1 Energy function for a piecewise linear twist angle field

I will define the piecewise linear twist angle as

$$\psi(r) = \begin{cases} \psi_c' r & 0 \leq r \leq R_c \\ \psi_s' r + \psi_1 & R_c < r \leq R_s \\ \psi_R' r + \psi_2 & R_s < r \leq R. \end{cases} \tag{1}$$

which have two constraints,

$$\psi_1 = (\psi_c' - \psi_s')R_c, \tag{2a}$$

$$\psi_2 = (\psi_s' - \psi_R')R_s + (\psi_c' - \psi_s')R_c, \tag{2b}$$

to ensure continuity of $\psi(r)$.

Next, I will insert this into the free energy per unit volume,

$$\begin{aligned} E(R,\eta,\delta;\psi(r)) = {} & \frac{2}{R^2}\int_0^R r dr \left[\frac{1}{2}\left(\psi' + \frac{\sin 2\psi}{2r} - 1\right)^2 + \frac{1}{2}K_{33}\frac{\sin^4\psi}{r^2}\right] \\ & + \frac{\Lambda\delta^2}{2R^2}\int_0^R r dr \big(4\pi^2 - \eta^2\cos^2\psi(r)\big)^2 + \frac{\omega\delta^2}{2}\left(\frac{3}{4}\delta^2 - 1\right) \\ & - \frac{(1+k_{24})}{R^2}\sin^2\psi(R) + \frac{2\gamma}{R}. \end{aligned} \tag{3}$$

The resulting equation can be written as

$$
\begin{aligned}
&E(R,\eta,\delta,R_c,R_s,\psi'_c,\psi'_s,\psi'_R) = \\
&\qquad \frac{2}{R^2}\Bigg(\frac{1}{4}(u(0,R_c,\psi'_c)+u(R_c,R_s,\psi'_s)+u(R_s,R,\psi'_R))+\frac{1}{8}(f_1(0,R_c,0,\psi'_c) \\
&\qquad + f_1(R_c,R_s,\psi_1,\psi'_s)+f_1(R_s,R,\psi_2,\psi'_R))+\frac{1}{2}K_{33}(f_2(0,R_c,0,\psi'_c)+f_2(R_c,R_s,\psi_1,\psi'_s) \\
&\qquad + f_2(R_s,R,\psi_2,\psi'_R))+\frac{1}{4}(v(0,R_c,0,\psi'_c)+v(R_c,R_s,\psi_1,\psi'_s)+v(R_s,R,\psi_2,\psi'_R))\Bigg) \\
&\qquad + \frac{\Lambda\delta^2}{2R^2}\Bigg(8\pi^4R^2-8\pi^2\eta^2(g_1(0,R_c,0,\psi'_c)+g_1(R_c,R_s,\psi_1,\psi'_s)+g_1(R_s,R,\psi_2,\psi'_R)) \\
&\qquad + \eta^4(g_2(0,R_c,0,\psi'_c)+g_2(R_c,R_s,\psi_1,\psi'_s)+g_2(R_s,R,\psi_2,\psi'_R))\Bigg) \\
&\qquad + \frac{\omega\delta^2}{2}\left(\frac{3}{4}\delta^2-1\right)-\frac{(1+k_{24})}{R^2}\sin^2(\psi'_R R+\psi_2)+\frac{2\gamma}{R} \qquad (4)
\end{aligned}
$$

which utilizes a derivation shown in the appendix (see eqns 13 and 14, as well as the definitions of the functions u, v, f_α, and g_α in equations 15a, 15b, 15c, and 15d, respectively). I will minimize this equation subject to the constraint equations 2a and 2b to determine the equilibrium configuration of the fibril.

2 Differentiation of the energy

The derivative with respect to the fibril radius, R, is

$$
\begin{aligned}
&\frac{\partial E}{\partial R} = \\
&\quad - \frac{4}{R^3}\Bigg(\frac{1}{4}(u(0,R_c,\psi'_c) + u(R_c,R_s,\psi'_s) + u(R_s,R,\psi'_R)) + \frac{1}{8}(f_1(0,R_c,0,\psi'_c) \\
&\quad + f_1(R_c,R_s,\psi_1,\psi'_s) + f_1(R_s,R,\psi_2,\psi'_R)) + \frac{1}{2}K_{33}(f_2(0,R_c,0,\psi'_c) + f_2(R_c,R_s,\psi_1,\psi'_s) \\
&\quad + f_2(R_s,R,\psi_2,\psi'_R)) + \frac{1}{4}(v(0,R_c,0,\psi'_c) + v(R_c,R_s,\psi_1,\psi'_s) + v(R_s,R,\psi_2,\psi'_R))\Bigg) \\
&\quad + \frac{2}{R^2}\Bigg(\frac{1}{4}\frac{\partial u(R_s,x_2,\psi'_R)}{\partial x_2}\bigg|_{x_2=R} + \frac{1}{8}\frac{\partial f_1(R_s,x_2,\psi_2,\psi'_R)}{\partial x_2}\bigg|_{x_2=R} + \frac{1}{2}K_{33}\frac{\partial f_2(R_s,x_2,\psi_2,\psi'_R)}{\partial x_2}\bigg|_{x_2=R} \\
&\quad + \frac{1}{4}\frac{\partial v(R_s,x_2,\psi_2,\psi'_R)}{\partial x_2}\bigg|_{x_2=R}\Bigg) \\
&\quad - \frac{\Lambda\delta^2}{R^3}\big(8\pi^4R^2 - 8\pi^2\eta^2(g_1(0,R_c,0,\psi'_c) + g_1(R_c,R_s,\psi_1,\psi'_s) + g_1(R_s,R,\psi_2,\psi'_R)) \\
&\quad + \eta^4(g_2(0,R_c,0,\psi'_c) + g_2(R_c,R_s,\psi_1,\psi'_s) + g_2(R_s,R,\psi_2,\psi'_R))\big) \\
&\quad + \frac{\Lambda\delta^2}{2R^2}\Bigg(16\pi^4R - 8\pi^2\eta^2\frac{\partial g_1(R_s,x_2,\psi_2,\psi'_R)}{\partial x_2}\bigg|_{x_2=R} + \eta^4\frac{\partial g_2(R_s,x_2,\psi_2,\psi'_R)}{\partial x_2}\bigg|_{x_2=R}\Bigg) \\
&\quad + \frac{2(1+k_{24})}{R^3}\sin^2(\psi'_R R + \psi_2) - \frac{2\psi'_R(1+k_{24})}{R^2}\sin(\psi'_R R + \psi_2)\cos(\psi'_R R + \psi_2) - \frac{2\gamma}{R^2}. \qquad (5)
\end{aligned}
$$

The derivative with respect to the inverse period of density modulations, η, is

$$
\begin{aligned}
\frac{\partial E}{\partial \eta} =& \frac{\Lambda\delta^2}{2R^2}\big(-16\pi^2\eta(g_1(0,R_c,0,\psi'_c) + g_1(R_c,R_s,\psi_1,\psi'_s) + g_1(R_s,R,\psi_2,\psi'_R)) \\
&+ 4\eta^3(g_2(0,R_c,0,\psi'_c) + g_2(R_c,R_s,\psi_1,\psi'_s) + g_2(R_s,R,\psi_2,\psi'_R))\big) \qquad (6)
\end{aligned}
$$

The derivative with respect to the size of the density modulations, δ, is

$$
\begin{aligned}
\frac{\partial E}{\partial \delta} =& \frac{\Lambda\delta}{R^2}\big(8\pi^4R^2 - 8\pi^2\eta^2(g_1(0,R_c,0,\psi'_c) + g_1(R_c,R_s,\psi_1,\psi'_s) + g_1(R_s,R,\psi_2,\psi'_R)) \\
&+ \eta^4(g_2(0,R_c,0,\psi'_c) + g_2(R_c,R_s,\psi_1,\psi'_s) + g_2(R_s,R,\psi_2,\psi'_R))\big) + \omega\delta\left(\frac{3}{2}\delta^2 - 1\right). \qquad (7)
\end{aligned}
$$

The derivative with respect to the core radius size, R_c, is

$$
\begin{aligned}
&\frac{\partial E}{\partial R_c} = \\
&\quad \frac{2}{R^2}\Bigg(\frac{1}{4}\Bigg(\frac{\partial u(0,x_2,\psi'_c)}{\partial x_2}\bigg|_{x_2=R_c} + \frac{\partial u(x_1,R_s,\psi'_s)}{\partial x_1}\bigg|_{x_1=R_c}\Bigg) + \frac{1}{8}\Bigg(\frac{\partial f_1(0,x_2,0,\psi'_c)}{\partial x_2}\bigg|_{x_2=R_c} \\
&\quad + \frac{\partial f_1(x_1,R_s,\psi_1,\psi'_s)}{\partial x_1}\bigg|_{x_1=R_c} + \frac{\partial f_1(R_c,R_s,\xi,\psi'_s)}{\partial \xi}\bigg|_{\xi=\psi_1}\frac{\partial \psi_1}{\partial R_c} + \frac{\partial f_1(R_s,R,\xi,\psi'_R)}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial R_c}\Bigg) \\
&\quad + \frac{1}{2}K_{33}\Bigg(\frac{\partial f_2(0,x_2,0,\psi'_c)}{\partial x_2}\bigg|_{x_2=R_c} + \frac{\partial f_2(x_1,R_s,\psi_1,\psi'_s)}{\partial x_1}\bigg|_{x_1=R_c} + \frac{\partial f_2(R_c,R_s,\xi,\psi'_s)}{\partial \xi}\bigg|_{\xi=\psi_1}\frac{\partial \psi_1}{\partial R_c} \\
&\quad + \frac{\partial f_2(R_s,R,\xi,\psi'_R)}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial R_c}\Bigg) + \frac{1}{4}\Bigg(\frac{\partial v(0,x_2,0,\psi'_c)}{\partial x_2}\bigg|_{x_2=R_c} + \frac{\partial v(x_1,R_s,\psi_1,\psi'_s)}{\partial x_1}\bigg|_{x_1=R_c} \\
&\quad + \frac{\partial v(R_c,R_s,\xi,\psi'_s)}{\partial \xi}\bigg|_{\xi=\psi_1}\frac{\partial \psi_1}{\partial R_c} + \frac{\partial v(R_s,R,\xi,\psi'_R)}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial R_c}\Bigg)\Bigg) \\
&\quad + \frac{\Lambda\delta^2}{2R^2}\Bigg(-8\pi^2\eta^2\Bigg(\frac{\partial g_1(0,x_2,0,\psi'_c)}{\partial x_2}\bigg|_{x_2=R_c} + \frac{\partial g_1(x_1,R_s,\psi_1,\psi'_s)}{\partial x_1}\bigg|_{x_1=R_c} \\
&\quad + \frac{\partial g_1(R_c,R_s,\xi,\psi'_s)}{\partial \xi}\bigg|_{\xi=\psi_1}\frac{\partial \psi_1}{\partial R_c} + \frac{\partial g_1(R_s,R,\xi,\psi'_R)}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial R_c}\Bigg) \\
&\quad + \eta^4\Bigg(\frac{\partial g_2(0,x_2,0,\psi'_c)}{\partial x_2}\bigg|_{x_2=R_c} + \frac{\partial g_2(x_1,R_s,\psi_1,\psi'_s)}{\partial x_1}\bigg|_{x_1=R_c} \\
&\quad + \frac{\partial g_2(R_c,R_s,\xi,\psi'_s)}{\partial \xi}\bigg|_{\xi=\psi_1}\frac{\partial \psi_1}{\partial R_c} + \frac{\partial g_2(R_s,R,\xi,\psi'_R)}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial R_c}\Bigg)\Bigg) \\
&\quad - \frac{2(1+k_{24})}{R^2}\sin(\psi'_R R+\psi_2)\cos(\psi'_R R+\psi_2)\frac{\partial \psi_2}{\partial R_c}.
\end{aligned}
\tag{8}
$$

The derivative with respect to the shelf radius size, R_s, is

$$
\begin{aligned}
\frac{\partial E}{\partial R_s} &= \\
&\frac{2}{R^2}\Bigg(\frac{1}{4}\Bigg(\frac{\partial u(R_c,x_2,\psi_s')}{\partial x_2}\bigg|_{x_2=R_s}+\frac{\partial u(x_1,R,\psi_R')}{\partial x_1}\bigg|_{x_1=R_s}\Bigg)+\frac{1}{8}\Bigg(\frac{\partial f_1(R_c,x_2,\psi_1,\psi_s')}{\partial x_2}\bigg|_{x_2=R_s} \\
&+\frac{\partial f_1(x_1,R,\psi_2,\psi_R')}{\partial x_1}\bigg|_{x_1=R_s}+\frac{\partial f_1(R_s,R,\xi,\psi_R')}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial R_s}\Bigg) \\
&+\frac{1}{2}K_{33}\Bigg(\frac{\partial f_2(R_c,x_2,\psi_1,\psi_s')}{\partial x_2}\bigg|_{x_2=R_s}+\frac{\partial f_2(x_1,R,\psi_2,\psi_R')}{\partial x_1}\bigg|_{x_1=R_s}+\frac{\partial f_2(R_s,R,\xi,\psi_R')}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial R_s}\Bigg) \\
&+\frac{1}{4}\Bigg(\frac{\partial v(R_c,x_2,\psi_1,\psi_s')}{\partial x_2}\bigg|_{x_2=R_s}+\frac{\partial v(x_1,R,\psi_2,\psi_R')}{\partial x_1}\bigg|_{x_1=R_2}+\frac{\partial v(R_s,R,\xi,\psi_R')}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial R_s}\Bigg)\Bigg) \\
&+\frac{\Lambda\delta^2}{2R^2}\Bigg(-8\pi^2\eta^2\Bigg(\frac{\partial g_1(R_c,x_2,\psi_1,\psi_s')}{\partial x_2}\bigg|_{x_2=R_s}+\frac{g_1(x_1,R,\psi_2,\psi_R')}{\partial x_1}\bigg|_{x_1=R_s}+\frac{\partial g_1(R_s,R,\xi,\psi_R')}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial R_s}\Bigg)\Bigg) \\
&+\eta^4\Bigg(\frac{\partial g_2(R_c,x_2,\psi_1,\psi_s')}{\partial x_2}\bigg|_{x_2=R_s}+\frac{\partial g_2(x_1,R,\psi_2,\psi_R')}{\partial x_1}\bigg|_{x_1=R_s}+\frac{\partial g_2(R_s,R,\xi,\psi_R')}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial R_s}\Bigg) \\
&-\frac{2(1+k_{24})}{R^2}\sin(\psi_R' R+\psi_2)\cos(\psi_R' R+\psi_2)\frac{\partial \psi_2}{\partial R_s}. \qquad (9)
\end{aligned}
$$

The derivative with respect to the twist angle gradient in the core, ψ_c', is

$$
\begin{aligned}
\frac{\partial E}{\partial \psi_c'} &= \\
&\frac{2}{R^2}\Bigg(\frac{1}{4}\frac{\partial u(0,R_c,\zeta)}{\partial \zeta}\bigg|_{\zeta=\psi_c'}+\frac{1}{8}\Bigg(\frac{\partial f_1(0,R_c,0,\zeta)}{\partial \zeta}\bigg|_{\zeta=\psi_c'}+\frac{\partial f_1(R_c,R_s,\xi,\psi_s')}{\partial \xi}\bigg|_{\xi=\psi_1}\frac{\partial \psi_1}{\partial \psi_c'} \\
&+\frac{\partial f_1(R_s,R,\xi,\psi_R')}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial \psi_c'}\Bigg)+\frac{1}{2}K_{33}\Bigg(\frac{\partial f_2(0,R_c,0,\zeta)}{\partial \zeta}\bigg|_{\zeta=\psi_c'}+\frac{\partial f_2(R_c,R_s,\xi,\psi_s')}{\partial \xi}\bigg|_{\xi=\psi_1}\frac{\partial \psi_1}{\partial \psi_c'} \\
&+\frac{\partial f_2(R_s,R,\xi,\psi_R')}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial \psi_c'}\Bigg)+\frac{1}{4}\Bigg(\frac{\partial v(0,R_c,0,\zeta)}{\partial \zeta}\bigg|_{\zeta=\psi_c'}+\frac{\partial v(R_c,R_s,\xi,\psi_s')}{\partial \xi}\bigg|_{\xi=\psi_1}\frac{\partial \psi_1}{\partial \psi_c'} \\
&+\frac{\partial v(R_s,R,\xi,\psi_R')}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial \psi_c'}\Bigg)\Bigg)+\frac{\Lambda\delta^2}{2R^2}\Bigg(-8\pi^2\eta^2\Bigg(\frac{\partial g_1(0,R_c,0,\zeta)}{\partial \zeta}\bigg|_{\zeta=\psi_c'} \\
&+\frac{\partial g_1(R_c,R_s,\xi,\psi_s')}{\partial \xi}\bigg|_{\xi=\psi_1}\frac{\partial \psi_1}{\partial \psi_c'}+\frac{\partial g_1(R_s,R,\xi,\psi_R')}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial \psi_c'}\Bigg) \\
&\eta^4\Bigg(\frac{\partial g_2(0,R_c,0,\zeta)}{\partial \zeta}\bigg|_{\zeta=\psi_c'}+\frac{\partial g_2(R_c,R_s,\xi,\psi_s')}{\partial \xi}\bigg|_{\xi=\psi_1}\frac{\partial \psi_1}{\partial \psi_c'} \\
&+\frac{\partial g_2(R_s,R,\xi,\psi_R')}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial \psi_c'}\Bigg)\Bigg)-\frac{2(1+k_{24})}{R^2}\sin(\psi_R' R+\psi_2)\cos(\psi_R' R+\psi_2)\frac{\partial \psi_2}{\partial \psi_c'}. \qquad (10)
\end{aligned}
$$

The derivative with respect to the twist angle gradient in the shelf, ψ'_s, is

$$
\begin{aligned}
&\frac{\partial E}{\partial \psi'_s} = \\
&\quad \frac{2}{R^2}\Bigg(\frac{1}{4}\frac{\partial u(R_c,R_s,\zeta)}{\partial \zeta}\bigg|_{\zeta=\psi'_s} + \frac{1}{8}\Bigg(\frac{\partial f_1(R_c,R_s,\psi_1,\zeta)}{\partial \zeta}\bigg|_{\zeta=\psi'_s} + \frac{\partial f_1(R_c,R_s,\xi,\psi'_s)}{\partial \xi}\bigg|_{\xi=\psi_1}\frac{\partial \psi_1}{\partial \psi'_s} \\
&\quad + \frac{\partial f_1(R_s,R,\xi,\psi'_R)}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial \psi'_s}\Bigg) + \frac{1}{2}K_{33}\Bigg(\frac{\partial f_2(R_c,R_s,\psi_1,\zeta)}{\partial \zeta}\bigg|_{\zeta=\psi'_s} + \frac{\partial f_2(R_c,R_s,\xi,\psi'_s)}{\partial \xi}\bigg|_{\xi=\psi_1}\frac{\partial \psi_1}{\partial \psi'_s} \\
&\quad + \frac{\partial f_2(R_s,R,\xi,\psi'_R)}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial \psi'_s}\Bigg) + \frac{1}{4}\Bigg(\frac{\partial v(R_c,R_s,\psi_1,\zeta)}{\partial \zeta}\bigg|_{\zeta=\psi'_s} + \frac{\partial v(R_c,R_s,\xi,\psi'_s)}{\partial \xi}\bigg|_{\xi=\psi_1}\frac{\partial \psi_1}{\partial \psi'_s} \\
&\quad + \frac{\partial v(R_s,R,\xi,\psi'_R)}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial \psi'_s}\Bigg)\Bigg) + \frac{\Lambda\delta^2}{2R^2}\Bigg(-8\pi^2\eta^2\Bigg(\frac{\partial g_1(R_c,R_s,\psi_1,\zeta)}{\partial \zeta}\bigg|_{\zeta=\psi'_s} \\
&\quad + \frac{\partial g_1(R_c,R_s,\xi,\psi'_s)}{\partial \xi}\bigg|_{\xi=\psi_1}\frac{\partial \psi_1}{\partial \psi'_s} + \frac{\partial g_1(R_s,R,\xi,\psi'_R)}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial \psi'_s}\Bigg) \\
&\quad + \eta^4\Bigg(\frac{\partial g_2(R_c,R_s,\psi_1,\zeta)}{\partial \zeta}\bigg|_{\zeta=\psi'_s} + \frac{\partial g_2(R_c,R_s,\xi,\psi'_s)}{\partial \xi}\bigg|_{\xi=\psi_1}\frac{\partial \psi_1}{\partial \psi'_s} \\
&\quad + \frac{\partial g_2(R_s,R,\xi,\psi'_R)}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial \psi'_s}\Bigg)\Bigg) - \frac{2(1+k_{24})}{R^2}\sin(\psi'_R R+\psi_2)\cos(\psi'_R R+\psi_2)\frac{\partial \psi_2}{\partial \psi'_s}. \qquad (11)
\end{aligned}
$$

Finally, the derivative with respect to the twist angle gradient in surface reconstruction region, ψ'_R, is

$$
\begin{aligned}
&\frac{\partial E}{\partial \psi'_R} = \\
&\quad \frac{2}{R^2}\Bigg(\frac{1}{4}\frac{\partial u(R_s,R,\zeta)}{\partial \zeta}\bigg|_{\zeta=\psi'_R} + \frac{1}{8}\Bigg(\frac{\partial f_1(R_s,R,\psi_2,\zeta)}{\partial \zeta}\bigg|_{\zeta=\psi'_R} + \frac{\partial f_1(R_s,R,\xi,\psi'_R)}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial \psi'_R}\Bigg) \\
&\quad + \frac{1}{2}K_{33}\Bigg(\frac{\partial f_2(R_s,R,\psi_2,\zeta)}{\partial \zeta}\bigg|_{\zeta=\psi'_R} + \frac{\partial f_2(R_s,R,\xi,\psi'_R)}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial \psi'_R}\Bigg) \\
&\quad + \frac{1}{4}\Bigg(\frac{\partial v(R_s,R,\psi_2,\zeta)}{\partial \zeta}\bigg|_{\zeta=\psi'_R} + \frac{\partial v(R_s,R,\xi,\psi'_R)}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial \psi'_R}\Bigg)\Bigg) + \frac{\Lambda\delta^2}{2R^2}\Bigg(-8\pi^2\eta^2\Bigg(\frac{\partial g_1(R_s,R,\psi_2,\zeta)}{\partial \zeta}\bigg|_{\zeta=\psi'_R} \\
&\quad + \frac{\partial g_1(R_s,R,\xi,\psi'_R)}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial \psi'_R}\Bigg) + \eta^4\Bigg(\frac{\partial g_2(R_s,R,\psi_2,\zeta)}{\partial \zeta}\bigg|_{\zeta=\psi'_R} \\
&\quad + \frac{\partial g_2(R_s,R,\xi,\psi'_R)}{\partial \xi}\bigg|_{\xi=\psi_2}\frac{\partial \psi_2}{\partial \psi'_R}\Bigg)\Bigg) - \frac{2(1+k_{24})}{R^2}\sin(\psi'_R R+\psi_2)\cos(\psi'_R R+\psi_2)\Bigg(R+\frac{\partial \psi_2}{\partial \psi'_R}\Bigg).
\end{aligned}
\qquad (12)
$$

3 Detailed calculations

For a general linear function of the form $\psi(r) = \psi'_{ab} r + \psi_0$ in the region $a < r < b$, the two integrals in eqn 3 become

$$\begin{aligned}
&\int_a^b r dr \left[\frac{1}{2} \left(\psi'_{ab} + \frac{\sin(2(\psi'_{ab} r + \psi_0))}{2r} - 1 \right)^2 + \frac{1}{2} K_{33} \frac{\sin^4(\psi'_{ab} r + \psi_0)}{r^2} \right] \\
&\quad = \int_a^b dr \left(\frac{(1-\psi'_{ab})^2}{2} r + \frac{1}{8} \frac{\sin^2(2(\psi'_{ab} r + \psi_0))}{r} - \frac{(1-\psi'_{ab})}{2} \sin(2(\psi'_{ab} r + \psi_0)) + \frac{1}{2} K_{33} \frac{\sin^4(\psi'_{ab} r + \psi_0)}{r} \right) \\
&\quad = \left(\frac{1}{4} u(a, b, \psi'_{ab}) + \frac{1}{8} f_1(a, b, \psi_0, \psi'_{ab}) + \frac{1}{2} K_{33} f_2(a, b, \psi_0, \psi'_{ab}) + \frac{1}{4} v(a, b, \psi_0, \psi'_{ab}) \right) \qquad (13)
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b r dr \big(4\pi^2 - \eta^2 \cos^2(\psi'_{ab} r + \psi_0) \big)^2 \\
&\qquad = \int_a^b dr \big(16\pi^4 r - 8\pi^2 r \cos^2(\psi'_{ab} r + \psi_0) \eta^2 + \eta^4 r \cos^4(\psi'_{ab} r + \psi_0) \big) \\
&\qquad = \big(8\pi^4 (b^2 - a^2) - 8\pi^2 \eta^2 g_1(a, b, \psi_0, \psi'_{ab}) + \eta^4 g_2(a, b, \psi_0, \psi'_{ab}) \big) \qquad (14)
\end{aligned}$$

where I have defined the functions

$$u(x_1, x_2, \zeta) = (1-\zeta)^2 (x_2^2 - x_1^2), \qquad (15a)$$

$$v(x_1, x_2, \xi, \zeta) = \frac{(1-\zeta)}{\zeta} (\cos(2(\zeta x_2 + \xi)) - \cos(2(\zeta x_1 + \xi))), \qquad (15b)$$

$$f_\alpha(x_1, x_2, \xi, \zeta) = \int_{x_1}^{x_2} du \frac{\sin^{2\alpha}\left(\frac{2}{\alpha}(\zeta u + \xi)\right)}{u}, \qquad (15c)$$

$$g_\alpha(x_1, x_2, \xi, \zeta) = \int_{x_1}^{x_2} u \cos^{2\alpha}(\zeta u + \xi) du \qquad (15d)$$

Note that eqn 15d can be integrated analytically, and will be in the computation, but it's form is not very illuminating and so I have left it in integral form.

For $\zeta \ll 1$, I can expand the final three of these equations up to $\mathcal{O}(\zeta^4)$ using trigonometric identities to get

$$
\begin{aligned}
v(x_1, x_2, \xi, \zeta) = {} & -2\sin(2\xi)(x_2 - x_1) + 2(\sin(2\xi)(x_2 - x_1) - \cos(2\xi)(x_2^2 - x_1^2))\zeta \\
& + \frac{2}{3}(3\cos(2\xi)(x_2^2 - x_1^2) + 2\sin(2\xi)(x_2^3 - x_1^3))\zeta^2 \\
& + \frac{2}{3}(-2\sin(2\xi)(x_2^3 - x_1^3) + \cos(2\xi)(x_2^4 - x_1^4))\zeta^3, \qquad (16a) \\
f_1(x_1, x_2, \xi, \zeta) = {} & \sin^2(2\xi)\ln\frac{x_2}{x_1} + 4\zeta(x_2 - x_1)\cos(2\xi)\sin(2\xi) \\
& + 2\zeta^2(x_2^2 - x_1^2)\left(\cos^2(2\xi) - \sin^2(2\xi)\right) - \frac{32}{9}\zeta^3(x_2^3 - x_1^3)\sin(2\xi)\cos(2\xi) \qquad (16b) \\
f_2(x_1, x_2, \xi, \zeta) = {} & \sin^4\xi\ln\frac{x_2}{x_1} + 4\zeta(x_2 - x_1)\sin^3\xi\cos\xi + \zeta^2(x_2^2 - x_1^2)\sin^2\xi(3\cos^2\xi - \sin^2\xi) \\
& + \frac{4}{3}\zeta^3(x_2^3 - x_1^3)\sin\xi\cos\xi\left(\cos^2\xi - 5\sin^2\xi\right) \qquad (16c) \\
g_1(x_1, x_2, \xi, \zeta) = {} & \frac{(x_2^2 - x_1^2)\cos^2\xi}{2} - \frac{2(x_2^3 - x_1^3)\cos\xi\sin\xi}{3}\zeta - \frac{(x_2^4 - x_1^4)(\cos^2\xi - \sin^2\xi)}{4}\zeta^2 \\
& + \frac{4(x_2^5 - x_1^5)\cos\xi\sin\xi}{15}\zeta^3 \\
g_2(x_1, x_2, \xi, \zeta) = {} & \frac{(x_2^2 - x_1^2)\cos^4\xi}{2} - \frac{4(x_2^3 - x_1^3)\cos^3\xi\sin\xi}{3}\zeta - \frac{(x_2^4 - x_1^4)\cos^2\xi(\cos^2\xi - 3\sin^2\xi)}{2}\zeta^2 \\
& - \frac{(x_2^5 - x_1^5)\cos\xi(5\cos^3\xi - 12\sin^3\xi)}{15}\zeta^3 \qquad (16d)
\end{aligned}
$$

The derivatives of these functions are listed below:

$$
\begin{aligned}
\frac{\partial u}{\partial x_1} &= -2(1-\zeta)^2 x_1 \qquad (17a) \\
\frac{\partial u}{\partial x_2} &= 2(1-\zeta)^2 x_2 \qquad (17b) \\
\frac{\partial u}{\partial \xi} &= 0 \qquad (17c) \\
\frac{\partial u}{\partial \zeta} &= -2(1-\zeta)(x_2^2 - x_1^2) \qquad (17d)
\end{aligned}
$$

$$\frac{\partial v}{\partial x_1} = 2(1-\zeta)\sin(2(\zeta x_1+\xi)) \tag{18a}$$

$$\frac{\partial v}{\partial x_2} = -2(1-\zeta)\sin(2(\zeta x_2+\xi)) \tag{18b}$$

$$\frac{\partial v}{\partial \xi} = \begin{cases} -4\cos(2\xi)(x_2-x_1)+(4\cos(2\xi)(x_2-x_1)+4\sin(2\xi)(x_2^2-x_1^2))\zeta, & \zeta=0 \\ -2\frac{(1-\zeta)}{\zeta}(\sin(2(\zeta x_2+\xi))-\sin(2(\zeta x_1+\xi))), & \zeta\neq 0 \end{cases} \tag{18c}$$

$$\frac{\partial v}{\partial \zeta} = \begin{cases} 2\sin(2\xi)(x_2-x_1)-2\cos(2\xi)(x_2^2-x_1^2)+4(\cos(2\xi)(x_2^2-x_1^2)+\frac{2}{3}\sin(2\xi)(x_2^3-x_1^3))\zeta, & \zeta=0 \\ \frac{-2(1-\zeta)}{\zeta}(x_2\sin(2(\zeta x_2+\xi))-x_1\sin(2(\zeta x_1+\xi)))-\frac{1}{\zeta^2}(\cos(2(\zeta x_2+\xi))-\cos(2(\zeta x_1+\xi))), & \zeta\neq 0 \end{cases} \tag{18d}$$

$$\frac{\partial f_\alpha}{\partial x_1} = \begin{cases} \infty, & x_1=0,\xi\neq 0 \\ -\left(\frac{2\zeta}{\alpha}\right)^{2\alpha} x_1^{2\alpha-1}, & x_1=0,\xi=0 \\ -\frac{\sin^{2\alpha}\left(\frac{2}{\alpha}(\zeta x_1+\xi)\right)}{x_1}, & x_1\neq 0 \end{cases} \tag{19a}$$

$$\frac{\partial f_\alpha}{\partial x_2} = \begin{cases} \infty, & x_1=0,\xi\neq 0 \\ \left(\frac{2\zeta}{\alpha}\right)^{2\alpha} x_2^{2\alpha-1}, & x_1=0,\xi=0 \\ \frac{\sin^{2\alpha}\left(\frac{2}{\alpha}(\zeta x_1+\xi)\right)}{x_1}, & x_1\neq 0 \end{cases} \tag{19b}$$

$$\frac{\partial f_\alpha}{\partial \xi} = \begin{cases} \infty, & x_1=0,\xi\neq 0, \\ 4\sin(2\xi)\cos(2\xi)\ln\frac{x_2}{x_1}+8(x_2-x_1)(\cos^2(2\xi)-\sin^2(2\xi))\zeta, & \zeta=0,\alpha=1, \\ 4\sin^3\xi\cos\xi\ln\frac{x_2}{x_1}+4(x_2-x_1)\sin^2\xi(3\cos^2\xi-\sin^2\xi)\zeta, & \zeta=0,\alpha=2, \\ \int_{x_1}^{x_2} du\frac{4\sin^{2\alpha-1}\left(\frac{2}{\alpha}(\zeta u+\xi)\right)\cos\left(\frac{2}{\alpha}(\zeta u+\xi)\right)}{u}, & x_1\neq 0 \end{cases} \tag{19c}$$

$$\frac{\partial f_\alpha}{\partial \zeta} = \begin{cases} 4(x_2-x_1)\cos(2\xi)\sin(2\xi)+4(x_2^2-x_1^2)(\cos^2(2\xi)-\sin^2(2\xi))\zeta, & \zeta=0,\alpha=1 \\ 4(x_2-x_1)\sin^3(\xi)\cos(\xi)+2(x_2^2-x_1^2)\sin^2(\xi)(3\cos^2(\xi)-\sin^2(\xi))\zeta, & \zeta=0,\alpha=2 \\ \frac{1}{\zeta}\left(\sin^{2\alpha}\left(\frac{2}{\alpha}(\zeta x_2+\xi)\right)-\sin^{2\alpha}\left(\frac{2}{\alpha}(\zeta x_1+\xi)\right)\right), & \zeta\neq 0 \end{cases} \tag{19d}$$

$$\frac{\partial g_\alpha}{\partial x_1} = -x_1 \cos^{2\alpha}(\zeta x_1 + \xi) \tag{20a}$$

$$\frac{\partial g_\alpha}{\partial x_2} = x_2 \cos^{2\alpha}(\zeta x_2 + \xi) \tag{20b}$$

$$\frac{\partial g_\alpha}{\partial \xi} = \begin{cases} -(x_2^2 - x_1^2)\cos\xi \sin\xi - \frac{2(x_2^3 - x_1^3)(\cos^2\xi - \sin^2\xi)}{3}\zeta, & \zeta = 0, \alpha = 1, \\ -2(x_2^2 - x_1^2)\cos^3\xi \sin\xi - \frac{4(x_2^3 - x_1^3)(\cos^4\xi - 3\cos^2\xi \sin^2\xi)}{3}\zeta, & \zeta = 0, \alpha = 2, \\ -2\alpha \int_{x_1}^{x_2} u \cos^{2\alpha-1}(\zeta u + \xi)\sin(\zeta u + \xi) du & \zeta \neq 0, \end{cases} \tag{20c}$$

$$\frac{\partial g_\alpha}{\partial \zeta} = \begin{cases} \frac{-2(x_2^3 - x_1^3)\cos\xi \sin\xi}{3} - \frac{(x_2^4 - x_1^4)(\cos^2\xi - \sin^2\xi)}{2}\zeta, & \zeta = 0, \alpha = 1, \\ \frac{-4(x_2^3 - x_1^3)\cos^3\xi \sin\xi}{3} - (x_2^4 - x_1^4)\cos^2\xi(\cos^2\xi - \sin^2\xi)\zeta, & \zeta = 0, \alpha = 2, \\ -2\alpha \int_{x_1}^{x_2} u^2 \cos^{2\alpha-1}(\zeta u + \xi)\sin(\zeta u + \xi) du, & \zeta \neq 0. \end{cases} \tag{20d}$$

4 References