infty}\sum_{s^t}k_{t+1}^{II}(s^t)\left\{-q_t^0(s^t)+\sum_{s^{t+1}|s^t}[r_{t+1}^0(s^{t+1})+q_{t+1}^0(s^{t+1})(1-\delta)]\right\}, \tag{3.18}$$

where the profit is a linear function of investments in capital. The profit maximizing level of capital stock $k_{t+1}^{II}(s^t)$ is equal to zero or infinity if the associated multiplicative term in brackets is either strictly negative or strictly positive. If this terms is zero, then the firm is indifferent between all levels of $k_{t+1}^{II}(s^t)$, as this implies zero-profit for all levels of investment. In equilibrium, k_0^{II} and $k_{t+1}^{II}(s^t)$ are strictly positive and finite, so each curly bracket must equal zero, therefore we must have the following two equations satisfied for the equilibrium prices:

$$p_{k0}=r_0^0(s_0)+q_0^0(s_0)(1-\delta), \tag{3.19a}$$

$$q_t^0(s^t)=\sum_{s^{t+1}|s^t}[r_{t+1}^0(s^{t+1})+q_{t+1}^0(s^{t+1})(1-\delta)]. \tag{3.19b}$$

3.4.4 Equilibrium prices and quantities

According to equilibrium conditions (3.15), each input in the production technology is paid its marginal product, hence profit maximization of the type I firm ensures an efficient allocation of labor services and capital. This, however, gives no information about the equilibrium quantities of labor and capital. Profit maximization of firm II imposes no-arbitrage restrictions (3.19) across prices p_{0k} and $\{r_t^0(s^t), q_t^0(s^t)\}$, but nothing is said about the specific equilibrium value of an individual price. To solve for equilibrium prices and quantities, we go back to the representative household's FOCs (3.9).

Substituting (3.15b) into the household's FOC (3.9b), we have

$$\beta^t u_\ell(s^t)\pi_t(s^t)=\eta q_t^0(s^t)A_t(s^t)F_n(s^t); \tag{3.20a}$$

and then substituting (3.19b) and (3.15a) into (3.9a), we get

$$\begin{aligned}\beta^t u_c(s^t)\pi_t(s^t)&=\eta\sum_{s^{t+1}|s^t}[r_{t+1}^0(s^{t+1})+q_{t+1}^0(s^{t+1})(1-\delta)]\\&=\eta\sum_{s^{t+1}|s^t}q_{t+1}^0(s^{t+1})[A_{t+1}(s^{t+1})F_k(s^{t+1})+(1-\delta)].\end{aligned} \tag{3.20b}$$

We use $q_t^0(s^t)=\beta^t u_c(s^t)\pi_t(s^t)/\eta$ as given by the household's FOC (3.9a) and the corresponding expression for $q_{t+1}^0(s^{t+1})$ to substitute into (3.20a) and (3.20b) respectively. This produces expressions identical to the planner's FOCs (3.6b) and (3.6c) respectively. Thus we have verified the equivalence of the allocations in the social planner case and the competitive equilibrium with time 0 trading.

Given this equivalence of allocations, we may compute the competitive equilibrium by computing the social planner allocations, since this is a simpler problem. We calculate the equilibrium prices by substituting the allocation from the planning problem into the FOCs of the households and firms. Relative prices are then determined, and in order to get the absolute prices we fix one good as numeraire. Any such normalization is as if we would set the multiplier η on the household's present value budget constraint equal to an arbitrary positive number. For example, if we set $\eta=1$, we are measuring prices in units of marginal utility of the time 0 consumption good. Alternatively, we may set $q_0^0(s_0)=1$ by setting $\eta=u_c(s_0)$. We can compute $q_t^0(s^t)$ from (3.9a), $w_t^0(s^t)$ from (3.9b), and $r_t^0(s^t)$ from (3.15a). Finally, we can compute p_{k0} from (3.19a).

3.4.5 Implied wealth dynamics

After any given history s^t, we convert all prices, wages, and rental rates that are associated with current and future deliveries so that they are expressed in terms of time t, history s^t consumption goods, i.e. we change the numeraire:

$$q_\tau^t(s^\tau) \equiv \frac{q_\tau^0(s^\tau)}{q_t^0(s^t)} = \beta^{\tau-t}\frac{u_c(s^\tau)}{u_c(s^t)}\pi_\tau(s^\tau \mid s^t), \tag{3.21a}$$

$$w_\tau^t(s^\tau) \equiv \frac{w_\tau^0(s^\tau)}{q_t^0(s^t)}, \tag{3.21b}$$

$$r_\tau^t(s^\tau) \equiv \frac{r_\tau^0(s^\tau)}{q_t^0(s^t)}. \tag{3.21c}$$

In contrast to the previous section where we wanted to calculate the implied wealth of a household at time t after history s^t excluding the endowment stream, here we ask the same but not we exclude the value of labor instead. For example, the household's net claim to delivery of goods in a future period $\tau \geq t$, contingent on history s^τ, is given by $[q_\tau^t(s^\tau)c_\tau(s^\tau) - w_\tau^t(s^\tau)n_\tau(s^\tau)]$, as expressed in terms of time t, history s^t consumption goods. Thus, the household's wealth, or the value of all its current and future net claims, expressed in terms of date t, history s^t consumption good, is

$$\begin{aligned}
\Upsilon_t(s^t) &= \sum_{\tau=t}^{\infty}\sum_{s^\tau|s^t}\left\{q_\tau^t(s^\tau)c_\tau(s^\tau) - w_\tau^t(s^\tau)n_\tau(s^\tau)\right\} \\
&= \sum_{\tau=t}^{\infty}\sum_{s^\tau|s^t}\Big\{q_\tau^t(s^\tau)\Big[A_\tau(s^\tau)F\big(k_\tau(s^{\tau-1}), n_\tau(s^\tau)\big) + (1-\delta)k_\tau(s^{\tau-1}) - k_{\tau+1}(s^\tau)\Big] \\
&\qquad\qquad - w_\tau^t(s^\tau)n_\tau(s^\tau)\Big\} \\
&= \sum_{\tau=t}^{\infty}\sum_{s^\tau|s^t}\Big\{q_\tau^t(s^\tau)\Big[A_\tau(s^\tau)\Big(F_k(s^\tau)k_\tau(s^{\tau-1}) + F_n(s^\tau)n_\tau(s^\tau)\Big) + (1-\delta)k_\tau(s^{\tau-1}) - k_{\tau+1}(s^\tau)\Big] \\
&\qquad\qquad - w_\tau^t(s^\tau)n_\tau(s^\tau)\Big\} \\
&= \sum_{\tau=t}^{\infty}\sum_{s^\tau|s^t}\Big\{r_\tau^t(s^\tau)k_\tau(s^{\tau-1}) + q_\tau^t(s^\tau)\Big[(1-\delta)k_\tau(s^{\tau-1}) - k_{\tau-1}(s^\tau)\Big]\Big\} \\
&= r_t^t(s^t)k_t(s^{t-1}) + q_t^t(s^t)(1-\delta)k_t(s^{t-1}) \\
&\qquad + \sum_{\tau=t+1}^{\infty}\sum_{s^{\tau-1}|s^t}\left\{\sum_{s^\tau|s^{\tau-1}}\left[r_\tau^t(s^\tau) + q_\tau^t(s^\tau)(1-\delta)\right] - q_{\tau-1}^t(s^{\tau-1})\right\}k_\tau(s^{\tau-1}) \\
&= [r_t^t(s^t) + (1-\delta)]k_t(s^{t-1}).
\end{aligned} \tag{3.22}$$

The first equality used the equilibrium outcome that consumption is equal to the difference between production and investment in each period, the second one uses Euler's theorem on linearly homogeneous functions, i.e. $F(k,n) = F_k(k,n)k + F_n(k,n)n$. The third one invokes equilibrium input prices from (3.15), the fourth one simply rearranges these terms, and the last equality acknowledges that $q_t^t(s^t) = 1$ and that each bracket must be zero due to the equilibrium price condition (3.19b).*

3.5 Sequential trading: Arrow securities

We guess that at time t and after history s^t, there exist a wage rate $\tilde{w}_t(s)^t$, a rental rate $\tilde{r}_t(s^t)$, and Arrow security prices $\tilde{Q}_t(s_{t+1} \mid s^t)$. The pricing kernel may be interpreted as giving the price of one unit of $t+1$ consumption, contingent on the realization s_{t+1} at $t+1$, when the history at t is s^t.

*Wealth is return on capital?

3.5.1 Household

At each $t \geq 0$ and after s^t, the representative household buys consumption goods $\tilde{c}_t(s^t)$, sells labor $\tilde{n}_t(s^t)$ and trades claims to date $t+1$ consumption, whose payment is contingent on the realization of s_{t+1}. Let $\tilde{a}_t(s^t)$ denote the claims to time t consumption that the household brings into time t in history s^t. The household faces a sequence of budget constraints for $t \geq 0$, where the time t history s^t budget constraint is

$$\tilde{c}_t(s^t) + \sum_{s_{t+1}} \tilde{a}_{t+1}(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1} \mid s^t) \leq \tilde{w}_t(s^t)\tilde{n}_t(s^t) + \tilde{a}_t(s^t), \tag{3.23}$$

where $\{\tilde{a}_{t+1}(s_{t+1}, s^t)\}$ is a vector of claims on time $t+1$ consumption, one element of the vector for each possible time $t+1$ realization of s_{t+1}.

In order to rule out Ponzi schemes, we must impose borrowing constraints on the household's asst position. We impose that the household's indebtedness in any state next period, $-\tilde{a}_{t+1}(s_{t+1}, s^t)$, is bounded by some arbitrary constant. As long as the household is constrained in such a way that they cannot run a true Ponzi scheme, equilibrium forces will ensure willingness to hold a market portfolio. We may for example set the debt limit equal to zero.

Now, let $\eta_t(s^t)$ and $\upsilon_t(s^t; s_{t+1})$ be the nonnegative Lagrange multipliers on the budget constraint (3.23) and the borrowing constraint with arbitrary debt limit zero, respectively, for time t and history s^t. The Lagrangian can then be formed as

$$\begin{aligned}\mathfrak{L} = \sum_{t=0}^{\infty}\sum_{s^t} \Big\{ &\beta^t u(\tilde{c}_t(s^t), 1-\tilde{n}_t(s^t))\pi_t(s^t) \\ &+ \eta_t(s^t)\big[\tilde{w}_t(s^t)\tilde{n}_t(s^t) + \tilde{a}_t(s^t) - \tilde{c}_t(s^t) - \sum_{s_{t+1}} \tilde{a}_{t+1}(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1} \mid s^t)\big] \\ &+ \upsilon_t(s^t; s_{t+1})\tilde{a}_{t+1}(s^{t+1}) \Big\},\end{aligned}$$

for a given initial wealth $\tilde{a}_0$. In equilibrium we assume the sequence of consumption and labor vectors to be interior solutions due to Inada conditions. These imply that the household will set $\tilde{c}_t(s^t), \ell_t(s^t) > 0$, i.e. $\tilde{n}_t(s^t) < 1$. The Inada conditions on the production function also imply a desirability to supply a positive amount of labor $\tilde{n}_t(s^t) > 0$. Given these interior solutions, the FOCs are

$$\frac{\partial}{\partial \tilde{c}_t(s^t)}\mathfrak{L} = \beta^t u_c(\tilde{c}_t(s^t), 1-\tilde{n}_t(s^t))\pi_t(s^t) - \eta_t(s^t) = 0, \tag{3.24a}$$

$$\frac{\partial}{\partial \tilde{n}_t(s^t)}\mathfrak{L} = -\beta^t u_\ell(\tilde{c}_t(s^t), 1-\tilde{n}_t(s^t))\pi_t(s^t) + \eta_t(s^t)\tilde{w}_t(s^t) = 0, \tag{3.24b}$$

$$\frac{\partial}{\partial \tilde{a}_{t+1}(s_{t+1}, s^t)}\mathfrak{L} = -\eta_t(s^t)\tilde{Q}_t(s_{t+1} \mid s^t) + \upsilon_t(s^t; s_{t+1}) + \eta_{t+1}(s_{t+1}, s^t) = 0, \tag{3.24c}$$

for all s_{t+1}, t, and s^t. Under the conjecture of zero debt limits, we may set $\upsilon_t(s^t; s_{t+1})$ to zero, and then rearranging the FOCs such that we have

$$\tilde{w}_t(s^t) = \frac{u_\ell(\tilde{c}_t(s^t), 1-\tilde{n}_t(s^t))}{u_c(\tilde{c}_t(s^t), 1-\tilde{n}_t(s^t))}, \tag{3.25a}$$

$$\tilde{Q}_t(s_{t+1} \mid s^t) = \beta\frac{u_c(\tilde{c}_{t+1}(s^{t+1}), 1-\tilde{n}_{t+1}(s^{t+1}))}{u_c(\tilde{c}_t(s^t), 1-\tilde{n}_t(s^t))}\pi_t(s^{t+1} \mid s^t), \tag{3.25b}$$

again for all s_{t+1}, t, and s^t.

3.5.2 Firm of type I

A type I firm is a production firm that chooses a quadrupel $\{\tilde{c}_t(s^t), \tilde{x}_t(s^t), \tilde{k}_t^I(s^t), \tilde{n}_t(s^t)\}$ for each date $t \geq 0$ and after history s^t. This is used to solve a static optimization problem:

$$\max\ \tilde{c}_t(s^t) + \tilde{x}_t(s^t) - \tilde{r}_t(s^t)\tilde{k}_t I(s^t) - \tilde{w}_t(s^t)\tilde{n}_t(s^t) \tag{3.26}$$

subject to

$$\tilde{c}_t(s^t) + \tilde{x}_t(s^t) \leq A_t(s^t)F(\tilde{k}_t^I(s^t), \tilde{n}_t(s^t)). \tag{3.27}$$

The zero-profit conditions are given by

$$\tilde{r}_t(s^t) = A_t(s^t)F_k(s^t), \tag{3.28a}$$
$$\tilde{w}_t(s^t) = A_t(s^t)F_n(s^t). \tag{3.28b}$$

If conditions (3.28) are satisfied, the firm makes zero-profit and its size is indeterminate. The firm of type I is willing to produce any quantities of $\tilde{c}_t(s^t)$ and $\tilde{x}_t(s^t)$ that the market demands, provided the mentioned conditions hold.

3.5.3 Firm of type II

A type II firm transforms output into capital, stores this capital, and earns revenue by renting out this capital to the type I firm. W.l.o.g. we can consider a two-period optimization problem where the firm decides on how much capital $\tilde{k}_{t+1}^{II}(s^t)$ to store at the end of period t after history s^t in order to earn a stochastic revenue of $\tilde{r}_{t+1}(s^{t+1})\tilde{k}_{t+1}^{II}(s^t)$ and liquidation value $(1-\delta)\tilde{k}_{t+1}^{II}(s^t)$ in the following period. The firm finances itself by issuing state-contingent debt to the households, implying that future income streams can be expressed in today's values by using prices $\tilde{Q}_t(s_{t+1} \mid s^t)$. A firm chooses $\tilde{k}_{t+1}^{II}(s^t)$ at date $t \geq 0$ after history s^t in order to solve the maximization problem

$$\max\ \tilde{k}_{t+1}^{II}(s^t) \left\{ -1 + \sum_{s_{t+1}} \tilde{Q}_t(s_{t+1} \mid s^t)[\tilde{r}_{t+1}(s^{t+1}) + (1-\delta)] \right\}. \tag{3.29}$$

The zero-profit condition is

$$1 = \sum_{s_{t+1}} \tilde{Q}_t(s_{t+1} \mid s^t)[\tilde{r}_{t+1}(s^{t+1}) + (1-\delta)] \tag{3.30}$$

We again have a firm of indeterminate size if condition (3.30) is satisfied.

3.5.4 Equilibrium prices and quantities

We propose that the allocations attained in the sequential equilibrium are the same ones as in the time 0 equilibrium, i.e.

$$\{\tilde{c}_t(s^t), \tilde{\ell}_t(s^t), \tilde{x}_t(s^t), \tilde{n}_t(s^t), \tilde{k}_{t+1}(s^t)\}_{t=0}^{\infty} = \{c_t(s^t), \ell_t(s^t), x_t(s^t), n_t(s^t), k_{t+1}(s^t)\}_{t=0}^{\infty}.$$

We guess that the prices in the sequential equilibrium satisfy

$$\tilde{Q}_t(s_{t+1} \mid s^t) = q_{t+1}^t(s^{t+1}), \tag{3.31a}$$
$$\tilde{w}_t(s^t) = w_t^t(s^t), \tag{3.31b}$$
$$\tilde{r}_t(s^t) = r_t^t(s^t). \tag{3.31c}$$

The other set of guesses is that the representative household chooses asst portfolios given by $\tilde{a}_{t+1}(s_{t+1}, s^t) = \Upsilon_{t+1}(s^{t+1}) \forall s_{t+1}$. When we show that the household can afford these portfolios together with the prescribed consumption-leisure combination, we find the initial wealth to be equal to

$$\tilde{a}_0 = [r_0^0(s_0) + (1-\delta)]k_0 = p_{k0}k_0,$$

i.e. the household in the sequential equilibrium starts out owning the initial capital stock, which they sell to a type II firm at the same competitive price as in the time 0 trading equilibrium.

3.5.5 Financing a type II firm

A type II firm purchases $\tilde{k}^{II}_{t+1}(s^t)$ in period t after history s^t by issuing one-period state-contingent claims that promise to pay

$$[\tilde{r}_{t+1}(s^{t+1}) + (1-\delta)]\tilde{k}^{II}_{t+1}(s^t)$$

consumption goods tomorrow in state s_{t+1}. We can price these consumption good via the pricing kernel and summing over all states s_{t+1} to get

$$\sum_{s_{t+1}} \tilde{Q}_t(s_{t+1} \mid s^t)[\tilde{r}_{t+1}(s^{t+1}) + (1-\delta)]\tilde{k}^{II}_{t+1}(s^t).$$

The firm breaks even by issuing these claims, implying an ownership structure of the firm where the creditors own the firm, i.e. the households earn the firm and earn zero profits.

Note that the economy's end-of-period wealth as embodied in $\tilde{k}^{II}_{t+1}(s^t)$ in period t after history s^t is willingly held by the representative household. This follows from the household's desired beginning-of-period wealth next period given by $\tilde{a}_{t+1}(s^{t+1})$ being equal to $\Upsilon_{t+1}(s^{t+1})$ in (3.22). Prices entice the household to enter each future period with a strictly positive net asset level equal to the value of the type II firm. This also shows how the debt limit of zero is non-binding in the optimization problem the household faces.

3.6 Recursive formulation

The finding of equivalence in allocations for both trading schemes holds for an arbitrary technology process $A_t(s^t)$, defined as a measurable function of the history of events s^t which in turn are governed by some arbitrary probability measure $\pi_t(s^t)$. At this general level, all prices $\{\tilde{Q}_t(s_{t+1} \mid s^t), \tilde{w}_t(s^t), \tilde{r}_t(s^t)\}$ and the capital stock $k_{t+1}(s^t)$ in the sequential-trading economy depend on the history s^t. In other words, the prices and the capital stock are time-varying functions of all past events $\{s_\tau\}_{\tau=0}^t$. We want to obtain a recursive formulation and solution to not only the sequential-trading equilibrium, but also to the benchmark social planner, so we make the following specification of the exogenous process forcing process for the technology process.

3.6.1 Technology is governed by a Markov process

The stochastic event is governed by a Markov process $[s \in S, \pi(s' \mid s), \pi_0(s_0)]$. We keep earlier assumptions about the deterministic nature of the period 0 state s_0, i.e. $\pi_0(s_0) = 1$ for some particular $s_0 \in S$. We then have the probability measures being given by

$$\pi_t(s^t) = \pi(s_t \mid s_{t-1})\pi(s_{t-1} \mid s_{t-2}) \dots \pi(s_1 \mid s_0)\pi_0(s_0).$$

Furthermore, we assume the aggregate technology level in period t, $A_t(s^t)$, is a time-invariant measurable function of its level in the last period and the current stochastic event s_t, i.e. $A_t(s^t) = A\big(A_{t-1}(s^{t-1}), s_t\big)$. For example, here we will use the multiplicative version in the sense of

$$A_t(s^t) = s_t A_{t-1}(s^{t-1}) = s_0 s_1 \dots s_t A_{-1},$$

given some initial A_{-1}.

3.6.2 Aggregate state of the economy

A recursive formulation allows us to incorporate more components of the state of the economy into our model. We need to know last period's technology level, denoted A, in order to determine the current state of technology sA, so we also need the current realization of the state of the world. This additional element A in the aggregate state vector does not constitute any conceptual change from our previous work. We merely include one more state variable that is a direct mapping from exogenous stochastic events, and it does not depend on any exogenous outcomes.

But we need to expand the aggregate state vector with an endogenous component, the beginning-of-period capital stock K. Thus, we have $X \equiv [K\ A\ s]$. This state vector is a complete summary of the economy's current position; we need only this state vector to compute an optimal allocation and it is all that it needed for the *invisible hand* to call out prices and implement the first-best allocation as a competitive equilibrium.

First we display the Bellman equation associated with a recursive formulation of the planning problem. Then we use the same state vector X for the planner's problem as a state vector in which to cast the Arrow securities in a competitive economy with sequential trading. Afterwards we define a competitive equilibrium and show how the induced prices are embedded in the decision rules and the value function of the planning problem.

3.7 Recursive formulation of the planning problem

The Bellman equation for the planning problem, using the respective capital letters, is

$$v(K, A, s) = \max_{C,N,K'} \left\{ u(C, 1-N) + \beta \sum_{s'} \pi(s' \mid s) v(K', A', s') \right\}, \tag{3.32}$$

subject to

$$K' + C \leq AsF(K, N) + (1-\delta)K, \tag{3.33a}$$
$$A' = As. \tag{3.33b}$$

Using the definition of the state vector $X = \begin{bmatrix} K & A & s \end{bmatrix}$, we denote the optimal policy functions

$$C = \Omega^C(X), \tag{3.34a}$$
$$N = \Omega^N(X), \tag{3.34b}$$
$$K' = \Omega^K(X). \tag{3.34c}$$

Equations (3.34b), (3.34c), and the Markov transition density $\pi(s' \mid s)$ induce a transition density $\Pi(X' \mid X)$ on the state X.

For convenience, we define

$$U_c(X) \equiv u_c(\Omega^C(X), 1 - \Omega^N(X)), \tag{3.35a}$$
$$U_\ell(X) \equiv u_\ell(\Omega^C(X), 1 - \Omega^N(X)), \tag{3.35b}$$
$$F_k(X) \equiv F_k(K, \Omega^N(X)), \tag{3.35c}$$
$$F_n(X) \equiv F_n(K, \Omega^N(X)). \tag{3.35d}$$

The FOCs of the planner's problem can be represented, using the envelope condition

$$v_K(K, A, s) = U_c(X)[AsF_k(X) + (1-\delta)],$$

as

$$U_\ell(X) = U_c(X) AsF_n(X) \tag{3.36a}$$
$$1 = \beta \sum_{X'} \Pi(X' \mid X) \frac{U_c(X')}{U_c(X)} [A's'F_K(X') + (1-\delta)]. \tag{3.36b}$$

3.8 Recursive formulation of sequential trading

3.8.1 A "Big K, little k" device

We have augmented the time t state of the economy by the last period's technology level A_{t-1} and the current aggregate value of the endogenous state variable K_t. We assume agents are price takers, they act as if their decisions do not influence prices in the current period and in the future. The sequential setting implies prices depending on the state of the world, partly captured by K_t, which is - on aggregate - influenced by the agents. But we want the individual agents to ignore their influence on the motion of K_t.

We do so by including big K and little k in our problem. Big K can be a vector of endogenous state variables influencing equilibrium prices, and is used for forecasting prices. Agents treat this as given when solving their optimization problem, and k is the value chosen by firms and consumers. This again is similar to the atomic agents, where we normalize the set of agents to 1, implying that for infinite agents each agent has a measure of zero. No single agent influences aggregate variables, but in aggregate they do influence the state variables. However, this distinction between the big and little k is only imposed during the optimization problem solving for households and firms, afterwards we set $K = k$.

3.8.2 Price system

The price system described here is fairly general, imposing either no, or very weak restrictions. As we move on from the centralized social planner problem to our one-period-ahead Arrow setting, we need to describe the aggregate state such that we can define the payoffs of the state-contingent one-period claims. We make a guess that we can use the same state vector X as before and for now ignore the optimal policy functions. We specify the price as a function of this state vector X, containing K, A, and s. These prices are, also similar to previous notation defined as $r(X)$ for the rental rate of capital, $w(X)$ as the wage rate, and $Q(X' \mid X)$ as the price of a one-unit consumption claim in the next period, contingent on the economy moving from state X to X'. Generally, prices are measured in units of the respective period's consumption good. Assume a law of motion for the endogenous state variable as follows:

$$K' = G(X). \tag{3.37}$$

Combining this with $A' = As$ and a subjective transition density $\hat{\pi}(s' \mid s)$ give a state-to-state transition density $\hat{\Pi}(X' \mid X)$ for the state X. For now we do not further restrict these functions.

3.8.3 Household problem

The perceives law of motion (3.37) for K and the transition density $\hat{\Pi}(X' \mid X)$ describe the beliefs of a household. We can now write the Bellman equation of the household as

$$J(a, X) = \max_{c,n,\bar{a}(X')} \left\{ u(c, 1-n) + \beta \sum_{X'} J(\bar{a}(X'), X')\hat{\Pi}(X' \mid X) \right\}, \tag{3.38}$$

subject to

$$c + \sum_{X'} Q(X' \mid X)\bar{a}(X') \leq w(X)n + a. \tag{3.39}$$

a has the same meaning as in previous sections, i.e. it captures the wealth of a household in units of current consumption goods, while $\bar{a}(X')$ denotes the next-period wealth in units of next period's consumption good. We now denote the household's optimal policy functions as

$$c = \sigma^c(a, X), \tag{3.40a}$$
$$n = \sigma^n(a, X), \tag{3.40b}$$
$$\bar{a}(X') = \sigma^n(a, X; X') \tag{3.40c}$$

Now we define the derivatives of the utility function w.r.t. consumption and leisure as their optimal policy functions:

$$\bar{u}_c(a, X) \equiv u_c(\sigma^c(a, X), 1 - \sigma^n(a, X)), \tag{3.41a}$$
$$\bar{u}_\ell(a, X) \equiv u_\ell(\sigma^c(a, X), 1 - \sigma^n(a, X)). \tag{3.41b}$$

These formulations imply that we can express the household's FONCs as

$$\bar{u}_\ell(a, X) = \bar{u}_c(a, X)w(X), \tag{3.42a}$$
$$Q(X' \mid X) = \beta \frac{\bar{u}_c(\sigma^a(a, X; X'), X')}{\bar{u}_c(a, X)} \hat{\Pi}(X' \mid X). \tag{3.42b}$$

3.8.4 Firm of type I

We now describe the static problem a firm of type 1 faces in the recursive formulation of the equilibrium. It can be written as

$$\max_{c,x,k,n} \{c + x - r(X)k - w(X)n\} \tag{3.43}$$

subject to

$$c + x \leq AsF(k, n). \tag{3.44}$$

Plugging the production function into the maximization problem and taking derivatives w.r.t. labor and capital, we get the zero-profit conditions:

$$r(X) = AsF_k(k, n), \tag{3.45a}$$

$$w(X) = AsF_n(k, n). \tag{3.45b}$$

3.8.5 Firm of type II

The recursive formulation of the firm type II problem can be written as

$$\max_{k'} k' \left\{ -1 + \sum_{X'} Q(X' \mid X)[r(X') + (1 - \delta)] \right\}. \tag{3.46}$$

This implies the zero-profit condition even without taking any derivatives:

$$1 = \sum_{X'} Q(X' \mid X)[r(X') + (1 - \delta)]. \tag{3.47}$$

The economic reasoning for the zero-profit conditions is the same as before. Positive profits result in firms producing infinite amounts of output, negative profits shut the firms down. Therefore any finite, nonzero output, the output demanded by the households, is induced by zero profits.

3.9 Recursive competitive equilibrium

We now impose equilibrium conditions on the price functions, the perceived law of motion for the endogenous state variable K', and the associated induced state transition probability. Imposing rational expectations removes the agent's expectations (the transition densities) from the free parameters, i.e. they become *fix* values.

3.9.1 Equilibrium restrictions across decision rules

We will define an equilibrium as a set of pricing functions, a perceived law of motion for K', and an associated transition density $\hat{\Pi}(X' \mid X)$ such that when the agents take these as give, the decision rules imply the law of motion for K after substituting $k = K$ and market clearing. This removes the arbitrary aspects of the function G, and the transition probability $\hat{\pi}$ implying also the removal of the arbitrary aspect of $\hat{\Pi}$, imposing rational expectations.

We need to find restrictions that induce the above mentioned. If state-contingent debt issued by firm II is to match the debt demand by households, i.e. if the debt market clears, then we must have

$$\bar{a}(X') = [r(X') + (1 - \delta)]K', \tag{3.48a}$$

and as a consequence the beginning-of-period assets in the budget constraint of the household in (3.39) need to satisfy

$$a = [r(X) + (1 - \delta)]K. \tag{3.48b}$$

We can substitute these two equations into the household budget constraint and get

$$\sum_{X'} Q(X' \mid X)[r(X') + (1-\delta)]K' \leq w(X)n - c + [r(X) + (1-\delta)]K \tag{3.49}$$

Recall the zero-profit condition for firm II, and K' being predetermined when entering the next period, the left side is simply K'. Furthermore, we can substitute the equilibrium factor prices from firm I, yielding

$$\begin{aligned} K' &= \big[AsF_k(k,n) + (1-\delta)\big]K + AsF_n(k,n)n - c \\ &= AsF(K, \sigma^n(a,X)) + (1-\delta)K - \sigma^c(a,X), \end{aligned} \tag{3.50}$$

where we use Euler's theorem on linearly homogeneous functions and the equilibrium conditions

$$\begin{aligned} K &= k \\ N &= n = \sigma^n(a,X) \\ C &= c = \sigma^c(a,X). \end{aligned}$$

We want to express the RHS of the above K' as a function of the current aggregate state X, so we impose the equilibrium condition $a = [r(X) + (1-\delta)K]$:

$$K' = AsF\big(K, \sigma^n([r(X) + (1-\delta)K], X)\big) + (1-\delta)K - \sigma^c\big([r(X) + (1-\delta)]K, X\big). \tag{3.51}$$

Note that this simply is plugging in the equilibrium condition in the optimal policy functions for labor and consumption.

We used some arbitrary perceived law of motion previously, but this formulation is the *actual* law of motion for K', as implied by the household's and firms' optimal decisions. In equilibrium, we want $K' = G(X)$ to be an outcome, rather than some arbitrary functional. Thus we need to find an equilibrium perceived law of motion. Considering rational expectations, we have the implication of the perceived and actual law of motion to be identical. Equating (3.51) and (3.37) gives

$$G(X) = AsF\big(K, \sigma^n([r(X) + (1-\delta)]K, X)\big) + (1-\delta)K - \sigma^c\big([r(X) + (1-\delta)]K, X\big). \tag{3.52}$$

The RHS itself is implicitly a function of G, such that the equation can be viewed as an instruction for finding a fixed point equation of a mapping from the perceived G and the price system to the actual G. This functional equation requires that the perceived and actual law of motion correspond, the latter is jointly determined by competitive equilibrium decisions of the household and firms.

Definition 3.6. A *recursive competitive equilibrium with Arrow securities* is a price system $r(X)$, $w(X)$, $Q(X' \mid X)$, a perceived law of motion $K' = G(X)$ and associated induced transition density $\hat{\Pi}(X' \mid X)$, a household value function $J(a,X)$, and decision rules $\sigma^c(a,X)$, $\sigma^n(a,X)$, $\sigma^a(a,X;X')$ such that

a. Given $r(X)$, $w(X)$, $Q(X' \mid X)$, $\hat{\Pi}(X' \mid X)$, the functions $\sigma^c(a,X)$, $\sigma^n(a,X)$, $\sigma^a(a,X;X')$ and the value function $J(a,X)$ solve the household's optimum problem;

b. For all X, the following two conditions hold:

$$\begin{aligned} r(X) &= AF_k\big(K, \sigma^n([r(X) + (1-\delta)]K, X)\big), \\ w(X) &= AF_n\big(K, \sigma^n([r(X) + (1-\delta)]K, X)\big); \end{aligned}$$

c. $Q(X' \mid X)$ and $r(X)$ satisfy the zero-profit condition of firm II: $1 = \sum_{X'} Q(X' \mid X)[r(X') + (1-\delta)]$;

d. The functions $G(X)$, $r(X)$, $\sigma^c(a,X)$, $\sigma n(a,X)$ satisfy (3.52).

e. $\hat{\pi} = \pi$.

Item a enforces optimization by the household, given the prices and the expectations. Item b requires break even at every capital stock and the households' labor choice for a firm I. Item c requires firm II to break even. Market clearing is implicit when item d requires the actual and perceived law of motion being equal. Item e and the equality of perceived and actual G imply $\hat{\Pi} = \Pi$. This implies that e and d impose rational expectations.

3.9.2 Using the planning problem

We do not directly try to solve the fixed point problem (3.52), but we use a guess-and-verify approach. We guess some candidate for G and a price system and use these to describe how to verify that these form an equilibrium. We use the decision rule for K' from the planning problem, i.e. we use $K' = \Omega^K(X)$. Similarly, we choose the pricing functions from the planning problem and then guess

$$r(x) = AF_k(X), \tag{3.53a}$$

$$w(X) = AF_n(X), \tag{3.53b}$$

$$Q(X' \mid X) = \beta\Pi(X' \mid X)\frac{U_c(X')}{U_c(X)}\left[A's'F_K(X') + (1-\delta)\right] \tag{3.53c}$$

In an equilibrium it will turn out that the household's decision rules for consumption and labor will match those chosen by the planner:

$$\Omega^C(X) = \sigma^c\big([r(X) + (1-\delta)]K, X\big), \tag{3.54a}$$

$$\Omega^N(X) = \sigma^n\big([r(X) + (1-\delta)]K, X\big). \tag{3.54b}$$

To actually show how these hold, we need to show that the FOCs for both types and firms are satisfied at these guesses. This is left as an exercise, here we exploit some consequences of the welfare theorems.

4 Asset Prices in an Exchange Economy (Lucas, 1978)

Remarks on Generalized Expectations of Random Variables

In this section we deviate from the often used definition of an expected value that takes on the form

$$\mathbb{E}x = \int x f_X(x)\,\mathrm{d}x.$$

Instead, we use a generalized version that does not require absolutely continuous distribution, i.e. we do not require a PDF at this point. We may use the generalized version for any discrete or continuous expectation, as long as the first moment is well defined. The importance of this generalized version in this context comes from the states of the world, where we do not want to make any assumptions on the state-space for now. This generalized expectation takes on the form

$$\mathbb{E}x = \int x\,\mathrm{d}\mathcal{F}(x).$$

Note that this equivalence comes from

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{F}(x) = f(x) \Leftrightarrow \mathrm{d}\mathcal{F}(x) = f(x)\,\mathrm{d}x.$$

Of course this expectation exhibits the same properties as the original version:

$$\mathbb{E}h(x) = \int h(x)\,\mathrm{d}\mathcal{F}(x),$$

$$\mathbb{E}x^n = \int x^n\,\mathrm{d}\mathcal{F}(x).$$

Notable, this type of integral is called a Riemann-Stieltjes integral, and it allows for integration by parts:

$$\int_a^b f(x)\,\mathrm{d}\mathcal{F}(x) = f(b)\mathcal{F}(b) - f(a)\mathcal{F}(a) - \int_a^b \mathcal{F}(x)\,\mathrm{d}f(x).$$

Short Introduction to Contraction Mappings

Before we start with contraction mappings, consider the following theorem, which is used throughout the next section:

> **Theorem.** If X is a set and M is a complete metric space, then the set $B(X, M)$ of all bounded functions $f : X \mapsto M$ is a complete metric space. Here we define the distance in $B(X, M)$ in terms of the distance M with the supremum norm
>
> $$d(f, g) \equiv \sup_{x \in X} \{d[f(x), g(x)]\}$$

> **Definition** (Contraction Mapping, Contraction)**.** Let (X, d) be a metric space. A mapping $T : X \mapsto X$ is a *contraction mapping*, or *contraction*, if there exists a constant $c \in [0, 1)$ such that
>
> $$d\big(T(x), T(y)\big) \leq cd(x, y)$$
>
> for all $x, y \in X$.

Thus, a contraction maps points closer together. In particular, for every $x \in X$ and any $r > 0$, all points $y \in B_r(x)$ are mapped into a ball $B_s(Tx)$ with $s < r$. A contraction mapping is uniformly continuous. If $T : X \mapsto X$, then a point $x \in X$ such that

$$T(x) = x$$

is called a *fixed point* of T. The contraction mapping theorem states that any strict contraction ($c < 1$) on a complete metric space has a unique fixed point. Note that we may freely choose the metric, the results do not depend on our choice. Thus we need to find *any* metric on X such that X is complete, and then T being a contraction on X implies existence and uniqueness of a fixed point.

> **Theorem** (Contraction Mapping Theorem)**.** If $T : X \mapsto X$ is a contraction mapping on a complete metric space (X, d), then there is exaclty one solution $x \in X$ of $T(x) = x$

Proof. Let $x_0 \in X$. We define a sequence $(x_n) \in X$ by

$$x_{n+1} = Tx_n \forall n \geq 0.$$

Let $x_n = T^n x_0$ be x_n after iteratively applying the contraction mapping. We need to show that (x_n) is a Cauchy sequence. Assume $n \geq m \geq 1$, then from the definition of a contraction mapping and the triangle inequality, we have

$$\begin{aligned}
d(x_n, x_m) &= d(T^n x_0, T^m x_0) \\
&\leq c^m d(T^{n-m} x_0, x_0) \\
&\leq c^m \Big[d(T^{n-m} x_0, T^{n-m-1} x_0) + d(T^{n-m-1} x_0, T^{n-m-2} x_0) + \cdots + d(T x_0, x_0)\Big] \\
&\leq c^m \left[\sum_{k=0}^{n-m-1} c^k\right] d(x_1, x_0) \\
&\leq c^m \left[\sum_{k=0}^{\infty} c^k\right] d(x_1, x_0) \\
&\leq \frac{c^m}{1-c} d(x_1, x_0).
\end{aligned}$$

As the multiplicative term converges to zero for $m \to \infty$, we have (x_n) being Cauchy. Furthermore, X is complete, thus (x_n) converges to a limit $x \in X$. The limit x being a fixed point of the contraction mapping follows from the continuity of the contraction mapping:

$$Tx = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x.$$

Finally, assume x and y to be fixed points with $x \neq y$. Then we have

$$0 \leq d(x,y) = d(Tx,Ty) \leq cd(x,y).$$

Since we have $c < 1$, then $d(x,y) = 0$, so we have a contradiction and need $x = y$, thus the fixed point is unique. □

Fixed points of dynamical systems

A dynamical system is defined by a state space X, whose elements describe the different states the system can be in, and a prescription that relates the state $x_t \in X$ to the state the system was in in a previous period. The dynamics may be described by a map $T : X \mapsto X$ that relates x_{n+1} at time $t = n+1$ to the state x_n at time $t = n$:

$$x_{n+1} = Tx_n.$$

Now, T may not be invertible, implying that the dynamics are only defined forward in time. If it is invertible, then we may describe the system both forwards and backwards in time. A fixed point of the map T corresponds to an equilibrium state of the discrete dynamical system. If the state space is a complete metric space and T is a contraction, then the contraction mapping theorem implies the existence of a unique equilibrium state, and that the system approaches this state as time tend to infinity, starting from any initial state. In such a case, this fixed point is called globally asymptotically stable.

We now consider a simple example, the logistic equation of population dynamics, defined as

$$x_{n+1} = 4\mu x_n(1 - x_n),$$

where $\mu \in [0,1]$ and $x_n[0,1]$. This equation can be written using $T : [0,1] \mapsto [0,1]$

$$Tx = 4\mu x(1 - x).$$

As for the interpretation, we may think of x_n denoting the n^{th} generation of a reproducing species. The mapping consists of two parts. The first part is a linear equation $x_{n+1} = 4\mu x_n$ and describes exponential growth for $\mu > 0.25$, and decay for $\mu < 0.25$, assuming a constant birth or death rate. The nonlinear part simply models a species in which overcrowding leads to declining birth rates as the population increases.

When we have $\mu \in [0, 0.25]$, the only fixed point for T in $[0,1]$ is the point 0, and T is a contraction on $[0,1]$. The proof of the contraction mapping theorem therefore implies that $\forall x_0 \in [0,1]; T^n x_0 \to 0$, i.e. regardless of the initial population size, the population always dies out over time. Assuming $\mu \in (0.25, 1]$. we have a second fixed point at $x = \frac{4\mu-1}{4\mu}$. Furthermore, for $\mu \to 1$, we have more and more complex dynamics.

Now consider a second application. Consider the solution of an equation $f(x) = 0$. We can obtain a solution by recasting the equation in the form of a fixed point equation $x = Tx$ and then construct approximations x_n starting from some initial guess x_0 by the iterations scheme

$$x_{n+1} = Tx_n.$$

We try to find a solution as the time-asymptotic state of an discrete dynamical system which has the solution as a stable fixed point. There are many ways to rewrite an equation $f(x) = 0$ as a fixed point problem. Ideally, we write it in such a way that T us a contraction on the whole space, or at least a contraction on some set that contains the solution we seek. To illustrate this, we show convergence of an algorithm that computes square roots. If $a > 0$, then $x = \sqrt{a}$ is the positive solution of the equation

$$x^2 - a = 0.$$

This can be rewritten as a fixed problem of the form

$$x = \frac{1}{2}\Big(x + \frac{a}{x}\Big).$$

The associated iteration scheme is then described by

$$x_{n+1} = \frac{1}{2}\Big(x_n + \frac{a}{x_n}\Big),$$

corresponding to a map $T : (0, \infty) \mapsto (0, \infty)$ given by

$$Tx = \frac{1}{2}\Big(x + \frac{a}{x}\Big).$$

The obvious fixed point is $x = \sqrt{a}$ of T. Furthermore, given an approximation x_n of $\sqrt{a}$, the average of x_n and a/x_n should be a better approximation, given x_n not being too small. It is therefore reasonable to expect the sequence of approximations obtained by iteration of the fixed point equation converging to $\sqrt{a}$. In order to prove this, we find an interval on which T is a contraction w.r.t. the usual absolute value metric on $\mathbb{R}$.

First we check whether T is a contraction at all. For $x_1, x_2 > 0$, we estimate that

$$|Tx_1 - Tx_2| = \left|\frac{1}{2}\Big(x_1 + \frac{a}{x_1}\Big) - \frac{1}{2}\Big(x_2 + \frac{a}{x_2}\Big)\right| = \frac{1}{2}\Big|1 - \frac{a}{x_1 x_2}\Big||x_1 - x_2|.$$

T contracts distances when $3x_1x_2 > a$. In order to satisfy this, we need to exclude too small x. Consider the action of T on an interval $[b, \infty)$ where $b > 0$. This is a complete metric space because $[b, \infty)$ is a closed subset of $\mathbb{R}$ and $\mathbb{R}$ is complete. We need to choose b wisely. First, we observe that $\forall x > 0$

$$Tx = \sqrt{a} + \frac{(x - \sqrt{a})^2}{2x} \geq \sqrt{a}.$$

Therefore, the restriction $[\sqrt{a}, \infty)$ is well defined, since

$$T\Big([\sqrt{a}, \infty)\Big) \subset [\sqrt{a}, \infty),$$

and T is a contraction on this interval with $c = 0.5$. It follows that for any $x_0 \geq \sqrt{a}$, the sequence $x_n = T^n x_0$ converges to $\sqrt{a}$ as $n \to \infty$. Moreover, this convergence speeds is exponentially fast, namely

$$\begin{aligned}
|T^n x_0 - \sqrt{a}| &\leq |T^n x_0 - T^m x_0| \\
&\leq \frac{c^n}{1-c}|Tx_0 - x_0| \\
&\leq \frac{1}{2^{n-1}}\Big|\frac{a}{x_0} - x_0\Big|.
\end{aligned}$$

If $x_0 \in (0, \sqrt{a})$, then $x_1 > \sqrt{a}$, and subsequent iterations remain in $[\sqrt{a}, \infty)$, so the iterations converge for any starting guess $x_0 \in (0, \infty)$.

Integral equations

We start by defining some equations which will be useful later on. A linear *Fredholm integral equation of the second kind* for an unknown function $f : [a, b] \mapsto \mathbb{R}$ is an equation of the form

$$f(x) - \int_a^b k(x, y) f(y)\, \mathrm{d}y = g(x), \tag{*}$$

where $k : [a, b] \times [a, b] \mapsto \mathbb{R}$ and $g : [a, b] \mapsto \mathbb{R}$ are given functions. A *Fredholm integral equation of the first kind* is an equation of the form

$$\int_a^b k(x, y) f(y)\, \mathrm{d}y = g(x).$$

The integral equation (*) may be written as a fixed point equation $Tf = f$, where the map T is defined by

$$(Tf)(x) = g(x) + \int_a^b k(x,y)f(y)\,\mathrm{d}y. \tag{**}$$

Theorem. Suppose that $k : [a,b] \times [a,b] \mapsto \mathbb{R}$ is a continuous function such that

$$\sup_{x\in[a,b]} \int_a^b |k(x,y)|\,\mathrm{d}y < 1, \tag{***}$$

and $g : [a,b] \mapsto \mathbb{R}$ is a continuous function. Then there is a unique continuous function $f : [a,b] \mapsto \mathbb{R}$ that satisfies (*).

Proof. We prove this theorem by showing that, when (***) is satisfied, the map T is a contraction on the normed space $C([a,b])$ with the uniform norm $\|\cdot\|_\infty$. The normed space $C([a,b])$ is complete (this is not shown here). Moreover, T is a contraction since for any $f_1, f_2 \in C([a,b])$ we have

$$\begin{aligned}
\|Tf_1 - Tf_2\|_\infty &= \sup_{x\in[a,b]} \left| \int_a^b k(x,y)\big(f_1(y) - f_2(y)\big)\,\mathrm{d}y \right| \\
&\leq \sup_{x\in[a,b]} \int_a^b |k(x,y)||f_1(y) - f_2(y)|\,\mathrm{d}y \\
&\leq \|f_1 - f_2\|_\infty \sup_{x\in[a,b]} \int_a^b |k(x,y)|\,\mathrm{d}y \\
&\leq c\|f_1 - f_2\|_\infty,
\end{aligned}$$

where

$$c = \sup_{x\in[a,b]} \int_a^b |k(x,y)|\,\mathrm{d}y < 1$$

The result then follows from the contraction mapping theorem. □

From the proof of the contraction mapping theorem, we can obtain the fixed point f as a limit,

$$f = \lim_{n\to\infty} T^n f_0, \tag{****}$$

for any $f_0 \in C([a,b])$. We may interpret this limit as a series. For this, we define a map $K : C([a,b]) \mapsto C([a,b])$ by

$$Kf = \int_a^b k(x,y)f(y)\,\mathrm{d}y.$$

The map K is called a *Fredholm integral operator*, and the function k is called the *kernel* K. We can then rewrite (*) as

$$(I - K)f = g. \tag{*****}$$

where I is simply the identity map. The contraction mapping T is given by $Tf = g + Kf$, which implies

$$T^n f_0 = g + K\big(g + \cdots + K(g + Kf_0)\big)$$

$$= g + Kg + \cdots + K^n g + K^{n+1} f_0.$$

Using this in (****), we find

$$f = \sum_{n=0}^{\infty} K^n g.$$

Since $f = (I - K)^{-1} g$, we can write this as

$$(I - K)^{-1} = \sum_{n=0} {}^{\infty} K^n,$$

which is called the *Neumann series.* We may use partial sums t approximate the inverse, called the *Born approximation.* Explicitly, we have

$$(I + K + K^2 + \dots) f(x) = f(x) + \int_a^b k(x,y) f(y) \, \mathrm{d}y + \iint_a^b k(x,y) k(y,z) f(z) \, \mathrm{d}y \, \mathrm{d}z + \dots$$

Note that $(I - K)^{-1}$ is really a geometric series that is absolutely convergent with respect to a suitable operator norm when $\|K\| < 1$. Thus we do not need a condition on g; (***) is a condition that ensures $I - K$ is invertible, and this only involves k.

Introduction

This paper looks at how asset prices for claims to output in an exchange economy fluctuate stochastically, where the fluctuation comes from the productivity in each production unit being a stochastic process evolving over time. The focus lies on the derivation of a functional equation in the vector of equilibrium asset prices which is solved for the prices as a function of the physical state of the world. This equation is a generalization of the Martingale property. The Martingale property is the property of a random variable for which the expectation in the next period is simply the realized value in the current period, i.e. $\mathbb{E}[X_{t+1} \mid X_t, X_{t-1}, \dots X_0] = X_t$, in other words - it could be viewed as a random walk with no drift, although this is of course not a technical description. This property of stochastic price sequences serves the role of being the defining characteristic of market *efficiency.* We examine the conditions under which a prices' failure to posses the Martingale property can be viewed as evidence of non-competitive or irrational behavior.

We assume the market efficiency hypothesis in the sense that we assume prices being a reflection of *all* available information, i.e. rational expectations. This property is rather unspecified, as we simply assume it to be the outcome of the process of learning and adapting, rather than viewing it as a description of how agents think about their environment, process information, and so on. Thus it would be beneficial to find some stability theory which gives insight into the forces which move an economy towards an equilibrium.

Description of the Economy

We assume the existence of a representative consumer who maximizes the expected discounted lifetime utility

$$\mathbb{E}\left\{\sum_{t=0}^{\infty} \beta^t U(c_t)\right\}, \tag{4.1}$$

where c_t is a stochastic process representing the consumption of a single good. Everything else is as in previous chapters. The consumption good is produced on n distinct productive units (firms), and we denote the output of firm i in period t by y_{it}, implying the period t output vector to be $y_t = (y_{1t}, \dots, y_{nt})$. We assume output to not only be perishable, but even fully depreciate over one period, implying the feasibility constraint on consumption

$$0 \le c_t \le \sum_{i=1}^{n} y_{it}.$$

The production is assumed to be entirely exogenous, i.e. we neglect any resource use and inputs in general. The output follows a law of motion described by a Markov process with the transition function

$$\mathcal{F}(y', y) = \mathbb{P}\mathrm{r}\{y_{t+1} \leq y' \mid y_t = y\}$$

Firm ownership is determined in each period in a competitive stock market. The equity share is assumed to be perfectly divisible. A share entitles its owner at the beginning of period t to all of the unit's output in the respective period. Shares are traded after payments of real dividend*, at a competitively determined price vector p_t. Let z_t denote the vector of the beginning-of-period share holdings of the representative consumer. It is easy to determine the equilibrium quantities of consumption and asset holds. All output will be consumed, i.e. $c_t = \sum_i y_{it}$ and all shares will be held, i.e. $z_t = \vec{1} \forall t$.

Note that we assume that all relevant information on the current and future physical state is summarized by the current output vector y. As we can write the problem recursively, note that the asset market solves a problem of the same form in each period. Equilibrium asset prices should (if we capture the appropriate behavior) be expressed as some fixed function $p(\cdot)$ of the state of the economy. So in each period we have $p_t = p(y_t)$, where the $p_i(y_t)$ is the price of a share of the ith firm when the economy is in state y_t. If this is the case, then it is sufficient to know this pricing function $p(y)$ and the transition function $\mathcal{F}(y', y)$ to determine the stochastic character of the price process $\{p_t\}$.

Similarly, for the household's decision problem, we expect the current consumption c_t and portfolio decisions z_{t+1} to be dependent on the beginning-of-period portfolio z_t, the prices p_t, and the relevant information of current and future state of the economy y_t. The behavior can again be described as fixed decision rules $c_t = c(z_t, y_t, p_t)$ and $z_{t+1} = g(z_t, y_t, p_t)$.

Given these perceived, future price behavior $\mathcal{F}(y', y)$ and $p(y)$, consumers will determine their decision rules optimally. Thus, a price function determines consumer behavior. We could also look at it from a different direction. Given decision rules $c(\cdot)$ and $g(\cdot)$, we are able to determine a price function p, due to market clearing conditions in each period. Rational expectations in this context then refers to these two approaches leading to the same pricing function.

Definition of Equilibrium

Assume $\beta \in (0, 1)$, and $U : \mathbb{R}^+ \mapsto \mathbb{R}^+$ being continuously differentiable, bounded, increasing, and strictly concave with $U(0) = 0$. Furthermore, let $\mathcal{F} : \mathbb{S}^{n+} \times \mathbb{S}^{n+} \mapsto \mathbb{R}$, where S^n is some n-dimensional space, and $\mathcal{F}$ being continuous. $\mathcal{F}(\cdot, y)$ is a distribution function for each fixed y and $\mathcal{F}(0, y) = 0$. Assume that the process defined by $\mathcal{F}$ has a stationary distribution $\phi(\cdot)$, the unique solution to

$$\phi(y') = \int \mathcal{F}(y', y) \, \mathrm{d}\phi(y),$$

and that for any continuous $g(y)$ we have

$$\int g(y') \, \mathrm{d}\mathcal{F}(y', y)$$

being a continuous function in y.

An equilibrium will be a pair of function. A price function $p(y)$ and an optimum value function $v(z, y)$. The value $v(z, y)$ will be interpreted as the value of (4.1) for a consumer beginning in state y with holdings z, following the optimum consumption portfolio policy afterwards.

Definition 4.1. An *equilibrium* is a continuous function $p(y) : \mathbb{S}^{n+} \mapsto \mathbb{S}^{n+}$ and a continuous, bounded function $v(z, y) : \mathbb{S}^{n+} \times \mathbb{S}^{n+} \mapsto \mathbb{R}^+$ such that

$$v(z, y) = \max_{c,x} \left\{ U(c) + \beta \int v(x, y') \, \mathrm{d}\mathcal{F}(y', y) \right\}, \tag{i}$$

*So the current dividend is not priced in, but the next period dividend is? Are we again at one-period-ahead claims to consumption then?

subject to

$$c + p(y) \cdot x \leq y \cdot z + p(y) \cdot z,\ c \geq 0,\ x \in [0, \bar{z}],$$

where $\bar{z}$ is a vector with components exceeding one;

$$\text{for each } y,\ v(\underline{1}, y) \text{ is attained by } c = \sum_{i=1}^{n} y_i \text{ and } x = \underline{1}. \tag{ii}$$

Condition (i) says that, given price behavior, the consumer allocated their resources $y \cdot z + p(y) \cdot z$ optimally among current consumption c and the end-of-period share holdings x. Condition (ii) requires that these consumption and portfolio choices are market clearing. Since we have market clearing in each period, we never observe any state except $z = \underline{1}$. The consumer has the option to choose security holdings $x \neq \underline{1}$. To evaluate these options, he needs to know $v(z, y)$ for all z.

Construction of the Equilibrium

Consider the maximization problem (i) for the consumer, given some price behavior $p(y)$.

Proposition 4.1. For each continuous price function $p(\cdot)$ there is a unique, bounded, continuous, nonnegative function $v(z, y; p)$ satisfying (i). For each y, $v(z, y; p)$ is an increasing, concave function of z.

Proof. Define the operator T_p on functions $v(z, y)$ such that (i) is equivalent to $T_p v = v$. The domain of T_p is the nonnegative orthant $\mathbb{L}^{2n+} \subset \mathbb{L}^{2n}$ of continuous, bounded functions $u : \mathbb{S}^{n+} \times \mathbb{S}^{n+} \mapsto \mathbb{R}$, normed by

$$\|u\| = \sup_{z,y} |u(z, y)|.$$

Applying T_p involves maximizing a continuous function over a compact set, $T_p u$ is then well defined for any $u \in \mathbb{L}^{2n+}$. $U(c)$ is bounded, thus $T_p u$ is bounded as well, and furthermore continuous. Hence we have $T_p : \mathbb{L}^{2n+} \mapsto \mathbb{L}^{2n+}$. T_p is monotone, i.e. $u \geq v \Rightarrow T_p u \geq T_p v$, and for any constant A we have $T_p(u + A) = T_p u + \beta A$. T_p is a contraction mapping*. It follows that $T_p v = v$ has a unique solution v in $\mathbb{L}^{2n+}$, as was to be shown. Furthermore, $\lim_{n\to\infty} T_p^n u = v$ for any $u \in \mathbb{L}^{2n+}$.

To prove that v is increasing in z, note that $T_p u$ is increasing in z for any u. Since $v = T_p v$, this implies that v is increasing in z.

To prove that v is concave in z, we show that $u(z, y)$ is concave in z, then it follows that $(T_p u)(z, y)$ is so as well. Let z_0, z_1 be chosen, $\theta \in [0, 1]$, and let $z^\theta = \theta z^0 + (1 - \theta) z^1$. Now let (c_i, x_i) attain $(T_p u)(z^i, y)$ for $i = 0, 1$. Now $(c^\theta, x^\theta) = (\theta c^0 + (1 - \theta) c^1, \theta x^0 + (1 - \theta) x^1)$ satisfies $c^\theta + p(y) \cdot x^\theta \leq y \cdot z^\theta + p(y) \cdot z^\theta$, so that

$$\begin{aligned} (T_p u)(z^\theta, y) &\geq U(c^\theta) + \beta \int u(x^\theta, y') \, \mathrm{d}\mathcal{F}(y', y) \\ &\theta (T_p u)(z^0, y) + (1 - \theta)(T_p u)(z^1, y), \end{aligned}$$

using the concavity of u and U. Hence $(T_p u)(z, y)$ is concave in z. It follows by induction that $T_p^n u$ is concave in z for all $n = 1, 2, \ldots$. Then, since $\lim_{n\to\infty} T_p^n u = v$, v is concave- □

The derivatives of v w.r.t. z are described in the following proposition.

Proposition 4.2. If $v(z, y; p)$ is attained at (c, x) with $c > 0$, then v is differentiable w.r.t. z at (z, y) and

$$\frac{\partial}{\partial z_i} v(z, y; p) = U'(c)[y_i + p_i(y)] \text{ for } i = 1, 2, \ldots, n. \tag{4.2}$$

*A contraction mapping can be defined for maps between metric spaces. If (M, d) and (N, d') are two metric spaces, then $f : M \mapsto N$ is a contraction mapping if $\forall x, y \in M : \exists k \in [0, 1) : d'\big(f(x), f(y)\big) \leq k d(x, y)$.

Proof. Define $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ by

$$f(\mathbf{A}) = \max_{c,x} \left\{ U(c) + \beta \int v(x, y') \, \mathrm{d}\mathcal{F}(y', y) \right\}$$

subject to

$$c + p(y) \cdot x \leq \mathbf{A}, \; c, x \geq 0.$$

For each $\mathbf{A}$, $f(\mathbf{A})$ is attained at $c(\mathbf{A})$, $x(\mathbf{A})$, and since the term which is to be maximized is strictly concave in c, $c(\mathbf{A})$ is unique and varying continuously in $\mathbf{A}$. If $c(\mathbf{A}) > 0$ and h is sufficiently small, $c(\mathbf{A}) + h$ is feasible at "income" $\mathbf{A} + h$, and $c(\mathbf{A} + h) - h$ is feasible at income $\mathbf{A}$. Thus

$$\begin{aligned} f(\mathbf{A} + h) &\geq u(c(\mathbf{A}) + h) + \underbrace{\beta \int v(x(\mathbf{A}), y') \, \mathrm{d}\mathcal{F}(y', y)}_{= f(\mathbf{A}) - u(c(\mathbf{A}))} \\ &= u(c(\mathbf{A}) + h) - u(c(\mathbf{A})) + f(\mathbf{A}) \end{aligned}$$

and

$$\begin{aligned} f(\mathbf{A}) &\geq u(c(\mathbf{A} + h) - h) + \beta \int v(x(\mathbf{A} + h), y', y) \\ &= u(c(\mathbf{A} + h) - h) - u(c(\mathbf{A} + h)) + f(\mathbf{A} + h). \end{aligned}$$

Combining both inequalities gives

$$\begin{aligned} U(c(\mathbf{A}) + h) - U(c(\mathbf{A})) &\leq f(\mathbf{A} + h) - f(\mathbf{A}) \\ &\leq U(c(\mathbf{A} + h)) - U(c(\mathbf{A} + h) - h). \end{aligned}$$

Dividing by h and taking the limit for $h \to 0$ provides the formal definition of the derivative, and when utilizing the continuity of $c(\cdot)$ gives

$$f'(\mathbf{A}) = U'(c(\mathbf{A})).$$

Now, plugging in $\mathbf{A} = y \cdot z + p(y) \cdot z$, so that we have $v(z, y; p) = f(\mathbf{A})$, we obtain what was to be shown:

$$\frac{\partial}{\partial z_i} v = f'(\mathbf{A}) \left[\frac{\partial}{\partial z_i} \mathbf{A} \right].$$

□

We now continue with the maximization problem described in (i), while not further relaxing the assumption of asset prices p being described by an arbitrary continuous function. The FOCs (necessary *and* sufficient) are

$$U'(c) p_i(y) = \beta \int \frac{\partial}{\partial x_i} v(x, y') \, \mathrm{d}\mathcal{F}(y', y) \text{ for } i = 1, 2, \ldots, n, \tag{4.3}$$

$$c + p(y) \cdot x = y \cdot z + p(y) \cdot z, \tag{4.4}$$

assuming $c, x > 0$. If the next period's optimal consumption is also strictly positive, then Proposition 4.2 additionally implies

$$\frac{\partial}{\partial x_i} v(x, y') = U'(c')[y'_i + p_i(y')] \text{ for } i = 1, 2, \ldots, n. \tag{4.5}$$

In equilibrium, i.e. when (ii) holds, $z = x = \underline{1}$, $c = \sum_j y_j$ and of course $c' = \sum_j y'_j$. Combining (4.3) and (4.5) and using these facts, we have for $i = 1, 2, \ldots, n$ that

$$U'\Big(\sum_{j=1}^n y_j \Big) p_i(y) = \beta \int U'\Big(\sum_{j=1}^n y'_j \Big) (y'_i + p_i(y')) \, \mathrm{d}\mathcal{F}(y', y). \tag{4.6}$$

But what does this equation mean? Firstly, it is a stochastic Euler equation as we had in many other places of these lecture notes. Secondly, and more economically speaking, it is the equating of the marginal rate of intertemporal substitution of consumption (always current to future) to the market rate of transformation, as given by the market rate on the security i. Note the independence of the value function $v(\cdot)$ which we used to derive it, thus it must hold for any such function. If $p^*(y)$ solves (4.6) and $v(z, y; p^*)$ is constructed as in Proposition 4.1, then the pair $\{p^*(y), v(z, y; p^*)\}$ constitutes an equilibrium. Thus we have coincident solutions, as mentioned earlier about the equivalence of the constructions, due to rational expectations.

Now define the following three functions for all $i = 1, 2, \ldots, n$:

$$g_i(y) = \beta \int U'\Big(\sum_j y'_j\Big) y'_i \, \mathrm{d}\mathcal{F}(y', y),$$

then if the n independent functional equations

$$f(y) = g_i(y) + \beta \int f(y') \, \mathrm{d}\mathcal{F}(y', y) \tag{4.7}$$

have solutions $\{f_1(y), f_2(y), \ldots, f_n(y)\}$, the price functions

$$p_i(y) = \frac{f_i(y)}{U'\Big(\sum_j y_j\Big)} \tag{4.8}$$

will solve (4.6), and $p(y) = \{p_1(y), p_2(y), \ldots, p_n(y)\}$ will be the equilibrium price function.

If f is any continuous, bounded, nonnegative function on $\mathbb{S}^{n+}$, the function $T_i f : \mathbb{S}^{n+} \mapsto \mathbb{R}^+$ given by

$$(T_i f)(y) = g_i(y) + \beta \int f(y') \, \mathrm{d}\mathcal{F}(y', y) \tag{4.9}$$

is well-defined and continuous in y. Now we pick some placeholder bound for U, which is concave, and then we can write for any c:

$$0 = U(0) \le U(c) + U'(c)(-c) \le B - cU'(c),$$

such that $cU'(c) \le B$ for all c. Thus $g_i(y)$ are bounded. Furthermore, since they are nonnegative, their sum is also bounded by βB. The (contraction) operators T_i in (4.9) take elements of the space $\mathbb{L}^{n+}$ of continuous, bounded functions into the same space. Solutions to $T_i f = f$ are solutions to (4.7), and solutions to (4.7) are solutions to $T_i f = f$.

Proposition 4.3. There is exactly one continuous, bounded solution f_i to (4.7), or to $T_i f = f$. For any $f_0 \in \mathbb{L}^{n+}$, we have $\lim_{n\to\infty} T^n f_0 = f_i$.

The proof follows from T_i being a contraction and then follows a similar pattern as the one for Proposition 4.1. Summarizing, this subsection elaborated on the existence of a unique price function in equilibrium of the characterized economy, where we have found functions such as (4.6)-(4.8) that allow us to characterize this price function.

A Duality Theorem

This subsection gives a more general way to construct the equilibrium price function, one in which we do not require differentiability of U. Since we may treat U as an unknown function for now this may be very useful as it is not hard to imagine utility functions which are not differentiable. Consider the functional equation

$$r(z, y) = \inf_{q \in \mathbb{S}^{n+}} \left\{ \sup_{c,x} \left[U(c) + \beta \int r(x, y)' \, \mathrm{d}\mathcal{F}(y', y) \right] \right\}, \tag{4.10}$$

subject to the feasibility constraint $c + q \cdot x \leq y \cdot z + q \cdot z$. As we will show, evaluating the optimal policy functions $q(z, y)$ at $(\underline{1}, y)$ is equivalent to the price functions from the previous subsection. Let $\mathbb{B}$ be the space of bounded, integrable functions on $\mathbb{S}^{n+} \times \mathbb{S}^{n+}$, and let $M : \mathbb{B} \mapsto \mathbb{B}$ be the operator such that we can rewrite (4.10) as $r = Mr$.

Proposition 4.4. There is exactly one bounded integrable function r satisfying $r = Mr$, and for any $u \in \mathbb{B}$, $\lim_{n\to\infty} M^n u = r$.

The proof is similar to the one of proposition 4.1.

Proposition 4.5. The solution r to (4.10) satisfies

$$r(z, y) = U(y \cdot z) + \beta \int r(z, y') \,\mathrm{d}\mathcal{F}(y', y). \tag{4.11}$$

Furthermore, r is continuous, nondecreasing, and concave in z for any fixed y.

Proof. Define $R : \mathbb{L}^{2n+} \mapsto \mathbb{L}^{2n+}$ by

$$(Rw)(z, y) = U(y \cdot z) + \beta \int w(z, y') \,\mathrm{d}\mathcal{F}(y', y)$$

such that (4.11) can be read as $r = Rr$. We need to show two aspects, under the assumption of w being continuous, nondecreasing, and concave in z for any y:

1. Rw has these properties
2. $Mw = Rw$.

Proof of the first aspect is omitted, as the arguments are the same as in Proposition 4.1 again. To prove the second property, note that $(c, x) = (y \cdot z, z)$ satisfies $c + q \cdot x \leq y \cdot z + q \cdot z$ for all q, so that $Mw \geq Rw$. Since we have w concave, for any (z, y) the set

$$A = \left\{(c, x) : U(c) + \beta \int w(x, y') \,\mathrm{d}\mathcal{F}(y', y) \geq (Rw)(z, y)\right\}$$

is convex. From the separation theorem for convex sets*, there is a number a_0 and a vector $a \in \mathbb{S}^n$ (not both zero), such that $(c, x) \in A$ implies $a_0 c + a \cdot x \geq a_0 y \cdot z + a \cdot z$. As $U(c)$ is strictly increasing, it follows that $a_0 > 0$ and a $a \geq 0$, so we can define $q = a/a_0$ and write

$$(c, x) \in A \Rightarrow c + q \cdot x \geq y \cdot z + q \cdot z. \tag{4.12}$$

Suppose that for the vector q there is a $(c, x) \in A$ (interior) with $c + q \cdot x = y \cdot z + q \cdot z$. Then, reducing c by an arbitrary small amount, we have a point $(c', x) \in A$ such that $c' + q \cdot x < y \cdot z + q \cdot z$; a contradiciton to the above implication (4.12). Thus q attains Mw, or $Mw = Rw$. □

Proposition 4.6. For all y, $r(\underline{1}, y) = v(\underline{1}, y)$.

Proof. From the definition of the equilibrium, v is the solution to (4.11) with $z = \underline{1}$. □

*Let A and B be two disjoint, nonempty convex subsets of $\mathbb{R}^n$, then there exists a nonzero vector v and a real number c such that $\langle x, v\rangle \geq c$ and $\langle y, v\rangle \leq c$ for all $x \in A$ and $y \in B$. I.e. the hyperplane $\langle \cdot, v\rangle = c$, v the normal vector, separates A and B. Note that this theorem does not apply if any of the two sets is not convex, and furthermore the vector separating them may share points with these sets, as we only have a weak inequality. E.g. think of a closed $\epsilon = 1$ ball in $\mathbb{R}^2$ around the points $(0, 0)$ and $(2, 0)$, then these two balls can be (weakly) separated by the vertical line going through $(0, 1)$. If they were open balls this would hold with a strict inequality.

Proposition 4.7. If $p(y)$ is an equilibrium price function, then $q(\underline{1}, y) = p(y)$ attains $r(\underline{1}, y)$.

And conversely, we have the following due to the equivalence:

Proposition 4.8. If $q(\underline{1}, y)$ attains $r(\underline{1}, y)$, then $p(y) = q(\underline{1}, y)$ is an equilibrium price function.

Proof. Let $q(\underline{1}, y)$ attain $r(\underline{1}, y)$ and suppose that (c^0, x^0) uniquely attains

$$\max_{c,x} \left\{ U(c) + \beta \int r(x, y') \,\mathrm{d}\mathcal{F}(y', y) \right\}, \tag{4.13}$$

subject to

$$c + q(\underline{1}, y) \cdot x \leq y \cdot \underline{1} + q(\underline{1}, y) \cdot 1.$$

If we have $(c^0, x^0) = \left(\sum_i y_i, \underline{1}\right)$, then the proposed statement follows from Proposition 4.6 and the definition of an equilibrium. If $(c^0, x^0) \neq \left(\sum_i y_i, \underline{1}\right)$, then a convex combination of the two points is feasible for the maximization problem, and it yields a higher value to the objective function. This is clearly a contradiction to the assumption of (c^0, x^0) solving the problem. Note that this is true due to r being concave in z, and U being strictly concave. $\square$

Stability of the Equilibrium

There is only one way for the economy to be in a competitive equilibrium, namely when all output is consumed, all asset shares are held, and those assets are priced according to (4.6), or equivalently the solution comes from solving the dynamic program in (4.10). But obviously the number of ways in which the economy can be out of equilibrium is large, thus the way we treat these cases is somewhat arbitrary, it is not readily resolvable by simple economic reasoning. Of course the described model makes very strong assumptions about what the agents know not only about the structure of the economy, but also on the ability of the agents to perform non-trivial calculations in order to maximize their utility. So what if agents are only assumed to have some more sensible knowledge and ability, will the agents revise their decision rules over time in order to converge to the described equilibrium? We look at three different stability questions for now.

Firstly, in each period some static market clearing occurs, setting current prices. This sense of stability is trivial and always obtained.* Secondly, agents may not have information (or just not enough information) on the distribution $\mathcal{F}(y', y)$ of the production shocks. This is a well understood problem that can be readily explained with Bayesian decision theory, i.e. agents update their prior given the new information obtained in each period. More information makes those priors better, leading over time to convergence to this stable system (the equilibrium). Lastly, consumers may have incorrect knowledge, about the pricing function, the distribution of future prices conditional on the current state of the world, or equivalently, about how to evaluate their end-of-period portfolio x.

Now we consider the last case, the one in which the agents do not know how to properly evaluate the value of their end-of-period portfolio. The correct way is to use the equilibrium value function given the preferences $v : \int v(x, y') \,\mathrm{d}\mathcal{F}(y', y)$. For us to be in equilibrium we need agents to know this. But now assume that agents do not know the proper value function, while still evaluating it over the correct distribution of shocks: $\int u(x, y') \,\mathrm{d}\mathcal{F}(y', y)$, where u is arbitrary but still satisfies being continuous, concave, and increasing in z. Now, as we do not assume heterogeneous agents, we can view this as some general bias in the population, i.e. a systematic error across agents which is perfectly correlated (as we have a representative consumer this must be the case)†. Let there be asset demand being drawn from the arbitrary portfolio evaluation formula, and assume that current period market clearing implies a price vector q at which $x = \underline{1}$, and $c = \sum_i y_i$. Given these prices, the realized utility needs to be examined.

*This would more be an issue of market structure rather than something on the individual level.

†There would be a trivial case where all agents have some bias, but for odd numbered agents this bias is the bias from the even numbered agents switched in sign, such that the bias systematically cancels out.

This utility is clearly given by $(Mu)(z, y)$, where M is defined as before. Going back to Proposition 4.5, we see that the price q which attains the RHS of (4.10) is precisely the price inducing market clearing, given this arbitrary portfolio valuation function u. Now the fun part: the agents experience this utility u, and then update their utility according to Mu given the prices, then M^2u, and so on. This is what learning entails in this context. And since we have shown that $M^n u \to v$, where v is the *true* valuation, prices will need to converge such that this original equilibrium holds. Successive approximation and the implied asymptotic convergence constitutes some form of stability. However, it should be noted that these approximations are not a description of observed behavior, but rather an arguments for theoretical reasoning for expecting an equilibrium, i.e. this does not capture some creative process in which agents discover and gain from not being in equilibrium, thus we should treat this mechanism very carefully and not rely on it when trying to explain the stability to people not very familiar with our mathematical, and economic methods. Case in point here is that we chose some arbitrary u, thus it pretty much cannot be some *mechanically* specified process.

Also of note should be that agents need not actually know much about Markov processes or dynamic programming. They also don't need to perform well at predicting price movements. The only requirement are consistent preferences on consumption and asset holdings. This is somewhat needed for the market system we often use in our economic models anyway, so not a particularly specific requirement in the grand scheme of things. However, preferences should be revised into a direction of the utility actually yielded by asset holding, so these agents cannot be completely oblivious to the world around them and have some vague idea of what the underlying issue is. Vague ideas here would describe some intuition in which direction their errors may be, and some intuition on how to (at least partially) resolve this error. No mechanical approximation process can be expected to capture the creativity required to put into discovery of ways to gain knowledge from deviation of the equilibrium.

Examples

Linear Utility

We now examine a special case which deviates from the previously described assumptions we need on utility functions. Constant marginal utility clearly violates the assumption of boundedness, but it is nonetheless a good example. Using $U(c) = c$ and thus $U'(c) = 1$, (4.6) becomes

$$p_i(y) = \beta \int \left(y_i' + p_i(y')\right) \mathrm{d}\mathcal{F}(y', y),$$

and when making use of the additive property of random variables, we can write this as

$$\begin{aligned} &= \beta \int y_i' \,\mathrm{d}\mathcal{F}(y', y) + \beta \int p_i(y') \,\mathrm{d}\mathcal{F}(y', y), \\ &= \beta \mathbb{E}[y_i' \mid y] + \beta \mathbb{E}[p_i(y') \mid y], \end{aligned} \tag{4.14}$$

which in turn can be solved for the pricing function of firm i

$$p_i(y) = \sum_{s=1}^{\infty} \beta^s \mathbb{E}[y_{i,t+s} \mid y_{it} = y].$$

So the firm i's asset price is the expected, discounted present value of its real dividend stream, conditional on the current state of the world which summarizes all relevant information. It is worth remembering at this point that y' captures the information over the entire future, not only the immediate future - that of the one period ahead.

One Asset

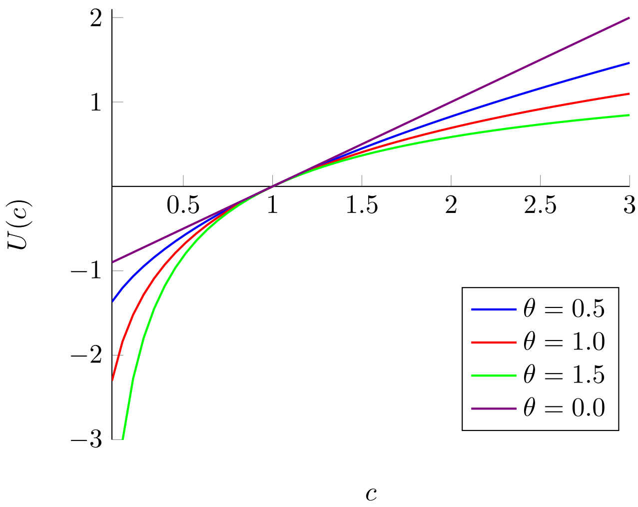
Now we consider an economy with one asset only. We may use equations (4.6) or (4.7) in order to solve for the pricing function. So what are the important aspects here? Clearly the information that the current physical state y captures, the way the CDF $\mathcal{F}(y', y)$ varies with y, but also the curvature of the utility function U, i.e.

the risk aversion. To see the immediate connection between the curvature and the risk aversion, consider the CRRA utility function

$$\tilde{U}(c) = \begin{cases} \frac{c^{1-\theta}-1}{1-\theta}, & \text{if } \theta \neq 1 \\ \log(c), & \text{if } \theta = 1 \end{cases}.$$

depicted for differing values of θ in Figure 4.1. Furthermore, note that for $\theta = 0$ we have risk-neutrality, i.e. $\mathbb{E}[U(c)] = U(\mathbb{E}[c])$. Letting $\theta < 0$ then gives risk-loving individuals.

Figure 4.1: $\tilde{U}(c)$



Now consider $\{y_t\}_{t=0}^{\infty}$ to be i.i.d. RVs: $\mathcal{F}(y', y) = \phi(y')$, the stationary distribution from the beginning of the section. Then $g(y)$ is the constant described as previously, but we drop the subscripts and replace the distribution over which we take the integral such that

$$\begin{aligned} \bar{g} &= \beta \int y'U'(y')\,\mathrm{d}\phi(y') \\ &= \beta\mathbb{E}[yU'(y)], \end{aligned}$$

where (4.9) now gives us, as we let $\lim_{n\to\infty} T^n 0$, and note that this expands due to the recursive formulation to a sum of the form $x + \beta x + \beta^2 x + \dots$, such that

$$f(y) = \frac{\bar{g}}{1-\beta},\ f'(y) = 0.$$

Now we plug this into (4.9) and get the following:

$$p_i(y) = \frac{(1-\beta)^{-1}\beta\mathbb{E}[yU'(y)]}{U'(y)},$$

which yields the derivative w.r.t. y as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}y} p_i(y) &= \frac{\overbrace{f'(y)}^{=0} U'(y) - U''(y)(1-\beta)^{-1}\beta\mathbb{E}[yU'(y)]}{U'(y)^2} \\ &= -\frac{\beta\mathbb{E}[yU'(y)]U''(y)}{(1-\beta)[U'(y)]^2} = p(y)\frac{-U''^{(y)}}{U'(y)} > 0. \end{aligned}$$

This can be rearranged and expanded by y to form an elasticity term

$$\frac{yp'(y)}{p(y)} = -\frac{yU''(y)}{U'(y)}.$$

This term measures the elasticity of the price w.r.t. income, equal to the Arrow-Pratt measure of relative risk aversion. Using the above example of the CRRA utility functions, the RHS of the previous equation (and of course this LHS as well) corresponds to θ.

Assume a period with high transitory income, agents attempt to distribute the windfall over future periods. This is done by purchasing securities. Since we do not allow for storage, the increased demand in a period is then met with increased asset prices.

Now we consider auto-correlated disturbances in the production. However, we restrict these correlations in the following sense, the stochastic difference equation governing y_t must have its root between zero and one. Assume that the CDF $\mathcal{F}$ is differentiable, i.e. assume the existence of a PDF w.r.t. the two inputs y' and y. Assume that these derivatives satisfy

$$0 < -\mathcal{F}_2 < \mathcal{F}_1. \tag{4.15}$$

Lemma 4.1. Let $\mathcal{F}$ satisfy (4.15), and let $h(y)$ have a derivative bounded by 0 and $h'_M > 0$. Then

$$0 \le \frac{\mathrm{d}}{\mathrm{d}y} \int h(y') \,\mathrm{d}\mathcal{F}(y', y) \le h'_M. \tag{4.16}$$

Proof. We use change-of-variable $u = \mathcal{F}(y', y)$, and invert this to get $y' = G(u, y)$ such that $G_2 = (-\mathcal{F}_2)/\mathcal{F}_1$. Then, the derivative in question is

$$\frac{\mathrm{d}}{\mathrm{d}y} \int_0^1 h\big(G(u, y)\big) \,\mathrm{d}u = \int_0^1 h'(G) G_2(u, y) \,\mathrm{d}u,$$

and the result follows from (4.15). □

From (4.9), taking the derivative, we have

$$\frac{\mathrm{d}}{\mathrm{d}y}(Tf)(y) = g'(y) + \beta \frac{\mathrm{d}}{\mathrm{d}y} \int f(y') \,\mathrm{d}\mathcal{F}(y', y). \tag{4.17}$$

Note that we need f to be differentiable in the first place. Furthermore, from the definition of $g(y)$ we have

$$g'(y) = \beta \frac{\mathrm{d}}{\mathrm{d}y} \int U'(y') y' \,\mathrm{d}\mathcal{F}(y', y). \tag{4.18}$$

We are interested in the slope of the solution $f(y)$ to (4.7), we need bounds for $U'(y)y$ and $U''(y)y + U'(y)$. Note that the latter is equivalent to $U'(y)[1 - R(y)]$, where R is the measure of relative risk aversion, which is constant for CRRA utility functions as the ones described previously. Going back, we assume 0 and $\bar{a}$ to be the lower and upper bounds on $U''(y)y + U'(y)$. Applying Lemma 4.1 to (4.18), we have

$$0 \le g'(y) \le \beta \bar{a}.$$

Repeatedly applying (4.17), using the same Lemma with each iteration, gives

$$0 \le f'(y) \le \frac{\beta \bar{a}}{1 - \beta}, \tag{4.19}$$

where $f(y)$ solves (4.8) given that we have one asset, and we again have this discounted infinite sum simplifying. From the pricing functions (4.8) we can derive an elasticity for the pricing function:

$$\frac{y p'(y)}{p(y)} = \frac{y f'(y)}{f(y)} - \frac{y U''(y)}{U'(y)}. \tag{4.20}$$

So now we have three terms worth examining. We start from the back with the last term on the RHS, a term which describes the income effect. Clearly this term is positive due to the assumptions laid on the utility function. The first terms is somewhat more interesting, we may call it *information effect*. This term has the sign of $f'(y)$, which we have shown to be weakly positive but bounded. Looking at (4.19) and (4.20), clearly our knowledge on the functional form of the utility function, more specifically its curvature is important in examining asset prices.

Assume relative risk aversion less than unity (e.g. CRRA utility: $\theta \in (0, 1)$), $f'(y)$ is strictly positive. Thus we can say that we have a positive information effect, new (good) information on future states (more pragmatic: information on dividends) leads to increased prices for the assets.

We can give a description for $p'(y)$; it is the change in the ratio of a general stock price index to the CPI, as the real output changes. The relationship between asset prices and real output is thus not simplistic, or even monotone, in this very simplified model economy.

Many, Independent Assets

Assume a large number of firms, and assume these firms are (at least mostly) independent. Then we may replace $U'\Big(\sum_j y_j\Big)$ in the pricing function (4.6) with $U'(\mu)$, where we use

$$\mu = \sum_j \mu_j = \sum_j \int y_i \phi(y) \, \mathrm{d}y,$$

which - in mean total output (sum over the expected values given the stationary distribution) - yields a good approximation of the pricing function in equilibrium.

Lemma 4.2. Let $S, T : \mathbb{L} \mapsto \mathbb{L}$ be contractions with modulus β and fixed points $f_S, f_T \in \mathbb{L}$. Suppose that

$$\|Sf - Tf\| \leq \mathbf{A} \forall f \in \mathbb{L},$$

then

$$\|f_S - f_T\| \leq \frac{\mathbf{A}}{1-\beta}.$$

Proof. For any f

$$\begin{aligned} \|S^2 f - T^2 f\| &\leq \|S^2 f - TSf\| + \|TSf - T^2 f\| \\ &\leq \|S(Sf) - T(Sf)\| + \beta \| \|Sf - Tf\| \\ &\leq \mathbf{A} + \beta \mathbf{A}, \end{aligned}$$

and more generally

$$\|S^n f - T^n f\| \leq \mathbf{A}(1 + \beta + \cdots + \beta^{n-1})$$

Letting $n \to \infty$ gives the desired result, as we need $\beta < 1$ for a contraction mapping. □

Now let $\tilde{g}_i(y)$ be an approximation for $g_i(y)$, and define the contraction $\tilde{T}_i$ as

$$\tilde{T}_i f = \tilde{g}_i(y) + \beta \int f(y') \, \mathrm{d}\mathcal{F}(y', y),$$

we have

$$\|\tilde{T}_i f - T_i f\| = \|\tilde{g}_i(y) - g_i(y)\|.$$

Assume f_i and $\tilde{f}_i$ to be the fixed points of the respective contractions, then Lemma 4.2 gives the boundary*

$$\|\tilde{f}_i - f_i\| \leq \frac{\|\tilde{g}_i - g_i\|}{1-\beta}. \tag{4.21}$$

*of something resembling an approximation error

We now want to properly define $\tilde{T}_i$, using $\tilde{f}_i$ as the fix point and the above approximation

$$\tilde{g}_i(y) = U'(\mu) \int y_i \, \mathrm{d}\mathcal{F}(y', y).$$

This implies an approximation for the pricing function that is defined as follows:

$$\tilde{p}_i(y) = \frac{\tilde{f}_i(y)}{U'(\mu)}.$$

Wanting to examine this approximation, we need furthermore a bound on the approximation error $\|\tilde{g}_i - g_i\|$. We assume $U''(y) : \|U''\| \leq \mathbf{M}$. Then, we have

$$\begin{aligned} \|g_i(y) - \tilde{g}_i(y)\| &= \beta \left\| \int \Big(U'\big(\sum_j y'_j\big) - U'\big(\mu\big)\Big) y'_i \, \mathrm{d}\mathcal{F}(y', y) \right\| \\ &\leq \beta \mathbf{M} \left\| \int \Big(\sum_j y'_j - \mu \Big) y'_i \, \mathrm{d}\mathcal{F}(y', y) \right\|, \end{aligned}$$

where we use the mean value theorem*, which gave us

$$\|U''(y)\| = \left\| \frac{U'\Big(\sum_j y'_j\Big) - U'(\mu)}{\sum_j y'_j - \mu} \right\| \leq \mathbf{M} \Leftrightarrow \left\| U'\Big(\sum_j y'_j\Big) - U'(\mu) \right\| \leq \left\| \mathbf{M}\Big(\sum_j y'_j - \mu\Big) \right\|.$$

Now assume y_i's to be independent, or equivalently assume the CDF to be multiplicatively separable, i.e. $\mathcal{F}(y', y) = \prod_k \mathcal{F}_k(y'_k, y_k)$, then we can evaluate the integral inside the norm

$$\begin{aligned} \int \Big(\sum_j y'_j - \mu \Big) y'_i \, \mathrm{d}\mathcal{F}(y', y) &= \int \Big(\sum_j (y'_j - \mu_j) \Big)(y'_i - \mu_i) \, \mathrm{d}\mathcal{F}(y', y) + \mu_i \int \sum_j (y'_i - \mu_j) \, \mathrm{d}\mathcal{F}(y', y) \\ &= \mathbb{V}\mathrm{ar}[y'_i \mid y_i] + \mu_i \sum_j \mathbb{E}[y'_j - \mu_j \mid y_j]. \end{aligned}$$

We can combine this term with the above result, and get

$$\|g_i(y) - \tilde{g}_i(y)\| \leq \sup_y \Big[\mathbb{V}\mathrm{ar}[y'_i \mid y_i] + \mu_i \sum_j \mathbb{E}[y'_j - \mu_j \mid y_j] \Big].$$

Consider a sequence of economies of same total size, but the number of independent production units of same size increases (at some point they can be viewed as a uniform distribution over the total size), $\mathbb{V}\mathrm{ar}[y'_i \mid y_i]$ and $\mu_i \sum_j \mathbb{E}[y'_j - \mu_j \mid y_j]$ will converge to zero, and therefore the approximation error of $\tilde{f}_i$ from (4.21) will become close. Note that $(1-\beta)^{-1}$ results in a multiplicative factor greater 1, thus the convergence of the approximation error will be slower than the one for $\tilde{g}_i$.

The Martingale Property

We might say that martingales are processes for which the best estimate of their future value is their current value. We have shown that equation (4.6) *makes work of* the assumption that prices are in equilibrium and reflect all available information. In this setting asset prices themselves do not have the Martingale property. But it is straightforward to construct a series w_{it} where $i = 1, \ldots, n$ that does have this property. This series is defined by

$$w_{i,t+1} - w_{it} = \beta U'\Big(\sum_j y_{j,t+1}\Big)(y_{i,t+1} + p_{i,t+1}) - U'\Big(\sum_j y_{jt}\Big) p_{it}, \tag{4.22}$$

*Assume f being continuous on the interval $[a, b]$, and differentiable on the open interval (a, b), then $\exists c \in (a, b) : f'(c) = \frac{f(b)-f(a)}{b-a}$.

since from (4.6) we have the RHS of (4.22) to be zero, conditional on all available information y_t.

We need to make two corrections in order to induce the Martingale property, at least *almost*. The first correction is more along the lines of a special case on the terms $U'(\sum_j y_{jt})$, which is not allowed to vary much. This is either because agents are indifferent to risk, for example the CRRA utility with $\theta = 1$ (linear utility), or because there is little aggregate risk (large number of independent assets, see the remark on how these dividends evolve as we approach a unit interval of independent firms, implying variance tending to zero). Furthermore, we need to correct for the discount factor β in order for dividends to almost have the Martingale property. Either way, no plausible argument can be made for a constant $U'(\cdot)$, at least as long as we do not want to deviate too much from somewhat realistic assumption on risk- or time-preferences.

The requirement of "conditions of market equilibrium can be stated in terms of expected returns" is not something particularly new, proponents of efficient market theories have talked about this for a while. But this paper gives an explicit framework which allows for the judgment of this requirement in meaning and plausibility. So we could describe under which conditions a price sequence might plausibly posses the Martingale property. In this context, we have shown that diminishing marginal rate of substitution of future for current consumption is inconsistent with this property.

Conclusion

One key takeaway from this discussion should be that knowledge of the true probability distribution of future prices may not be as important as one would assume it to be. Agents do not describe their behavior or decision making in such technical terms. Use of hindsight, combined with a reasonably stationary physical world will lead to behavior that can be well approximated by rational expectations.

It is possible to construct rigorous economic models which would either induce a Martingale property on the price sequence of assets, but as we have seen we can just as easily construct models where this property is not exhibited. Simply testing actual prices for this property may therefore not answer the question whether or not market efficiency exists. Thus, this paper was mainly concerned with bringing some convergence between finance and economics.

The assumed structure of time-additive utility is of course fairly restrictive, but provided sufficient impatience is assumed, we can use recursive, non-additive preferences as well. Of course this adds some complications to the model. Capital accumulation would also be a valuable addition. (4.6) would then be a dead-end, but (4.10) may be more promising, assuming capital enters in a non-trivial way. In this context Euler equations provide very little use aside from special cases.

Foundations of Asset Pricing (Lecture)

Throughout this subsection we assume a competitive environment with no information frictions where agents are forward looking, i.e. form expectation on the future. The Lucas Asset pricing model studies an endowment economy with infinitely many identical consumers i, where we have

$$\int i \, \mathrm{d}i = L_t.$$

Instead of firms however, we have identical infinitely-lived trees which constitute the only available assets and we have

$$\int k_t^i \, \mathrm{d}i = K_t,$$

which constitutes the aggregate supply. Utility again depends on the individual consumption and the marginal utility in every period is greater zero for all values of consumption. The aggregate output of the supply side of the economy is the perishable (i.e. fully depreciating) fruit which is harvested from any tree, denoted d_t (the dividends). Combining the above, we have the aggregate resource constraint in our tree-based economy

$$c_t L_t = d_t K_t \tag{1}$$

So what is sold in this economy? In equilibrium, the price of trees must be such that in each period each consumer does not want to change their holdings of trees (the tree holdings must be always optimal). There are no arbitrage opportunities of any kind in equilibrium, so no agent can secure positive profits without involving any risk.

Remark 4.1 (Arbitrage in the Context of Financial Markets)**.** It might be useful to clarify at this point what is meant by "arbitrage". Intuitively, many people would associate arbitrage with "buying low and selling high". More formally, an arbitrageur purchases a set of financial assets at a low price and sells them at a high price simultaneously. This timing element is important: namely, because of simultaneity, arbitrageurs require no outlay of personal endowment but only need to set up a set of simultaneous contracts such that the revenue generated from the selling contract pays off the costs of the buying contract, i.e. construct a portfolio consisting of purchased assets and short-sold assets which yields positive returns with no commitment. The simultaneity ensures that the arbitrageur carries no risk as none of his own personal resources are ever on the line.

One can argue that, in the "real" world, there is rarely a case of pure arbitrage. Indeed, most of what the financial community calls "arbitrage" is really just some very fast or short-term speculation. When speculating, agents usually purchase the assets first and sell them afterwards (or short-sell the assets first and purchase them afterwards), thus they must commit some of their own resources, at least temporarily – and they still run the risk that they will not get to dispose of the second half of their operation at the anticipated price. Pure arbitrage, in contrast, is simultaneous and riskless - the quintessential "free lunch".

The reason why pure arbitrage is not commonly observed is precisely the reasoning for the "no-arbitrage" assumption in financial market equilibrium: if there were arbitrage opportunities, these would be eliminated immediately. Specifically, if asset prices allow for arbitrage opportunities, then because of strong monotonicity of preferences and no bounds to short-selling, agents would immediately hone in on a portfolio position that yielded arbitrage profits. The "no-commitment" nature of arbitrage opportunities imply that agents can replicate this arbitrage portfolio infinitely with no personal resource constraint. If this begins to happen, then at some point (i.e. almost instantly), the price differences which enabled the arbitrageur to hold such a position would close.*

Returning to the model, let P_t denote the equilibrium price of trees after payment of dividends (*ex-dividend*). Any agent in the economy faces the following resource constraint:

$$k^i_{t+1}P_t + c^i_t = d_t k^i_t + P_t k^i_t.$$

Each consumer faces a decision problem where they choose a consumption path in order to maximize their expected present discounted value of future utility, where we assume the typical neoclassical properties on the functional form:

$$v(m^i_t) = \max \mathbb{E}^i_t \sum_{n=0}^{\infty} \beta^n u(c^i_{t+n}), \tag{2}$$

subject to

$$k^i_{t+1} = \Big(1 + \frac{d_t}{P_t}\Big)k^i_t - \frac{c^i_t}{P_t},$$
$$m^i_{t+1} = (P_{t+1} + d_{t+1})k^i_{t+1}$$

The first term of k^i_{t+1} consist of the previous period individual tree holdings, plus the tree holdings multiplied by the fruit produced by those trees, divided by the price of trees in the period, minus the current period consumption of fruit divided by the price for trees in that period. The net-holdings of trees, so to speak. The second constrains is then simply this next-period tree holding multiplied by the value of reselling after the dividend payment (the fruit harvest), plus the dividend (which can be understood as the wealth).

We can rewrite this problem in the form of a Bellman equation

$$v(m^i_t) = \max_{\{c^i_t\}} u(c^i_t) + \beta \mathbb{E}^i_t v(m^i_{t+1}). \tag{2'}$$

**from* a collection of essays, no author mentioned

We then have the FONCs for an interior solution

$$0 = u'(c_t^i) + \beta \mathbb{E}_t^i v'(m_{t+1}^i)\frac{\mathrm{d}m_{t+1}^i}{\mathrm{d}c_t^i},$$

$$u'(c_t^i) = \beta \mathbb{E}_t^i v'(m_{t+1}^i)\frac{P_{t+1}+d_{t+1}}{P_t} \equiv \beta \mathbb{E}_t^i R_{t+1} v'(m_{t+1}^i), \tag{3}$$

where R_{t+1} is the return on the asset, i.e. the value change from one period to the next, given by the price for which we could sell in period $t+1$ plus the dividend, divided by the price for which we could sell in period t.

From the *Envelope Theorem*, we have $v'(m_{t+1}^i) = u'(c_{t+1}^i)$. Thus (3) can be written as the typical stochastic Euler equation

$$u'(c_t^i) = \beta \mathbb{E}_t^i u'(c_{t+1}^i)\frac{P_{t+1}+d_{t+1}}{P_t}, \tag{4}$$

$$P_t = \beta \mathbb{E}_t^i \frac{u'(c_{t+1}^i)}{u'(c_t^i)}(P_{t+1}+d_{t+1}). \tag{4'}$$

The price in period t for an asset is the discounted expectation for the intertemporal marginal rate of substitution of consumption, multiplied by the next period price after dividend payments plus the dividends.

Assume we have identical consumers, so we have $c_t^j = c_t^i \equiv c_t \forall i, j$. We then normalize the stock of trees and the population to one, i.e. $L_t = K_t = 1 \forall t$. Thus we have the aggregate resource constraint $c_t = d_t$. In this case, we can simplify (4') so

$$P_t = \beta \mathbb{E}_t \frac{u'(d_{t+1})}{u'(d_t)}(P_{t+1}+d_{t+1}). \tag{4''}$$

Now we define a *stochastic discount factor* of the form $M_{t,t+n} = \beta^n \frac{u'(d_{t+n})}{u'(d_t)}$, so we have

$$P_t = \mathbb{E}_t M_{t,t+1}(P_{t+1}+d_{t+1}). \tag{5}$$

Since we of course have this for every period, e.g. for the next one

$$P_{t+1} = \mathbb{E}_{t+1} M_{t+1,t+2}(P_{t+2}+d_{t+2}),$$

we can use repeated substitution to get

$$P_t = \mathbb{E}_t M_{t,t+1} d_{t+1} + \mathbb{E}_t M_{t,t+1} \mathbb{E}_{t+1} M_{t+1,t+2} d_{t+2} + \dots. \tag{6}$$

Now we remember from the law of iterated expectations that the smaller information set dominates, i.e. $\mathbb{E}_t \mathbb{E}_{t+1} P_{t+2} = \mathbb{E}_t P_{t+2}$. Furthermore, we simplify the notation such that $M_{t,t+2} = M_{t,t+1} M_{t+1,t+2}$. Then we rewrite (6) as

$$P_t = \mathbb{E}_t \Big[M_{t,t+1} d_{t+1} + M_{t,t+2} d_{t+2} + M_{t,t+3} d_{t+3} + \dots \Big] = \mathbb{E}_t \sum_{s=t}^{\infty} M_{t,t+s} d_{t+s}. \tag{7}$$

Assume we hold the asset indefinitely after buying it in period t. Then we have the price of the asset being the sum of the dividends multiplied by their respective-period stochastic discount factor, and technically the reselling price - which as we do not sell the asset, or rather have $\lim_{t\to\infty} \beta^t = 0$, is zero. So the only relevant aspect for pricing the asset should be the dividends.

Translation of the setting: trees are shares of firms, and the fruits are dividends. The price of shares of course depends on the dividends.

In equilibrium no shares of trees are traded, this is no unique property. But the setting of homogeneous agents, no informational frictions, is a competitive setting. Consider the neoclassical labor market; the model represents a similar setting. We face a spot wage, we have no contracts. In such settings we talk about a Walrasian equilibrium (price vector such that the excess demand is zero). At the intersection point of supply and demand we have the equilibrium. In the Lucas tree model, the quantity is fixed (the trees are constant) and only the price changes (vertical supply curve). As demand shifts, prices change. The price an agent pays reflects future claims to consumption. The amount of fruit varies over time, which is modeled as a random variable. In the benchmark model, our investor is willing to pay for the trees because of the yields which generate utility. Equilibrium is reached via some kind of auction model, firms over-cut each other but workers undercut each other.

Remark 4.2 (CRRA Utility)**.** Assume that individual preferences exhibit the CRRA property, furthermore assume the CRRA functional form without the -1 term at the end. In this case, (4") can be rewritten as

$$P_t = \beta d_t^{\rho} \mathbb{E}_t d_{t+1}^{-\rho}(P_{t+1} + d_{t+1}), \tag{8}$$

where we use $u'(c) = c^{-\rho}$ and the fact that in aggregate we have $c_t = d_t$. If we now assume logarithmic utility, which is the case for $\rho = 1$ (as the function would otherwise be undefined for this value, this is a standard assumption), then we have (8) simplifying to

$$P_t = \beta d_t \mathbb{E}_t d_{t+1}^{-1}(P_{t+1} + d_{t+1}).$$

Again, rewriting this as the price to dividend ratio, which is an important measure in finance, we have

$$\frac{P_t}{d_t} = \beta \mathbb{E}_t d_{t+1}^{-1}(P_{t+1} + d_{t+1}) = \beta \mathbb{E}_t \left(1 + \frac{P_{t+1}}{d_{t+1}}\right),$$

and when repeatedly substituting the next period, we then have

$$\frac{P_t}{d_t} = \beta\left(1 + \beta\left[1 + \mathbb{E}_t \frac{P_{t+2}}{d_{t+2}}\right]\right) = \frac{\beta}{1-\beta} + \beta \mathbb{E}_t \lim_{n\to\infty} \beta^n \frac{P_{t+n}}{d_{t+n}}.$$

Assuming prices and dividends are bounded, the last term converges to zero as we let time run towards infinity.

We now explicitly define the discount factor $\beta = (1+\varphi)^{-1}$, where φ represents the time preference rate. Using this, the equilibrium price, or alternatively the dividend-price ration yields

$$P_t = d_t \frac{\beta}{1-\beta} = \frac{d_t}{\varphi}, \text{ or } \frac{d_t}{P_t} = \varphi.$$

Say now that φ is increasing, this implies that β decreases, i.e. the time preference rate increasing implies the discount factor decreasing, implying increasing impatience. We value the same amount of consumption higher than the future utility of the same consumption in a future period. Of course this raises the question as to why the price is decreasing in the time preference rate. The answer here is somewhat simple. Buying the asset in any period t decreases the consumption in that same period, as we are constrained by the wealth in that period. And while that asset would give claims to future consumption, we do value current consumption more highly, so we are less willing to buy an asset for future consumption, i.e. the demand lowers, yielding a lower price for a vertical supply (fixed supply).

Example 4.1 (CRRA Utility and Three States of the World)**.** We again assume a CRRA utility for the investor, but $\rho \in (0,1)$ such that we can write without a case distinction

$$u(c) = \frac{c^{1-\rho}}{1-\rho}.$$

Dividends exist in three different states, and so take on three values: $d_t \in \{d^1, d^2, d^3\} = \{0.5, 1.0, 1.5\}$. Furthermore, we assume a transition matrix of the form

$$\Pi = \begin{pmatrix} 0.5 & 0.25 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{pmatrix},$$

thus the states display some *stickiness*. However it should be noted that this is a stationary Markov chain, and as we let $t \to \infty$, independent of the initial state of the world, we have equal probability for all states, namely $\frac{1}{3}$.

Now, plugging in $c = d$, and making use of the derivative of the utility function, we can write (8) as

$$d_t^{-\rho} P_t = \beta \mathbb{E}_t d_{t+1}^{-\rho} P_{t+1} + d_{t+1}^{1-\rho}.$$

Now define the prices P^1, P^2, P^3 as the prices for shares in the cases of the dividends d^2, d^2 or d^3. We can now explicitly writing the expectations implied by the previous equation:

$$(d^1)^{-\rho} P_1 = \beta\Big\{\pi_{11}[(d^1)^{-\rho}P_1 + (d^1)^{1-\rho}] + \pi_{12}[(d^2)^{-\rho}P_2 + (d^2)^{1-\rho}] + \pi_{13}[(d^3)^{-\rho}P_3 + (d^3)^{1-\rho}]\Big\}$$

$$(d^2)^{-\rho}P_2 = \beta\Big\{\pi_{21}[(d^1)^{-\rho}P_1 + (d^1)^{1-\rho}] + \pi_{22}[(d^2)^{-\rho}P_2 + (d^2)^{1-\rho}] + \pi_{23}[(d^3)^{-\rho}P_3 + (d^3)^{1-\rho}]\Big\}$$
$$(d^3)^{-\rho}P_3 = \beta\Big\{\pi_{31}[(d^1)^{-\rho}P_1 + (d^1)^{1-\rho}] + \pi_{32}[(d^2)^{-\rho}P_2 + (d^2)^{1-\rho}] + \pi_{33}[(d^3)^{-\rho}P_3 + (d^3)^{1-\rho}]\Big\}$$

If we use the specific dividend values and the transition probabilities given by Π, then we can obtain a set of equations, with three different unknown share prices. These equations are linear in the share price, so solving this numerically is fairly straightforward. Now assume that $\beta = 0.96$ (which is a value fairly consistent with experimental evidence), then we obtain the following: Note that prices are pro-cyclical, they increase with the

Table 1: Numerically Solved Price System

CRRA	Implied prices		
ρ	P^1	P^2	P^3
0.5	16.5	23.5	28.8
1.0	12	24	36
2.0	7.4	29.3	65.6

dividends. Say the dividend increases, then the price for an asset must increase with a fixed supply, as the quantity cannot change due to it simply being a vertical supply curve. With increasing ρ, the range of prices increases, the expansion goes into both directions, with the main driver being the change in prices for the third state. For $\rho = 0$, we have risk-neutrality. The higher the coefficient, the more risk averse the agent is. Note that the parameter ρ is also the elasticity of prices w.r.t. dividends, provided these are i.i.d..

> **Remark 4.3.** Any definition of a competitive equilibrium, regardless of the type of economy studied, requires a statement on market clearing.

> **Remark 4.4.** The optimal policy function depends on *all* state variables.

> **Remark 4.5.** β falls under the term preferences, as it reflects the agents' time preference rates.

5 Time-Series Tools

5.1 Markov Chains

Markov chains are an ideal tool to study recursive model. With their help we can reduce the infinite horizon model to recursive models.

> **Definition 5.1** (Markov property)**.** Let $\{x_t\}$ denote a stochastic process. This process is said to have the *Markov property* if for all $k \geq 1$ and for all t, we have
>
> $$\Pr(x_{t+1} = j \mid x_t = i, x_{t-1} = l, \dots) = \Pr(x_{t+1} = j \mid x_t = i) = p_{ij},$$
>
> where $i, j, l \in \{1, 2, \dots, N\}$ are elements of the set of states.

Note that a Markov process is a memoryless process. In the above definition $p_{ij} \in [0, 1]$ denotes the transition probability from state i to state j. A stochastic process which has the Markov property is called a *Markov chain*. It is important to distinguish between conditional and unconditional probabilities of a stochastic process taking on a specific value from the set of possible states. A Markov matrix (transition matrix) features conditional probabilities.

Definition 5.2 (Time-invariant Markov chain)**.** A *time-invariant Markov chain* is defined by the following three features:

(i) a random $(N \times 1)$ vector $\mathbf{e}_t$ whose jth element equals 1 if $x_t = j$ and whose jth element equals otherwise. For example:

$$\mathbf{e}_t = \begin{cases} (1,0,0,\ldots,0)' & \text{if } x_t = 1 \\ (0,1,0,\ldots,0)' & \text{if } x_t = 2 \\ \cdots & \cdots \\ (0,0,0,\ldots,1)' & \text{if } x_t = N, \end{cases}$$

or if $x_t = j$, $\mathbf{e}_t$ equals the jth column of a $N \times N$ identity matrix $\mathbb{I}_N$.

(ii) $(N \times N)$ transition matrix $\mathbf{P}$, where $\mathbf{P}_{(i,j)} = p_{ij} = \mathbb{P}\mathrm{r}(x_{t+1} = j \mid x_t = i)$. For example, assuming N states, we have

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ p_{21} & p_{22} & \cdots & p_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N1} & p_{N2} & \cdots & p_{NN} \end{pmatrix}.$$

(iii) $\vec{\pi}_0$ is a $(N \times 1)$ vector of unconditional probabilities with ith element $\pi_{0i} = \mathbb{P}\mathrm{r}(x_0 = i)$, where $i \in \{1, 2, \ldots N\}$. For example:

$$\vec{\pi}_0' = \begin{pmatrix} \pi_{01} & \pi_{02} & \ldots & \pi_{0N} \end{pmatrix}.$$

This vector may represent some initial income distribution, for example when income is measured within N different bin sizes.

For this *time-invariant* interpretation to work, we need to make certain assumptions on the initial distributions (the unconditional probability vector), and the transition matrix (the matrix of conditional probabilities). We make the following assumptions:

(i) for $i = 1, 2, \ldots, N$, the matrix $\mathbf{P}$ satisfies $\sum_{j=1}^{N} p_{ij} = 1$, i.e. $\mathbf{P}$ is a stochastic matrix, which requires the row sum to be one for every row. For example,

$$\mathbf{P} = \begin{pmatrix} 0.7 & 0.3 & 0 \\ 0.2 & 0.5 & 0.3 \\ 0 & 0.5 & 0.5 \end{pmatrix}.$$

(ii) the vector of unconditional probabilities $\vec{\pi}_0$ satisfies $\sum_{i=1}^{N} \pi_{0i} = 1$.

Note that generally, the probability of moving from one state to another over two periods is fairly easily found, and generally given by $\mathbf{P}^2$:

$$\mathbb{P}\mathrm{r}(x_{t+2} = j \mid x_t = i) = \sum_{h=1}^{N} \mathbb{P}\mathrm{r}(x_{t+1} = h \mid x_t = i) \cdot \mathbb{P}\mathrm{r}(x_{t+2} = j \mid x_{t+1} = h) = \sum_{h=1}^{N} p_{ih} p_{hj} = p_{ij}^{(2)},$$

where it is important to remember that this does not correspond to simply taking the square of the ij element, but rather take that element of the squared transition matrix. More generally, we have

$$\mathbb{P}\mathrm{r}(x_{t+k} = j \mid x_t = i) = p_{ij}^{(k)}.$$

The (repeated) squaring of the transition matrix captures all ways in which a state can be reached over any given amount of time. Consider a 2×2 transition matrix, and assume you are interested in the two-period transition from one state to itself. Then we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + b \cdot c & \dots \\ \dots & \dots \end{pmatrix},$$

So from the first to the first state across two periods can be reached in two ways: either by staying in the first state, or by moving to the second and then back. Of course this gets more involved when we have more states, but the principle stays the same.

Let $\vec{\pi}_t = \mathbb{P}\mathrm{r}(\mathbf{x}_t)$ be the $N \times 1$ vector with the ith element being $\mathbb{P}\mathrm{r}(x_t = i)$, then we have the *unconditional* long-run probability distribution of $\mathbf{x}_t$ being determined by

$$\begin{aligned} \vec{\pi}_1 &= \mathbb{P}\mathrm{r}(\mathbf{x}_1) = \mathbf{P}'\vec{\pi}_0 \\ \vec{\pi}_2 &= \mathbb{P}\mathrm{r}(\mathbf{x}_2) = (\mathbf{P}')^2\vec{\pi}_0 \\ &\vdots \\ \vec{\pi}_k &= \mathbb{P}\mathrm{r}(\mathbf{x}_k) = (\mathbf{P}')^k\vec{\pi}_0 \end{aligned}$$

Definition 5.3 (Stationary Distributions)**.** Unconditional probability distributions evolve by being fed into the transition matrix:

$$\vec{\pi}_{t+1} = \mathbf{P}'\vec{\pi}_t.$$

An unconditional distribution is called *stationary* if it holds that

$$\vec{\pi}_{t+1} = \vec{\pi}_t = \vec{\pi},$$

i.e. if the unconditional distribution remains unchanged as time passes, put differently, if feeding some initial distribution into the transpose of the transition matrix gives back that same initial distribution.

Remark 5.1 (Formal Description of Stationarity)**.** We know that a stationary distribution must satisfy

$$\mathbf{P}'\vec{\pi} = \pi \Leftrightarrow (\mathbf{P}' - 1 \cdot \mathbb{I}_N) \cdot \vec{\pi} = \mathbf{0},$$

where $\mathbf{0}$ is the null vector. This represents N homogeneous linear equations. A more general representation of a matrix equation is given by

$$(\mathbf{P}' - r \cdot \mathbb{I}_N) \cdot \vec{\pi} = \mathbf{0}, \tag{*}$$

where r is a scalar which is allowed to either be complex or real values. We are interested in a non-trivial solution for $\vec{\pi}$ and a scalar r satisfying (*). Assuming they exist, then r characterizes an eigenvalue of $\mathbf{P}'$, and $\vec{\pi}$ characterizes an eigenvector. If such an eigenvector $\vec{\pi} \neq \mathbf{0}$ exists, then the coefficient matrix $\mathbf{P}' - r\mathbb{I}_N$ is required to be singular, i.e. the determinant must be zero:

$$|\mathbf{P}' - r \cdot \mathbb{I}_N| = \begin{vmatrix} p_{11} - r & p_{21} & \dots & p_{N1} \\ p_{12} & p_{22} - r & \dots & p_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1N} & p_{2N} & \dots & p_{NN} - r \end{vmatrix} = 0. \tag{**}$$

(**) is the *characteristic equation* of $\mathbb{P}'$ and an Nth degree polynomial, it has N roots, each of which is an eigenvalue. For each of these roots there exist infinitely many possible eigenvectors, and in order to have a unique eigenvector for any given root we need to normalize the vector elements.

As for the interpretation of the eigenvalues r, they tell us whether the input vector $\vec{\pi}$ is stretched or shrunk, whether it reverses its direction, or whether it remains unchanged. Thus, the eigenvalues help pin down the dynamics of our system, whether or not it is stable. If we have an eigenvalue $r = 1$, then we have a unit root, so it leaves the vector $\vec{\pi}$ unchanged.

Example 5.1. Consider the transition matrix, and its transpose

$$\mathbf{P} = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}, \ \mathbf{P}' = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}.$$

We want to solve $(\mathbf{P}' - r \cdot \mathbb{I}_N)\vec{\pi} = \mathbf{0}$ for the r's and $\vec{\pi}$'s. This is done in the following steps:

Step 1: First we find the eigenvalues, for this we solve for the determinant $|\mathbf{P}' - r \cdot \mathbb{I}_N| = 0$.

$$\begin{vmatrix} 0.8 - r & 0.3 \\ 0.2 & 0.7 - r \end{vmatrix} = 0.$$

This gives us the following equation:

$$\begin{aligned} (0.8 - r)(0.7 - r) - 0.06 &= 0 \\ 0.56 - 1.5r + r^2 - 0.06 &= 0 \\ r^2 - 1.5r + 0.5 &= 0 \\ (r - 1)(r - 0.5) &= 0. \end{aligned}$$

This is clearly solved by $r_1 = 1$ and $r_2 = 0.5$.

Step 2: To find the eigenvector $\vec{\pi}_1$ associated with the first eigenvalue, we solve

$$(\mathbf{P}' - 1 \cdot \mathbb{I}_N) \cdot \vec{\pi}_1 = \begin{pmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{pmatrix} \begin{pmatrix} \pi_1^1 \\ \pi_1^2 \end{pmatrix} = \mathbf{0}.$$

The solution is given by $\pi_1^1 = 1.5\pi_1^2$. So from the infinitely many solutions we normalize in order to have a unique solution. While there are many ways to normalize, like normalizing to unit length, we choose the solution such that $\pi_1^1 + \pi_1^2 = 1$, which when used above yields $(0.6, 0.4)'$ as the eigenvector. We can understand this vector as the steady-state of the matrix $\mathbf{P}'$, i.e.

$$\lim_{k\to\infty} (\mathbf{P}')^k = \begin{pmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{pmatrix}.$$

We now consider the application of this to the social planner's version of the stochastic growth model. We have the planner's optimization problem given by

$$\max_{\{c_t, i_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to the constraints

$$c_t + i_t = f(k_t, z_t) \tag{5.1a}$$
$$k_{t+1} = (1 - \delta)k_t + i_t \tag{5.1b}$$
$$z_{t+1} = \rho z_t + \epsilon_{t+1}, \tag{5.1c}$$

with $\epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$, and k_0, z_0 given. Suppose this problem had been solved for every possible pair of starting values $\{k_0, z_0\}$. Now define a value function as follows:

$$V(k_0, z_0) \equiv \max_{\{k_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\big[f(k_t, z_t) + (1 - \delta)k_t - k_{t+1}\big]$$

subject to

$$z_{t+1} = \rho z_t + \epsilon_{t+1}.$$

We can rewrite this as the Bellman's equation

$$V(k,z) = \max_{k'} \Big\{ u\big[f(k,z) + (1-\delta)k - k'\big] + \beta \mathbb{E}\big[V(k',z') \mid k,z\big] \Big\},$$

subject to

$$z' = \rho z + \epsilon'.$$

What we implicitly assume is that the capital stock is a continuous variable, and z follows a continuous stochastic process.

Remark 5.2. Value iteration procedures require the capital stock to lie on a discrete grid, so they cannot be immediately applied to the social planner's problem. Choosing a sufficiently fine grid for k, we can achieve an arbitrarily accurate approximation to the underlying continuous problem.

Remark 5.3. If z follows a continuous stochastic process, then we can write the expectations operator in the Bellman's equation in integral form:

$$\begin{aligned} V(k,z) &= \max_{k'} \Big\{ u\big[f(k,z) + (1-\delta)k - g(k,z)\big] + \beta \mathbb{E}\big[V(g(k,z), \rho z + \epsilon') \mid k,z\big] \Big\} \\ &= \max_{k'} \Big\{ u\big[f(k,z) + (1-\delta)k - g(k,z)\big] + \beta \int_{-\infty}^{\infty} V(g(k,z), \rho z + \epsilon') h(\epsilon') \, \mathrm{d}\epsilon' \Big\}, \end{aligned}$$

where k, z are given, and $h(\epsilon)$ denotes the probability density function of the standard normal distribution.

This issue can be avoided by approximating the continuous stochastic process by a discrete process, which is where Markov chains come into play. For a short primer on this topic, Tauchen (1986) describes the performance and process behind such approximations. Tauchen develops a procedure which allows the specification of a Markov chain, based on the moments of the underlying continuous process. $\{z_t^d\}$ approximates the continuous normal distribution in the following sense:

$$\{z_t\} \cong \{z_t^d\}.$$

This means that the processes are approximately equal, they have identical first and second moments, as well as the same autocorrelation structure.

Example 5.2. Let $\{z_t^d\}$ be a discrete Markov chain with N possible states, i.e. $z_t \in \{1, 2, \dots, N\}$. The transition probabilities are given by $p_{ij} = \Pr(z_{t+1} = j \mid z_t = i)$, and the matrix of transition probabilities is given by

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{1N} \\ p_{N1} & p_{NN} \end{pmatrix},$$

which allows us to write the Bellman's equation as

$$V(k, z = z^i) = \max_{k'} \Big\{ u\Big[f\big(k, z^i\big) + (1-\delta)k - g(k, z^i)\Big] + \beta \sum_{j=1}^{n_z} p_{ij} V\Big[g\big(k, z^i\big), z^j\Big] \Big\}.$$

Going back to our specific problem at hand, where we have $\{z_t\} : z_{t+1} = \rho z_t + \epsilon_{t+1}$ with $\epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$ and $|\rho| < 1$. We approximate this by a discrete two-state Markov chain using $N = 2$, $z_1 = -\sigma_z$, $z_2 = \sigma_z$, where σ_z denotes the standard deviation of z, which in our case is given by the square-root of the second moment

$$\sigma_z^2 = \mathbb{E} z^2 = \frac{\sigma_\epsilon^2}{1-\rho^2}.$$

Now assume that

$$p_{ii} \equiv \phi = \frac{1+\rho}{2}, \ p_{ij} \equiv 1 - \phi = \frac{1-\rho}{2},$$

such that

$$\mathbf{P} = \begin{pmatrix} \phi & 1-\phi \\ 1-\phi & \phi \end{pmatrix}.$$

Remark 5.4 (Tauchen's goal)**.** Choose the elements of $\mathbf{P}$ such that the first two moments and the autocorrelation structure coincide with the underlying stochastic process:

$$\mathbb{E}z_t^d = \mathbb{E}z_t = 0, \tag{5.2a}$$
$$\mathbb{E}\{(z_t^d)^2\} = \mathbb{E}z_t^2 = \sigma_z^2, \tag{5.2b}$$
$$\frac{\mathbb{E}z_t^d z_{t-\tau}^d}{\mathbb{E}\{(z_t^d)^2\}} = \rho^\tau = \frac{\mathbb{E}z_t z_{t-\tau}}{\mathbb{E}z_t^2},\ \tau = 0, 1, 2, \dots \tag{5.2c}$$

Back in our example, we need to check whether or not these conditions are fulfilled. First we note that $\mathbb{P}\mathrm{r}(z_t^d = z^1) = \mathbb{P}\mathrm{r}(z_t^d = z^2) = 0.5$. Now, using this we verify the three conditions:

$$\mathbb{E}z_t^d = 0.5 \cdot (-\sigma_z) + 0.5 \cdot \sigma_z = 0, \tag{5.3a}$$
$$\mathbb{E}\{(z_t^d)^2\} = 0.5\mathbb{E}\{(z^1)^2\} + 0.5\mathbb{E}\{(z^2)^2\} = \sigma_z^2, \tag{5.3b}$$
$$\mathbb{E}\{z_t^d z_{t-1}^d\} = \mathbb{E}\big\{\mathbb{E}\{z_t^d z_{t-1}^d \mid z_{t-1}^d\}\big\} = \mathbb{E}z_{t-1}^d \mathbb{E}\{z_t^d \mid z_{t-1}^d\}. \tag{5.3c}$$

The last one still need to be verified:

$$\begin{aligned}\mathbb{E}\{z^1[\phi z^1 + (1-\phi)z^2] + z^2[(1-\phi)z^1 + \phi z^2]\} &= 0.5z^1[\phi z^1 + (1-\phi)z^2] + 0.5[(1-\phi)z^1 + \phi z^2] \\ &= 0.5\phi(z^1)^2 + 0.5\phi(z^2)^2 + 0.5 \cdot 2(1-\phi)z^1 z^2 \\ &= \phi\sigma_z^2 + (1-\phi)\sigma_z^2 = (2\phi - 1)\sigma_z^2 = \rho\sigma_z^2.\end{aligned}$$

Remark 5.5. While our discretized version of the continuous stochastic process matches the continuous counterpart in the first two moments and the autocorrelation structure, it performs badly in other aspects. Generally it is somewhat difficult to approximate the integral over a normal distribution well, but with newer numerical methods it becomes less computationally expensive to improve upon this result. We can increase the number of states, but of course this also comes with a computational burden.

Note 5.1.

$$\mathbf{P} = \begin{pmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{pmatrix}, \mathbf{P}' = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix}$$

We have the matrices of eigenvectors as

$$\sigma(\mathbf{P}) = \begin{pmatrix} e_{1,1} & e_{2,1} \\ e_{1,2} & e_{2,2} \end{pmatrix}, \sigma(\mathbf{P}) = \begin{pmatrix} e_{2,2} & -e_{1,2} \\ -e_{2,1} & e_{1,1} \end{pmatrix}.$$

This relationship only seems to hold for 2×2 matrices.

5.2 Linear Difference Equations

Definition 5.4 (Stochastic Process)**.** A *stochastic process* is a collection of random variables indexed by a variable t. We require t to be an integer, i.e. we assume discrete time. Then, the stochastic process y_t is a collection of random variables $\dots y_{-1}, y_0, y_1, \dots$, where we have a random variable for each point in time.

We are interested in the probability distribution of any such sequence which is relevant to us, for example a time-series of the GDP. A single drawing of a sequence $\{y_t^1\}$ is called a particular realization of the stochastic process y_t. Our goal is to characterize the probability law governing he collection of RV's that make up the stochastic process by a list of means of y_t and the covariances between y's at different t's. But why is this? For one, we want to analyze the stability of whatever we are investigating. We usually try to find empirical regularities as they allow us to make some more general statements.

Remark 5.6. The difference between economic historians and macroeconomics is broadly described by historians looking at the why and how things happened for some particular historic episode, while macroeconomists are generally concerned with general validity of models - regardless of the particular episode - and the resulting policy implications.

Say we have a $n \times 1$ vector of random variables $\mathbf{x}_t \in \mathbb{R}^n$.

Example 5.3. Let $\{y_t^1\}_{t=-\infty}^{\infty}$ be a realization of a time series process. We construct a vector $\{x_t^1\}$ associated with time t. This vector is made up of the $j+1$ recent observations on $\mathbf{y}$ at time t.

$$\mathbf{x}_t^1 = \begin{pmatrix} y_t^1 \\ y_{t-1}^1 \\ \vdots \\ y_{t-j}^1 \end{pmatrix}.$$

We can view $\{y_t^1\}_{t=-\infty}^{\infty}$ as generating *one* particular value of vector $\mathbf{x}_t$. We calculate the probability distribution of the vector $\mathbf{x}_t^i$ across realizations i. This is called the *joint distribution* of random variables $(Y_t, Y_{t-1}, \dots, Y_{t-j})$. The joint density is denoted by

$$f_{Y_t, Y_{t-1}, \dots, Y_{t-j}}(y_t, y_{t-1}, \dots, y_{t-j}).$$

Using this distribution we can calculate the jth auto-covariance of Y_t, denoted γ_{jt}:

$$\gamma_{jt} = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} (y_t - \mu_t)(y_{t-j} - \mu_{t-j}) \times f_{Y_t, Y_{t-1}, \dots, Y_{t-j}}(y_t, y_{t-1}, \dots, y_{t-j}) \, \mathrm{d}y_t \dots \mathrm{d}y_{t-j} = \mathbb{E}\big[(Y_t - \mu_t)(Y_{t-j} - \mu_{t-j})\big]$$

Note that $\mu_t = \mathbb{E}Y_t$, i.e. it denotes the unconditional mean. Furthermore, the auto-covariance has the form of a covariance between two variables. If we set $j = 0$ this term trivially equals $\mathbb{V}\mathrm{ar}Y_t$.

We consider a random vector $\mathbf{x}_t \in \mathbb{R}^n$, and a first-order vector difference equation (*vector AR(1) representation*)

$$\mathbf{x}_t = \mathbf{A}_0 \mathbf{x}_{t-1} + \mathbf{C}\boldsymbol{\omega}_t, \ t = 0, 1, \dots, \tag{1}$$

where $\mathbf{x}_0$ is a vector of initial conditions, $\mathbf{A}_0$ is a $n \times n$ matrixx, $\mathbf{C}$ is a $n \times m$ matrix, and $\boldsymbol{\omega}_t$ is a $m \times 1$ vector of random processes. We now make a few general assumptions on these components:

(i) $\mathbf{x}_0 \sim \mathcal{N}(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$

(ii) $\mathbf{x}' \mid \mathbf{x} \sim \mathcal{N}(\mathbf{A}_0\mathbf{x}, \mathbf{C}\boldsymbol{\Sigma}\mathbf{C}')$, the transition probability density with

$$\pi(\mathbf{x}' \mid \mathbf{x}) \geq 0 \forall \mathbf{x}, \ \sum_{x'} \pi(\mathbf{x}' \mid \mathbf{x}) \, \mathrm{d}\mathbf{x}' = 1.$$

Now we start with more precise assumptions:

(I) $\boldsymbol{\omega}_t$ is an i.i.d. process satisfying $\boldsymbol{\omega} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$,

(II) Less strict alternative: $\boldsymbol{\omega}_t$ is an $m \times 1$ random vector with

$$\mathbb{E}[\boldsymbol{\omega}_t \mid \mathcal{I}_{t-1}] = \mathbf{0} \tag{2a}$$

$$\mathbb{E}[\boldsymbol{\omega}_t \boldsymbol{\omega}' \mid \mathcal{I}_{t-1}] = \boldsymbol{\Sigma}, \tag{2b}$$

where $\mathcal{I}_{t-1}$ denotes the period-specific information set capturing all relevant information.

(III) In addition to the first moment being zero, it holds that $\forall t$

$$\mathbb{E}[\boldsymbol{\omega}_t \boldsymbol{\omega}_\tau] = \begin{cases} \sigma_{\boldsymbol{\omega}}^2 & \text{if } t = \tau, \\ 0 & \text{if } t \neq \tau. \end{cases}$$

A process satisfying (III) is called **white noise**. From now on we assume for all error terms that they are white noise processes.

Consider the **linear state-space system**

$$\mathbf{x}_{t+1} = \mathbf{A}_0\mathbf{x}_t + \mathbf{C}\boldsymbol{\omega}_{t+1} \tag{3a}$$

$$\mathbf{y}_t = \mathbf{G}\mathbf{x}_t \tag{3b}$$

(3a) describes the **state equation**, it describes how the state variables evolve over time. Within our framework, it describes the law of motion for the capital stock or the TFP. (3b) is called the **observational equation**, it describes how variables other than the state variables are linked to the state vector. For example, the private consumption, in the functional form which describes how it depends on the capitals stock, is linked to a vector of state variables.

Example 5.4. We consider a second-order vector difference equation with a constant α:

$$z_{t+1} = \alpha + \rho_1 z_t + \rho_2 z_{t-1} + \omega_{t+1}.$$

We can rewrite this as

$$\begin{pmatrix} z_{t+1} \\ z_t \\ 1 \end{pmatrix} = \begin{pmatrix} \rho_1 & \rho_2 & \alpha \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_t \\ z_{t-1} \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \omega_{t+1},$$
$$z_t = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_t \\ z_{t-1} \\ 1 \end{pmatrix},$$

which is a linear state-space system.

Example 5.5. Let $\mathbf{z}_t$ be a $n \times 1$ vector of random variables. We represent a VAR process of order four, when cutting away the observational equation, as

$$\mathbf{z}_{t+1} = \sum_{j=1}^{4} \mathbf{A}_j \mathbf{z}_{t+1-j} + \mathbf{C}_y \boldsymbol{\omega}_{t+1}. \tag{4}$$

We can represent (4) as a linear state space system:

$$\begin{pmatrix} \mathbf{z}_{t+1} \\ \mathbf{z}_t \\ \mathbf{z}_{t-1} \\ \mathbf{z}_{t-2} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 & \mathbf{A}_4 \\ \mathbb{I}_n & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{I}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{I}_n & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{z}_t \\ \mathbf{z}_{t-1} \\ \mathbf{z}_{t-2} \\ \mathbf{z}_{t-3} \end{pmatrix} + \begin{pmatrix} \mathbf{C}_y \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \boldsymbol{\omega}_{t+1}, \tag{5}$$

where $\mathbf{A}_j$ is a $n \times n$ matrix and $\mathbf{z}_0' = \begin{pmatrix} \mathbf{z}_0 & \mathbf{z}_{-1} & \mathbf{z}_{-2} & \mathbf{z}_{-3} \end{pmatrix}$ is a vector of initial conditions.

Consider the random variable X_t.

Definition 5.5 (Covariance-stationarity)**.** If neither the mean μ_t nor the autocovariance γ_{jt} depend on the date t, then the process for X_t is said to be **covariance-stationary**, or **weakly stationary**:

$$\mathbb{E}X_t = \mu \forall t,$$
$$\mathbb{E}(X_t - \mu)(X_{t-j} - \mu) = \gamma_j \forall t, \text{ any } j.$$

This means that the sequence of auto-covariance matrices depends on the separation between dates, but not the date itself.

Definition 5.6 (Stable Matrix)**.** A square real-valued matrix $\mathbb{A}$ is **stable** if all of its eigenvalues have real parts that lie *inside* the unit circle.

We take a special form of (3):

$$\begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{A} \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{C} \end{pmatrix} \omega_{t+1}. \tag{6}$$

$\tilde{\mathbf{A}}$ is a stable matrix, i.e. the only solution to $(\tilde{\mathbf{A}} - \mathbb{I})\boldsymbol{\mu}_2 = 0$ is given by $\boldsymbol{\mu}_2 = \mathbf{0}$. In other words, 1 is *not* an eigenvalue of $\tilde{\mathbf{A}}$. The matrix given by

$$\mathbf{A}_0 = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{A} \end{pmatrix}$$

has one eigenvector associated with a single unit eigenvalue. Alternatively,

$$(\mathbf{A}_0 - 1\mathbb{I}) \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{A} - 1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

implies that $\begin{pmatrix} \mu_2 & \mu_1 \end{pmatrix}'$ with $\mu_2 = 0$ gives μ_1 being any arbitrary scalar that is the eigenvector of matrix $\mathbf{A}_0$ associated with the unit eigenvalue.

Note that the first equation $x_{1,t+1} = x_{1,t} + 0 \cdot \omega_{t+1}$ in (6) implies that $x_{1,t+1} = x_{1,0} \forall t \geq 0$. Then we can describe the following result:

The initial condition $x_{1,0}$ helps pin down a particular eigenvector $\begin{pmatrix} x_{1,0} & 0 \end{pmatrix}'$ of $\mathbf{A}_0$. We use this eigenvector as the unconditional mean of $\mathbf{x}$ that renders the stochastic process covariance stationary.

Remembering our goal of finding the first and second moment of the random vector $\mathbf{x}_t$, we assume that the initial condition $\mathbf{x}_0 \sim \mathcal{N}(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$, where $\boldsymbol{\Sigma}_0 = \mathbb{E}(\mathbf{x} - \boldsymbol{\mu}_0)(\mathbf{x} - \boldsymbol{\mu}_0)'$. If we now reconsider

$$\mathbf{x}_t = \mathbf{A}_0 \mathbf{x}_{t-1} + \mathbf{C}\boldsymbol{\omega}_t, \; t = 0, 1, \ldots. \tag{1}$$

We can take the unconditional expectation on both sides and obtain

$$\boldsymbol{\mu}_t = \mathbf{A}_0 \boldsymbol{\mu}_{t-1}. \tag{7}$$

We make the assumption that all eigenvalues of $\mathbf{A}_0$ lie strictly within the unit circle, expect for possible the one associated with a constant term. Then $\mathbf{x}_t$ has a **stationary** mean, i.e. $\boldsymbol{\mu}_t = \boldsymbol{\mu}_{t-1} = \boldsymbol{\mu}$. From (7) it follows that

$$(\mathbf{A}_0 - 1 \cdot \mathbb{I})\boldsymbol{\mu} = 0.$$

From (1) and (7) we have the implication that

$$\mathbf{x}_{t+1} - \boldsymbol{\mu}_{t+1} = \mathbf{A}_0(\mathbf{x}_t - \boldsymbol{\mu}_t) + \mathbf{C}\boldsymbol{\omega}_{t+1}.$$

The stationary variance matrix is given by

$$\mathbb{E}(\mathbf{x}_{t+1} - \boldsymbol{\mu})(\mathbf{x}_{t+1} - \boldsymbol{\mu})' = \mathbf{A}_0 \mathbb{E}\big[(\mathbf{x}_t - \boldsymbol{\mu})(\mathbf{x}_t - \boldsymbol{\mu})'\big]\mathbf{A}_0' + \mathbf{C}\mathbb{E}[\boldsymbol{\omega}_{t+1}\boldsymbol{\omega}_{t+1}'],$$

or

$$\mathbf{C}_x(0) = \mathbf{A}_0 \mathbf{C}_x(0) \mathbf{A}_0' + \mathbf{C}\boldsymbol{\Sigma}\mathbf{C}'.$$

If we want to compute the auto-covariance sequence, we can do so by taking

$$\mathbf{x}_{t+j} - \boldsymbol{\mu}_{t+j} = \mathbf{A}_0^j(\mathbf{x}_t - \boldsymbol{\mu}_t) + \mathbf{C}\boldsymbol{\omega}_{t+j} + \cdots + \mathbf{A}_0^{j-1}\mathbf{C}\boldsymbol{\omega}_{t+1}$$

then multiplying it by $(\mathbf{x}_t - \boldsymbol{\mu}_t)'$ and then taking the unconditional expectations. This yields the auto-covariance sequence

$$\mathbf{C}_x(j) = \mathbf{A}_0^j \mathbf{C}_x(0).$$

The lag operator is defined as

$$LX_t = X_{t-1},$$

and allows for

$$L^n X_t = X_{t-n},\ n = \dots, -2, -1, 0, 1, 2, \dots.$$

Multiplying a variable X_t by the lag operator L^n gives the value of X shifted back n periods. If we have $n < 0$ then we have a shift forwards by $-n$ periods. The lag operator is considered a *mapping* in the sense that if we operate on the sequence $\{X_t\}$ with L^n to obtain a new sequence $\{y_t\}_{t=-\infty}^{\infty} = \{X_{t-n}\}_{t=-\infty}^{\infty}$ the operator L^n maps one sequence into another one.

We often multiply $(1 - \lambda L)^{-1}$ by X_t in order to obtain the value for a converging sum:

$$(1 - \lambda L)^{-1} X_t = (1 + \lambda L + \lambda^2 L^2 + \dots) X_t = \sum_{i=0}^{\infty} \lambda^i X_{t-i}.$$

Note that this requires the series to be converging, i.e. we exclude the case of a unit root and explosive processes. In other words, we require $|\lambda| < 1$. Why are we doing this? We are interested in **impulse response functions**.

Assume the case discussed earlier where the eigenvalues of $\mathbf{A}_0$ not associated with a constant are bounded from above in modulus by unity. We then rewrite (1) using the lag operator as

$$(\mathbb{I} - \mathbf{A}_0 L)\mathbf{x}_{t+1} = \mathbf{C}\boldsymbol{\omega}_{t+1}, \tag{8a}$$

and then

$$\mathbf{x}_{t+1} = (\mathbb{I} - \mathbf{A}_0 L)^{-1} \mathbf{C}\boldsymbol{\omega}_{t+1}. \tag{8b}$$

Now, using $(\mathbb{I} - \mathbf{A}_0 L)^{-1} = (\mathbb{I} + \mathbf{A}_0 L + \mathbf{A}_0^2 L^2 + \dots)$, we can apply this to both sides to obtain the MA(∞) representation

$$\mathbf{x}_{t+1} = \sum_{j=0}^{\infty} \mathbf{A}_0^j \mathbf{C}\boldsymbol{\omega}_{t+1-j}, \tag{9}$$

which solves (1) and expresses the invertible AR(1) process in the MA(∞) form. From this, we have the **impulse response** of lag j being defined as

$$h_j = \mathbf{A}_0^j \mathbf{C} = \frac{\partial}{\partial \boldsymbol{\omega}_{t-j}} \mathbf{x}_t.$$

As a motivating example, consider the Keynesian spending multiplier. Assume the government increases spending in period $t - j$, what happens in period t?

6 Numerical Solution Methods: Perturbation Methods

Perturbation is defined as

> [..] a deviation of a system, moving object, or process from its regular or normal state or path, caused by an outside influence.

In other words, we are talking about deviations from the steady-state. We once again consider the solving procedure of the one-sector Ramsey growth model.

Remark 6.1 (First Procedure)**.** The *deterministic* version reduces to a 2^{nd} order non-linear difference equation. This equation is not trivially solved, we do not have a general closed-form solution, so we need to either resort to numerical methods, or to very special cases.

The Lagrangain from the planner's problem with inelastic full employment is given by

$$\mathfrak{L}(c_t, k_{t+1}, \lambda_t) = \sum_{t=}^{\infty} \beta^t \Big\{ u(c_1, 1) + \lambda_t[(k_t) + (1-\delta)k_t - k_{t+1} - c_t] \Big\}.$$

After taking the FONCs, the last of which simply gives the feasibility constraint, we can consolidate the first two FONCs to obtain

$$u'(c_t) = \beta u'(c_{t+1})[f'(k_{t+1}) + (1-\delta)] \forall t \geq 1,$$
$$c_t \leq f(k_t) + (1-\delta)k_t - k_{t+1} \forall t \geq 0.$$

This can be even further reduced by plugging in the consumption values for the second term holding with equality:

$$u'\Big(f(k_t) + (1-\delta)k_t - k_{t+1}\Big) = \beta u'\Big(f(k_{t+1}) + (1-\delta)k_{t+1} - k_{t+2}\Big)\big[f'(k_{t+1}) + (1-\delta)\big]$$

Note that this holds in expectation in the stochastic case. But since it only holds in expectation, we can define the **Euler error**

$$v_t = u'\Big(f(k_t) + (1-\delta)k_t - k_{t+1}\Big) - \beta u'\Big(f(k_{t+1}) + (1-\delta)k_{t+1} - k_{t+2}\Big)\big[f'(k_{t+1}) + (1-\delta)\big] \qquad (*)$$

That being said, as mentioned in an earlier section, we have a special case if we assume $\delta = 1$, logarithmic utility, and Cobb-Douglas production. This was shown by Brock and Mirman (*JET* 1972) to yield an analytical solution for the Euler equation (the one before (*)):

$$k_{t+1} = \alpha\beta k_t^{\alpha}$$

or

$$\log k_{t+1} = \log \alpha\beta + \alpha \log k_t \equiv g(k_t),$$

given some initial capital stock k_0.

Remark 6.2 (Planner's stochastic dynamic optimization problem)**.** The social planner seeks to maximize the period-zero expected discounted lifetime utility

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \qquad (1)$$

with $\beta \in (0,1)$, subject to a resource constraint and the law of motion for the TFP:

$$k_{t+1} = z_t k_t^{\alpha} + (1-\delta)k_t - c_t \qquad (2)$$

$$z_{t+1} = \rho \cdot z_t + \epsilon_t, \qquad (3)$$

where we assume $\alpha, \rho \in (0,1)$ and $\epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$. We can express this problem recursively in two ways:

$$V(k_t, \epsilon_t) = \max_{c_t, k_{t+1}} \Big\{ u(c_t) + \beta \mathbb{E}_t\big[V(k_{t+1}, \epsilon_{t+1}) \mid \epsilon_t\big] \Big\} \text{ subject to (2) and (3),}$$

or alternatively

$$V(k, z) = \max_{k'} \Big\{ u(zk^{\alpha} + (1-\delta)k - k') + \beta \mathbb{E}\big[V(k', z') \mid z\big] \Big\} \text{ subject to (3).}$$

The takeaway here is that solving the dynamic stochastic model is equivalent to finding a time-invariant function $k' = g(k, z)$ that links today's state to tomorrow's state.

What are we learning from DSGE models when we expose them to exogenous shocks? Basically how agents adjust their behavior and how we return to the equilibrium. The social planner is a shortcut for the market economy. The novelty from the DSGE model, and using impulse responses, is that it gives us insights into how consumers react, how prices react, etc.. **Interactions** are the added value.

So what is the plan from here on? Study methods designed to help find such a function.

In any linearization approach we have five steps:

1. Find necessary equations characterizing the equilibrium, i.e. constraints, FONCs, etcs.
2. Pick parameters and find a steady state.
3. Linearize necessary equations characterizing the equilibrium of the system to make equations approximately linear in the deviations from the steady state.
4. Solve for the recursive equilibrium law of motion using one of the linearization methods.
5. Analyze the solution via impulse-response analysis and second-order-properties, possible taking into account some filtering of the data. This can be done without simulating the model.

Steps 1 and 2 are easy and we have done them quite often so far. So we are focusing on two methods which are used in step 3 and 4. As a first step, always find the deterministic steady state.

Step 3: replace the **Euler error** v by V, where V is the first-order Taylor series expansion of v around the steady-state of the system $\boldsymbol{\theta} = [k, z]$. This comes from the stochastic equivalent to (*). The Euler equations in deterministic steady-states always hold exactly, but in the stochastic case we may have small deviations, these deviations are called the Euler error. Thus the true Euler error is given by v and the approximation is denoted by V. Generally, using slightly sloppy notation, the vector valued Taylor approximation of first-order is given by

$$F(\boldsymbol{\theta}_t) = f(\boldsymbol{\theta}) + f'(\boldsymbol{\theta}_t - \boldsymbol{\theta}) + \boldsymbol{\xi}_t,$$

where $\boldsymbol{\xi}_t$ is some approximation error that occurs. In our case we have

$$V(\boldsymbol{\theta}_t) = v(\boldsymbol{\theta}) + v'(\boldsymbol{\theta})(\boldsymbol{\theta}_t - \boldsymbol{\theta}).$$

Note that $\mathbb{E}v(\boldsymbol{\theta}) = 0$, so we have the one-sector stochastic growth model Euler error approximation given by

$$V = V_1(k_t - k) + V_2(k_{t+1} - k) + V_3(k_{t+2} - k) + V_4(z_t - z) + V_5(z_{t+1} - z) + \eta_{t+1},$$

where

$$V_1 = \frac{\partial}{\partial k_t} v \bigg|_{k_{t+i}=k, z_{t+i}=z} \forall i \in \{0, 1, 2\}.$$

k satisfies $v(k, k, k, z, z) = 0$ and the Taylor residual is $\mathbb{E}_t \eta_{t+1} = 0$.

Step 4: find $\hat{g}$ such that

$$\mathbb{E}_t V\Big(k_t, \hat{g}(k_t, z_t), \hat{g}\big(\hat{g}(k_t, z_t), z_{t+1}\big), z_t, z_{t+1}\Big) = 0 \forall k_t, z_t.$$

There are two different linearization methods yielding similar results. Firstly the method of undetermined coefficients, and secondly the state-space approach.

6.1 Method of Undetermined Coefficients

We postulate the existence of a $\hat{g}$ which is close to the true g. We make a guess based on the assumption that g is **linear** in the state variables. Then we have

$$k_{t+1} - k = \hat{g}(k_t, z_t) = \beta_1(k_t - k) + \beta_2(z_t - z), \tag{6.1}$$
$$z_{t+1} - z = \rho(z_t - z) + \epsilon_{t+1}. \tag{6.2}$$

So now we need to find coefficients β_1 and β_2 which satisfy the condition that the expected Euler error is zero. To find that out, plug our guess into the equation for the linearized Euler error. This yields the expression

$$\begin{aligned} V = {} & V_1(k_t - k) + V_2\big(\beta_1(k_t - k) + \beta_2(z_t - z)\big) + V_3\big(\beta_1^2(k_k - k) + \beta_1\beta_2(z_t - z) + \beta_2(\rho[z_t - z] + \epsilon_{t+1})\big) \\ & + V_4(z_t - z) + V_5\big(\rho(z_t - z) + \epsilon_{t+1}\big) + \eta_{t+1}. \end{aligned}$$

Assume that the error term ϵ_{t+1} is uncorrelated with all previous terms. At any given time t we have $\mathbb{E}_t\epsilon_{t+1} = 0$. We can take the **expected Euler error** and collect terms such that

$$\mathbb{E}_t[V \mid k_t, z_t] = \tilde{\beta}_1(k_t - k) + \tilde{\beta}_2(z_t - z),$$

where $\tilde{\beta}_1 = \tilde{\beta}_1(\beta_1, \beta_2)$ and $\tilde{\beta}_2 = \tilde{\beta}_2(\beta_1, \beta_2)$. Since we require that the conditional expectation of the Euler error is zero for all realizations z_t and k_t, it follows that

$$\tilde{\beta}_1(\beta_1, \beta_2) = 0 \text{ and } \tilde{\beta}_2(\beta_1, \beta_2) = 0$$

We get **two** solutions for β_1, β_2 satisfying the requirements. Note that bigger shocks mean that the approximation error gets bigger, thus our approximation gets worse. We have two solutions since our method generates a *characteristic quadratic equation* in one of the two parameters (the first one). We then get to choose between the two solutions. As we want a stable solution, we pick the one that corresponds to a *stable root* of the equation, i.e. we choose the one for which $|\beta_1| \leq 1$.

We use these in the two equations (6.1) and (6.2), and use the steady state values as the initial values k_0 and z_0.

6.2 The State-Space Approach

This approach follows the method developed by Blanchard and Kahn (*Econometrica* 1980). We rewrite the linearized model in the **state-space form**, where $\mathbf{Y}_t$ is the period t state-space vector, more precisely, the deviation from the steady-state. Furthermore we have the error vector $\boldsymbol{\mu}_t$ with $\mathbb{E}_t\boldsymbol{\mu}_{t+1} = \mathbf{0}$.

$$\mathbf{Y}_t = \begin{pmatrix} k_{t+1} - k \\ k_t - k \\ z_t - z \end{pmatrix}, \; \mathbf{Y}_{t+1} = \begin{pmatrix} k_{t+2} - k \\ k_{t+1} - k \\ z_{t+1} - z \end{pmatrix}, \; \boldsymbol{\mu}_{t+1} = \begin{pmatrix} \eta_{t+1} \\ 0 \\ \epsilon_{t+1} \end{pmatrix}$$

We want a system of linearized equations and find 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ such that it holds that

$$\mathbf{A}\mathbf{Y}_{t+1} + \mathbf{B}\mathbf{Y}_t = \boldsymbol{\mu}_{t+1}$$

We now have three equations which will be represented in this system:

$$\begin{aligned} V_1(k_t - k) + V_2(k_{t+1} - k) + V_3(k_{t+2} - k) + V_4(z_t - z) + V_5(z_{t+1} - z) &= \eta_{t+1}, \\ (k_{t+1} - k) - (k_{t+1} - k) &= 0, \\ (z_{t+1} - z) - \rho(z_t - z) &= \epsilon_{t+1}. \end{aligned}$$

This implies

$$\mathbf{A} = \begin{pmatrix} V_3 & V_2 & V_5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; \mathbf{B} = \begin{pmatrix} 0 & V_1 & V_4 \\ -1 & 0 & 0 \\ 0 & 0 & -\rho \end{pmatrix},$$

and then we can rewrite the original equaiton as

$$\mathbf{Y}_{t+1} = -\mathbf{A}^{-1}\mathbf{B}\mathbf{Y}_t + \mathbf{A}^{-1}\boldsymbol{\mu}_{t+1} \equiv \boldsymbol{\Pi}\mathbf{Y}_t + \mathbf{A}^{-1}\boldsymbol{\mu}_{t+1}$$

It is important to note that for many interesting economies, such as those featuring money, the $\mathbf{A}$ matrix is not invertible. That is one of the reasons why the approach is difficult to get started.

An **issue** is that uniqueness of the decision rule $g(\cdot)$ needs to be guaranteed, but this may not be trivial. To motivate why this is not trivial, we consider two scenarios.

Example 6.1 ($\boldsymbol{\mu}_t = 0$; deterministic linearized system)**.** The **solution path** is given by

$$\begin{aligned}
\mathbf{Y}_1 &= \boldsymbol{\Pi}\mathbf{Y}_0, \\
\mathbf{Y}_2 &= \boldsymbol{\Pi}\boldsymbol{\Pi}\mathbf{Y}_0 \\
\vdots &= \vdots \\
\mathbf{Y}_t &= \boldsymbol{\Pi}^t Y_0.
\end{aligned}$$

In period zero, we cannot fully determine the initial vector $\mathbf{Y}_0$ since we do not know k_1 in period zero. There is **one degree of freedom**. Consequently, there is a continuum of $\mathbf{Y}_1$ vectors consistent with the optimality conditions described in the matrix equations above. For any k_1 we pick, there is a vector $(k_0,\ z_0)$ and thus a vector $\mathbf{Y}_0$ that is consistent with the above solution path.

Example 6.2 ($\mathbf{Y}_0$ known and $\boldsymbol{\mu}_t \neq 0$)**.** The general solution to the dynamic system equals $\mathbf{Y}_{t+1} = \boldsymbol{\Pi}\mathbf{Y}_t + \mathbf{A}^{-1}\boldsymbol{\mu}_{t+1}$. There still exist many solutions since the error term on the linearized deviations from the steady state technology is only restricted to be mean zero: $\mathbb{E}_t\eta_{t+1} = 0$, which is satisfied by many shock paths since the shock term is only uniquely pinned down *in expected value*, so we have no unique solution. Hence we need to find restrictions that help us pin down $\mathbf{Y}_0$ and η_{t+1}.

We may address the issue from example 6.1 by imposing a **transversality condition**:

$$\lim_{t\to\infty} \mathbb{E}_0\beta^t\mathbf{Y}_t = \mathbf{0}.$$

Does the condition guarantee a **unique** path of $\mathbf{Y}_0$? Does it eliminate all but one value of the first element of $\mathbf{Y}_0$? Yes, it works if the matrix $\boldsymbol{\Pi}$ has as many explosive roots as there are degrees of freedom in the vector $\mathbf{Y}_0$. In order to illustrate this, consider the following:

$$\begin{aligned}
\mathbf{Y}_1 &= \boldsymbol{\Pi}\mathbf{Y}_0 + \mathbf{A}^{-1}\boldsymbol{\mu}_1 \\
\mathbf{Y}_2 &= \boldsymbol{\Pi}^2\mathbf{Y}_0 + \boldsymbol{\Pi}^1\mathbf{A}^{-1}\boldsymbol{\mu}_1 + \mathbf{A}^{-1}\boldsymbol{\mu}_2 \\
\vdots &\vdots \\
\mathbf{Y}_t &= \boldsymbol{\Pi}^t\mathbf{Y}_0 + \boldsymbol{\Pi}^{t-1}\mathbf{A}^{-1}\boldsymbol{\mu}_1 + \boldsymbol{\Pi}^{t-2}\mathbf{A}^{-1}\boldsymbol{\mu}_2 + \cdots + \boldsymbol{\Pi}^0\mathbf{A}^{-1}\boldsymbol{\mu}_t,
\end{aligned}$$

in expectation the error terms go to zero vectors. And then we have

$$\mathbb{E}_0\beta^t\mathbf{Y}_t = (\boldsymbol{\Pi}\beta)^t\mathbf{Y}_0, \text{ or } \mathbb{E}_0\mathbf{Y}_t = \boldsymbol{\Pi}^t\mathbf{Y}_0.$$

This is because $\mathbb{E}_0\boldsymbol{\mu}_t = 0\forall t \geq 1$.

Remark 6.3 (Some Notes on Eigen-Decomposition (Spectral Decomposition))**.** Note that $\boldsymbol{\Pi}\mathbf{P} = \mathbf{P}\boldsymbol{\Lambda}$, where $\mathbf{P}$ denotes the matrix of right eigenvectors of $\boldsymbol{\Pi}$, and $\boldsymbol{\Lambda}$ denotes the corresponding matrix of eigenvalues.

$$\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \ \mathbf{P} = \begin{pmatrix} \mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3 \end{pmatrix}.$$

Multiplying the above equation by $\mathbf{P}^{-1}$ yields

$$\boldsymbol{\Pi} = \mathbf{P}\boldsymbol{\Lambda}\mathbf{P}^{-1}.$$

Similarly, using the matrix of left-eigenvectors we can write

$$\mathbf{P}^{-1}\mathbf{\Pi} = \mathbf{\Lambda}\mathbf{P}^{-1}$$

For our procedure to find a unique solution, we require the **left** eigenvectors of $\mathbf{\Pi}$. Matlab as standard gives out the *right* eigenvectors. It can be shown that the left eigenvectors of $\mathbf{\Pi}$ are equal to the right eigenvectors of $\mathbf{\Pi}'$, the transpose of $\mathbf{\Pi}$. Furthermore, we have

$$\mathbf{\Pi}^t = \mathbf{P}\mathbf{\Lambda}^t\mathbf{P}^{-1}.$$

From the transversality condition it then follows that

$$\begin{aligned} \mathbb{E}_0\mathbf{Y}_t &= \mathbf{\Pi}^t\mathbf{Y}_0 = \mathbf{P}\mathbf{\Lambda}^t\mathbf{P}^{-1}\mathbf{Y}_0, \\ \mathbf{E}_0\mathbf{P}^{-1}\mathbf{Y}_t &= \mathbf{\Lambda}^t\mathbf{P}^{-1}\mathbf{Y}_0, \end{aligned}$$

or more explicitly expressed as

$$\mathbf{E}_0\begin{pmatrix}\mathbf{P}_1^{-1}\mathbf{Y}_t\\ \mathbf{P}_2^{-1}\mathbf{Y}_t\\ \mathbf{P}_3^{-1}\mathbf{Y}_t\end{pmatrix} = \begin{pmatrix}\lambda_1^t & 0 & 0\\ 0 & \lambda_2^t & 0\\ 0 & 0 & \lambda_3^t\end{pmatrix}\begin{pmatrix}\mathbf{P}_1^{-1}\mathbf{Y}_0\\ \mathbf{P}_2^{-1}\mathbf{Y}_0\\ \mathbf{P}_3^{-1}\mathbf{Y}_0\end{pmatrix} = \begin{pmatrix}\lambda_1^t\mathbf{P}_1^{-1}\mathbf{Y}_0\\ \lambda_2^t\mathbf{P}_2^{-1}\mathbf{Y}_0\\ \lambda_3^t\mathbf{P}_3^{-1}\mathbf{Y}_0\end{pmatrix}.$$

By rotating the eigenvectors, we are left with three independent equations. This rotation helps us to see which restriction the requirement that $\beta^t\mathbf{Y}_t \to \mathbf{0}$ as $t \to \infty$ imposes on $\mathbf{Y}_0$.

Going back to our state-space approach, we consider some cases and evaluate whether or not these cases yield a **stable and unique** solution path. This is done by looking whether or not the transversality condition imposes any restrictions.

(i) All three eigenvalues lie inside the unit-root circle: $|\lambda_i| < 1 \forall i$. The transversality condition imposes no restriction, note that $\lambda_i^t \to 0$ implies that we have a **continuum of possible solutions**, so we have it satisfied by all possible paths.

(ii) All three eigenvalues lie outside of the unit-root circle: $|\lambda_i| > 1 \forall i$. There is **no stable solution**, no path for $\{\mathbf{Y}_t\}_{t=0}$ is consistent with the transversality condition as there will always be explosive behavior over time.

(iii) There is exactly one explosive root and two stable roots: $|\lambda_1| > 1,\ |\lambda_i| < 1,\ i = 2, 3$. There is a **unique solution** consistent with the transversality condition. We have exactly as many unstable eigenvectors (or associated eigenvalues) as our dynamic system has degrees of freedom. This rule can be generalized for non-one-sector stochastic growth models: if we had two degrees of freedom in a more complex model: two explosive roots needed.

So what good is this result in our search for the policy function? Assume λ_1 to be the explosive root, then we have

$$\mathbf{P}_1^{-1}\mathbf{\Pi} = \lambda_1\mathbf{P}_1^{-1}.$$

The transversality condition $\lim_{r\to\infty}\mathbb{E}_0\mathbf{Y}_t = \mathbf{0}$ imposes a restriction on the dynamic system:

$$\mathbf{P}_1^{-1}\mathbf{Y}_0 = \mathbf{0}.$$

This means that we need to choose an initial vector $\mathbf{Y}_0$, and therefore also k_1, such that the above condition is fulfilled. Only in this way we are able to get a stable solution path. Imposing the restriction gives us

$$\begin{aligned} \mathbf{P}_1^{-1} &= \begin{pmatrix}\alpha & \beta & \gamma\end{pmatrix}, \\ \mathbf{P}_1^{-1}\mathbf{Y}_0 &= \alpha(k_1 - k) + \beta(k_0 - k) + \gamma(z_0 - z) = 0. \end{aligned}$$

The second equation can be rearranged to

$$k_1 - k = \beta_1(k_0 - k) + \beta_2(z_0 - z)$$

where we have $\beta_1 = \beta\alpha^{-1}$ and $\beta_2 = \gamma\alpha^{-1}$. So now we have a time-invariant policy function which should give more stable, reliable forecast methods. We could do policy analysis where the policy change does not change the policy function.

Remark 6.4 (Closing Remarks)**.** What Lucas criticized was forecasting rules in old-fashioned macro models which change over time. Also old fashioned were rational expectations, which in our models mean that agents are assumed to know the underlying laws of motion, the laws of probability, the distribution of shocks, and just generally understand the workings of their environment. Looking ahead, we have *behavioral meets macro: expectation formation.*

What are the alternatives for rational expectation, and what do they imply for policy analysis? Rational expectations were the only way to close models at the time of the Lucas' critique. What do (or did) rational expectations help with? It is viewed it as powerful concept for two reasons: closing the model (in a consistent way such that no internal contradictions) and as a benchmark powerful enough for consistent predictions and then contrast reality against these predictions. This means we can test the model predictions. Many predictions were or are not far off, but details matter. This assumption does not imply that all agents in real-life form expectations rationally. Think about the following: many of us have changed universities, and that changed the Macro professors. This is an environment change, where we have new rules. What good would it have been to stick to the precise study methods for preparing exams? Rational expectations would mean that we are well advised to adapt to the new environment, to adjust our behavior. This illustrates the difference between adaptive expectations and rational expectations. If the environment changes and you do not adjust you will make losses would the proponent of rational expectations tell you.

Notes on Voluntary Exercises

These notes are in no way fully correct solutions, but rather talking points for presenting the proposed solution.

Exercise 1: AD setting with only initial endowment

H households and each has a time 0 discounted lifetime utility

$$\sum_{t=0}^{\infty} \beta^t u(c_t^h),$$

with $\beta \in (0,1)$, c_t^h denoting household h consumption at time t. Household endowment is $\omega_i \geq 0$ in period 0 with $\sum_i^H \omega_i = Y > 0$.

(a) Define a CE allocation in this setup

A competitive equilibrium is an allocation which maps from the initial endowment $\{\omega_i\}_{t=0}^{\infty}$ onto a sequence of consumption allocations $\{c_t^i\}_{t=0}^{\infty}$ in such a way that we maximize the lifetime utility subject to $\sum_{t=0}^{\infty} c_t^h \leq \omega_h$.

(b) Describe what AD securities are in this setup

We can view this world as a world of perfectly correlated determinate histories, only one good state (the initial state), and the entire future are bad states (no further endowments). Trading across households does not make sense, as the natural debt limit is of course ω_h for each household, and since each household faces the same determinant future and they know this, the household only trades with itself across time, and since they know the future, they set this trading at time zero already, where the price should simply be the marginal rate of intertemporal substitution of consumption.

(c) Give the Pareto weights in this setup

The Pareto weights should be $\lambda_i = \frac{\omega_i}{\sum_i^H \omega_i}$. This satisfies $\sum_i^H \lambda_i = 1$. If a social planner were to assign other weights, and as the households are identical (time preference rate and utility function do not differ). Since we have a fixed limit on the overall endowment of the economy, every household should be assigned the consumption good in the same extend as their proportion to the overall endowment is. If we shifted consumption from any one household to the other, one would be strictly worse off.

Exercise 2.1: Lucas tree model

We have two kinds of trees denoted $i = 1, 2$. The first kind is ugly and gives no direct utility but yields a fruit dividend denoted d_{1t}. d_{1t} is a random first-order Markov process. The second kind of tree is beautiful and gives direct utility plus a fruit dividend d_{2t} in each period. $\forall t : d_{1t} = d_{2t} = 0.5(d_{1t} + d_{2t})$. We have N individuals, and one tree of each kind for each consumer. The consumers maximize the expected discounted lifetime utility

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t, k_{2.t}),$$

where $\beta \in (0,1)$, $u(c_t, k_{2,t}) = \log c_t + \gamma \log k_{2,t}$, and $\gamma \geq 0$. $k_{2,t}$ is the stock of beautiful trees and $p_{i,t}$ denotes the price of a type i tree in the period t.

(a) Consumer problem in sequential form

The maximization problem is described by

$$V(c_0, k_{1,0}, k_{2,0}) = \log c_0 + \gamma k_{2,0} + \mathbb{E}_0 \Big[\beta(\log c_1 + \gamma \log k_{2,1}) + \beta^2(\log c_2 + \gamma \log k_{2,2}) + \dots \Big]$$

$$= \max_{\{c_t, k_{1,t+1}, k_{2,t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big[\log c_t + \gamma \log k_{2,t} \Big]$$

subject to the budget constraint sequence

$$\begin{aligned} c_t + k_{1,t+1}p_{1,t+1} + k_{2,t+1}p_{2,t+1} &\leq k_{1,t}p_{1,t} + k_{2,t}p_{2,t} + d_{1t}k_{1,t} + d_{2t}k_{2,t} \\ &= k_{1,t}(p_{1,t} + d_{1,t}) + k_{2,t}(p_{2,t} + d_{2,t}). \\ \text{given } c_0, k_{1,0}, k_{2,0}. \end{aligned}$$

The LHS can be viewed as the consumption plus the investment (for next period consumption and investment), the RHS is the dividend on the trees plus the selling of said trees (or rather their current value).

(b) Consumer problem in recursive form

We have the choice variables $c_t, k_{1,t+1}, k_{2,t+1}$ in each period $t \geq 0$ ($k_{1,t}$ and $k_{2,t}$ are determined in period $t-1$ and thus state variables). Other state variables are given by the dividends $d_{1,t}, d_{2,t}$ (exogenous). Writing the above equation recursively, we have

$$V(c_t, k_{1,t}, k_{2,t}) = \max_{c_t, k_{1,t+1}, k_{2,t+1}} \Big\{ \log c_t + \gamma \log k_{2,t} + \beta \mathbb{E}_t V(c_{t+1}, k_{1,t+1}, k_{2,t+1}) \}$$

subject to the same constraint sequence as above.

(c) Equilibrium price guess

We need to make a differentiation as we allow $\gamma \geq 0$. If $\gamma = 0$, the direct utility from the beautiful tree is zero, and since both trees yields the same fruit dividends the prices should be equal. For $\gamma > 0$, the utility derived from the beautiful tree consists of the direct utility from it being beautiful, and indirect utility from the fruit dividend. So overall, the relationship between the prices in each period should be

$$p_{1,t} \leq p_{2,t}.$$

Exercise 2.2: Dynamic Programming - Ramsey Problem

We consider a standard Ramsey problem of the form

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to

$$c_t + k_{t+1} \leq f(k_t)$$

with a given k_0. u and f satisfy the typical neoclassical properties on such functions.

(a) Change in notation and Bellman equation

We define $c_t = c$, $k_t = k$, and $k_{t+1} = k'$. Then the constraint can, at equality be written as $c = f(k) - k'$. We then have the Bellman equation, when substituting the rewritten constraint into the maximization problem:

$$V(k) = \max_{k'} \Big\{ u(f(k) - k') + \beta V(k') \Big\}$$

(b)-(c) Benveniste-Scheinkman formula and Euler condition

Due to optimality of the policy function $g(k)$, we have $V'(k) = u_1(k, g(k))$. Using our defined $u(\cdot)$, we have

$$V'(k) = \left[\frac{\partial}{\partial k}(f(k) - k')\right] u_k\big(f(k) - k'\big) = f_k(k) u_k\big(f(k) - k'\big)$$

Furthermore, we have

$$\frac{\partial}{\partial k'} u\big(f(k) - k'\big) = (-1) u_{k'}\big(f(k) - k'\big).$$

See Remark 3.6 on page 30 for the derivation.