1 Piecewise linear definition

I will define the piecewise linear twist angle as

$$\psi(r) = \begin{cases} \psi_c' r & 0 \le r \le R_c \\ \psi_s' r + (\psi_c' - \psi_s')R_c & R_c < r \le R_s \\ \psi_R' r + (\psi_s' - \psi_R')R_s + (\psi_c' - \psi_s')R_c & R_s < r \le R. \end{cases} \tag{1}$$

Next, I will insert this into the free energy per unit volume,

$$\begin{aligned} E(R,\eta,\delta;\psi(r)) = & \frac{2}{R^2}\int_0^R r dr \left[\frac{1}{2}\left(\psi' + \frac{\sin 2\psi}{2r} - 1\right)^2 + \frac{1}{2}K_{33}\frac{\sin^4\psi}{r^2}\right] \\ & + \frac{\Lambda\delta^2}{2R^2}\int_0^R r dr \left(\frac{4\pi^2}{\cos^2\psi} - \eta^2\right)^2 + \frac{\omega\delta^2}{2}\left(\frac{3}{4}\delta^2 - 1\right) \\ & - \frac{(1+k_{24})}{R^2}\sin\psi(R) + \frac{2\gamma}{R}. \end{aligned} \tag{2}$$

2 Detailed calculations

For a general linear function of the form $\psi(r) = \psi_{ab}' r + \psi_0$ in the region $a < r < b$, the two integrals in eqn 2 become

$$\begin{aligned} & \int_a^b r dr \left[\frac{1}{2}\left(\psi_{ab}' + \frac{\sin(2(\psi_{ab}' r + \psi_0))}{2r} - 1\right)^2 + \frac{1}{2}K_{33}\frac{\sin^4(\psi_{ab}' r + \psi_0)}{r^2}\right] \\ & = \int_a^b dr \left(\frac{(1-\psi_{ab}')^2}{2} r + \frac{1}{8}\frac{\sin^2(2(\psi_{ab}' r + \psi_0))}{r} - \frac{(1-\psi_{ab}')}{2}\sin(2(\psi_{ab}' r + \psi_0)) + \frac{1}{2}K_{33}\frac{\sin^4(\psi_{ab}' r + \psi_0)}{r}\right) \\ & = \left(\frac{1}{4}u(a,b,\psi_{ab}') + \frac{1}{8}f_1(a,b,\psi_0,\psi_{ab}') + \frac{1}{2}K_{33}f_2(a,b,\psi_0,\psi_{ab}') + \frac{1}{4}v(a,b,\psi_0,\psi_{ab}')\right) \end{aligned} \tag{3}$$

and

$$\begin{aligned} & \int_a^b r dr \left(\frac{4\pi^2}{\cos^2(\psi_{ab}' r + \psi_0)} - \eta^2\right)^2 \\ & \qquad = \int_a^b dr \left(\frac{16\pi^4 r}{\cos^4(\psi_{ab}' r + \psi_0)} - \frac{8\pi^2 r}{\cos^2(\psi_{ab}' r + \psi_0)}\eta^2 + \eta^4 r\right) \\ & \qquad = \left(16\pi^4 g_2(a,b,\psi_0,\psi_{ab}') - 8\pi^2\eta^2 g_1(a,b,\psi_0,\psi_{ab}') + \frac{\eta^4}{2}(b^2 - a^2)\right) \end{aligned} \tag{4}$$

where I have defined the functions

$$u(x_1,x_2,\zeta) = (1-\zeta)^2(x_2^2 - x_1^2), \tag{5a}$$

$$v(x_1,x_2,\xi,\zeta) = \frac{(1-\zeta)}{\zeta}(\cos(2(\zeta x_2+\xi)) - \cos(2(\zeta x_1+\xi))), \tag{5b}$$

$$f_\alpha(x_1,x_2,\xi,\zeta) = \int_{x_1}^{x_2} du \frac{\sin^{2\alpha}\left(\frac{2}{\alpha}(\zeta u+\xi)\right)}{u}, \tag{5c}$$

$$g_\alpha(x_1,x_2,\xi,\zeta) = \int_{x_1}^{x_2} du \frac{u}{\cos^{2\alpha}(\zeta u+\xi)}. \tag{5d}$$

For $\zeta \ll 1$, I can expand the final three of these equations up to $\mathcal{O}(\zeta^4)$ using trigonometric identities to get

$$\begin{aligned} v(x_1,x_2,\xi,\zeta) &= -2(1-\zeta)\sin(2\xi)(x_2-x_1) - 2(1-\zeta)\cos(2\xi)(x_2^2-x_1^2)\zeta \\ &\quad + \frac{4}{3}(1-\zeta)\sin(2\xi)(x_2^3-x_1^3)\zeta^2 + \frac{2}{3}\cos(2\xi)(x_2^4-x_1^4)\zeta^3, \end{aligned} \tag{6a}$$

$$\begin{aligned} f_1(x_1,x_2,\xi,\zeta) &= \sin^2(2\xi)\ln\frac{x_2}{x_1} + 4\zeta(x_2-x_1)\cos(2\xi)\sin(2\xi) \\ &\quad + 2\zeta^2(x_2^2-x_1^2)\left(\cos^2(2\xi)-\sin^2(2\xi)\right) - \frac{32}{9}\zeta^3(x_2^3-x_1^3)\sin(2\xi)\cos(2\xi) \end{aligned} \tag{6b}$$

$$\begin{aligned} f_2(x_1,x_2,\xi,\zeta) &= \sin^4\xi\ln\frac{x_2}{x_1} + 4\zeta(x_2-x_1)\sin^3\xi\cos\xi + \zeta^2(x_2^2-x_1^2)\sin^2\xi(\cos^2\xi-\sin^2\xi) \\ &\quad + \frac{4}{3}\zeta^3(x_2^3-x_1^3)\sin\xi\cos\xi\left(\cos^2\xi - 5\sin^2\xi\right) \end{aligned} \tag{6c}$$

$$\begin{aligned} g_1(x_1,x_2,\xi,\zeta) &= \frac{1}{\cos^2\xi}\left(\frac{x_2^2-x_1^2}{2} + \frac{2\zeta(x_2^3-x_1^2)}{3}\tan\xi + \frac{\zeta^2(x_2^4-x_1^4)(3\tan^2\xi+1)}{4}\right. \\ &\quad \left. + \frac{4\zeta^3(x_2^5-x_1^5)(4+3\tan^2\xi)\tan\xi}{15}\right) \end{aligned} \tag{6d}$$

$$\begin{aligned} g_2(x_1,x_2,\xi,\zeta) &= \frac{1}{\cos^4\xi}\left(\frac{x_2^2-x_1^2}{2} + \frac{4\zeta(x_2^3-x_1^3)\tan\xi}{3} + \frac{\zeta^2(x_2^4-x_1^4)(1+5\tan^2\xi)}{2}\right. \\ &\quad \left. + \frac{\zeta^3(x_2^5-x_1^5)(60\tan^2\xi+28)\tan\xi}{15}\right) \end{aligned} \tag{6e}$$

The derivatives of these functions are listed below:

$$\frac{\partial u}{\partial x_1} = -2(1-\zeta)^2 x_1 \tag{7a}$$

$$\frac{\partial u}{\partial x_2} = 2(1-\zeta)^2 x_2 \tag{7b}$$

$$\frac{\partial u}{\partial \xi} = 0 \tag{7c}$$

$$\frac{\partial u}{\partial \zeta} = -2\zeta(1-\zeta)(x_2^2 - x_1^2) \tag{7d}$$

$$\frac{\partial v}{\partial x_1} = 2(1-\zeta)\sin(2(\zeta x_1 + \xi)) \tag{8a}$$

$$\frac{\partial v}{\partial x_2} = -2(1-\zeta)\sin(2(\zeta x_1 + \xi)) \tag{8b}$$

$$\frac{\partial v}{\partial \xi} = \begin{cases} -4\cos(2\xi)(x_2 - x_1) + (4\cos(2\xi)(x_2 - x_1) + 4\sin(2\xi)(x_2^3 - x_1^3))\zeta, & \zeta = 0 \\ -2\frac{(1-\zeta)}{\zeta}(\sin(2(\zeta x_2 + \xi)) - \sin(2(\zeta x_1 + \xi))), & \zeta \neq 0 \end{cases} \tag{8c}$$

$$\frac{\partial v}{\partial \zeta} = \begin{cases} 2\sin(2\xi)(x_2 - x_1) - 2\cos(2\xi)(x_2^2 - x_1^2) + 4(\cos(2\xi)(x_2^2 - x_1^2) + \frac{2}{3}\sin(2\xi)(x_2^3 - x_1^3))\zeta, & \zeta = 0 \\ \frac{-2(1-\zeta)}{\zeta}(x_2\sin(2(\zeta x_2 + \xi)) - x_1\sin(2(\zeta x_1 + \xi))) - \frac{1}{\zeta^2}(\cos(2(\zeta x_2 + \xi)) - \cos(2(\zeta x_1 + \xi))), & \zeta \neq 0 \end{cases} \tag{8d}$$

$$
\frac{\partial f_\alpha}{\partial x_1} = \begin{cases} \infty, & x_1 = 0, \xi \neq 0 \\ -\left(\frac{2\zeta}{\alpha}\right)^{2\alpha} x_1^{2\alpha-1}, & x_1 = 0, \xi = 0 \\ -\frac{\sin^{2\alpha}\left(\frac{2}{\alpha}(\zeta x_1+\xi)\right)}{x_1}, & x_1 \neq 0 \end{cases} \tag{9a}
$$

$$
\frac{\partial f_\alpha}{\partial x_2} = \begin{cases} \infty, & x_1 = 0, \xi \neq 0 \\ \left(\frac{2\zeta}{\alpha}\right)^{2\alpha} x_2^{2\alpha-1}, & x_1 = 0, \xi = 0 \\ \frac{\sin^{2\alpha}\left(\frac{2}{\alpha}(\zeta x_1+\xi)\right)}{x_1}, & x_1 \neq 0 \end{cases} \tag{9b}
$$

$$
\frac{\partial f_\alpha}{\partial \xi} = \begin{cases} \infty, & x_1 = 0, \xi \neq 0 \\ \int_{x_1}^{x_2} du \frac{4\sin\left(\frac{2}{\alpha}(\zeta u+\xi)\right)\cos\left(\frac{2}{\alpha}(\zeta u+\xi)\right)}{u}, & x_1 \neq 0 \end{cases} \tag{9c}
$$

$$
\frac{\partial f_\alpha}{\partial \zeta} = \begin{cases} 4(x_2-x_1)\cos(2\xi)\sin(2\xi) + 4(x_2^2-x_1^2)(\cos^2(2\xi)-\sin^2(2\xi))\zeta, & \zeta = 0, \alpha = 1 \\ 4(x_2-x_1)\sin^3(\xi)\cos(\xi) + 2(x_2^2-x_1^2)\sin^2(\xi)(3\cos^2(\xi)-\sin^2(\xi))\zeta, & \zeta = 0, \alpha = 2 \\ \frac{1}{4\zeta}\left(\sin\left(\frac{2}{\alpha}(\zeta x_2+\xi)\right) - \sin\left(\frac{2}{\alpha}(\zeta x_1+\xi)\right)\right), & \zeta \neq 0 \end{cases} \tag{9d}
$$

3 References