# The Artificial Intelligence Toolbox Part II – CS26210

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# Using Qwizdom QVR

- On any web-enabled device go to:
- http://qvr.qwizdom.com
- Select **I have a Session Key**
- Enter the code **Q5VN94**

If you aren't already using AU Eduroam wireless have a look at

http://www.inf.aber.ac.uk/advisory/faq/253

# Program

Week 1

7/02 Set Theory, Fuzzy Logic (319)
8/02 Fuzzy Logic (B20) - Hand-out Assignment 1

Week 2

14/02 Fuzzy Logic - Further Exercises (319)
15/02 Theory of Probability (B20)

Week 3

21/02 Conditional Probability (319)
22/02 Probabilistic Methods (B20) - Hand-in Assignment 1 (Blackboard)

Week 4

28/02 Bayesian Networks (B20)
1/03 In Class Test (319) (Set Theory, Theory of Probability, Conditional Probability)

Week 5

7/03 Bayesian networks (319)
8/03 Discussion, further exercises (B20) - Hand-out Assignment 2
22/03 Hand-in Assignment 2 (Blackboard)

# Friday 15st, February 2013

- Sample Space
- Events
- Probability
- Disjoint Events
- Joint Events
- How to calculate probabilities
- Conditional probability

# Thursday 21$^{st}$, February 2013

- $P(A|B)$, $P(A \cup B)$, and $P(A \cap B)$
- Cond. Independence
- Product Rule
- Discrete Random Variable

# Friday 22nd, February 2013

- Joint Probability Distribution
- Full Joint Probability Distribution Table
- First-order Markov Chain
- Dempster-Shafter Theory of Evidence

# True or False

- An event is not a subset of the sample space

# True or False

Experiment = drawing the top card from a deck of 52

Sample space S = {1,..., 52};

The probability of picking up a queen or a king is:

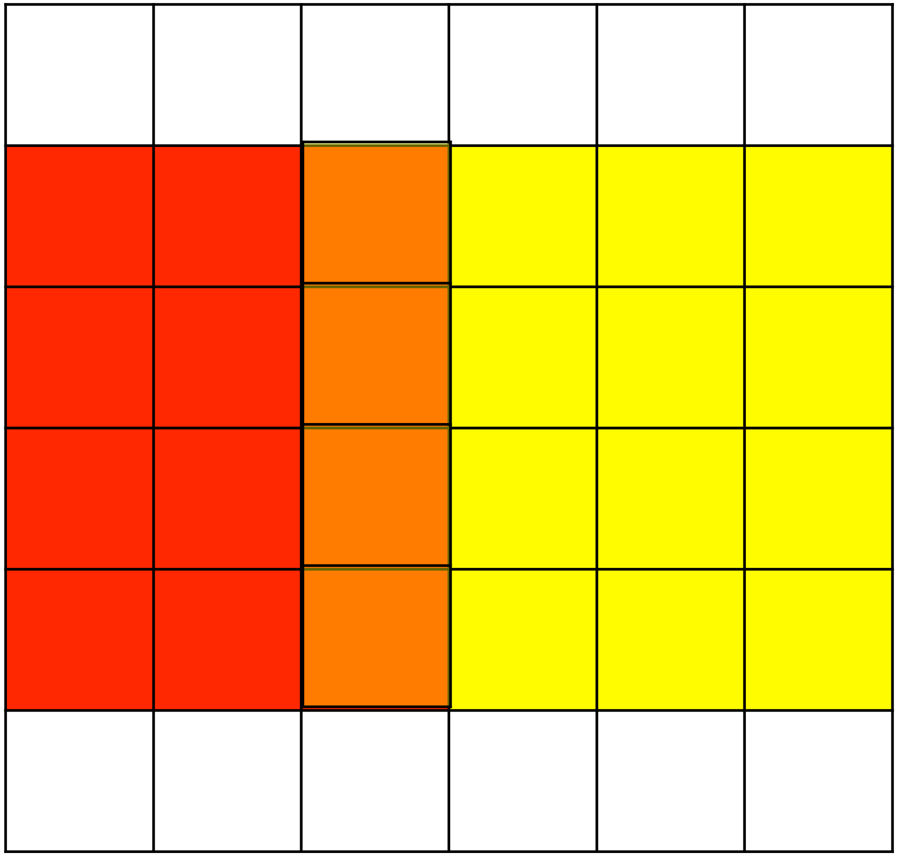
$$P(Q \cap K)$$

# Choose among the options

If two fair coins are flipped, what is the probability that the two faces are different?

A) 1/2
B) 1/3
C) 1
D) 2

# $P(A \mid B)$, $P(A \cup B)$, and $P(A \cap B)$

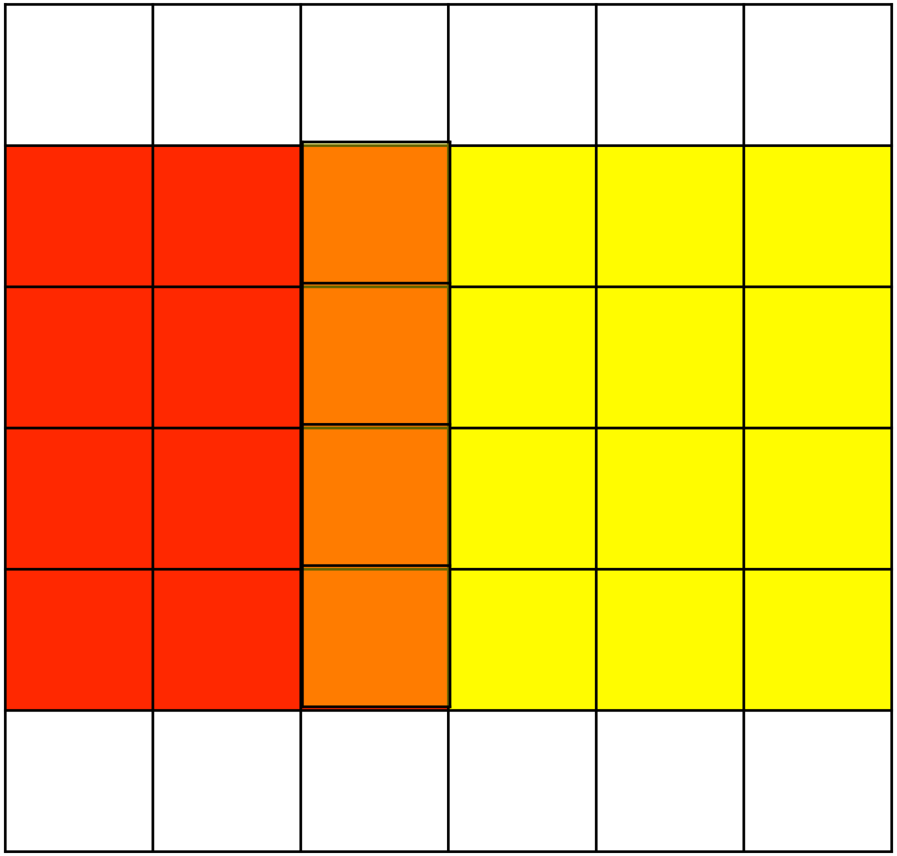


Conditional Probability

A = red $P(A) = 12/36 = 1/3$
B = yellow $P(B) = 16/36=4/9$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = 1/4$$

# $P(A|B)$, $P(A \cup B)$, and $P(A \cap B)$



Probability

Joint events $(A \cap B) \neq \emptyset$

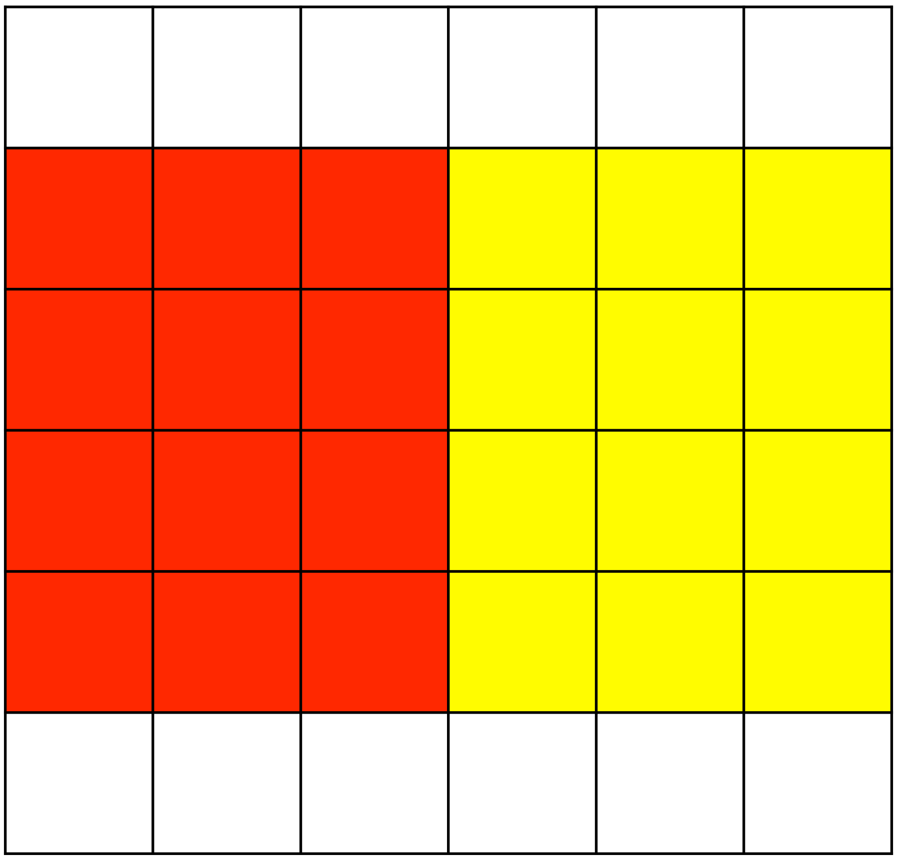
A = red $P(A) = 12/36 = 1/3$

B = yellow $P(B) = 16/36 = 4/9$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A) + P(B) - P(A \cup B)$

# $P(A \mid B)$, $P(A \cup B)$, and $P(A \cap B)$



Probability

Disjoint events $(A \cap B) = \emptyset$

A = red $P(A) = 12/36 = 1/3$

B = yellow $P(B) = 12/36=1/3$

$P(A \cup B) = P(A) + P(B)$

$P(A \cap B) = \emptyset$

# Conditional Independence

To say that two events (A and B) are **independent** means that the occurrence of one event (B) makes it neither more nor less probable that the other event (A) occurs

$$P(A|B) = P(A)$$

# Conditional Independence
# $P(A|B) = P(A)$

Experiment:
Drawing the top card from a 52-card deck.

We are told that the card is a spades?

What is the probability that the card is a queen?

# Conditional Independence

## $P(A|B) = P(A)$

Event B – the card is a spades
B={x: x=1,...,13; spades}
P(B) = 13/52

Event A – the card is a queen A={x: x = queen}
P(A) = 4/52 = 1/13

$P(A \cap B) = 1/52$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{(1/52)}{(13/52)} = 1/13$$

Finding out that the card is a spade does
not make it more or less probably that it is a queen.

# Conditional Independence

Experiments: deck of 52 cards

If two cards are drawn *with* replacement from the deck of cards, the event of drawing a red card on the first trial P(B) and that of drawing a red card on the second trial P(A) are INDEPENDENT

$$P(B) = 26/52$$
$$P(A|B) = 26/52$$

# Conditional Independence

Experiments: deck of 52 cards

If two cards are drawn *without* replacement from a deck of cards, the event of drawing a red card on the first trial P(B) and that of drawing a red card on the second trial P(A) are NOT INDEPENDENT

$$P(B) = 26/52$$
$$P(A|B) = 25/51$$

# Conditional Independence

Two events A and B are conditionally independent given C if

$$P(A \mid B \cap C) = P(A \mid C)$$

$$P(A \cap B \mid C) = P(A \mid C)\, P(B \mid C)$$

# Conditional Independence


$$P(\text{one}) = 5/13$$

$$P(\text{one} \mid \text{square}) = 3/8$$

one and square are not conditionally independent

$$P(\text{one} \mid \text{black}) = 3/9 = 1/3$$

$$P(\text{one} \mid \text{square} \cap \text{black}) = 2/6 = 1/3$$

$$P(\text{one} \mid \text{white}) = 1/4 = 1/2$$

$$P(\text{one} \mid \text{square} \cap \text{white}) = 1/2$$

one and square are conditionally independent given black/white

$P(A \mid C) = P(A \mid B \cap C)$ A and B are Cond. Indep. Given C

# Conditional Independence

Two events A and B are conditionally independent given C if

$$P(A \mid C) = P(A \mid B \cap C)$$

We have just seen, in our example that

One (A) and square (B) are conditionally independent given black/white (C)

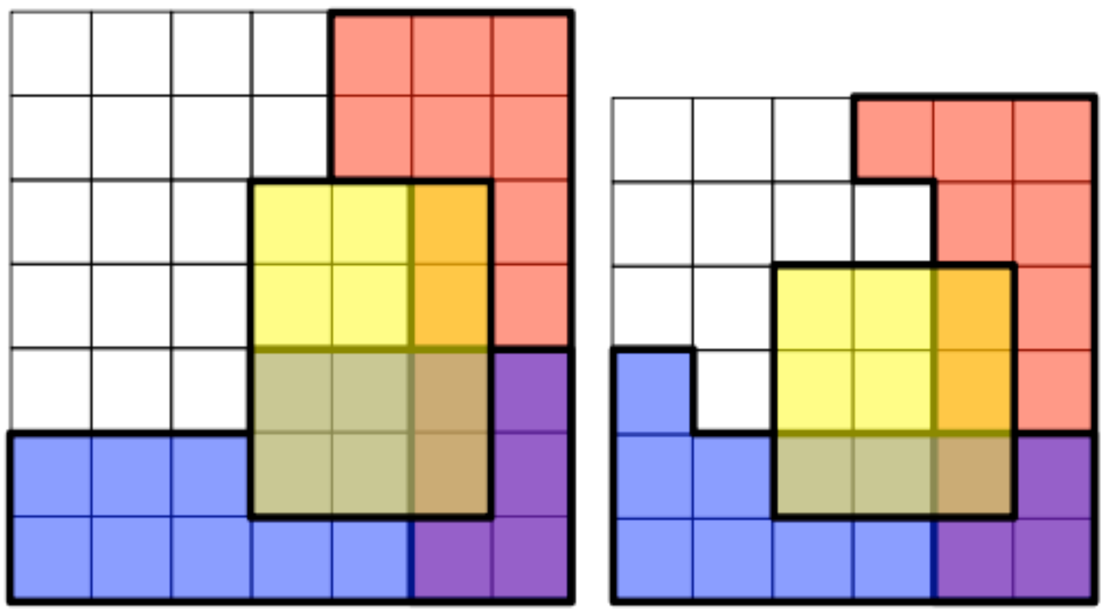
# Conditional Independence

# $P(A \cap B \mid C) = P(A \mid C)\, P(B \mid C)$

# A and B are independent given C

R(red), Y(yellow), B (blue) examples

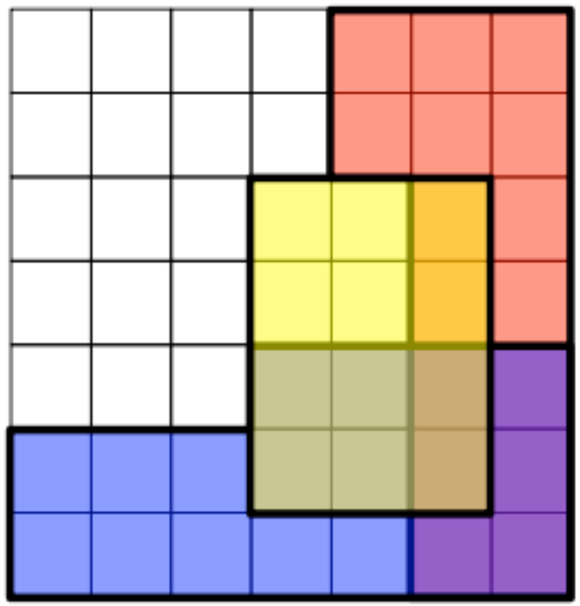
$P(R \cap B \mid Y) = P(R \mid Y)\, P(B \mid Y)$

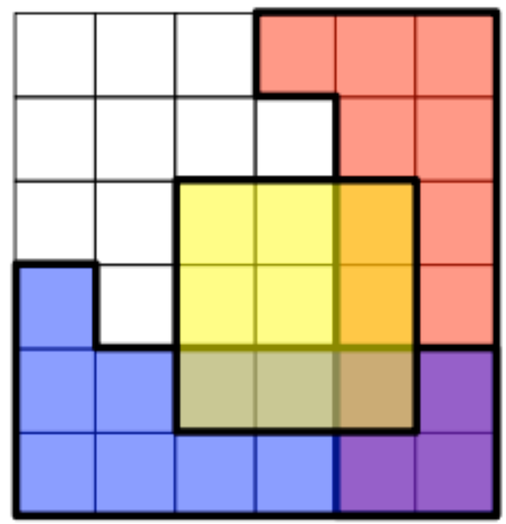


R (A), and B (B) are independent given Y (C)

# Conditional Independence

$$P(R \cap B \mid Y) = P(R \mid Y)P(B \mid Y)$$





P(R) = 16/49
P(B) = 18/49
P(Y) = 12/49
P(R|Y) = 4/12 = 1/3;
P(B|Y) = 6/12 = 1/2;
P(R|Y)P(B|Y) = 1/6;
P(R∩B|Y) = 2/12=1/6;

P(R) = 13/36
P(B) = 13/36
P(Y) = 6/36
P(R|Y) = 3/9 = 1/3;
P(B|Y) = 3/9 = 1/3;
P(R|Y)P(B|Y) = 1/9;
P(R∩B|Y) = 1/9;

# Conditional Independence

First case

$P(A|B) = P(A)$

A and B Ind.

Second case

$P(A|B \cap C) = P(A|C)$

A and B Indep. given C

Third case

$P(A \cap B|C)=P(A|C)\ P(B|C)$

A and B Indep. given C

# Product Rule

$$P(a|b) = \frac{P(a \cap b)}{P(b)}$$

$$P(b)P(a|b) = \frac{P(a \cap b)P(b)}{P(b)}$$

$$P(b)P(a|b) = P(a \cap b)$$

**Product rule**

$$P(a \cap b) = P(a|b)P(b) = P(b|a)P(a)$$

It allows easy computation of the probabilities of intersections of events

# Conditional Independence

Experiments: deck of 52 cards

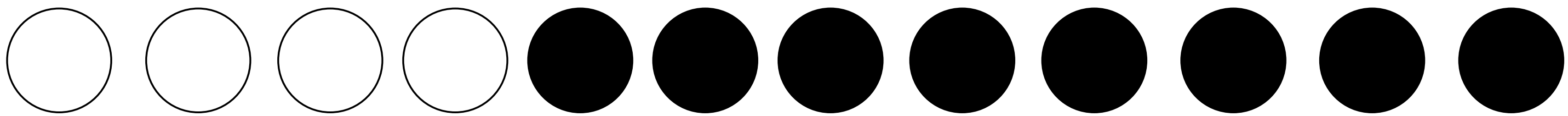
If two cards are drawn *without* replacement from a deck of cards, the event of drawing a red card on the first trial P(B) and that of drawing a red card on the second trial P(A) are NOT INDEPENDENT

$$P(B) = 26/52$$
$$P(A|B) = 25/51$$

If the question is:
What is the probability that the first card is red and the second card is red

$$P(A \cap B) = P(B)P(A|B) = 26/52 * 25/51$$

# Product Rule – An example



We select two balls at random without replacement from an urn which contains 4 white balls and 8 black balls.

Compute the probability that both balls are white.

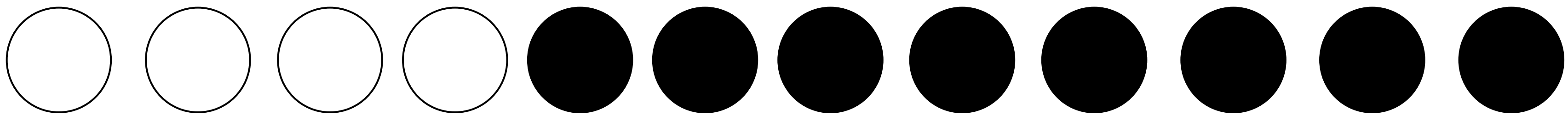
Total 12 balls, 4 white and 8 black

Event B: the first ball is white;
Event A: the second ball is white;
Event C: both balls are white;

Using the Product Rule we can write:

$$P(C) = P(A \cap B) = P(A|B)P(B)$$

# Product Rule – An example



Event B: the first ball is white; $P(B) = 4/12 = 1/3$

Second white given first white; $P(A|B) = 3/11$

Event A: the second ball is white; $P(A) = ?$

Event C: both balls are white; $P(C) = P(A \cap B)$

Using the Product Rule we can write:

$$P(C) = P(A \cap B) = P(A|B)P(B) = 3/11 * 1/3 = 1/11$$

# P(A)

Compute the probability that the second ball is white P(A).

A = (A ∩ B) ∪ (A ∩ Ƅ) with Ƅ the complement of B

Ƅ -> first ball not white

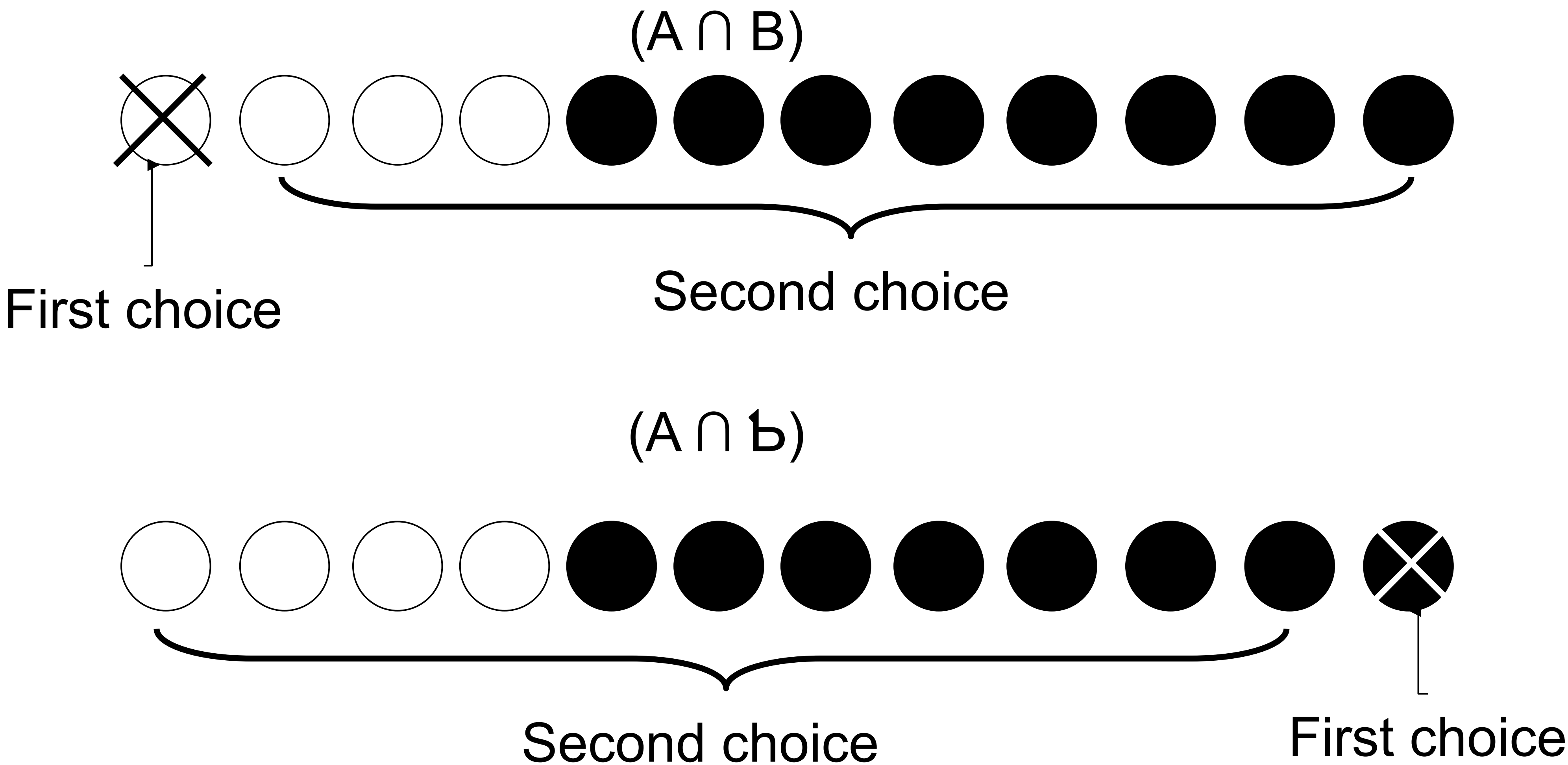
A is given by the intersection of A and B

plus the union of the

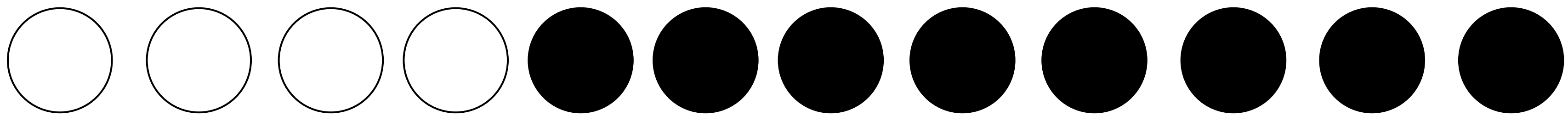
intersection of Ƅ and A

Ƅ - The complement of B is the set of all elements not belonging to B

# P(A)


(A ∩ B)
First choice
Second choice
(A ∩ Ь)
Second choice
First choice

# P(A)



Compute the probability that the second ball is white P(A).

A = (A ∩ B) ∪ (A ∩ Ь) with Ь the complement of B

P(A) = (A ∩ B) ∪ (A ∩ Ь)

= P(B)P(A|B) + P(Ь)P(A|Ь)

= 4/12*3/11 + 8/12*4/11

= 1/3

# True or False

**Is this the Product rule**

$$P(a \cap b) = P(a|b)P(b)$$

# Discrete Random Variable

# Discrete Random Variables

Formally, a random variable is a real-valued function of the elements of the sample space

# Discrete Random Variables

We roll a pair of fair dice 1 time. The sample space is:

$S = \{(x_1, x_2): x_1=1,...,6; x_2=1,...,6;\}$

$S=\{(1,1),(1,2), ..., (6,6)\}$; 36 elements

Let X be the sum of the two numbers that occur

$X(\omega) = (x_1 + x_2)$; for $\omega = (x_1, x_2) \in S$

The domain of $X = \{2,..., 12\}$

X is a discrete random variable because it is a function (+) of the elements of the sample space S

# Discrete Random Variables (notation)

- Greek letters ($\omega$) are used to represent a generic element of the sample space (e.g., $\omega = (x_1, x_2) \in S$)

- Capital letters from the end of the alphabet (X, Y, Z, U, V, W) represent random variables

- Lower case letters (*x, y, z, u, v, w*) stand for particular values in the domain of the random variable (e.g., sum = 2)

- $X(\omega)$ is the functional representation of the random variables
$X(\omega) = (x_1 + x_2)$

# Probability function for a Discrete Random Variables

$P(X(\omega)=x)$ or $P(X)^{(x)}$

represents the probability function for X

# Probability Function

We roll a pair of fair dice. The sample space is:

$S = \{(x_1, x_2): x_1=1,\ldots,6; x_2=1,\ldots,6;\}$

$S=\{(1,1),(1,2), \ldots, (6,6)\}$; 36 elements

Discrete Random Variable X

$X(\omega) = (x_1 + x_2)$; for $\omega = (x_1, x_2) \epsilon S$

The domain of $X = \{2,\ldots, 12\}$

The probability function for

$P(X(\omega)=11)$ or $P(X)^{(11)}$

$P(X)^{(11)} = 2/36$; $S=\{\ldots, (5,6),\ldots (6,5)\ldots\}$

# Probability Function

# True or False

We roll a pair of fair dice.

Sample space $S = \{(x_1, x_2): x_1=1,...,6; x_2=1,...,6;\}$

Random variable $X(\omega) = (x_1 + x_2)$; for $\omega = (x_1, x_2) \in S$

Domain of $X=\{2,...,12\}$

The sum is 2 - $P(X)^{(2)} = 1/36$

The sum is 3 - $P(X)^{(3)} = 2/36$

The sum is 4 - $P(X)^{(4)} = 3/36$

The sum is 5 - $P(X)^{(5)} = 4/36$

The sum is 6 - $P(X)^{(6)} = 5/36$

The sum is 7 - $P(X)^{(7)} = 6/36$

The sum is 8 - $P(X)^{(8)} = 5/36$

The sum is 9 - $P(X)^{(9)} = 4/36$

The sum is 10 - $P(X)^{(10)} = 3/36$

The sum is 11 - $P(X)^{(11)} = 2/36$

The sum is 12 - $P(X)^{(12)} = 1/36$