1

Set Theory

In the study of probability theory and statistics an exact medium of communication is extremely important; if the meaning of the question that is asked is confused by semantics, the solution is all the more difficult, if not impossible, to find. The usual exact language employed to state and solve probability problems clearly is that of set theory. The amount of set theory that is required for relative ease and comfort in probability manipulations is easily acquired. We shall take a brief look at some of the simpler definitions, operations, and concepts of set theory, not because these ideas are necessarily a part of probability theory but rather because the time needed to master them is more than compensated for by later simplifications in the study of probability.

1.1. Set Notation, Equality, and Subsets

A set is a collection of objects. The objects themselves can be anything from numbers to battleships. An object that belongs to a particular set is called an element of that set. We shall commonly use capital letters from the beginning of the alphabet to denote sets (A, B, C, etc.) and lower case letters from the end of the alphabet to denote elements of sets (x, y, z, etc.).

To specify that certain objects belong to a given set, we shall use braces { } (commonly called set builders) and either the roster (complete listing of all elements) or the rule method. For example, if we want to write that the set A consists of the letters a, b, c and that the set B consists of the first 10 integers, we may write

$A = \{a, b, c\}$ (roster method of specification)

$B = \{x: x = 1, 2, 3, \ldots, 10\}$ (rule method of specification).

The above two sets can easily be read as "A is the set of elements a, b, c" and "B is the set of elements x such that $x = 1$ or $x = 2$ or $x = 3$ and so on up to or $x = 10$." We shall use the symbol $\in$ as shorthand for "belongs to" and thus can write for the two sets defined above that $a \in A$, $7 \in B$. Just as a line drawn through an equals sign is taken as negation of the equality, we shall use $\notin$ to mean "does not belong to"; thus $a \notin B$, $9 \notin A$, $f \notin A$, $102 \notin B$, etc., where A and B are the sets defined above.

DEFINITION 1.1.1. Two sets A and B are *equal* if and only if every element that belongs to A also belongs to B and every element that belongs to B also belongs to A.

Then two sets are equal only if both contain exactly the same elements. Notice in particular that the order of listing of elements of a set is of no importance and that the sets

$$A = \{1, 2, a, 3\}, \qquad B = \{a, 1, 2, 3\}$$

are equal. Also, the number of times that an element is listed in the roster specifying the set is of no concern; the two sets

$$C = \{1, 2, 3\}, \qquad D = \{1, 2, 2, 1, 3, 1\}$$

are equal. We shall have no use for the redundancy exhibited in the roster of D and thus shall generally assume that if the roster of a particular set contains n elements, the elements are distinguishable and that the same element does not occur more than once.

DEFINITION 1.1.2. A is a *subset* of B (written $A \subset B$) if and only if every element that belongs to A also belongs to B.

In a sense, A is a subset of B if A is contained in B. For example, $\{a\}$ is a subset of $\{a, z\}$ and both of these are subsets of $\{a, b, z\}$. The set of all people residing in California is a subset of the set of all people residing in the United States; the set of all pine trees is a subset of the set of all trees.

A word regarding the difference between something belonging to a set A and being a subset of A may be in order. For example, define

$$A = \{1, 2, 3\}, \qquad B = \{1, 3\}, \qquad C = \{1\}.$$

Then it is correct to say $B \subset A$ and $C \subset A$ since every element that belongs to B also belongs to A, as does every element that belongs to C. However, it is not correct to say that $B \in A$ or that $C \in A$ since B and C are not specified in the roster of elements that belong to A. Similarly, it is correct to say that $1 \in A$ and $1 \in C$, not $1 \subset A$ and $1 \subset C$ since 1 is not a set. Thus we can say that the set of all married people is a subset of the set of all people, but we do not say that the set of all married people belongs to the set of all people. A particular married person belongs to the set of all married people (as well as to the set of all people), but this person is not a subset of either (since a person is not a set).

An alternative definition of set equality can be given by saying that $A = B$ if and only if $A \subset B$ and $B \subset A$. This, in fact, provides a very useful way of demonstrating that two sets are equal, as we shall see. Most simple set equalities can be rigorously and easily proved in this way.

From the definitions, it is easily seen that every set is equal to itself and that every set is a subset of itself. That is, we can say

$$A = A, \qquad \text{for all } A,$$
$$A \subset A, \qquad \text{for all } A.$$

In textbooks on algebra, these two statements are summarized by saying that both the equality and the subset relations are reflexive.

EXERCISE 1.1.

1. Show that the subset relationship is transitive; that is, $A \subset B$ and $B \subset C$ imply $A \subset C$.
2. Show that the subset relationship is not symmetric; that is, $A \subset B$ and $B \subset A$ are logically different statements and one is not implied by the other. Give an example illustrating this nonsymmetry.
3. Give an example showing that if $E \subset F$ and $D \subset F$, we do not necessarily have $D \subset E$ or $E \subset D$.
4. Show that set equality is transitive. (Transitivity is defined in problem 1 above.)
5. If $A = \{x: x = 1, 0, 1\}$, $B = \{-1, 0, 1\}$, $C = \{0, 1, -1\}$, $D = \{-1, 1, -1\}$ mark each of the following either true or false.

(a) $A = B$	(h) $C \subset B$
(b) $A = C$	(i) $0 \in B$
(c) $B = C$	(j) $0 \subset C$
(d) $A \subset B$	(k) $A = D$
(e) $A \subset C$	(l) $D \subset A$
(f) $B \subset C$	(m) $B \subset D$
(g) $C \subset A$	(n) $D \in C$.

6. Define $E = \{x: 0 \leq x \leq 1\}$, $F = \{y: 0 \leq y \leq 1\}$, $G = \{x: 0 < x < 1\}$, $H = \{1\}$. Mark each of the following either true or false.

(a) $E = F$
(b) $F = G$
(c) $F \subset G$
(d) $G \subset F$
(e) $H \subset G$
(f) $H \subset E$
(g) $H \in F$.

7. Is the set of all students a subset of the set of all people? Is it a subset of the set of all people under 36 years of age? Under 50 years of age?

1.2. Union and Intersection

We shall now study some of the algebra of sets. Just as the algebra of real numbers is concerned with operations performed with numbers and their consequences, the algebra of sets is concerned with operations performed with sets and their consequences. The first set operation we define is called the union of two sets.

DEFINITION 1.2.1. The *union* of A and B (written $A \cup B$) is the set which consists of all the elements that belong to A or to B or to both; i.e., $A \cup B = \{x: x \in A \text{ or } x \in B\}$.

For example, if $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{0\}$, then $A \cup B = \{1, 2, 3\}$, $A \cup C = \{0, 1, 2\}$, and $B \cup C = \{1, 3, 0\}$. Notice from the definition of the union that $A \cup B$ and $B \cup A$ are identical sets (the operation is commutative), since the collection of objects that belong to A or to B is the same; it does not matter whether we first list those that belong to A or those that belong to B. We also note that the operation of forming unions is associative; i.e.,

$$(A \cup B) \cup C = A \cup (B \cup C).$$

For this reason it is not necessary to place parentheses about pairs of sets when we take the union of several sets; it doesn't matter which union is formed first.

We notice a number of interesting relationships linking unions of sets and the concept of subsets. For example, for any A and B we would know that $A \subset A \cup B$ and $B \subset A \cup B$ (verification of these facts is asked for in problem 1 below). Not quite so obviously, $A \cup B = A$ if $B \subset A$, since the set which consists of all those elements that belong to A or to B or to both would merely consist of those elements that belong to A originally if every element belonging to B also belongs to A. Thus the union of the set of people living in California with the set of people living in the United States is the set of people living in the United States; the union of the set of all cows with the set of all animals is the set of all animals.

A second set operation which we shall find useful is called the intersection of two sets.

DEFINITION 1.2.2. The *intersection* of A and B (written $A \cap B$) is that set which consists of all elements that belong both to A and to B; i.e., $A \cap B = \{x: x \in A \text{ and } x \in B\}$.

For example, if $A = \{0, 1\}$, $B = \{1, 3\}$, $C = \{0, 1, 3\}$, then $A \cap B = \{1\}$, $A \cap C = \{0, 1\}$, $B \cap C = \{1, 3\}$. If A is the set of married people and B is the set of people living in California, then $A \cap B$ is the set of married people living in California.

It is easy to see that the intersection operation is also commutative and associative; that is,

$$A \cap B = B \cap A \quad \text{and} \quad A \cap (B \cap C) = (A \cap B) \cap C.$$

We note as well that

$$A \cap B \subset B, \qquad A \cap B \subset A,$$

no matter what the sets A and B may be. Similarly, if $B \subset A$, then

$$A \cap B = B.$$

Thus the intersection of the set of all buildings in California with the set of all buildings in the United States is the set of all buildings in California.

A handy way of picturing sets, relationships between sets, and operations with sets is to employ a Venn diagram. Various geometric shapes can be drawn on a plane and, if we assume that the points interior to and on the boundary of each figure constitute a set, many of the concepts already discussed can be visually displayed. In Figure 1.1 are given sets A, B, C, D, and $A \cup B$ and $C \cap D$.

There are two distributive laws linking the operations of unions and

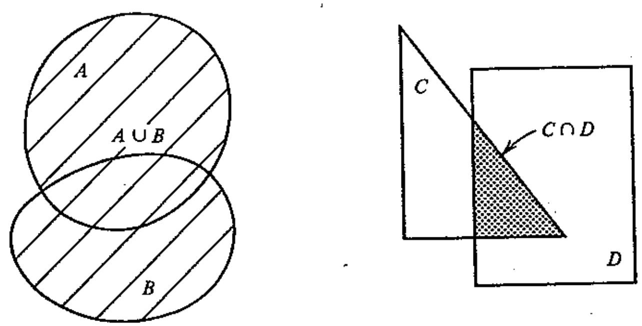

Figure 1.1.

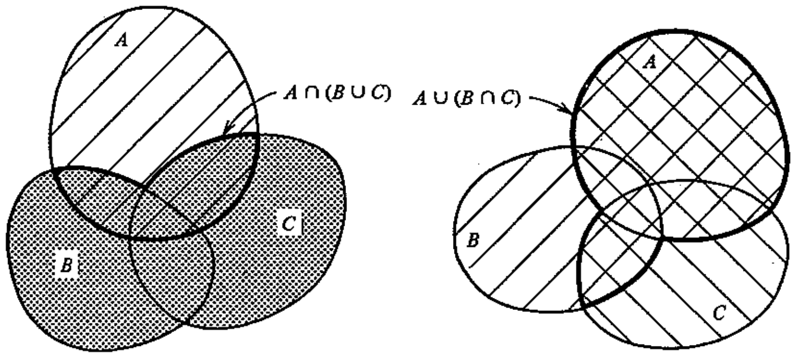

Figure 1.2.

intersections. These are:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{1.1}$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C). \tag{1.2}$$

Figure 1.2 gives a Venn diagram which the reader can use to verify that both of these statements are true.

Note that there exists a rather natural analog between unions of sets and addition of numbers since, in some senses, the set $A \cup B$ is the result of "adding" the set B to the set A. If we also pretend that there is an analog between set intersection and multiplication of numbers, then distributive law 1.1 above is analogous to $a(b + c) = ab + ac$, which is true for all real numbers a, b, and c. However, distributive law 1.2 would be analogous to $a + b \cdot c = (a + b) \cdot (a + c)$, which is not true for all a, b, and c.

EXERCISE 1.2.

1. Show that $A \cap B \subset A \subset A \cup B$ and that $A \cap B \subset B \subset A \cup B$.

2. If $A = \{1, 0\}$, $B = \{x: 0 < x < 1\}$, $C = \{\frac{1}{2}\}$, compute $A \cup B$, $A \cup C$, $B \cup C$, $A \cap B$, $A \cap C$, $B \cap C$.

3. Show that $B \cup B = B \cap B = B$ for all B.

4. Prove distributive law number 1.2.

5. What is the implication of the equation $E \cap F = F$?

6. What is the implication of the equation $E \cup F = E$?

7. Define:

$$A = \{x: x = 1, 2, 3, \ldots, 10\}$$
$$B = \{x: 1 \le x \le 10\}$$
$$C = \{x: x = 0, 1, 2, 3, 4, 5, 6\}$$
$$D = \{0, 10, 20, 30\}$$

and compute

(a) $A \cup B$
(b) $A \cap B$
(c) $A \cup C$
(d) $A \cap C$
(e) $A \cup D$
(f) $A \cap D$
(g) $B \cup C$
(h) $B \cap C$
(i) $B \cup D$
(j) $B \cap D$
(k) $C \cup D$
(l) $C \cap D$
(m) $A \cup B \cup C$
(n) $A \cap (B \cup C)$
(o) $A \cup (B \cap C)$
(p) $A \cap B \cap C$
(q) $C \cup (A \cap D)$
(r) $(A \cup B) \cap (C \cup B)$.

1.3. Universal Set, Complement, and Cartesian Product

Generally, all of the sets entering into particular discussions will have certain things in common. For example, a group of botanists might discuss sets of trees, in which event all of the sets entering into their discussion would have trees as elements. A group of mathematicians might discuss sets of real numbers, in which event all of the sets entering into their discussion would have real numbers as elements. The universal set for a particular discussion is that set which has as elements all elements of every set entering into the discussion. Thus the universal set for the botanists' discussion could be the set of all trees; the universal set for the mathematicians' discussion might be the set of all real numbers. Note that different discussions may have different universal sets.

If we have in mind a particular universal set, then definition of a set A automatically also specifies a second set $\bar{A}$, called the complement of A, which is defined as follows.

DEFINITION 1.3.1. The *complement* of A (with respect to a given universal set) is the set of all elements not belonging to A; i.e., $\bar{A} = \{x: x \notin A\}$.

If, for example, our universal set is

$$U = \{x: 1 \le x \le 10\}$$

and we define

$$A = \{x: 1 \le x \le 2\}$$
$$B = \{1, 10\}$$

then

$$\bar{A} = \{x: 2 < x \le 10\}$$
$$\bar{B} = \{x: 1 < x < 10\}.$$

Or, if our universal set is the set of all people living in the United States and we define A as the set of people living in California and B as the set of people living in San Francisco, then $\bar{A}$ is the set of people living in the United States outside of California and $\bar{B}$ is the set of people living in the United States outside of San Francisco.

Figure 1.3 gives a Venn diagram illustrating a universal set U, a set A, a set $B \subset A$, $\bar{A}$, and $\bar{B}$. Note that $A \cup \bar{A} = U$ and, as is always the case, that if $B \subset A$, then $\bar{A} \subset \bar{B}$.

The null (or empty) set is frequently of use and is defined as follows:

Definition 1.3.2. The *null set* $\varnothing$, is the set with no elements; i.e., $\varnothing = \{\ \}$.

$\varnothing$ occurs as the intersection of two sets that have no elements in common. For example, if A is the set of all frogs and B is the set of all people, then $A \cap B = \varnothing$. More generally, from the definition of $\bar{A}$, $A \cap \bar{A} = \varnothing$ for any A.

The empty set plays a role analogous to the real number 0 in some (not all) manipulations. If a is any real number, we know that

$$a + 0 = a, \qquad a \cdot 0 = 0.$$

If A is any set, we can easily see that

$$A \cup \varnothing = A, \qquad A \cap \varnothing = \varnothing.$$

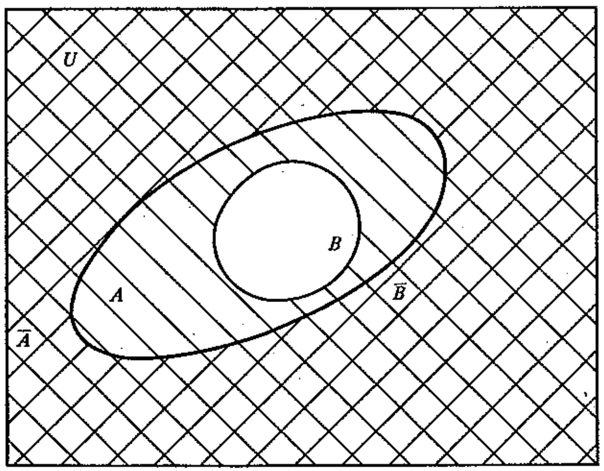

Figure 1.3.

Note, however, that we can have $A \cap B = \varnothing$ where neither A nor B is empty, whereas $a \cdot b = 0$ implies that a or b (or both) is 0. From our definition of subsets, we must admit that $\varnothing \subset A$ for all A, since every element belonging to $\varnothing$ (none) also belongs to A. Alternatively, every x belonging to $\varnothing$ also belongs to A is a true statement, vacuously, since there is no x belonging to $\varnothing$ to make the statement false. Unfortunately, $\varnothing$ cannot be pictured on a Venn diagram like other sets, because if we draw any shape at all we don't have a picture of $\varnothing$ since our shape is not empty.

As was discussed earlier, a set of elements is unchanged if we merely rearrange the ordering of its elements. To have at our disposal collections of n elements where order is of importance, we define an n-tuple as follows.

Definition 1.3.3. An n-tuple is an ordered array of n elements written $(x_1, x_2, \ldots, x_n)$.

For example, (1, 2), (2, 1), (0, 100), (a, b) are all 2-tuples; (a, b, c), (b, a, c), (1, 1, 1), (2, 1, 2) are all 3-tuples; etc. Two n-tuples are different, even if they contain the same elements, if they are written in a different order and, of course, an n-tuple and an $(n + 1)$-tuple cannot be equal since they have different numbers of elements. Thus

$$(1, 2) \neq (2, 1)$$

$$(1, 3, 1) \neq (1, 1, 3)$$

$$(1, 1) \neq (1, 1, 1).$$

We shall have use for sets of n-tuples as we proceed. For example, if we want to discuss the set of married couples living in Texas, then we want each couple to be an element of the set; for clarity we list the husband's name first when specifying the couple. Each element of the set then is a couple or a 2-tuple. Note that a married individual living in Texas does not belong to the set; he is a component of one of the 2-tuples that belongs to the set. As a second example, the set of points lying in the first quadrant of the usual Cartesian plane is a set of 2-tuples. Each point is represented by a 2-tuple (x, y) where x is the horizontal coordinate and y the vertical coordinate.

One other set operation, the Cartesian product of two sets will be of interest to us. It is defined as follows.

Definition 1.3.4. The *Cartesian product* $A \times B$ of A and B is the set of all possible 2-tuples (x_1, x_2) where $x_1 \in A$, $x_2 \in B$. That is, $A \times B = \{(x_1, x_2): x_1 \in A, x_2 \in B\}$.

Let us consider a number of examples of Cartesian products. Suppose that $A = \{0, 1, 2\}$, $B = \{3, 5\}$, $C = \{0\}$. Then

$$\begin{aligned}
A \times B &= \{(0, 3), (0, 5), (1, 3), (1, 5), (2, 3), (2, 5)\} \\
A \times C &= \{(0, 0), (1, 0), (2, 0)\} \\
B \times C &= \{(3, 0), (5, 0)\} \\
B \times A &= \{(3, 0), (3, 1), (3, 2), (5, 0), (5, 1), (5, 2)\} \\
C \times A &= \{(0, 0), (0, 1), (0, 2)\} \\
C \times B &= \{(0, 3), (0, 5)\} \\
A \times A &= \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\} \\
B \times B &= \{(3, 3), (3, 5), (5, 3), (5, 5)\} \\
C \times C &= \{(0, 0)\}.
\end{aligned}$$

If D is the set of all positive real numbers, then $D \times D$ is the set of points in the first quadrant of the plane.

Notice that $A \times B$ and $B \times A$ are not equal in general. The operation is not commutative. Problem 9 below asks you to specify when these two would be equal. We can form the Cartesian product of any number of sets; $A \times B \times C$ is a set of 3-tuples, $A_1 \times A_2 \times \cdots \times A_n$ is a set of n-tuples. Taking A, B, and C as defined above,

$$\begin{aligned}
A \times B \times C &= \{(0, 3, 0), (0, 5, 0), (1, 3, 0), (1, 5, 0), (2, 3, 0), (2, 5, 0)\} \\
A \times C \times B &= \{(0, 0, 3), (0, 0, 5), (1, 0, 3), (1, 0, 5), (2, 0, 3), (2, 0, 5)\}.
\end{aligned}$$

Strictly speaking, we should denote the first element listed in $A \times B \times C$ above as $((0, 3), 0)$ if we are thinking of $(A \times B) \times C$, or as $(0, (3, 0))$ if we are thinking of $A \times (B \times C)$. We have no reason to distinguish between $((0, 3), 0)$, $(0, (3, 0))$, and $(0, 3, 0)$, so we simply define $A \times B \times C$, as noted, to be the collection of 3-tuples which results when the innermost pair of parentheses is deleted. Thus Cartesian products are associative.

EXERCISE 1.3.

1. Name two distinct universal sets for a discussion involving sets whose elements are all people residing in a particular town.
2. Given the universal set
$$U = \{1, 2, 3, \ldots, n\}$$
and the sets
$$A = \{1, 2, 3\}, \qquad B = \{2, 3, n\},$$
compute $\bar{A} \cap \bar{B}$, and $\bar{A} \cup \bar{B}$. Compare this latter set with $\overline{A \cap B}$.
3. Can we define a set A such that $A = \bar{A}$? Such that $A \subset \bar{A}$?

4. Show that $\bar{\bar{A}} = A$.
5. Show that $\overline{A \cup B} = \bar{A} \cap \bar{B}$. (This is one of De Morgan's laws.)
6. Show that $\overline{A \cap B} = \bar{A} \cup \bar{B}$. (This is also one of De Morgan's laws.)

Hint: In problem 5, replace A and B by their complements and then take the complement of both sides.

7. Draw Venn diagrams illustrating the sets $A \cap \bar{B}$, $\bar{A} \cap \bar{B}$, $A \cup B \cup C$, $A \cap B \cap C$, $(A \cup B) \cap C$, $(A \cap C) \cup (B \cap C)$.
8. Define $A = \{1, 2\}$, $B = \{2, 1\}$, $C = \{10, 12\}$, and form the Cartesian products $A \times B$, $A \times C$, $B \times C$, $B \times A$, $C \times A$, $C \times B$, $A \times A$, $B \times B$, $C \times C$, $A \times B \times C$, $C \times B \times A$, $C \times A \times B$.
9. What are the conditions under which $A \times B = B \times A$?
10. If A is the set of married men living in Texas and B is the set of married women living in Texas, is $A \times B$ the set of married couples living in Texas?
11. If A is the set of people with United States citizenship and B is the set of people with Canadian citizenship, does $A \cap B = \varnothing$?

1.4. Element Functions and Set Functions

When discussing variables whose values are determined by an experiment or some chance mechanism, we shall have use for the definition of a function of the elements of a set.

DEFINITION 1.4.1. A (real-valued) *element function* f defined on a set S is a rule which associates a (real) number with every element of the set. The number associated with a particular element is called the value of the function for that element and is denoted by $f(\omega)$ for $\omega \in S$.

Many different functions could be defined for the elements of the same set (that is, many different rules could be used). Note that the same number could be associated with more than one element.

For example, consider the set of people residing in a certain town. There are many different rules that we could use to associate a real number with each element of the set, such as the age of the individual, the weight of the individual, the height of the individual, the street number of the individual's residence, the distance of the individual's residence from the city hall, etc. Each of these rules—age, height, weight, etc.—is called an element function defined on the set.

As a second example, suppose that we are given the set of 2-tuples

$$S = \{(x_1, x_2): x_1 = 1, 2, 3;\ x_2 = 1, 2, 3\}.$$

The rule $f[(x_1, x_2)] = x_1 + x_2$ for $(x_1, x_2) \in S$ associates the sum of the two numbers making up the 2-tuple with each element of S. Thus,

$$f[(1, 1)] = 1 + 1 = 2, \qquad f[(2, 3)] = 2 + 3 = 5,$$

etc. A second-element function which we might define for the same set S is

$$g[(x_1, x_2)] = \frac{x_1}{x_2} \qquad \text{for} \qquad (x_1, x_2) \in S.$$

Then

$$g[(1, 1)] = \tfrac{1}{1} = 1, \qquad g[(2, 2)] = \tfrac{2}{2} = 1,$$
$$g[(1, 3)] = \tfrac{1}{3}, \qquad g[(3, 1)] = \tfrac{3}{1} = 3,$$

and so on.

Two sets which are of interest for any function are its domain of definition and its range. These are defined as follows.

DEFINITION 1.4.2. The *domain of definition* for an element function defined on a set S is simply the set S itself; the *range* of an element function defined on a set S is the collection of real numbers it associates with the elements of S (the range is the set of values of the function).

Thus, for the first example discussed above, the domain of definition of each of the functions given is the set of people residing in the town; the range of the function which associates ages with people is the set of ages of the residents of the town. The range of the weight function is the set of weights of the residents; the range of the height function is the set of heights of residents; the range of the street number function is the set of street numbers in the town; the range of the distance function is the set of distances. In the second example, the domain of definition for both f and g is the set S; the range of f is the set $\{2, 3, 4, 5, 6\}$ and the range of g is the set $\{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1, \frac{3}{2}, 2, 3\}$.

Let us now briefly discuss sets, each of whose elements is a set. Such a set is called a class.

DEFINITION 1.4.3. A *class* is a set, each of whose elements is a set.

Generally we shall use script letters to denote classes. Thus,

$$\mathcal{H} = \{\{1\}, \{2\}\}$$
$$\mathcal{K} = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$$
$$\mathcal{A} = \{\{a\}, \{b\}, \{a, b\}\}$$

are all classes. The set $\mathcal{B}$ which has as elements all the subsets of $S = \{1, 2, 3\}$ is a class.

DEFINITION 1.4.4. A class $\mathcal{F}$ is *closed* with respect to unions and intersections if and only if $A \cup B \in \mathcal{F}$ and $A \cap B \in \mathcal{F}$ for all A and B belonging to $\mathcal{F}$.

The class $\mathcal{H}$ defined above is not closed since $\{1\} \cup \{2\} = \{1, 2\} \notin \mathcal{H}$. The class $\mathcal{K}$ defined above is closed since the union or the intersection of any two of its elements again belongs to $\mathcal{K}$. The class $\mathcal{A}$ is closed with respect to unions since the union of any two of its elements again belongs to $\mathcal{A}$, but it is not closed with respect to intersections since $\{a\} \cap \{b\} = \varnothing$ does not belong to $\mathcal{A}$. Thus we do not say that $\mathcal{A}$ is closed. The class $\mathcal{B}$ is closed since the union of any two subsets of S is again a subset of S, as is the intersection of any two subsets of S.

Given a class $\mathcal{F}$ that is closed with respect to unions and intersections, a set function on $\mathcal{F}$ is defined as follows.

DEFINITION 1.4.5. A rule f which associates a real number (denoted by $f(A)$) with each $A \in \mathcal{F}$ is called a (real-valued) *set function* defined on $\mathcal{F}$.

Consider the class $\mathcal{F}$ of subsets of $S = \{0, 1, 2\}$. Then

$$\mathcal{F} = \{\varnothing, \{0\}, \{1\}, \{0, 1\}, \{2\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

As in the case of element functions, there are many different set functions that could be defined on $\mathcal{F}$. For example,

$$f(A) = 0, \qquad \text{if } 1 \in A$$
$$= 1, \qquad \text{if } 1 \notin A$$

is a rule which can be used to associate a number with every set $A \in \mathcal{F}$. Thus f is called a set function defined on $\mathcal{F}$. Similarly,

$$n(A) = \text{number of elements in } A, \; A \in \mathcal{F}$$
$$s(A) = \text{square of the number of elements belonging to } A, \; A \in \mathcal{F}$$

are each set functions defined on $\mathcal{F}$. The domain of definition of each of them then is $\mathcal{F}$. The range of f is the set $\{0, 1\}$, the range of n is the set $\{0, 1, 2, 3\}$, and the range of s is the set $\{0, 1, 4, 9\}$.

EXERCISE 1.4.

1. Let A be the set of individuals in your family and define the element function $f(\omega)$ = age of ω for $\omega \in A$. Specify the range of f.

2. Let

$$B = \{1, 2, 3, 4, 5, 6\}$$
$$C = B \times B$$

and define the element function

$$g[(x_1, x_2)] = x_1 + x_2 \qquad \text{for} \qquad (x_1, x_2) \in C.$$

What is the range of g?

3. Let $D = \{(x_1, x_2, x_3): x_1 = 0, 1;\ x_2 = 0, 1;\ x_3 = 0, 1\}$ and define

$$h[(x_1, x_2, x_3)] = x_1 + x_2 - x_3.$$

What is the range of h?

4. Define $C = \{(x_1, x_2): x_1 = 1, 2, 3, 4, 5, 6;\ x_2 = 1, 2, 3, 4, 5, 6\}$ and let

$$g[(x_1, x_2)] = x_1 + x_2, \qquad \text{for } (x_1, x_2) \in C.$$

Specify

$$A_7 = \{(x_1, x_2): g[(x_1, x_2)] = 7\}$$
$$A_3 = \{(x_1, x_2): g[(x_1, x_2)] = 3\}$$
$$A_{10} = \{(x_1, x_2): g[x_1, x_2)] = 10\}.$$

5. Let $\mathcal{F}$ be the class of all subsets of $S = \{1, 2, 3, \ldots, k\}$. Define $n(A)$ = number of elements in A for $A \in \mathcal{F}$. What is the range of n?

6. Let $\mathcal{K}$ be the class of all subsets of $S = \{1, 2, 3, \ldots, k\}$. Define $p(A) = 1/k$ times the number of elements in A for $A \in \mathcal{K}$. What is the range of p?

7. Let S and n be as defined in problem 5. Show that

$$n(A) \leq n(B), \qquad \text{if } A \subset B,$$
$$n(A \cup B) = n(A) + n(B), \qquad \text{if } A \cap B = \varnothing.$$

2

Probability

We are now ready to begin our study of probability theory itself. The earliest known applications of probability theory occurred in the seventeenth century. A French nobleman of that time was interested in several games then played at Monte Carlo; he tried unsuccessfully to describe mathematically the relative proportion of the time that certain bets would be won. He was acquainted with two of the best mathematicians of the day, Pascal and Fermat, and mentioned his difficulties to them. This began a famous exchange of letters between the two mathematicians concerning the correct application of mathematics to the measurement of relative frequencies of occurrences in simple gambling games. Historians generally agree that this exchange of letters was the beginning of probability theory as we now know it.

For many years a simple relative frequency definition of probability was all that was known and was all that many felt was necessary. This definition proceeds roughly as follows. Suppose that a chance experiment is to be performed (some operation whose outcome cannot be predicted in advance); thus there are several possible outcomes which can occur when the experiment is performed. If an event A occurs with m of these outcomes, then the probability of A occurring is the ratio m/n where n is the total number of outcomes possible. Thus, if the experiment consists of one roll of a fair die and A is the occurrence of an even number, the probability of A is $\frac{3}{6}$.

There are many problems for which this definition is appropriate, but such a heuristic approach is not conducive to a mathematical treatment of the theory of probability. The mathematical advances in probability theory were