Dynamic Macroeconomics with Numerics: Project II

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Part 1: Blanchard Kahn Approach

First note that we have

$$z_{t+1} = \exp(\epsilon_{t+1}) z_t^{\rho},$$

but we may also use

$$z_{t+1} = \exp(x_{t+1}) = \exp(\rho x_t + \epsilon_{t+1}),$$

However the derivatives evaluated at the steady state remain the same in either notation. Now we summarize the equilibrium conditions in the following four equations:

$$\begin{aligned}
\mathbb{E}c_{t+1} &= \beta c_t \mathbb{E}\Big[\alpha \exp(x_{t+1})^{1-\alpha} U_{t+1}^{\alpha} k_{t+1}^{\alpha-1} + 1 - \delta U_{t+1}^{\phi}\Big] \\
k_{t+1} &= \exp\big([1-\alpha]x_t\big)(k_t U_t)^{\alpha} + (1-\delta U_t^{\phi})k_t - c_t \\
U_{t+1} &= \left(\frac{\alpha}{\delta\phi}\right)^{\frac{1}{\phi-\alpha}} \exp\left(\frac{1-\alpha}{\phi-\alpha} x_{t+1}\right) k_{t+1}^{\frac{\alpha-1}{\phi-\alpha}} \\
x_{t+1} &= \rho x_t + \epsilon_{t+1}
\end{aligned}$$

For ease of notation we have

$$\forall i \in \{0,1,2,\dots\} : \mathbf{s}_{t+i} = \begin{pmatrix} c_{t+i} \\ U_{t+i} \\ k_{t+i} \\ x_{t+i} \end{pmatrix}.$$

Now we write this system of equations as follows:

$$\mathbf{F}(\mathbf{s}_{t+1}, \mathbf{s}_t) = \begin{pmatrix} c_{t+1} - \beta c_t \Big[\alpha \exp(x_{t+1})^{1-\alpha} U_{t+1}^{\alpha} k_{t+1}^{\alpha-1} + 1 - \delta U_{t+1}^{\phi}\Big] \\ U_{t+1} - \left(\frac{\alpha}{\delta\phi}\right)^{\frac{1}{\phi-\alpha}} \exp\left(\frac{1-\alpha}{\phi-\alpha} x_{t+1}\right) k_{t+1}^{\frac{\alpha-1}{\phi-\alpha}} \\ \exp\big([1-\alpha]x_t\big)(k_t U_t)^{\alpha} + (1-\delta U_t^{\phi})k_t - c_t - k_{t+1} \\ x_{t+1} - \rho x_t \end{pmatrix},$$

where we have $\mathbb{E}_t \mathbf{F}(\mathbf{s}_{t+1}, \mathbf{s}_t) = \mathbf{0}$.

Now we make use of the Jacobian being defined as

$$\mathbf{J} = -[D\mathbf{F}_2(\mathbf{s}^*, \mathbf{s}^*)]^{-1} D\mathbf{F}_1(\mathbf{s}^*, \mathbf{s}^*),$$

so we can look at two separate matrices. We start by taking the derivatives w.r.t. the period t variables of $\mathbf{s}_t$:

$$D\mathbf{F}_1(\mathbf{s}^*, \mathbf{s}^*) = \begin{pmatrix} -\beta[\alpha U_*^{\alpha} k_*^{\alpha-1} + 1 - \delta U_*^{\phi}] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & \delta\phi U_*^{\phi-1} k_* + \alpha k_*^{\alpha} U_*^{\alpha-1} & (1-\delta U_*^{\phi}) + \alpha k_*^{\alpha-1} U_*^{\alpha} & (1-\alpha)(k_* U_*)^{\alpha} \\ 0 & 0 & 0 & -\rho \end{pmatrix}$$

Next up is the second matrix containing the partial derivatives w.r.t. the period $t+1$ variables of $\mathbf{s}_{t+1}$:

$$D\mathbf{F}_2(\mathbf{s}^*, \mathbf{s}^*) = \begin{pmatrix} 1 & -\beta c_* \Big[\alpha^2 (U_* k_*)^{\alpha-1} + \delta\phi U_*^{\phi-1}\Big] & -\beta c_* \alpha(\alpha-1) U_*^{\alpha} k_*^{\alpha-2} & -\beta c_* \Big[\alpha(1-\alpha) U_*^{\alpha} k_*^{\alpha-1}\Big] \\ 0 & 1 & -\left(\frac{\alpha}{\delta\phi}\right)^{\frac{1}{\phi-\alpha}} \left(\frac{\alpha-1}{\phi-\alpha} k_*^{\frac{\alpha-1}{\phi-\alpha}-1}\right) & -\left(\frac{\alpha}{\delta\phi}\right)^{\frac{1}{\phi-\alpha}} \frac{1-\alpha}{\phi-\alpha} k_*^{\frac{\alpha-1}{\phi-\alpha}} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We use this matrix in MATLAB.

Alternative: 3-Equation System

$$U^{\alpha}_{t+1} = \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} \exp\left(\frac{(1-\alpha)\alpha}{\phi-\alpha}x_{t+1}\right)k_{t+1}^{\frac{(\alpha-1)\alpha}{\phi-\alpha}}$$
$$U^{\phi}_{t+1} = \left(\frac{\alpha}{\delta\phi}\right)^{\frac{\phi}{\phi-\alpha}} \exp\left(\frac{(1-\alpha)\phi}{\phi-\alpha}x_{t+1}\right)k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}}$$

Now we have

$$\mathbf{F}(\mathbf{s}_{t+1}, \mathbf{s}_t) = \begin{pmatrix}(1)\\(2)\\(3)\end{pmatrix},$$

with the elements being given by the equations below.

$$c_{t+1} - \beta c_t\Bigg[\alpha \exp\left([1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}]x_{t+1}\right)k_{t+1}^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}k_{t+1}^{\frac{(\alpha-1)\phi}{\phi-\alpha}}\exp\left(x_{t+1}\frac{(1-\alpha)\phi}{\phi-\alpha}\right)\Bigg] \tag{1}$$

$$\exp\left(\Big[1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}\Big]x_t\right)k_t^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} + k_t - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}\exp\left(\frac{(1-\alpha)\phi}{\phi-\alpha}x_t\right)k_t^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}} - c_t - k_{t+1} \tag{2}$$

$$x_{t-1} - \rho x_t \tag{3}$$

Now we take the partial derivatives w.r.t. $c_t, c_{t+1}, k_t, k_{t+1}, x_t, x_{t+1}$ for all three equations evaluated at the steady state. First the Euler error:

$$\begin{aligned}
\frac{\partial}{c_t}(1) &= -\beta\Bigg[\alpha k_*^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}k_*^{\frac{(\alpha-1)\phi}{\phi-\alpha}}\Bigg]\\
\frac{\partial}{k_t}(1) &= 0\\
\frac{\partial}{x_t}(1) &= 0\\
\frac{\partial}{c_{t+1}}(1) &= 1\\
\frac{\partial}{k_{t+1}}(1) &= -\beta c_*\Bigg[\alpha\Big(\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}\Big)k_*^{\alpha-2+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}k_*^{\frac{(\alpha-1)\phi}{\phi-\alpha}-1}\Big(\frac{(\alpha-1)\phi}{\phi-\alpha}\Big)\Bigg]\\
\frac{\partial}{x_{t+1}}(1) &= -\beta c_*\Bigg[\alpha\big[1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}\big]k_*^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}k_*^{\frac{(\alpha-1)\phi}{\phi-\alpha}}\frac{(1-\alpha)\phi}{\phi-\alpha}\Bigg]
\end{aligned}$$

Now the resource constraint:

$$\begin{aligned}
\frac{\partial}{c_t}(2) &= -1\\
\frac{\partial}{k_t}(2) &= \Big(\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}\Big)k_*^{\alpha-1+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} + 1 - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}\Big(1+\frac{(\alpha-1)\phi}{\phi-\alpha}\Big)k_*^{\frac{(\alpha-1)\phi}{\phi-\alpha}}\\
\frac{\partial}{x_t}(2) &= \Big[1-\alpha+\frac{(1-\alpha)\alpha}{\phi-\alpha}\Big]k_*^{\alpha+\frac{(\alpha-1)\alpha}{\phi-\alpha}}\left(\frac{\alpha}{\delta\phi}\right)^{\frac{\alpha}{\phi-\alpha}} - \delta\Big(\frac{\alpha}{\delta\phi}\Big)^{\frac{\phi}{\phi-\alpha}}\frac{(1-\alpha)\phi}{\phi-\alpha}k_*^{1+\frac{(\alpha-1)\phi}{\phi-\alpha}}\\
\frac{\partial}{c_{t+1}}(2) &= 0\\
\frac{\partial}{k_{t+1}}(2) &= -1\\
\frac{\partial}{x_{t+1}}(2) &= 0
\end{aligned}$$

And lastly the error term from the law of motion for x:

$$\frac{\partial}{c_t}(3) = 0$$
$$\frac{\partial}{k_t}(3) = 0$$
$$\frac{\partial}{x_t}(3) = -\rho$$
$$\frac{\partial}{c_{t+1}}(3) = 0$$
$$\frac{\partial}{k_{t+1}}(3) = 0$$
$$\frac{\partial}{x_{t+1}}(3) = 1$$

We have the two matrices

$$D\mathbf{F}_1(\cdot) = \begin{pmatrix} \frac{\partial}{\partial c_t}(1) & \frac{\partial}{\partial k_t}(1) & \frac{\partial}{\partial x_t}(1) \\ \frac{\partial}{\partial c_t}(2) & \frac{\partial}{\partial k_t}(2) & \frac{\partial}{\partial x_t}(2) \\ \frac{\partial}{\partial c_t}(3) & \frac{\partial}{\partial k_t}(3) & \frac{\partial}{\partial x_t}(3) \end{pmatrix}, \ D\mathbf{F}_2(\cdot) = \begin{pmatrix} \frac{\partial}{\partial c_{t+1}}(1) & \frac{\partial}{\partial k_{t+1}}(1) & \frac{\partial}{\partial x_{t+1}}(1) \\ \frac{\partial}{\partial c_{t+1}}(2) & \frac{\partial}{\partial k_{t+1}}(2) & \frac{\partial}{\partial x_{t+1}}(2) \\ \frac{\partial}{\partial c_{t+1}}(3) & \frac{\partial}{\partial k_{t+1}}(3) & \frac{\partial}{\partial x_{t+1}}(3) \end{pmatrix}$$