Riemann surfaces

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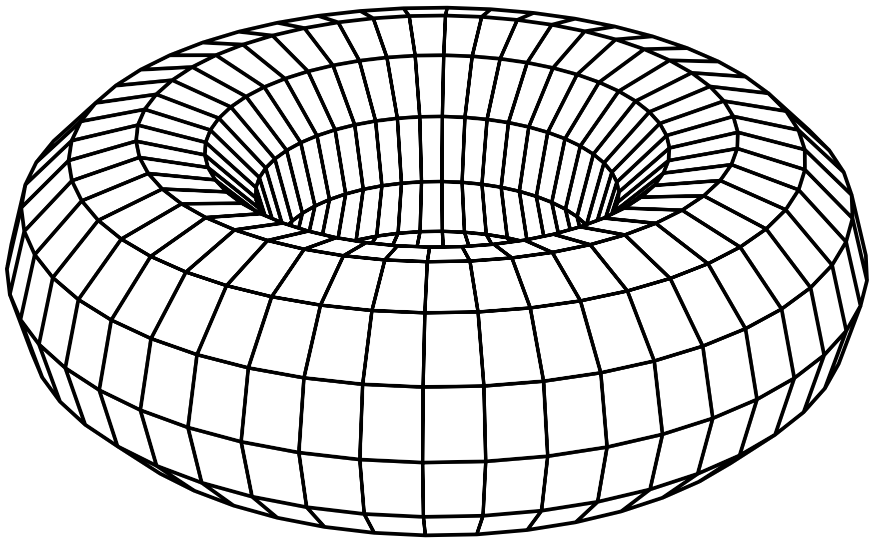


Project III

Department of Mathematical Sciences

Durham University

For Mum,
for Dad,
за Девче

This piece of work is a result of my own work and I have complied with the Department's guidance on multiple submission and on the use of AI tools. Material from the work of others not involved in the project has been acknowledged, quotations and paraphrases suitably indicated, and all uses of AI tools have been declared.

Contents

Introduction

First introduced in 1851, as part of Bernhard Riemann's doctoral thesis[1], Riemann surfaces have been central to the development of mathematics in the 20th and 21st centuries. This publication sparked the rigorous and methodical study of topology, and was responsible for drawing significant links between complex analysis and algebraic geometry, an area essentially born of the thesis. Riemann's discovery of these objects is representative of the unification of many previously explored branches of theory, and this influence is evident by the naming of key theorems throughout, for example with Abel's addition theorem, and the Jacobi inversion theorem.

[1] Bernhard Riemann. "Grundlagen für eine allgemeine Theorie Functionen einer veränderlichen complexen Großen". PhD thesis. University of Göttingen, 1851

The first exploration of the concept, by Riemann himself, was concerned with branches of multivalued complex functions, for example, the square root. Riemann had the idea ("one of the most illuminating in the history of mathematics"[2]) to represent the relation $p(x, y) = 0$ between $x, y \in \mathbb{C}$ by covering a plane representing the values of x, by a surface representing the values of y. We will explore this idea and those related in Section 3.4.

[2] John. Stillwell. *Mathematics and its history.* 2nd ed. Undergraduate texts in mathematics. Springer, 2002

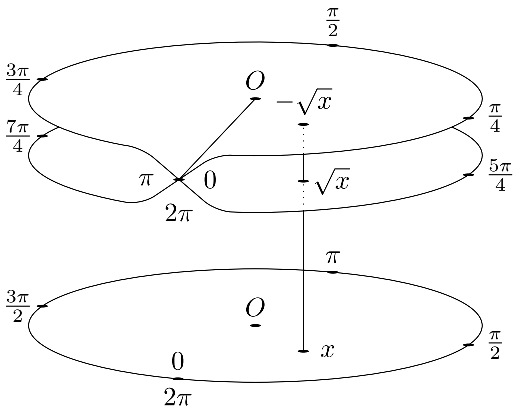


Figure 1: Freely deforming the unit disk such that the points $y = \pm\sqrt{x}$ lie above x.

The study of Riemann surfaces has behind it two remarkably distinct trails of research and theory, paved respectively by complex manifold theory, and algebraic geometry. We tend to explore the subject through the former characterisation, but ommission of key results and ideas arising from the latter would be misguided. As a result, there are chapters and sections of the report which take more explicit notice of the algebraic geometric development, although these are somewhat less integral to our main aim; the Riemann–Roch theorem for compact Riemann surfaces.

Acknowledgement

I am indebted to my project supervisor Raphael Zentner for his numerous resource recommendations, for pushing me and my project in a direction both interesting and challenging, and for the provision of herbal tea and chocolate.

1

Preliminaries

The following chapter contains well-known definitions, and results from analysis and topology on which the content of this report will be built.

1.1 Topology

Let X and Y be topological spaces.

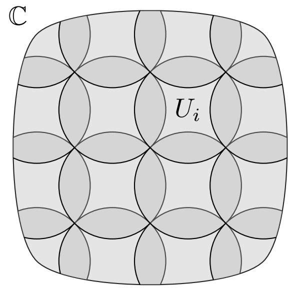


Figure 1.1: The subsets U_i provide an open cover of $\mathbb{C}$.

DEFINITION 1.1 (Punctured neighbourhood). A *punctured neighbourhood* P of $x \in X$ is a set of the form $P = Q \setminus \{x\}$ where Q is an open neighbourhood of x.

DEFINITION 1.2 (Continuous map). A map $f : X \to Y$ is called a *continuous map* if $f^{-1}(V) \subseteq X$ is open for all open $V \subseteq Y$.

DEFINITION 1.3 (Open map). A map $f : X \to Y$ is called an *open map* if $f(U) \subseteq Y$ is open for all open $U \subseteq X$.

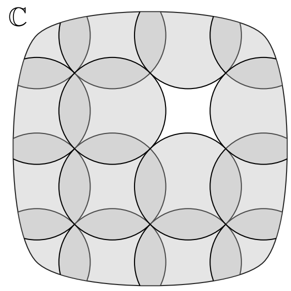


Figure 1.2: This open cover does not admit a finite subcover, so $\mathbb{C}$ is non-compact.

DEFINITION 1.4 (Open cover). An *open cover* of X is a collection $(U_i)_{i \in I}$ of open subsets of X such that $X = \bigcup_{i \in I} U_i$.

DEFINITION 1.5 (Compactness). X is *compact* if every open cover of X admits a finite subcover.

1.2 Real Analysis

Our work will primarily be in complex analysis, although preliminary understanding of real analytic concepts will be useful. The following definitions are due to Rudin[3].

[3] Walter Rudin. *Real and complex analysis.* 1987

DEFINITION 1.6 (C^∞). A real-valued function is C^∞ if it has continuous derivatives to all orders on its domain. We sometimes refer to the set of C^∞ functions on domain X by $C^\infty(X; \mathbb{R})$.

DEFINITION 1.7 (Support). The *support* of a function f on a topological space X is the set

$$\operatorname{supp} f = \overline{\{x \in X : f(x) \neq 0\}}.$$

Definition 1.8 (Convolution). For two functions $f, g : \mathbb{R}^2 \to \mathbb{R}$, the *convolution* of f and g is denoted and defined by

$$(f * g)(x) = \int_{\mathbb{R}^2} f(y)g(x-y)\ \mathrm{d}y.$$

Definition 1.9 (Cauchy sequence). Let H be an inner product space, and let $\|\cdot\| = \langle \cdot, \cdot \rangle$ be the induced norm. A sequence $(h_i) \in H$ is called a *Cauchy sequence* if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n, m \geq N \implies \|h_n - h_m\| < \varepsilon.$$

Definition 1.10 (Hilbert space). A *Hilbert space* $\mathcal{H}$ is an inner product space which is *complete*, i.e., a space in which every Cauchy sequence converges.

1.3 Complex Analysis

Let $U \subseteq \mathbb{C}$ be open.

Definition 1.11 (Holomorphicity). A function $f : U \to \mathbb{C}$ is called *holomorphic on* $U' \subseteq U$ if it is complex differentiable at every point $z \in U'$. The function is called *holomorphic at* $z_0 \in U$ if it is holomorphic on some open neighbourhood of z_0.

Definition 1.12 (Meromorphicity). A function $f : U \to \mathbb{C}$ is called *meromorphic at* $z_0 \in U$ if it is holomorphic on a punctured neighbourhood of z_0, and has either a removable singularity, or pole at z_0.

We call D a *domain* if it is an open, connected subset of $\mathbb{C}$. For the following two results, D is a domain, and $f : D \to \mathbb{C}$ is holomorphic.

Theorem 1.13 (Identity theorem). f is identically zero in D if the zero set of f has an accumulation/limit point in D.

Theorem 1.14 (Maximum modulus principle). If there exists a point $z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$ for all points $z \in D$, f is constant.

We refer to holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ as *entire*, which allows us to state the following theorems succinctly.

Theorem 1.15 (Open mapping theorem). A non-constant, entire map is open.

Theorem 1.16 (Liouville's theorem). A bounded, entire map is constant.

2

Manifold Theory

The following chapter serves as an introduction to manifold theory which, through specification, will arrive us at the notion of a Riemann surface. The definitions are similar to those found in Lee[4].

2.1 Topological manifolds

[4] John. Lee. *Introduction to Topological Manifolds*. Graduate Texts in Mathematics, 202. Springer New York, 2011

With an understanding of the direction in which our consideration is headed, we introduce the following concepts in their complex version. There are times however, when further intuition may be gained by consideration of the real analogues of the definitions, and examples will be given when this is the case. We start with a basic topological idea.

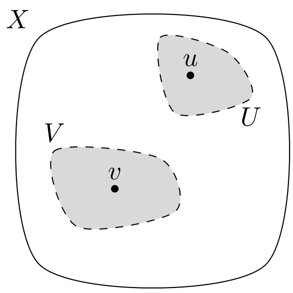


Figure 2.1: A Hausdorff space.

Definition 2.1 (Hausdorff). A topological space X is *Hausdorff* if for all distinct points $u, v \in X$ there exist open sets $U \ni u$, $V \ni v$ such that $U \cap V = \varnothing$.

Example 2.2. Let $\mathbb{C}$ have the discrete topology, that is, the topology with the basis of singleton sets in $\mathbb{C}$. It is trivial that this space is Hausdorff; two distinct points $z, \omega \in \mathbb{C}$ can be separated by their respective singletons, $\{z\} \cap \{\omega\} = \varnothing$.

Definition 2.3 (Locally Euclidean). Let X be a topological space. We say that X is *locally Euclidean* of dimension n, if for each $x \in X$ there exists an open neighbourhood U_x of x homeomorphic, via homeomorphism ϕ_x, to an open subset $\tilde{U}_x \subseteq \mathbb{C}^n$.

Remark. It is important to mention that reference to dimension will be implicit reference to *complex* dimension, unless otherwise specified.

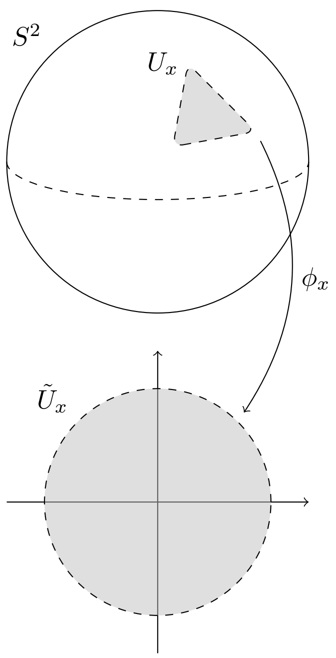
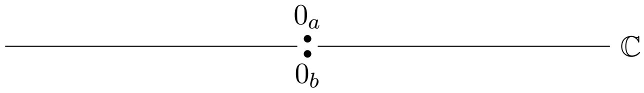


Figure 2.2: S^2 is (real) locally Euclidean of dimension 2.

The homeomorphisms, ϕ_x, are called *coordinate functions* (sometimes coordinates) at x, motivated by the fact that they provide a local Euclidean coordinate system on the space. Together with the domains on which they are defined, coordinate functions are called *charts*, denoted by (U_x, ϕ_x). Any collection of chart domains which provides an open cover of the space is called an *atlas*, $\{U, \phi\}$.

Example 2.4 (Complex plane with two origins). Consider the space L, constructed by taking two copies of $\mathbb{C}$ and identifying all identical points apart from the origins. With greater formality, let

$$L = (0_a \times \mathbb{C} \sqcup 0_b \times \mathbb{C}) / \sim$$
$$(0_a, z) \sim (0_b, z) \quad \forall z \in \mathbb{C} \setminus \{0\}.$$



Describing the quotient topology on this space is straightforward; we can form a basis as follows. Firstly, we take the open sets of $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, and join this collection with sets $(U \setminus \{0\}) \cup \{0_i\}$, for open $U \subseteq \mathbb{C}$ containing 0.

Since $\mathbb{C}^* \subseteq \mathbb{C}$, we need only check if the two origin points 0_a and 0_b are locally Euclidean. The open neighbourhoods $\mathbb{C}_i := \mathbb{C}^* \cup \{0_i\}$ of 0_i are both homeomorphic to $\mathbb{C}$ via

$$\phi_i : \mathbb{C}_i \to \mathbb{C} : z \mapsto \begin{cases} z & z \in \mathbb{C}^* \\ 0 & z = 0_i \end{cases}$$

and hence L is locally Euclidean of dimension 1.

Motivated in part by the existence of a few pathological counterexamples, the definition of a topological manifold is often stated with the extra condition of second-countability[5].

[5] Lee, *Introduction to Topological Manifolds*, p. 36

DEFINITION 2.5 (Second countability). A topological space X is called *second countable* if it admits a countable basis for its topology.

EXAMPLE 2.6. Let $X = \left\{(x, y) \in \mathbb{C}^2 : xy = 0\right\}$. We aim to show that this set has a countable basis for the subset topology from $\mathbb{C}^2$. Firstly, we note that $\mathbb{R}$ admits a basis

$$\mathcal{B} = \{(r, s) : r, s \in \mathbb{Q}\},$$

i.e., the collection of open intervals in $\mathbb{R}$ with rational endpoints. Further, $\mathcal{B}$ has the same cardinality as $\mathbb{Q}^2 = \mathbb{Q} \times \mathbb{Q}$ which is countable as the cartesian product of two countable sets (Figure 2.3). In particular, $\mathbb{R}$ is second countable.

$\mathbb{R}^4$ is second countable when viewed as the fourth product of $\mathbb{R}$ endowed with the product topology and since $\mathbb{R}^4 \cong \mathbb{C}^2$, $\mathbb{C}^2$ is also second countable. Finally, since second countability is hereditary, our claim is justified.

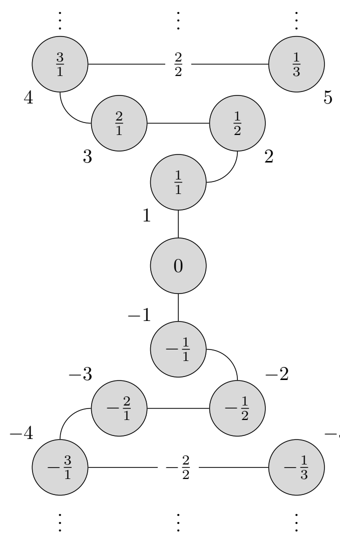


Figure 2.3: $\mathbb{Q}$ is countable.

It is interesting to note that a restriction to topological spaces with countable topologies is too heavy-handed. We would like to be able to consider $\mathbb{C}$ as a manifold, but even this space has uncountably many open sets.

DEFINITION 2.7 (Topological manifold). An n-dimensional *topological manifold* is a second-countable, Hausdorff space, which is locally Euclidean of dimension n.

NONEXAMPLE 2.8. Example 2.6 is not locally Euclidean.

Example 2.4 is not Hausdorff.

Example 2.2 is not second countable.

Example 2.9. Trivial examples of topological manifolds are $\mathbb{C}^n$ with the standard topology, and subsets $U \subseteq \mathbb{C}^n$ with the induced subset topology. The identity map over the whole space provides a suitable atlas in both cases.

Before examining further non-trivial examples in detail, we consider the imposition of further structure on a manifold, as this leads the way to more interesting results.

2.2 Complex analytic manifolds

Further restrictions on manifolds are imposed via the charts which define them.

Definition 2.10 (Compatible charts). The charts (U_x, ϕ_x) and (U_y, ϕ_y) on a topological space X are said to be *compatible* if

$$\phi_{x,y} := \phi_x \circ \phi_y^{-1} : \phi_y(U_x \cap U_y) \to \phi_x(U_x \cap U_y)$$

is holomorphic. Charts are *vacuously compatible* if $U_x \cap U_y \neq \varnothing$.

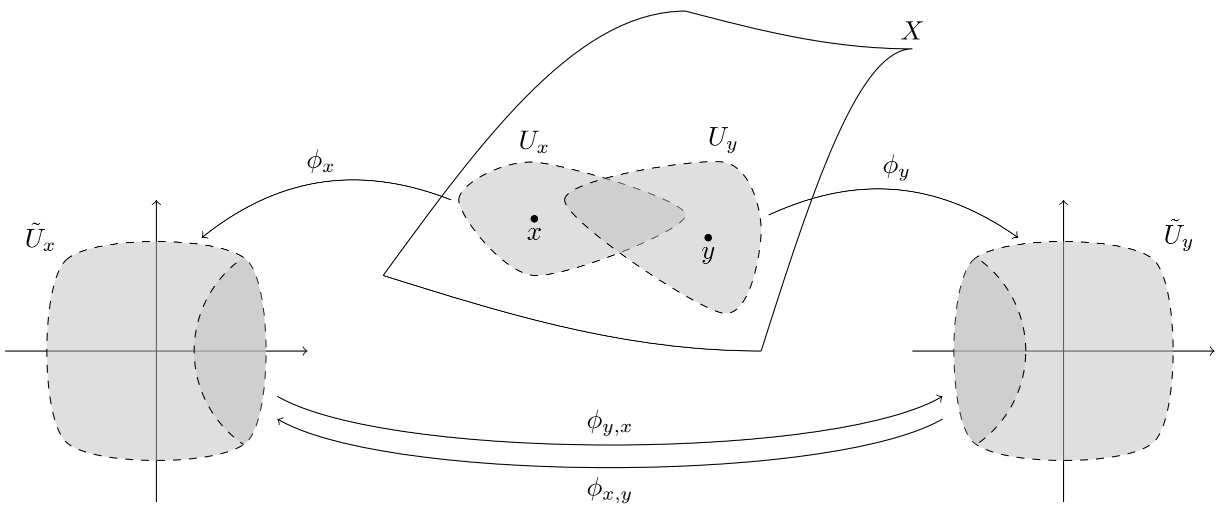


Figure 2.4: We call the functions $\phi_{x,y}$ and $\phi_{y,x}$ *transition functions.*

Remark. From initial inspection it is clear that this definition is symmetric[6]. If $\phi_{x,y}$ is holomorphic on $\phi_y(U_x \cap U_y)$ then $\phi_{y,x}$ will also be holomorphic on $\phi_x(U_x \cap U_y)$. Reflexivity of the relation is also clear, as a result of the holomorphicity of the identity function.

[6] Rick Miranda. *Algebraic curves and Riemann surfaces.* Graduate studies in mathematics ; v. 5. American Mathematical Society, 1995, p. 2

We cannot *a priori* make an analogous statement on the transitivity of this relation. Suppose that (U_1, ϕ_1) is compatible with (U_2, ϕ_2) which is in turn compatible with (U_3, ϕ_3). The composition

$$\phi_3 \circ \phi_1^{-1} = \left(\phi_3 \circ \phi_2^{-1}\right) \circ \left(\phi_2 \circ \phi_1^{-1}\right)$$

is holomorphic on the domain $\phi_1(U_1 \cap U_2 \cap U_3)$ but not necessarily $\phi_1(U_1 \cap U_3)$.

Definition 2.11 (Complex analytic manifold). A manifold is *analytic* if it has pairwise compatible coordinate charts.

Remark. It is interesting at this point[7] to consider the different structures which we could have imposed on a manifold.

- If we decided that our transition functions should be differentiable, we would arrive at the definition of a differentiable manifold.
- If we asserted that the transition functions should be C^∞, we would recover the notion of a smooth manifold.
- If we required the transition functions to be C^∞ with positive Jacobian, we would consider oriented smooth manifolds.

all of which have interesting associated theory and applications.

[7] Simon Donaldson. *Riemann Surfaces*. Vol. 22. Oxford Graduate Texts in Mathematics. Oxford University Press, 2011, p. 30

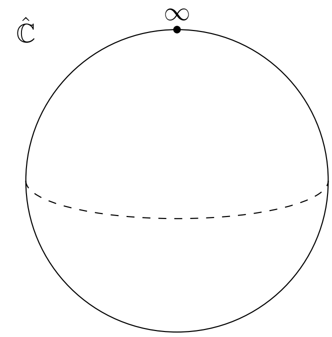


Figure 2.5: $\hat{\mathbb{C}} \cong S^2$

We will now consider the Riemann sphere $\hat{\mathbb{C}}$, which we define via Alexandroff's[8] one-point compactification to be

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}\,.$$

[8] P. Alexandroff. "Über die Metrisation der im Kleinen kompakten topologischen Räume". In: *Mathematische Annalen* 92 (1924), pp. 294–301

We take the topology of $\hat{\mathbb{C}}$ to be the collection of open sets in $\mathbb{C}$ together with unions $\{\infty\} \cup (\mathbb{C} \setminus K)$ for compact $K \subseteq \mathbb{C}$.

Example 2.12. We now aim to impose the structure of a complex analytic manifold on the Riemann sphere. To begin, consider the following subsets of $\hat{\mathbb{C}}$,

$$U_0 = \hat{\mathbb{C}} \setminus \{\infty\} = \mathbb{C},$$
$$U_\infty = \hat{\mathbb{C}} \setminus \{0\}\,.$$

We note that $U_0 \cup U_\infty = \hat{\mathbb{C}}$ and furthermore, $U_0 \cap U_\infty = \mathbb{C}^*$. Both of these subsets are homeomorphic to $\mathbb{C}$ via the coordinate functions,

$$\phi_0 : U_0 \to \mathbb{C} : z \mapsto z,$$
$$\phi_\infty : U_\infty \to \mathbb{C} : z \mapsto \begin{cases} \frac{1}{z} & z \in \mathbb{C}, \\ 0 & z = \infty. \end{cases}$$

It remains to verify that the transition functions are holomorphic, in particular, holomorphic on $\mathbb{C}^*$. The transition functions are given by,

$$\phi_{0,\infty} = \phi_0 \circ \phi_\infty^{-1} = \frac{1}{z} = \phi_\infty \circ \phi_0^{-1} = \phi_{\infty,0}.$$

Clearly, $z \mapsto 1/z$ is holomorphic on $\mathbb{C}^*$, and hence $\{(U_0, \phi_0), (U_\infty, \phi_\infty)\}$ is a suitable atlas.

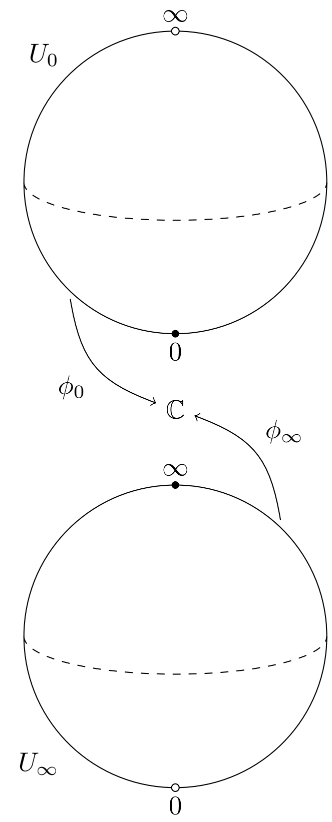


Figure 2.6: $\hat{\mathbb{C}} = U_0 \cup U_\infty$

It is reasonable at this point to ask whether this is the only possible atlas by which the Riemann sphere is seen to be a complex analytic manifold. Under our current understanding of the definition, this is clearly not the case. We may trivially add charts to this atlas which are contained by the current charts, without fundamentally changing the structure of the manifold. We would like for manifolds obtained in this manner to be considered equivalent.

Definition 2.13 (Compatible atlases). Two atlases $\mathcal{A}$ and $\mathcal{B}$ are said to be *compatible* if $\mathcal{A} \cup \mathcal{B}$ is an atlas.

We note that this condition essentially makes a statement about the compatibility of the charts in each of the atlases. In particular, every chart in $\mathcal{A}$ is compatible with every chart in $\mathcal{B}$. In this case, we often say that the charts of $\mathcal{A}$ are compatible with $\mathcal{B}$.

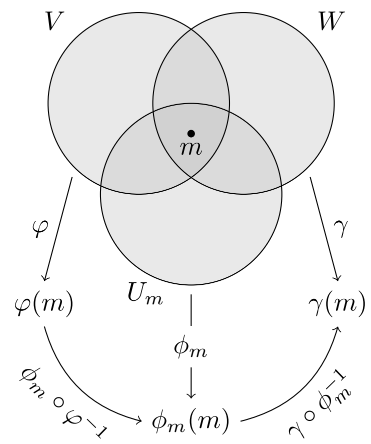


Figure 2.7: The composition of two holomorphic functions $\gamma \circ \phi_m^{-1}$ and $\phi_m \circ \varphi^{-1}$.

Lemma 2.14. If two charts are individually compatible with the atlas $\mathcal{A}$, they are compatible with eachother.

Proof. Let $\{U_\alpha, \phi_\alpha\}$ be an atlas for the manifold M, and consider two additional charts on M, (V, φ) and (W, γ) compatible with this atlas. If $V \cap W = \varnothing$, we are done, and we hence make the contrary assumption. Let $m \in V \cap W$. Since $m \in M$, there exists some chart, say (U_m, ϕ_m), such that $m \in U_m$. Therefore, $m \in V \cap W \cap U_m$.

We remarked on the transitivity of compatibility on the intersection of three chart domains, and by this remark the transition function $\varphi \circ \gamma^{-1}$ is holomorphic on $\gamma(V \cap W \cap U_m)$. Specifically, it is holomorphic at the point $\gamma(m)$, and the initially arbitrary choice of m determines that the transition function is holomorphic on $\gamma(V \cap W)$. □

An atlas $\mathcal{M}$ is said to be *maximal* if it cannot be contained by any other atlas. The following result allows us to define the notion of equivalence of analytic manifolds.

Proposition 2.15. Every atlas $\mathcal{A}$ is contained in a unique maximal atlas $\mathcal{M}$.

Proof. Let $\mathcal{C}$ denote the set of charts compatible with the atlas $\mathcal{A}$, and let $\mathcal{M} = \mathcal{A} \cup \mathcal{C}$. The charts in $\mathcal{C}$ are pairwise compatible by Lemma 2.14 and hence $\mathcal{M}$ is an atlas. This atlas is maximal since any chart compatible with $\mathcal{M}$ is compatible with the sub-atlas $\mathcal{A}$ and is hence contained by $\mathcal{M}$.

To see that this atlas is unique, let $\mathcal{M}'$ denote another maximal atlas containing $\mathcal{A}$. Every chart in $\mathcal{M}'$ is compatible with $\mathcal{A}$ and hence the method of construction of $\mathcal{M}$ determines that $\mathcal{M}' \subseteq \mathcal{M}$. Since both atlases are maximal, it must be the case that $\mathcal{M}' = \mathcal{M}$. □

As a finalé to the chapter, we introduce the definition of equivalence of manifolds. We could have, and it is often seen in the literature[9], included this notion of equivalence directly in the definition of the manifold.

[9] Miranda, *Algebraic curves and Riemann surfaces*

Definition 2.16 (Equivalent manifolds). Let X and Y be two complex analytic manifolds, with respective atlases, $\mathcal{X}$ and $\mathcal{Y}$. X and Y are said to be *equivalent* if $\mathcal{X}$ and $\mathcal{Y}$ are contained in the same maximal atlas.

As our study progresses, we will see an alternative formulation of equivalence, which is somewhat more intuitive and follows more readily from complex analysis. The formulation presented is however the standard in modern manifold theory, and is therefore useful for future consideration.

3
Riemann Surfaces

We now define Riemann surfaces, considering initial examples.

3.1 Definition

Definition 3.1 (Riemann surface). A *Riemann surface* is a connected, one-dimensional complex analytic manifold.

Remark. Since we made the inclusion of second countability in our definition of a topological manifold, it is important to make comment on the fact that this inclusion is redundant for Riemann surfaces. In particular, every connected Riemann Surface is second countable; a highly non-trivial result due to Tibor Radó[10].

We also note that an assertion of connectedness is not universal in the literature, but for our considerations, it is convenient.

[10] T Radó. "Uber den Begriff Riemannschen Fläche, Szeged Univ". In: *Act* 2 (1925), pp. 101–121

The concision of this definition is maybe unexpected, and it is common[11] to define Riemann surfaces with a more verbose definition without the abstraction to manifolds, although our approach is equivalent. A keen eye will note that this definition also determines that we have already encountered an example of a Riemann surface — the Riemann sphere $\hat{\mathbb{C}}$, is one dimensional and a suitable atlas was given in Example 2.12.

[11] Donaldson, *Riemann Surfaces*, p. 29

Example 3.2. With reference to Example 2.9, $\mathbb{C}$ is a Riemann surface. We call this Riemann surface the *complex plane*.

We now consider some further examples in full detail as these will be useful in providing future examples as we develop the theory in parallel.

3.2 Projective line

Definition 3.3 (Complex Projective Line). The *complex projective line*, denoted $\mathbb{CP}^1$ is the set of one dimensional subspaces of $\mathbb{C}^2$.

We know from linear algebra that one dimensional subspaces are spanned by a single element. For an element $(x, y) \in \mathbb{C}^2 \setminus \{0\}$, we denote the span of (x, y) by,

$$[x : y] = \{\lambda(x, y) : \lambda \in \mathbb{C}\}.$$

Hence, an alternative expression of $\mathbb{CP}^1$ is given as,

$$\mathbb{CP}^1 = \left\{[x : y] : (x, y) \in \mathbb{C}^2 \setminus \{0\}\right\}.$$

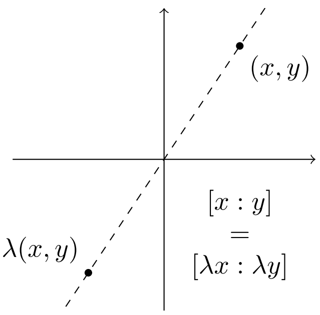


Figure 3.1: It is useful to consider these notions via their real analogues.

We refer to the elements $[x : y] \in \mathbb{CP}^1$ as *homogeneous coordinates* since $\forall \lambda \in \mathbb{C}^*$,

$$[x : y] = [\lambda x : \lambda y].$$

We now aim to give topological structure to $\mathbb{CP}^1$, and from this, Riemann surface structure. For this process, we take inspiration from Miranda[12].

[12] Miranda, *Algebraic curves and Riemann surfaces*, p. 8

Consider the following two subsets of $\mathbb{CP}^1$,

$$U_x = \{[x : y] : x \neq 0\},$$
$$U_y = \{[x : y] : y \neq 0\},$$

which are clearly such that $\mathbb{CP}^1 = U_x \cup U_y$. We consider the following functions on these subsets;

$$\phi_x : U_x \to \mathbb{C} : [x : y] \mapsto \frac{y}{x},$$
$$\phi_y : U_y \to \mathbb{C} : [x : y] \mapsto \frac{x}{y}.$$

It is clear that these functions are respectively bijective. With this we can give each U_i a topology; a set $U \subseteq U_i$ is open if and only if $\phi_i(U)$ is open in $\mathbb{C}$. Since the subsets U_i cover $\mathbb{CP}^1$ we can equally well give this space a topology, asserting that $U \subseteq \mathbb{CP}^1$ is open if and only if $U \cap U_i$ is open for each $i = x, y$.

This definitively determines that (U_x, ϕ_x) and (U_y, ϕ_y) are charts on $\mathbb{CP}^1$, and furthermore, these charts are compatible, since

$$\phi_x \circ \phi_y^{-1}(z) = \frac{1}{z} = \phi_y \circ \phi_x^{-1}(z)$$

is holomorphic on $\phi_x(U_x \cap U_y) = \mathbb{C}^* = \phi_y(U_x \cap U_y)$.

It remains to show that $\mathbb{CP}^1$ is Hausdorff. As usual, we consider two points $p, q \in \mathbb{CP}^1$. If these points are simultaneously contained by either U_x or U_y, they may be separated by disjoint open sets since the U_i are individually Hausdorff. The remaining case is that of the points $p = [1 : 0]$ and $q = [0 : 1]$. These points are respectively contained by the disjoint sets $\phi_x^{-1}(D)$ and $\phi_y^{-1}(D)$ where $D \subseteq \mathbb{C}$ is the open unit disc[13].

[13] Miranda, *Algebraic curves and Riemann surfaces*, p. 9

> Remark. There is a remarkable similarity between the approach seen here and the one we took for the Riemann sphere, which is, as usual in mathematics, no coincidence. We can show that as topological spaces $\hat{\mathbb{C}}$ and $\mathbb{CP}^1$ are homeomorphic via the map
>
> $$g : \hat{\mathbb{C}} \to \mathbb{CP}^1 : z \mapsto \begin{cases} [z : 1] & z \neq \infty \\ [1 : 0] & z = \infty \end{cases}$$
>
> which is continuous and has continuous inverse,
>
> $$h : \mathbb{CP}^1 \to \hat{\mathbb{C}} : [x : y] \mapsto \begin{cases} \frac{x}{y} & y \neq 0 \\ \infty & y = 0. \end{cases}$$
>
> Furthermore, we will see in Section 7.3 that there is only one possible holomorphic structure on $\hat{\mathbb{C}}$, the same as the one we have given to $\mathbb{CP}^1$.

3.3 Complex Tori

We now aim to give Riemann surface structure to complex tori. We will, in fact, consider one specific torus, although the approach is easily generalisable. Firstly, we recall the definition of the Gaussian integers,

$$\mathbb{Z} \oplus i\mathbb{Z} = \{a + ib : a, b \in \mathbb{Z}\}$$

which we choose to refer to by Λ.

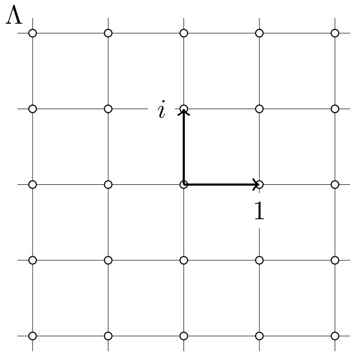


Figure 3.2: We refer to discrete additive subgroups like Λ as lattices.

We can construct the quotient space $\mathbb{C}/\Lambda$, which is more explicitly expressed as,

$$\begin{aligned} \mathbb{C}/\!\sim &= \{[z] : z \in \mathbb{C}\}, \\ [z] &= \{\omega \in \mathbb{C} : z \sim \omega\}, \\ z \sim \omega &\iff z - \omega \in \Lambda. \end{aligned}$$

This space has a natural quotient topology via the projection map $\pi : \mathbb{C} \to \mathbb{C}/\Lambda : z \mapsto [z]$. Further to this, we can show that π is open — considering an open set $U \subseteq \mathbb{C}$, showing that $\pi(U)$ is open is simply noticing that

$$\pi^{-1}(\pi(U)) = \bigcup_{\lambda \in \Lambda} (\lambda + U)$$

is a union of translations of U by elements in Λ; hence open.

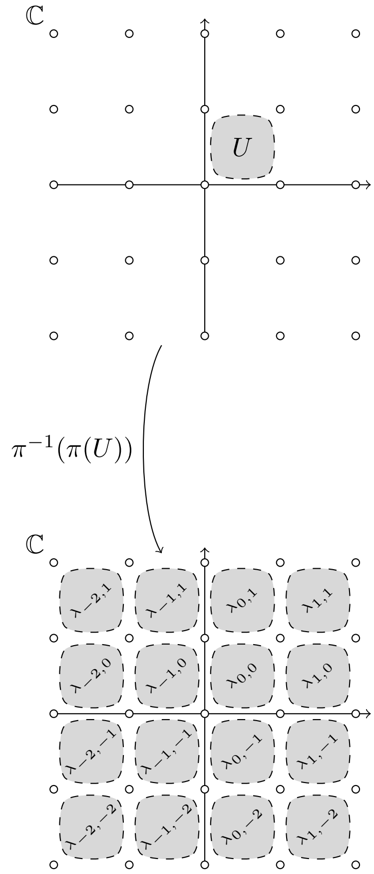


Figure 3.3: $\lambda_{a,b} = U + a + ib$

In order to find a covering set of charts, we consider sets of the form

$$B(z_0; \varepsilon) = \{\omega \in \mathbb{C} : |z_0 - z| < \varepsilon\},$$

and note that for $0 < \varepsilon < \frac{1}{2}$, the map $\pi|_{B(z_0;\varepsilon)} : B(z_0; \varepsilon) \to \pi(B(z_0; \varepsilon))$ is injective. Furthermore, as a restriction of π, this map is continuous, surjective, and open, and this is sufficient to state $\pi|_{B(z_0;\varepsilon)}$ as a homeomorphism between these two sets. Setting ϕ_{z_0} to be the inverse of this homeomorphism gives us a complex chart at the point $\pi(z_0) = [z_0]$. This shows $\mathbb{C}/\Lambda$ to be locally Euclidean, and the compatibility of these charts is all that stands between the realisation of a Riemann surface structure.

For brevity, let U_α denote the domain of the coordinate ϕ_α, that is, the set $\pi(B(\alpha; \varepsilon))$. The transition function $\phi_{z,\omega} : \phi_\omega(U_z \cap U_\omega) \to \phi_z(U_z \cap U_\omega)$ is such that

$$\begin{aligned} \pi(\phi_{z,\omega})(u) &= \pi(\phi_z \circ \phi_\omega^{-1})(u) \\ &= \phi_\omega^{-1}(u) \\ &= \pi(u) \end{aligned}$$

for every $u \in \phi_\omega(U_z \cap U_\omega)$, and notably, this determines that $\phi_\omega(u) \sim u$. We now observe that $\phi_{z,\omega}$ is continuous, and that Λ is discrete. $\phi_{z,\omega}(u) - u$ is therefore constant on the connected components of $\phi_\omega(U_z \cap U_\omega)$ and $\phi_{z,\omega}$ is holomorphic. The charts are compatible.

To justify our naming of the section as 'Complex Tori' we claim that the above constructed quotient space is homeomorphic to the torus, defined as $S^1 \times S^1$ for $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Finding an explicit homeomorphism is not difficult, and we do this now. Consider the point $[z_0] \in \mathbb{C}/\Lambda$ and let one of the representatives of this point be given as $z_0 = a + ib$ for $a, b \in \mathbb{R}$. Then, the map

$$\begin{aligned} f &: \mathbb{C}/\Lambda \to S^1 \times S^1 \\ &: [z_0] \mapsto \left(e^{2a\pi i}, e^{2b\pi i}\right). \end{aligned}$$

is a homeomorphism. The well-definedness of this map is seen by considering the periodicity of each of the coordinates in the image, and the other conditions for a homeomorphism follow easily from properties of the exponential function. We can get better intuition for this homeomorphism if we think about the set $P_0 = \{z \in \mathbb{C} : 0 \leq \mathfrak{Im}(z) < 1, 0 \leq \mathfrak{Re}(z) < 1\}$. Each point in $\mathbb{C}/\Lambda$ has a unique representative in this set, and furthermore, the closure of the set, has double representatives only on its boundary. Identifying these points in the way described by the equivalence relation gives us the following mental picture.

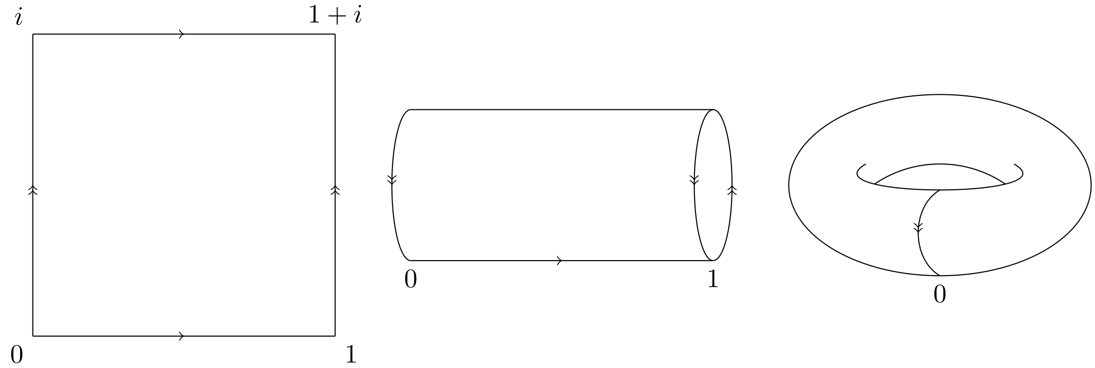


Figure 3.4: The intuition behind the homeomorphism.

Statement can then be made on the topological properties of the Riemann surface we constructed; $\mathbb{C}/\Lambda$ is a compact Riemann surface of genus one.

3.4 Affine curves

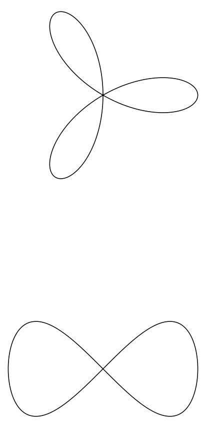


Figure 3.5: The trifolium, and lemniscate, two real affine curves.

> Definition 3.4 (Affine curve). An *affine curve* is the zero locus of a polynomial $f(x, y)$ in $\mathbb{C}^2$,
>
> $$X = \left\{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\right\}.$$

The following section is concerned with giving Riemann surface structure to affine curves which have a certain property; smoothness.

> Definition 3.5 (Smooth affine curve). An affine curve X defined by polynomial $f(x, y) \in \mathbb{C}[x, y]$ is called smooth if at all points $x \in X$,
>
> $$\frac{\partial f}{\partial x} \neq 0 \quad \text{or} \quad \frac{\partial f}{\partial y} \neq 0.$$

We introduce this restriction for good reason. The Implicit Function Theorem[14], tells us that under this restriction, there always exists a function g such that the affine curve is locally the graph of g. This is particularly useful since it allows us to give Riemann surface structure to these curves.

[14] Miranda, *Algebraic curves and Riemann surfaces*, p. 10

Suppose that X is a smooth affine curve; the zero locus of a polynomial $f(x, y)$, and take a point $p \in X$. If $\partial f/\partial x(p) \neq 0$, we can take the projection map $\pi_x : (x, y) \mapsto x$ to be a complex chart in a neighbourhood of the point p, which has a continuous inverse $\pi_x^{-1} : x \mapsto (x, g_p(x))$ where g_p is the function which exists by the Implicit Function Theorem. Alternatively, if $\partial f/\partial x(p) = 0$, the smoothness of X forces that $\partial f/\partial y(p) \neq 0$ and we can make an identical construction.

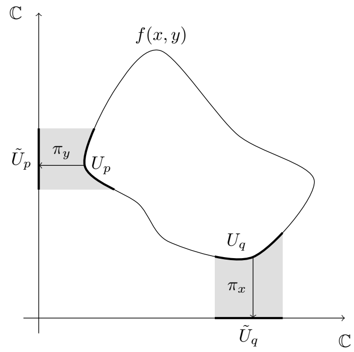


Figure 3.6: Finding charts on smooth affine curves.

Of course, we still need to check that these charts are compatible, and there are two main cases to consider. Let U_p and U_q be neighbourhoods of two points $p, q \in X$. If the coordinate functions defined on both of these domains, are the first projection, that is π_x, the transition function is nothing more than the identity, which is holomorphic. If, without loss of generality, the coordinate function on U_p is π_x and on U_q, π_y, the transition function, has the action

$$\pi_{x,y} = \pi_x \circ \pi_y^{-1} : y \mapsto (y, g(y)) \mapsto g(y),$$

where $g(y)$ is the holomorphic local representation of $f(x, y)$ whose existence is assured by the Implicit Function Theorem.

The second countability and Hausdorffness of X follow hereditarily from $\mathbb{C}^2$, and if we assert that the function f is irreducible, then X is also connected[15].

[15] I. R. Shafarevich. *Basic algebraic geometry*. Springer, 1977 - 1974, p. 127

Remark. We note that smooth affine curves give us our first non-trivial example of non-compact Riemann surfaces — the Heine–Borel theorem states that compact subsets of $\mathbb{C}^2$ must be bounded, but for all $x_0 \in \mathbb{C}$, there exists $y \in \mathbb{C}$ such that $f(x_0, y) = 0$.

3.5 Projective curves

The previous remark gives motivation for this section on projective curves. In the same way that the compactification of the complex plane arrived us at the Riemann sphere, compactification of algebraic curves will arrive us at a collection of examples central to the theory of compact Riemann surfaces.

Definition 3.6 (Complex Projective Space). Let $u, v \in \mathbb{C}^n$, and let $u \sim v \iff \exists \lambda \in \mathbb{C}^*$ such that $u = \lambda v$. *Complex projective space*, denoted by $\mathbb{CP}^n$ is defined as,

$$\mathbb{CP}^n = \mathbb{C}^{n+1}/\sim,$$

i.e., the quotient space of $\mathbb{C}^{n+1}$ by the multiplicative action of $\mathbb{C}^*$.

It is easy to see that for $n = 1$, this definition coincides with our original definition for the complex projective line. This alternative formulation is useful since it makes the Hausdorff nature of the space more overtly obvious — the quotient topology from the natural projection map will be Hausdorff. Complex projective space has other desirable features, such as compactness.

Proposition 3.7. $\mathbb{CP}^n$ is compact.

Proof. Let $\pi : \mathbb{C}^n \setminus \{0\} \to \mathbb{CP}^n$ be the quotient map described above, and also let $S^n = \{z \in \mathbb{C}^n : |z| = 1\}$. The restriction of the quotient map, $\pi|_{S^n} : S^n \to \mathbb{CP}^n$ is continuous, since π is continuous.

Furthermore, since any point in projective space can be represented, in homogeneous coordinates, by one which has unit norm, the map is surjective. Finally, the closedness and boundedness of S^n determines that $\mathbb{CP}^n$ is compact as claimed. □

Recalling that Riemann surfaces are one-dimensional, it is clear that further restrictions on complex projective space are necessary such that the existence of Riemann surfaces is possible. We will primarily dedicate focus to the case of $n = 2$, i.e., the projective plane, although the cases of higher dimension follow similarly[16]. As with the case of affine curves, restrictions come in the form of zero loci of polynomials.

[16] Donaldson, *Riemann Surfaces*, p. 33

Definition 3.8 (Homogeneity). The *degree* of a monomial is the sum of the exponents of its variables.

The *degree* of a polynomial is the maximal degree of its monomials.

A polynomial is *homogeneous* if its degree equals the degree of all its monomials.

The identification of points in $\mathbb{C}^3$ under the multiplicative action of $\mathbb{C}^*$ means that the notion of a polynomial in $\mathbb{CP}^2$ is not well-defined[17]. For example, take $F(x, y, z)$ to be a homogeneous polynomial of degree d and $[x_0 : y_0 : z_0] \in \mathbb{CP}^2$. Then,

[17] Miranda, *Algebraic curves and Riemann surfaces*, p. 14

$$F(x_0, y_0, z_0) = F(\lambda x_0, \lambda y_0, \lambda z_0) = \lambda^d F(x_0, y_0, z_0)$$

for all $\lambda \in \mathbb{C}^*$. This will not pose a problem however, since we are actually interested in the points where this polynomial F is zero.

Definition 3.9 (Projective curves). A *projective curve* is the zero locus of a homogeneous polynomial $F(x, y, z)$ in $\mathbb{CP}^2$,

$$X = \left\{[x : y : z] \in \mathbb{CP}^2 : F(x, y, z) = 0\right\}.$$

We aim to show, that under further assumptions, similar to those seen in Section 3.4, projective curves are Riemann surfaces. The argument relies on the fact that projective curves can be locally represented by affine curves, as we shall see.

Definition 3.10 (Smooth projective curve). A projective curve X defined by polynomial $F(x, y, z)$ is called *smooth* if,

$$F = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$$

has no common solutions in $\mathbb{CP}^2$.

We arm ourselves with a famous result due to Euler before showing that these smooth projective curves are indeed Riemann surfaces.

Lemma 3.11 (Euler's formula). Let $F(x_1, ..., x_n)$ be a homogeneous polynomial of degree d. Then,

$$F = \frac{1}{d} \sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i}.$$

Let X be a smooth projective curve defined by polynomial $F(x, y, z)$. We consider the following subset of X,

$$X_0 = \left\{(a, b) \in \mathbb{C}^2 : F(1, a, b) = 0\right\}$$

which is exactly an affine curve, with defining polynomial $f(a, b) = F(1, a, b)$. Noting that the restriction to $x = 1$ is nothing more than restricting to $x \neq 0$, under a consideration of homogeneous coordinates, we see that X can be covered by three sets of this form; one for each variable of the polynomial. Euler's identity asserts that,

$$\frac{1}{d}\left(x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z}\right) = F$$

and this is sufficient to show that F is not divisible by any of x, y, z. To see this, suppose otherwise, and let $F = xG$ for some polynomial G. There exists a non-zero point $(0, y, z)$ such that $G(0, y, z)$ vanishes[18], and this is a contradiction to our assumption of the smoothness of X. This determines, that X_0 (and the respective sets X_1, X_2 for the other variables) is actually a *smooth* affine curve, and hence a Riemann surface.

[18] Donaldson, *Riemann Surfaces*, p. 35

It therefore remains to show that the complex structures on these subsets of X are compatible. Consider a point $p \in X_0 \cap X_1$, which is equivalent to the restriction that for $p = [p_0, p_1, p_2]$, $p_0, p_1 \neq 0$. Let $\phi_0 : U_0 \to \tilde{U_0}$ be a chart about p in X_0, which acts as $[x : y : z] \mapsto y/x$ and let ϕ_1 be the corresponding chart in X_1. Then,

$$\phi_1 \circ \phi_0^{-1} : \omega \mapsto [1 : \omega : h(\omega)] \mapsto \frac{h(\omega)}{\omega}$$

where h is holomorphic as dictated by the Implicit Function Theorem. Finally, since $p \in X_1$ it cannot be the case that $\omega = 0$ and hence the transition function is holomorphic.

We note also that Riemann surfaces arising from smooth projective curves are compact, since they are closed subsets of compact spaces.

Remark. We mentioned earlier that the case for higher dimensional projective spaces follows similarly (a thorough treatment can be found in Griffiths–Harris[19]).

Suppose we are working in $\mathbb{CP}^n$, and that we have a collection of smooth projective curves $X_1, ..., X_{n-1}$. If we find the intersection of these $n - 1$ curves, we have will have a subset of dimension one, and therefore potential to realise a Riemann surface structure. Given certain properties of the intersection, we can indeed state this to be a Riemann surface. This is, in fact, the construction of a *projective variety* which are the objects of interest in projective algebraic geometry.

[19] Phillip Griffiths and Joe. Harris. *Principles of algebraic geometry.* 1978

4

Functions & Maps

We now spend some time considering the different classes of functions and maps on Riemann surfaces. This will allow for extensive generalisations from the theory of complex analysis.

4.1 Holomorphic functions

Definition 4.1 (Holomorphicity). The complex-valued function $f : X \to \mathbb{C}$ is said to be *holomorphic at* $x \in X$ if $f \circ \phi^{-1}$ is holomorphic at $\phi(x)$ for every chart $\phi : U \to \tilde{U}$ with $x \in U$.

f is *holomorphic on* $W \subseteq X$ if it is holomorphic at every point $w \in W$.

f is *holomorphic* if it is holomorphic on X.

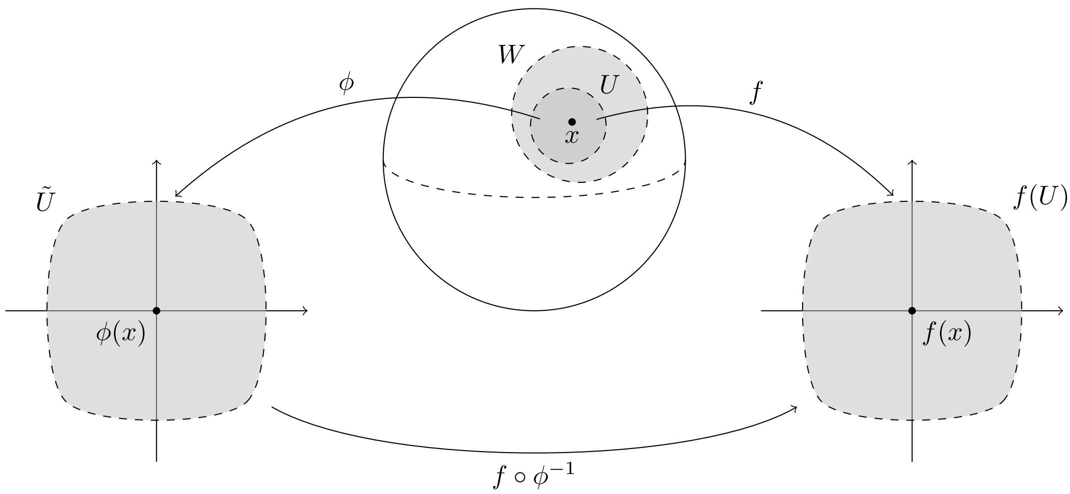


Figure 4.1: The function f is holomorphic (at/on) in line with the function $f \circ \phi^{-1}$.

Example 4.2. For a Riemann surface X, with atlas $\{U_\alpha, \phi_\alpha\}$, the charts $\phi_\alpha : U_\alpha \to \tilde{U}_\alpha$ are holomorphic on U_α.

Example 4.3. Let $f : X \to \mathbb{C} : x \to z_0$ for $z_0 \in \mathbb{C}$ constant. Since translative functions are holomorphic in the complex plane, the function f is holomorphic at each point $x \in X$, and hence holomorphic.

Notation. It is common to denote the set of holomorphic functions on an open subset $W \subseteq X$ of a Riemann surface X by $\mathcal{O}(W)$.

PROPOSITION 4.4. $\mathcal{O}(W)$ is a $\mathbb{C}$–algebra.

Proof. We recall that for this to be the case, we need that constant functions on W are holomorphic, and that sums of holomorphic functions are holomorphic. Since the former was shown in Example 4.3 we proceed with the latter.

Let $f, g : W \to \mathbb{C}$ be holomorphic on $W \subseteq X$, and let $w \in W$. Considering a chart (U, ϕ) at w, and calling on the right-distributivity of function composition,

$$(f+g) \circ \phi^{-1}(w) = (f \circ \phi^{-1})(w) + (g \circ \phi^{-1})(w)$$

is a sum of functions which are holomorphic at $\phi(w)$. The arbitrary choice of w ensures that $f+g$ is indeed holomorphic on W. □

It is not at all surprising that with the basic definitions supplied, we can find natural extensions of many results from complex analysis[20].

[20] Miranda, *Algebraic curves and Riemann surfaces*, p. 29

THEOREM 4.5 (Maximum modulus principle). Let the function $f : X \to \mathbb{C}$ be holomorphic on Riemann surface X. If there exists a point $x_0 \in X$ such that, $|f(x)| \leq |f(x_0)|$ for all points $x \in X$ then f is constant.

Proof. Suppose that the point x_0 exists, and consider the set $\mathcal{X} = \{x \in X : f(x) = f(x_0)\}$. We can alternatively formulate this set as the pre-image of the point $f(x_0)$, i.e., $\mathcal{X} = f^{-1}(f(x_0))$, which is closed by the continuity of the function f. Further to this, $\mathcal{X}$ is non-empty since $x_0 \in \mathcal{X}$.

Let $\xi \in \mathcal{X}$, and let a chart about the point ξ be given by $\phi : U \to \tilde{U}$. From the holomorphicity of f we know that the function $f \circ \phi^{-1}$ is holomorphic between open subsets of the complex plane. Additionally, we have that

$$|f(u)| \leq |f(\xi)| \quad \forall u \in U,$$

which implies,

$$|(f \circ \phi^{-1})(\phi(u))| \leq |(f \circ \phi^{-1})(\phi(\xi))| \quad \forall u \in U.$$

Then, by the maximum modulus principle (Theorem 1.14) the function $f \circ \phi^{-1}$ is constant on the connected open subset $\tilde{U}$. Consequently,

$$f(u) = (f \circ \phi^{-1})(\phi(u)) = (f \circ \phi^{-1})(\phi(\xi)) = f(\xi) = f(x_0)$$

for all $u \in U$, which determines that $u \in \mathcal{X}$ for all $u \in U$. This is equivalent to $U \subseteq \mathcal{X}$, and the arbitrary choice of u gives the openness of $\mathcal{X}$. Since X is connected $\mathcal{X} = X$, and the statement is proved. □

COROLLARY 4.6 (Liouville). A holomorphic function $f : X \to \mathbb{C}$ on a compact Riemann surface is constant.

Proof. We know that $|f|$ is a continuous function since f is holomorphic, and the compactness of X assures that $|f|$ attains its maximum at some point in X. Theorem 4.5 determines that the function is constant, since X is connected. $\square$

Example 4.7. Consider the torus examined in Section 3.3, $\mathbb{C}/\Lambda$, and the related projection map $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$. A function $f : W \subseteq \mathbb{C}/\Lambda \to \mathbb{C}$ is holomorphic at a point $w \in W$ if and only there exists a point $z \in \pi^{-1}(w)$ such that $f \circ \pi$ is holomorphic at z, a fact fairly easily observed by considering how we defined the complex charts on this space. Furthermore, f is holomorphic on W if and only if $f \circ \pi$ is holomorphic on the pre-image of this subset, $\pi^{-1}(W)$.

We now consider the case where $W = \mathbb{C}/\Lambda$ referring to Varolin[21]. Considering an entire function f^* we aim to determine the necessary criteria such that there exists a function f, as above, with $f^* = f \circ \pi$. In order for this to be the case, we need f^* to be invariant under the action of the lattice Λ. In other words, we need $f^*(z) = f^*(z + \lambda)$ for all $\lambda \in \Lambda$. Functions with this property are referred to as doubly periodic, and by the constancy of f as a holomorphic function on a compact Riemann surface, these functions are constant also.

[21] Dror Varolin. *Riemann surfaces by way of complex analytic geometry.* Vol. 125. Graduate Studies in Mathematics. American Mathematical Society, 2011

Corollary 4.6 is perhaps indicative that there is less to explore in the realm of holomorphic functions on Riemann surfaces, than there was in the complex plane. Generalisation of another class of functions from complex analysis proves more fruitful.

4.2 Meromorphic functions

Definition 4.8 (Meromorphicity). The complex-valued function $f : X \to \mathbb{C}$ is said to be *meromorphic at* $x \in X$ if $f \circ \phi^{-1}$ is meromorphic at $\phi(x)$, for every chart $\phi : U \to \tilde{U}$ with $x \in U$.

f is *meromorphic on* $W \subseteq X$ if it is meromorphic at every point $w \in W$.

f is *meromorphic* if it is meromorphic on X.

Notation. It is common to denote the set of meromorphic functions on an open subset $W \subseteq X$ of a Riemann surface X by $\mathcal{M}(W)$.

Example 4.9. From our definition it is clear that every holomorphic function is meromorphic, and hence, for an open subset $W \subseteq X$, $\mathcal{O}(W) \subseteq \mathcal{M}(W)$.

Example 4.10. Rational functions on $\hat{\mathbb{C}}$ are meromorphic.

We recall that rational functions are those which can be expressed as the quotient of polynomials, with denominator not identically 0. Denoting the ring of polynomials functions over $\mathbb{C}$ by $\mathbb{C}[z]$, any rational function $f : \hat{\mathbb{C}} \to \mathbb{C}$ can be expressed as

$$f(z) = \frac{p(z)}{q(z)},$$

where $p \in \mathbb{C}[z]$ and $q \in \mathbb{C}[z] \setminus \{0\}$. The algebraic closedness of $\mathbb{C}$

determines that both p and q are entirely factorable;

$$\frac{p(z)}{q(z)} = \frac{(z - z_{p_1}) \cdots (z - z_{p_m})}{(z - z_{q_1}) \cdots (z - z_{q_n})},$$

and hence the function has poles at the points z_{q_i}, while being holomorphic away from these points, with some nuance related to cancellation of factors.

We can again generalise some well-known results from complex analysis.

Theorem 4.11 (Principle of Isolation). The zeroes and poles of a meromorphic function $f : X \to \mathbb{C}$ which is not identically zero form a discrete subset of X.

Proof. Suppose that this wasn't the case, and let $\{x_i\}$ be a sequence of roots of f in X with a limit point $x \in X$. Then $f(x) = 0$ and this is a limit point of the sequence $\{f(x_i)\}$ in $\mathbb{C}$. By the Identity theorem, the function f must be identically 0. A similar argument for the function $1/f$ gives proof for the case of the poles. □

Corollary 4.12. A meromorphic function $f : X \to \mathbb{C}$ on a compact Riemann surface X which is not identically zero has a finite number of zeros and poles.

Example 4.13. Meromorphic functions on $\hat{\mathbb{C}}$ are rational.

Let $f : \hat{\mathbb{C}} \to \mathbb{C}$ be a meromorphic function. Corollary 4.12 allows us to enumerate the zeroes and poles of this function. Let there be n zeroes, with orders e_i, and m poles with orders f_j. If $m = 0$, f is holomorphic and hence constant by the compactness of $\hat{\mathbb{C}}$.

We assume that $n \geq 1$ and construct the function

$$g(z) = f(z) \cdot \frac{\prod_{i=1}^{n} (z - z_i)^{e_i}}{\prod_{j=1}^{m} (z - z_j)^{f_j}},$$

which is a meromorphic function of the Riemann sphere without zeroes or poles in the complex plane. It must be the case that either g or $1/g$ is holomorphic at $z = \infty$ and hence the $g(z) = c \in \mathbb{C}$, which via algebraic manipulation gives

$$f(z) = c \cdot \frac{\prod_{i=1}^{n} (z - z_i)^{e_i}}{\prod_{j=1}^{m} (z - z_j)^{f_j}},$$

and f is rational.

The Laurent series of a meromorphic function is an important construction in complex analysis, and turns out to also be important for meromorphic functions on Riemann surfaces. If we have a function $f : X \to \mathbb{C}$ which is meromorphic at a point $x \in X$, we can expand the corresponding locally read meromorphic function $f \circ \phi^{-1}$, as a Laurent series. It is important to note that contrary to the concepts which we have thus far encountered, the Laurent series of $f \circ \phi^{-1}$ *is* dependent on the choice of chart.

Definition 4.14 (Order). Let $f : X \to \mathbb{C}$ be meromorphic at $x \in X$. The *order* of f at x is

$$\operatorname{ord}(f; x) = \min \{n : c_n \neq 0\},$$

where c_i are the coefficients of the local Laurent series of f, i.e., $f \circ \phi_x^{-1}$.

Initially, the notion of order maybe seems more contrived than it does useful. It will however have great importance as we reach the latter stages of our progression, in particular in Chapter 7. To continue our exposition of the meromorphic functions on the Riemann sphere, we consider the following example[22], which will undergo generalisation later.

[22] Miranda, *Algebraic curves and Riemann surfaces*, p. 27

Example 4.15. A meromorphic function $f : \hat{\mathbb{C}} \to \mathbb{C}$ is such that

$$\sum_{z \in \hat{\mathbb{C}}} \operatorname{ord}(f; z) = 0.$$

We know from Example 4.13 that all meromorphic functions can be expressed rationally, and complete factorisation of this rational expression gives us that

$$f(z) = c \cdot \frac{\prod_{i=1}^{n} (z - z_i)^{e_i}}{\prod_{j=1}^{m} (z - z_j)^{f_j}},$$

where $e_i, f_j \in \mathbb{N}$, and $c \in \mathbb{C} \setminus \{0\}$. Alternatively, letting indices range over the integers gives us the expression,

$$f(z) = c \cdot \prod_k (z - z_k)^{\alpha_k}.$$

We observe that $\operatorname{ord}(f; z_k) = \alpha_k$, and away from these points the only non-zero order is found at $z = \infty$. In particular, the order at this point is given by $\operatorname{ord}(f; \infty) = \sum_i e_i - \sum_j f_j = -\sum_k \alpha_k$. Then,

$$\sum_{z \in \hat{\mathbb{C}}} \operatorname{ord}(f; z) = \sum_i \alpha_i - \sum_i \alpha_i = 0.$$

4.3 Holomorphic maps

Category theory tells us that a natural next step in our study is the consideration of maps between our objects of interest. In many ways, the remaining sections in the chapter provide generalisation of those already seen. Let X and Y be Riemann surfaces with atlases $\{U, \phi\}$ and $\{V, \psi\}$ respectively.

Definition 4.16 (Holomorphicity). The map $F : X \to Y$ is said to be *holomorphic at* $x \in X$ if, for all charts $\phi : U \to \tilde{U}$ with $x \in U$ and $\psi : V \to \tilde{V}$ with $F(x) \in V$, the composition

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

is holomorphic at $\phi(x)$.

F is *holomorphic on* $W \subseteq X$ if it is holomorphic at every point $w \in W$.

F is *holomorphic* if it is holomorphic on X.

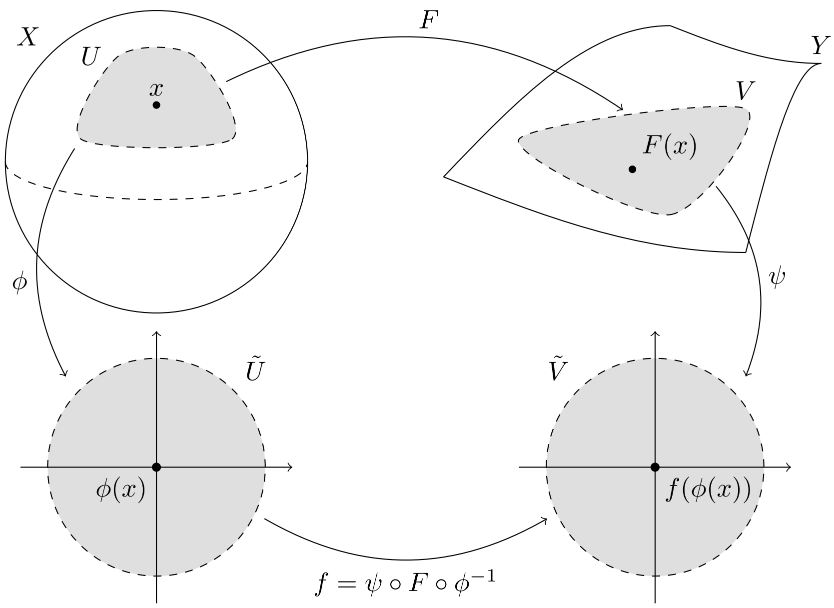


Figure 4.2: The map F is holomorphic if and only if the map f is holomorphic.

REMARK. We have seen that $\mathbb{C}$ is a non-compact Riemann surface, and hence a holomorphic function is simply a holomorphic map to the complex plane.

It is also worth noting the convention seen in Figure 4.2. We tend to refer to maps specifically between Riemann surfaces by capital letters, e.g., F, and the corresponding local representation of the map by the corresponding lower case letter, e.g., f.

EXAMPLE 4.17. The most basic example is that of the identity map $\mathrm{id} : X \to X : x \mapsto x$. This map is easily seen to be holomorphic since, for any charts ϕ_i and ϕ_j defined on a neighbourhood of the point x we have that

$$\begin{aligned} \phi_i \circ \mathrm{id} \circ \phi_j^{-1} &= \phi_i \circ \phi_j^{-1} \\ &= \phi_{i,j} \end{aligned}$$

is a transition function, and hence definitively holomorphic.

The fact that identity functions are holomorphic in this framework allows us to make a further statement, of category theoretic flavour[23].

[23] Miranda, *Algebraic curves and Riemann surfaces*, p. 39

PROPOSITION 4.18. Riemann surfaces, and the holomorphic maps between them, form a category.

Proof. Example 4.17 showed the identity map to be holomorphic, and hence it remains to show that the composition of holomorphic maps is holomorphic. Let $F : X \to Y$ and $G : Y \to Z$ be holomorphic maps between Riemann surfaces, X, Y, Z with atlases $\{U, \phi\}, \{V, \psi\}, \{W, \sigma\}$

respectively.

Consider a point $x \in X$, and let $f = \psi \circ F \circ \phi^{-1}$ which is holomorphic at $\phi(x)$. Let $y = F(x)$, and let $g = \sigma \circ G \circ \psi^{-1}$ which is holomorphic at $\psi(y)$. The composition

$$\begin{aligned} g \circ f &= \left(\sigma \circ G \circ \psi^{-1}\right) \circ \left(\psi \circ F \circ \phi^{-1}\right) \\ &= \sigma \circ G \circ F \circ \phi^{-1} \end{aligned}$$

is holomorphic at $\phi(x)$, and the arbitrary choice of x means that $G \circ F$ is holomorphic. $\square$

Tying the loose ends which we left at the end of Chapter 2, we can consider Riemann surfaces to be equivalent if there exists a bijective holomorphic map with holomorphic inverse between them[24]. In this case, we say that the Riemann surfaces are *biholomorphic*.

[24] Donaldson, *Riemann Surfaces*, p. 10

Remark. An earlier remark made mention of the fact that holomorphic functions are nothing more than a specification of holomorphic maps, made by choosing the codomain to be $\mathbb{C}$. A very similar remark can be made for meromorphic functions and the Riemann sphere, $\hat{\mathbb{C}}$.

Let $f : X \to \mathbb{C}$ be a meromorphic function, and define a map $F : X \to \hat{\mathbb{C}}$ such that

$$F(x) = \begin{cases} f(x) & \text{if } x \text{ is not a pole of } f, \\ \infty & \text{if } x \text{ is a pole of } f. \end{cases}$$

This function is holomorphic away from the poles of f since f is definitively as such. Furthermore, F is holomorphic at the poles of f and thus holomorphic.

4.4 The local model

It is interesting to note that in our definition of holomorphicity we required that the local representation of the map was holomorphic with respect to *every* pair of charts. There is a natural question which arises from this: are there charts in which the local form is more 'nicely' represented than others? This is indeed the case, and in fact we can make a guarantee on the form of these local charts in their 'nicest' form. Before stating this, we consider two lemmas which give a complete local description of holomorphic maps on Riemann surfaces[25]. Let $f : U \to \mathbb{C}$ be a holomorphic function defined on an open neighbourhood U of $0 \in \mathbb{C}$, such that $f(0) = 0$.

[25] Donaldson, *Riemann Surfaces*, p. 43

Lemma 4.19. If $f'(0) \neq 0$, there exists a neighbourhood $U' \subset U$ of 0 such that U' is homeomorphic to its image under f, and f has holomorphic inverse on U'.

Lemma 4.20. If $f \not\equiv 0$, there exists a unique integer $m \geq 1$ and a holomorphic function $g : U' \to \mathbb{C}$ defined on some neighbourhood $U' \subset U$ of 0 such that $f(z) = g(z)^m$ on U' and $g'(0) \neq 0$.

Theorem 4.21 (Local Normal Form). Let $F : X \to Y$ be a non-constant holomorphic map between Riemann surfaces X and Y. For any $x \in X$ there are charts around x, and $F(x) \in Y$ such that the local expression of F under these charts is given by $z \mapsto z^m$ for some unique integer $m \geq 1$.

Proof. Suppose we have a holomorphic map F as described, and consider the local representation of this map in arbitrary charts about a point $x \in X$ and the point $F(x) \in Y$. We denote this local representation by $f = \psi \circ F \circ \phi^{-1}$, and apply Lemma 4.20 to find a function g defined on a smaller domain such that $f = g^m$. We know that ther derivative of g is non-vanishing at 0, and we can hence state, with reference to Lemma 4.19, that g is a biholomorphism. We can therefore compose g with the arbitrary charts we started with to find charts which have the desired property. □

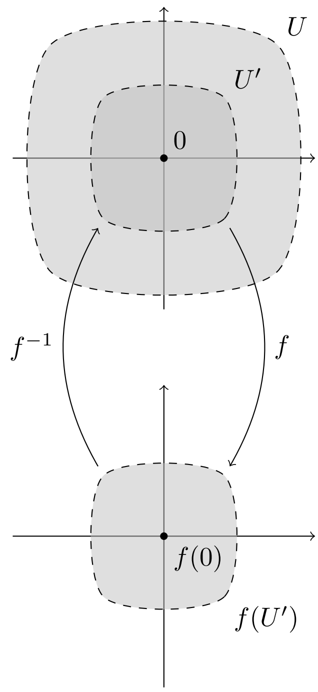


Figure 4.3: Lemma 4.19 is an inverse function theorem type statement.

We now have a full understanding of the possible holomorphic maps between Riemann surfaces locally, and the aim of globalising this idea is a natural next step. Before completing this process, we state yet another generalisation from complex analysis.

Corollary 4.22 (Open Mapping Theorem). A non-constant holomorphic map $F : X \to Y$ is open.

4.5 Multiplicity and Degree

In order to globalise this local model of holomorphic maps, we must restrict our consideration to compact Riemann surfaces.

Definition 4.23 (Multiplicity). The *multiplicity* of a point $x \in X$ under a map $F : X \to Y$ is the unique integer m such that there is a local representation of the map as $z \mapsto z^m$. We denote this by $\mathrm{mult}(F; x)$.

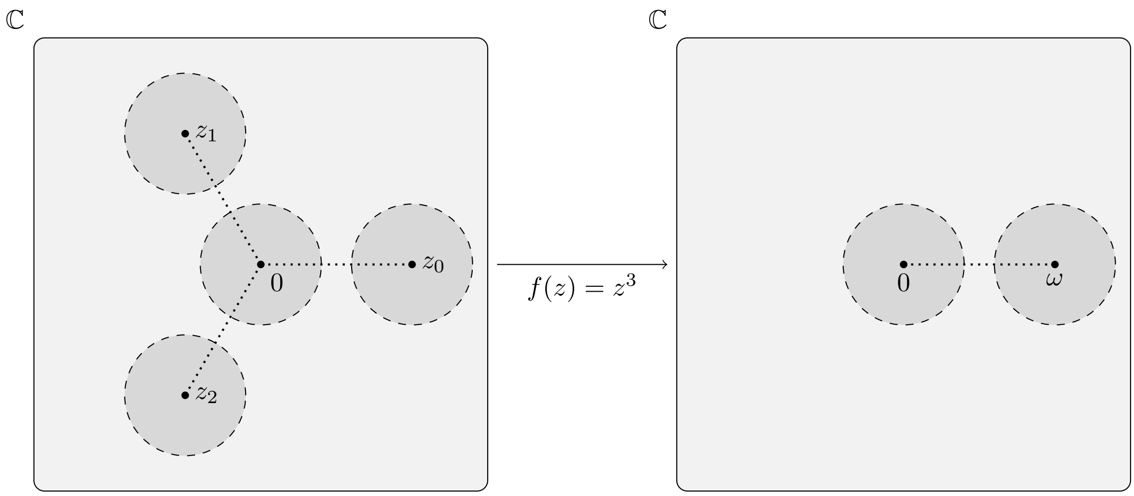


Figure 4.4: The only ramification point is at $z = 0$.

If $\mathrm{mult}(F; x) > 1$, we call x a *ramification point*, and the image of a ramification point, is called a *branch point*. Intuition for these concepts is not hard to come by; consider the entire map $f : z \mapsto z^3$, for example. The following explanation will be best understood with relation to Figure 4.4.

In neighbourhoods of each of the points $z_i \in \mathbb{C}^*$, z_i is the only point such that $f(z_i) = \omega$, and so locally, we can represent this map as constant, and in particular we have that $\mathrm{mult}(f; z_i) = 1$. Now consider the point $0 \in \mathbb{C}$, and furthermore a neighbourhood U of 0. For every point $u \in U \setminus \{0\}$ there are two other points which all, under f, map to the same point. Hence the local normal form at 0 is exactly $z \mapsto z^3$, and $\mathrm{mult}(f; 0) = 3$. From this we can see that 0 is the only ramification point of the map f.

> Remark. This is a basic, but important example, especially since Theorem 4.21 told us that any map $F : X \to Y$ can be read locally in this way.

Definition 4.24 (Degree). The *degree* of a point $y \in Y$ under a map $F : X \to Y$ is the sum of the multiplicities of points in the pre-image of y. Notatively,

$$\deg(F; y) = \sum_{x \in F^{-1}(y)} \mathrm{mult}(F; x).$$

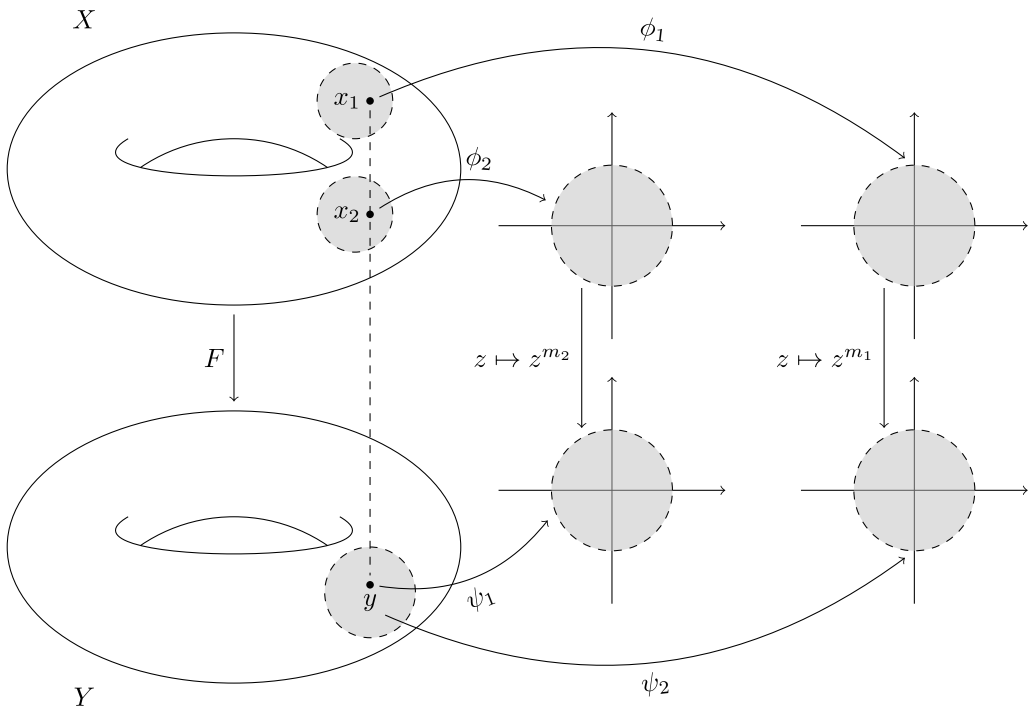


Figure 4.5: We have that $\mathrm{mult}(F; x_i) = m_i$ and that $\deg(F; y) = m_1 + m_2$.

Referring back to the example of $f : z \mapsto z^3$, we note that the degree of the map is constant, since $\deg(F; \omega) = 3$ for all $\omega \in \mathbb{C}$. We can follow a similar line of argument to show the constancy of the degree under a map $g : z \mapsto z^n$ for $n \geq 1$, and it is natural to question whether this is a general phenomenon, for all holomorphic maps. This is indeed the case, as summarised in the following proposition.

PROPOSITION 4.25. For a non-constant holomorphic map $F : X \to Y$ between compact Riemann surfaces X and Y, the degree $\deg(F; y)$ is constant over all $y \in Y$.

NOTATION. Motivated by this result, we alter our notation, and refer to the degree of a map $F : X \to Y$ by $\deg(F)$, removing the dependence on the point in the codomain.

To finish the chapter, we find a relation between the degree of a holomorphic map between compact Riemann surfaces, and their respective genera. The result is the Riemann–Hurwitz formula, stated by Riemann, and later proved by Adolf Hurwitz[26]. There are a number of different proofs of this result, in flavours topological[27], algebraic[28], and analytic[29].

[26] Adolf Hurwitz. "Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten". In: (1932)

[27] Donaldson, *Riemann Surfaces*, p. 100

[28] Griffiths and Harris, *Principles of algebraic geometry*, p. 90

[29] Otto Forster. *Lectures on Riemann surfaces.* Graduate texts in mathematics ; 81. Springer-Verlag, 1981, p. 140

THEOREM 4.26 (Riemann–Hurwitz Formula). Let $F : X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces. Then,

$$2g(X) - 2 = \deg(F)(2g(Y) - 2) + \sum_{x \in X} [\mathrm{mult}(F; x) - 1].$$

EXAMPLE 4.27. Let $F(x, y, z) = x^d + y^d + z^d$ and let X be the associated zero locus in projective space. Explicitly,

$$X = \left\{ [x : y : z] \in \mathbb{CP}^2 : F(x, y, z) = 0 \right\}.$$

We know from Section 3.5, the homogeneity and non-singularity of F determine that X is a compact Riemann surface, and we aim to compute the genus of this Riemann surface. With π as a projection map defined by

$$\pi : X \to \mathbb{CP}^1 : [x : y : z] \mapsto [x : y],$$

we call on the fact[30] that a point $p \in X$ is a ramification point under π if and only if $\partial F / \partial z = 0$. If we take $\alpha = [1 : 0] \in \mathbb{CP}^1$ it cannot be the case that this is a branch point since $z \neq 0$ in order that $\pi^{-1}(\alpha) \subseteq X$. We must have that $1 + z^d = 0$, and this occurs only at the d roots of unity, all of which are non-zero. Therefore,

$$\deg(\pi) = d.$$

Proposition 4.25 tells us that this is the degree of any point in $\mathbb{CP}^1$. To proceed we consider the point $\beta = [1 : y] \in \mathbb{CP}^1$ such that $1 + y^d = 0$. For points of this form, it must be the case that pre-images have $z = 0$, and in particular, that $|\pi^{-1}(\beta)| = 1$. There are d solutions to the equation $1 + y^d = 0$, and since every point in $\mathbb{CP}^1$ can be represented in the form $[1 : \gamma]$ for some $\gamma \in \mathbb{C}$, we see that these d solutions are the only possible branch points in the codomain. Applying (and rearranging) the formula, we find that

$$g(X) = \frac{(d-1)(d-2)}{2}.$$

5

Differential forms

The following chapter gives an exposition of the basics of differential forms on manifolds, following Bott–Tu[30]. In this way, it is written with the same motivation as Chapter 2; we will conclude with specification to Riemann surfaces.

5.1 Differential forms on $\mathbb{R}^2$

Before considering what it means to integrate a function on a Riemann surface, or even a smooth manifold, we develop the theory of differential forms in the real plane. Differential forms provide an alternative framework to that of vector calculus; one which is often considered to be more simple and flexible[31]. Let x_1, x_2 be the standard coordinates on $\mathbb{R}^2$ and denote by Ω^* the space generated by $\mathrm{d}x_1$, $\mathrm{d}x_2$, governed by the relations,

$$\mathrm{d}x_i \,\mathrm{d}x_i = 0,$$
$$\mathrm{d}x_1 \,\mathrm{d}x_2 = -\,\mathrm{d}x_2 \,\mathrm{d}x_1.$$

[30] Raoul Bott and Loring W. Tu. *Differential Forms in Algebraic Topology.* Springer New York, NY, 2011

[31] Loring W. Tu. *An introduction to manifolds.* Universitext. Springer, 2008

Remark. The space Ω^* can be regarded as a vector space over $\mathbb{R}$, and has the basis,

$$1,\ \mathrm{d}x_1,\ \mathrm{d}x_2,\ \mathrm{d}x_1 \,\mathrm{d}x_2.$$

Definition 5.1 (Differential forms[32]). With Ω^* as above, the C^∞ *differential forms* are elements of,

$$\Omega^*\left(\mathbb{R}^2\right) = \left\{C^\infty \text{ functions on } \mathbb{R}^2\right\} \otimes_{\mathbb{R}} \Omega^*$$

where $\otimes_{\mathbb{R}}$ denotes the tensor product over $\mathbb{R}$.

[32] Bott and Tu, *Differential Forms in Algebraic Topology*, p.13

Remark. A differential form is simply an expression of the form,

$$\omega = \sum_I f_I \,\mathrm{d}x_I$$

where I denotes a multi-index, and the f_I are smooth.

There is also a natural decomposition (or grading[33]) of the space $\Omega^*(\mathbb{R}^2)$ as,

$$\Omega^*(\mathbb{R}^2) = \bigoplus_{i=0}^{2} \Omega^i(\mathbb{R}^2)$$

[33] Bott and Tu, *Differential Forms in Algebraic Topology*, p. 13

where the space $\Omega^i(\mathbb{R}^2)$ contains the i-forms, over $\mathbb{R}^2$. To dispense with abstraction, the space $\Omega^1(\mathbb{R}^2)$ can be recognised as,

$$\Omega^1(\mathbb{R}^2) = \left\{ f(x,y)\ \mathrm{d}x + g(x,y)\ \mathrm{d}y : f, g \in C^\infty\left(\mathbb{R}^2\right) \right\}.$$

Furthermore, we can naturally define a differential operator d which acts between these subspaces.

DEFINITION 5.2 (Exterior derivative). We define the *exterior derivative* to be the differential operator,

$$\mathrm{d} \colon \Omega^i(\mathbb{R}^2) \to \Omega^{i+1}(\mathbb{R}^2)$$

with the following conditions.

1. If $f \in \Omega^0(\mathbb{R}^2)$, then,
$$\mathrm{d}f = \frac{\partial f}{\partial x}\ \mathrm{d}x + \frac{\partial f}{\partial y}\ \mathrm{d}y.$$

2. If $\omega \in \Omega^*(\mathbb{R}^2)$ can be expressed as $\omega = \sum f_I\ \mathrm{d}x_I$ for multi-index I,
$$\mathrm{d}\omega = \sum \mathrm{d}f_I\ \mathrm{d}x_I.$$

PROPOSITION 5.3. $\mathrm{d}^2 = 0$.

Proof. Let us first consider concretely the form of the operator d in our context. We are given the form of the operator $d : \Omega^0(\mathbb{R}^2) \to \Omega^1(\mathbb{R}^2)$ in the definition, so it remains to consider the form of the operator as a function between $\Omega^1(\mathbb{R}^2)$ and $\Omega^2(\mathbb{R}^2)$.

Suppose that $\omega \in \Omega^1(\mathbb{R}^2)$, expressible as $f_1\ \mathrm{d}x_1 + f_2\ \mathrm{d}x_2$ for smooth 0-forms f_1, f_2. Then,

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}(f_1\ \mathrm{d}x_1 + f_2\ \mathrm{d}x_2) \\
&= \left(\frac{\partial f_1}{\partial x_1}\ \mathrm{d}x_1 + \frac{\partial f_1}{\partial x_2}\ \mathrm{d}x_2\right)\ \mathrm{d}x_1 + \left(\frac{\partial f_2}{\partial x_1}\ \mathrm{d}x_1 + \frac{\partial f_2}{\partial x_2}\ \mathrm{d}x_2\right)\ \mathrm{d}x_2 \\
&= \frac{\partial f_1}{\partial x_2}\ \mathrm{d}x_2\ \mathrm{d}x_1 + \frac{\partial f_2}{\partial x_1}\ \mathrm{d}x_1\ \mathrm{d}x_2 \\
&= \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)\ \mathrm{d}x_1\ \mathrm{d}x_2
\end{aligned}$$

using the fact that $\mathrm{d}x_1\ \mathrm{d}x_2 = -\ \mathrm{d}x_2\ \mathrm{d}x_1$.

At this point, we also note that for $\mathbb{R}^2$ the operator d^2 can only act between Ω^0 and Ω^2. Hence, we consider a smooth function f,

$$\begin{aligned}
\mathrm{d} \circ \mathrm{d}f &= \mathrm{d}\left(\frac{\partial f}{\partial x_1}\ \mathrm{d}x_1 + \frac{\partial f}{\partial x_2}\ \mathrm{d}x_2\right) \\
&= \left(\frac{\partial f}{\partial x_1 \partial x_2} - \frac{\partial f}{\partial x_2 \partial x_1}\right) = 0
\end{aligned}$$

with the final equality a consequence of the interchangeabiliy of partial derivatives for smooth functions (Clairaut's Theorem). $\square$

There is another important operation on differential forms, called the wedge/exterior product, which is easily defined in our framework.

Definition 5.4 (Exterior product). Let $\sigma, \tau \in \Omega^*(\mathbb{R}^2)$, be representable as $\sum_I f_I \, \mathrm{d}x_I$ and $\sum_J g_J \, \mathrm{d}x_J$ for multi-indices I, J respectively. Then, we define the *exterior product* of σ and τ, to be the operation,

$$\sigma \wedge \tau = \sum_I f_I \, \mathrm{d}x_I \wedge \sum_J g_J \, \mathrm{d}x_J = \sum_{I,J} f_I g_J \, \mathrm{d}x_I \, \mathrm{d}x_J.$$

Example 5.5. We consider the case of $\sigma, \tau \in \Omega^1(\mathbb{R}^2)$. Let $\sigma = f_1 \, \mathrm{d}x_1 + f_2 \, \mathrm{d}x_2$ and $\tau = g_1 \, \mathrm{d}x_1 + g_2 \, \mathrm{d}x_2$. Then,

$$\begin{aligned}
\sigma \wedge \tau &= (f_1 \, \mathrm{d}x_1 + f_2 \, \mathrm{d}x_2) \wedge (g_1 \, \mathrm{d}x_1 + g_2 \, \mathrm{d}x_2) \\
&= f_1 g_1 \, \mathrm{d}x_1 \, \mathrm{d}x_1 + f_1 g_2 \, \mathrm{d}x_1 \, \mathrm{d}x_2 + f_2 g_1 \, \mathrm{d}x_2 \, \mathrm{d}x_1 + f_2 g_2 \, \mathrm{d}x_2 \, \mathrm{d}x_2 \\
&= f_1 g_2 \, \mathrm{d}x_1 \, \mathrm{d}x_2 + f_2 g_1 \, \mathrm{d}x_2 \, \mathrm{d}x_1 \\
&= (f_1 g_2 - f_2 g_1) \, \mathrm{d}x_1 \, \mathrm{d}x_2
\end{aligned}$$

with each simplification justified by the original defining relations of the $\mathrm{d}x_i$.

5.1.1 The de Rham complex

$$\begin{array}{c}
\Omega^0 \left(\mathbb{R}^2\right) \\
\big\downarrow \mathrm{d} \\
\Omega^1 \left(\mathbb{R}^2\right) \\
\big\downarrow \mathrm{d} \\
\Omega^2 \left(\mathbb{R}^2\right)
\end{array}$$

Figure 5.1: The de Rham complex on $\mathbb{R}^2$.

In a somewhat implicit manner, we have introduced a cohomological concept, with further reaching consequences, than the one we initial desired. The differential operator d together with Ω^*, defines the *de Rham complex* on $\mathbb{R}^2$ (and by extension $\mathbb{R}^n$). In a more general treatment, the de Rham complex is an example of a *differential complex* since it can be expressed as a direct sum of vector spaces $\Omega^* = \bigoplus_{i=0}^2 \Omega^i$ where there are sequential homomorphisms defined by the operator d such that $\mathrm{d}^2 = 0$.

Definition 5.6 (de Rham cohomology groups). We define the ith *de Rham cohomology group* on $\mathbb{R}^n$ to be the vector space,

$$H^i\left(\mathbb{R}^n\right) = \frac{\ker\left(\mathrm{d} : \Omega^i\left(\mathbb{R}^2\right) \to \Omega^{i+1}\left(\mathbb{R}^2\right)\right)}{\operatorname{im}\left(\mathrm{d} : \Omega^{i-1}\left(\mathbb{R}^2\right) \to \Omega^i\left(\mathbb{R}^2\right)\right)}.$$

Of course, we are mostly interested in $\mathbb{R}^2$, and the definition can be easily specified to this account. We refer to the differential forms contained in the above kernel as *exact*, and those contained in the image as *closed*.

5.2 Differential forms on smooth manifolds

Since manifolds are by definition locally Euclidean, we aim to transport our ideas about differential forms in $\mathbb{R}^2$ into the manifold setting using the charts and coordinate functions which define the manifold structure.

Definition 5.7 (Pullback). Let u_1, u_2 and v_1, v_2 be standard coordinate systems on two subsets $U, V \subseteq \mathbb{R}^2$. For a C^∞ function $f : U \to V$ we define the pullback map f^* on 0-forms by,

$$\begin{aligned} f^* &: \Omega^0(V) \to \Omega^0(U) \\ &: g \mapsto g \circ f. \end{aligned}$$

So for smooth maps, the pullback is nothing more than pre-composition by a smooth function. We now aim to extend this idea to forms of higher order, that is 1- and 2-forms. If we impose[34] that the differential operator d should commute with the operation of 'pulling-back', there is a unique expression of f^*. Letting U, V be as in the definition, consider a 1-form $\omega \in \Omega^1(V)$, expressible as $\omega = \alpha \ \mathrm{d}v_1 + \beta \ \mathrm{d}v_2$, for smooth functions α, β. Then the pullback is defined as,

[34] Bott and Tu, *Differential Forms in Algebraic Topology*, p. 36

$$\begin{aligned} f^* &: \Omega^1(V) \to \Omega^1(U) \\ &: \omega \mapsto (\alpha \circ f) \left(\frac{\partial v_1}{\partial u_1} \ \mathrm{d}u_1 + \frac{\partial v_1}{\partial u_2} \ \mathrm{d}u_2 \right) \\ &\qquad + (\beta \circ f) \left(\frac{\partial v_2}{\partial u_1} \ \mathrm{d}u_1 + \frac{\partial v_2}{\partial u_2} \ \mathrm{d}u_2 \right). \end{aligned}$$

Remark. Imposing that the operation of pulling-back should commute with the differential operator d is equivalent to the assertion that d is independent of the system of coordinates.

Definition 5.8 (Differential form). For a smooth manifold M, defined by an atlas $\{U_i, \phi_i\}$, a C^∞ *differential form* is a collection of differential forms $\{\omega_i\}$, each defined on $\tilde{U}_i$ such that

$$U_i \cap U_j \neq \varnothing \implies \phi_i^* \omega_i = \phi_j^* \omega_j$$

on the intersection.

This is an intuitively logical extension of the theory, and while we won't consider smooth manifolds anymore, C^∞ forms will remain important, even in the complex setting.

5.3 Differential forms on Riemann surfaces

Thus far, our development of differential forms has been independent of complex structure. This complex structure is, however, integral to the definition of Riemann surfaces and we can use it to further explore differential forms on manifolds.

To begin with, we change our notation to one more in line with our complex considerations. Consider a local coordinate $z = x + iy$, and also the complex conjugate $\overline{z} = x - iy$. We can easily invert these relations,

$$x = \frac{z + \overline{z}}{2}, \qquad y = \frac{z - \overline{z}}{2i}$$

and can equally well express any C^∞ 1-form as,

$$\omega = f(z, \overline{z}) \ \mathrm{d}z + g(z, \overline{z}) \ \mathrm{d}\overline{z}.$$

Further to this, we can define the partial differential operators in this context by,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

which have associated differential operators,

$$\partial = \frac{\partial}{\partial z}\,\mathrm{d}z, \qquad \overline{\partial} = \frac{\partial}{\partial \overline{z}}\,\mathrm{d}\overline{z}.$$

Remark. The idea is that the additional complex structure can be summarised as a map $\star : \Omega^1 \to \Omega^1$ such that

$$\begin{aligned} \star\,\mathrm{d}x &= \mathrm{d}y, \\ \star\,\mathrm{d}y &= -\,\mathrm{d}x. \end{aligned}$$

From this, we can see that $\star^2 = -1$, and hence we can decompose the space of C^∞ 1-forms into the eigenspaces of this operator.

Lemma 5.9. For the differential operators $\mathrm{d}, \partial, \overline{\partial}$, we have the following identities,

$$\begin{aligned} \mathrm{d} &= \partial + \overline{\partial}, \\ \partial^2 = \overline{\partial}^2 &= \partial\overline{\partial} + \overline{\partial}\partial = 0. \end{aligned}$$

$$\begin{tikzcd} \Omega^{0,1} \arrow[r, "\partial"] & \Omega^2 \\ \Omega^0 \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"'] & \Omega^{1,0} \arrow[u, "\overline{\partial}"'] \end{tikzcd}$$

Figure 5.2: Visualising the action of ∂ and $\overline{\partial}$.

Notation. We denote by $\Omega^{1,0}(X)$ the space of differential 1-forms which are locally expressible as $f(z, \overline{z})\,\mathrm{d}z$, and by $\Omega^{0,1}(X)$ the space of differential forms locally expressible as $g(z, \overline{z})\,\mathrm{d}\overline{z}$.

This notation is particularly well chosen since it makes clear the fact that we are decomposing the space of smooth 1-forms $\Omega^1(X)$.

Definition 5.10 (Holomorphic/Meromorphic form). A *holomorphic 1-form* (*meromorphic 1-form*) on a Riemann surface X is a form which can be locally expressed by,

$$\omega = f(z)\,\mathrm{d}z$$

where f is holomorphic (meromorphic).

Definition 5.11 (Anti-holomorphic form). An *anti-holomorphic 1-form* on a Riemann surface X is a form which can be locally expressed by,

$$\omega = g(z)\,\mathrm{d}\overline{z}$$

where $\overline{g(z)}$ is holomorphic.

Notation. It is common to denote the holomorphic 1-forms on a Riemann surface X by $\mathcal{O}^1(X)$, the meromorphic 1-forms by $\mathcal{M}^1(X)$, and the anti-holomorphic 1-forms by $\overline{\mathcal{O}^1}(X)$.

There are two results, closely related, which we will need later.

Proposition 5.12. A smooth function $f \in \Omega^0(X)$ is holomorphic if and only if $\overline{\partial} f = 0$.

Proof. Fixing a local coordinate $z = x + iy$, we see that

$$\overline{\partial} f = 0 \iff \frac{\partial f}{\partial \overline{z}}\, \mathrm{d}\overline{z} = 0 \iff \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = 0,$$

which are exactly the Cauchy–Riemann equations for f. □

Proposition 5.13. A smooth function $g \in \Omega^0(X)$ is anti-holomorphic if and only if $\partial g = 0$.

These results, can be extended to the holomorphic and anti-holomorphic 1-forms, and in many cases[35] the condition that $\overline{\partial}\alpha = 0$ is the defining condition for $\alpha \in \mathcal{O}^1(X)$.

[35] Forster, *Lectures on Riemann surfaces*

Recall that a key integer value associated to a meromorphic map was the order. We can extend this notion to a meromorphic 1-forms as follows.

Definition 5.14 (Order). Let $p \in X$, $\omega \in \mathcal{M}^1(X)$. Given a local coordinate z centred on p, and a local representation of ω as $f(z)\, \mathrm{d}z$, the *order* of ω is defined as,

$$\operatorname{ord}(\omega; p) = \operatorname{ord}(f; 0).$$

There is a final result[36] in the realm of meromorphic forms and functions which will prove useful.

[36] Miranda, *Algebraic curves and Riemann surfaces*, p.132

Lemma 5.15. Let $\omega_1, \omega_2 \in \mathcal{M}^1(X)$ with $\omega_1 \not\equiv 0$. Then, there exists a unique function $f \in \mathcal{M}(X)$ such that

$$\omega_2 = f\omega_1.$$

Proof. Suppose that there exists a chart $\phi : U \to \tilde{U}$ which gives a local coordinate z, and suppose also that for $i = 1, 2$, $\omega_i = g_i(z)\, \mathrm{d}z$, where g_i are meromorphic functions on $\tilde{U}$. If we take $f = (g_2/g_1) \circ \phi$, this is a meromorphic function on U, and has the desired property. Showing that this function is independent of coordinate chart is straightforward, and this is indeed the function we desired. □

5.3.1 The Dolbeault complex

As for the de Rham complex with the differential operator d, we can define the Dolbeault complex with the operator $\overline{\partial}$, which satisfies the conditions for a differential complex since $\overline{\partial} \circ \overline{\partial} = 0$. For our consideration, there will be four particularly important vector spaces.

$$\begin{aligned}H^{0,0}(X) &= \ker\left(\overline{\partial} : \Omega^0 \to \Omega^{0,1}\right)\\ H^{1,0}(X) &= \ker\left(\overline{\partial} : \Omega^{1,0} \to \Omega^{2}\right)\\ H^{0,1}(X) &= \operatorname{coker}\left(\overline{\partial} : \Omega^0 \to \Omega^{0,1}\right) = \Omega^0 \big/ \operatorname{im}\left(\overline{\partial} : \Omega^0 \to \Omega^{0,1}\right)\\ H^{1,1}(X) &= \operatorname{coker}\left(\overline{\partial} : \Omega^{1,0} \to \Omega^{2}\right) = \Omega^{1,0} \big/ \operatorname{im}\left(\overline{\partial} : \Omega^{1,0} \to \Omega^{2}\right)\end{aligned}$$

The importance of the last two of these spaces is not at all obvious, but the first two can be easily understood as the spaces of holomorphic functions and holomorphic 1-forms.

5.4 Integration on Riemann surfaces

The work of this chapter has all been motivated by the prospect of defining integration on a Riemann surface, which is what we do now. As in complex analysis, there are some fundamental results contained in this area. We have defined the objects we want to integrate, i.e. 1- and 2-forms, so we now define[37] the sets over which we want to integrate.

[37] Miranda, *Algebraic curves and Riemann surfaces*, p. 118

Definition 5.16 (Path). A *path* on a Riemann surface X is a continuous, piecewise C^∞ function,

$$\gamma : [a, b] \subseteq \mathbb{R} \to X$$

where the interval $[a, b]$ is closed.

Remark. If $\gamma(a) = \gamma(b)$, we call the path *closed*. We call a path *simple* if it is either closed and injective on $[a, b)$, or injective.

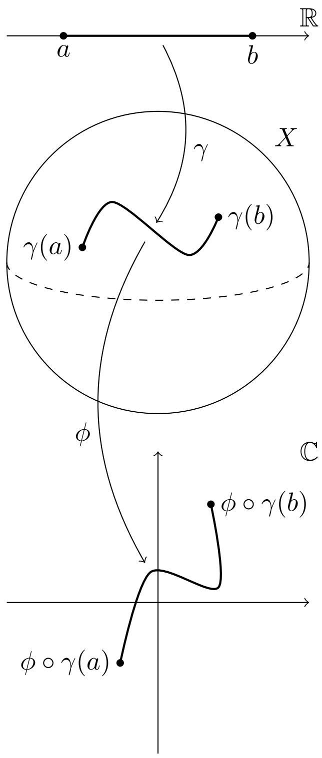


Figure 5.3: The integral of 1-form over a path on a Riemann surface.

Definition 5.17 (1-form integration). Let X be a Riemann surface, ω a C^∞ 1-form on X, and γ a path whose image is contained by the domain of a single chart in the atlas of X, $\phi : U \to \tilde{U}$. Suppose that ω can be locally represented by $f \ \mathrm{d}z + g \ \mathrm{d}\overline{z}$ in U. Then, the *integral* of ω along γ is,

$$\int_\gamma \omega = \int_{\phi\gamma} f \ \mathrm{d}z + g \ \mathrm{d}\overline{z}$$

where the right-hand side integral is the standard contour integral.

It is fairly clear that any path on a X may be arbitrarily partitioned, and in particular, can be partitioned such that each segment of γ, is contained in a single chart of the atlas, i.e., $\gamma_i \subseteq U_i$. In this way, we can define the integral of a 1-form over arbitrary paths on Riemann surfaces. We also note that the value of this integral is independent of the choice of partition, since every partition will have some common refinement.

Definition 5.18 (Residue). Let $\omega \in \mathcal{M}^1(X)$, which has local representation $f(z) \ \mathrm{d}z$ at a point $p \in X$. Consider a small loop γ in X

encircling p, then the *residue* of ω at p is,

$$\operatorname{Res}(\omega; p) = \frac{1}{2\pi i} \int_\gamma \omega.$$

Remark. To be exact, by a 'small loop' encircling p, we mean a path in X with p in its interior, containing no other poles of ω.

Donaldson[38] remarks that we may equally well consider the Laurent series expansion of the local representation of ω as

$$\omega = f(z)\ \mathrm{d}z = \sum_{j=-k}^{\infty} a_j z^j\ \mathrm{d}z,$$

and take $\operatorname{Res}(\omega; p) = a_{-1}$. This is the route which Miranda[39] takes. We can of course show that these two definitions give equivalent formulations. From complex analysis we know that the residue of a meromorphic function is an important quantity, and we will soon make a statement of similar intent. First, we consider what it means to integrate a 2-form on a Riemann surface.

[38] Donaldson, *Riemann Surfaces*, p. 76

[39] Miranda, *Algebraic curves and Riemann surfaces*, p. 121

Definition 5.19 (2-form integration). Let X be a Riemann surface, ρ a C^∞ 2-form on X, and T a triangular region contained by the domain of a single chart in the atlas of X, $\phi : U \to \tilde{U}$. Suppose that ρ has local representation $f(z, \overline{z})\ \mathrm{d}z\ \mathrm{d}\overline{z}$ in U. Then, the *integral* of ρ over T is,

$$\int_T \rho = \int_{\phi(T)} f(z, \overline{z})\ \mathrm{d}z\ \mathrm{d}\overline{z}.$$

Remark. To be pedantic, when we say a triangular region T on X, we really mean the homeomorphic image of some triangular (in the obvious sense) region in the complex plane.

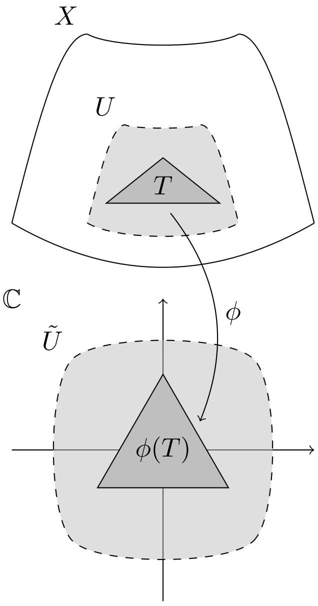


Figure 5.4: A integral over the triangular region T on a Riemann surface.

In the multivariate calculus setting there is a relation between the integral over a region and the integral along the boundary of this region. With the language of differential forms, we can make analogous statements on Riemann surfaces. We retain the meaning of T as a triangular region contained by a chart domain U on Riemann surface X in each of the following results.

Theorem 5.20 (Stokes' Theorem). Let $\omega \in \Omega^1(X)$. Then,

$$\oint_{\partial T} \omega = \int_T \mathrm{d}\omega.$$

Proof. Considering the left-hand side integral as the traversal of the boundary counter-clockwise as usual, we can use the definitions of each integral, combined with the local representation of the 1-form

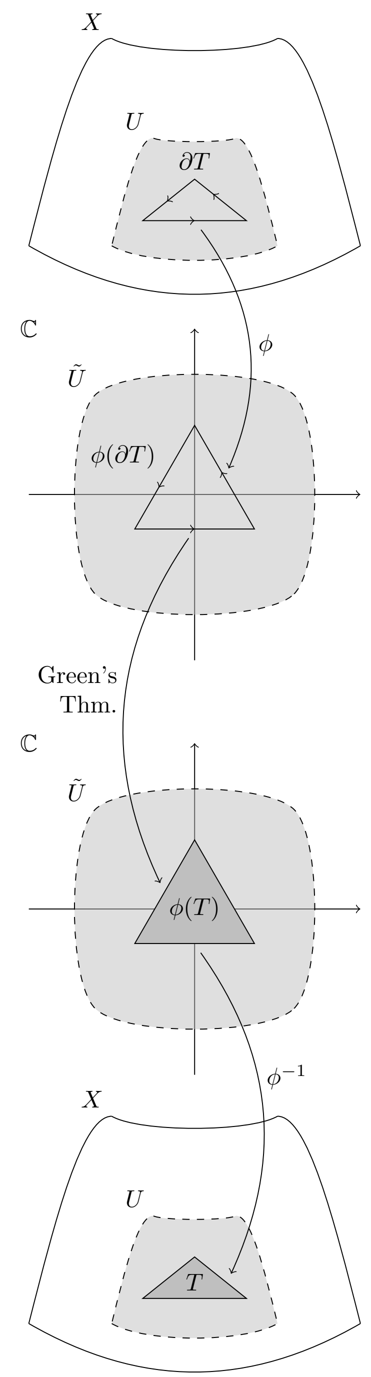


Figure 5.5: Visualizing the proof of Stokes' theorem.

$\omega = f(z, \overline{z})\ \mathrm{d}z + g(z, \overline{z})\ \mathrm{d}\overline{z}$,

$$\begin{aligned}\oint_{\partial T} \omega &= \oint_{\partial \phi(T)} f(z, \overline{z})\ \mathrm{d}z + g(z, \overline{z})\ \mathrm{d}\overline{z} \\ &= \int_{\phi(T)} \left(\frac{\partial g}{\partial z} - \frac{\partial f}{\partial \overline{z}} \right) \mathrm{d}z\ \mathrm{d}\overline{z} = \int_T \mathrm{d}\omega\end{aligned}$$

where the second equality is justified by Green's theorem in the plane. □

Theorem 5.21 (Residue Theorem). Let $\omega \in \Omega^1(X)$ for a compact Riemann surface X. Then,

$$\sum_{x \in X} \mathrm{Res}(\omega; x) = 0.$$

Proof. We know that the points at which $\mathrm{Res}(\omega; x) \neq 0$ are finite, since X is compact, and we hence label these points by $p_1, ..., p_n$. If we construct small, simple, closed paths about each of these poles, and denote these by γ_i, we have by definition that,

$$\mathrm{Res}(\omega; p_i) = \oint_{\gamma_i} \omega.$$

Furthermore, denote by Γ_i the interior of γ_i, and let $\Gamma = \bigsqcup_i \Gamma_i$. Then, $\partial(X \setminus \Gamma)$ is the disjoint union of the γ_i, and,

$$\begin{aligned}\sum_{i=1}^{n} \mathrm{Res}(\omega; p_i) &= \frac{1}{2\pi i} \sum_{i=1}^{n} \int_{\gamma_i} \omega \\ &= \frac{1}{2\pi i} \int_{\partial(X \setminus \Gamma)} \omega \\ &= \frac{1}{2\pi i} \int_{X \setminus \Gamma} \mathrm{d}\omega = 0\end{aligned}$$

with the final equality justified by the fact that $\mathrm{d}\omega = 0$ on $X \setminus \Gamma$, where ω is holomorphic, i.e., away from its poles. □

Corollary 5.22. Let f be a non-constant meromorphic function on compact Riemann surface X. Then,

$$\sum_{x \in X} \mathrm{ord}(f; x) = 0.$$

Proof. It can[40] be shown that for a non-constant meromorphic function f,

$$\mathrm{ord}(f; x) = \mathrm{Res}\left(\frac{\mathrm{d}f}{f}; x \right)$$

for all $x \in X$. Application of the Residue theorem to the 1-form $\mathrm{d}f/f$ gives the result. □

40 Forster, *Lectures on Riemann surfaces*, p. 80

6

Poisson's Equation

We reach the main chapter of the report, and prove[41] *a deep, analytic result for the to-be-defined Laplacian operator.*

6.1 Laplace operator

To give context to the developments of the chapter, it makes sense to state the result which we ultimately aim to prove. In order to state this result however, it is necessary to define the Laplacian operator.

[41] Donaldson, *Riemann Surfaces*

Definition 6.1 (Laplacian). For a Riemann surface X, we define the *Laplacian* to be the differential operator

$$\Delta : \Omega^0(X) \to \Omega^2(X) : f \mapsto 2i\overline{\partial}\partial f,$$

and say that the function f is *harmonic* if it satisfies,

$$\Delta f = 0.$$

Remark. This definition is chosen specifically such that it agrees, in local coordinates, with the familiar representation of the Laplace operator as

$$\begin{aligned}\Delta f &= \frac{2i}{4}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f \;\mathrm{d}z\; \mathrm{d}\overline{z} \\ &= -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f \;\mathrm{d}x\; \mathrm{d}y.\end{aligned}$$

Theorem 6.2. Let X be a compact Riemann surface and let $\rho \in \Omega^2(X)$. There exists a function $f \in \Omega^0(X)$ such that $\Delta f = \rho$ if and only if

$$\int_X \rho = 0$$

and this solution is unique up to an additive constant.

The above theorem is referred to as the 'Main Theorem' in Donaldson[42]. Intuitively, we can think of this statement as giving the necessary and sufficient conditions to invert the Laplacian, i.e., solve Poisson's equation, on a compact Riemann surface.

[42] Donaldson, *Riemann Surfaces*, p. 113

Initial progress on the proof of Theorem 6.2 is easy to come by, in particular in the 'if' direction.

Lemma 6.3. If there exists a function f such that $\Delta f = \rho$,

$$\int_X \rho = 0.$$

Proof. This is a corollary to Stokes' theorem.

$$\int_X \rho = \int_X \Delta f = 2i \int_X \overline{\partial}\partial f = 2i \int_X \mathrm{d}(\partial f) = 2i \int_\varnothing \partial f = 0$$

□

Lemma 6.4. If there exists a function f such that $\Delta f = \rho$, this function is unique up to addition by a constant.

Proof. A simple argument based on maximum modulus principle. □

So, the interesting content in Theorem 6.2 can be summarised as the following.

Theorem 6.5. If $\rho \in \Omega^2(X)$ is such that

$$\int_X \rho = 0,$$

there exists a function f such that $\Delta f = \rho$.

This is the result we aim to prove in the proceeding sections.

6.2 Dirichlet norm

Definition 6.6 (L^2 inner product). Let $\alpha, \beta \in \Omega^{1,0}(X)$. Then, we define the L^2 *inner product* to be,

$$\langle \alpha, \beta \rangle_{L^2} = \int_X i\alpha \wedge \overline{\beta}.$$

To show that this is definition is more familiar than is superficially obvious, let us consider a local coordinate $z = x + iy$ in which α and β are expressible as $f(z)\ \mathrm{d}z$ and $g(z)\ \mathrm{d}z$ respectively. Then,

$$\begin{aligned}\langle \alpha, \beta \rangle_{L^2} &= \int_X i\alpha \wedge \overline{\beta} \\ &= i \int_X (f(z)\ \mathrm{d}z) \wedge \overline{(g(z)\ \mathrm{d}z)} \\ &= i \int_X f(z)\overline{g(z)}\ \mathrm{d}z\ \mathrm{d}\overline{z} \\ &= 2 \int_X f\overline{g}\ \mathrm{d}x\ \mathrm{d}y,\end{aligned}$$

and we can define the associated norm,

$$\|\alpha\|_{L^2}^2 = \langle \alpha, \alpha \rangle_{L^2} = \int_X i\alpha \wedge \overline{\alpha} = 2 \int_X |f|^2\ \mathrm{d}x\ \mathrm{d}y,$$

which we refer to as the L^2 *norm*.

Lemma 6.7. $\|\cdot\|_{L^2}$ defines a norm on the compactly supported $(1,0)$-forms.

Proof. There is little to note apart from the fact that compactness ensures the finiteness of the defining integral. $\square$

Remark. The main observation here is that the L^2 inner product (and also L^2 norm) is coordinate independent.

We can easily extend our definition for the L^2 inner product (and consequently norm) to the C^∞ 1-forms, by associating these with their $(1,0)$ component. In particular, let $A, B \in \Omega^1(X)$, then,

$$\langle A, B\rangle_{L^2} = 2\left\langle A^{1,0}, B^{1,0}\right\rangle_{L^2} = 2i\int_X A^{1,0} \wedge B^{0,1},$$
$$\|A\|_{L^2}^2 = 2\left\|A^{1,0}\right\|_{L^2}^2.$$

Definition 6.8 (Dirichlet inner product). Let $f, g : X \to \mathbb{R}$ be C^∞, at least one compactly supported. The *Dirichlet inner product* is defined as

$$\langle f, g\rangle_{\mathcal{D}} = \langle\, \mathrm{d}f,\, \mathrm{d}g\rangle_{L^2},$$

and similarly, the *Dirichlet norm* is defined as,

$$\|f\|_{\mathcal{D}} = \|\, \mathrm{d}f\|_{L^2}.$$

Remark. We note that the Dirichlet 'norm' is in fact a 'semi-norm', since it is positive semi-definite rather than definite. We can see this by noting that the constant functions all have zero derivative. As a result of this, it is common to consider the space of functions $C^\infty(X,\mathbb{R})/\mathbb{R}$, i.e., the quotient of the space of C^∞ functions on X by the constant functions.

Lemma 6.9. If at least one of f, g have compact support,

$$\langle f, g\rangle_{\mathcal{D}} = \int_X f\Delta g = \int_X g\Delta f.$$

Proof. Following from the definitions outlined above, and recalling that f and g are real valued gives,

$$\begin{aligned}\langle f, g\rangle_{\mathcal{D}} &= \langle\, \mathrm{d}f,\, \mathrm{d}g\rangle_{L^2}\\ &= 2i\int_X \partial f \wedge \overline{\partial g}\\ &= 2i\int_X \partial f \wedge \overline{\partial} g.\end{aligned}$$

Then, integrating by parts gives,

$$2i\int_X \partial f \wedge \overline{\partial} g = 2i\int_X \partial\left(f\overline{\partial} g\right) - f\partial\overline{\partial} g.$$

Finally, applying Stokes' theorem to the first term in this expression,

$$\begin{aligned}2i\int_X \partial\left(f\overline{\partial}g\right) - 2i\int_X f\partial\overline{\partial}g &= -2i\int_X f\partial\overline{\partial}g \\ &= 2i\int_X f\overline{\partial}\partial g \\ &= \int_X f\Delta g.\end{aligned}$$

□

6.3 Changing viewpoint

It is at this point natural to ask how the constructions introduced in the previous section help us in proving Theorem 6.5. We now dedicate some time to explore this question in detail, ultimately repackaging the information contained in Theorem 6.5 to something more appropriate.

To begin, let us denote by H, the space $C^\infty(X;\mathbb{R})/\mathbb{R}$, which is an inner product space with respect to the Dirichlet inner product.

Remark. In particular, the inner product $\langle\cdot,\cdot\rangle_{\mathcal{D}}$ is positive definite in this case, since the constant functions are identified with the class which may be represented by the zero function, $[0]$. Moving forward, we will not make explicit distinction between a function $\psi \in C^\infty(X;\mathbb{R})$ and the equivalence class of functions represented by $[\psi] \in C^\infty(X;\mathbb{R})/\mathbb{R}$, since this distinction does little more than contribute notative complexity to the discussion.

It is clear that for $f \in H$, $\rho \in \Omega^2(X)$, the statement $\Delta f = \rho$ is equivalent to

$$\int_X g\left(\rho - \Delta f\right) = 0$$

for every function $g \in C^\infty(X;\mathbb{R})$. Furthermore,

$$\begin{aligned}\int_X g\left(\rho - \Delta f\right) &= \int_X g\rho - \int_X g\Delta f \\ &= \int_X g\rho - \langle g, f\rangle_{\mathcal{D}}\,.\end{aligned}$$

Consider now the functional defined by

$$\begin{aligned}\tilde{\rho} &: C^\infty(X;\mathbb{R}) \to \mathbb{R} \\ &: g \mapsto \int_X \rho g,\end{aligned}$$

and suppose that $\int_X \rho = 0$ as in the hypothesis of Theorem 6.5. We claim that this functional descends to a linear functional on H, which is justifiable by the following. Let $g \sim g'$ in $C^\infty(X;\mathbb{R})/\mathbb{R}$, that is let $g = g' + c$ for some $c \in \mathbb{R}$. Then,

$$\tilde{\rho}(g) = \int_X \rho g = \int_X \rho\left(g' + c\right) = \int_X \rho g' + \int_X \rho c = \int_X \rho g' = \tilde{\rho}(g').$$

Hence, we define the descended functional,

$$\begin{aligned}\hat{\rho} &: H \to \mathbb{R}\\ &: [g] \mapsto \int_X \rho g\end{aligned}$$

which allows us to repackage the information contained in Theorem 6.5.

THEOREM 6.5. If $\rho \in \Omega^2(X)$ is such that

$$\int_X \rho = 0,$$

there exists a function f such that

$$\hat{\rho}(g) = \langle g, f \rangle_{\mathcal{D}}$$

for all $g \in H$.

6.4 Riesz representation theorem

The previous section seems to have done little for progress aside from change the framework in which our problem is stated. This is, however, an important shift, and brings us to a well known result from functional analysis, which will prove integral.

THEOREM 6.10 (Riesz representation theorem). Let $\sigma : \mathcal{H} \to \mathbb{R}$ be a bounded linear function on a real Hilbert space $\mathcal{H}$. Then, there exists an element $z \in \mathcal{H}$, such that

$$\sigma(x) = \langle z, x \rangle$$

for all $x \in \mathcal{H}$.

Let us now consider the obstacles between our current position and application of this theorem.

1. The functional $\hat{\rho}$ may not be bounded.
2. The space H is not a Hilbert space.

We address these issues in turn.

6.4.1 Boundedness of $\hat{\rho}$

In order to show the boundedness of $\hat{\rho}$, we aim to apply a well known result due to Poincaré.

PROPOSITION 6.11 (Poincaré inequality). Let $D \subset \mathbb{R}^2$ be a circular disc of area A. Then for ψ defined on a superset of $\overline{D}$, there exists a constant C such that

$$\|\psi - \psi_D\|_{L^2(D)} \leq C \|\nabla \psi\|_{L^2(D)},$$

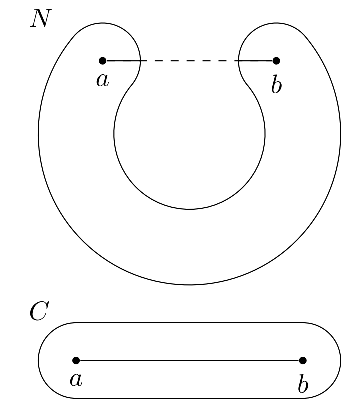


Figure 6.1: Proposition 6.11 actually holds for any bounded convex set C, but we need only consider the case of the disc. Above is a convex set C, and a non-convex set N.

where ψ_D denotes the average of ψ over D,

$$\psi_D = \frac{1}{A} \int_D \psi(y) \, \mathrm{d}y.$$

Remark. For reasons related to *partitions of unity* and an isomorphism between the second de Rham group and $\mathbb{R}$, it suffices to show that $\hat{\rho}$ is bounded when it is supported in a given chart of the Riemann surface[43].

Let (U, ϕ) be a chart, such that $\operatorname{supp} \rho \subset U$, and by homeomorphism identify its image $\tilde{U}$ with a circular disc D of area A. We have a local coordinate system (x, y) on D afforded to us by the coordinate map ϕ, and in this system, we can consider ρ as a function of integral 0, supported on D, and ψ as a function (initially on X) defined on some superset of $\overline{D}$.

These identifications allow us to express the functional in this local coordinate as

$$\hat{\rho}(\psi) = \int_D \rho\psi \, \mathrm{d}x \, \mathrm{d}y,$$

which is equivalently expressed as

$$\hat{\rho}(\psi) = \int_D \rho(\psi - \psi_D) \, \mathrm{d}x \, \mathrm{d}y,$$

since the integral of ρ is zero. This is exactly, the form (up to scaling) of the L^2 norm introduced earlier, and as such, we can utilise the Cauchy-Schwarz inequality to state that,

$$\left| \int_D \rho(\psi - \psi_D) \, \mathrm{d}x \, \mathrm{d}y \right| \leq \|\rho\|_{L^2(D)} \|\psi - \psi_D\|_{L^2(D)}.$$

A final series of relations, the first justified by Proposition 6.11, and the last by definition,

$$|\hat{\rho}(\psi)| \leq C\|\nabla\psi\|_{L^2(D)} \leq \|\nabla\psi\|_{L^2(X)} = C\|\psi\|_{\mathcal{D}}$$

proves the boundedness of the functional in U and, by the above remark, on X.

6.4.2 Completion

We know that the functional $\hat{\rho}$ is bounded, and hence it remains to find a Hilbert space which is related to H. We do this via the abstract completion.

Definition 6.12 (Completion). For an inner product space H, with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$, the *completion* of H with respect to $\| \cdot \|$, is a set of equivalence classes of Cauchy sequences $(g_i) \in H$ where,

$$(g_i) \sim (g_i') \iff \|g_i - g_i'\| \to 0.$$

43 Donaldson, *Riemann Surfaces*, p. 125

We denote by $\mathcal{H}$ the completion of H with respect to the Dirichlet norm $\|\cdot\|_{\mathcal{D}}$. At this stage, application of the Riesz representation theorem is obstructed solely by the fact that our functional $\hat{\rho}$ is defined on H, rather than $\mathcal{H}$. The following result helps us solve this minor issue.

LEMMA 6.13. Let U, V be two normed spaces with respective norms $\|\cdot\|_U$ and $\|\cdot\|_V$, and consider the linear map $\sigma : U \to V$. If σ is bounded, then $\sigma(u_i)$ is a Cauchy sequence in V for any Cauchy sequence (u_i) in U.

Proof. We recall that the boundedness of the linear map σ is definitively equivalent to the existence of a constant C such that

$$\|\sigma(u)\|_V \leq C\|u\|_U$$

for all $u \in U$. Since (u_i) is Cauchy,

$$\|u_n - u_m\|_U \to 0$$

as $n, m \to \infty$, and hence

$$\begin{aligned} \|\sigma(u_n) - \sigma(u_m)\|_V &= \|\sigma(u_n - u_m)\|_V \\ &\leq C\|u_n - u_m\|_U \to 0 \end{aligned}$$

as $n, m \to \infty$. □

In our setting, this result determines that the sequence $\hat{\rho}(g_i)$ is Cauchy in $\mathbb{R}$, and hence convergent. As a result, we can define the extension of $\hat{\rho}$ by,

$$\begin{aligned} \hat{\rho} &: \mathcal{H} \to \mathbb{R} \\ &: [(g_i)] \mapsto \lim_{i\to\infty} \hat{\rho}(g_i), \end{aligned}$$

which is a bounded linear map as needed. We choose to denote the extended functional in the same way as the original functional, since these are so closely related. Finally, we are in a position to apply Theorem 6.10, and can assert the existence of $f \in \mathcal{H}$ such that

$$\hat{\rho}(g) = \langle f, g\rangle_{\mathcal{D}}$$

for all $g \in \mathcal{H}$. A solution of this type is called a *weak solution*, and for the proof of Theorem 6.5 we must show that this weak solution is in fact a valid, smooth solution.

6.5 Weyl's Lemma

The following section concerns itself with the proof of a version of Weyl's lemma, which will tie up the proof of Theorem 6.5, in that it will determine that any weak solution is smooth.

PROPOSITION 6.14 (Weyl's Lemma). Let $D \subseteq \mathbb{C}$ be a bounded, open set, and let $\rho \in \Omega^2(D)$. If $\phi \in L^2(D)$, such that

$$\int_D \chi\Delta\phi = \int_D \chi\rho$$

for all compactly supported, smooth functions χ, ϕ is smooth, and satisfies $\Delta\phi = \rho$.

6.5.1 Relation to an L^2 function

To reiterate our current position, we have found a weak solution to the Posson equation, $f \in \mathcal{H}$ which is a Cauchy sequence (f_i) with respect to $\|\cdot\|_{\mathcal{D}}$ such that, for any g,

$$\langle f_i, g\rangle \to \hat{\rho}(g)$$

as $i \to \infty$. In order to apply Proposition 6.14 we aim to associate the Cauchy sequence with a locally L^2 function. We first consider this association in a single coordinate chart (U, ϕ), and identify $\phi(U)$ with a bounded open set $D \subseteq \mathbb{C}$. Since the underlying collection of functions H of our completion $\mathcal{H}$ is $C^\infty(X;\mathbb{R})/\mathbb{R}$, we may freely add constants to the functions f_i such that their integral vanishes over the region D. In particular, we can ensure that the average $[f_i]_D = 0$. Then,

$$\begin{aligned}\|f_n - f_m\|_{L^2(D)} &= \|f_n - f_m - [f_n - f_m]_D\|_{L^2(D)}\\ &\leq \|\nabla(f_n - f_m)\|_{L^2(D)}\\ &\leq \|f_n - f_m\|_{\mathcal{D}} \to 0\end{aligned}$$

as $n, m \to \infty$,which is enough to declare (f_i) as Cauchy in L^2. Furthermore, since $L^2(D)$ is complete, (f_i) converges to some function $f \in L^2(D)$.

Lemma 6.15. The sequence of functions (f_i) converges locally in L^2 on X.

Proof. We now aim to extend our above argument to account for all coordinate charts.

Let A be the collection of points $x \in X$ such that there exists a coordinate chart U_x in which $\phi_i \to \phi$ with respect to $\|\cdot\|_{L^2}$. A is by nature open, and non-empty by the previous argument. Furthermore, X is connected, and hence, we aim to show that A is closed, determining that $A = X$.

To proceed by contradiction, let us suppose that A is not closed. Taking, $x \in \overline{A} \setminus A$, and a neighbourhood U_x of x, we see that in the same way as above, there exists a sequence of real numbers (c_i) such that $(f_i - c_i)$ converges in $L^2(U_x)$. For any $y \in A \cap U_x$, both $(f_i - c_i)$ and (f_i) converge, and this forces that $c_i \to 0$, since these limits must equate as $i \to \infty$. This determines that x is in fact in A, which gives the desired contradiction. □

The outcome of the subsection is that we now have a function f on X which is locally square-integrable, and a weak solution to $\Delta f = \rho$.

6.5.2 The Newton potential

To proceed with the proof of Proposition 6.14, we consider the case where $\rho = 0$, i.e., the case where the L^2 function f is harmonic. We note also that since smoothness is a local property[44], we can further reduce our consideration to any interior set D' such that the ε-neighbourhood of D' is contained by D. For this case, a first necessary introduction is the Newton potential.

[44] Donaldson, *Riemann Surfaces*, p. 127

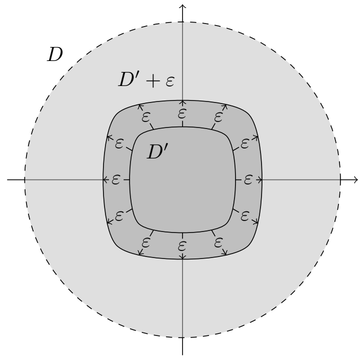


Figure 6.2: The ε-neighbourhood of D' is contained by D.

Definition 6.16 (Newton potential). The *Newton potential* is the function

$$K(x) = \frac{1}{2\pi} \log |x|.$$

Remark. We make a brief note of the fact that for any smooth function g which is compactly supported in $\mathbb{C}$, the convolution $K * g$ is smooth.

There is a very close relation between $K(x)$, and the operator Δ. This is well summarised by the following standard result.

Lemma 6.17. Let σ and τ be compactly supported in $\mathbb{C}$. Then,

$$K * (\Delta \sigma) = \sigma,$$
$$\Delta(K * \tau) = \tau.$$

Remark. This result is pertinent since it essentially states that convolution with the Newton potential K, is inverse to operation by Δ.

We recall the mean value property of harmonic functions[45], states that a smooth harmonic function ϕ defined on the neighbourhood of D' centered at 0, is such that,

[45] Rudin, *Real and complex analysis*, p. 237

$$\phi(0) = \phi_{\partial D'}.$$

Consider a smooth *cutoff* function $\beta : [0, \infty) \to \mathbb{R}$ defined such that it is constant for small r, and zero for $r > \varepsilon$, where $\varepsilon > 0$ is fixed. Furthermore, let β be normalized such that,

$$2\pi \int_0^\infty r\beta(r) \; \mathrm{d}r = 2\pi \int_0^\varepsilon r\beta(r) \; \mathrm{d}r = 1.$$

We now define a related function $B : z \mapsto \beta(|z|)$. It is clear that B is smooth, and we can also show that it has integral 1 over $\mathbb{C}$,

$$\int_{\mathbb{C}} B(z) \; \mathrm{d}z = \int_{\mathbb{C}} \beta(|z|) \; \mathrm{d}z = \int_0^{2\pi} \int_0^\infty r\beta(r) \; \mathrm{d}r \; \mathrm{d}\theta = 1.$$

Lemma 6.18. If f is a smooth, harmonic function on D', then $B * f = f$.

Proof. Translation invariance allows us to treat only the case where

$z = 0$. In this scenario,

$$\begin{aligned} B * f(0) &= \int_{\mathbb{C}} B(-z) f(z)\ \mathrm{d}z \\ &= \int_0^\infty \int_0^{2\pi} r\beta(r) f(re^{i\theta})\ \mathrm{d}\theta\ \mathrm{d}r \\ &= 2\pi f(0) \int_0^\infty r\beta(r)\ \mathrm{d}r \\ &= f(0). \end{aligned}$$

□

COROLLARY 6.19. Let $J \subset \mathbb{C}$ be compact, and let f be a smooth function on $\mathbb{C}$ such that $\operatorname{supp} \Delta f \subset \mathbb{C}$. Then $B * f = f$ outside the ε-neighbourhood of J.

At this point, the remainder of the argument is clear. We can combine the previous lemma, with what we know about the convolution as follows. If the function f is smooth on D, we must have that $B * f = f$ on any smooth interior domain D' of D. Furthermore, we know that for any L^2 function g, $B * g$ is smooth. As a result, there is an equivalence between proving the smoothness of f in D' and showing that $B * f = f$ in this domain.

Let χ be a smooth function such that $\operatorname{supp} \chi \subset D'$. We aim to show that given this arbitrary choice of function,

$$\langle \chi, f - B * f \rangle_{L^2(D')} = 0,$$

using the fact[46] that $\langle a, b * c \rangle = \langle b * a, c \rangle$. Then, dropping the subscript for brevity,

[46] Donaldson, *Riemann Surfaces*, p. 129

$$\langle \chi, f - B * f \rangle = \langle \chi, f \rangle - \langle B * \chi, f \rangle = \langle \chi - B * \chi, f \rangle.$$

We know from Lemma 6.17 that $\Delta(K * \chi) = \chi$, and hence $\operatorname{supp} \Delta(K * \chi) = \operatorname{supp} \chi \subseteq D'$. As a result, we can apply Corollary 6.19 to state that $B * (K * \chi) = K * \chi$ outside D'. Now, let $h = K * \chi - B * K * \chi = K * (\chi - B * \chi)$, which by the previous arguments is compactly supported in D. Since f is a weak solution, we have that, $\langle \Delta h, f \rangle = 0$, and rearranging, we have that,

$$\langle \Delta h, f \rangle = 0 \iff \langle \Delta(K * (\chi - B * \chi)), f \rangle = 0 \iff \langle \chi - B * \chi, f \rangle = 0.$$

The final piece in the proof of Proposition 6.14 is to show that this is sufficient to state that f is smooth on D for any $\rho \in \Omega^2(D)$. For any ρ, we can choose some ρ' which is equal to ρ on a neighbourhood of D' and compactly supported on D. We can find a weak solution $f' = K * \rho'$ to $\Delta f' = \rho'$ on D, which we know to be smooth. Then, the smoothness of f is equivalent to the smoothness of $f - f'$, and since we know that $\Delta(f - f') = \rho - \rho' = 0$ on D, this must be the case.

With the proof of Proposition 6.14 comes also the proof of Theorem 6.5, and we can move onto the consequences of this fundamental analytic result.

7

Riemann–Roch Theorem

With the conditions for inversion of the Laplacian behind us, we now prove the famous Riemann–Roch theorem.

7.1 Divisors

In order to state the Riemann–Roch theorem, we first introduce the notion of a divisor[47]. While initially an abstract construction, divisors will allow us to jointly consider the zeros and poles of a meromorphic function.

[47] Miranda, *Algebraic curves and Riemann surfaces*, p. 129

Definition 7.1 (Divisor). A *divisor* on a Riemann surface X is a function $D : X \to \mathbb{Z}$ which has discrete support in X.

With the introduction of a formal summation notation, the idea of a divisor become clearer. In particular, it is common to denote a divisor D by

$$D = \sum_{p \in X} D(p) \cdot p,$$

where $D(p) \neq 0$ only on a discrete subset of X.

Remark. We make an initial remark that in the same way as the discreteness of zeroes of meromorphic functions on all Riemann surfaces determined the finiteness of zeroes on *compact* Riemann surfaces, the support of a divisor on a compact Riemann surface is necessarily finite.

This remark is important, since it allows us to remove the formality of the sum in construction of the following.

Definition 7.2 (Degree). Let D be a divisor on a compact Riemann surface. The *degree* of D is the sum of its values,

$$\deg(D) = \sum_{p \in X} D(p).$$

There will be two particularly important examples of divisors for our progression towards the Riemann–Roch theorem; those related to meromorphic functions and 1-forms.

DEFINITION 7.3 (Principal divisor). A divisor is said to be *principal* if it can be associated to the order of a meromorphic function,

$$(f) = \sum_{p \in X} \mathrm{ord}(f; p) \cdot p.$$

EXAMPLE 7.4. We have completely classified the meromorphic functions on the Riemann sphere as the rational functions, and aim to determine a general form for the principal divisor associated to these functions. Let

$$f(z) = c \cdot \frac{\prod_{i=1}^{n} (z - z_i)^{e_i}}{\prod_{j=1}^{m} (z - z_j)^{f_j}}$$

be a general rational function on $\hat{\mathbb{C}}$. Then,

$$(f) = \sum_{i=1}^{n} e_i \cdot z_i - \left(\sum_{j=1}^{m} f_j \right) \cdot \infty,$$

since the function has zeroes of order e_i at the points $\{z_i\}$ and poles of order f_j at the points $\{z_j\}$.

DEFINITION 7.5 (Canonical divisor). A divisor is said to be *canonical* if it can be associated to the order of a meromorphic 1-form,

$$(\omega) = \sum_{p \in X} \mathrm{ord}(\omega; p) \cdot p.$$

REMARK. It is common to denote the set of all divisors on a Riemann surface X by $\mathrm{Div}(X)$, and this is in fact a group under pointwise addition. Furthermore, it follows directly from the definition that for $f, g \in \mathcal{M}(X) \setminus \{0\}$,

$$\begin{aligned}(fg) &= (f) + (g),\\ (f/g) &= (f) - (g),\\ (1/f) &= -(f),\end{aligned}$$

and in particular, the principal divisors form a subgroup of $\mathrm{Div}(X)$.

It is logical now to make an attempt to organise the divisors, and in this vein, we introduce a partial ordering. In particular, $D \geq 0$ if $D(p) \geq 0$ for all p, and $D \geq D'$ if $D - D' \geq 0$. From this partial ordering, we can define[48] some notable spaces of divisors.

[48] Miranda, *Algebraic curves and Riemann surfaces*, p. 146

DEFINITION 7.6. The space of meromorphic functions which have poles bounded by the divisor D is denoted and defined as,

$$L(D) = \{f \in \mathcal{M}(X) : (f) + D \geq 0\}.$$

Understanding of this space is most easily obtained by considering a single point $x \in X$. Suppose the divisor D is such that $D(x) = n > 0$. Then, the condition that $f \in L(D)$, is that $(f) + D \geq 0$ at x, which is in turn that $\operatorname{ord}(f; x) + n \geq 0$, i.e., the function f can have a pole of at *most* order n at the point x.

Now consider the alternative, that $D(x) = -n < 0$. Then, with the same logic as previously, $\operatorname{ord}(f; x) - n \geq 0$, and in particular, f must have a zero of at *least* order n at the point x.

> Remark. If we take D to be the trivial divisor, then the space of meromorphic functions bounded by this divisor is exactly the space of holomorphic functions. That is,
>
> $$L(0) = \mathcal{O}(X)$$
>
> which is the space of constant functions for any compact X.

We could choose to define an analogous space for the meromorphic 1-forms, although this would be unnecessary. Recall that by Lemma 5.15, two non-zero 1-forms are multiplicatively related by a unique meromorphic function. Therefore, for a canonical divisor $K = (\omega)$, the space

$$\begin{aligned} L(K - D) &= \{f \in \mathcal{M}(X) : (f) + K - D \geq 0\} \\ &= \{f \in \mathcal{M}(X) : (f) + (\omega) - D \geq 0\} \\ &= \{f \in \mathcal{M}(X) : (f\omega) - D \geq 0\} \\ &= \{f\omega \in \mathcal{M}^1(X) : (f\omega) - D \geq 0\} \end{aligned}$$

is exactly the analogous construction we wanted.

7.2 Cohomological precursors

Historically, the Riemann–Roch theorem came to fruition in two parts[49]. Riemann found a lower bound for the dimension of $L(D)$, and it was Riemann's student, Roch, who provided the term needed for equality. We will approach our consideration of this result in the same manner, but it is first necessary to prove some results related to the cohomological concepts we introduced in Chapter 5.

[49] Miranda, *Algebraic curves and Riemann surfaces*, p. 192

Lemma 7.7. The map

$$i : \overline{\mathcal{O}^1}(X) \to H^{0,1}(X) : \omega \mapsto [\omega]$$

is an isomorphism.

Proof. Firstly, we show that i is injective; in particular that $\ker(i)$ is trivial. To do this, consider an element $[\omega] \in H^{0,1}(X)$ such that $[\omega] = 0$. By the definition of the cokernel, this means that there exists some smooth function $f \in \Omega^0(X)$ such that

$$\begin{aligned} \overline{\partial} f &= \omega, \\ \partial\overline{\partial} f &= \partial\omega. \end{aligned}$$

By Proposition 5.13, we know that $\partial\omega = 0$, and in particular,

$$\Delta f = 0.$$

By Theorem 6.5 this function f must be unique up to additive constant, and since $\Delta 0 = 0$, f must be constant. This determines that

$$0 = \overline{\partial} f = \omega,$$

hence $\ker(i)$ is trivial, and i is injective.

Now, to show that i is surjective, consider an element $[\theta] \in H^{0,1}(X)$. We need to find an element $\omega \in \overline{\mathcal{O}^1}(X)$ such that $i(\omega) = [\theta]$ that is, we need to find a smooth function ϕ such that,

$$\omega = \theta + \overline{\partial}\phi.$$

This is equivalent to,

$$\begin{aligned} \partial\omega &= \partial\theta + \partial\overline{\partial}\phi \\ 0 &= \partial\theta + \partial\overline{\partial}\phi \\ \Delta\phi &= 2i\partial\theta. \end{aligned}$$

Since

$$\int_X \partial\theta = \int_X \mathrm{d}\theta = \int_{\partial X} \theta = 0,$$

Theorem 6.5 determines the existence of ϕ. $\square$

Remark. This provides some unification of our understanding. It was clear that $\mathcal{O}^1(X) \cong H^{1,0}(X)$, and we know have that $\overline{\mathcal{O}^1}(X) \cong H^{0,1}(X)$.

Lemma 7.8. The map

$$v : \mathcal{O}^1(X) \oplus \overline{\mathcal{O}^1}(X) \to H^1(X) : (\omega_1, \omega_2) \mapsto [\omega_1 + \omega_2]$$

is an isomorphism.

Proof. The proof uses a similar strategy to that employed for the previous lemma, and we choose not to spell out the details. $\square$

With these two lemmas, both of which are essentially consequences of Theorem 6.5, we obtain the following result.

Theorem 7.9 (Hodge decomposition theorem). For a compact Riemann surface X,

$$H^1(X) \cong H^{1,0}(X) \oplus H^{0,1}(X).$$

It can be shown[50], that the dimension of the first de Rham cohomology group $H^1(X)$ for a compact Riemann surface X is equal to $2g$ where g is the genus. We are most interested in the dimension of the Dolbeault cohomology groups, for reasons which will soon become clear, and the following result quenches this interest.

[50] Griffiths and Harris, *Principles of algebraic geometry*, p. 80

LEMMA 7.10. For a compact Riemann surface X, $H^{1,0}(X) \cong H^{0,1}(X)$.

Proof. Recognising that the conjugation map,

$$\overline{\cdot} : \mathcal{O}^1(X) \to \overline{\mathcal{O}^1}(X) : \omega \mapsto \overline{\omega}$$

is an invertible, linear transformation and equivalently an isomorphism, gives,

$$H^{1,0}(X) \cong \mathcal{O}^1(X) \cong \overline{\mathcal{O}^1}(X) \cong H^{0,1}(X)$$

as needed. □

In particular, a basic linear algebraic argument allows us to state that the dimension of both of $H^{1,0}(X)$ and $H^{0,1}(X)$ is g for a compact Riemann surface of genus g, i.e., half of the dimension of $H^1(X)$.

COROLLARY 7.11. For a compact Riemann surface X, the dual space $(H^{0,1}(X))^*$ is isomorphic to $H^{1,0}(X)$.

Proof. Since vector spaces of the same finite dimension are isomorphic,

$$H^{1,0}(X) \cong H^{0,1}(X) \cong (H^{0,1}(X))^*.$$

□

7.3 Riemann–Roch theorem

In the first part of this section, we aim to solve the problem of existence of meromorphic functions. Given the results, we have obtained in the previous section, it will be useful to rephrase our problem to relate to the Dolbeault cohomological groups.

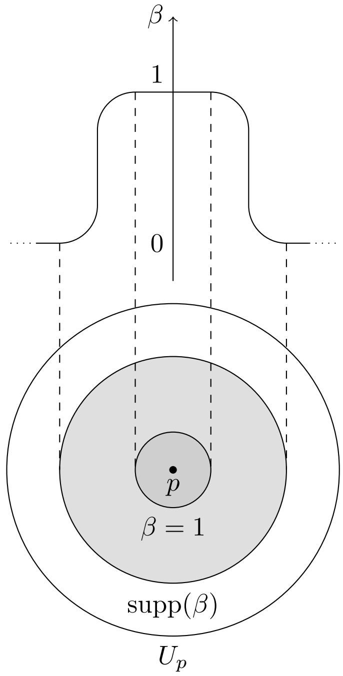


Figure 7.1: Constructing the smooth cutoff function β.

LEMMA 7.12. $\hat{\mathbb{C}}$ admits a non-constant meromorphic function.

Proof. Let us consider a single point $p \in \hat{\mathbb{C}}$. We aim to construct a meromorphic function on $\hat{\mathbb{C}}$ which has a simple pole at p. In order to do this, consider the chart domain U_p containing p, and let z be a local coordinate in this chart. In this chart domain, $1/z$ is a meromorphic function, and we can extend this globally by introducing a *cutoff function* β. We define β such that $\mathrm{supp}(\beta) \subset U_p$, and equal to 1 near p.

The product β/z can be regarded as a function over X; extending by zero outside of U_p. Hence, the problem of finding a meromorphic function with a simple pole at p is equivalent to finding a function $g \in \Omega^0(X)$ such that $g + \beta/z \in \mathcal{O}(X \setminus \{p\})$.

By Proposition 5.12 we know that this is equivalent to,

$$\overline{\partial}\left(g + \beta \cdot \frac{1}{z}\right) = 0 \iff \overline{\partial} g = -\overline{\partial}\left(\beta \cdot \frac{1}{z}\right).$$

A further application of Proposition 5.12, and relabelling $A = \overline{\partial}(\beta)1/z$, gives the simplification,

$$\overline{\partial} g = -A.$$

If we consider the corresponding class $[A] \in H^{0,1}(X)$, we can see that the existence of a smooth function g, is equivalent to the condition that $A \in \mathrm{im}(\overline{\partial})$, which is in turn equivalent to $[A] = [0] \in H^{0,1}(X)$. Our work in the previous section allows us to state that $\dim H^{0,1}(X) = 0$, which is enough to guarantee the existence of the function g. □

This is the most basic case in the proof of the following theorem. While we will not provide a proof, it is intuitively clear how to proceed given a Riemann surface of higher genus; adding sufficiently many poles, or in particular sufficiently increasing $\deg(D)$, forces a linear dependence between the elements of the $H^{0,1}(X)$, which allows for the construction of the promised meromorphic function.

THEOREM 7.13 (Riemann–Roch inequality). For a compact Riemann surface X of genus g, and a divisor D,

$$\dim L(D) \geq \deg(D) + 1 - g.$$

COROLLARY 7.14. Every compact Riemann surface has a non-constant meromorphic function.

Proof. This follows directly from Riemann's inequality. If we take $\deg(D) > g$, then

$$\dim L(D) \geq \deg(D) + 1 - g > g + 1 - g = 1,$$

and hence there must be a non-constant function in $L(D)$. □

COROLLARY 7.15. If X is a compact Riemann surface of genus 0, then $X \cong \hat{\mathbb{C}}$.

Proof. If we take $\deg(D) = 1$,

$$\dim L(D) \geq \deg(D) + 1 - g = 2,$$

and in particular, there exists a meromorphic function which has a single, simple pole. Let p be this point, let $\varphi : X \to \mathbb{C}$ be this function, and $\Phi : X \to \hat{\mathbb{C}}$ the corresponding holomorphic representation. Then $\deg(\Phi) = \deg(\Phi; \infty) = \mathrm{mult}(\Phi; p) = 1$, and hence Φ is bijective, and further to this, a biholomorphism. □

As mentioned previously, it was Gustav Roch, a student of Riemann, who strengthened the inequality to an equality by incorporating the space of meromorphic 1-forms associated to a canonical divisor. We will prove the first case of the theorem, that is in the case where D is a non-negative divisor, following Donaldson[51]. We need a final definition to prove the theorem in this restricted form.

[51] Donaldson, *Riemann Surfaces*, p. 115

Definition 7.16 (Tangent residue). Let $p \in X$ for a compact Riemann surface X. Let f have a local Laurent series expansion about the point p as,

$$f = \sum_{i=-1}^{\infty} a_i z^i$$

where z is the local coordinate centred at p. Then, the *tangent residue* of f is $a_{-1}\frac{\partial}{\partial z} \in TX_p$ where TX_p denotes the tangent space at p.

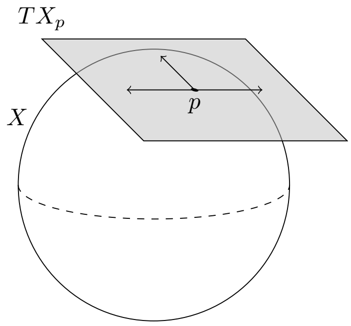


Figure 7.2: A visualisation of $T\hat{\mathbb{C}}_p$.

Theorem 7.17 (Riemann–Roch theorem). Let X be a compact Riemann surface, and K a canonical divisor on X. Then, for any divisor D,

$$\dim L(D) - \dim L(K - D) = \deg(D) + 1 - g.$$

Proof. As stated above, we will consider the case where $D = p_1 + \cdots + p_d$ for p_i distinct points of X. In this case, $L(D)$ is the space of meromorphic functions which have at worst simple poles at each of the p_i. Furthermore, the space $L(K - D)$ can be recognised as the space of holomorphic 1-forms which vanish at each of the p_i. This is understood by considering the fact that $\omega \in L(K-D) \implies (\omega) \geq D$, which imposes that $\operatorname{ord}(\omega; p_i) \geq 1$ for each $1 \leq i \leq d$.

We aim to build the proof in stages, in an argument which relies heavily on results from linear algebra. Firstly, consider the inclusion defined as

$$\begin{aligned} I : \mathbb{C} \to L(D) : c \mapsto f_c &: \mathbb{C} \to \mathbb{C} \\ &: z \mapsto c, \end{aligned}$$

i.e., a map which takes the constants in $\mathbb{C}$ to their respective constant maps f_c. We can make initial note of the fact that the image of I, $\operatorname{im} I$, is the set of complex valued constant functions.

We consider also the map

$$R : L(D) \to \bigoplus_{i=1}^{d} TX_{p_i},$$

which maps the meromorphic function f to its tangent residues in the local coordinates z_i of each of the p_i. The kernel of the map, is the collection of functions which have zero residue at every one of the p_i, and since we are considering the case of $D(p) \in \{0, 1\}$, this is exactly the set of holomorphic functions. Furthermore, we know that holomorphic functions are constant, and therefore, $\ker R = \operatorname{im} I$. In particular, we have an exact sequence of vector spaces,

$$\mathbb{C} \xrightarrow{I} L(D) \xrightarrow{R} \bigoplus_{i=1}^{d} TX_{p_i}.$$

We can relate the dimensions of these spaces using the rank nullity theorem,

$$\begin{aligned}\dim L(D) &= \dim \operatorname{im} R + \dim \ker R \\ &= \dim \operatorname{im} R + \dim \ker I \\ &= \dim \operatorname{im} R + 1,\end{aligned}$$

hence our interest now lies in the space $\operatorname{im} R$.

In the same vein as our previous work, we aim to find a map whose kernel is the same as $\operatorname{im} R$. With this in mind, we consider the family of maps

$$A_i : TX_{p_i} \to H^{0,1}(X),$$

which for a local coordinate z_i centered on p_i, maps $\partial/\partial z_i$ to the cohomology class $[A_i]$ where A_i is defined in the same way as in the proof of Lemma 7.12, that is, as the global extension of $\overline{\partial}(\beta_i \cdot 1/z_i)$. We therefore have a map

$$\underline{A} : \bigoplus_{i=1}^{d} TX_{p_i} \to H^{0,1}(X) : (t_1, ..., t_d) \mapsto \sum_{i=1}^{d} A_i(t_i).$$

and aim to show that $\operatorname{im} R = \ker \underline{A}$. To begin, consider an element $R(f) = (\lambda_1 \cdot \partial/\partial z_1, ..., \lambda_d \cdot \partial/\partial z_d)$, and the map

$$F = f - \sum_{i=1}^{d} \lambda_i \beta_i \frac{1}{z_i},$$

which is, by design a smooth function on X. Therefore, $[\overline{\partial} F] = 0$ in $H^{0,1}(X)$, and furthermore, since f is holomorphic away from the points p_i, $\overline{\partial} f = 0$ away from these points, and a natural extension to 0 over these points gives,

$$\begin{aligned}\left[\overline{\partial}\left(\sum_{i=1}^{d} \lambda_i \beta_i \frac{1}{z_i}\right)\right] &= [\overline{\partial} f] \\ \left[\overline{\partial}\left(\sum_{i=1}^{d} \lambda_i \beta_i \frac{1}{z_i}\right)\right] &= 0 \\ \sum_{i=1}^{d} \lambda_i [A_i] &= 0,\end{aligned}$$

and in particular, $\operatorname{im} R \subseteq \ker \underline{A}$. To show the other inclusion, consider $\underline{\alpha} = (\alpha_1 \partial/\partial z_1, ..., \alpha_d \partial/\partial z_d) \in \ker \underline{A}$. By the definition of $\underline{A}$, this is equivalent to

$$\underline{A}(\underline{\alpha}) = 0 \in H^{0,1}(X),$$

which is equivalent to the existence of a function $g \in \Omega^0(X)$ such that,

$$\overline{\partial} g = -\underline{A}(\underline{\alpha}).$$

Rearranging this expression gives us that

$$\underbrace{\overline{\partial}\left(g+\alpha_1\beta_1\frac{1}{z_1}+\cdots+\alpha_d\beta_d\frac{1}{z_d}\right)}_{G\in L(D)}=0,$$

and since $R(G)=\underline{\alpha}$, $\ker \underline{A}\subseteq \operatorname{im} R$, and in particular $\operatorname{im} R=\ker \underline{A}$. Hence we can extend our exact sequence,

$$\mathbb{C}\xrightarrow{I}L(D)\xrightarrow{R}\bigoplus_{i=1}^{d}TX_{p_i}\xrightarrow{\underline{A}}H^{0,1}(X),$$

and rephrase our previous expression as,

$$\dim L(D)=\dim\ker\underline{A}+1.$$

In order to compute $\dim\ker\underline{A}$, we consider the transpose of this map,

$$\underline{A}^T:\left(H^{0,1}(X)\right)^*\to\left(\bigoplus_{i=1}^{d}TX_{p_i}\right)^*=\bigoplus_{i=1}^{d}T^*X_{p_i},$$

and use elementary linear algebraic relations, together with the fact that $\dim H^{0,1}(X)=g$,

$$\begin{aligned}\dim\bigoplus TX_{p_i}-\dim H^{0,1}(X)&=\dim\operatorname{im}\underline{A}+\dim\ker\underline{A}\\&\quad-\dim\operatorname{im}\underline{A}^T-\dim\ker\underline{A}^T\\ \deg(D)-g&=\dim\ker\underline{A}-\dim\ker\underline{A}^T.\end{aligned}$$

From Corollary 7.11 we know that $(H^{0,1}(X))^*\cong H^{1,0}(X)$ and consequently, we define a final map

$$E:H^{1,0}\to\bigoplus_{i=1}^{d}T^*X_{p_i}:\omega\mapsto(\omega(p_1),...,\omega(p_d)).$$

It can be shown[52] that $\underline{A}^T=2\pi iE$, and noticing that $\ker E$ is the space of holomorphic 1-forms which vanish at every point p_i gives,

$$\begin{aligned}\dim L(D)&=1+\dim\ker\underline{A}\\ \dim L(D)-\dim\ker\underline{A}^T&=1+\deg(D)-g\\ \dim L(D)-\dim\ker E&=1+\deg(D)-g\\ \dim L(D)-\dim L(K-D)&=1+\deg(D)-g.\end{aligned}$$

□

[52] Donaldson, *Riemann Surfaces*, p. 116

Conclusion

With the proof of the Riemann–Roch theorem in its restricted form comes the end of this report. Discussion of the full proof of the theorem would involve some preliminary work with sheaf theory, and unfortunately, any attempt to introduce the necessary constructions would be either rushed or incomplete. A full treatment of the Riemann–Roch theorem can be found in Forster[53], and there are clear similarities between this proof and the one which we have presented.

[53] Forster, *Lectures on Riemann surfaces*

A natural next step in the study of Riemann surfaces would be to explore the Uniformisation theorem, which is a result aimed at the classification of the simply connected Riemann surfaces, conjectured in 1882 by Felix Klein "in the midst of an asthma attack"[54].

[54] William Abikoff. "The Uniformization Theorem". In: *The American mathematical monthly* 88 (1981), pp. 574–

THEOREM 7.18 (Uniformisation Theorem). Every simply connected Riemann surface is biholomorphic to one of $\mathbb{C}$, $\hat{\mathbb{C}}$, or the unit disc $\mathbb{D}$.

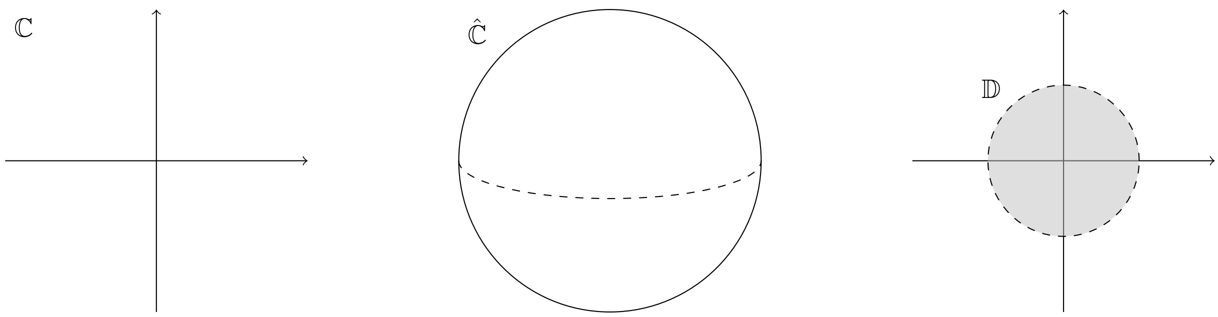


Figure 7.3: Drawing every simply connected Riemann surface.

Donaldson[55] explores this result in full detail, and the proof ultimately hinges on an analogue of the main analytic result of this report, Theorem 6.5.

[55] Donaldson, *Riemann Surfaces*, Ch. 10

There are some further reaching consequences of the existence of meromorphic functions which we did not explore. The most famous of these consequences was proved by Riemann, and was the initial motivation for his pursuit of a bound on the number of linearly independent meromorphic functions. Riemann proved that all compact Riemann surfaces can be embedded in projective space, and hence the study of compact Riemann surfaces can be rephrased as the study of one dimensional projective varieties.

This duality in the theory of projective geometry and analytic manifold theory is extended in higher dimensions, and one may consider further exploration in the direction of closely related concepts such as Riemannian manifolds, and further to this Kähler manifolds. In fact, fixing some metric on the Riemann surface gives some element of simplification to the arguments presented in Chapter 6, and many of the results presented admit logical generalisations.

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