Dynamic Macroeconomics with Numerics: Project II

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One-Sector Stochastic Growth Model

Notes on optimality conditions

We have

$$\mathbf{X}_{t+1} = \begin{pmatrix} c_{t+1} \\ k_{t+1} \\ U_{t+1} \\ z_{t+1} \end{pmatrix} = f(\mathbf{X}_t) = \begin{pmatrix} c_t\beta\Big\{\alpha(\rho x_t+\epsilon_{t+1})^{1-\alpha}U_{t+1}^{\alpha}k_{t+1}^{1-\alpha}+1-\delta U_{t+1}^{\phi}\Big\} \\ k_t-\delta U_t^{\phi}k_t+i_t \\ \Big\{(\delta\phi)^{-1}\alpha\exp(\rho x_t+\epsilon_{t+1})^{1-\alpha}(k_t-\delta U_t^{\phi}k_t+i_t)^{\alpha-1}\Big\}^{\frac{1}{\phi-\alpha}} \\ \rho x_t+\epsilon_{t+1} \end{pmatrix}$$

We now express all terms in terms of past values:

$$\begin{aligned}
z_{t+1} &= \exp(\rho x_t+\epsilon_{t+1}) \\
k_{t+1} &= (1-\delta U_t^{\phi})k_t + z_t^{1-\alpha}k_t^{\alpha}U_t^{\alpha} - \beta c_t\Big\{\alpha U_t z_t^{1-\alpha}k_t^{\alpha-1}U_t^{\alpha}+1-\delta U_t^{\phi}\Big\} \\
&= \left(1-\left\{\frac{\alpha}{\delta\phi}z_t^{1-\alpha}k_t^{\alpha-1}\right\}^{\frac{1}{\phi-\alpha}}\right)k_t + z_t^{1-\alpha}k_t^{\alpha}U_t^{\alpha} - \beta c_t\Big\{\alpha U_t z_t^{1-\alpha}k_t^{\alpha-1}U_t^{\alpha}+1-\delta U_t^{\phi}\Big\} \\
&= k_t - k_t^{\frac{\phi-1}{\phi-\alpha}}\left(\frac{\alpha}{\delta\phi}\right)^{\frac{1}{\phi-\alpha}}z_t^{\frac{1-\alpha}{\phi-\alpha}} + z_t^{1-\alpha}k_t^{\alpha}U_t^{\alpha} - \beta c_t\Big\{\alpha U_t z_t^{1-\alpha}k_t^{\alpha-1}U_t^{\alpha}+1-\delta U_t^{\phi}\Big\} \\
U_{t+1} &= \left(\frac{\alpha}{\delta\phi}\right)^{\frac{1}{\phi-\alpha}}\exp(\rho x_t+\epsilon_{t+1})^{\frac{1-\alpha}{\phi-\alpha}}\left(k_t - k_t^{\frac{\phi-1}{\phi-\alpha}}\left(\frac{\alpha}{\delta\phi}\right)^{\frac{1}{\phi-\alpha}}z_t^{\frac{1-\alpha}{\phi-\alpha}} + z_t^{1-\alpha}k_t^{\alpha}U_t^{\alpha} - \beta c_t\Big\{\alpha U_t z_t^{1-\alpha}k_t^{\alpha-1}U_t^{\alpha}+1-\delta U_t^{\phi}\Big\}\right)^{\frac{\alpha-1}{\phi-\alpha}} \\
c_{t+1} &= c_t\beta\Big\{\alpha(\rho x_t+\epsilon_{t+1})^{1-\alpha}U_{t+1}^{\alpha}k_{t+1}^{1-\alpha}+1-\delta U_{t+1}^{\phi} \\
i_{t+1} &= z_{t+1}^{1-\alpha}k_{t+1}^{\alpha}U_{t+1}^{\alpha} - \beta c_t\Big\{\alpha U_{t+1}z_{t+1}^{1-\alpha}k_{t+1}^{\alpha-1}U_{t+1}^{\alpha}+1-\delta U_{t+1}^{\phi}\Big\} \\
y_{t+1} &= z_{t+1}^{1-\alpha}(k_{t+1}U_{t+1})^{\alpha}
\end{aligned}$$

Important here are the partial derivatives of the the first three terms w.r.t. all past terms:

$$\begin{aligned}
\frac{\partial}{\partial z_t}k_{t+1} &= -\frac{1-\alpha}{\phi-\alpha}k_t^{\frac{\phi-1}{\phi-\alpha}}\left(\frac{\alpha}{\delta\phi}\right)^{\frac{1}{\phi-\alpha}}z_t^{\frac{1-\phi}{\phi-\alpha}} + (1-\alpha)\left(\frac{k_tU_t}{z_t}\right)^{\alpha} - \beta c_t(1-\alpha)z_t^{-\alpha}U_t^{1+\alpha}k_t^{\alpha-1} \\
\frac{\partial}{\partial k_t}k_{t+1} &= 1-\frac{\phi-1}{\phi-\alpha}k_t^{\frac{\alpha-1}{\phi-\alpha}}\left(\frac{\alpha}{\delta\phi}\right)^{\frac{1}{\phi-\alpha}}z_t^{\frac{1-\alpha}{\phi-\alpha}} + \alpha z_t^{1-\alpha}k_t^{\alpha-1}U_t^{\alpha} - \beta c_t\alpha U_t^{1+\alpha}z_t^{1-\alpha}k_t^{\alpha-2} \\
\frac{\partial}{\partial U_t}k_{t+1} &= \alpha z_t^{1-\alpha}k_t^{\alpha}U_t^{\alpha-1} - \beta c_t\Big\{\alpha(1+\alpha)z_t^{1-\alpha}k_t^{\alpha-1}U_t^{\alpha} - \delta\phi U_t^{\phi-1}\Big\} \\
\frac{\partial}{\partial c_t}k_{t+1} &= \beta\Big\{\alpha z_t^{1-\alpha}k_t^{\alpha-1}U_t^{1+\alpha}+1-\delta U_t^{\phi}\Big\} \\
\frac{\partial}{\partial i_t}k_{t+1} &= 1 \\
\frac{\partial}{\partial y_t}k_{t+1} &= 1-\beta c_t(??)
\end{aligned}$$

as we have $k_{t+1} = [\dots] + y_t - \beta c_t\frac{\partial}{\partial k_t}y_t + [\dots]$????

Next we have

$$\begin{aligned}
\frac{\partial}{\partial z_t}U_{t+1} &= \left(\frac{\alpha}{\delta\phi}\right)^{\frac{1}{\phi-\alpha}}z_{t+1}^{\frac{1-\alpha}{\phi-\alpha}}\left\{(1-\alpha)\left(\frac{k_tU_t}{z_t}\right)^{\alpha} - \frac{1-\alpha}{\phi-\alpha}k_t^{\frac{\phi-1}{\phi-\alpha}}\left(\frac{\alpha}{\delta\phi}\right)^{\frac{1}{\phi-\alpha}}z_t^{\frac{1-\phi}{\phi-\alpha}} - \beta c_t\alpha(1-\alpha)\left(\frac{U_t}{z_t}\right)^{\alpha}k_t^{\alpha-1}\right\} \\
&\qquad \left\{k_t - k_t^{\frac{\phi-1}{\phi-\alpha}}\left(\frac{\alpha}{\delta\phi}\right)^{\frac{1}{\phi-\alpha}}z_t^{\frac{1-\alpha}{\phi-\alpha}} + z_t^{1-\alpha}k_t^{\alpha}U_t^{\alpha} - \beta c_t\Big\{\alpha U_t z_t^{1-\alpha}k_t^{\alpha-1}U_t^{\alpha}+1-\delta U_t^{\phi}\Big\}\right\}^{\frac{2\alpha-1-\phi}{\phi-\alpha}}\frac{\alpha-1}{\phi-\alpha} \\
\frac{\partial}{\partial k_t}U_{t+1} &= \left(\frac{\alpha}{\delta\phi}\right)^{\frac{1}{\phi-\alpha}}z_{t+1}^{\frac{1-\alpha}{\phi-\alpha}}\ \dots \text{fuck me}
\end{aligned}$$

Dynare: Policy Function using Parameter Set (i)

$$k_t = \underbrace{60.62}_{=G_0} + \underbrace{0.974}_{=G_1}(k_{t-1} - \bar{k}) + \underbrace{2.055}_{=G_2}(x_t - \bar{x})$$