1 Limits

Limit of $f(x)$ is L as x approaches a:

$$\lim_{x \to a} f(x) = L \tag{1}$$

Right handed limit, i.e. a limit that is approaching from the positive direction and converging on a is denoted by:

$$\lim_{x \to a^+} f(x) = L \tag{2}$$

Similarly a left handed limit, i.e. one that is approaching from the negative direction and converging on a is denoted by:

$$\lim_{x \to a^-} f(x) = L \tag{3}$$

A couple of additional points about handed limits:

- They are useful in the case of "difficult" functions such as step functions.
- If both the left and right limits are equal then a normal limit exists.

1.1 Properties of Limits

1. Constants can be factored out:

$$\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x) \tag{4}$$

2. Limits of a sum or difference is just the limit of their parts:

$$\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) \tag{5}$$

3. Similarly, limits applied to the product and quotient of two functions is just the limit of their parts:

$$\lim_{x \to a} f(x)g(x) = \lim_{x \to a} f(x) \lim_{x \to a} g(x) \tag{6}$$

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \tag{7}$$

4. Powers can be factored out:

$$\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n \tag{8}$$

5. Limits of constants are just themselves:

$$\lim_{x \to a} c = c \tag{9}$$

6. Limits of the variable are simply the limit value itself:

$$\lim_{x \to a} x = a \tag{10}$$

1.2 Continuity

A function is continuous at $x = a$ if:

$$\lim_{x \to a} f(x) = f(a) \tag{11}$$

A function is continuous over an interval $[a, b]$ if it is continuous at each point in the interval.

If $f(x)$ is continuous at $x = a$ then:

$$\begin{gathered} \lim_{x \to a} f(x) \\ \lim_{x \to a^+} f(x) \\ \lim_{x \to a^-} f(x) \end{gathered} \tag{12}$$

Two types of discontinuity:

- **Jump Discontinuity:** occurs where graphs have a break in them
- **Removable Discontinuity:** occurs where there is a hole in the graph

2 Derivatives

The derivative of $f(x)$ with respect to x is $f'(x)$. This is formally defined as being:

$$f'(x) = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h} \tag{13}$$

Intuitively this is form the function takes as the difference between two points (x and $x + h$ shrinks to zero).

2.1 Rules for Computing Derivatives

- The derivative of the sum and difference of two functions is simply the derivative of the respective functions:

$$(f(x) \pm g(x))' = f'(x) \pm g'(x) \tag{14}$$

- Constants may be factored out of the derivative:

$$(cf(x))' = cf'(x) \tag{15}$$

- The derivative of a constant is always zero:

$$f(x) = c \implies f'(x) = 0 \tag{16}$$

- The power rule can be used to compute the derivative of terms raised to a power:

$$f(x) = x^n \implies f'(x) = nx^{(n-1)} \tag{17}$$

- The product rule can be used to compute the derivative of a the product of two functions:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x) \tag{18}$$

- Similarly the quotient rule can be used to compute the derivative of the quotient of two functions:

$$\left(\frac{f(x)}{g(x)} \right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \tag{19}$$

- The chain rule can be used to compute the derivative of more complicated functions where the derivative is a composition of two functions:

$$(f \circ g)'(x) = f'(g(x))g'(x) \tag{20}$$

2.2 Table of Useful Derivatives

$\frac{d}{dx} cos(x)$	$-sin(x)$
$\frac{d}{dx} e^x$	e^x
$\frac{d}{dx} a^x$	$a^x ln(a)$
$\frac{d}{dx} ln(x)$	$\frac{1}{x}$
$\frac{d}{dx} log_a(x)$	$\frac{1}{x ln(a)}$

3 Applications of Derivatives

3.1 Critical Points

$x = c$ is a critical point if $f(c)$ exists and the following is true:

$$\begin{gathered} f'(c) = 0 \\ f'(c) = \text{doesn't exists} \end{gathered} \tag{21}$$

3.2 Minimum and Maximum Values

- **Global minimum:** $f(x) \geq f(c)$ for every x in a domain.
- **Local minimum:** $f(x) \geq f(c)$ for every x over an interval.
- **Global maximum:** $f(x) \leq f(c)$ for every x in a domain.
- **Local maximum:** $f(x) \leq f(c)$ for every x over an interval.

3.3 Extreme Value Theorem

If $f(x)$ is continuous on $[a, b$ then there exist two numbers such that ac, $d \leq b$ such that $f(c)$ is the global maximum and $f(d)$ is the global minimum.

3.4 Finding Absolute Extrema

- Verify the function is continuous over the interval.
- Find all critical points in the interval
- Evaluate critical points and end points.
- Identify the extrema